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*On separately subharmonic functions (Lelong’s problem)*


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ABSTRACT. — The main result of the present paper is: every separately-subharmonic function $u(x, y)$, which is harmonic in $y$, can be represented locally as a sum two functions, $u = u^* + U$, where $U$ is subharmonic and $u^*$ is harmonic in $y$, subharmonic in $x$ and harmonic in $(x, y)$ outside of some nowhere dense set $S$.

RÉSUMÉ. — Le résultat essentiel de ce papier est le suivant: toute fonction séparément sous-harmonique $u(x, y)$ qui est harmonique en $y$ peut être représentée localement comme la somme de deux fonctions $u = u^* + U$, où $U$ est sous-harmonique et $u^*$ est harmonique en $y$, sous-harmonique en $x$ et harmonique en $(x, y)$ en dehors d’une ensemble nulle part dense $S$.

1. Introduction

We will consider functions $u(x, y)$ of two groups of variables $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. If $u$ is separately harmonic, i.e., harmonic in $x$ for fixed $y$ and harmonic in $y$ for fixed $x$, then $u$ will be harmonic in both variables (Lelong [2], see also [1]). Lelong investigated also separately subharmonic functions, and proved a series of special results in this area. Here originates the question about subharmonicity of separately subharmonic functions.

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However, Wiegerinck, [3] (see also [4]) has shown that a separately subharmonic function need not to be subharmonic in general. He constructed a separately subharmonic function \( u(x, y) \) in the bidisk \( U^2 = \{|x| < 1\} \times \{|y| < 1\} \subset R_x^2 \times R_y^2 \approx \mathbb{C} \times \mathbb{C} \), which is not bounded above near 0.

The problem of subharmonicity of a separately subharmonic function \( u(x, y) \) that is in addition harmonic in \( y \), is still open.

In the present paper we will study the class of these functions. Let us begin by recalling the following well-known results:

1. If a separately subharmonic function is bounded above, then it is subharmonic (Lelong [2], Avanissian [5]);

2. If \( u^+ \in L^1_{loc} \), then \( u \) is subharmonic (Arsove [6]);

3. If \( u^+ \in L^p_{loc}, p > 0 \), then \( u \) is subharmonic (Riihentaus [7]);

4. There are also positive results under weak growth conditions (see [8], [9]).

We note that the conditions in the above results are not separated in \( x \) and \( y \). The following results demand separate conditions:

5. Suppose that \( u(x, y) \) is defined on the product domain \( B = B_1 \times B_2 \subset R_x^n \times R_y^m \). If \( u \) is subharmonic in \( x \) and harmonic in \( y \), then there are nowhere dense closed sets \( S_1 \subset B_1, S_2 \subset B_2 \) such that \( u \) is subharmonic in \( G = (B_1 \times B_2) \setminus (S_1 \times S_2) \) (Cegrell and Sadullaev [10]);

6. If \( u(x, y) \) real analytic, subharmonic in \( x \), and harmonic in \( y \), then \( u \) is subharmonic (Imomkulov [11]);

7. There exists a separately subharmonic function \( u(x, y) \), which is real analytic in \( x \), but which is not subharmonic (Cegrell and Sadullaev [10]);

8. If \( u(x, y) \) is \( C^2 \) and subharmonic in \( x \), harmonic in \( y \), then \( u \) is subharmonic (Kolodziej and Thorbjörnson [12]).

2. Results

Let \( u(x, y) \) be a separately subharmonic function in the product domain \( B = B_1 \times B_2 \), which is harmonic in \( y \). We will assume that \( u \) satisfies
this condition in a slightly larger domain \( \tilde{B} = \tilde{B}_1 \times \tilde{B}_2 \) such that \( \tilde{B} \supset \tilde{B} \). Then \( u(x, y) \) is subharmonic in a domain \( \tilde{B}_1 \times \tilde{B}_2 \setminus (S_1 \times S_2) \), where \( S_1 \subset \tilde{B}_1, S_2 \subset \tilde{B}_2 \) are closed, nowhere dense sets. Moreover, for every fixed \( y \in \tilde{B}_2 \) the Laplacian \( \Delta_x u(x, y) \) defines a positive distribution as follows

\[
F(\varphi) = \int_{\tilde{B}_1} u(x, y) \Delta_x \varphi(x) \, dx \quad \varphi \in C_0^\infty,
\]

thus for every test function \( \varphi(x) \in C_0^\infty(B_1) \), \( \text{supp} \varphi \subset B_1 \), \( \varphi \geq 0 \) we have \( F(\varphi) \geq 0 \). Hence, \( \Delta_x u(x, y) \) is a Borel measure, depending on the parameter \( y \).

**Theorem 2.1.** — For every test-function \( \varphi(x) \in C_0^\infty(B_1) \) \( F(\varphi) \) is harmonic in \( y \) for \( y \in B_2 \setminus S_2 \). Moreover, if \( \text{supp} \varphi \cap S_1 = \emptyset \) then \( F(\varphi) \) is harmonic in \( y \) for all \( y \in B_2 \).

We say that the measure \( \Delta_x u(x, y) \) has the harmonic property with respect to \( y \) in the domain \( G = (B_1 \times B_2) \setminus (S_1 \times S_2) \).

**Proof.** — The result 5) above states that \( u(x, y) \) is subharmonic and therefore \( u \) is locally bounded above in \( G = (B_1 \times B_2) \setminus (S_1 \times S_2) \). Hence the integral

\[
F(\varphi)(y) = \int_{B_1} \varphi(x) \Delta_x u(x, y) = \int_{B_1} u(x, y) \Delta_x \varphi(x)
\]

is harmonic in \( B_2 \setminus S_2 \). If \( \text{supp} \varphi \cap S_1 = \emptyset \), then this integral is harmonic in all \( B_2 \). □

**Corollary 2.2** The measure \( F_E(y) = \int_E \Delta_x u(x, y) \) is harmonic in \( B_2 \) for any \( E \subset \subset B_1 \setminus S_1 \).

**Corollary 2.3.** — The total measure \( \|\Delta_x u(x, y)\|_{B_1} = \int_{B_1} \Delta_x u(x, y) \) is finite \((\neq \infty)\) for every fixed \( y \in B_2 \) and is harmonic function in \( B_2 \setminus S_2 \).

**Theorem 2.4.** — The function \( F_{B_1 \setminus S_1}(y) = \int_{B_1 \setminus S_1} \Delta_x u(x, y) \) is bounded and positive harmonic in \( B_2 \).

**Proof.** — Let us take an increasing sequence of compacts \( E_j \subset E_{j+1} \subset \subset B_1 \setminus S_1 \) such that \( \bigcup_j E_j = B_1 \setminus S_1 \). Then the functions \( F_{E_j}(y) = \int_{E_j} \Delta_x u(x, y) \)
are harmonic in $B_2$ and form an increasing sequence in $j$. By Harnack’s theorem either $F_{E_j}(y) \nearrow +\infty$ or $(F_{E_j})_j$ converges to a harmonic function. The first possibility is ruled out, because Corollary 2.3 provides a bound on the $F_{E_j}(y)$ for every $y \in B_2 \setminus S_2$.

Thus $\lim_{j \to \infty} F_{E_j}(y) = \int_{B_1 \setminus S_1} \Delta_x u(x, y)$ is harmonic in $B_2$, which completes the proof. □

Now we consider the potential

$$U(x, y) = \int_{B_1 \setminus S_1} K(x - w) \Delta_w u(w, y),$$

where $K$ is the Newtonian kernel,

$$K(w) = \begin{cases} 
\frac{1}{2\pi} \ln |w|, & \text{if } n = 2 \\
\frac{1}{(n - 2)\sigma_n |w|^{n-2}}, & \text{if } n > 2.
\end{cases}$$

The measure $\Delta_x u(x, y)$ has the harmonic property in $(B_1 \setminus S_1) \times B_2$. Moreover, for some constant $C$ the total measure $\int_{B_1 \setminus S_1} \Delta_x u(x, y) \leq C$, $y \in B_2$. It follows that the integral $\int_{B_1 \setminus S_1} \varphi(w) \Delta_w u(w, y)$ is harmonic in $y$ for every continuous function $\varphi \in C(\bar{B}_1)$. Let $K_j(w) \in C^\infty(\mathbb{R}^n)$ approximate $K$ from above, $K_j(w) \downarrow K(w)$. Then for every fixed $x \in B_1$ we have

$$\int_{B_1 \setminus S_1} K_j(x - w) \Delta_w u(w, y) \downarrow \int_{B_1 \setminus S_1} K(x - w) \Delta_w u(w, y)$$

for $j \to \infty$, hence $U(x, y)$ is harmonic in $y$ for fixed $x \in B_1$. Moreover, $U$ is subharmonic in $x$ and bounded above in $B_1 \times B_2$. It follows by the theorem of Lelong and Avanissian (1), that $U$ is subharmonic in $B_1 \times B_2$.

Now we take the difference $u^*(x, y) = u(x, y) - U(x, y)$. The function $u^*(x, y)$ is separately subharmonic and is harmonic in $y$. Moreover, $u^*(x, y)$ is harmonic in $x$ outside $S_1$. Thus we have

**Theorem 2.5.** — Every separately subharmonic function, which is harmonic in $y$, can locally be represented as a sum of two functions:

$$u(x, y) = u^*(x, y) + U(x, y),$$

where $U$ is a subharmonic function and $u^*$ is separately subharmonic and harmonic in $y$, such that the associated measure $\Delta_x u^*(x, y)$ is supported on $S_1$ for every fixed $y \in B_2$. 

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Problem 2.6. — We finish our discussion by recalling an open problem on the definition of plurisubharmonic functions: in this definition one demands two conditions.

a. The function \( u(z) \) is upper semicontinuous;

b. For each complex line \( l \) the restriction \( u|_l \) is subharmonic.

The above results on separately subharmonic functions seem to indicate, that the condition a. may be implied by b. But this is still open.

Bibliography