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*An inequality for local unitary Theta correspondence*


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An inequality for local unitary Theta correspondence

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ABSTRACT. — Given a representation $\pi$ of a local unitary group $G$ and another local unitary group $H$, either the Theta correspondence provides a representation $\theta_H(\pi)$ of $H$ or we set $\theta_H(\pi) = 0$. If $G$ is fixed and $H$ varies in a Witt tower, a natural question is: for which $H$ is $\theta_H(\pi) \neq 0$?

For given dimension $m$ there are exactly two isometry classes of unitary spaces that we denote $H^\pm_m$. For $\varepsilon \in \{0, 1\}$ let us denote $m^\varepsilon_\pi$ the minimal $m$ of the same parity of $\varepsilon$ such that $\theta_{H^\varepsilon_m}(\pi) \neq 0$, then we prove that $m^+_\pi + m^-_\pi \geq 2n + 2$ where $n$ is the dimension of $\pi$.

RÉSUMÉ. — Étant donnée une représentation $\pi$ d’un groupe unitaire local $G$ et un autre groupe unitaire local $H$, on sait que soit la correspondance Theta fournit une représentation $\theta_H(\pi)$ de $H$ soit on pose $\theta_H(\pi) = 0$. Si on fixe $G$ et on laisse $H$ varier dans une tour de Witt, une question naturelle est: pour quels $H$ a-t-on $\theta_H(\pi) \neq 0$ ? Pour chaque dimension $m$ il y a exactement deux classes d’équivalence d’espaces unitaires que nous dénotons $H^\pm_m$. Pour $\varepsilon \in \{0; 1\}$, dnotons $m^\varepsilon_\pi$ le plus petit $m$ de la parité de $\varepsilon$ tel que $\theta_{H^\varepsilon_m}(\pi) \neq 0$, alors nous montrons que $m^+_\pi + m^-_\pi \geq 2n + 2$ où $n$ est la dimension de $\pi$.

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1. Introduction

The Theta correspondence is a powerful tool for the study of automorphic and local representations. It has been studied and used in the global and in the local case by various authors, see for instance [Har07], [HKS96], [How], [Kud86], [KR05], [MVW87], [Ral84], [Wal90]. We will restrict ourselves to the local case: we suppose that the base field is a \( p \)-adic field with \( p \neq 2 \). The Theta correspondence builds a duality between the representations of two reductive groups forming a dual pair inside a given symplectic (or metaplectic) group. The theory will be explained in greater detail in section 2. We will be interested in the so-called unitary case where both groups are unitary. To an irreducible representation \( \pi \) of the first group \( G \) corresponds at most one representation of the second group \( H \) that we denote \( \theta(\pi) = \theta(G, H, \pi) \) where \( \theta(\pi) = 0 \) if there is no representation of \( H \) corresponding to \( \pi \) (in the unitary case, \( \theta \) depends on the choice of a auxiliary character \( \chi \), we will thus write \( \theta_\chi \) instead of \( \theta \) in that case). One can fix a representation \( \pi \) of an unitary group \( G = U(W) \) and vary the second group \( H = U(V) \), where \( W \) and \( V \) are Hermitian spaces and \( G \) and \( H \) are their respective unitary groups. One way to vary the space \( V \) is to start from a given irreducible space \( V_0 \) and to add hyperbolic planes \( V_1 \). One obtains a so-called Witt tower of spaces \( V_r = V_0 \oplus (V_1)^r \) and groups \( H_r = H(V_r) \). We have (up to isometry) four such towers depending on the parity of \( r \) and on the sign of the Hasse invariant (see below for its definition). We denote them, with a slight notation shift, \( V_{r \pm m_0} \) where \( m_0 = 0 \) or \( 1 \), the dimension of \( V_{r \pm m_0} \) is \( 2r + m_0 \) and \( \pm \) is the sign of the Hasse invariant. It is now well known that if \( \theta_\chi(G, H(V_{2r+2m_0}), \pi) \neq 0 \) then \( \theta_\chi(G, H(V_{2r+2+m_0}), \pi) \neq 0 \). We can thus consider, for a given \( m_0 \), the two integers \( m_{\chi}^{\pm}(\pi) \) which are the minimal \( m = 2r + m_0 \) such that \( \theta_\chi(G, H(V_m^\pm), \pi) \neq 0 \).

We prove here a part of a conjecture of Harris, Kudla and Sweet (see Conjecture 2.7), namely

**Theorem 3.10.** — Let \( \pi \) be an irreducible admissible representation of \( G(W) \) where \( \dim W = n \). Then

\[
m_{\chi}^+(\pi) + m_{\chi}^-(\pi) \geq 2n + 2.
\]

The conjecture (the Conservation Relation, see Conjecture 2.7) asserts that the inequality is in fact an equality.

In some important cases, Theorem 3.10, combined with the results of [HKS96] on local zeta integrals, suffices to prove stronger results. In parti-
cicular, it is known, thanks to [HKS96], that
\[ m = \inf(m^+(\pi), m^-_X(\pi)) \leq n. \]

When \( m = n \) Harris and Kudla use this inequality and Theorem 3.10 to prove the **Dichotomy Conjecture** of [HKS96] ([Har07][Theorem 2.1.7]), which determines whether \( m = m^+_X(\pi) \) or \( m = m^-_X(\pi) \) in terms of local root numbers.

The (still-conjectural) Conservation Relation, the Dichotomy Conjecture (now proved), and Kudla’s Persistence Principle (Proposition 2.6) go a long way toward providing a complete explicit determination of the local theta correspondence. Resolving the remaining ambiguities will require a better understanding of the poles of local zeta integrals. A key step in the present paper, as in [KR05], is to prove simplicity of these poles for unramified representations. This implies the Conservation Relation when \( \pi \) is the trivial representation, and a doubling argument that goes back to Kudla and Rallis, together with a cocycle calculation, then implies Theorem 3.10.

The inequality proved in Theorem 3.10 is applied in a global situation in [Har07] to study special values of \( L \)-functions.

While we were writing this manuscript, Harris brought to our attention that a proof in his article [Har07] was incomplete. Since the arguments are related to the ones explained here, we have added that proof as an appendix to this paper.

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2. **Notations**

This section recalls the local Theta correspondence as in [Kud96] and cites some of the results of [HKS96].

We fix once and for all a non archimedean local field \( F \) of residual characteristic different from 2.
The mapping $\Delta$ will always be a diagonal embedding, usually from $G$ to $G \times G$ except in one point where it will be precised.

### 2.1. Heisenberg group

Let $W$ be a vector space with a symplectic form $\langle ., . \rangle$ on which the group $\text{GL}(W)$ will act on the right -- accordingly, if $f$ and $g$ are two endomorphisms of $W$, we will denote $f \circ g$ the endomorphism such that $(f \circ g)(w) = g(f(w))$. We will denote, as usual,

$$\text{Sp}(W) = \{ g \in \text{GL}(W) | \forall (x, y) \in W^2, \langle xg, yg \rangle = \langle x, y \rangle \}$$

its isometry group.

**Definition 2.1.** — The Heisenberg group of $W$ if the group $H(W) = W \ltimes F$ with product

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2} \langle w_1, w_2 \rangle).$$

The centre of $H(W)$ is $\{ (0, t) | t \in F \}$ and $\text{Sp}(W)$ acts on $H(W)$ via its action on $W$:

$$(w, t)^g = (wg, t).$$

We recall

**Theorem 2.2 (Stone–von Neumann).** — Let $\psi$ be a non trivial unitary character of $F$. There exists, up to isomorphism, one smooth irreducible representation $(\rho_\psi, S)$ of $H(W)$ such that

$$\rho_\psi((0, t)) = \psi(t) \cdot \text{id}_S.$$

If we fix such a representation $(\rho_\psi, S)$, then for any $g \in \text{Sp}(g)$, the representation $h \mapsto \rho_\psi^g(h) = \rho_\psi(h^g)$ is a representation of $H(W)$ with the same central character, which means that it must be isomorphic to $\rho_\psi$. Hence there is an isomorphism $A(g) \in \text{GL}(S)$, unique up to a scalar, such that

$$\forall h \in H, \quad A(g)^{-1} \rho_\psi(h) A(g) = \rho_\psi^g(h). \quad (2.1)$$

The group

$$\text{Mp}(W) = \{ (g, A(g)) | \text{equation (1) holds} \}$$

is independent of the choice of $\psi$ and is a central extension of $\text{Sp}(W)$ by $\mathbb{C}^\times$:

$$0 \rightarrow \mathbb{C}^\times \rightarrow \text{Mp}(W) \rightarrow \text{Sp}(W) \rightarrow 1.$$
The group $\text{Mp}(W)$ has a natural representation, called the Weil representation, $\omega_\psi$, on $S$ given by

$$
\omega_\psi : \text{Mp}(W) \rightarrow \text{End}(S)
$$

$$
(g, A(g)) \mapsto A(g)
$$

2.2. The Schrödinger model of the Weil representation

The natural mapping $(g, A(g)) \mapsto A(g)$ defines a representation of $\text{Mp}(W)$ which has several models. We are interested in the so-called Schrödinger model.

Let $Y$ be a Lagrangian of $W$, i.e. a maximal isotropic subspace of $W$ and $W = X \oplus Y$ a complete polarisation of $W$. We consider $Y$ as a degenerate symplectic space and see $H(Y) = Y \ltimes F$ as a maximal abelian subgroup of $H(W)$. We consider the extension $\psi_Y$ of the character $\psi$ from $F$ to $H(Y)$ defined by $\psi_Y(y, t) = \psi(t)$. Let

$$
S_Y = \text{Ind}_{H(Y)}^{H(W)} \psi_Y.
$$

We recall that $S_Y$ is the space of the functions $f : H(W) \rightarrow \mathbb{C}$ such that

$$
\forall h \in H, \forall h_1 \in H(Y), f(h_1 h) = \psi_Y(h_1) f(h)
$$

and such that there exists a compact open subgroup $L$ of $W$ satisfying

$$
\forall h \in H, \forall l \in L, f(h(l, 0)) = f(h).
$$

We fix an isomorphism of $S_Y$ with the space $S(X)$ of Schwartz functions on $X$ by

$$
S_Y \rightarrow S(X),
$$

$$
f \mapsto \varphi : X \rightarrow \mathbb{C},
$$

$$
x \mapsto \varphi(x) = f(x, 0).
$$

The group $H(W)$ acts on $S_Y$ by right translation while it acts on $\varphi \in S(X)$ by

$$
(\rho(x + y, t) \varphi)(x_0) = \psi \left( t + \langle x_0, y \rangle + \frac{1}{2} \langle x, y \rangle \right) \varphi(x_0 + x)
$$

where $x + y \in W$ is with $x \in X$ and $y \in Y$. Then (see [MVW87]) $(\rho, S(X))$ is a model for the Weil representation.

We specify the operator $\omega_\psi$ as follows. We identify an element $w \in W$ with the row vector $(x, y) \in X \oplus Y$. An element $g \in \text{Sp}(W)$ will be of
the form \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( a \in \text{End}(X) \), \( b \in \text{Hom}(X,Y) \), \( c \in \text{Hom}(Y,X) \) and \( d \in \text{End}(Y) \). Let \( P_Y = \{ g \in \text{Sp}(W) | c = 0 \} \) be the maximal parabolic subgroup of \( \text{Sp}(W) \) that stabilises \( Y \) and \( N_Y = \{ g \in P_Y | d = \text{id}_Y \} \) its unipotent radical. We have a Levi subgroup \( M_Y = \{ g \in P_Y | b = 0 \} \) of \( P_Y \) and \( P_Y = M_Y N_Y \).

We define the following natural mappings:

\[
m : \text{GL}(X) \longrightarrow M_Y
\]
\[
a \longrightarrow m(a) = \begin{pmatrix} a & 0 \\ 0 & a^\vee \end{pmatrix}
\]

\[
n : \text{Her}(X,Y) \longrightarrow N_Y
\]
\[
b \longrightarrow n(b) = \begin{pmatrix} \text{id}_X & b \\ 0 & \text{id}_Y \end{pmatrix}
\]

where \( a^\vee \) is the inverse of the dual of \( a \) and \( \text{Her}(X,Y) \) is the subset of those \( b \in \text{Hom}(X,Y) \) which are Hermitian (in both cases we identify the dual of \( X \oplus Y \) with \( Y \oplus X \) using \( \langle \cdot, \cdot \rangle \)).

**Proposition 2.3** ([Kud96, Proposition 2.3, p.8). — Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(g) \). The operator \( r(g) \) of \( S(X) \) defined by

\[
r(g)(\varphi)(x) = \int_{\text{Ker}c \setminus Y} \varphi \left( \frac{1}{2} \langle xa, xb \rangle - \langle xb, yc \rangle + \frac{1}{2} \langle yc, yd \rangle \right) d\mu_g(y)
\]

is proportional to \( A(g) \) and moreover is unitary for a unique Haar measure \( d\mu_g(y) \) on \( \text{Ker} \ c \setminus Y \).

**2.3. Dual reductive pairs**

**Definition 2.4.** — A dual reductive pair \((G,G')\) in \( \text{Sp}(W) \) is a pair of subgroups of \( \text{Sp}(W) \) such that both \( G \) and \( G' \) are reductive and

\[
\text{Cent}_{\text{Sp}(W)}(G) = G' \quad \text{and} \quad \text{Cent}_{\text{Sp}(W)}(G') = G.
\]

If \((G,G')\) is a dual reductive pair in \( \text{Sp}(W) \), we denote \( \tilde{G} \) and \( \tilde{G}' \) the pullbacks of the subgroups in \( \text{Mp}(W) \). As seen in [MVW87], there exists a natural morphism

\[
j : \tilde{G} \times \tilde{G}' \longrightarrow \text{Mp}(W)
\]

such that the restriction of \( j \) to \( \mathbb{C}^\times \times \mathbb{C}^\times \) is the product.

We consider the pullback \((j^*(\omega_\psi), S)\) of \( \omega_\psi \) to \( \tilde{G} \times \tilde{G}' \). We note that the central character for both \( \tilde{G} \) and \( \tilde{G}' \) is the identity:

\[
\omega_\psi(j(z_1,z_2)) = z_1 z_2 \cdot \text{id}_S.
\]
Let $\pi$ be an irreducible admissible representation of $\tilde{G}$ such that the central character of $\pi$ is the identity. If

$$N(\pi) = \bigcap_{\lambda \in \text{Hom}_{\tilde{G}}(S,\pi)} \text{Ker} \lambda$$

then $S(\pi) = S/N(\pi)$ is the largest quotient of $S$ on which $\tilde{G}$ acts by $\pi$. The action of $\tilde{G}'$ on $S$ commutes with the action of $\tilde{G}$ so that $\tilde{G}'$ acts on $S(\pi)$ and thus $S(\pi)$ is a representation of $\tilde{G} \times \tilde{G}'$. There exists (see [MVW87]) a smooth representation $\Theta_\psi(\pi)$ of $G'$, unique up to isomorphism, such that

$$S(\pi) \simeq \pi \otimes \Theta_\psi(\pi).$$

The principal result of the theory is the following

**Theorem 2.5** (Howe duality principle). — Let $F$ be a non archimedean local field with residual characteristic different from 2 and let $\pi$ be an irreducible admissible representation of $G$.

i) If $\Theta_\psi(\pi) \neq 0$, then it is an admissible representation of $\tilde{G}'$ of finite length.

ii) If $\Theta_\psi(\pi) \neq 0$, there exists a unique $\tilde{G}'$-submodule $\Theta^0_\psi(\pi)$ such that the quotient

$$\theta_\psi(\pi) = \Theta_\psi(\pi)/\Theta^0_\psi(\pi)$$

is irreducible. If $\Theta_\psi(\pi) = 0$, we let $\theta_\psi(\pi) = 0$.

iii) If two irreducible admissible representations $\pi_1$ and $\pi_2$ of $\tilde{G}$ are such that $\theta_\psi(\pi_1) \simeq \theta_\psi(\pi_2) \neq 0$ then $\pi_1 \simeq \pi_2$.

2.4. The unitary case

Let $E/F$ be a quadratic extension and $\epsilon_{E/F}$ the corresponding quadratic character of $F^\times$.

We fix a quadratic space $W$ of dimension $n$ with skew-Hermitian form

$$\langle ., . \rangle : W \times W \rightarrow E$$

(linear in the second argument). We will denote $G(W)$ its isometry group.

Let $V$ be a quadratic space of dimension $m$ with Hermitian form

$$\langle . | . \rangle : V \times V \rightarrow E$$
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(linear in the second argument). We will denote

\[ G(V) = \{ g \in \text{GL}(V) | \forall v, w \in V, (gv|gw) = (v|w) \} \]

the isometry group of \( V \). The space \( V \) will vary in the remaining of the paper.

Let \( W = R_{E/F}(V \otimes E \ W) \) with symplectic form

\[ \langle \langle \cdot, \cdot \rangle \rangle : W \otimes W \longrightarrow F \]

\[ (v_1 \otimes w_1, v_2 \otimes w_2) \longmapsto \langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \frac{1}{2} \text{Tr}_{E/F}( (v_1, v_2) \langle w_1, w_2 \rangle ) . \]

The pair \((G(V), G(W))\) is a dual reductive pair in \( \text{Sp}(W) \). We have a natural inclusion

\[ \iota : G(V) \times G(W) \longrightarrow \text{Sp}(W) \]

\[ (g, h) \longmapsto \iota(g, h) = g \otimes h . \]

For any pair of characters \( \chi = (\chi_m, \chi_n) \) of \( E^\times \) such that

\[ \chi_n |_{F^\times} = \epsilon_{E/F}^n , \quad \chi_m |_{F^\times} = \epsilon_{E/F}^m , \]

one can define, see [Kud94, Proposition 4.8, p.396], a homomorphism

\[ \tilde{\iota}_\chi : G(V) \times G(W) \longrightarrow \text{Mp}(W) \]

lifting \( \iota \) (the homomorphism \( \tilde{\iota}_\chi \) does depend on \( \chi \)). Since the context will usually make clear which of \( \chi_m \) and \( \chi_n \) is considered, we will often use \( \chi \) instead of \( \chi_m \) or \( \chi_n \). Moreover we define \( \iota_{V, \chi} \) (resp. \( \iota_{W, \chi} \)) the restriction of \( \iota_\chi \) to \( G(V) \times 1 \) (resp. \( 1 \times G(W) \)).

We will denote \( \omega_\psi \) the Weil representation of \( \text{Mp}(W) \) and \( \omega_\chi \) its pullback through \( \tilde{\iota}_\chi \). As before, if \( \pi \) is an irreducible admissible representation of \( G(V) \), we get a representation \( \Theta_\chi(\pi, V) \) of \( G(W) \) such that

\[ S(\pi) \simeq \pi \otimes \Theta_\chi(\pi, V) \]

and if \( \Theta_\chi(\pi, V) \neq 0 \), we say that \( \pi \) appears in the local Theta correspondence for the pair \((G(V), G(W))\). This condition depends on \( \chi_m \) but not on \( \chi_n \). As above we define \( \theta_\chi(\pi, V) \) to be the unique irreducible quotient of \( \Theta_\chi(\pi, V) \) (or 0 if \( \Theta_\chi(\pi, V) = 0 \)).
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**Witt towers.** For a fixed dimension $m$, there are two equivalence classes of Hermitian spaces of dimension $m$ over $E$. These two classes are distinguished by their Hasse invariant

$$\epsilon(V) = \epsilon_{E/F}((-1)^{\frac{m(m-1)}{2}} \det V).$$

We thus get two families of spaces $V^\pm_m$ where the sign is the sign of the Hasse invariant. As Hermitian spaces we have

$$V^+_m + 2 \cong V^+_m \oplus V^1_1,$$

where $V^1_1$ is an hyperbolic plane and the direct sum is orthogonal. We thus get four so-called Witt towers

$$V^+_{2r} = V^+_0 \oplus (V^+_1)^r, \quad V^-_{2r+2} = V^-_2 \oplus (V^-_1)^r,$$

$$V^+_r = V^+_1 \oplus (V^+_1)^r, \quad V^-_{2r+1} = V^-_1 \oplus (V^-_1)^r,$$

where $V^+_0$ is the null vector space, $V^-_2$ is an anisotropic 2-dimensional Hermitian space and $V^+_1$ are one dimensional anisotropic Hermitian spaces. In each case the integer $r$ is the Witt index of the corresponding Hermitian space.

We have

**Proposition 2.6** [HKS96], [Kud96]. — Consider a Witt tower $\{V^\epsilon_m\}$ with $\epsilon = \pm$.

i) (Persistence) If $\theta^\chi(\pi, V^\epsilon_m) \neq 0$ then $\theta^\chi(\pi, V^\epsilon_{m+2}) \neq 0$.

ii) (Stable range) We have $\theta^\chi(\pi, V^\epsilon_m) \neq 0$ if the Weil index $r_0$ of $V^\epsilon_m$ is such that $r_0 \geq n$.

We fix $m_0 \in \{0, 1\}$ and a character $\chi$ of $E^\times$ such that $\chi_{|F^\times} = \epsilon_{E/F}^{m_0}$ and we consider the two towers $V^\pm_m$ with $m$ of the parity of $m_0$ (if $m_0 = 0$ we disregard $V^-_0$ which does not exist). Let $m^\chi_\pm(\pi)$ be the smallest $m$ such that

$$\theta^\chi(\pi, V^\pm_m) \neq 0.$$

Based on several examples, we have

**Conjecture 2.7** (Conservation relation, [HKS96, Speculations 7.5 and 7.6], [KR05, Conjecture 3.6]). — If $\pi$ is an irreducible admissible representation of $G(W)$, then

$$m^\chi_+(\pi) + m^\chi_-(\pi) = 2n + 2.$$

(1) We recall that the Witt index of a quadratic space is the dimension of a maximal totally isotropic subspace
2.5. Degenerate principal series

Let $W_+$ and $W_-$ be two copies of $W$ with respectively the same form as $W$ and its opposite. We keep the pair of characters $\chi = (\chi_m, \chi_n)$. We fix for the space $W_+ \oplus W_-$ the complete polarisation $X \oplus Y$ where $X = \{ (w, -w) | w \in W \}$ and $Y = \{ (w, w) | w \in W \} = \Delta(W)$ (recall that $\Delta$ is the diagonal embedding of $W$ in $W_+ \oplus W_-\)$. We let then

$$
W_+ = R_{E/F}(V \otimes E W_+), \quad W_- = R_{E/F}(V \otimes E W_-),
$$

$$
X = R_{E/F}(V \otimes E X), \quad Y = R_{E/F}(V \otimes E Y).
$$

and we consider the representation $\omega_{V,W_+\oplus W_-,\chi}$ of $G(V) \times G(W_+ \oplus W_-)$ induced by the Weil representation of $W_+ \oplus W_-$ on $S = S(X) \simeq S(V^n)$. Let $R_n(V, \chi)$ be the maximal quotient of $S$ on which $G(V)$ acts by the character $\chi_m$. The space $R_n(V, \chi)$ can be seen as a representation of $G(W) \times G(W)$ via the natural embedding

$i : G(W) \times G(W) = G(W_+) \times G(W_-) \hookrightarrow G(W_+ \oplus W_-)$.

From now on, we will denote $G = G_n = G(W)$ and $\tilde{G} = \tilde{G}_n = G(W_+ \oplus W_-)$ so that $i : G \times G \hookrightarrow \tilde{G}$.

We then have

**Proposition 2.8 ([HKS96], Proposition 3.1 and discussion before).** — If $\pi$ be an irreducible admissible representation of $G(W)$, then

$$
\Theta_\chi(\pi, V) \neq 0 \iff \text{Hom}_{G \times G}(R_n(V, \chi), \pi \otimes (\chi_m \cdot \pi^\vee)) \neq 0.
$$

Let $P_Y$ be the parabolic subgroup of $\tilde{G}$ stabilising $Y$. We will denote $M_Y$ its maximal Levi subgroup and $N_Y$ its unipotent radical. As for the symplectic case, $M_Y$ and $N_Y$ are parametrised respectively by $\text{GL}(X)$ and $\text{Her}(X,Y)$.

For $s \in \mathbb{C}$ and $\chi$ a character of $E^\times$, let

$$I_n(s, \chi) = \text{Ind}_{P_Y}^{\tilde{G}}(\chi) \cdot |^s$$

be the degenerate principal series (the induction is unitary and the elements of $I_n(s, \chi)$ are locally constant functions $\Phi(g, s)$).

We can identify $R_n(V, \chi)$ as a subspace of some $I_n(s, \chi)$ by sending an element $\varphi \in S(X)$ to the function $g \mapsto \omega_\chi(g) \varphi(0)$ — (we recall that we denote $\omega_\chi = \omega_\psi \circ \tilde{i}_{V,\chi}$). The spaces $R_n(V^\pm_m, \chi)$ allows us to decompose the various $I_n(s, \chi)$ as explained by the following proposition.
Proposition 2.9 ([KS97, Theorem 1.2, p.257]).—Let \( V^\pm_m \) be an \( m \)-dimensional unitary space and Hasse invariant \( \pm \). Let \( s_0 = \frac{m-n}{2} \) and \( \chi \) a character of \( E^\times \) such that \( \chi|_{F^\times} = \epsilon_{E/F}^m \).

i) If \( m \leq n \), i.e. if \( s_0 \leq 0 \), then the modules \( R_n(V^\pm_m, \chi) \) are irreducible and \( R_n(V^+_m, \chi) \oplus R_n(V^-_m, \chi) \) is the maximal completely reducible submodule of \( I_n(s_0, \chi) \).

ii) If \( m = n \), i.e. if \( s_0 = 0 \), then \( I_n(0, \chi) = R_n(V^+_m, \chi) \oplus R_n(V^-_m, \chi) \).

iii) If \( n < m < 2n \), i.e. if \( 0 < s_0 < \frac{n}{2} \), then \( I_n(s_0, \chi) = R_n(V^+_m, \chi) + R_n(V^-_m, \chi) \) and \( R_n(V^+_m, \chi) \cap R_n(V^-_m, \chi) \) is the unique irreducible submodule of \( I_n(s_0, \chi) \).

iv) If \( m = 2n \), i.e. if \( s_0 = \frac{n}{2} \), then \( I_n(s_0, \chi) = R_n(V^+_m, \chi), \) of codimension 1 and is the unique irreducible submodule of \( I_n(s_0, \chi) \).

v) If \( m > 2n \), i.e. if \( s_0 > \frac{n}{2} \), then \( I_n(s_0, \chi) = R_n(V^+_m, \chi) \) is irreducible.

In all other cases \( I_n(s, \chi) \) is irreducible.

To refine the aforementioned decompositions we begin with the Bruhat decomposition of \( \tilde{G} \):

\[
\tilde{G} = \prod_{j=0}^{n} P_Y \omega_j P_Y, \quad \text{with} \quad \omega_j = \begin{pmatrix} I_{n-j} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_j \\ 0 & 0 & I_{n-j} & 0 \\ 0 & -I_j & 0 & 0 \end{pmatrix}
\]

and let us introduce, as in [Kud96, p.19] and [Rao93] the mapping

\[
x : \quad \tilde{G} \quad \longrightarrow \quad E^\times / N_{E/F} E^\times \\
p_1 \omega_j^{-1} p_2 \quad \longmapsto \quad \det(p_1 p_2|_Y) \mod N_{E/F} E^\times
\]

Whenever \( \chi|_{F^\times} = 1 \) we can introduce the character \( \chi_{\tilde{G}} \) of \( \tilde{G} \)

\[
\chi_{\tilde{G}}(g) = \chi(x(g))
\]

We extend the definition of \( R_n \) as follows:

\[
R_n(V^+_0, \chi) = R_n(0, \chi) = C \cdot \chi_{\tilde{G}}
\]

and \( R_n(V^+_0, \chi) \) is a submodule of dimension 1 of \( I_n(-\frac{n}{2}, \chi) \) (we are, at least formally, in the case i) of Proposition 2.9). As a last step, we define the intertwining operators

\[
M_n(s, \chi) : I_n(s, \chi) \longrightarrow I_n(-s, \chi)
\]
by the integral
\[ M_n(s, \chi)(\Phi) = \int_{\mathcal{N}_Y} \Phi(w_n u g, s) \, du = \int_{\text{Her}(X,Y)} \Phi(w_n n(b) g, s) \, db, \]
which is convergent for \( \text{Re} \, s > \frac{n}{2} \) and by meromorphic continuation for \( s \in \mathbb{C} \). The Haar measure \( db \) is chosen self-dual with respect to the Fourier transform
\[ \hat{\phi}(y) = \int \phi(b) \psi(\text{Tr}(by)) \, db. \]
We normalise \( M_n(s, \chi) \) using
\[ a(s, \chi) = \prod_{j=0}^{n-1} L_F \left( 2s + j - (n - 1), \chi e_j^2 \right) \]
and then \( M_n^*(s, \chi) = \frac{1}{a(s, \chi)} M_n(s, \chi) \) is holomorphic and non zero (see [KS97, Proposition 3.2]).

**Proposition 2.10** [KS97].—Let \( V_m^\pm \) be the \( m \)-dimensional unitary space of dimension \( m \) and Hasse invariant \( \pm \). Let \( s_0 = \frac{m-n}{2} \) and \( \chi \) a character of \( E^\times \) such that \( \chi|_{F^\times} = \epsilon_{E/F}^n \).

i) If \( m = 0 \), i.e. if \( s_0 = -\frac{n}{2} \), then \( \text{Ker}(M_n^*(-\frac{n}{2}, \chi)) = R_n(V_0^+, \chi) \) and \( \text{Im}(M_n^*(-\frac{n}{2}, \chi)) = R_n(V_{2n}^+, \chi) \).

ii) If \( 1 \leq m < n \), i.e. if \(-\frac{n}{2} < s_0 < 0 \), then \( \text{Ker}(M_n^*(s_0, \chi)) = R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi) \) and \( \text{Im}(M_n^*(s_0, \chi)) = R_n(V_{2m-n}^+, \chi) \cap R_n(V_{2m-n}^-, \chi) \).

iii) If \( n \leq m < 2n \), i.e. if \( 0 \leq s_0 < \frac{n}{2} \), then \( \text{Ker}(M_n^*(s_0, \chi)) = R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi) \), \( M_n^*(s_0, \chi)(R_n(V_m^+, \chi)) = R_n(V_{2m-n}^+, \chi) \) thus we have \( \text{Im}(M_n^*(s_0, \chi)) = R_n(V_{2m-n}^+, \chi) \oplus R_n(V_{2m-n}^-, \chi) \).

iv) If \( m = 2n \), i.e. if \( s_0 = \frac{n}{2} \), then \( \text{Ker}(M_n^*(\frac{n}{2}, \chi)) = R_n(V_{2n}^-, \chi) \) and \( \text{Im}(M_n^*(\frac{n}{2}, \chi)) = M_n^*(\frac{n}{2}, \chi)(R_n(V_{2n}^-, \chi)) = R_n(V_0^+, \chi) \).

### 2.6. Local Zeta integral

The last element we will use is the local Zeta integral of a representation. We fix \( \pi \) an irreducible admissible representation of \( G(W) \).
Definition 2.11. — A matrix coefficient of $\pi$ is a linear combination of functions of the form

$$\phi(g) = \langle \pi(g)\xi, \xi^\vee \rangle$$

where $\xi$ and $\xi^\vee$ are vectors of the space of $\pi$ and $\pi^\vee$ respectively.

Moreover if $\xi_\circ$ and $\xi_\circ^\vee$ are preassigned spherical vectors of $\pi$ and $\pi^\vee$, we let

$$\phi^\circ(g) = \langle \pi(g)\xi_\circ, \xi_\circ^\vee \rangle.$$ 

We parametrise the space of matrix coefficients with the space of $\pi \otimes \pi^\vee$ through the obvious projection. If $s \in \mathbb{C}$ with $\text{Re} \ s$ large enough, $\xi \in \pi$, $\xi^\vee \in \pi^\vee$, $\Phi \in I_n(s, \chi)$, let

$$Z(s, \chi, \pi, \xi \otimes \xi^\vee, \Phi) = \int_G \langle \pi(g)\xi, \xi^\vee \rangle \Phi(i(g, I_n), s)dg$$

and extend it linearly to the space of matrix coefficients of $\pi$. We fix a maximal compact subgroup $K$ of $\tilde{G}$.

Definition 2.12. — A standard section $\Phi$ is a mapping from $\mathbb{C}$ to the set of functions from $\tilde{G}$ to $\mathbb{C}$ such that $\forall s \in \mathbb{C}, \Phi(g, s) = \Phi(s)(g) \in I_n(s, \chi)$ and, moreover, $\Phi(s)|_K$ is independent of $s$.

It is rather obvious that any element $\Phi(g, s) \in I_n(s, \chi)$ can be inserted in a (unique) standard section. The Zeta integral above defines, for $\text{Re} \ s$ sufficiently large, an intertwining operator

$$Z(s, \chi, \pi) \in \text{Hom}_{G \times G}(I_n(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)).$$

If $\Phi$ is a standard section, this operator can be meromorphically extended for all $s \in \mathbb{C}$ to an operator

$$Z^*(s, \chi, \pi) \in \text{Hom}_{G \times G}(I_n(s, \chi), \pi \otimes (\chi \cdot \pi^\vee)).$$

3. Our results

3.1. Decomposition of the degenerate principal series

Let $\Omega(W_+ \oplus W_-)$ be the Grassmannian of the Lagrangians of $W_+ \oplus W_-$. We can identify

$$P_Y\backslash G(W_+ \oplus W_-) \simeq \Omega(W_+ \oplus W_-)$$
using the map $P_Y \cdot g \longmapsto Yg$. There is a right action of $i(G(W) \times G(W))$ on $\Omega(W_+ \oplus W_-)$ which orbits are parametrised by the elements of the decomposition

$$G(W_+ \oplus W_-) = \prod_{r=0}^{r_0} P_Y \delta_r i(G(W) \times G(W))$$

where $r_0$ is the Witt index of $W$. The aforementioned orbits are of the form

$$\Omega_r = P_Y \backslash P_Y \delta_r i(G(W) \times G(W)).$$

The orbit $\Omega_r$ is made of the Lagrangians $Z$ such that $\dim Z \cap W_+ = \dim Z \cap W_- = r$. The only open orbit is that of $Y$, which is $\Omega_0$, while the only closed one is that of $\Omega_{r_0}$ and the closure of the orbit $\Omega_r$ is

$$\overline{\Omega}_r = \bigsqcup_{j \geq r} \Omega_j.$$

We consider the filtration

$$I_n(s, \chi) = I_n^{(r_0)}(s, \chi) \supset \cdots \supset I_n^{(1)}(s, \chi) \supset I_n^{(0)}(s, \chi),$$

where

$$I_n^{(r)}(s, \chi) = \{ \Phi \in I_n(s, \chi) | \Phi|_{\Omega_{r+1}} = 0 \}.$$

Let

$$Q_n^{(r)}(s, \chi) = I_n^{(r)}(s, \chi)/I_n^{(r-1)}(s, \chi)$$

be the successive quotients of the filtration. All $I_n^{(r)}(s, \chi)$ and $Q_n^{(r)}(s, \chi)$ are $G \times G$-stable.

Let $T_W$ be the Witt tower containing $W$. For any $W' \in T_W$ of dimension $n' = n - 2r \leq n$, let $G_{n'} = G(W')$. We identify $W'$ with a subspace of $W$ isomorphic to $W'$. There is a Witt decomposition

$$W = U' \oplus W' \oplus U$$

where $U$ and $U'$ are dual isotropic subspaces of dimension $r$. Let $P_r$ be the parabolic subgroup of $G$ stabilising $U$. The Levi subgroup of $P_r$ is isomorphic to $GL(U) \times G_{n'}$, so that, if we denote $M_r$ its Levi component and $N_r$ its unipotent radical, we have isomorphisms

$$M_r \simeq GL(U) \times G_{n'},$$

$$P_r \simeq (GL(U) \times G_{n'}) \ltimes N_r.$$

Note in particular for $r = 0$ that $U = U' = \{0\}$, $W' = W$ and $P_0 = G_n = G$. 

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An inequality for local unitary Theta correspondence

Let
\[
St_r = i^{-1}(\delta_r^{-1} P_Y \delta_r \cap i(G \times G))
\]
be the stabiliser of \( P_Y \delta_r \) in \( i^{-1}(P_Y) \backslash G \times G \).

**Lemma 3.1.** — For a convenient choice of \( \delta_r \) (specified in Equation (3.3) below), we have
\[
St_r = (\text{GL}(U) \times \text{GL}(U) \times \Delta(G'_{n'})) \ltimes (N_r \times N_r) \subset P_r \times P_r.
\]
Moreover
\[
Q_r(s, \chi) \cong \text{Ind}_{F_r \times F_r}^{G \times G} \left( \chi \cdot |s + \frac{r}{2} \otimes \chi| \cdot |s + \frac{r}{2} \otimes (S(G'_{n'}) \cdot (1 \otimes \chi)) \right)
\]
where the action of \( G'_{n'} \times G'_{n'} \) on the space \( S(G'_{n'}) \cdot (1 \otimes \chi) \) is given by \((g_1, g_2)\varphi(g) = \chi(\det g_2)\varphi(g_2^{-1}gg_1)\).

**Proof.** — We let \( G' = G'_{n'} \).

Recall the Witt decomposition \( W = U' \oplus W' \oplus U \) and consider the Lagrangian
\[
Z = U \times \{0\} \oplus \Delta(W') \oplus \{0\} \times U
\]
in \( W_+ \oplus W_- \). Since the action of \( \tilde{G} \) on \( \Omega(W_+ \oplus W_-) \) is transitive, there exists \( \delta_r \in \tilde{G} \) such that \( Z = Y\delta_r \). Since any linear map from \( Y \) to \( Z \) can be extended to an element of \( \tilde{G} \), we can furthermore require that
\[
\forall v \in U', \delta_r|_{\Delta(U')} (v, v) = (0, vd) \in \{0\} \times U
\]
\[
\delta_r|_{\Delta(W')} = \text{id}_{\Delta(W')}
\]
\[
\forall u \in U, \delta_r|_{\Delta(U)} (u, u) = (u, 0) \in U \times \{0\}
\]
(3.3)

where \( d : U' \rightarrow U \) is any isomorphism. Note in particular that \( \delta_0 = \text{id}_G \). Following [Kud96, Proof of Proposition 2.1, p.68], we find that there is a bijection between the orbit \( \Omega_r \) of \( Z \) and the set
\[
\{(Z_+, Z_-, \lambda)\}
\]
where \( Z_\pm \) is an isotropic subspace of \( W_\pm \) of dimension \( r \) and
\[
\lambda : Z_\pm / Z_\mp \rightarrow Z_- / Z_-
\]
is an isometry\(^2\). The action of \((g_+, g_-) \in G \times G\) on this set is given by

\[
(g_+, g_-)(Z_+, Z_-, \lambda) = (Z_+g_+, Z_-g_-, g_+^{-1}\circ \lambda \circ g_-).
\]

The stabiliser of \((Z_+, Z_-, \lambda)\) is

\[
\{(g_+, g_-) \in G \times G | g_\pm \text{ stabilises } Z_\pm \text{ and } g_+^{-1}\circ \lambda \circ g_- = \lambda \}.
\]

In our situation and with our choice of \(\delta_r\), we have \(Z_+ = Z_- = U, Z_+/Z_+ = W'\) and \(\lambda = \text{id}_{W'}\). Hence, denoting \(\text{pr}_{W'}\) the projection on \(W'\) parallel to \(U' \oplus U\),

\[
\text{St}_r = \{(g_+, g_-) \in P_r \times P_r | g_+|_{W'+U} \circ \text{pr}_{W'} = g_-|_{W'+U} \circ \text{pr}_{W'}\} = (\text{GL}(U) \times \text{GL}(U)) \times \Delta(G') \times (N_r \times N_r).
\]

For further reference, an element of \(P_r\) has the form

\[
\begin{pmatrix}
a & b & c \\
0 & e & b^* \\
0 & 0 & a^\vee
\end{pmatrix}
\]

where \(b^*\) depends on \(b, a\) and \(e\) and where \(c\) satisfies an equation depending on \(a, b\) and \(e\). We thus have

\[
g_\pm = \begin{pmatrix}
a_\pm & b_\pm & c_\pm \\
0 & e_\pm & b^*_\pm \\
0 & 0 & a^\vee_\pm
\end{pmatrix}
\]\n
(3.4)

and the condition \(g_+|_{W'+U} \circ \text{pr}_{W'} = g_-|_{W'+U} \circ \text{pr}_{W'}\) is simply \(e_+ = e_-\).

The description of the stabiliser allows us to describe the induced representations. If \(\tilde{g} \in \text{St}_r\), then \(p(\tilde{g}) = \delta_r i(\tilde{g})\delta_r^{-1} = n \cdot m(a_r(\tilde{g})) \in P_r\). Let \(\xi_{s,r}\) be the character of \(\text{St}_r\) defined by \(\xi_{s,r}(\tilde{g}) = \chi(a_r(\tilde{g}))|\det a_r(\tilde{g})|^{s+\frac{r}{2}}\). Consider the morphism of \(G \times G\)-modules

\[
Q_n^{(r)}(s, \chi) \longrightarrow \text{Ind}_G^G(\xi_{s,r})
\]

\[
\phi_f(g_1, g_2) = \int_{N_r} f(\delta_r, n(u) i(g_1, g_2)) \, du
\]

where \(f \in I_n^{(r)}(s, \chi)\) is a representative of \(\overline{f}\). This morphism is an isomorphism (see [HKS96, Equation (4.9), p.963]). Let \(\tilde{g} = (g_+, g_-)\) be an element of \(\text{St}_r\) decomposed as in (3.4). Then \(\det(a_r(\tilde{g})) = \det a_+ \det a_- \det e_+\) (where we recall that \(e_+ = e_-\)). Since \(e_+ \in G'\), \(|\det e_+| = 1\) hence

\[
Q_n^{(r)}(s, \chi) \cong \text{Ind}_G^G(\chi| \cdot |^{s+\frac{r}{2}} \otimes \chi) \cdot |^{s+\frac{r}{2}} \otimes \chi)
\]

\[
\cong \text{Ind}_{P_r \times P_r}^G(\text{Ind}_{St_r}^{P_r \times P_r}(\chi| \cdot |^{s+\frac{r}{2}} \otimes \chi) \cdot |^{s+\frac{r}{2}} \otimes \chi))
\]

\(^{(2)}\) in [Kud96] it is an anti-isometry but, since \(W_\pm\) has the opposite form of \(W_+\), here \(\lambda\) is an isometry.
The induction from \( \text{St}_r \) to \( P_r \times P_r \) is an induction from \( \Delta(G') \) to \( G' \times G' \). Moreover, if \( f \in \text{Ind}^{G' \times G'}_{\Delta(G')} \chi \) then \( f(h_1, h_2) = \chi(h_2)f(h_2^{-1}h_1, 1) \). Hence
\[
\text{Ind}^{G' \times G'}_{\Delta(G')} \chi \simeq S(G') \cdot (1 \otimes \chi)
\]
where the action of \( G' \times G' \) on \( S(G') \cdot (1 \otimes \chi) \) is given by
\[
\rho(g_1, g_2)\varphi(g) = \chi(\det g_2)\varphi(g_2^{-1}gg_1). \]
Hence
\[
\text{Ind}^{P_r \times P_r}_{\text{St}_r}(\chi \mid \cdot \mid s + r_2 \otimes \chi \mid \cdot \mid s + r_2 \otimes \chi) \simeq \chi \mid \cdot \mid s + r_2 \otimes \chi \mid \cdot \mid s + r_2 \otimes (S(G') \cdot (1 \otimes \chi)).
\]
The result follows. □

### 3.2. Simplicity of poles

We prove in our case the result of [KR05, section 5]. We follow the same method. We denote \( \chi_0 \) the trivial character of \( F^\times \).

**Proposition 3.2.**—Let \( \delta_s \in \mathcal{H}(G \vert K) \otimes \mathbb{C}[q^s, q^{-s}] \) be the element defined by
\[
\delta_s = \prod_{i=1}^{r_0} (1 - q^{-s-\frac{1}{2}}t_i)(1 - q^{-s-\frac{1}{2}}t_i^{-1}).
\]
where we recall that \( \mathcal{H}(G \vert K) \simeq \mathbb{C}[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]^{W_G} \). For an unramified representation \( \pi \) of \( G \), let \( \pi(\delta_s) \) be the scalar by which \( \delta_s \) acts on the unramified vector in \( \pi \). Then for all matrix coefficients \( \phi \) of \( \pi \) and all standard sections \( \Phi(s) \in I_n(s) \), the function
\[
\pi(\delta_s) \cdot Z(s, \chi_0, \pi, \phi, \Phi)
\]
is an entire function of \( s \).

**Proof of Proposition 3.2.**—We divide the proof into four steps.

### 3.2.1. Step 1

By linearity of \( Z \), we can limit ourselves to the case where \( \phi \) is of the form
\[
\phi(g) = \langle \pi(g)\pi(g_1)\xi_0, \pi^\vee(g_2)\xi_\vee \rangle
\]
where $\xi_\circ$ and $\xi_\vee^\circ$ are spherical vectors in $\pi$ and $\pi_\vee$ and $g_1, g_2 \in G$. Then we have

\[
Z(s, \chi_0, \pi, \phi, \Phi) = \int_G \langle (\pi(g)\pi(g_1)\xi_\circ, \pi_\vee(g_2)\xi_\vee^\circ) \Phi_s(i(g, I_n)) \rangle dg
\]

(3.5)

\[
= \int_G \langle \pi(g)\xi_\circ, \xi_\vee^\circ \rangle \Phi_s(i(g, I_n)i(g_1^{-1}, g_2^{-1})) dg
\]

\[
= |\det g_2|^{s + r_0 - \frac{3}{2}} \int_G \phi^\circ(g) \Phi_s(i(g, I_n)i(g_1^{-1}, g_2^{-1})) dg
\]

since $|\det g_2| = 1$ and $\phi^\circ$ is bi-$K$-invariant, for all $k_1, k_2 \in K$,

\[
= \int_G \phi^\circ(g) \Phi_s(i(k_2^{-1}gk_1, I_n)i(g_1^{-1}, g_2^{-1})) dg
\]

\[
= \int_G \phi^\circ(g) \Phi_s(i(g, I_n)i(k_1, k_2)i(g_1^{-1}, g_2^{-1})) dg
\]

and thus

\[
= \int_G \phi^\circ(g) \Psi_s(i(g, I_n)) dg
\]

where, for any $h \in H = G_{2n}$,

\[
\Psi_s(h) := \int_{K \times K} \Phi_s(hi(k_1, k_2)i(g_1^{-1}, g_2^{-1})) dk_1dk_2.
\]

(3.6)

Note that $\Psi_s$ is $K \times K$-invariant section of $I_n(s)$ which is not necessarily standard.

3.2.2. Step 2

We consider the algebra

\[
\mathcal{A} = \mathbb{C}[X, X^{-1}] \otimes \mathcal{H}(G // K) \simeq \mathbb{C}[X, X^{-1}] \otimes \mathbb{C}[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]^{W_G},
\]

where $\mathcal{H}(G // K)$ is the $K$-spherical Hecke algebra of $G$ and the element $\mathfrak{z} \in \mathcal{A}$ defined as:

\[
\mathfrak{z} = \prod_{i=1}^{r_0} (1 - Xq^{\frac{i}{2}}t_i)(1 - Xq^{-\frac{i}{2}}t_i^{-1}).
\]

We let $G \times G$ act on $I_n(s)$ through $i$. We extend this action to $\mathcal{H}(G // K) \times \mathcal{H}(G // K)$ and we let any $\phi \in \mathcal{H}(G // K)$ act as $(\phi, 1) \in \mathcal{H}(G // K) \times \mathcal{H}(G // K)$. We define the action of $\mathcal{A}$ on the space $I_n(s)^{K \times 1}$ of $K \times 1$-fixed vectors of $I_n(s)$ by the aforementioned action of $\mathcal{H}(G // K)$ and by $X \cdot \varphi = q^{-s}\varphi$ for any $\varphi \in I_n(S)$. Note that action of $1 \times G$ commutes with the action of $\mathcal{A}$.
Proposition 3.3. — For any standard section $\Phi_s$ with associated section $\Psi_s$ defined by (3.6), we have

$$\Psi_s \ast \zeta \in I_n^{(0)}(s)^K \times K.$$ 

Proof of Proposition 3.3. — We want to show the the image of $\Psi_s \ast \zeta$ in each $Q_n^{(r)}(s) = Q_n^{(r)}(s, \chi_0)$ is 0 for $0 < r \leq r_0$. As an illustration, we will do the first step separately in the case of a split Hermitian space (in particular $n = 2r_0$). Consider the projection induced by restriction to the closed orbit:

$$\text{pr}_{r_0} : I_n(s) = I_n^{(r_0)}(s) \rightarrow Q_n^{(r_0)}(s) \simeq \text{Ind}_{P_{r_0}}^G(\varphi \cdot |s + \frac{r_0}{2}\rangle) \otimes \text{Ind}_{P_{r_0}}^G(\varphi \cdot |s + \frac{r_0}{2}\rangle)$$

$$\Phi_s \mapsto ((g_1, g_2) \mapsto \Phi_s(i(g_1, g_2))).$$

If we let $\zeta$ act only on the first term of the tensor product on the right side, we have

$$\text{pr}_{r_0}(\Psi_s \ast \zeta) = \text{pr}_{r_0}(\Psi_s) \ast \zeta.$$ 

On the other hand, we have

$$\text{Ind}_{P_{r_0}}^G(\varphi \cdot |s + \frac{r_0}{2}\rangle) \subset \text{Ind}_{B}^G(\lambda)$$

where $B$ is the standard Borel subgroup of $G$ and $\lambda$ is the unramified principal series representation with Satake parameter

$$(q^{s+\frac{r_0}{2} - \frac{1}{2}}, q^{s+\frac{r_0}{2} - \frac{3}{2}}, \ldots, q^{s+\frac{1}{2}}).$$

The element $\zeta$ acts on the $K$-fixed vector of this representation by the scalar

$$\prod_{i=1}^{r_0} (1 - q^{-s - \frac{1}{2}} q^{s+\frac{r_0}{2} - \frac{1}{2} - i}) (1 - q^{-s - \frac{1}{2}} q^{s} - s - r_0 - \frac{1}{2} + i) = 0.$$ 

This means that $\text{pr}_{r_0}(\Psi_s \ast \zeta) = 0$ i.e. that $\Psi_s \ast \zeta \in I_n^{(r_0-1)}(s)$.

More generally, if we restrict the orbit of a section to $\Omega_r$, we obtain a map

$$\text{pr}_r : I_n(s) \rightarrow \text{Ind}_{P_r \times P_r}^{G \times G}(\varphi \cdot |s + \frac{r}{2}\rangle \otimes \varphi \cdot |s + \frac{r}{2}\rangle \otimes C(G_{n-2r})) =: B_r(s)$$

where $C(G_{n-2r})$ is the space of smooth functions on $G_{n-2r}$. There is a non-degenerate pairing between $Q_n^{(r)}(s)$ and $B_r(-s - r)$ given by

$$\langle f_1, f_2 \rangle = \int_{P_r \times P_r \times G \times G} \langle f_1(g_1, g_2), f_2(g_1, g_2) \rangle_{G_{n-r}} d\mu(g_1) d\mu(g_2),$$

where the internal pairing is the integration over $G_{n-r}$ and the external integral is the invariant functional for functions which transform on the
left according to the square of the modulus character. A straightforward
density argument shows that \( \phi \in Q_n^{(r)}(s) \) is 0 if and only if it pairs to zero
against all elements of the subspace \( Q_n^{(r)}(-s-r) \subset B_r(-s-r) \). In addition if
\( \phi \in Q_n^{(r)}(s) \) we can limit ourselves to the elements of \( Q_n^{(r)}(-s-r)K \times K \).
Let \( f_s \in Q_n^{(r)}(-s-r)K \times K \) and \( s = \frac{q - s - r}{2}, \ldots, \frac{q - s + r}{2} \).

\[ \langle \text{pr}_r(\Psi_s * \tilde{\lambda}_s), f_s \rangle = \langle \text{pr}_r(\Psi_s) * \tilde{\lambda}_s, f_s \rangle = \langle \text{pr}_r(\Psi_s), f_s * \tilde{\lambda}_s \rangle. \]

**Lemma 3.4.** — For any \( f_s \in Q_n^{(r)}(-s-r)K \times K \) we have

\[ f_s * \tilde{\lambda}_s = 0. \]

**Proof of Lemma 3.4.** — Since \( f_s \) is an element of a parabolic induction
and is fixed by a maximal compact, it is determined by its value at the
identity element \( I_n \). It is not difficult to see that \( f_s(I_n) \in \mathcal{S}(G)K \times K_{n-r}K \)
where \( K_{n-r} = G_{n-r} \cap K \). Let \( \tau \) be an irreducible admissible representation
of \( G_{n-r} \). The action of \( \mathcal{S}(G_{n-r}) \) on \( \tau \) determines a \( G_{n-r} \times G_{n-r} \)-equivariant
map

\[ \mu_\tau : \mathcal{S}(G_{n-r}) \longrightarrow \text{Hom}^{\text{smooth}}(\tau, \tau) \cong \tau^\vee \otimes \tau \]

where \( \text{Hom}^{\text{smooth}}(\tau, \tau) \) is the space of vector-space homomorphisms fixed by
a compact open subgroup of \( G_{n-r} \times G_{n-r} \). The two factors of \( G_{n-r} \times G_{n-r} \) act respectively by pre- and post-multiplication on the elements of \( \text{Hom}^{\text{smooth}}(\tau, \tau) \) so that each has finite dimensional image. A function \( \varphi \in \mathcal{S}(G_{n-r})K \times K_{n-r} \) is nonzero if and only if there exists an irreducible admissible representation \( \tau \) such that \( \tau(\varphi) \neq 0 \), i.e. such that \( \mu_\tau(\varphi) \neq 0 \).

Consider \( f_s * \tilde{\lambda}_s \). Let \( \tau \) be, as above, an irreducible admissible representation
of \( G_{n-r} \). The map \( \mu_\tau \) induces

\[ \text{Ind}(\mu_\tau) : \text{Ind}_{P_r \times P_r}^{G \times G}(\| \cdot |^{-s-rac{r}{2}} \otimes \| \cdot |^{-s-rac{r}{2}} \otimes \mathcal{S}(G_{n-r})) \]

\[ \longrightarrow \text{Ind}_{P_r \times P_r}^{G \times G}(\| \cdot |^{-s-rac{r}{2}} \otimes \| \cdot |^{-s-rac{r}{2}} \otimes \tau^\vee \otimes \tau) \]

which satisfies \( \text{Ind}(\mu_\tau)(f_s)(I_n) = \mu_\tau(f_s(I_n)) \). The latter induced representation is isomorphic to

\[ \text{Ind}_{P_r}^{G}(\| \cdot |^{-s-rac{r}{2}} \otimes \tau^\vee) \otimes \text{Ind}_{P_r}^{G}(\| \cdot |^{-s-rac{r}{2}} \otimes \tau) \]

which can be embedded in

\[ \text{Ind}_{B}^{G} \lambda_1 \otimes \text{Ind}_{B}^{G} \lambda_2 \]

where the Satake parameters are

\[ \lambda_1 = (q^{-s-rac{r}{2}}, q^{-s-rac{r}{2}}, \ldots, q^{-s+rac{r}{2}}, q^{-\nu_1}, \ldots, q^{-\nu_{n-r}}) \]
\[ \lambda_2 = (q^{-s-rac{r}{2}}, q^{-s-rac{r}{2}}, \ldots, q^{-s+rac{r}{2}}, q^{-\nu_1}, \ldots, q^{-\nu_{n-r}}) \]

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(\text{where } (q^{\nu_1}, \ldots, q^{\nu_{n-r}}) \text{ is the Satake parameter of } \tau). \text{ The operator } z_s^\vee \text{ acts on the unique line of } K \times K\text{-invariant vectors of this representation by the scalar}

\[ \prod_{i=1}^{r} (1 - q^{-s} q^{-\frac{i}{2}} q^{s-\frac{i}{2}})(1 - q^{-s} q^{-\frac{i}{2}} q^{s+\frac{i}{2}}) \cdot \text{(factor)} = 0. \]

But \text{Ind}(\mu_\tau)(f_s) \text{ is a } K \times K\text{-invariant vector in this representation so that}

\[ \text{Ind}(\mu_\tau)(f_s)^{*} z_s^\vee(I_n) = 0 \]

Since this is true for all \( \tau \), we have \( f_s^* z_s^\vee(I_n) = 0 \) and thus \( f_s^* z_s^\vee = 0. \)

□

We have \text{pr}_r(\Psi_s \ast \mathfrak{z}) = 0 \text{ for all } r > 0, \text{ which means that the support of } \Psi_s \ast \mathfrak{z}

3.2.3. Step 3

Consider the isomorphism

\[ \text{pr}_0 : I_n(s) \longrightarrow Q_n^{(0)}(G) \simeq S(G). \]

Proposition 3.3 shows that, for a fixed \( s \), we have \( \text{pr}_0(\Psi_s \ast \mathfrak{z}) \in S(G)^{K \times K}. \)

Its support could vary with \( s \). The following proposition shows that the support of \( \text{pr}_0(\Psi_s \ast \mathfrak{z}) \) is bounded uniformly in \( s \).

**Lemma 3.5.** — We have \( \text{pr}_0(\Psi_s \ast \mathfrak{z}) \in C[q^s, q^{-s}] \otimes S(G)^{K \times K} = C[q^s, q^{-s}] \otimes \mathcal{H}(G // K). \)

**Proof of Lemma 3.5.** — Using the Cartan decomposition, write

\[ \text{pr}_0(\Psi_s \ast \mathfrak{z}) = \sum_{\lambda \in \Lambda} c_\lambda(s) L_\lambda, \]

where \( L_\lambda \) is the characteristic function of the double coset \( Kg_\lambda K \) and \( \Lambda \) is the usual semigroup.

**Lemma 3.6.** — We have

\[ c_\lambda(s) \in C[q^s, q^{-s}] \]

and thus is an entire function of \( s \).
Proof. — We have
\[ c_\lambda(s) \cdot \|L_\lambda\|^2 = \int_G (\Psi_s * z)(i(g, I_n)) \cdot L_\lambda(g) dg. \] (3.7)
The integral on the right is a (finite) linear combination, with coefficients in \( C[q^s, q^{-s}] \) of integrals of the form
\[
\int_G \int_G (\Psi_s * z)(i(g, I_n)) \cdot L_\mu(g_0) \cdot L_\lambda(g) dg_0 \cdot L_\lambda(g) dg
\] (3.8)
\[
= \int_G \int_G (\Psi_s * z)(i(g_0, I_n)) \cdot L_\mu(g^{-1} g_0) \cdot L_\lambda(g_0) dg_0 dg d g
\]
where \( \varphi \) is a function depending on \( \lambda \) and \( \mu \). Since this function is a (finite) linear combination of characteristic functions of cosets \( gK \), the integral in the last line of (3.8) is a (finite) linear combination with coefficients in \( C[q^s, q^{-s}] \) of integrals of the form
\[
\int_K \int_{K \times K} \Phi_s(i(gk, I_n)i(k_1, k_2)i(g_1^{-1}, g_2^{-1})) dk_1 dk_2 dk.
\]
But \( \Phi_s \) is standard, hence it is right-invariant under a fixed compact open subgroup \( H \), uniformly in \( s \). This means that the set of \( g \) necessary to obtain the full integral (3.7) is finite and fixed. The elements \( g_1 \) and \( g_2 \) are fixed by the matrix coefficient \( \phi \) we are considering and thus the integral (3.7) is a (finite) linear combination of \( q^{\ell s} \) with \( \ell \in \mathbb{Z} \). \( \square \)

Let then \( \Lambda_1 \) be the set of \( \lambda \in \Lambda \) such that \( c_\lambda \neq 0 \) and for \( \lambda \in \Lambda \) let
\[
D_\lambda = \{ s \in \mathbb{C} : c_\lambda(s) = 0 \}.
\]
If \( \lambda \in \Lambda_1 \) then \( D_\lambda \) is a numerable subset of \( \mathbb{C} \). Hence \( \bigcup_{\lambda \in \Lambda_1} D_\lambda \) is numerable and thus different from \( \mathbb{C} \). Let \( s_0 \in \mathbb{C} \) be such that \( \forall \lambda \in \Lambda_1, c_\lambda(s_0) \neq 0 \). Since
\[
\text{pr}_0(\Psi_{s_0} * z) = \sum_{\lambda \in \Lambda_1} c_\lambda(s_0) \cdot L_\lambda
\]
has compact support, \( \Lambda_1 \) is finite and thus for all \( s \in \mathbb{C} \), \( \text{pr}_0(\Psi_s * z) \) has support in \( \bigcup_{\lambda \in \Lambda_1} L_\lambda \). \( \square \)

3.2.4. Step 4

Going back to the Zeta integral in (3.5), we define
\[
Z^*(s, \chi_0, \pi, \phi, \Phi) = \int_G \phi^0(g)(\Psi_s * z)(i(g, I_n)) dg.
\]
This integral is equal to the scalar by which $\text{pr}_0(\Psi_s \ast \xi)$ acts on $\xi_0$ and is thus an entire function of $s$ because it is an element of $\mathbb{C}[q^s, q^{-s}]$. On the other hand, if $\text{Re}(s)$ is large enough we can unfold

$$Z^*(s, \chi_0, \pi, \phi, \Phi) = \pi(\xi_0) \int_G \hat{\phi}^\circ(g) \Psi_s(i(g, I_n)) dg$$

$$= \pi(\xi_0) Z(s, \chi_0, \pi, \phi, \Phi)$$

where $\pi(\xi_0)$ is the scalar by which $\xi_0 = \xi|_{X=q^{-s}}$ acts on the spherical vector of $\pi$. Since $Z^*(s, \chi_0, \pi, \phi, \Phi)$ is an entire function of $s$, this completes the proof or Proposition 3.2. \hfill \square

### 3.3. The conjecture holds for the trivial representation in the even dimensional tower

**Definition 3.7** ([HKS96, Definition 4.6, p.963]). — For $s_0 \in \mathbb{C}$, $\chi$ a character and $\pi$ and irreducible admissible representation of $G$, we say that $\pi$ occurs in the boundary at the point $s = s_0$ if

$$\text{Hom}_{G \times G}(Q_n^{(r)}(s_0, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0$$

for some $r > 0$.

**Proposition 3.8.** — Let $\pi = 1$ the trivial representation of $G$, $\varpi_E$ an uniformiser of $E$ and $Q_E = |\varpi_E|$. We will denote $X^u(E^\times)$ the set of unramified characters of $E^\times$. Let

$$X(1) \neq \left\{ (s, \chi) \in \mathbb{C} \times X^u(E^\times) \left| \chi(\varpi_E) = (-1)^k, s = \frac{n}{2} - r - \frac{ki\pi}{\log q_E}, 1 \leq r \leq r_0 \right. \right\}$$

with $1 \leq r \leq r_0$ and $k \in \mathbb{Z}$.

Then 1 appears in the boundary at $s$ if and only if $(s, \chi) \in X(1)$. Moreover if $(s_0, \chi) \notin X(1)$, for any standard section $\Phi$ the operator $Z(s, \chi, 1)$ is holomorphic at $s = s_0$ and

$$\text{Hom}_{G \times G}(I_n(s_0, \chi), 1 \otimes \chi) = C \cdot Z(s, \chi, 1).$$

**Proof.** — We know from Lemma 3.1 that

$$\text{Hom}_{G \times G}(Q_n^{(r)}(s, \chi), 1 \otimes \chi)$$

$$= \text{Hom}_{G \times G} \left( \text{Ind}_P^G \chi^s \otimes \chi \otimes \text{St}(G) \cdot (1 \otimes \chi) \right) \cdot (1 \otimes \chi)$$

$$\approx \text{Hom}_{G \times G} \left( 1 \otimes \chi^{-1}, \text{Ind}_P^G \chi^s \otimes \text{St}(G) \cdot (1 \otimes \chi^{-1}) \right)$$

$$\approx \text{Hom}_{M_r \times M_r} \left( 1 \otimes \chi^{-1} \cdot \chi^{-1} \cdot \text{St}(G) \cdot (1 \otimes \chi^{-1}) \right)$$

where

$$\chi^s \otimes \chi \otimes \text{St}(G) \cdot (1 \otimes \chi^{-1}) \approx \text{St}(G) \cdot (1 \otimes \chi^{-1})$$

and

$$\chi^s \otimes \chi \otimes \text{St}(G) \cdot (1 \otimes \chi^{-1}) \approx \text{St}(G) \cdot (1 \otimes \chi^{-1})$$

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because the Jacquet module for $1 \otimes \chi^{-1}$ is $1 \otimes \chi^{-1}$ (as a representation of $M_r$).

Now if $g$ corresponds to $(a, g')$ in Equation (3.2) then $\det g = \det \frac{a}{\det a^{-1}}$ $\det g'$ so that $\chi(\det g) = \chi(\det a)^2 \chi(\det g')$ but $\dim \Hom_{G' \times G'}(1 \otimes \chi^{-1}, C^\infty(G') \cdot (1 \otimes \chi^{-1})) = 1$ (see [HKS96, end of section 4, p.964] for general $\pi$). Thus

$$\simeq \Hom_{\GL(U) \times \GL(U)}(1 \otimes \chi^{-2}, \chi^{-1} | \cdot |^{-s+\frac{n}{2} - r} \otimes \chi^{-1} | \cdot |^{-s+\frac{n}{2} - r})$$

It follows that $\pi$ occurs in the boundary at $s$ if and only if $\chi$ is unramified, $\chi(\varpi_E) = (-1)^k$ and $(s - \frac{n}{2} + r) \log q_E + ki\pi = 0$, as required.

Suppose $(s_0, \chi) \not\in X(1)$, i.e. $1$ does not appear in the boundary. Let $k$ be the maximum order of the $Z$ integral in $s = s_0$ (as $\Phi$ varies). Thus

$$Z(s, \chi, 1, \Phi) = \frac{\tau_{-k}(s, \chi, 1, \Phi)}{(s - s_0)^k} + \cdots + \tau_0(s, \chi, 1, \Phi) + \cdots$$

where the $\tau_i$ are holomorphic functions of $s$ in a neighbourhood of $s_0$ and $\tau_{-k}$ is non-zero. The leading term $\tau_{-k}$ is itself an intertwining operator. If we had $k > 0$, that is, if the $Z$ integral had a pole in $s = s_0$, the restriction of $\tau_{-k}$ to $I_n^{(0)}(s_0, \chi)$ would be zero because the $Z$ integral is convergent on

$$I_n^{(0)}(s_0, \chi) = Q_n^{(0)}(s, \chi) \simeq S(G) \cdot (1 \otimes \chi)$$

thus convergent for every standard section $\Phi(s)$ such that $\Phi \in I_n^{(0)}(s, \chi)$. This means that we would have a non-zero intertwining operator in $\Hom_{G \times G}(Q_n^{(r)}(s, \chi), 1 \otimes \chi)$ for some $r > 0$, which is impossible by hypothesis. Thus $k \leq 0$, i.e. the integral is entire for any $\Phi \in I_n(s_0, \chi)$. Moreover, $Z(s_0, \chi, 1)$ is a non-zero intertwining operator between $I_n^{(0)}(s_0, \chi)$ and $1 \otimes \chi$, which means that $\Hom_{G \times G}(I_n^{(0)}(s_0, \chi), 1 \otimes \chi)$ is non zero, thus has dimension 1, and that $Z(s_0, \chi, 1)$ is its basis.

Let $\lambda \in \Hom_{G \times G}(I_n(s_0, \chi), 1 \otimes \chi)$. Its restriction $\tilde{\lambda}$ to $I_n^{(0)}(s_0, \chi)$ is a multiple of $Z(s_0, \chi, 1)$. Since $1$ is supposed not to appear in the boundary, if $\lambda \neq 0$, then $\tilde{\lambda} \neq 0$, i.e. $\tilde{\lambda} = cZ(s_0, \chi, 1)$ for some $c \neq 0$. Since $\lambda - cZ(s_0, \chi, 1)$ is zero on $I_n^{(0)}(s_0, \chi)$, it must be zero everywhere, i.e. $\lambda = cZ(s_0, \chi, 1)$. \qed

**Theorem 3.9.** — Let $m$ be an even integer and $\chi_0$ the trivial character of $E^\times$, then

$$\forall m \leq 2n, \quad \Hom_{G \times G}(R_n(V_m^-, \chi_0), 1) = 0,$$
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so that by (ii) of Proposition 2.6

\[ \text{Hom}_{G \times G} (R_n (V_{2n+2}^{-}, \chi_0), 1) = 0 \]

and thus \( m_{\chi_0}^{-} (1) = 2n + 2 \). Since \( m_{\chi_0}^{+} (1) = 0 \), we have

\[ m_{\chi_0}^{+} (1) + m_{\chi_0}^{-} (1) = 2n + 2 \]

\( \text{Proof. — By (i) of Proposition 2.6, it suffices to prove that} \)

\[ \text{Hom}_{G \times G} (R_n (V_{2n}^{-}, \chi_0), 1) = 0 \]

From Proposition 3.8 we know that

\[ \text{Hom}_{G \times G} (I_n (-n, \chi_0), 1) \]

is non zero and is generated by

\[ Z \left( -\frac{n}{2}, \chi_0, 1 \right) \]

which is holomorphic at \( -\frac{n}{2} \). The element of \( I_n (-\frac{n}{2}, \chi_0) \) equal to 1 on \( K \) is \( \chi_{0, \tilde{G}} \). As seen in [Li92, Theorem 3.1, p.186] and [LR05, Proposition 3, p.333] we have

\[ Z \left( -\frac{n}{2}, \chi_0, 1, \phi_0, \chi_{0, \tilde{G}} \right) \neq 0 \]

and thus \( Z \left( -\frac{n}{2}, \chi_0, 1 \right) (\chi_{0, \tilde{G}}) \neq 0 \). Let

\[ \phi \in \text{Hom}_{G \times G} (R_n (V_{2n}^{-}, \chi_0), 1) \]

and

\[ \tilde{\phi} = \phi \circ M_n^* \left( -\frac{n}{2}, \chi_0 \right) \in \text{Hom}_{G \times G} \left( I_n \left( -\frac{n}{2}, \chi_0 \right), 1 \right) \]

We have \( \chi_{0, \tilde{G}} \in R_n (V_0^+, \chi_{0, \tilde{G}}) = \ker M_n^* \left( -\frac{n}{2}, \chi_0 \right) \) so that \( \tilde{\phi}(\chi_{0, \tilde{G}}) = 0 \). This means that \( \tilde{\phi} = 0 \) because it is a multiple of \( Z \left( -\frac{n}{2}, \chi_0, 1 \right) \). We know from Proposition 2.10 that the mapping

\[ M_n^* \left( -\frac{n}{2}, \chi_0 \right) : I_n \left( -\frac{n}{2}, \chi_0 \right) \rightarrow R_n (V_{2n}^{-}, \chi_0) \]

is surjective so that \( \phi = 0 \). \( \square \)
3.4. Half of the conjecture

**Theorem 3.10.** — Let $\pi$ be an irreducible admissible representation of $G(W)$, then

$$m^+_\chi(\pi) + m^-_{\chi}(\pi) \geq 2n + 2.$$ 

**Proof.** — Fix $m_0 \in \{0, 1\}$, a character $\chi$ of $E^\times$ such that $\chi|_{F^\times} = \epsilon_{E/F}^{m_0}$ and suppose we have two Hermitian spaces $V^+_a$ and $V^-_b$ such that

$$\theta_\chi(\pi, V^+_a) \neq 0 \quad \text{and} \quad \theta_\chi(\pi, V^-_b) \neq 0,$$

with $\dim V^+_a = a$, $\dim V^-_b = b$, $a$ and $b$ of the parity of $m_0$, $\epsilon(V^+_a) = 1$ and $\epsilon(V^-_b) = -1$. Let $V^-_{b,-}$ be the same space as $V^-_b$ with opposite form and

$$\mathbb{W}_a = V^+_a \otimes W, \quad \mathbb{W}_b = V^-_b \otimes W, \quad \mathbb{W}_{b,-} = V^-_{b,-} \otimes W.$$ 

We denote $\omega_{a,\chi}$ (resp. $\omega_{b,\chi}$, $\omega_{b,-,\chi}$) the representations of $G$ induced by the representations $\omega_{a,\psi}$ (resp. $\omega_{b,\psi}$, $\omega_{b,-,\psi}$) of $\text{Mp}(\mathbb{W}_a)$ (resp. $\text{Mp}(\mathbb{W}_b)$, $\text{Mp}(\mathbb{W}_{b,-})$). By hypothesis on $V^+_a$ and $V^-_b$ we have two non-zero (and thus surjective) elements

$$\lambda \in \text{Hom}_G(\omega_{a,\chi}, \pi), \quad \mu \in \text{Hom}_G(\omega_{b,\chi}, \pi).$$

Let $g_0 \in \text{GL}_F(W)$ be an $F$-automorphism of $W$ which is conjugate-linear as an $E$-morphism. Then $\text{Ad}(g_0)$ is a MVW involution on $G$. Conjugating $\mu$ and $\pi$ by $\text{Ad}(g_0)$ we get a non-zero morphism

$$\mu^\vee \in \text{Hom}_G(\omega_{b,\chi}^\vee, \pi^\vee)$$ 

and thus a surjective

$$\nu_0 = \lambda \otimes \mu^\vee \in \text{Hom}_{G \times G}(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, \pi \otimes \pi^\vee).$$

We consider the projection of $\nu_0$ on the trivial subquotient and see it as a $G$-homomorphism through the diagonal action of $G$. We get a non-zero element

$$\nu \in \text{Hom}_G(\omega_{a,\chi} \otimes \omega_{b,\chi}^\vee, 1).$$

We have

$$\omega_{b,\psi} \simeq \omega_{b,-,\psi} \simeq \omega_{b,-,\psi}.\quad (3)$$

On the other hand we can identify $\text{Mp}(\mathbb{W}_b)$ and $\text{Mp}(\mathbb{W}_{b,-})$ in which case we get the following

---

(3) The first isomorphism holds true because $\omega_{b,\psi}$ is unitary, the second because of the definition of $r(g)$ in 2.3
Lemma 3.11. — We have
\[ \tilde{i}_{b,\chi} \cong \tilde{i}_{b,-,\chi^{-1}} , \]
where we added a subscript to \( \tilde{i} \) to remember which Hermitian space is involved.

Proof. — The space \( V^-_b \) can be decomposed as an orthogonal direct sum of a split space and zero, one or two anisotropic lines. Since the splitting \( \tilde{i} \) is additive, we consider separately the split and the anisotropic case.

We first consider the case in which \( V^-_b \) is split. We will need some additional notations (see [HKS96, n.10, p.950]). For any additive character \( \eta \) of \( F \) and \( a \in F \) we will let \( \eta_a \) be the character such that \( \eta_a(x) = \eta(ax) \), \( \gamma_F(\eta) \in \mu_8 \) is the Weil index of the quadratic character \( x \mapsto -\eta(x^2) \) and \( \gamma_F(a, \eta) = \frac{\gamma_F(\eta_a)}{\gamma_F(\eta)} \). Recall that (see [HKS96, n.11, p.950])

\[ \gamma_F(ab, \eta) = (a, b)_F \gamma_F(a, \eta) \gamma_F(b, \eta) . \]

Let \( \eta \) be the character such that \( \eta(x) = \psi\left(\frac{1}{2}x\right) \) (i.e. \( \eta = \psi_2 \)). For \( g \in G \), we denote \( j(g) \) the integer such that \( i(g, I_n) \in P_Y \delta_{j(g)} i(G \times G) \). Since \( V^-_b \) is split we have (see [HKS96, 1.15, p.953]),

\[ \tilde{i}_{b,\chi}(g) = (\nu_b(g), \beta_{V^-_b,\chi}(g)) \]

with
\[ \beta_{V^-_b,\chi}(g) = \chi(x(g)) \gamma_F(\eta \circ RV)^{-j(g)} \]

where
\[ \gamma_F(\eta \circ RV) = (\Delta, \det V^-_b) F \gamma_F(-\Delta, \eta)^b \gamma_F(-1, \eta)^{-b} . \]

Let \( \varphi : \text{Sp}(\mathbb{W}_b) \times \mathbb{C}^1 \cong \text{Mp}(\mathbb{W}_b) \rightarrow \text{Sp}(\mathbb{W}_{b,-}) \times \mathbb{C}^1 \cong \text{Mp}(\mathbb{W}_{b,-}) \)

be the identification. Then \( \chi(x(g)) = \chi^{-1}(x(g)) \) and

\[ \frac{\gamma_F(-\Delta, \eta) \gamma_F(-1, \eta)^{-1}}{\gamma_F(\eta-\Delta) \gamma_F(\eta-1)^{-1}} = \frac{\gamma_F(\eta \Delta)}{\gamma_F(\eta_1)} = \frac{\gamma_F(\Delta, \eta) \gamma_F(1, \eta)^{-1}}{\gamma_F(-\Delta, \eta)(-1)^{-1} \gamma_F(-1, \eta)^{-1}} \]

\[ = (\Delta, -1)_F \gamma_F(-\Delta, \eta)(-1, -1)_F \gamma_F(-1, \eta)^{-1} \]

\[ = (\Delta, -1)_F \gamma_F(-\Delta, \eta) \gamma_F(-1, \eta)^{-1} \]

\[ \text{(4) for this single proof, we fix } \delta \in F^\times - F^\times \text{ such that } \Delta = \delta^2 \in F^\times \text{ and use it to identify the Hermitian and skew-Hermitian spaces} \]
thus, since \( \det V_{b,-}^- = (-1)^b \det V_b^- \), we have \( \beta_{V_b^-}^- \chi(g) = \beta_{V_{b,-}^-}^- \chi^{-1}(g) \) and

\[
\varphi \circ \tilde{\iota}_{b,-} = \tilde{\iota}_{b,-} \chi^{-1}
\]
as claimed.

We now consider the case in which \( V_{b,-}^- \) is an anisotropic line. We identify \( V_{b,-}^- \) with \( E \) and if \( (x,y) \in E^2 \), we have \( \langle x,y \rangle = axy \) for some \( a \in F \). If \( g \in G(V_{b,-}^-) = E^1 \), we decompose \( g = x + \delta y \) (with \( x, y \in F \)) and we have (see [Kud94, Proposition 4.8, p.396])

\[
\beta_{V_{b,-}^-} \chi(g) = \chi(\delta(g - 1)) \gamma_F(2ay(x - 1), \eta) \gamma_F(\eta)(\Delta, -2y(1 - x))_F
\]
and

\[
\beta_{V_{b,-}^-}^- \chi(g) = \chi(\delta(g - 1)) \gamma_F(\eta_{2ay(x-1)})(\Delta, -2y(1 - x))_F .
\]

It is immediate that \( \beta_{V_{b,-}^-}^- \chi^{-1}(g) = \beta_{V_{b,-}^-}^- \chi(g) \) and

\[
\varphi \circ \tilde{\iota}_{b,-} = \tilde{\iota}_{b,-} \chi^{-1}
\]
as claimed. \( \square \)

Let

\[
V_{a,b,-} = V_{a}^+ \oplus V_{b,-}^- , \quad W_{a,b,-} = W_{a} \oplus W_{b,-}
\]
and let, as before, \( \chi_0 \) be the trivial character of \( E^\times \). We denote, as above, \( \omega_{a,b,-,\chi_0} \) the representation of \( G \) induced by the Weil representation \( \omega_{a,b,-,\psi} \).

Let

\[
\tilde{i} : \text{Mp}(W_a) \times \text{Mp}(W_{b,-}) \longrightarrow \text{Mp}(W_{a,b,-})
\]
be the natural map whose restriction to \( C^1 \) is the product. Then

\[
\tilde{i}^* \omega_{a,b,-,\psi} = \omega_{a,\psi} \otimes \omega_{b,-,\psi} .
\]

According to [HKS96, Lemma 5.2, p.964],

\[
\tilde{i}_{a,b,-,\chi_0} = \tilde{i} \circ (\tilde{i}_{a,\chi} \times \tilde{i}_{b,-,\chi^{-1}}) \circ \Delta : G \longrightarrow \text{Mp}(W_{a,b,-}) .
\]

Thus as a representation of \( G \) we have

\[
\omega_{a,\chi} \otimes \omega_{b,-,\chi^{-1}} \simeq \omega_{a,b,-,\chi_0} .
\]

We thus have a non-zero element

\[
\nu \in \text{Hom}_G(\omega_{a,\chi} \otimes \omega_{b,-,\chi}^\vee, 1) \simeq \text{Hom}_G(\omega_{a,b,-,\chi_0}, 1) .
\]
We have \( \dim V_{a,b,-} = a + b \) even. Let us compute \( \epsilon(V_{a,b,-}) \):

\[
\epsilon(V_{a,b,-}) = (-1)^{\frac{(a+b)(a+b-1)}{2}} \det V_{a,b,-} = (-1)^{\frac{(a-1)+ab+b(a+b-1)}{2}} \det V_{a}^{+} \det V_{b}^{-} = (-1)^{\frac{a(a-1)+ab}{2}+ab} \det V_{a}^{+}(-1)^{b} \det V_{b}^{-} = (-1)^{ab+b}(-1)^{\frac{(a-1)}{2}} \det V_{a}^{+}(-1)^{\frac{b(b-1)}{2}} \det V_{b}^{-} = (-1)^{ab+b} \epsilon(V_{a}^{+}) \epsilon(V_{b}^{-}) .
\]

Since both \( ab \) and \( b \) have the parity of \( m_{0} \) we have \( \epsilon(V_{a,b,-}) = \epsilon(V_{a}^{+}) \epsilon(V_{b}^{-}) = -1 \). Thus, according to Theorem 3.9

\[
a + b \geq 2n + 2
\]
as needed. \( \square \)

3.5. Criterion

**Definition 3.12.**— For a given \( m \in \{0,...,2n\} \), let \( m' = 2n - m \). The space \( V_{m}^{\pm} \) is said to be complementary to \( V_{m}^{\pm} \) (the space \( V_{2n}^{-} \) has no complementary).

**Remark 3.13.**— If \( V_{m}^{\pm} \) is complementary of \( V_{m}^{\pm} \), then \( s_{0}' = \frac{m'-n}{2} = \frac{2n-m-n}{2} = \frac{n-m}{2} = -s_{0} \).

**Theorem 3.14.**— Fix \( m_{0} \in \{0,1\} \) and a character \( \chi \) of \( E^{\times} \) such that \( \chi|_{F^{\times}} = \epsilon^{m_{0}}_{E/F} \). Suppose that

\[
\dim \text{Hom}_{G \times G}(I_{n}(s_{0}, \chi), \pi \otimes (\chi \cdot \pi^{\vee})) = 1
\]

for all \( s_{0} \) in

\[
\begin{cases} 
\{ -\frac{n}{2}, 1 - \frac{n}{2}, ..., \frac{n}{2} - 1, \frac{n}{2} \} & \text{if } m_{0} = 0 \\
\{ \frac{1-n}{2}, \frac{3-n}{2}, ..., \frac{n-3}{2}, \frac{n-1}{2} \} & \text{if } m_{0} = 1,
\end{cases}
\]
i.e. for all \( s_{0} \in \frac{m_{0}}{2} + \mathbb{Z} \) such that \( |s_{0}| \leq \frac{n}{2} \). Then

\[
m_{\chi}^{+}(\pi) + m_{\chi}^{-}(\pi) = 2n + 2 .
\]

To prove the theorem, we will need the composition series for \( I_{n}(s_{0}, \chi) \) in each case where it is reducible. Using [KS97], we give here those series explicitly with indication of the action of the operators \( M^{*}(s_{0}, \chi) \). In the diagram we have implicitly \( m' = 2n - m \). Note that \( V_{0}^{\pm} \) does not exist,
but we define the space $R_n(V_0^-,\chi)$ as the zero-dimensional subspace in $R_n(V_0^+,\chi)$.

\[
\begin{array}{c}
0 \subset R_n(V_0^+,\chi) \subset I(-\frac{n}{2},\chi) \\
\| \\
R_n(V_0^+,\chi) \quad M^*(\frac{n}{2},\chi)(R_n(V_{2n}^+,\chi)) = M^*(\frac{n}{2},\chi)(I(\frac{n}{2},\chi)) \\
\| \\
M^*(\frac{n}{2},\chi)(R_n(V_{2n}^+,\chi)) \quad \text{Ker} M^*(-\frac{n}{2},\chi) \quad m = 0, \quad s_0 = -\frac{n}{2} \\
\end{array}
\]

\[
\begin{array}{c}
0 \subset R_n(V_{m}^+,\chi) \subset R_n(V_{m}^+,\chi) \oplus R_n(V_{m}^-,\chi) \subset I_n(s_0,\chi) \\
\| \\
R_n(V_{m}^+,\chi) \quad R_n(V_{m}^+,\chi) \oplus R_n(V_{m}^-,\chi) = I(0,\chi) \\
\| \\
\text{Ker} M^*(0,\chi) \quad R_n(V_{m}^-,\chi) \\
\| \\
M^*(0,\chi)(R_n(V_{m}^-,\chi)) \quad m = n, \quad s_0 = 0 \\
\end{array}
\]

\[
\begin{array}{c}
0 \subset R_n(V_{m}^+,\chi) \cap R_n(V_{m}^-,\chi) \subset R_n(V_{m}^+,\chi) + R_n(V_{m}^-,\chi) = I_n(s_0,\chi) \\
\| \\
\text{Im} M^*(-s_0,\chi) \quad R_n(V_{m}^-,\chi) \\
\| \\
\text{Ker} M^*(s_0,\chi) \quad n < m < 2n, \quad 0 < s_0 < \frac{n}{2} \\
\end{array}
\]

\[
\begin{array}{c}
0 \subset R_n(V_{2n}^-,\chi) \subset R_n(V_{2n}^+,\chi) = I_n(\frac{n}{2},\chi) \\
\| \\
\text{Im} M^*(-\frac{n}{2},\chi) \\
\| \\
\text{Ker} M^*(\frac{n}{2},\chi) \quad m = 2n, \quad s_0 = \frac{n}{2} \\
\end{array}
\]

In each case an inclusion sign means that the quotient is non-zero and irreducible.
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Proof. — Fix $m_0 \in \{0,1\}$ and a character $\chi$ of $E^\times$ such that $\chi|_{F^\times} = e^{m_0}_{E/F}$. For $0 \leq m' \leq 2n$, we put $m = 2n - m'$ and recall that $s_0 = \frac{m-n}{2}$.

The case $m^+_\chi(\pi) = 0$ is immediate because it implies $\pi = 1$ and Theorem 3.9 says that $m^-_\chi(\pi) = 2n + 2$.

If $s_0 \geq 0$ we have $I_n(s_0, \chi) = R_n(V^+_m, \chi) + R_n(V^-_m, \chi)$ and thus, thanks to the hypothesis of the theorem, at least one of

$$\text{Hom}_{G \times G}(R_n(V^+_m, \chi), \pi \otimes (\chi \cdot \pi^\vee))$$

is non zero. Thanks to Proposition 2.8 this in turn means that

$$\min(m^+_\chi(\pi), m^-_\chi(\pi)) \leq n + 1$$

(the bound is $n + 1$ and not $n$ in case $m$ and $n$ have opposite parity). If $s_0 < \frac{n}{2}$ then $I_n(s_0, \chi)$ is irreducible and thus

$$R_n(V^+_m, \chi) = I_n(s_0, \chi).$$

Since we have $m > 2n \geq \min(m^+_\chi(\pi), m^-_\chi(\pi))$, by the persistence principle (see Proposition 2.6, point (1.)) we have

$$\text{Hom}_{G \times G}(R_n(V^+_m, \chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0$$

for one and thus both signs $\pm$. This means $\max(m^+_\chi(\pi), m^-_\chi(\pi)) \leq 2n + 2 - m_0$.

Let $\epsilon = \pm$ be such that $m^\epsilon_\chi(\pi) = \min(m^+_\chi(\pi), m^-_\chi(\pi))$. We let $m'$ be $m^\epsilon_\chi(\pi)$ (and choose $m$ and $s_0$ accordingly). As observed above, the case $m' = 0$ has already been proved. If $m' = 1$, then from Theorem 3.10 we have $m^-_\chi(\pi) \geq 2n + 1$ and thus, thanks to the preceding bound, $m^-_\chi(\pi) = 2n + 1$ (observe that if $m' = 1$ then $m_0 = 1$).

We now suppose $2 \leq m' \leq n + 1$, i.e. $-\frac{1}{2} \leq s_0 \leq \frac{n}{2} - 1$. By Theorem 3.10 we thus have $m^-_\chi(\pi) \geq 2n + 2 - m' \geq n + 1$. Since $m'$ is the minimum of $m^\pm_\chi(\pi)$, we have

$$\text{Hom}_{G \times G}(R_n(V^+_m, \chi) \oplus R_n(V^-_{m-2}, \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 0 \quad (3.9)$$

(here $R_n(V^-_0, \chi) = 0$ as defined above). This means that any element of $\text{Hom}_{G \times G}(I_n(-s_0 - 1, \chi), \pi \otimes (\chi \cdot \pi^\vee))$ factors through

$$I_n(-s_0 - 1, \chi)/R_n(V^+_m, \chi) \oplus R_n(V^-_m, \chi) \simeq \text{Im}M^*(-s_0 - 1, \chi)$$

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and thus

$$\dim \text{Hom}_{G \times G}(\text{Im} M^*(-s_0 - 1, \chi), \pi \otimes (\chi \cdot \pi^\vee)) = 1.$$  

On the other hand, let

$$\mu \in \text{Hom}_{G \times G}(I_n(s_0 + 1, \chi), \pi \otimes (\chi \cdot \pi^\vee))$$

with $\mu \neq 0$. Suppose

$$\mu\big|_{R_n(V_{m+2}^{-\epsilon})} = 0.$$  

Then, since $\mu \neq 0$ we have

$$\mu\big|_{R_n(V_{m+2}^{\epsilon})} = 0,$$

and thus

$$\text{Hom}_{G \times G}(R_n(V_{m+2}^{-\epsilon})/R_n(V_{m+2}^{\epsilon}) \cap R_n(V_{m+2}^{\epsilon}), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0.$$  

But $M^*(s_0 + 1)$ identifies

$$R_n(V_{m+2}^{-\epsilon})/R_n(V_{m+2}^{\epsilon}) \cap R_n(V_{m+2}^{\epsilon})$$

with $R_n(V_{m'-2}^{\epsilon})$. This means that

$$\text{Hom}_{G \times G}(R_n(V_{m'-2}^{\epsilon}), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0.$$  

From (3.9), we know that this is impossible. Hence $\mu$ must be non-zero on $R_n(V_{m+2}^{\epsilon})$ thus

$$m_{\chi}^{-\epsilon}(\pi) \leq m + 2 = 2n + 2 - m'.$$

We thus have $m_{\chi}^+(\pi) + m_{\chi}^{-}(\pi) = 2n + 2$ as claimed. \hfill $\Box$
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APPENDIX

A. Completion of a proof

As announced in the introduction, we want to add a missing statement in the proof of [Har07, Theorem 3.4, p.128]. In the proof of the theorem, one should check that the spherical vector of the representation $I_n(s,\alpha^*)$ belongs to $R_n(V^+_m,\chi)$ for almost all places $v$. We prove it here in the following lemma.

**Lemma A.1.** — We suppose $E/F$, $V$, $m$, $n$, $G$, $H$, $W$, $\chi$, $\psi$ and $\psi$ are as above. We suppose in addition that $E/F$, $\chi$ and $\psi$ are unramified. Then for any $s = m - \frac{n^2}{2}$ the spherical vector of $I_n(s,\chi)$ is in $R_n(V^+_m,\chi)$.

**Proof.** — The spherical vector of $I_n(s,\chi)$ is the unique element $\Phi$ such that $\Phi(K) = \{1\}$. Thus one only needs to check that there is an element in $\Phi \in R_n(V^+_m,\chi)$ such that $\Phi(K) = \{1\}$. Remember that $R_n(V^+_m,\chi) = \{g \mapsto \omega(\chi(g))\varphi(0) : \varphi \in S(V)\}$.

We let $V$ be any of the two spaces $V^\pm$. The action of $G$ over the space $S(V)$ can be summarised by (see [KS97, top of p.280]):

$$
\begin{align*}
\omega(\chi(m(a))\varphi(x) &= \chi(\det a)|\det a|^\frac{n}{2}E\varphi(x) \\
\omega(\chi(n(b))\varphi(x) &= \psi(\text{tr}((x,x)b))\varphi(x) \\
\omega(\delta_r)\varphi(x) &= \gamma^{-r} \int_{V^r} \psi\left(\text{Tr}_{E/F}(x'')z\right) \varphi(x' + z)dz
\end{align*}
$$

with the following conventions for the last integral: $V$ is decomposed as $V^r \oplus V^n$, $x = x' + x''$ according to this decomposition and the Haar measure $dz$ is the $r$-power of the Haar measure of $V$ which is self-dual for the Fourier transform defined by the pairing $\psi\circ\text{Tr}_{E/F}(,)$ and $\gamma$ is a quotient of Weil indexes of quadratic forms.

If $k \in P \cap K$, we obviously have $\omega(\chi(k))\varphi(0) = \varphi(0)$. An element $f \in I_n(0,\chi)$ is spherical if and only if $\forall k \in K$, $f(k) = f(I_n) \neq 0$. Thus the spherical vector of $I_n(0,\chi)$ will be in $R_n(V,\chi)$ if and only if $\omega(\delta_r)\varphi(0) = \varphi(0)$ for all $r$ (and $\varphi(0) \neq 0$).

We now suppose that $V = V^+$; remember that the uniformiser $\varpi$ of $F$ is an uniformiser for $E$. We choose an orthonormal basis $(v_1,\ldots,v_n)$ of $V$.

We first compute the Haar measure of $V$. Let $V_\varphi$ be the $O_E$-module generated by $(v_1,\ldots,v_n)$ in $V$ and $\varphi$ its characteristic function. After identification of $V^*$ with $V$ thanks to $\psi\circ\text{Tr}_{E/F}(,)$, the Fourier transform of $\varphi$
is
\[ \hat{\varphi}(y) = \int_V \psi(\text{Tr}_{E/F}(x,y))\varphi(x)\,dx. \]
We readily see that \( \hat{\varphi} = \mu(O)\varphi \) so that
\[ \hat{\hat{\varphi}} = \mu(O)\varphi \]
which means that the measure has to be normalised by \( \mu(O) = 1 \).

We now compute \( \gamma \) in both cases for \( W \): Hermitian or skew-Hermitian.
Its precise definition, taken from [Kud94, Theorem 3.1, p.378, case 3], is as follows. Fix \( \delta \in E^\times \) be such that \( E = F(\delta) \) and \( \Delta = \delta^2 \in F^\times \). Then
\[ \gamma = (\det V, \Delta)_F \gamma_F(-\Delta, \eta)^m \gamma_F(-1, \eta)^{-m}. \]
Since \( E/F \) is unramified, \( \Delta \) has valuation 0. Looking at [Rao93, Prop A.11, p.369] we readily see that \( \gamma_F(-\Delta, \eta) = \gamma_F(-1, \eta) = 1 \). One should note that the correct formula for \( \gamma_F(a, \eta) \) in Proposition A.11 should be
\[ \gamma_F(a, \eta) = (\bar{u}_F)^{\alpha(\eta)} \cdot \left\{ \left( \frac{\bar{u}_F}{F} \right) \gamma_F(\bar{\eta}) \right\}^{\alpha(a)} \]
but that does not change anything for us because \( \alpha(\eta) = 0 \) anyway. Since \( V = V^+ \), we have \( (\det V, \Delta)_F = 1 \) and thus \( \gamma = 1 \). Observe that this remains true if \( W \) is skew-Hermitian (case 3 of [Kud94]) because the definition of \( \gamma \) differs between the two cases by a scaling by \( \delta \) for \( V \) and the product by \( \chi(\delta) \); since \( \delta \) has valuation 0 this does not change \( \gamma \). \( \square \)

This allows us to slightly reformulate [Har07, Theorem 3.2, p.125], since one hypothesis is now proved.

**Th. 3.2 (Harris).**—Let \( G = GU(W) \), a unitary group with signature \( (r, s) \) at infinity, and let \( \pi \) be a cuspidal automorphic representation of \( G \). We assume \( \pi \otimes \chi \) occurs in anti-holomorphic cohomology \( H^{rs}(\text{Sh}(W), E_\mu) \) where \( \mu \) is the highest weight of a finite-dimensional representation of \( G \). Let \( \chi, \alpha \) be algebraic Hecke characters of \( K^\times \) of type \( \eta_k \) and \( \eta_k^{-1} \), respectively. Let \( s_0 \) be an integer which is critical for the \( L \)-function \( L^{\text{mot},S}(s, \pi \otimes \chi, St, \alpha) \); i.e. \( s_0 \) satisfies the inequalities (3.3.8.1) of [Har97]:

\[ \frac{n - \kappa}{2} \leq s_0 \leq \min(q_{s+1}(\mu) + k - \kappa - Q(\mu), p_s(\mu - k - \mathcal{P}(\mu)), \]

Define \( m = 2s_0 - \kappa \). Let \( \alpha^* \) denote the unitary character \( \alpha/|\alpha| \) and assume

\[ \alpha^*|_{A_q^\times} = \varepsilon_{K}^m. \]
Suppose there is a positive-definite hermitian space $V$ of dimension $m$ and a finite set $S$ of finite primes such that

(a) For every finite $v$ in $S$, $\pi_v$ does not occur in the boundary at $s_0$ for $\alpha_v^*$, and $\pi_v$ is ambiguous for $m$ and $\alpha_v^*$;
(b) For every finite $v$, $\Theta^{\alpha^*}(\pi_v \otimes \chi_v, V_v) \neq 0$;
(c) For every finite $v$ outside $S$, all data $(\pi_v, \chi_v, \alpha_v, \text{and the additive character } \psi_v)$ are unramified.

Then

(i) One can find a factorizable vector $\phi_f \in I_n(s, \alpha^*)_f$ such that for every finite $v$, $\phi_v \in R_n(V_v, \alpha^*)$ and $\phi_f$ takes values in $(2\pi i)^{(s_0+\kappa)n}L \cdot \mathbb{Q}^{ab}$ and two factorizable vectors $\varphi \in \pi \otimes \chi$, $\varphi' \in \alpha^* \cdot (\pi \otimes \chi)^\vee$ arithmetic over the field of definition $E(\pi)$ of $\pi_f$.

(ii) Suppose $\varphi$ is as in (i). Then
$$L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) \sim E(\pi, \chi^{(2)} \cdot \alpha); \kappa \ P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$$
where $P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$ is the period
$$(2\pi i)^{s_0 n - \frac{n \kappa}{2} + k(r-s) + \kappa s} g(\varepsilon \frac{[n]}{\kappa}) \cdot \pi^c P(s)(\pi, *, \varphi) g(\alpha_0)^* p((\chi^{(2)} \cdot \alpha)^\vee, 1)^{r-s}$$
appearing in Theorem 3.5.13 of [Har97].

Proof. — With respect to the original theorem we just removed the existence of factorizable vectors in $\pi \otimes \chi$ and $\alpha^* \cdot (\pi \otimes \chi)^\vee$, the existence of $\phi_f$ and, accordingly, condition (a). The fact that there are factorizable vectors in $\pi \otimes \chi$ and $\alpha^* \cdot (\pi \otimes \chi)^\vee$ is well known. We know that for any $v$ such that no data ramifies (neither the extension nor the characters), then the spherical vector $\phi_v^\circ$ is in $R_n(V_{m,v}^+)$. However for all but finitely many $v$, we have $V_v \simeq V_{m,v}^+$. Denote $S'$ the set of primes that are either infinite or such that some data ramify or such that $V_v \neq V_{m,v}^+$. Then for $v \notin S'$, let $\phi_v = \phi_v^\circ$ the spherical vector. For any finite $v \in S'$, let $\phi_v$ be any element of $\text{Soc}_{n,m}(s)$. Then $\phi_f = \otimes \phi_v \in I_n(s, \alpha^*)_f$ satisfies condition (a) of [Har07, Theorem 3.2]. Thus the hypotheses of Harris’ Theorem are verified. □
Bibliography


