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The Lane-Emden Function and Nonlinear Eigenvalues Problems


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The Lane-Emden Function and Nonlinear Eigenvalues Problems

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RÉSUMÉ. — Nous considérons un problème aux valeurs propres, semi-linéaire elliptique, sur une boule de $\mathbb{R}^n$ et montrons que ces valeurs et fonctions propres peuvent s’obtenir à partir de la fonction de Lane-Emden.

ABSTRACT. — We consider a semilinear elliptic eigenvalues problem on a ball of $\mathbb{R}^n$ and show that all the eigenfunctions and eigenvalues, can be obtained from the Lane-Emden function.

1. Introduction

We consider the problem

\[
(P_\lambda^\alpha) \begin{cases}
\Delta u + \lambda (1 + u)^\alpha = 0, & \text{in } B_1 \\
u > 0, & \text{in } B_1 \\
u = 0, & \text{on } \partial B_1 
\end{cases}
\]

where $B_1$ is the unit ball of $\mathbb{R}^n$, $n \geq 3$, $\lambda > 0$ and $\alpha > 1$.

This problem arises in many physical models like the nonlinear heat generation and the theory of gravitational equilibrium of polytropic stars (cf. [2] and [11]). It is well known (cf. [2], [10], [12]) that there exists a critical constant $\lambda^*(\alpha)$, such that $(P_\lambda^\alpha)$ admits, at least, one solution if $0 < \lambda < \lambda^*(\alpha)$ and no solution if $\lambda > \lambda^*(\alpha)$. We deal here with these critical constants and the corresponding eigenfunctions.

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Let $\phi$ be the Lane-Emden function (cf. [1], [5], [6], [15]) in the $n$-dimensional space and $r_0$ the first "zero" of $\phi$, we show that

$$\lambda^*(\alpha) = \max_{r \in [0, r_0]} r^2 \phi^{\alpha-1}(r).$$

We use this formula to compute $\lambda^*(\alpha)$, when $\alpha$ is the Critical Sobolev Exponent. We also extend, to the subcritical case, an estimate of $\lambda^*(\alpha)$ given in [10] and show qualitative properties of the eigenfunctions.

In the Appendix, we show how to approximate $\phi$, so one can use numerical approaches (Maple or Matlab) to get estimates of $\lambda^*(\alpha)$.

2. Scalings of the Lane-Emden function as solutions

When $0 < \lambda \leq \lambda^*(\alpha)$, it is known that any regular solution of $(P^\alpha_\lambda)$ is radial and the minimal one is stable and analytical (cf.[8], [12]).

**Proposition 2.1.** — Let $u$ be a regular solution of $(P^\alpha_\lambda)$, then

$$u(r) = (1 + u(0)) \phi \left( \sqrt{\lambda}(1 + u(0))^{\frac{\alpha-1}{2}} r \right) - 1, \ \forall \ r \in [0, 1]$$

where $\phi$ is the Lane-Emden function, in the $n$-dimensional space.

**Proof.** — The Lane-Emden function (cf. [1], [5], [6], [15]) is the solution of

$$(L - E) \left\{ 
\frac{\phi''(r)}{r} + \frac{n-1}{r} \phi'(r) + \phi(r) |\phi(r)|^{\alpha-1} = 0, \\
\phi(0) = 1, \ \phi'(0) = 0.
\right\}$$

The proof of the proposition is quite immediate.

3. The Subcritical Case

Let us consider the problem $(P^\alpha_\lambda)$, with $1 < \alpha < \frac{n+2}{n-2}$. Let $\phi$ be the Lane-Emden function.

**Proposition 3.1.** — There exists $r_0 > 0$, such that $\phi(r_0) = 0$, $\phi(r) > 0$, $\forall r \in [0, r_0]$ and

$$\lambda^*(\alpha) = \max_{\rho \in [0, r_0]} \rho^2 \phi^{\alpha-1}(\rho).$$

We also have

$$\lambda^*(\alpha) \geq \frac{2}{(\alpha - 1)^2} (\alpha(n-2) - n), \ i f \ \frac{n}{n-2} < \alpha < \frac{n+2}{n-2}.$$
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\textbf{Proof.} — As \( \phi(0) > 0 \), we infer that \( \phi > 0 \), on a maximal interval \([0, r_0[\). The problem
\[
\begin{aligned}
\begin{cases}
\Delta u + u^\alpha = 0, & \text{in } \mathbb{R}^n \\
u > 0, & \text{in } \mathbb{R}^n
\end{cases}
\end{aligned}
\]
does not admit a solution (cf.\([4]\)), so we infer that \( r_0 < \infty \) and \( \phi(r_0) = 0 \).

Let us put
\[
\psi_\rho(r) = \frac{\phi(\rho r) - \phi(\rho)}{\phi(\rho)}, \quad \forall \ r \in [0, 1],
\]
with \( 0 < \rho < r_0 \), then \( \psi_\rho \) is a solution of \((P^\alpha_\lambda)\), with \( \lambda = \rho^{2\alpha - 1}(\rho) \). We infer that
\[
\max_{\rho \in [0, r_0]} \rho^2 \phi_{\alpha - 1}(\rho) \leq \lambda^* (\alpha).
\]
Let us suppose that
\[
\max_{\rho \in [0, r_0]} \rho^2 \phi_{\alpha - 1}(\rho) < \lambda^* (\alpha),
\]
if \( u_{\lambda^*(\alpha)} \) is the unique solution of \((P^\alpha_\lambda^*(\alpha))(cf.\([10]\))\), one can use Proposition
1 to show that
\[
\begin{aligned}
u_{\lambda^*(\alpha)}(r) &= \left(1 + u_{\lambda^*(\alpha)}(0)\right) \left(\frac{\phi \left(\lambda^*(\alpha)\right)^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha - 1}{2}} r}{1 + u_{\lambda^*(\alpha)}(0)}\right).
\end{aligned}
\]
Let us put \( \rho_{\lambda^*(\alpha)} = \left(\lambda^*(\alpha)\right)^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha - 1}{2}} \). As \( u_{\lambda^*(\alpha)} \geq 0 \), we infer that \( \rho_{\lambda^*(\alpha)} < r_0 \). As \( u_{\lambda^*(\alpha)}(1) = 0 \), we infer that
\[
\frac{1}{1 + u_{\lambda^*(\alpha)}(0)} = \phi \left(\lambda^*(\alpha)\right)^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha - 1}{2}}.
\]
So we get
\[
u_{\lambda^*(\alpha)}(r) = \frac{\phi(\rho_{\lambda^*(\alpha)} r) - \phi(\rho_{\lambda^*(\alpha)})}{\phi(\rho_{\lambda^*(\alpha)})} \quad \text{and} \quad \lambda^*(\alpha) = \left(\rho_{\lambda^*(\alpha)}\right)^2 \phi_{\alpha - 1}(\rho_{\lambda^*(\alpha)}).
\]
The last equality leads to a contradiction.

To prove the last statement, we use the fact that the maximum here is achieved at a unique \( r_\alpha \) (see the next lemma). So we get
\[
\phi'(r_\alpha) = -\frac{2}{(\alpha - 1)r_\alpha} \phi(r_\alpha), \quad \text{and}
\]
\[
\phi_{\alpha - 3}(r_\alpha) \left(2\phi^2(r_\alpha) + 4r_\alpha(\alpha - 1)\phi(r_\alpha)\phi'(r_\alpha) + (\alpha - 1)r_\alpha^2 \left(\phi'(r_\alpha)\right)^2 + \phi(r_\alpha)\phi''(r_\alpha)\right) \leq 0.
\]
We first replace $\phi''(r_\alpha)$ by its value from $(L - E)$ and then $\phi'(r_\alpha)$, from the previous equality, to get

$$\phi^{\alpha-1}(r_\alpha) \left(- (\alpha - 1) \lambda^*(\alpha) + 2(n - 4) + 4\frac{\alpha - 2}{\alpha - 1}\right) \leq 0.$$  

Simplifying, one gets the estimate.

**Remark 3.2.** — The last statement in Proposition 2 is also true for $\alpha \geq \frac{n + 2}{n - 2}$, with the same proof, provided that $\sup_{r \in \mathbb{R}_+} r^2 \phi^{\alpha-1}(r)$ is attained (see the next Proposition 6); this has been proved in [10], using sophisticated arguments.

**Lemma 3.3.** — Let us put $g(r) = r^2 \phi^{\alpha-1}(r)$, $r \in [0, r_0]$, there exists $\rho_0 \in [0, r_0]$ such that $g$ is increasing on $[0, \rho_0]$ and decreasing on $[\rho_0, r_0]$.

**Proof.** — Let $\rho$ be an arbitrary positive constant with $\rho < r_0$, then, as we have already mentioned $\psi_\rho$ is a solution of $(P_\gamma^\alpha)$, where $\gamma = g(\rho)$. As $g'(r) = r \phi^{\alpha-2}(r) (2 \phi(r) + (\alpha - 1) r \phi'(r))$, we infer that $g$ is increasing on a maximal interval $I_0 \subset [0, r_0]$ with $0 \in I_0$.

Using Proposition 2, there exists $\rho_0 \in [0, r_0]$, such that $g(\rho_0) = \max_{r \in [0, r_0]} g(r) = \lambda^*(\alpha)$. This $\rho_0$ is unique, otherwise, if there exists $\lambda \in (0, r_0]$, such that $g(\lambda) = \max_{r \in [0, r_0]} g(r) = \lambda^*(\alpha)$, then $\psi_{\rho_0}$ and $\psi_\lambda$ are both solutions of the problem $(P_{\lambda^*(\alpha)}^\alpha)$. As $\phi$ is decreasing on $[0, r_0]$, we infer that $\psi_{\rho_0}(0) = \frac{1 - \phi(\rho_0)}{\phi(\rho_0)} \neq \frac{1 - \phi(\lambda)}{\phi(\lambda)} = \psi_\lambda(0)$. So we get two different solutions of the problem $(P_{\lambda^*(\alpha)}^\alpha)$. This leads to a contradiction (cf. [10]).

As $g(r_0) = 0$, we infer that $I_0 \neq [0, r_0]$. Let us put $\delta = \sup I_0$. The function $g$ can’t be constant on a nontrivial interval $J \subset [\delta, r_0]$, for if $g(r) = c$ in $J$, then for every $\lambda \in J$, $\psi_\lambda$ is a solution of $(P_{c}^\alpha)$. As $\psi_{\lambda_1}(0) \neq \psi_{\lambda_2}(0)$, if $\lambda_1, \lambda_2 \in J$ and $\lambda_1 \neq \lambda_2$, we infer that the problem $(P_{c}^\alpha)$ admits an infinity of solutions. This leads again to a contradiction (cf. [10]).

So if $g$ is not decreasing on $[\delta, r_0]$, then there exists $\beta_1$ and $\beta_2$ with $r_0 > \beta_2 > \beta_1 > \delta$, such that $g$ is decreasing on $[\delta, \beta_1]$ and increasing on $[\beta_1, \beta_2]$. Let us put $c_0 = \min(g(\beta_1), g(\beta_2))$, then $c_0 > g(\beta_1)$. Let us choose $c \in ]g(\beta_1), c_0[$, so the problem $g(t) = c$ admits at least three different solutions $\lambda_i \in [0, \beta_2]$, $1 \leq i \leq 3$. As $\psi_{\lambda_i}(0) \neq \psi_{\lambda_j}(0)$, if $i \neq j$, $1 \leq i, j \leq 3$, we obtain three solutions for the problem $(P_{c}^\alpha)$. So we get a contradiction.

We conclude that $g$ is increasing on $[0, \delta]$, decreasing on $[\delta, r_0]$ and $\delta = \rho_0$.

**Proposition 3.4.** — If $\lambda = \lambda^*(\alpha)$, there exists a unique $\rho_{\lambda^*(\alpha)} \in [0, r_0]$, such that
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\[ \lambda^*(\alpha) = \left( \rho_{\lambda^*(\alpha)} \right)^2 \phi^{\alpha - 1}(\rho_{\lambda^*(\alpha)}) \] and the unique solution \( u_{\lambda^*(\alpha)} \) of \( (P^\alpha_{\lambda^*(\alpha)}) \)

is

\[ u_{\lambda^*(\alpha)}(r) = \frac{\phi(\rho_{\lambda^*(\alpha)} r) - \phi(\rho_{\lambda^*}(\alpha))}{\phi(\rho_{\lambda^*(\alpha)})} = \psi_{\rho_{\lambda^*(\alpha)}}(r), \quad \forall \ r \in [0, 1]. \]

When \( 0 < \lambda < \lambda^*(\alpha) \), there exist exactly two constants \( r_\lambda \) and \( \rho_\lambda \), such that \( 0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0 \), \( \lambda = r_\lambda^2 \phi^{\alpha - 1}(r_\lambda) = r_\lambda^2 \phi^{\alpha - 1}(\rho_\lambda) \)

and the only two solutions of \( (P^\alpha_\lambda) \) are

\[ u_\lambda = \psi_{r_\lambda}, \quad v_\lambda = \psi_{\rho_\lambda}; \]

the minimal one (cf. [2]) is \( u_\lambda, \lim_{\lambda \to 0} u_\lambda = 0 \) in \( C^0(B_1) \) and

\[ \lim_{\lambda \to 0} v_\lambda(r) = \infty, \quad \forall \ r \in [0, 1]. \]

**Proof.** — Using Proposition 2 and Lemma 1, one infers that the only solution of \( (P^\alpha_{\lambda^*(\alpha)}) \) is \( \psi_{\rho_0} \). We put \( \rho_{\lambda^*(\alpha)} = \rho_0 \). If \( 0 < \lambda < \lambda^*(\alpha) \), using the lemma again, we infer that \( g(t) = \lambda \) admits exactly two solutions \( r_\lambda \) and \( \rho_\lambda \), with \( 0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0 \). Let us put \( u_\lambda = \psi_{r_\lambda} \) and \( v_\lambda = \psi_{\rho_\lambda} \), \( u_\lambda(0) \neq v_\lambda(0) \). These two functions \( u_\lambda \) and \( v_\lambda \) are solutions of the the problem \( (P^\alpha_\lambda) \), which admits only two ones (cf. [10]).

As \( \phi \) is decreasing on \( [0, r_0] \), one can verify that \( u_\lambda(0) < v_\lambda(0) \), so we infer that the minimal solution (cf. [2]) is \( u_\lambda \).

As \( \lambda = r_\lambda^2 \phi^{\alpha - 1}(r_\lambda) = r_\lambda^2 \phi^{\alpha - 1}(\rho_\lambda), \) \( 0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0 \), we get

\[ \lim_{\lambda \to 0} r_\lambda = 0, \quad \lim_{\lambda \to 0} \rho_\lambda = r_0, \quad \lim_{\lambda \to 0} u_\lambda(r) = \lim_{r_\lambda \to 0} \frac{\phi(r_\lambda r) - \phi(\rho_\lambda)}{\phi(\rho_\lambda)} - 1 = 0, \quad \text{and} \quad \lim_{\lambda \to 0} v_\lambda(r) = \lim_{r_\lambda \to 0} \frac{\phi(r_\lambda r) - \phi(\rho_\lambda)}{\phi(\rho_\lambda)} = \phi(r_\lambda r) \left( \lim_{r_\lambda \to 0} \frac{1}{\phi(\rho_\lambda)} \right) = \infty, \quad \forall \ r \in [0, 1]. \]

**4. The Critical Sobolev Exponent Case**

In this section, we suppose that \( \alpha = \frac{n+2}{n-2} \) and \( n \geq 3 \).

Let us consider the following problem

\[ (P^\alpha) \left\{ \begin{array}{ll} \Delta u + u^\alpha = 0, & \text{in } \mathbb{R}^n \\ u > 0, & \text{in } \mathbb{R}^n. \end{array} \right. \]

**Remark 4.1.** — Every radially symmetrical solution of \( (P^\alpha) \) verifies \( \lim_{r \to \infty} u(r) = 0 \) (cf. [9]).

Following the method of Pohozaev in [14], the problem

\[ (Q^\alpha) \left\{ \begin{array}{ll} u''(r) + \frac{n-1}{r} u'(r) + u^\alpha(r) = 0, & \forall \ r > 0 \\ u > 0, \ u(0) = 1, \ u'(0) = 0 \end{array} \right. \]

admits a solution \( \phi \).
Lemma 4.2. — Let \( u \) be a radially symmetrical regular solution of \((P^\alpha)\), then
\[
u(r) = u(0) \phi \left( u(0) \frac{\alpha-1}{2} r \right).
\]

Proof. — This proof is immediate.

Lemma 4.3. — Let us put \( g(r) = r^2 \phi^{\alpha-1}(r), \ r \in \mathbb{R}_+ \), then there exists \( r_0 > 0 \), such that \( g \) is increasing on \([0, r_0]\), decreasing on \([r_0, \infty[\), with \( \lim_{r \to \infty} g(r) = 0 \).

Proof. — As we have already mentioned, \( g \) is increasing near 0. Let us assume that \( g \) is nondecreasing on \([0, \infty[\), then we have two possibilities
\[
\lim_{r \to \infty} g(r) = \infty \text{ or } \lim_{r \to \infty} g(r) = c, \quad 0 < c < \infty.
\]
For every \( \rho > 0 \), \( \psi_\rho \) is a solution of \((P^\alpha_\gamma)\), with \( \gamma = \rho^2 \phi^{\alpha-1}(\rho) = g(\rho) \). We infer (cf. [2], [10]) that \( g(r) \leq \lambda^*(\alpha), \ \forall \ r > 0 \), so the first limit becomes impossible.

In the second case, we have two subcases: \( c \) is achieved or not.

If \( c \) is not achieved, then \( \forall \ l \) such that \( 0 < l < c \), there exists \( r_l > 0 \) such that \( g(r_l) = l \). One can verify that \( \forall \ 0 < l < c \), the problem \((P^\alpha_l)\) admits the solution \( \psi_{r_l} \), so we infer that \( c \leq \lambda^*(\alpha) \). Let \( u \) be a radially symmetrical solution (cf. [2], [10] and [3]) of \((P^\alpha_0)\). As in the proof of Proposition 2, one can verify that
\[
u = \psi_\rho, \ \rho = \sqrt{c} \left( 1 + u(0) \right)^{\frac{\alpha-1}{2}} \text{ and } \frac{1}{1 + u(0)} = \phi(\rho).
\]
As \( c = \rho^2 \phi^{\alpha-1}(\rho) = g(\rho) \), we get a contradiction.

Let us suppose that \( c \) is achieved, as \( g \) is assumed to be nondecreasing, there exists \( r_0 \) such that \( g(r) = c, \ \forall \ r \geq r_0 \). Let us choose, an arbitrary constant \( \rho > 0 \) such that \( \rho \geq r_0 \). The function \( \psi_\rho \) is a solution of the problem \((P^\alpha_\gamma)\), where \( \gamma = \rho^2 \phi^{\alpha-1}(\rho) = g(\rho) = c, \ \forall \ \rho \geq r_0 \). This means that this problem, with such a \( \gamma \), admits an infinity of solutions \( \psi_\rho \); this leads to a contradiction (cf. [2], [10]). So \( g \) is not nondecreasing on \([0, \infty[\). As \( g \) can’t be constant on a nontrivial interval, we deduce that there exists positive constants \( r_1 \) and \( r_2 \), such that \( r_1 < r_2 \), with \( g \) is increasing on \([0, r_1]\) and decreasing on a maximal interval \([r_1, r_2]\). Let us suppose that \( g \) increases again on \([r_2, r_3]\), with \( r_2 < r_3 \). If \( \gamma \in \text{[}\ g(r_2), \min (g(r_1), g(r_3))\text{[} \), then \( g(r) = \gamma \) admits, at least, three roots, so the problem \((P^\alpha_\gamma)\) admits, at least, three solutions; this gives again a contradiction (cf. [10]).
Finally, we get the existence of \( r_0 > 0 \), such that \( g \) is increasing on \([0, r_0]\) and decreasing on \([r_0, \infty[\). As \( g > 0 \), we infer that \( \lim_{r \to \infty} g(r) = c_0 \geq 0 \). If \( c_0 > 0 \), then for every \( c \in ]0, c_0[ \), there exists a unique \( \rho_c \in \mathbb{R}_+ \), verifying \( g(\rho_c) = c \). As \( c < \lambda^*(\alpha) \), the problem \((P^\alpha_c)\) admits exactly two solutions (cf. [10]). One of these two solutions is \( \psi_{\rho_c} \). Let \( u_c \) be the other one, then, using Proposition 2 again, we get

\[
u_c(r) = \psi_\gamma, \quad \gamma = c^{\frac{1}{2}} (1 + u_c(0))^{\frac{\alpha - 1}{2}} = c^{\frac{1}{2}} \phi^{1 - \alpha} \left( c^{\frac{1}{2}} (1 + u_c(0))^{\frac{\alpha - 1}{2}} \right).
\]

So we infer that \( c = g(\gamma) \). As the two solutions are different, \( \rho_c \neq \gamma \) and \( \gamma \) is another root of \( g(r) = c \). This gives a contradiction and proves that necessarily \( c = 0 \). This ends the proof of the lemma.

**PROPOSITION 4.4.** — **Let us assume** \( \alpha = \frac{n+2}{n-2}, \ n \geq 3 \), **then**

\[
\lambda^*(\alpha) = \max_{r \in [0, \infty[} g(r).
\]

**Proof.** — Let \( \gamma = g(\rho) = \rho^2 \phi^{\alpha - 1}(\rho), \rho \in \mathbb{R}_+ \), we have seen that \( \psi_\rho \) is a solution of \((P^\alpha_\gamma)\). So we infer that \( g(\rho) \leq \lambda^*(\alpha) \), \( \forall \rho \in \mathbb{R}_+ \).

Let us suppose that

\[
\max_{r \in [0, \infty[} g(r) < \lambda^*(\alpha)
\]

and let \( u \) be the unique solution (cf. [10]) of \((P^\alpha_{\lambda^*(\alpha)})\). As in the proof of Proposition 2, we get that \( u = \psi_\rho \) and \( \lambda^*(\alpha) = g(\rho) \). This gives a contradiction.

**PROPOSITION 4.5.** — **We have** \( \lambda^*(\alpha) = \frac{n(n-2)}{4} \). **There exists a unique** \( r_{\lambda^*(\alpha)} = \sqrt{n(n-2)} \), **such that** \( \lambda^*(\alpha) = r_{\lambda^*(\alpha)}^2 \phi^{\alpha - 1}(r_{\lambda^*(\alpha)}) \) **and a unique solution of** \((P^\alpha_{\lambda^*(\alpha)})\)

\[
u_{\lambda^*(\alpha)} = \psi_{r_{\lambda^*(\alpha)}}.
\]

**If** \( 0 < \lambda < \lambda^*(\alpha) \), **there exist exactly two constants**

\[
r_{\lambda} = \frac{1 - \frac{2\lambda}{n(n-2)}}{(n(n-2))^{-1} \sqrt{2\lambda}} \quad \text{and} \quad \rho_{\lambda} = \frac{1 - \frac{2\lambda}{n(n-2)} + \sqrt{1 - \frac{4\lambda}{n(n-2)}}}{(n(n-2))^{-1} \sqrt{2\lambda}}
\]

such that \( 0 < r_{\lambda} < r_{\lambda^*(\alpha)} < \rho_{\lambda}, \ \lambda = g(r_{\lambda}) = g(\rho_{\lambda}) \) **and the only two solutions of** \((P^\alpha_{\lambda})\) **are**

\[
u_{\lambda} = \psi_{r_{\lambda}} \quad \text{and} \quad \nu_{\lambda} = \psi_{\rho_{\lambda}},
\]

**the minimal one** (cf. [2]) **is** \( u_\lambda; \lim_{\lambda \to 0} u_\lambda = 0, \ \text{in} \ C^0(\overline{B_1}) \) **and** \( \lim_{\lambda \to 0} u_\lambda(r) = r^{2-n} - 1, \ \forall \ r \in [0, 1] \).
Proof. — One can use Lemma 3 to get the existence (and the uniqueness) of $r_{\lambda^*(\alpha)} = r_0$, $r_\lambda$ and $\rho_\lambda$. It is then easy to verify that $\psi_{r_{\lambda^*(\alpha)}}$ is a solution of $(P_{\lambda^*(\alpha)}^\alpha)$, $u_\lambda = \psi_{r_\lambda}$ and $v_\lambda = \psi_{\rho_\lambda}$ are solutions of $(P_\lambda^\alpha)$. The problem $(P_\alpha)$ admits only two solutions (cf. [10]), as $\phi$ is decreasing on $\mathbb{R}_+$, one can verify that $u_\lambda(0) < v_\lambda(0)$, so $u_\lambda \neq v_\lambda$. We conclude that $u_\lambda$ and $v_\lambda$ are the only solutions of $(P_\alpha)$ and the minimal one (cf. [2]) is $u_\lambda$.

Let us compute the constants $r_{\lambda^*(\alpha)}$, $r_\lambda$ and $\rho_\lambda$.

It is well known (cf. [13]) that, if $\alpha = \frac{n+2}{n-2}$, the problem $(Q^\alpha)$ admits the continuum of spherically symmetrical "instantons"

$$u_\gamma(r) = \gamma^{\frac{n+2}{2}} (n(n-2))^{\frac{n+2}{4}} (\gamma^2 + r^2)^{-\frac{2n}{2}} \cdot \gamma > 0.$$ 

Let us fix $\gamma > 0$, so $u_\gamma(0) = \gamma^{\frac{2-n}{2}} (n(n-2))^{\frac{n+2}{4}}$. Using Lemma 2, we get the expression of the Lane-Emden function

$$\phi(r) = \frac{1}{u_\gamma(0)} u_\gamma (u_\gamma(0)^{-\frac{n-2}{2}} r) = \left( 1 + \frac{r^2}{n(n-2)} \right)^{\frac{2-n}{2}}.$$ 

As $\alpha - 1 = \frac{n+2}{n-2} - 1 = \frac{4}{n-2}$, we infer that

$$g(r) = r^2 \phi^{\alpha-1}(r) = r^2 \left( 1 + \frac{r^2}{n(n-2)} \right)^{-2}.$$ 

Using Proposition 4, a direct calculation gives

$$\lambda^*(\alpha) = \max_{r>0} r^2 \left( 1 + \frac{r^2}{n(n-2)} \right)^{-2} = r^2 \left( 1 + \frac{r^2}{n(n-2)} \right)^{-2} \bigg|_{r=r_{\lambda^*(\alpha)} = \sqrt{n(n-2)}} = \frac{n(n-2)}{4}.$$ 

In [7], the previous constant has been computed, using the Pohozaev Identity. If $0 < \lambda < \lambda^*(\alpha)$, the equation $g(r) = \lambda$ admits two positive roots

$$r_\lambda = \sqrt{1 - \frac{2\lambda}{n(n-2)} - \sqrt{1 - \frac{4\lambda}{n(n-2)}}} \quad \text{and} \quad \rho_\lambda = \sqrt{1 - \frac{2\lambda}{n(n-2)} + \sqrt{1 - \frac{4\lambda}{n(n-2)}}}.$$ 

This gives us $u_\lambda = \psi_{r_\lambda}$ and $v_\lambda = \psi_{\rho_\lambda}$; as $r_\lambda < \rho_\lambda$, we get $u_\lambda(0) < v_\lambda(0)$, so $u_\lambda$ is the minimal solution.
As \( \lambda = r^2 \phi^{\alpha-1}(r_\lambda) = \rho^2 \phi^{\alpha-1}(\rho_\lambda) \), 0 < \( r_\lambda < \lambda^*(\alpha) < \rho_\lambda < \infty \), one can verify that 
\[ \lim_{\lambda \to 0} r_\lambda = 0, \lim_{\lambda \to 0} \rho_\lambda = \infty, \lim_{\lambda \to 0} u_\lambda = 0, \text{ in } C^0(B_1) \text{ and} \]
\[ \lim_{\lambda \to 0} \psi_\lambda(0) = \lim_{\rho_\lambda \to \infty} \frac{\phi(\rho_\lambda r)}{\phi(\rho_\lambda)} = 1 = r^{2-n} - 1, \forall r \in [0,1]. \]

5. The Supercritical Case

We consider here the case \( \alpha > \frac{n+2}{n-2}, n \geq 3 \). Let us put 
\[ f(\alpha) = \frac{4\alpha}{\alpha - 1} + 4\sqrt{\frac{\alpha}{\alpha - 1}}, \forall \alpha > 1. \]
Let’s first detail a condition, \( f(\alpha) > n - 2 \), used in [10].

**Lemma 5.1.** — If \( 3 \leq n \leq 10 \) and \( \alpha > \frac{n+2}{n-2} \) or \( n > 10 \) and \( \frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}+4} \), then \( f(\alpha) > n - 2 \). If \( n > 10 \) and \( \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}+4} \leq \alpha \), then \( f(\alpha) \leq n - 2 \).

**Proof.** — Let us put \( p(t) = 4t^2 + 4t \) and \( u = \sqrt{\frac{\alpha}{\alpha - 1}} \), so we get \( f(\alpha) = p(u) \). The only positive root of \( p(t) = n - 2 \), is \( t_0 = \frac{\sqrt{n-1}-1}{\sqrt{2}} \) and the equation \( u = \frac{\sqrt{n-1}-1}{\sqrt{2}} \) has the only solution \( \alpha_0 = \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}+4} \). But \( \alpha_0 > 0 \), if and only if \( n > 10 \).

For every \( \alpha > \frac{n+2}{n-2} \), we have \( \alpha > 1 \) so we get \( \sqrt{\frac{\alpha}{\alpha - 1}} > 1 > \frac{\sqrt{n-1}-1}{\sqrt{2}} \), if \( 3 \leq n \leq 10 \). We infer that \( f(\alpha) > n - 2 \), if \( 3 \leq n \leq 10 \).

If \( n > 10 \), we have \( \alpha_0 > \frac{n+2}{n-2} > 1 \), one can verify that if \( \frac{n+2}{n-2} < \alpha < \alpha_0 \), then \( f(\alpha) > n - 2 \) and \( f(\alpha) \leq n - 2 \), if \( \alpha \geq \alpha_0 \).

**Proposition 5.2.** — Let us put \( \lambda_s = \frac{2}{(\alpha-1)^2} (\alpha(n-2) - n) \).
If \( 3 \leq n \leq 10 \) and \( \frac{n+2}{n-2} < \alpha \) or \( n > 10 \) and \( \frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}+4} \) then
\[ \lambda^*(\alpha) = \max_{\mathbb{R}^+_+} g(r), \lambda^*(\alpha) > \lambda_s \text{ and } \phi(r) \sim \lambda_s^{\frac{1}{2-\alpha}} r^{\frac{2}{1-\alpha}}, \text{ as } r \to \infty. \]

If \( (\rho_i) \) is an increasing sequence of positive reals, such that \( (\psi_{\rho_i}) \) are solutions of \( (P^n_{\lambda_s}) \) and \( \lim_{i \to \infty} \rho_i = \infty \), then \( \lim_{i \to \infty} \psi_{\rho_i}(r) = \lambda_s^{\frac{1}{2-\alpha}} (r^{\frac{2}{1-\alpha}} - 1), \forall r \in [0,1]. \)
If $n > 10$ and $\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leq \alpha$ then

$$
\lambda^*(\alpha) = \sup_{\mathbb{R}_+^s} g(r) = \lambda_s \text{ and } \phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}}, \text{ as } r \to \infty.
$$

If $(\lambda_i)$ is an increasing positive sequence such that $\lim_{i \to \infty} \lambda_i = \lambda_s$ and $\forall i, w_i$ is the unique solution of $(P_{\lambda_i}^\alpha)$, then

$$
\lim_{i \to \infty} w_i(r) = \lambda_s^{\frac{1}{\alpha-1}} (r^{\frac{2}{1-\alpha}} - 1), \forall r \in [0, 1].
$$

**Proof.** — As in the proof of Proposition 4, one can verify that $\lambda^*(\alpha) = \sup_{\mathbb{R}_+^s} g(r)$, where $g(r) = r^2 \phi^{\alpha-1}(r)$.

If $(3 \leq n \leq 10$ and $\frac{n+2}{n-2} < \alpha)$ or $(n > 10$ and $\frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4})$, using Lemma 4, we get $f(\alpha) > n - 2$. So we can use Theorem 1 in [10] to infer that $\lambda^*(\alpha) > \lambda_s$, $(P_{\lambda^*(\alpha)}^\alpha)$ admits a unique solution and $(P_{\lambda_i}^\alpha)$ admits an infinity of solutions. Using the unique solution $u_{\lambda^*(\alpha)}$ of $(P_{\lambda^*(\alpha)}^\alpha)$, one can deduce from Proposition 1 that $u_{\lambda^*(\alpha)} = \psi_\rho$, where $\rho \in \mathbb{R}_+^s$ and $g(\rho) = \lambda^*(\alpha)$. We conclude that the supremum is achieved and $\lambda^*(\alpha) = \max_{\mathbb{R}_+^s} g(r)$.

Let us suppose that

$$
a = \liminf_{r \to \infty} g(r) < A = \limsup_{r \to \infty} g(r).
$$

For every $\lambda \in [a, A]$, the equation $g(r) = \lambda$ admits a sequence of roots $(r_i)$, with $\lim_{i \to \infty} r_i = \infty$. As for every $i$, $\psi_{r_i}$ is a solution of $(P_{\lambda_i}^\alpha)$, we get an infinity of solutions for this problem; but an infinity of solutions exists only when $\lambda = \lambda_s$ (cf. [10]). We get a contradiction and infer that

$$
a = A = \lambda_s = \lim_{r \to \infty} g(r), \text{ so } \phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}}, \text{ as } r \to \infty.
$$

If $(\rho_i)$ is an increasing sequence of positive constants, such that $(\psi_{\rho_i})$ are solutions of $(P_{\lambda_i}^\alpha)$ and $\lim_{i \to \infty} \rho_i = \infty$, then one can use the previous asymptotic behavior of $\phi$ to get $\lim_{i \to \infty} \psi_{\rho_i}(r) = \lambda_s^{\frac{1}{\alpha-1}} (r^{\frac{2}{1-\alpha}} - 1), \forall r \in [0, 1]$.

If $n > 10$ and $\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leq \alpha$, we get from Lemma 4 that $f(\alpha) \leq n - 2$. Using [10] again, we infer that $\lambda^*(\alpha) = \lambda_s$, $(P_{\lambda}^\alpha)$ admits a unique solution for every $\lambda \in [0, \lambda^*(\alpha)]$. As the function $g$ is increasing near $r = 0$, we infer that $g$ is increasing on a nontrivial open interval $I \subset \mathbb{R}_+^s$. For, on one hand, if $g$ decreases on a nontrivial open interval $I \subset \mathbb{R}_+^s$, then the equation $g(r) = \lambda$ admits at least two roots $r_1 < r_2$, if $\lambda \in \min_I g(r), \max_I g(r)$, and $\psi_{r_1}$ and $\psi_{r_2}$ are solutions of $(P_{\lambda}^\alpha)$.
with \( \psi_{r_1}(0) \neq \psi_{r_2}(0) \), this violates the uniqueness result of [10]. On another hand, the function \( g \) can’t be constant on a nontrivial interval, otherwise we get an infinity of solutions for some \( \lambda \). One can then see that
\[
\lim_{r \to \infty} g(r) = \sup_{\mathbb{R}_+^*} g(r) = \lambda^*(\alpha); \quad \lambda^*(\alpha) = \lambda_s \text{ (cf. [10])}.
\]

So \( \phi(r) \sim \lambda_s^{1-\frac{\alpha}{2}} r^{\frac{2}{1-\alpha}} \), as \( r \to \infty \).

Using this asymptotic behavior, one can show the last statement of the proposition.

Let us put
\[
(Q_\alpha^r) \begin{cases}
\Delta u + \lambda(1 + u)^\alpha = 0, & \text{in } B_{r_0} \\
u > 0, & \text{in } B_{r_0} \\
u = 0, & \text{on } \partial B_{r_0}
\end{cases}
\]
where \( B_{r_0} = \{ x \in \mathbb{R}^n, \|x\| < r_0 \} \). For every solution \( u \) of \( (Q_\alpha^r) \), we put \( v(r) = u(r_0 r) \) for every \( r \in [0, 1] \). Let \( \lambda^*_{r_0}(\alpha) \), be the maximal eigenvalue of \( (Q_\alpha^r) \).

**Lemma 5.3.** — A function \( u \) is a solution of \( (Q_\alpha^r) \), if and only if \( v \) is a solution of \( (P_{r_0}^\alpha) \). In particular, we get \( \lambda^*_{r_0}(\alpha) = r_0^2 \lambda^*(\alpha) \).

**Proof.** — The proof is easy.

**Remark 5.4.** — According to the previous lemma, the results obtained here for \( (P_{r_0}^\alpha) \) (on the unit ball \( B_1 \)), can be easily stated for \( (Q_\alpha^r) \) (on any ball \( B_{r_0} \)).

### 6. Appendix

Let \( S_{i}^k \) be the set of all the \( (k-i) \)-selections of \( \{1, \ldots, i\} \) and \( s(j) \) the multiplicity of the element \( j \), \( 1 \leq j \leq i \). If \( u \) is a analytical solution of \( (P_{r_0}^\alpha) \), with \( u(r) = \Sigma_{k=0}^\infty a_k r^k \) near \( r = 0, r_0 \) the convergence radius of this series, then

**Proposition 6.1.** —
\[
\forall \ k \geq 0, \ a_{2k+1} = 0, \quad a_2 = \frac{\lambda}{n-2}(1 + a_0)^\alpha \left( \frac{1}{n} - \frac{1}{2} \right)
\]
and \( \forall \ k > 1 \),
\[
a_{2k} = \frac{\lambda}{n-2} \left( \frac{1}{2k+n-2} - \frac{1}{2k} \right) \times \\
\Sigma_{i=1}^{k-1} (1 + a_0)^{\alpha-i} \frac{1}{i!} \Pi_{p=0}^{i-1} (\alpha - p) \Sigma_{s \in S_{i-1}^k} \Pi_{j=1}^{i} a_{2(1+s(j))}.
\]
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**Proof.** — Let us choose \(0 < r \leq \rho < r_0\), by standard integrations, we get

\[
 u(r) - u(\rho) = \frac{\lambda}{n-2} \times \left( r^{2-n} - \rho^{2-n} \right) \int_0^r t^{n-1}(1 + u(t))^\alpha dt + \int_\rho^r (t - \rho^{2-n} t^{n-1})(1 + u(t))^\alpha dt.
\]

Let us point out that

\[
 (1 + u(r))^\alpha = (1 + u(0) - u(0) + u(r))^\alpha
\]

\[
 = (1 + u(0))^{\alpha} \left( 1 + \frac{u(r) - u(0)}{1 + u(0)} \right)^{\alpha} = (1 + a_0)^{\alpha} \left( 1 + \sum_{i=1}^\infty \frac{a_i}{1 + a_0} r^i \right)^{\alpha},
\]

we infer that

\[
 (1 + u(r))^\alpha = (1 + a_0)^{\alpha} \left( 1 + \sum_{j=1}^\infty \frac{\alpha(\alpha - 1)\ldots(\alpha - j + 1)}{j!} \left( \sum_{i=1}^\infty \frac{a_i}{1 + a_0} r^i \right)^j \right).
\]

All these series are uniformly convergent on \([0, \rho]\). If we put \((1 + u(r))^\alpha = \sum_{j=0}^\infty c_j r^j\), we get

\[
 u(r) = \frac{\lambda}{n-2} \left( r^{2-n} - \rho^{2-n} \right) \int_0^r t^{n-1} \sum_{j=0}^\infty c_j t^j dt + \int_\rho^r (t - \rho^{2-n} t^{n-1}) \sum_{j=0}^\infty c_j t^j dt
\]

\[
 = \frac{\lambda}{n-2} \left( \sum_{j=0}^\infty c_j \frac{r^{2+j}}{j+n} - \sum_{j=0}^\infty c_j \frac{\rho^{2-n} r^{j+n}}{j+n} + \sum_{j=0}^\infty c_j \frac{\rho^{j+2}}{j+2} - \sum_{j=0}^\infty c_j \frac{\rho^{j+2}}{j+n} \right)
\]

\[
 + \frac{\lambda}{n-2} \left( -\sum_{j=0}^\infty c_j \frac{r^{j+2}}{j+2} + \sum_{j=0}^\infty c_j \frac{\rho^{2-n} r^{j+n}}{j+n} \right)
\]

\[
 = \frac{\lambda}{n-2} \left( \sum_{j=2}^\infty c_{j-2} \frac{r^j}{j+n-2} + \sum_{j=0}^\infty c_j \frac{\rho^{j+2}}{j+2} - \sum_{j=0}^\infty c_j \frac{\rho^{j+2}}{j+n} - \sum_{j=2}^\infty c_{j-2} \frac{r^j}{j} \right).
\]

We finally obtain

\[
 (2) \quad u(r) = \frac{\lambda}{n-2} \left( \sum_{j=2}^\infty c_{j-2} \left( \frac{1}{j+n-2} - \frac{1}{j} \right) r^j + \sum_{j=0}^\infty c_j \rho^{j+2} \left( \frac{1}{j+2} - \frac{1}{j+n} \right) \right).
\]

Using the previous identity, we obtain

\[
 a_1 = 0, \quad \forall \, k > 1, \quad a_k = \frac{\lambda}{n-2} \left( \frac{1}{k+n-2} - \frac{1}{k} \right) c_{k-2}.
\]
Using (1), we get
\[ c_0 = (1 + a_0)^\alpha, \quad c_1 = \alpha (1 + a_0)^{\alpha - 1} a_1 = 0 \]
and
\[ \forall k > 1, \quad c_k = (1 + a_0)^\alpha \sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) \frac{1}{(1 + a_0)^j} \sum_{s \in S_k^j} \prod_{i=1}^{j} a_{1 + s(i)} \]
\[ = \sum_{j=1}^{k} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S_k^j} \prod_{i=1}^{j} a_{1 + s(i)}. \]
Using the previous relation and the fact that \( a_1 = 0 \), one can verify (by induction) that \( a_{2k+1} = 0, \forall k > 0 \). We then obtain from (2) and the expression of \( c_k \)
\[ a_{2k} = \frac{\lambda}{n - 2} \left( \frac{1}{2k + n - 2} - \frac{1}{2k} \right) c_{2k-2} \]
\[ = \frac{\lambda}{n - 2} \left( \frac{1}{2k + n - 2} - \frac{1}{2k} \right) \sum_{j=1}^{k-1} \frac{1}{j!} \prod_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S_{k-1}^j} \prod_{i=1}^{j} a_{2(1 + s(i))}. \]
\[ \forall j \in [1, k - 1], \quad \text{Card}(S_{k-1}^j) = C_k^{j-1}. \]
Let us put
\[ d_2 = \frac{1}{2n} \quad \text{and} \quad \forall k > 1, \]
\[ d_{2k} = \frac{1}{(2k + n - 2)(2k)} \sum_{i=1}^{k-1} \frac{1}{i!} \prod_{p=0}^{i-1} (\alpha - p) \sum_{s \in S_{k-1}^i} \prod_{j=1}^{i} d_{2(1 + s(j))}, \]
then

**Lemma 6.2.** \(- a_{2k} = (-1)^k \lambda^k (1 + a_0)^{k(\alpha - 1) + 1} d_{2k}, \forall k > 1. \)

**Proof.**
\[ a_4 = \frac{\alpha \lambda^2}{(n - 2)^2} = (1 + a_0)^{2\alpha - 1} \left( \frac{1}{n + 2} - \frac{1}{4} \right) \left( \frac{1}{n} - \frac{1}{2} \right) \]
\[ = \lambda^2 (1 + a_0)^{2\alpha - 1} \frac{1}{4(n + 2)} \frac{\alpha}{2n} = \lambda^2 (1 + a_0)^{2(\alpha - 1) + 1} \frac{1}{4(n + 2)} \frac{\alpha}{2n}. \]
\[ d_4 = \frac{1}{4(n + 2)} \sum_{i=1}^{k-1} \frac{1}{i!} \prod_{p=0}^{i-1} (\alpha - p) \sum_{s \in S_i^j} \prod_{j=1}^{i} d_{2(1 + s(j))} \]
\[ = \frac{\alpha}{4(n + 2)} d_2 = \frac{1}{4(n + 2)} \frac{\alpha}{2n}. \]
so we infer that the formula is true for \( k = 2 \). Let us suppose it true for every \( j \), such that \( 2 \leq j \leq k \). From Proposition 7, we have

\[
a_{2(k+1)} = \frac{\lambda}{n-2} \left( \frac{1}{2k+n} - \frac{1}{2(k+1)} \right) \sum_{j=1}^{\lambda} \frac{1}{j!} \Pi_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S_k^j} \Pi_{i=1}^j a_2(1 + s(i))
\]

\[
= \frac{-\lambda}{(2(k+1) + n - 2)(2(k+1))} \sum_{j=1}^{\lambda} \frac{1}{j!} \Pi_{p=0}^{j-1} (\alpha - p) (1 + a_0)^{\alpha - j} \sum_{s \in S_k^j} \Pi_{i=1}^j a_2(1 + s(i)) \cdot
\]

\( \forall j \in [1,k], \forall s \in S_k^j, \) if \( i \in [1,j] \), then \( 1 \leq 1 + s(i) \leq k \),

so one can use the hypothesis to get \( \forall i \in [1,j] \),

\[
a_{2(1 + s(i))} = (-1)^{1+s(i)} \lambda^{1+s(i)} (1 + a_0)^{(s(i)+1)(\alpha-1)+1} d_2(1 + s(i))
\]

We then obtain

\[
\Pi_{i=1}^j a_{2(1+s(i))} = (-1)^j \lambda^j (1 + a_0)^{\alpha j + (\alpha - 1)(k-j)} \Pi_{i=1}^j d_2(1 + s(i))
\]

\[
= (-1)^j \lambda^j (1 + a_0)^{\alpha j + (\alpha - 1)s(i)} \Pi_{i=1}^j d_2(1 + s(i)).
\]

But for every \( s \in S_k^j \), we have \( \Sigma_{i=1}^j s(i) = k - j \).

We infer that

\[
\Pi_{i=1}^j a_{2(1+s(i))} = (-1)^k \lambda^k (1 + a_0)^{\alpha j + (\alpha - 1)(k-j)} \Pi_{i=1}^j d_2(1 + s(i))
\]

\[
= (-1)^k \lambda^k (1 + a_0)^{(\alpha-1)k+j} \Pi_{i=1}^j d_2(1 + s(i)).
\]

Substituting in the expression of \( a_{2(k+1)} \), we obtain

\[
a_{2(k+1)} = (-1)^{k+1} \lambda^{k+1} (1 + a_0)^{k(\alpha-1)+\alpha} \frac{1}{(2(k+1) + n - 2)(2(k+1))} \times \]

\[
\Sigma_{j=1}^{\lambda} \frac{1}{j!} \Pi_{p=0}^{j-1} (\alpha - p) \sum_{s \in S_k^j} \Pi_{i=1}^j d_2(1 + s(i))
\]

\[
= (-1)^{k+1} \lambda^{k+1} (1 + a_0)^{(k+1)(\alpha-1)+1} \frac{1}{(2(k+1) + n - 2)(2(k+1))} \times \]

\[
\Sigma_{j=1}^{\lambda} \frac{1}{j!} \Pi_{p=0}^{j-1} (\alpha - p) \sum_{s \in S_k^j} \Pi_{i=1}^j d_2(1 + s(i)).
\]

\[
= (-1)^{k+1} \lambda^{k+1} (1 + a_0)^{(k+1)(\alpha-1)+1} d_2(k+1).
\]
Let us compute the first terms of the Lane-Emden function,
\[ \phi(r) = \sum_{i=0}^{\infty} a_{2i} r^{2i}, \]
near \( r = 0 \), where \( a_0 = 1 \), and
\[ a_{2i} = (-1)^i 2^{i(\alpha - 1) + 1} d_{2i}, \quad \forall i > 1. \]

\[ d_0 = 1; \quad d_2 = \frac{1}{2n}; \quad d_4 = \frac{1}{4(n+2)} \alpha d_2 = \frac{\alpha}{(2n)(4(n+2))}; \]
\[ d_6 = \frac{1}{6(n+4)} \left( \alpha d_4 + \frac{1}{2} \alpha(\alpha - 1) d_2^2 \right) = \frac{1}{6(n+4)} \left\{ \frac{\alpha^2}{(2n)(4(n+2))} + \frac{\alpha(\alpha - 1)}{2(2n)^2} \right\}; \]
\[ d_8 = \frac{1}{8(n+6)} \left( \alpha d_6 + \alpha(\alpha - 1) d_4 d_2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6} d_2^3 \right) \]
\[ = \frac{1}{8(n+6)} \left\{ \frac{\alpha^3}{(2n)(4(n+2))(6(n+4))} + \frac{\alpha^2(\alpha - 1)}{2(2n)^2(6(n+4))} + \frac{\alpha^2(\alpha - 1)}{2(2n)^2(4(n+2))} \right. \]
\[ + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6(2n)^3} \left\} \right; \]
\[ d_{10} = \frac{1}{10(n+8)} \left\{ \alpha d_8 + \frac{\alpha(\alpha - 1)}{2} (2d_2 d_6 + d_4^3) + \frac{\alpha(\alpha - 1)(\alpha - 2)}{6} d_2^2 d_4 + \frac{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)}{24} d_2^4 \right\} \]
\[ = \frac{1}{10(n+8)} \left\{ \frac{\alpha^4}{(2n)(4(n+2))(6(n+4))(8(n+6))} + \frac{\alpha^3(\alpha - 1)}{2(2n)^2(6(n+4))(8(n+6))} \right. \]
\[ + \frac{\alpha^3(\alpha - 1)}{(2n)^2(4(n+2))(8(n+6))} + \frac{\alpha^2(\alpha - 1)(\alpha - 2)}{6(2n)^3(8(n+6))} + \frac{\alpha^3(\alpha - 1)}{(2n)^2(4(n+2))(6(n+4))} \]
\[ + \frac{\alpha^2(\alpha - 1)^2}{2(2n)^3(6(n+4))} + \frac{\alpha^3(\alpha - 1)}{2(2n)^2(4(n+2))^2} + \frac{\alpha^2(\alpha - 1)(\alpha - 2)}{2(2n)^3(4(n+2))} \]
\[ + \frac{\alpha(\alpha - 1)(\alpha - 2)(\alpha - 3)}{24(2n)^4} \left\} \right. \].

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