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<http://afst.cedram.org/item?id=AFST_2010_6_19_S1_215_0>
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Dedicated to Melvin Henriksen

ABSTRACT. — Using lattice-ordered algebras it is shown that a totally ordered field which has a unique total order and is dense in its real closure has the property that each of its positive semidefinite rational functions is a sum of squares.

RéSUMÉ. — En utilisant les algèbres réticulées, on montre qu’un corps totalement ordonné qui a un unique ordre total et qui est dense dans sa clôture réelle a la propriété que chacune des ses fonctions rationnelles positives semi-définies est une somme de carrés.

Hilbert’s seventeenth problem asks if a rational function with rational coefficients which is positive semidefinite over the field of real numbers is a sum of squares of rational functions with rational coefficients. Artin [1] (or [10]) showed that this is indeed the case and, in fact, proved the stronger theorem that any subfield of the reals which has a unique total order also has this property. In [8, p. 641] (also see [7, p. 295]), Jacobson presented this result for totally ordered fields that were not necessarily archimedean, and McKenna gave the converse of this theorem in [11]. In this note I will give a proof, using some aspects of the theory of lattice-ordered rings given in Henriksen and Isbell [6], of Jacobson’s version of Artin’s theorem. I believe this proof of Artin’s solution to Hilbert’s problem was known to Weinberg in 1968. One aspect of this approach is that it avoids any use of model theory.

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– 215 –
Let $K$ be a totally ordered field. A rational function $r(x_1,\cdots,x_n) \in K(x_1,\cdots,x_n)$ is positive semidefinite on $K$, abbreviated P.S.D., if $r(a_1,\cdots,a_n) \geq 0$ for all $a_1,\cdots,a_n$ in $K$ for which $r(a_1,\cdots,a_n)$ is defined. The positive cone of the partially ordered group $G$ will be denoted by $G^+$, and $S(R)$ denotes the set of sums of squares in the commutative ring $R$. If $F$ is an extension field of the totally ordered field $K$ it is well-known that $K^+S(F) = \{ \Sigma_i a_i f_i^2 : a_i \in K^+, f_i \in F \}$ is the intersection of those total orders of $F$ which contain $K^+$. The subfield $K$ of the totally ordered field $F$ is dense in $F$ if for all $a, b$ in $F$ with $a < b$ there exists some $c \in K$ with $a < c < b$. According to McKenna the totally ordered field $K$ has Hilbert’s property if, for every $n$, each rational function in $K(x_1,\cdots,x_n)$ that is P.S.D. on $K$ is a sum of squares in $K(x_1,\cdots,x_n)$. The theorem to be proved, as stated in [8, p. 641], is

**Theorem 0.1.** — (Artin [1]). Let $F$ be the real closure of the totally ordered field $K$. If $K$ has a unique total order and is dense in $F$, then $K$ has Hilbert’s property.

The cardinality of the set $X$ will be denoted by $|X|$. If $A$ and $B$ are subsets of the partially ordered set $X$, then $A < B$ (respectively, $A \leq B$) means $a < b (a \leq b)$ for every $a \in A$ and $b \in B$. For an ordinal number $\alpha$, $X$ is called an $\eta_\alpha$-set (respectively, an almost $\eta_\alpha$-set) if whenever $A$ and $B$ are subsets of $X$ with $A < B (A \leq B)$ and $|A \cup B| < \aleph_\alpha$, then $A < c < B (A \leq c \leq B)$ for some $c \in X$; in these definitions either $A$ or $B$ could be empty. The cardinal number $\aleph_\alpha$ is regular if $|\bigcup_{i \in I} A_i| < \aleph_\alpha$ provided $|I| < \aleph_\alpha$ and $|A_i| < \aleph_\alpha$ for every $i \in I$. We start with a well-known embedding theorem.

**Theorem 0.2.** — Suppose $\alpha \geq 1$ and $\aleph_\alpha$ is a regular cardinal. Let $K$ be a totally ordered subfield of the totally ordered field $L$ and let $F$ be a real closed $\eta_\alpha$-field. If $\sigma : K \rightarrow F$ is an embedding of totally ordered fields with $|K| < \aleph_\alpha$ and $|L| \leq \aleph_\alpha$, then $\sigma$ can be extended to an embedding of totally ordered fields $\tau : L \rightarrow F$.

**Proof.** — A proof for the case $K = \mathbb{Q}$ is contained in the proof of Theorem 2.1 of [3]. A slight modification of the proof of Theorem 4.4.3 in [13, p. 95] proves this stronger result.

Our construction of a totally ordered $\eta_1$-field will use the following fact about lattices.

**Lemma 0.3.** — ([14, p. II-62] ; also, see [4, p. 176]). Let $f : L \rightarrow M$ be a lattice homomorphism of the lattice $L$ onto the lattice $M$. If $S$ is a countable
subset of $M$ then there exists a subset $T$ of $L$ such that $f : T \rightarrow S$ is an order isomorphism.

Proof. — We assume that $S$ is infinite; the case that $S$ is finite is done similarly. Suppose $S = \{f(x_1), f(x_2), \cdots\}$. Let $t_1 = x_1$. Suppose $t_1, \cdots, t_{n-1}$ have been chosen so that $f : \{t_1, \cdots, t_{n-1}\} \rightarrow \{f(x_1), \cdots, f(x_{n-1})\}$ is an order isomorphism with $f(t_i) = f(x_i)$. Let $X = \{t_i : f(t_i) < f(x_n)\}$, $Y = \{t_j : f(x_n) < f(t_j)\}$, $x = \bigvee_i t_i$, $y = \bigwedge_j t_j$ and $t_n = (x \lor x_n) \land y$. If $X$ or $Y$ is empty just delete $x$ or $y$ from the definition of $t_n$; we will assume neither $X$ nor $Y$ is empty since the other cases follow in a similar way. Now, $X < Y$ since $f(t_i) < f(t_j)$ and hence $t_i < t_j$ for $t_i \in X$ and $t_j \in Y$. Thus $x \leq y$,

$$f(x) = \bigvee_i f(t_i) \leq f(x_n) \leq \bigwedge_j f(t_j) = f(y),$$

and

$$f(t_n) = (f(x) \lor f(x_n)) \land f(y) = f(x_n) \land f(y) = f(x_n).$$

Now, $t_i < t_n$ iff $f(t_i) < f(t_n)$ ($i = 1, \cdots, n - 1$). For, $t_i < t_n$ gives $f(x_i) = f(t_i) \leq f(t_n) = f(x_n)$ and hence $f(t_i) < f(t_n)$; and $f(t_i) < f(t_n) = f(x_n)$ gives $t_i \leq x \leq y$, $t_i \leq (x \lor x_n) \land y = t_n$, and hence $t_i < t_n$. Similarly, $t_n < t_j$ iff $f(t_n) < f(t_j)$ for $j = 1, \cdots, n - 1$. □

Theorem 0.4. — ([15]; also [14, p. II-63]). Let $\{M_n : n \in \mathbb{N}\}$ be a sequence of nonzero $\ell$-groups. Then $\overline{M} = \prod_n M_n / \oplus_n M_n$ and all of its homomorphic images are almost $\eta_1$-groups.

Proof. — The homomorphisms in “homomorphic images” are, of course, morphisms between $\ell$-groups. We will only consider $\overline{M}$ since the same proof works for $M/C$ where $C$ is a normal convex $\ell$-subgroup of $\Pi_n M_n$ which contains $\oplus_n M_n$. Suppose $\overline{A} < \overline{B}$ are countable subsets of $\overline{M}$. We assume $\overline{A}$ and $\overline{B}$ are infinite. From Lemma 0.3 we can find subsets $A = \{a_n : n \in \mathbb{N}\} \subset \{b_n : n \in \mathbb{N}\} = B$ of $\Pi_n M_n$ such that $\overline{A} = \{\overline{a_n} : n \in \mathbb{N}\}$, $\overline{B} = \{\overline{b_n} : n \in \mathbb{N}\}$ and $A \cup B \twoheadrightarrow \overline{A} \cup \overline{B}$ is an order isomorphism. For each $n \in \mathbb{N}$ take $g_n \in M_n$ with

$$\{a_1(n), \cdots, a_n(n)\} \leq g_n \leq \{b_1(n), \cdots, b_n(n)\},$$

and let $g \in \Pi_n M_n$ be defined by $g(n) = g_n$. Then $\overline{A} \leq \overline{g} \leq \overline{B}$. To see that $\overline{A} \leq \overline{g}$ fix $k \in \mathbb{N}$. If $n \in \mathbb{N}$ and $a_k(n) \not\leq g_n$, then $k > n$; that is, $n \in \{1, \cdots, k - 1\}$. So if $h_k \in \Pi_n M_n$ is defined by

$$h_k(n) = \begin{cases} -g_n + a_k(n) & \text{if } a_k(n) \not\leq g_n \\ 0 & \text{if } a_k(n) \leq g_n \end{cases}$$

then $h_k \in \oplus_n M_n$ and $a_k \leq g + h_k$; hence $\overline{a_k} \leq \overline{g}$. Similarly, $\overline{g} \leq \overline{B}$. □
The following well-known result follows quickly from Theorem 0.4.

**Corollary 0.5.** — Suppose $K$ is a real closed field and $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$ which contains all complements of finite subsets of $\mathbb{N}$. Then the ultraproduct $K^\mathbb{N}/\mathcal{F}$ is a real closed $\eta_1$-field.

**Proof.** — For $f \in K^\mathbb{N}$ let $Z(f) = \{n \in \mathbb{N} : f(n) = 0\}$. Recall that $K^\mathbb{N}/\mathcal{F} = K^\mathbb{N}/I(\mathcal{F})$ where $I(\mathcal{F}) = \{f \in K^\mathbb{N} : Z(f) \in \mathcal{F}\}$ is a maximal ideal of $K^\mathbb{N}$ which is an $\ell$-ideal (all of the ideals of $K^\mathbb{N}$ are $\ell$-ideals). Using the standard characterization of a real closed field as a totally ordered field in which each positive element is a square and each polynomial of odd degree has a root it is clear that $K^\mathbb{N}/\mathcal{F}$ is real closed. Since $I(\mathcal{F})$ contains $\oplus_n K$, $K^\mathbb{N}/\mathcal{F}$ is a totally ordered almost $\eta_1$-field. But a totally ordered almost $\eta_\alpha$-division ring $D$ is an $\eta_\alpha$-division ring. For suppose, for example, that $A \leq c \leq B$ with $|A \cup B| < \aleph_\alpha$, $c \in A$, and $B$ has no least element. Then $0 < B - c$ has no least element, $(B - c)^{-1} < u^{-1}$ for some $u \in D$ since $(B - c)^{-1}$ has no largest element, $u < B - c$, and $A < c + u < B$. \(\square\)

An $\ell$-ring $R$ which is an algebra over the partially ordered ring $C$ is called an $\ell$-algebra if $C^+R^+ \subseteq R^+$. Let $S$ be a set of words in the free $\ell$-algebra on a countably infinite free generating set. The variety of $\ell$-algebras determined by $S$ is the class $\mathcal{V}(S)$ consisting of all those $\ell$-algebras $R$ which satisfy each word in $S : g(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in R$ and all $g(x_1, \ldots, x_n) \in S$. According to Birkhoff’s theorem [2, p. 169] a class of $\ell$-algebras $\mathcal{V}$ is a variety if and only if each $\ell$-subalgebra and each homomorphic image of an $\ell$-algebra in $\mathcal{V}$ also belongs to $\mathcal{V}$, and the direct product of any set of $\ell$-algebras from $\mathcal{V}$ is in $\mathcal{V}$. If $K$ is an $\ell$-algebra, then $\mathcal{V}_C(K)$ denotes the variety of $\ell$-algebras generated by $K$. The $\ell$-algebra $R$ belongs to $\mathcal{V}_C(K)$ if and only if it satisfies each $\ell$-algebra identity that $K$ satisfies. A small extension of a result from [6] is crucial to this proof.

**Theorem 0.6** ([6, 3.8]). — Let $C$ be a common totally ordered subring of the totally ordered fields $K$ and $L$. If $K$ is real closed then $L \in \mathcal{V}_C(K)$.

**Proof.** — Suppose $g(x_1, \ldots, x_n)$ is a word in the free (commutative) $C$-$f$-algebra that $K$ satisfies. Let $\alpha_1, \ldots, \alpha_m$ be all the elements of $C$ which occur in $g(x_1, \ldots, x_n)$ and let $a_1, \ldots, a_n \in L$. If $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$ which contains the complement of each finite subset of $\mathbb{N}$, then by Corollary 0.5 and Theorem 0.2 the embedding

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_m) \rightarrow K \rightarrow K^\mathbb{N}/\mathcal{F}$$
An ℓ-algebra approach to Artin’s solution

can be extended to an embedding \( \psi : \mathbb{Q}(\alpha_1, \ldots, \alpha_m, a_1, \ldots, a_n) \rightarrow K_N/F. \) Since \( \psi \) fixes each \( \alpha_i \) we have \( \psi(g(a_1, \ldots, a_n)) = g(\psi(a_1), \ldots, \psi(a_n)) = 0. \) \( \square \)

We will now give the proof of Theorem 0.1.

Suppose \( r(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)g(x_1, \ldots, x_n)^{-1} \in K(x_1, \ldots, x_n) \) is P.S.D. on \( K \) and let \( h(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)g(x_1, \ldots, x_n). \) Then \( h(\alpha_1, \ldots, \alpha_n) \geq 0 \) for all \( \alpha_1, \ldots, \alpha_n \in F \) and hence \( h(x_1, \ldots, x_n)^{-} = 0 \) is an identity for the \( K-\ell \)-algebra \( F. \) Let \( P \) be a total order of \( K(x_1, \ldots, x_n) \) which extends \( K^{+} \) and let \( E \) be the real closure of \( (K(x_1, \ldots, x_n), P). \) Then \( \mathcal{V}_K(F) = \mathcal{V}_K(E) \) by Theorem 0.6 and hence \( h(x_1, \ldots, x_n)^{-} = 0 \) is also an identity for the \( K-\ell \)-algebra \( E. \) So \( h(x_1, \ldots, x_n) \in P \) and hence \( r(x_1, \ldots, x_n) \in K^{+}S(K(x_1, \ldots, x_n)) = S(K(x_1, \ldots, x_n)) \) since \( K^{+} = S(K). \) \( \square \)

The proof I have given of Theorem 0.1 also proves the following additional versions of Artin’s theorem. The first version is given in [5] and [7, p. 295] and the second version which, along with the reference [5], was kindly pointed out to me by Delzell, comes from Lang [9, p. 387]. Of course, for the second version one needs to use the well-known fact that for a field \( E \) whose characteristic is not 2, \( S(E) \) is the intersection of all of the total orders of \( E \) [7, p. 288].

Let \( K \) be a subfield of the real closed field \( F \) with the total order it inherits from \( F. \) If \( r(x_1, \ldots, x_n) \in K(x_1, \ldots, x_n) \) is P.S.D. on \( F, \) then \( r(x_1, \ldots, x_n) \in K^{+}S(K(x_1, \ldots, x_n)). \)

Let \( r(x_1, \ldots, x_n) \in K(x_1, \ldots, x_n) \) where \( K \) is a field whose characteristic is not 2. If \( r(x_1, \ldots, x_n) \) is P.S.D. on each algebraic extension \( L \) of \( K, \) for any total order of \( L, \) then \( r(x_1, \ldots, x_n) \) is a sum of squares in \( K(x_1, \ldots, x_n). \)

Bibliography


[10] Lang (S.) and Tate (J.T.). — The collected papers of Emil Artin, Addison-Wesley, Reading (1965).


