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Value distribution problem for $p$-adic meromorphic functions and their derivatives


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Value distribution problem for $p$-adic meromorphic functions and their derivatives

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**ABSTRACT.** — In this paper we discuss the value distribution problem for $p$-adic meromorphic functions and their derivatives, and prove a generalized version of the Hayman Conjecture for $p$-adic meromorphic functions.

**RéSUMÉ.** — Dans cet article on discute le problème de la distribution des valeurs pour des fonctions méromorphes $p$-adiques et ses dérivées, et démontre une version généralisée de la conjecture de Hayman pour des fonctions méromorphes $p$-adiques.

1. Introduction

In [11] Hayman proved the following well-known result:

**Theorem 1.1.** — Let $f$ be a meromorphic function on $\mathbb{C}$. If $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer $k$ and for all $z \in \mathbb{C}$, then $f$ is constant.

Hayman also proposed the following conjecture (see [12]).

**Hayman Conjecture.** — If an entire function $f$ satisfies $f^n(z)f'(z) \neq 1$ for a positive integer $n$ and all $z \in \mathbb{C}$, then $f$ is a constant.

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It has been verified for transcendental entire functions by Hayman himself for \( n > 1 \) ([12]), and by Clunie for \( n \geq 1 \) ([5]). These results and some related problems have become to be known as Hayman’s Alternative, and caused increasingly attentions (see [1], [2], [4], [14], [15], [17]).

In recent years the similar problems are investigated for functions in a non-Archimedean fields. In [16] J. Ojeda proved that for a transcendental meromorphic function \( f \) in an algebraically closed fields of characteristic zero, complete for a non-Archimedean absolute value \( K \), the function \( f' f^n - 1 \) has infinitely many zeros, if \( n \geq 2 \).

The aim of this paper is to establish a similar results for a differential monomial of the form \( f^n (f^{(k)})^m \), where \( f \) is a meromorphic function in \( \mathbb{C}_p \). Namely, we prove the following theorem.

**Theorem 1.2 (A generalized version of the Hayman Conjecture for \( p \)-adic meromorphic functions).** — Let \( f \) be a meromorphic function on \( \mathbb{C}_p \), satisfying the condition \( f^n (f^{(k)})^m (z) \neq 1 \) for all \( z \in \mathbb{C}_p \) and for some non-negative integers \( n, k, m \). Then \( f \) is a polynomial of degree \( < k \) if one of the following conditions holds:

1. \( f \) is an entire function.
2. \( k > 0 \), and either \( m = 1 \), \( n > \frac{1+\sqrt{1+4k}}{2} \), or \( m > 1, n \geq 1 \).
3. \( n \geq 0, m > 0, k > 0 \), and there are constants \( C, r_0 \) such that \( |f|_r < C \) for all \( r > r_0 \).

In the next section we first recall some facts of the \( p \)-adic Nevanlinna theory ([6-10], [13]) for later use. Theorem 1.2 is proved in Section 3 by using several Lemmas.

### 2. Value distribution of \( p \)-adic meromorphic functions

Let \( f \) be a non-constant holomorphic function on \( \mathbb{C}_p \). For every \( a \in \mathbb{C}_p \), expanding \( f \) around \( a \) as \( f = \sum P_i (z - a) \) with homogeneous polynomials \( P_i \) of degree \( i \), we define

\[
v_f (a) = \min \{ i : P_i \neq 0 \}.
\]

For a point \( d \in \mathbb{C}_p \) we define the function \( v_f^d : \mathbb{C}_p \to \mathbb{N} \) by

\[
v_f^d (a) = v_{f-d} (a).
\]
Fix a real number $\rho$ with $0 < \rho \leq r$. Define

$$N_f(a, r) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n_f(a, x)}{x} dx,$$

where $n_f(a, x)$, as usually, is the number of the solutions of the equation $f(z) = a$ (counting multiplicity) in the disk $D_x = \{z \in \mathbb{C}_p : |z| \leq x\}$.

If $a = 0$, then set $N_f(r) = N_f(0, r)$.

For $l$ a positive integer, set

$$N_{l,f}(a, r) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n_{l,f}(a, x)}{x} dx,$$

where

$$n_{l,f}(a, r) = \sum_{|z| \leq r} \min \{v_{f-a}(z), l\}.$$

Let $k$ be a positive integer. Define the function $v_f^{\leq k}$ from $\mathbb{C}_p$ into $\mathbb{N}$ by

$$v_f^{\leq k}(z) = \begin{cases} 0 & \text{if } v_f(z) > k \\ v_f(z) & \text{if } v_f(z) \leq k \end{cases},$$

and

$$n_f^{\leq k}(r) = \sum_{|z| \leq r} v_f^{\leq k}(z), \quad n_f^{\leq k}(a, r) = n_f^{\leq k}(r).$$

Define

$$N_f^{\leq k}(a, r) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n_f^{\leq k}(a, x)}{x} dx.$$

If $a = 0$, then set $N_f^{\leq k}(r) = N_f^{\leq k}(0, r)$.

Set

$$N_{l,f}^{\leq k}(a, r) = \frac{1}{\ln p} \int_{\rho}^{r} \frac{n_{l,f}^{\leq k}(a, x)}{x} dx,$$

where

$$n_{l,f}^{\leq k}(a, r) = \sum_{|z| \leq r} \min \{v_{f-a}^{\leq k}(z), l\}.$$

In a like manner to used for holomorphic functions we define

$$N_f^{\leq k}(a, r), N_{l,f}^{\leq k}(a, r), N_f^{\geq k}(a, r), N_{l,f}^{\geq k}(a, r), N_f^{= k}(a, r), N_{l,f}^{= k}(a, r).$$
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Recall that for a holomorphic function \( f(z) \) in \( \mathbb{C}_p \), represented by the power series
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n,
\]
for each \( r > 0 \), we define \( |f|_r = \max\{|a_n| r^n, 0 \leq n < \infty\} \).

Now let \( f = \frac{f_1}{f_2} \) be a non-constant meromorphic function on \( \mathbb{C}_p \), where \( f_1, f_2 \) be holomorphic functions on \( \mathbb{C}_p \) having no common zeros, we set \( |f|_r = \frac{|f_1|_r}{|f_2|_r} \). For a point \( d \in \mathbb{C}_p \cup \{\infty\} \) we define the function \( v^d_f : \mathbb{C}_p \to \mathbb{N} \) by
\[
v^d_f(a) = v_{f_1 - df_2}(a)
\]
with \( d \neq \infty \), and
\[
v^\infty_f(a) = v_{f_2}(a).
\]

For a point \( a \in \mathbb{C} \) define:
\[
m_f(\infty, r) = \max\{0, \log |f|_r\}, m_f(a, r) = m_{\frac{1}{f-a}}(\infty, r),
\]
\[
N_f(a, r) = N_{f_1 - a f_2}(r), N_f(\infty, r) = N_{f_2}(r),
\]
\[
T_f(r) = \max_{1 \leq i \leq 2} \log |f_i|_r.
\]

In a like manner we define
\[
N_{l,f}(a, r), N_{f}^{<k}(a, r), N_{l,f}^{<k}(a, r), N_{f}^{<k}(a, r), N_{l,f}^{>k}(a, r),
\]
\[
N_{f}^{>k}(a, r), N_{l,f}^{>k}(a, r), N_{l,f}^{>k}(a, r),
\]
with \( a \in \mathbb{C}_p \cup \{\infty\} \).

Then we have (see [11])
\[
N_f(a, r) + m_f(a, r) = T_f(r) + O(1)
\]
with \( a \in \mathbb{C}_p \cup \{\infty\} \),
\[
T_f(r) = T_{\frac{1}{f}}(r) + O(1),
\]
\[
|f^{(k)}|_r \leq \frac{|f|_r}{r^k},
\]
\[
m_{\frac{f^{(k)}}{f}}(\infty, r) = O(1).
\]

The following two lemmas were proved in [11] (see also [3], [6]).
Lemma 2.1. — Let $f$ be a non-constant holomorphic function on $\mathbb{C}_p$. Then
\[ T_f(r) - T_f(\rho) = N_f(r), \]
where $0 < \rho \leq r$.

Notices that $N_f(r)$ depends on fixed $\rho$.

Lemma 2.2. — Let $f$ be a non-constant meromorphic function on $\mathbb{C}_p$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C}_p$. Then
\[ (q - 1)T_f(r) \leq N_{1,f}(\infty, r) + \sum_{i=1}^{q} N_{1,f}(a_i, r) - N_{0,f'}(r) - \log r + O(1), \]
where $N_{0,f'}(r)$ is the counting function of the zeros of $f'$ which occur at points other than roots of the equations $f(z) = a_i, i = 1, ..., q$, and $0 < \rho \leq r$.

3. A Generalized Hayman-Conjecture for $p$-adic meromorphic functions

We are going to prove Theorem 1.2. We need the following Lemmas.

Lemma 3.1. — Let $f$ be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \not\equiv 0$ and $n, k, m$ be positive integers. Then
1. $T_f(r) \leq T_{f^n(f^{(k)})^{m-1}}(r) + O(1),$
2. $T_f(r) \leq T_{f^n(f^{(k)})^m}(r) + O(1),$

In particular $f^n(f^{(k)})^m$ is non-constant.

Proof. —
1. Set $A = f^n(f^{(k)})^m - 1$. Then we have
\[ A + 1 = f^n(f^{(k)})^m, \]
\[ N_f(0, r) \leq N_{A+1}(0, r), \]
\[ \frac{1}{f^{n+m}} = \frac{1}{A+1} \left( \frac{f^{(k)}}{f} \right)^m. \]
Moreover
\[ m_{\frac{f^{(k)}}{f}}(\infty, r) = O(1). \]
Therefore

\[ m_f(0, r) \leq (n + m)m_f(0, r) = m_{f^n+m}(0, r) \leq m_{A+1}(0, r) + O(1). \]

Thus

\[ T_f(r) = N_f(0, r) + m_f(0, r) \leq N_{A+1}(0, r) + m_{A+1}(0, r) = T_{f^n(f^{(k)})^m-1} + O(1). \]

2. Since \( T_{f^n(f^{(k)})^m}(r) = T_{f^n(f^{(k)})^m-1}(r) + O(1) \) we have

\[ T_f(r) \leq T_{f^n(f^{(k)})^m}(r) + O(1). \]

From this it follows that \( f^n(f^{(k)})^m \) is non-constant.

Lemma 3.1 is proved. □

**Lemma 3.2.** — Let \( f \) be a non-constant meromorphic function on \( \mathbb{C}_p \) such that \( f^{(k)} \not\equiv 0 \), and let \( m, n > 1, k > 0 \) be integers, \( a \in \mathbb{C}_p, a \neq 0 \). Then we have:

1. \[ \frac{n(n - 2) + k(mn - m - n) + m(n - 1)}{(n + k)(n + m + km)} T_f(r) \leq N_{1,f^n(f^{(k)})^m}(a, r) - \log r + O(1), \]

2. If \( n^2 - n - k > 0 \),

\[ \frac{n^2 - n - k - 1}{(n + k)(n + 1 + k)} T_f(r) \leq N_{1,f^n(f^{(k)})^m}(a, r) - \log r + O(1). \]

**Proof.** —

1. Since \( m, n > 1 \) we have \( n(n - 2) + k(mn - m - n) + m(n - 1) \geq 0 \).

Because \( f^{(k)} \not\equiv 0 \), from Lemma 3.1 it follows that \( f^n(f^{(k)})^m \) is not constant.

Applying Lemma 2.2 to \( f^n(f^{(k)})^m \) with the values \( \infty, 0 \) and \( a \), we obtain

\[ T_{f^n(f^{(k)})^m}(r) \leq N_{1,f^n(f^{(k)})^m}(\infty, r) + N_{1,f^n(f^{(k)})^m}(0, r) + N_{1,f^n(f^{(k)})^m}(a, r) - \log r + O(1). \]

Denote by \( N_{f^{(k)}}(0, r; f \neq 0) \) the counting function of those zeros of \( f^{(k)} \) which are not the zeros of \( f \), where a zero of \( f^{(k)} \) is counted according to...
its multiplicity. Then we get

\[ N_{f(k)}(0, r; f \neq 0) = N_{f(k)}(0, r) \]

\[ \leq N_{f(k)}(\infty, r) + N_{f(k)}(\infty, r) + O(1) \]

\[ \leq kN_{1,f}(\infty, r) + N_{f}^{<k}(0, r) + kN_{1,f}^{>k}(0, r) + O(1). \]

Therefore,

\[ N_{f(k)}(0, r; f \neq 0) \leq kN_{1,f}(\infty, r) + N_{f}^{<k}(0, r) + kN_{1,f}^{>k}(0, r) + O(1). \]

From this it follows

\[ N_{1,f}^{n}(f(k))^{m}(0, r) \leq N_{1,f}(0, r) + N_{f(k)}(0, r; f \neq 0) \]

\[ \leq kN_{1,f}(\infty, r) + N_{f}^{<k}(0, r) + kN_{1,f}^{>k}(0, r) + O(1) \]

\[ \leq (k + 1)N_{1,f}(0, r) + kN_{1,f}(\infty, r) \]  \hspace{1cm} (3.3)

Again, we see that

\[ N_{f}^{n}(f(k))^{m}(0, r) - N_{1,f}^{n}(f(k))^{m}(0, r) \]

\[ \geq ((1 + k)n + m - 1)N_{1,f}^{>k+1}(0, r) + (n - 1)N_{1,f}^{<k}(0, r). \]  \hspace{1cm} (3.4)

On the other hand,

\[ N_{1,f}(0, r) = N_{1,f}^{<k}(0, r) + N_{1,f}^{>k+1}(0, r). \]

From this and (3.3), (3.4) we obtain

\[ N_{f}^{n}(f(k))^{m}(0, r) \leq (k + 1)N_{1,f}^{>k+1}(0, r) + kN_{1,f}(\infty, r) \]

\[ + \frac{k + 1}{n - 1} (N_{f}^{n}(f(k))^{m}(0, r) - N_{1,f}^{n}(f(k))^{m}(0, r)) \]

\[ - ((k + 1)n + m - 1)N_{1,f}^{>k+1}(0, r)) + O(1). \]

Thus

\[ \frac{n + k}{n - 1} N_{1,f}^{n}(f(k))^{m}(0, r) \leq \frac{k + 1}{n - 1} N_{f}^{n}(f(k))^{m}(0, r) + kN_{1,f}(\infty, r) \]

\[ + (k + 1 - \frac{(k + 1)n + m - 1}{n - 1})N_{1,f}^{>k+1}(0, r) \]

\[ + O(1). \]

Note that

\[ k + 1 - \frac{(k + 1)(k + 1)n + m - 1}{n - 1} < 0, \]

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we have

\[ N_{1,f^n(f(k))^m}(0,r) \leq \frac{k+1}{n+k} N_{f^n(f(k))^m}(0,r) + \frac{k(n-1)}{n+k} N_{1,f}(\infty, r) + O(1). \]

Moreover if \( a \) is a pole of \( f \) with multiplicity \( t \) then \( a \) is a pole of \( f^n(f(k))^m \) with multiplicity \( nt + (t + k)m \geq n + (1 + k)m \). Thus

\[ N_{f^n(f(k))^m}(\infty, r) \geq (n + (k + 1)m) N_{1,f}(\infty, r), \]

and

\[ N_{1,f^n(f(k))^m}(\infty, r) = N_{1,f}(\infty, r). \]

Therefore,

\[ T_{f^n(f(k))^m}(r) \leq \frac{k+1}{n+k} N_{f^n(f(k))^m}(0,r) \]
\[ + \left(1 + \frac{k(n-1)}{n+k}\right) N_{1,f^n(f(k))^m}(\infty, r) \]
\[ + N_{1,f^n(f(k))^m}(a, r) - \log r + O(1), \]

\[ T_{f^n(f(k))^m}(r) \leq \frac{k+1}{n+k} N_{f^n(f(k))^m}(0,r) \]
\[ + \frac{n(k+1)}{(n+k)(n+(k+1)m)} N_{f^n(f(k))^m}(\infty, r) \]
\[ + N_{1,f^n(f(k))^m}(a, r) - \log r + O(1), \]

From this and by Lemma 2.1, we have

\[ \frac{n(n-2) + k(mn - m - n) + m(n-1)}{(n+k)(n+m+km)} T_{f^n(f(k))^m}(r) \]
\[ \leq N_{1,f^n(f(k))^m}(a, r) - \log r + O(1). \]

By Lemma 3.1

\[ \frac{n(n-2) + k(mn - m - n) + m(n-1)}{(n+k)(n+m+km)} T_f(r) \leq N_{1,f^n(f(k))^m}(a, r) - \log r + O(1). \]

Applying the above arguments to case \( m = 1 \), and using \( n^2 - n - k > 0 \), we obtain 2.

Lemma 3.2 is proved. \qed

For simplicity we denote:

\[ B = f(f(k))^m, b = f(k), c = f(f(k))^m - 1, v = \frac{1}{(f(k))^m}, \]
Then we have the following lemma.

**Lemma 3.3.** — Let \( f \) be a non-constant meromorphic function on \( \mathbb{C}_p \) such that \( f^{(k)} \not\equiv 0 \), and let \( k > 0, m > 1 \) be integers. Then we have

\[
B^{(k+1)} + a_k B^{(k)} + \ldots + a_1 B^{(1)} + a_0 B \equiv 0.
\]

**Proof.** — We first prove that

\[
(Bv)^{(j)} \equiv \sum_{i=0}^{j} \binom{j}{i} B^{(i)} v^{(j-i)},
\]

\( j = 1, 2, \ldots, k + 1 \), by induction.

For \( j = 1 \), we have

\[
(Bv)^{(1)} \equiv \sum_{i=0}^{1} \binom{1}{i} B^{(i)} v^{(1-i)}.
\]

Assume

\[
(Bv)^{(j)} \equiv \sum_{i=0}^{j} \binom{j}{i} B^{(i)} v^{(j-i)},
\]

we will prove that

\[
(Bv)^{(j+1)} \equiv \sum_{i=0}^{j+1} \binom{j+1}{i} B^{(i)} v^{(j+1-i)}.
\]

Indeed, we have

\[
(Bv)^{(j+1)} \equiv ((Bv)^{(j)})^{(1)} \equiv \sum_{i=0}^{j} \binom{j}{i} (B^{(i)} v^{(j-i)})^{(1)}
\]

\[
\equiv \sum_{i=0}^{j} \binom{j}{i} (B^{(i+1)} v^{(j-i)} + B^{(i)} v^{(j+1-i)}) \equiv \sum_{i=0}^{j+1} \binom{j+1}{i} B^{(i)} v^{(j+1-i)}.
\]

Returning to the proof of Lemma 3.3, from \( b = f^{(k)} \), we have \( b' = f^{(k+1)} \). Therefore

\[
f^{(k+1)} - \frac{b'}{b} f^{(k)} \equiv 0
\]

(3.6)
Because $B = f(f^{(k)})^m$, $v = \frac{1}{(f^{(k)})^m}$, we obtain $f \equiv Bv$. Since (3.6) we have

$$ (Bv)^{(k+1)} - \frac{b'}{b}(Bv)^{(k)} \equiv 0 $$

(3.7)

From (3.5), (3.7) we obtain

$$ \sum_{i=0}^{k+1} \binom{k+1}{i} B^{(i)} v^{(k+1-i)} - \sum_{i=0}^{k} \binom{k}{i} B^{(i)} v^{(k-i)} \equiv 0. $$

Thus

$$ Bv^{(k+1)} + \binom{k+1}{1} B^{(1)} v^{(k)} + \binom{k+1}{2} B^{(2)} v^{(k-1)} + \ldots + \binom{k+1}{k} B^{(k)} v^{(1)} + B^{(k+1)} v $$

$$ - \frac{b'}{b} (Bv^{(k)} + \binom{k}{1} B^{(1)} v^{(k-1)} + \binom{k}{2} B^{(2)} v^{(k-2)} + \ldots + \binom{k}{k-1} B^{(k-1)} v^{(1)} + B^{(k)} v) $$

$$ \equiv 0. $$

Dividing the left hand side by $v$, we get

$$ \frac{v^{(k+1)} - \frac{b'}{b} v^{(k)}}{v} B + \frac{\binom{k+1}{1} v^{(k)} - \binom{k}{1} \frac{b'}{b} v^{(k-1)}}{v} B^{(1)} $$

$$ + \ldots + \frac{\binom{k+1}{k} v^{(1)} - \binom{k}{k} \frac{b'}{b} v^{(0)}}{v} B^{(k)} + B^{(k+1)} \equiv 0. $$

So

$$ B^{(k+1)} + a_k B^{(k)} + \ldots + a_1 B^{(1)} + a_0 B \equiv 0. $$

(3.8)

□

**Lemma 3.4.** — Let $f$ be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \neq 0$, and let $k > 0, m > 1$ be integers. Suppose that $f$ is not a polynomial of degree $k$. Then we have $a_0 \neq 0$, and

$$ \frac{m^2 k + m^2 - 2mk - m - 1}{m(k+1)(mk + m + 1)} T_f(r) \leq N_{1, f^{(k)}}^m (1, r) + O(1). $$

**Proof.** — Suppose $a_0 \equiv 0$. Because $a_0 = \frac{v^{(k+1)} - \frac{b'}{b} v^{(k)}}{v}$, we get

$$ v^{(k+1)} \equiv \frac{b'}{b} v^{(k)} $$

(3.9)

Consider following two cases.
Case 1. $v^{(k)} \equiv 0$. We have $v \equiv h$, a polynomial of degree $< k$, and $h \not\equiv 0$. Thus $(f^{(k)})^m h \equiv 1$. If $z_0$ is a pole of $f^{(k)}$, then $z_0$ is a pole of $f$ with multiplicity at least $k + 1$. So $z_0$ is a zero of $h$ with multiplicity at least $k + 1$, a contradiction. Thus $f^{(k)}$ has no poles, and from $(f^{(k)})^m h \equiv 1$ it follows that $f$ is a polynomial of degree $k$, a contradiction.

Case 2. $v^{(k)} \not\equiv 0$. From (3.8), we have

$$\frac{v^{(k+1)}}{v^{(k)}} \equiv \frac{b'}{b}.$$ 

So $v^{(k)} \equiv cb \equiv cf^{(k)}$, $c \not\equiv 0$. Solving this, we get

$$v \equiv c(f + t), t^{(k)} \equiv 0.$$ 

From this $t$ we see that $t$ is a polynomial of degree $< k$, and $\frac{1}{(f^{(k)})^m} \equiv c(f + t)$. Thus $c(f + t)(f^{(k)})^m \equiv 1$. Set $F = f + t$. Then $F^{(k)} \equiv f^{(k)}$ and $cF(F^{(k)})^m \equiv 1$. By Lemma 3.1, we get a contradiction, and then $a_0 \not\equiv 0$.

Now we are going to prove the inequality in the lemma. Since $k, m$ are positive integers and $m \geq 2$, it is easy to see that $m^2k + m^2 - 2mk - m - 1 \geq 0$. From (3.8) and $B \equiv c + 1$ we get

$$(c + 1)^{(k+1)} + a_k(c + 1)^{(k)} + \ldots + a_1(c + 1)^{(1)} + a_0(c + 1) \equiv 0,$$

$$c^{(k+1)} + a_k c^{(k)} + \ldots + a_1 c^{(1)} + a_0(c + 1) \equiv 0,$$

$$a_0 c + c^{(k+1)} + a_k c^{(k)} + \ldots + a_1 c^{(1)} \equiv -a_0,$$  \hspace{1cm} (3.10)

$$\frac{1}{a_0} \left( \frac{c^{(k+1)}}{c} + a_k \frac{c^{(k)}}{c} + \ldots + a_1 \frac{c^{(1)}}{c} \right) + \frac{1}{c} + 1 \equiv 0. \hspace{1cm} (3.11)$$

Since $a_0 = \frac{v^{(k+1)} - \frac{b'}{b} v^{(k)}}{v}$, we see that any pole of $a_0$ can occur only at poles or zeros of $b$, and each pole of $a_0$ has multiplicity at most $k + 1$. So

$$N_{a_0}(\infty, r) \leq (k + 1) \left( N_{1,b}(\infty, r) + N_{1,b}(0, r) \right)$$

$$\leq (k + 1) \left( N_{1,f}(\infty, r) + N_{1,b}(0, r) \right).$$

On the other hand, a zero of $b$ of multiplicity $s$ is a zero of $c'$ of multiplicity at least $ms - 1 \geq (m - 1)s$. Also, $c + 1 \not\equiv 0$ at such a zero of $b$.

$$N_{1,b}(0, r) \leq \frac{1}{m - 1} N_{c'}(\infty, r)$$

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\[
\leq \frac{1}{m-1} H_{\frac{1}{c}}(r) = \frac{1}{n-1} H_{\frac{1}{c}}(r)
\]

\[
= \frac{1}{m-1} \left( N_{\frac{1}{c}}(\infty, r) + m_{\frac{1}{c}}(\infty, r) \right)
\]

\[
= \frac{1}{m-1} N_{\frac{1}{c}}(\infty, r) + O(1)
\]

\[
= \frac{1}{m-1} \left( N_{1,c}(\infty, r) + N_{1,c}(0, r) \right) + O(1)
\]

\[
= \frac{1}{m-1} \left( N_{1,f}(\infty, r) + N_{1,c}(0, r) \right) + O(1).
\]

Thus

\[
N_{a_0}(\infty, r) \leq (k + 1) \left( N_{1,b}(\infty, r) + N_{1,b}(0, r) \right)
\]

\[
\leq (k + 1) \left( N_{1,f}(\infty, r) + \frac{1}{m-1} \left( N_{1,f}(\infty, r) + N_{1,c}(0, r) \right) \right) + O(1)
\]

\[
= \frac{m(k+1)}{m-1} N_{1,f}(\infty, r) + \frac{k+1}{m-1} N_{1,c}(0, r).
\]

Note that \( B \equiv c + 1 \equiv f(f^{(k)})^m \). Therefore a pole of \( f \) of multiplicity \( s \) is a pole of \( B \) of multiplicity \( s + (s + k)m \geq 1 + (1 + k)m \). So

\[
N_{1,f}(\infty, r) \leq \frac{1}{1 + m(k+1)} N_B(\infty, r) \leq \frac{1}{1 + m(k+1)} T_B(r) + O(1).
\]

Combining the above inequalities and note that \( T_B(r) = T_c(r) + O(1) \) we obtain

\[
N_{a_0}(\infty, r) \leq \frac{m(k+1)}{(m-1)(1 + m(k+1))} T_c(r) + \frac{k+1}{m-1} N_{1,c}(0, r) + O(1).
\]

Since (3.10), a zero of \( c \) of multiplicity \( s > k + 1 \) is a zero of \( a_0 \). From this and (3.11) we have

\[
N_c(0, r) \leq N_{a_0}(0, r) + (k + 1) N_{1,c}(0, r),
\]

\[
m_c(0, r) \leq m_{a_0}(0, r) + O(1).
\]

Then (3.8) and Lemma 2.1 give us

\[
T_c(r) = N_c(0, r) + m_c(0, r) + O(1)
\]

\[
\leq N_{a_0}(0, r) + (k + 1) N_{1,c}(0, r) + m_{a_0}(0, r) + O(1)
\]

\[
\leq T_{a_0}(r) + (k + 1) N_{1,c}(0, r) + O(1)
\]

\[
= N_{a_0}(\infty, r) + m_{a_0}(\infty, r) + (k + 1) N_{1,c}(0, r) + O(1)
\]

\[
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\]
\[
N a_0 (\infty, r) + m B^{(k+1)} + a_k B^{(k)} + \ldots + a_1 B^{(1)} + a_1 B(1) - B(\infty, r) + (k+1) N_{1,c}(0, r) + O(1)
\]
\[
\leq \frac{m(k+1)}{(m-1)(1+m(k+1))} T_c(r) + \frac{k+1}{m-1} N_{1,c}(0, r)(0, r)
\]
\[
+(k+1) N_{1,c} + O(1)
\]
\[
\leq \frac{m(k+1)}{(m-1)(1+m(k+1))} T_c(r) + \frac{m(k+1)}{m-1} N_{1,c}(0, r) + O(1).
\]

So
\[
(1 - \frac{m(k+1)}{(m-1)(1+m(k+1))}) T_c(r) \leq \frac{m(k+1)}{m-1} N_{1,c}(0, r) + O(1).
\]

From this and Lemma 3.1 we obtain
\[
\frac{m^2 k + m^2 - 2mk - m - 1}{m(k+1)(mk + m + 1)} T_f(r) \leq N_{1,f(f(k))}^m(1, r) + O(1).
\]

□

Now we use the above Lemmas to prove the main result of the paper.

**Proof of Theorem 1.2.** — Assume, on the contrary, that \( f \) is not a polynomial of degree \(< k \).

If \( f \) is an entire function, then from Lemma 3.1 it implies that \( (f^n(f^{(k)}))^m \) is not constant. Then \( (f^n(f^{(k)}))^m (z) - 1 \) must have a zero, a contradiction.

Assume \( k > 0 \). If \( m > 1, n > 1 \) then the condition 1. in Lemma 3.2 holds, and we see that \( (f^n(f^{(k)}))^m (z) - 1 \) is not constant, so it must have a zero, a contradiction.

If \( m = 1, n > 1 \), \( f(k) \not\equiv 0 \). Write \( f = \frac{f_1}{f_2} \), where \( f_1 \) and \( f_2 \) are holomorphic functions,

\[
\frac{m^2 k + m^2 - 2mk - m - 1}{m(k+1)(mk + m + 1)} T_f(r) \leq N_{1,f(f(k))}^m(1, r) + O(1).
\]

Now let \( m > 1, n = 1 \). Is is easy to see that in this case we have \( m^2 k + m^2 - 2mk - m - 1 > 0 \). If \( f \) is a polynomial of degree \( k \), then by Lemma 3.3, we see that \( (f(f^{(k)}))^m (z) - 1 \) has a zero, a contradiction. On the other hand, if \( f \) is a polynomial of degree \( k \), or \( f \) is a transcendental function, then it is obviously that \( (f(f^{(k)}))^m (z) - 1 \) also has a zero, a contradiction.

It remains to consider the case when the condition 3. is satisfied. Then \( f^{(k)} \not\equiv 0 \). Write \( f = \frac{f_1}{f_2} \), where \( f_1 \) and \( f_2 \) are holomorphic functions,
having no common zeros, and \( f^{(k)} = \frac{a_k}{f_2^{i_{k+1}}} \), where \( a_k \) is a polynomial of \( f_1, f_2, f_1', f_2', \ldots, f_1^{(k)}, f_2^{(k)} \). If \( f_2 \) is constant, then by \( |f_1|_r < C|f_2|_r \), we see that \( f_1 \) is constant, and therefore, \( f \) is constant, a contradiction. Suppose that \( f_2 \) is non-constant. Then \( f_2 \) has a zero. Let \( d \) denote the greatest common divisor of \( a_k \) and \( f_2^{k+1} \). Set \( h = \frac{a_k}{d} \) and \( l = \frac{f_2^{l_{k+1}}}{d} \). Let \( d_1 \) denote the greatest common divisor of \( h^m \) and \( f_2^m \). Set \( h_1 = \frac{h^m}{d_1} \) and \( l_1 = \frac{f_2^m}{d_1} \). Then

\[
\frac{f_2^m}{d_1} = \frac{f_1^m h^m - f_2^m l_1^m}{l_1^m} = \frac{f_1^m h_1 - l_1^m}{l_1^m}
\]

(3.11)

Note that \( f^n(f^{(k)})^m(z) \neq 1 \) for all \( z \in \mathbb{C}_p \). Thus \( f^n(f^{(k)})^m \neq 1 \). If \( l \) is constant, then \( f^{(k)} \) is an entire function. Thus \( f \) is an entire function, a contradiction. So \( l \) is non-constant. Therefore, \( l \) has a zero.

Next we are going to show by induction that \( |f_1^n|_r|a^m_k|_r < |f_2^{n+(k+1)m}|_r \), for all \( r \) satisfying \( r > R_0, r > r_0 \), where \( R_0 \) is a some constant. For \( k = 1 \), we have \( a_1 = f_1 f_2 - f_2 f_1 \). Since \( |f_1|_r \leq |f_1|_r |f_2|_r \), and \( |f_1|_r < C|f_2|_r \), we get \( |f_1 f_2|_r \leq |f_1|_r |f_2|_r \), \( |f_2 f_1|_r \leq |f_1|_r |f_2|_r \), and \( |f_1|_r |a^m_1|_r < |f_2|_r^{n+2m} \), for all \( r \) satisfying \( r > R_1, r > r_0 \), where \( R_1 \) is a some constant. Assume we have \( |f_1^n|_r|a^m_1|_r < |f_2^{n+(i+1)m}|_r \), for all \( r \) satisfying \( r > R_i, r > r_0 \), where \( R_i \) is a some constant. Now for \( k = i + 1 \) we get \( a_{i+1} = a_i f_2 - f_2 f_1 a_i \). By the induction hypothesis and \( i + 1 \leq 1, a_i |f_2|_r \leq |a^m_1|_r, |f_2|_r \leq |f_2|_r \), we have \( |f_2^n|_r|a^m_{i+1}|_r < |f_2^{n+(i+2)m}|_r \), for all \( r \) satisfying \( r > R_{i+1}, r > r_0 \). So \( |f_1^n|_r|a^m_k|_r < |f_2^{n+(k+1)m}|_r \), for all \( r \) satisfying \( r > R_0, r > r_0 \), where \( R_0 \) is a some constant. From this and (3.11) it follows \( N_{f^n_1(f^{(k)})^m}(1, r) = N_{f_1^n h_1^{-1} i^m}(r) \), and \( |f_1^n|_r |h^m_1|_r < |f_2^m|_r |l^m_1|_r, |f_1^n|_r |h_1^m|_r < |l_1^m|_r |l^m_1|_r \). Therefore \( |h^m_1|_r - l^m_1|_r = |l_1^m|_r \). So \( T_{f_1^n h_1^{-1} i^m}(r) = T_{l^m_1}(r) \). By Lemma 2.1 we get \( N_{f_1^n h_1^{-1} i^m}(r) = T_{l^m_1}(r) + O(1) \). Because \( l \) has a zero. Thus \( l^m_1 \) has a zero. Therefore, \( f^n(f^{(k)})^m - 1 \) has a zero, a contradiction.

Theorem 1.2 is proved.

By taking \( k = 1 \) we have a differential monomial like in Hayman results, and from Theorem 1.2 it follows the following

**Corollary 3.5.** — Let \( f \) be a meromorphic function on \( \mathbb{C}_p \), satisfying the condition \( f^n (f')^m(z) \neq 1 \) for all \( z \in \mathbb{C}_p \) and for some positive integers \( n, m \). Then \( f \) is a constant function if one of the following conditions holds:

1. \( f \) is an entire function,
2. \( \max\{m, n\} > 1 \),
3. There exist constants \( C, r_0 \) such that \( |f|_r < C \) for all \( r > r_0 \).
Remark. — Indeed, in [16], Theorem 3 shows that $f^4 + f'$ has at least one zero that is not a zero of $f$, hence setting $g(x) = \frac{1}{f(x)}$, we can check that $g^2 g'$ takes the value 1 at least one time. So the case $n = 2, m = k = 1$ of Theorem 1.2 has been established in [16].

Bibliography