TIEN DUC

*On some properties of three-dimensional minimal sets in* $\mathbb{R}^4$


<http://afst.cedram.org/item?id=AFST_2013_6_22_3_465_0>
On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

Tien Duc LUU\(^{(1)}\)

**Abstract.** — We prove in this paper the Hölder regularity of Almgren minimal sets of dimension 3 in $\mathbb{R}^4$ around a $Y$-point and the existence of a point of particular type of a Mumford-Shah minimal set in $\mathbb{R}^4$, which is very close to a $T$. This will give a local description of minimal sets of dimension 3 in $\mathbb{R}^4$ around a singular point and a property of Mumford-Shah minimal sets in $\mathbb{R}^4$.

**Résumé.** — On prouve dans cet article la régularité Höldérienne pour les ensembles minimaux au sens d’Almgren de dimension 3 dans $\mathbb{R}^4$ autour d’un point de type $Y$ et dans le cas d’un ensemble Mumford-Shah minimal dans $\mathbb{R}^4$ qui est très proche d’un $T$, l’existence d’un point avec une densité particulière. Cela donne une description locale des ensembles minimaux de dimension 3 dans $\mathbb{R}^4$ autour d’un point singulier et une propriété des ensembles Mumford-Shah minimaux dans $\mathbb{R}^4$.

1. Introduction

In this paper we will prove two theorems. The first theorem is about local Hölder regularity of three-dimensional minimal sets in $\mathbb{R}^4$ and the second theorem is about the existence of a point of a particular type of a Mumford-Shah minimal set, which is close enough to a cone of type $T$.

Let us give the list of notions that we will use in this paper.

\(^{(1)}\) Reçu le 22/03/2012, accepté le 20/12/2012

\(^{(1)}\) Bâtiment 430, Département de Mathématique, Université Paris Sud XI, 91405 Orsay
luutienduc@gmail.com

Article proposé par Gilles Carron.
\[ H^d \text{ the } d\text{-dimensional Hausdorff measure.} \]

\[ \theta_A(x, r) = \frac{H^d(A \cap B(x, r))}{r^d}, \text{ where } A \subset \mathbb{R}^n \text{ is a set of dimension } d \text{ and } x \in A. \]

\[ \theta_A(x) = \lim_{r \to 0} \theta_A(x, r), \text{ called the density of } A \text{ at } x, \text{ if the limit exists.} \]

Local Hausdorff distance \( d_{x,r}(E, F) \). Let \( E, F \subset \mathbb{R}^n \) be closed sets which meet the ball \( B(x, r) \). We define

\[ d_{x,r}(E, F) = \frac{1}{r} \left[ \sup \{ \text{dist}(z, F); x \in E \cap B(x, r) \} + \sup \{ \text{dist}(z, E); z \in F \cap B(x, r) \} \right]. \]

Let \( E, F \subset \mathbb{R}^n \) be closed sets and \( H \subset \mathbb{R}^n \) be a compact set. We define

\[ d_H(E, F) = \sup \{ \text{dist}(x, F); x \in E \cap H \} + \sup \{ \text{dist}(x, E); x \in F \cap H \}. \]

Convergence of a sequence of sets. Let \( U \subset \mathbb{R}^n \) be an open set, \( \{ E_k \} \subset U, k \geq 1 \), be a sequence of closed sets in \( U \) and \( E \subset U \). We say that \( \{ E_k \} \) converges to \( E \) in \( U \) and we write \( \lim_{k \to \infty} E_k = E \), if for each compact \( H \subset U \), we have

\[ \lim_{k \to \infty} d_H(E_k, E) = 0. \]

Blow-up limit. Let \( E \subset \mathbb{R}^n \) be a closed set and \( x \in E \). A blow-up limit \( F \) of \( E \) at \( x \) is defined as

\[ F = \lim_{k \to \infty} \frac{E - x}{r_k}, \]

where \( \{ r_k \} \) is any positive sequence such that \( \lim_{k \to \infty} r_k = 0 \) and the limit is taken in \( \mathbb{R}^n \).

Now we give the definition of Almgren minimal sets of dimension \( d \) in \( \mathbb{R}^n \).

**Definition 1.1.** — Let \( E \) be a closed set in \( \mathbb{R}^n \) and \( d \leq n - 1 \) be an integer. An Almgren competitor (Al-competitor) of \( E \) is a closed set \( F \subset \mathbb{R}^n \) that can be written as \( F = \varphi(E) \), where \( \varphi: \mathbb{R}^n \to \mathbb{R}^n \) is a Lipschitz mapping such that \( W_\varphi = \{ x \in \mathbb{R}^n; \varphi(x) \neq x \} \) is bounded.

An Al-minimal set of dimension \( d \) in \( \mathbb{R}^n \) is a closed set \( E \subset \mathbb{R}^n \) such that \( H^d(E \cap B(0, R)) < +\infty \) for every \( R > 0 \) and

\[ H^d(E \setminus F) \leq H^d(F \setminus E) \]

for every Al-competitor \( F \) of \( E \).
On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

Next, we give the definition of Mumford-Shah (MS) minimal sets in $\mathbb{R}^n$.

**Definition 1.2.** — Let $E$ be a closed set in $\mathbb{R}^n$. A Mumford-Shah competitor (also called MS-competitor) of $E$ is a closed set $F \subset \mathbb{R}^n$ such that we can find $R > 0$ such that

\[ F \setminus B(0, R) = E \setminus B(0, R) \]  (1.2.1)

and $F$ separates $y, z \in \mathbb{R}^n \setminus B(0, R)$ when $y, z$ are separated by $E$.

A Mumford-Shah minimal (MS-minimal) set in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that

\[ H^{n-1}(E \setminus F) \leq H^{n-1}(F \setminus E) \]  (1.2.2)

for any MS-competitor $F$ of $E$.

Here, $E$ separates $y, z$ means that $y$ and $z$ lie in different connected components of $\mathbb{R}^n \setminus E$.

It is easy to show that any MS-minimal set in $\mathbb{R}^n$ is also an Al-minimal set of dimension $n - 1$ in $\mathbb{R}^n$. Next, if $E$ is an MS-minimal set in $\mathbb{R}^n$, then $E \times \mathbb{R}$ is also an MS-minimal set in $\mathbb{R}^n \times \mathbb{R}$, by exercise 16, p 537 of [5].

We give now the definition of minimal cones of type $\mathbb{P}$, $\mathbb{Y}$ and $\mathbb{T}$, of dimension 2 and 3 in $\mathbb{R}^n$.

**Definition 1.3.** — A two-dimensional minimal cone of type $\mathbb{Y}$ is just a two-dimensional affine plane in $\mathbb{R}^n$. A three-dimensional minimal cone of type $\mathbb{P}$ is a three-dimensional affine plane in $\mathbb{R}^n$.

Let $S$ be the union of three half-lines in $\mathbb{R}^2 \subset \mathbb{R}^n$ that start from the origin 0 and make angles $120^\circ$ with each other at 0. A two-dimensional minimal cone of type $\mathbb{Y}$ is set of the form $Y' = j(S \times L)$, where $L$ is a line passing through 0 and orthogonal to $\mathbb{R}^2$ and $j$ is an isometry of $\mathbb{R}^n$. A three-dimensional minimal cone of type $\mathbb{Y}$ is a set of the form $Y = j(S \times P)$, where $P$ is a plane of dimension 2 passing through 0 and orthogonal to $\mathbb{R}^2$ and $j$ is an isometry of $\mathbb{R}^n$. We call $j(L)$ the spine of $Y'$ and $j(P)$ the spine of $Y$.

Take a regular tetrahedron $R \subset \mathbb{R}^3 \subset \mathbb{R}^n$, centered at the origin 0, let $K$ be the cone centered at 0 over the union of the 6 edges of $R$. A two-dimensional minimal cone of type $\mathbb{T}$ is of the form $j(K)$, a three-dimensional minimal cone of type $\mathbb{T}$ is a set of the form $T = j(K \times L)$, where $L$ is the line passing through 0 and orthogonal to $\mathbb{R}^3$ and $j$ is an isometry of $\mathbb{R}^n$. We call $j(L)$ the spine of $T$. 

– 467 –
We denote by $d_P, d_Y, d_T$ the densities at the origin of the 3-dimensional minimal cones of type $P$, $Y$ and $T$, respectively. It is clear that $d_P < d_Y < d_T$.

We can now define a Hölder ball for a set $E \subset \mathbb{R}^n$.

**Definition 1.4.** — Let $E$ be a closed set in $\mathbb{R}^n$. Suppose that $0 \in E$. We say that $B(0, r)$ is a Hölder ball of $E$, of type $P$, $Y$ or $T$ with exponent $1 + \alpha$, if there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ and a cone $Y$ of dimension 2 or 3, centered at the origin, of type $P$, $Y$ or $T$, respectively, such that

\begin{align*}
|f(x) - x| & \leq \alpha r \quad \text{for } x \in B(0, r) \quad (1.4.1) \\
(1 - \alpha)\frac{|x - y|}{r}^{(1+\alpha)} & \leq \frac{|f(x) - f(y)|}{r} \leq (1 + \alpha)\frac{|x - y|}{r}^{(1-\alpha)} \quad \text{for } x, y \in B(0, r) \quad (1.4.2)
\end{align*}

$E \cap B(0, (1 - \alpha)r) \subset f(Y \cap B(0, r)) \subset E \cap B(0, (1 + \alpha)r)$. \quad (1.4.3)

For the sake of simplicity, we will say that $E$ is Bi-Hölder equivalent to $Y$ in $B(0, r)$, with exponent $1 + \alpha$.

If in addition, our function $f$ is of class $C^{1,\alpha}$, then we say that $E$ is $C^{1,\alpha}$ equivalent to $Y$ in the ball $B(0, r)$. Here, $f$ is said to be of class $C^{1,\alpha}$ if $f$ is differentiable and its differential is a Hölder continuous function, with exponent $\alpha$.

J. Taylor in [11] has obtained the following theorem about local $C^1$-regularity of two-dimensional minimal sets in $\mathbb{R}^3$.

**Theorem 1.5.** [11]. — Let $E$ be a two-dimensional minimal set in $\mathbb{R}^3$ and $x \in E$. Then there exists a radius $r > 0$ such that in the ball $B(x, r)$, $E$ is $C^{1,\alpha}$ equivalent to a minimal cone $Y(x, r)$ of dimension 2, of type $P$, $Y$ or $T$. Here $\alpha$ is a universal positive constant.

As we know, any two-dimensional minimal cone in $\mathbb{R}^3$ is automatically of type $P$, $Y$ or $T$. This is a great advantage when we study two-dimensional minimal sets of dimension 2 in $\mathbb{R}^3$, because each blow-up limit at some point of a two-dimensional minimal set is a minimal cone of the same dimension. So we can approximate our minimal set by cones which we know the structure of.

The problem of two-dimensional minimal sets in $\mathbb{R}^n$ with $n > 3$ is more difficult. Here we don’t know the list of two-dimensional minimal cones. But G. David gives in section 14 of [3] a description of two-dimensional minimal
On some properties of three-dimensional minimal sets in \( \mathbb{R}^4 \). Thanks to this, he can prove the local Hölder regularity of two-dimensional minimal sets in \( \mathbb{R}^n \).

**Theorem 1.6.** [3].— Let \( E \) be a two-dimensional minimal set in \( \mathbb{R}^n \) and \( x \in E \). Then for each \( \alpha > 0 \), there exists a radius \( r > 0 \) such that in the ball \( B(x, r) \), \( E \) is Hölder equivalent to a two-dimensional minimal cone \( Y(x, r) \), with exponent \( \alpha \).

The \( C^1 \) regularity of two-dimensional minimal sets in \( \mathbb{R}^n \) needs more efforts. We have to prove that the local distance between \( E \) and a two-dimensional minimal cone in \( B(x, r) \) is of order \( r^a \), where \( a \) is a positive universal constant when \( r \) tends to 0. G. David in [4] shows the \( C^1 \) regularity of \( E \) locally around \( x \), but he needs to add an additional condition, called ”full length” to some blow-up limit of \( E \) in \( x \).

**Theorem 1.7.** [4].— Let \( E \) be a two-dimensional minimal set in the open set \( U \subset \mathbb{R}^n \) and \( x \in E \). We suppose that some blow-up limit of \( E \) at \( x \) is a full length minimal cone. Then there is a unique blow-up limit \( X \) of \( E \) at \( x \), and \( x + X \) is tangent to \( E \) at \( x \). In addition, there is a radius \( r_0 > 0 \) such that \( E \) is \( C^1, \alpha \) equivalent to \( x + X \) in the ball \( B(x, r_0) \), where \( \alpha > 0 \) is a universal constant.

Let us say more about the “full length” condition for a two-dimensional minimal cone \( F \) centered at the origin in \( \mathbb{R}^n \). As in [3, Sect 14], the set \( K = F \cap \partial B(0, 1) \) is a finite union of great circles and arcs of great circles \( C_j, j \in J \). The \( C_j \) can only meet when they are arcs of great circles and only by sets of 3 and at a common endpoint. Now for each \( C_j \) whose length is more than \( \frac{9\pi}{10} \), we cut \( C_j \) into 3 sub-arcs \( C_{j,k} \) with the same length so that we have a decomposition of \( K \) into disjoint arcs of circles \( C_{j,k}, (j, k) \in \bar{J} \) with the same length and for each \( C_{j,k} \), we have length\((C_{j,k}) \leq \frac{9\pi}{10} \). The full length condition says that if we have another net of geodesics \( K_1 = \bigcup_{(i,j) \in \bar{J}} C_{i,j}^1 \), for which the Hausdorff distance \( d(C_{j,k}, C_{i,j}^1) \leq \eta \), where \( \eta \) is a small constant which depends only on \( n \), and if \( H^1(K_1) > H^1(K) \), then we can find a Lipschitz function \( f : \mathbb{R}^n \to \mathbb{R}^n \) such that \( f(x) = x \) out of the ball \( B(0, 1) \) and \( f(B(0, 1)) \subset B(0, 1) \) such that \( H^2(f(F_1) \cap B(0, 1)) \leq H^2(F_1 \cap B(0, 1)) - C[H^1(K_1) - H^1(K)] \). Here \( C > 0 \) is a constant and \( F_1 \) is the cone over \( K_1 \). See [4, Sect 2] for more details.

It happens that all two-dimensional minimal cones in \( \mathbb{R}^3 \) satisfy the full length condition. So the theorem of G. David is a generalization of the theorem of J. Taylor.
For minimal sets of dimension $\geq 3$, little is known. Almgren in [1] showed that if $F$ is a three-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin and over a smooth surface in $S^3$, the unit sphere of dimension 3, then $E$ must be a 3-plane. Then J. Simon in [10] showed that this is true for hyperminimal cones in $\mathbb{R}^n$ with $n < 7$. That is, if $F$ is a minimal cone of dimension $n - 1$ in $\mathbb{R}^n$, centered at the origin and over a smooth surface in $S^{n-1}$, then $F$ must be an $n - 1$ plane. There is no theorem yet about the regularity of minimal sets of dimension $\geq 3$ with singularities.

Our first theorem is to prove a local Hölder regularity of three-dimensional minimal sets in $\mathbb{R}^4$. But we don’t know the list of three-dimensional minimal cones in $\mathbb{R}^4$ and we don’t have a nice description of three-dimensional minimal cones as we have for two-dimensional minimal cones. So we shall restrict to some particular type of points, at which we can obtain some information about the blow-up limits.

Now let $E$ be a three-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$. We want to show that $E$ is Bi-Hölder equivalent to a three-dimensional minimal cone of type $P$ or $Y$ in the ball $B(x, r)$, for some radius $r > 0$. If $\theta_E(x) = d_P$, then W. Allard in [2] showed that there exists a radius $r > 0$ such that in the ball $B(x, r)$, $E$ is $C^1$ equivalent to a 3-dimensional plane. We consider then the next possible density of $E$ at $x$, so we suppose that $\theta_E(x) = d_Y$. Since every blow-up limit of $E$ at $x$ is a 3-dimensional minimal cone of type $Y$, then for each $\epsilon > 0$, there exists a radius $r > 0$ and a 3-dimensional minimal cone $Y(x, r)$ of type $Y$ such that

$$d_{x,r}(E, Y(x, r)) \leq \epsilon. \quad (*)$$

By using $(*)$ and the minimality of $E$, we shall be able to approximate $E$ by 3-dimensional minimal cones of type $P$ or $Y$ at every point in $E \cap B(x, r/2)$ and at every scale $t \leq r/2$. We shall then use Theorem 1.1 in [6] to conclude that $E$ is Bi-Hölder equivalent to a 3-dimensional minimal cone of type $Y$ in the ball $B(x, r/2)$. Our first theorem is the following.

**Theorem 1.**— *Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$ and $x \in E$ such that $\theta_E(x) = d_Y$. Then for each $\alpha > 0$, we can find a radius $r > 0$, which depends also on $x$, such that $B(x, r)$ is a Hölder ball (see Def 1.4) of type $Y$ of $E$, with exponent $1 + \alpha$.*

Our second theorem concerns Mumford-Shah minimal sets in $\mathbb{R}^4$. In [3], G. David showed that there are only 3 types of Mumford-Shah minimal sets in $\mathbb{R}^3$, which are the cones of type $P$, $Y$ and $T$. The most difficult part is to show that if $F$ is a Mumford-Shah minimal set in $\mathbb{R}^3$, which is close enough in $B(0, 2)$ to a $T$ centered at 0, then there must be a $T$-point of $F$ in $B(0, 1)$. To prove this proposition, G. David used very nice techniques which involve
On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

the list of connected components. We want to obtain a similar result for a Mumford-Shah minimal set in $\mathbb{R}^4$ which is close enough to a $T$ of dimension 3. But we cannot obtain a result which is as good as in [3, 18.1]. The reason is that we don’t know if there exists a minimal cone $C$ of dimension 3 in $\mathbb{R}^4$, centered at 0, which satisfies $d_Y < \theta_C(0) < d_T$. Our second theorem is the following.

**Theorem 2.** There exists an absolute constant $\epsilon > 0$ such that the following holds. Let $E$ be an MS-minimal set in $\mathbb{R}^4$, $r > 0$ be a radius, and $T$ be a 3-dimensional minimal cone of type $T$ centered at the origin such that

$$d_{0,r}(E,T) \leq \epsilon.$$    

Then in the ball $B(0,r)$, there is a point of $E$ which is neither of type $P$ nor $Y$.

See Definition 2.5 for the definition of points of type $P$ and $Y$. We divide the paper into two parts. In the first part, we prove Theorem 1. In the second part, we prove Theorem 2.

I would like to thank Professor Guy David for many helpful discussions on this paper.

2. Hölder regularity near a point of type $Y$ for a 3-dimensional minimal set in $\mathbb{R}^4$

In this section we prove Theorem 1. We start with the following lemma.

**Lemma 2.1.** Let $F$ be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin, and let $x \in F \cap \partial B(0,1)$. Then each blow-up limit $G$ of $F$ at $x$ is a 3-dimensional minimal cone $G$ of type $P$, $Y$ or $T$ and centered at 0. The type of $G$ depends only on $x$ and $\theta_E(x) = \theta_G(0)$.

We define the type of $x$ to be the type of $G$.

**Proof.** We denote by $0x$ the line passing by 0 and $x$. Suppose that $G$ is a blow-up limit of $F$ at $x$. Then $G = \lim_{k \to \infty} \frac{F-x}{r_k}$ with $\lim_{k \to \infty} r_k = 0$. Let $y \in G$, we want to show that $y + 0x \subset G$. Setting $F_k = \frac{F-x}{r_k}$, as \{$F_k$\} converges to $G$, we can find a sequence $y_k \in F_k$ such that \{y_k\}_{k=1}^{\infty} converges to $y$. Setting $z_k = r_k y_k + x$, then $z_k \in F$ by definition of $F_k$, and $z_k$ converges to $x$ because $r_k$ converges to 0. We fix $\lambda \in \mathbb{R}$ and we set $v_k = (1 + \lambda r_k) z_k$. Then $v_k \in F$ as $F$ is a cone centered at 0. We have next that $w_k = r_k^{-1} (v_k - x) \in F_k$. On the other hand,
\[ w_k = r_k^{-1}((1 + \lambda r_k)z_k - x) \]
\[ = r_k^{-1}((1 + \lambda r_k)(r_ky_k + x) - x) \]
\[ = r_k^{-1}(r_ky_k + \lambda r_k^2 y_k + \lambda r_k x) \]
\[ = y_k + \lambda x + \lambda r_k y_k, \]

we see that \( \lim_{k \to \infty} w_k = y + \lambda x \). As \( \{F_k\} \) converges to \( G \), we see that \( y + \lambda x \in G \). Call \( H \) the tangent plane to \( \partial B(0, 1) \) at \( x \). Since for each \( y \in G \) and \( \lambda \in \mathbb{R} \), we have \( y + \lambda x \in G \), we have that \( G = G' \times O x \), with \( G' \subset G \cap H \).

Next, as \( F \) is a minimal set and \( G \) is a blow-up limit of \( F \) at \( x \), by [3, 7.31], \( G \) is a minimal cone centered at 0. But \( G = G' \times O x \), then by [3, 8.3], \( G' \) is a minimal cone in \( H \), centered at \( x \). Since \( H \) is a 3-plane, we must have that \( G' \) is a 2-dimensional minimal cone of type \( \mathbb{P}, \mathbb{Y} \) or \( \mathbb{T} \) and then \( G \) is also a 3-dimensional minimal cone of type \( \mathbb{P}, \mathbb{Y} \) or \( \mathbb{T} \). Next, as \( G \) is a blow-up limit of \( F \) at \( x \), by [3, 7.31], we have \( \theta_{F}(x) = \theta_{G}(0). \)

We see from this lemma that for each \( x \in F \setminus \{0\} \), where \( F \) is a 3-dimensional minimal cone in \( \mathbb{R}^4 \) centered at the origin,

\[ \theta_{F}(x) \text{ can take only one of the three values } d_{P}, d_{Y}, d_{T}. \]  

But we do not know the list of possible values of \( \theta_{F}(0) \). However, the following lemma says that for this cone \( F \), it is not possible that \( d_{P} < \theta_{F}(0) < d_{Y} \).

**Lemma 2.2.** — There does not exist a 3-dimensional minimal cone \( F \) in \( \mathbb{R}^4 \), centered at the origin such that \( d_{P} < \theta_{F}(0) < d_{Y} \).

**Proof.** — Suppose that there is a cone \( F \) as in the hypothesis and

\[ d_{P} < \theta_{F}(0) < d_{Y}. \]  

(2.2.1)

We first show that

for each \( x \in F \cap \partial B(0, 1) \), we have \( \theta_{F}(0) \geq \theta_{F}(x) \).

(2.2.2)

Indeed, since \( F \) is a minimal cone, for each \( z \in F \), the function \( \theta_{F}(z,t) \) is nondecreasing. So for \( r > 0 \), we have \( \theta_{F}(x,r) \geq \theta_{F}(x) \), which means that \( H^3(F \cap B(x,r))/r^3 \geq \theta_{F}(x) \). Since \( B(x,r) \subset B(0,r+1) \), we obtain \( H^3(F \cap B(x,r))/r^3 \leq H^3(F \cap B(0,r+1))/r^3 \) and thus \( H^3(F \cap B(0,r+1))/r^3 \geq \theta_{F}(x) \). We deduce that \( (H^3(F \cap B(0,r+1))/r^3)(r+1)^3/r^3 \geq \theta_{F}(x) \).

Since \( F \) is a cone centered at 0, \( H^3(F \cap B(0,r+1))/r^3 = \theta_{F}(0) \) for each \( r > 0 \). We deduce then \( \theta_{F}(0)((r+1)^3/r^3 \geq \theta_{F}(x) \) for each \( r > 0 \). We let \( r \to +\infty \) and we obtain then \( \theta_{F}(0) \geq \theta_{F}(x) \), which is (2.2.2).
Now (2.2.1) and (2.2.2) give us that $\theta_F(x) < d_Y$ for each $x \in F \cap \partial B(0,1)$. By (1), we have $\theta_F(x) = d_P$ for $x \in F \cap \partial B(0,1)$. So by [2, 8.1], there exists a neighborhood $U_x$ of $x$ in $\mathbb{R}^4$ such that $F \cap U_x$ is a 3-dimensional smooth manifold. We deduce that $F \cap \partial B(0,1)$ is a 2-dimensional smooth sub-manifold of $\partial B(0,1)$. By [1, Lemma 1], $F$ is a 3-plane passing through 0. But this implies that $\theta_F(0) = d_P$, we obtain then a contradiction, Lemma 2.2 follows. \hfill \Box

**Lemma 2.3.** — Let $F$ be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin 0. If $\theta_F(0) = d_Y$, then $F$ is a 3-dimensional cone of type $\Psi$.

**Proof.** — As in the argument for (2.2.2), we have that for each $x \in F \cap \partial B(0,1)$, $\theta_F(x) \leq \theta_F(0) = d_Y$. So $\theta_F(x)$ can only take one of the two values $d_P$ or $d_Y$. If all $x \in F \cap \partial B(0,1)$ are of type $\mathbb{P}$, then by the same argument as above, $F$ will be a 3-plane, and then $\theta_F(0) = d_P$, a contradiction. So there must be a point $y \in F \cap \partial B(0,1)$, such that $\theta_F(y) = d_Y$. By the same argument like above, $\theta_F(0)(r + 1)^3/r^3 \geq \theta_F(y,r)$ for each $r > 0$. Letting $r \to \infty$ and noting that $\theta_F(y,r)$ is non-decreasing in $r$, we have $d_Y \geq \lim_{r \to \infty} \theta_F(y,r)$. But $\theta_F(y,r) \geq \theta_F(y) = d_Y$ for each $r > 0$, so we must have $\theta_F(y,r) = d_Y$ for $r > 0$. By [3, 6.2], $F$ must be a cone centered at $y$. But we have also that $F$ is a cone centered at 0. So $F$ is of the form $F = F' \times 0y$, where $F'$ is a cone in a 3-plane $H$ passing through 0 and orthogonal to $0y$. Since $F$ is a minimal cone, by [3, 8.3], $F'$ is also a 2-dimensional minimal cone in $H$ and centered at 0. So $F'$ must be a cone of type $\mathbb{P}$, $\Psi$ or $T$. Since $\theta_F(0) = d_Y$, we must have that $F'$ is a 2-dimensional minimal cone of type $\Psi$ and we deduce that $F$ is a 3-dimensional minimal cone of type $\Psi$. \hfill \Box

We can now consider 3-dimensional minimal sets in $\mathbb{R}^4$. We start with the following lemma.

**Lemma 2.4.** — Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$. Then

(i) There does not exist a point $z \in E$ such that $d_P < \theta_E(z) < d_Y$.

(ii) If $x \in E$ such that $\theta_E(x) = d_P$, then each blow-up limit of $E$ at $x$ is a 3-dimensional plane.

(iii) If $\theta_E(x) = d_Y$, then each blow-up limit of $E$ at $x$ is a 3-dimensional minimal cone of type $\Psi$.

**Proof.** — The proof uses Lemmas 2.2 and 2.3. Take any point $z \in E$, let $F$ be a blow-up limit of $E$ at $z$. Then by [3, 7.31], $F$ is a cone and $\theta_F(0) = \theta_E(x)$. By Lemma 2.2, it is not possible that $d_P < \theta_F(0) < d_Y$, which means that it is also not possible that $d_P < \theta_E(x) < d_P$, (i) follows.
If \( x \in E \) such that \( \theta_E(x) = d_P \), then any blow-up limit \( F \) of \( E \) at \( x \) satisfies \( \theta_F(0) = \theta_E(x) = d_P \). By the same arguments as in Lemma 2.2, for each \( y \in F \cap \partial B(0,1) \), \( \theta_F(y) \leq \theta_F(0) = d_P \). We deduce that \( \theta_F(y) = d_P \) for each \( y \in F \cap \partial B(0,1) \), and then \( F \) will be a 3-dimensional minimal cone over a smooth sub-manifold of \( \partial B(0,1) \). By [1, Lemma 1], \( F \) must be a 3-dimensional plane, (ii) follows.

If \( x \in E \) such that \( \theta_E(x) = d_Y \), then any blow-up limit \( F \) of \( E \) at \( x \) satisfies \( \theta_F(0) = \theta_E(x) = d_Y \). By Lemma 2.3, \( F \) must be a 3-dimensional minimal cone of type \( Y \), (iii) follows. □

Lemma 2.4 allows us to define the points of type \( P \) and \( Y \) of a 3-dimensional minimal set in \( \mathbb{R}^4 \).

**Definition 2.5.** — Let \( E \) be a 3-dimensional minimal set in \( \mathbb{R}^4 \) and \( x \in E \). We call \( x \) a point of type \( P \) if \( \theta_E(x) = d_P \). We call \( x \) a point of type \( Y \) if \( \theta_E(x) = d_Y \).

The following proposition says that if a 3-dimensional minimal set \( E \) is close enough to a 3-dimensional plane \( P \) in the ball \( B(x,2r) \), then \( E \) is Bi-Hölder equivalent to \( P \) in \( B(x,r) \).

**Proposition 2.6.** — For each \( \alpha > 0 \), we can find \( \epsilon > 0 \) such that the following holds.

Let \( E \) be a 3-dimensional minimal set in \( \mathbb{R}^4 \) and \( x \in E \). Let \( P \) be a 3-dimensional plane such that

\[
d_{x,2^3r}(E,P) \leq \epsilon.
\] (2.6.1)

Then \( E \) is Bi-Hölder equivalent to \( P \) in the ball \( B(x,r) \), with Hölder exponent \( 1 + \alpha \).

**Proof.** — Take any point \( y \in B(x,r) \). Since \( B(y,2^4r) \subset B(x,2^5r) \), we have

\[
d_{y,2^4r}(E,P) \leq 2d_{x,2^5r}(E,P) \leq 2\epsilon.
\] (2.6.2)

By [3, 16.43], for each \( \epsilon_1 > 0 \), we can find \( \epsilon > 0 \) such that if (2.6.2) holds, then

\[
H^3(E \cap B(y,2^3r)) \leq H^3(P \cap B(y,(1+\epsilon_1)2^4r)) + \epsilon_1 r^3 \\
\leq d_P(2^3r)^3 + C\epsilon_1 r^3.
\] (2.6.3)

Now (2.6.3) implies that \( \theta_E(y,2^3r) \leq d_P + C\epsilon_1 \). If \( \epsilon_1 \) is small enough, then \( \theta_E(y) \leq \theta_E(y,2^3r) < d_Y \). We deduce that \( \theta_E(y) = d_P \) and \( y \) is a \( \mathbb{P} \) point.
Since $\theta_E(y, t)$ is a non-decreasing function in $t$, we have

$$0 \leq \theta_E(y, t) - \theta_E(y) \leq C\epsilon_1 \text{ for } 0 < t \leq 2^3r. \quad (2.6.4)$$

By [3, 7.24], for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that if (2.6.4) holds, then there exists a 3-dimensional minimal cone $F$, centered at $y$, such that

$$d_{y,t/2}(E, F) \leq \epsilon_2 \text{ for } 0 < t \leq 2^3r, \quad (2.6.5)$$

and

$$|\theta_E(y, 2^2r) - \theta_F(y, 2^2r)| \leq \epsilon_2. \quad (2.6.7)$$

Since $d_P \leq \theta_E(y, 2^2r) \leq d_P + C\epsilon_1$, we deduce from (2.6.7) that $\theta_F(y, 2^2r) \leq d_P + C\epsilon_1 + \epsilon_2$. So if $\epsilon_1$ and $\epsilon_2$ are small enough, then $\theta_F(y, 2^2r) < dy$. Which implies $\theta_F(y) < dy$. Since $F$ is a minimal cone centered at $y$, we deduce that $F$ must be a 3-dimensional plane, by the same arguments as in second part of Lemma 2.4.

Now we can conclude that for each $y \in E \cap B(x, r)$ and each $t \leq r$, there exists a 3-dimensional plane $P(y, t)$, which is $F$ in (2.6.5), such that $d_{y,t}(E, P(y, t)) \leq \epsilon_2$. By [6,2.2], for each $\alpha > 0$, we can find $\epsilon_2 > 0$, and then $\epsilon > 0$, such that $E$ is Bi-Hölder equivalent to a $P$ in the ball $B(x, r)$. \hfill \square

**Proposition 2.7.**— For each $\eta > 0$, we can find $\epsilon > 0$ with the following properties. Let $E$ be a minimal set of dimension 3 in $\mathbb{R}^4$ and $Y$ be a 3-dimensional minimal cone of type $\mathbb{P}$, centered at the origin. Suppose that $d_{0,1}(E, Y) \leq \epsilon$. Then in the ball $B(0, \eta)$, there must be a point $y \in E$, which is not of type $\mathbb{P}$.

**Proof.**— Suppose that the lemma fails. Then each $z \in B(0, \eta)$ is of type $\mathbb{P}$. We note $F_1, F_2, F_3$ the three half-plane of dimension 3 which form $Y$ and $L$ the spine of $Y$, which is a plane of dimension 2. Then $F_i, 1 \leq i \leq 3$ have common boundary $L$. Take $w_i \in F_i \cap \partial B(0, \eta/4), 1 \leq i \leq 3$, such that the distance $\text{dist}(w_i, L) = \eta/4$. We see that the $w_i$ lie in a 2-dimensional plane orthogonal to $L$. Since $d_{0,1}(E, Y) \leq \epsilon$, we have that for each $1 \leq i \leq 3$, there exists $z_i \in E$ such that $d(z_i, w_i) \leq \epsilon$. Now $d(z_i, 0) \leq d(w_i, 0) + \epsilon = \eta/4 + \epsilon < 3\eta/8$ and $\text{dist}(z_i, L) \geq \text{dist}(w_i, L) - \epsilon = \eta/4 - \epsilon > 3\eta/16$. So if $\epsilon$ is small enough, we have that for each $1 \leq i \leq 3$, the ball $B(z_i, \eta/8)$ does not meet $L$. As a consequence, $Y$ coincide with $F_i$ in the ball $B(z_i, \eta/8)$ for $1 \leq i \leq 3$. We have next

$$d_{z_i, \eta/8}(E, F_i) = d_{z_i, \eta/8}(E, Y) \leq \frac{8}{\eta}d_{0,1}(E, Y) \leq \frac{8\epsilon}{\eta}. \quad (2.7.1)$$
Take a very small constant $\alpha > 0$, say, $10^{-15}$. Then by Proposition 2.6, we can find $\epsilon > 0$ such that if (2.7.1) holds, then

$$E \text{ is Bi-Hölder equivalent to } F_i \text{ in the ball } B(z_i, \eta/2^k) \text{ for each } 1 \leq i \leq 3 \text{ with Hölder exponent } 1 + \alpha.$$  \hspace{1cm} (2.7.2)

Next, since we suppose that each $z \in B(0, \eta)$ is of type $\mathbb{P}$, we have that there exists a radius $r_z > 0$, such that

$$E \text{ is Bi-Hölder equivalent to a 3-dimensional plane in the ball } B(z, r_z), \text{ with exponent } 1 + \alpha.$$ \hspace{1cm} (2.7.3)

In the ball $B(0, \eta)$, we have

$$d_{0, \eta}(E, Y) \leq \frac{1}{\eta} d_{0, 1}(E, Y) \leq \frac{\epsilon}{\eta}.$$ \hspace{1cm} (2.7.4)

We can adapt the arguments in [3], section 17 to obtain that there does not exist a set $E$, which satisfies the conditions (2.7.2), (2.7.3) and (2.7.4). The idea is as follows, we construct a sequence of simple and closed curves $\gamma_0, \gamma_1, ..., \gamma_k$ such that $\gamma_k \cap E = \emptyset$ and $\gamma_0$ intersects $E$ transversally at exactly 3 points in the ball $B(z_i, \eta/2^k)$. For each $0 \leq i \leq k-1$, $\gamma_i$ intersects $E$ transversally at a finite number of points and $|\gamma_i \cap E| - |\gamma_{i+1} \cap E|$ is even, here $|\gamma_i \cap E|$ denotes the number of intersections of $\gamma_i$ with $E$. This is impossible since $|\gamma_0 \cap E| = 3$ and $|\gamma_k \cap E| = 0$. We obtain then a contradiction. Proposition 2.7 follows. \hfill $\square$

**Lemma 2.8.** — For each $\delta > 0$, we can find $\epsilon > 0$ such that the following holds.

Let $F$ be a 3-dimensional minimal cone in $\mathbb{R}^4$, centered at the origin. Suppose that $d_Y < \theta_F(0) < d_Y + \epsilon$. Then there exists a 3-dimensional minimal cone $Y_F$, of type $\mathbb{Y}$, centered at $0$ such that $d_{0, 1}(F, Y_F) \leq \delta$.

**Proof.** — Suppose that the lemma fails. Then there exists $\delta > 0$, such that we can find 3-dimensional minimal cones $F_1, ..., F_k, ...$ centered at $0$, satisfying $d_Y \leq \theta_{F_i} \leq d_Y + 1/2^k$, and for any 3-dimensional minimal cone $Y$ of type $\mathbb{Y}$, centered at $0$, we have $d_{0, 1}(Y, F_i) > \delta$.

Now we can find a sub-sequence $\{F_{j_k}\}_{k=1}^\infty$ of $\{F_i\}_{i=1}^\infty$ such that this subsequence converges to a closed set $G \subset \mathbb{R}^4$. By [3, 3.3], $G$ is also a minimal set. Since each $F_{j_k}$ is a cone centered at $0$, $G$ is also a cone centered at $0$. So $G$ is a 3-dimensional minimal cone centered at $0$. By [3, 3.3], we have

$$H^3(G \cap B(0, 1)) \leq \liminf_{k \to \infty} H^3(F_{j_k} \cap B(0, 1)), \hspace{1cm} (2.8.1)$$

- 476 -
which implies that
\[ \theta_G(0) \leq \liminf_{k \to \infty} \left( d_Y + 1/2^j_k \right) = d_Y. \quad (2.8.2) \]

By [3, 3.12], we have
\[ H^3(G \cap B(0,1)) \geq \limsup_{k \to \infty} H^3(F_{j,k} \cap B(0,1)), \quad (2.8.3) \]
which implies that
\[ \theta_G(0) \geq \limsup_{k \to \infty} \left( d_Y + 1/2^j_k \right) = d_Y. \quad (2.8.4) \]

From (2.8.2) and (2.8.4), we have that \( \theta_G(0) = d_Y \). Then by Lemma 2.3, \( G \) must be a 3-dimensional minimal cone of type \( \mathbb{Y} \), centered at 0. Since \( \lim_{k \to \infty} F_{j,k} = G \), there is \( k > 0 \) such that \( d_{0,1}(F_{j,k}, G) \leq \delta/2 \), which is a contradiction. The lemma follows. \( \square \)

The following lemma is similar to Lemma 2.8, but we consider minimal sets in general.

**Lemma 2.9.** — For each \( \delta > 0 \), we can find \( \epsilon > 0 \) such that the following holds.

Suppose that \( E \) is a 3-dimensional minimal set in \( \mathbb{R}^4 \) and \( 0 \in E \). Suppose that
\[ d_Y \leq \theta_E(0) \leq d_Y + \epsilon, \quad (2.9.1) \]
and
\[ \theta_E(0,4) - \theta_E(0) \leq \epsilon. \quad (2.9.2) \]

Then there exists a 3-dimensional minimal cone \( Y_E \), of type \( \mathbb{Y} \), centered at 0 such that
\[ d_{0,1}(E, Y_E) \leq \delta. \]

**Proof.** — By [3, 7.24], for each \( \epsilon_1 > 0 \), we can find \( \epsilon > 0 \) such that if (2.9.2) holds, then there is a 3-dimensional minimal cone \( F \) centered at the origin, such that
\[ d_{0,2}(F, E) \leq \epsilon_1, \quad (2.9.3) \]
and
\[ |\theta_F(0,2) - \theta_E(0,2)| \leq \epsilon_1. \quad (2.9.4) \]

Since \( E \) is minimal, \( \theta_E(0,4) \geq \theta_E(0,2) \geq \theta_E(0) \). So from (2.9.1) and (2.9.2), we have that \( d_Y \leq \theta_E(0,2) \leq d_Y + 2\epsilon \). With (2.9.4), we have
\[ d_Y - \epsilon_1 \leq \theta_F(0,2) \leq d_Y + 2\epsilon + \epsilon_1. \quad (2.9.5) \]

- 477 -
Now if we choose $\epsilon_1$ small enough, then $\theta_F(0) = \theta_F(0, 2) \geq d_Y - \epsilon_1 > d_P$, so by Lemma 2.2, we have $\theta_F(0) \geq d_Y$. Thus

$$d_Y \leq \theta_F(0) \leq d_Y + 2\epsilon + \epsilon_1.$$  \hfill (2.9.6)

By Lemma 2.8, for each $\epsilon_3 > 0$, we can find $\epsilon_1 > 0$, and then $\epsilon > 0$, such that if (2.9.6) holds, then there is a 3-dimensional minimal cone $Y_F$ of type $\mathbb{Y}$, centered at 0 such that

$$d_{0,2}(F, Y_F) \leq \epsilon_3.$$  \hfill (2.9.7)

From (2.9.3) and (2.9.7) we have

$$d_{0,1}(E, Y_F) \leq 2(d_{0,2}(E, F) + d_{0,2}(F, Y_F)) \leq 2(\epsilon_1 + \epsilon_3).$$  \hfill (2.9.8)

Now for each $\delta > 0$, we choose $\epsilon > 0$ such that $2(\epsilon_1 + \epsilon_3) < \delta$, we set then $Y_E = Y_F$ and the lemma follows. \hfill $\square$

We are ready to prove Theorem 1.

**Theorem 2.10.**— For each $\alpha > 0$, we can find $\epsilon > 0$ such that the following holds.

Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$, which contains the origin 0. Suppose that there exists a radius $r > 0$ such that

$$d_Y \leq \theta_E(0) \leq d_Y + \epsilon,$$  \hfill (2.10.1)

and

$$\theta_E(0, 2^{11}r) - \theta_E(0) \leq \epsilon.$$  \hfill (2.10.2)

Then $E$ is Bi-Hölder equivalent to a 3-dimensional minimal cone $Y$ of type $\mathbb{Y}$ and centered at 0 in the ball $B(0, r)$, with Hölder exponent $1 + \alpha$.

**Proof.**— By Lemma 2.9, for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.10.1) and (2.10.2) hold, then there exists a 3-dimensional minimal cone $Y$, of type $\mathbb{Y}$, centered at 0 such that

$$d_{0,2^n}(E, Y) \leq \epsilon_1.$$  \hfill (2.10.3)

We consider a point $y \in E \cap B(0, r)$. We set

$$E_Y = \{ z \in E \cap \overline{B}(0, 4r) \} \text{ is not a } \mathbb{P}\text{-point.}$$  \hfill (2.10.4)

We note that $E_Y$ is closed. Indeed, if $z$ is an accumulation point of $E_Y$, then if $z$ is a $\mathbb{P}$-point, then there exists a neighborhood $V_z$ of $z$ in $E$ such...
On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

that $V_z$ has only points of type $P$, as in the proof of Proposition 2.6, which is not possible. So $z$ cannot be a $P$-point and as a consequence, $z \in E_Y$.

**Case 1, $y \in E_Y$.**

Since $y$ is not a $P$-point, $\theta_E(x) \neq d_P$, then by Lemma 2.4, we have

\[
\theta_E(y) \geq d_Y; \quad (2.10.5)
\]

Next, $B(y, 2^8r) \subset B(0, 2^9r)$, by (2.10.3), we have

\[
d_{y,2^8r}(E,Y) \leq 2d_{0,2^9r}(E,Y) \leq 2\epsilon_1. \quad (2.10.6)
\]

By [3, 16.43], for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that if (2.10.6) holds, then

\[
H^3(E \cap B(y, 2^7r)) \leq H^3(Y \cap B(y, (1 + \epsilon_2)2^7r)) + \epsilon_2r^3, \quad (2.10.7)
\]

which, together with (2.10.5), imply

\[
d_Y \leq \theta_E(y, 2^7r) \leq d_Y + C\epsilon_2. \quad (2.10.8)
\]

But $E$ is a minimal set, so the function $\theta_E(y,.)$ is non-decreasing. So we have

\[
d_Y \leq \theta_E(y,t) \leq d_Y + C\epsilon_2 \text{ for } 0 < t \leq 2^7r. \quad (2.10.9)
\]

By Lemma 2.8, for each $\epsilon_3 > 0$, we can find $\epsilon_2, \epsilon_1 > 0$, and then $\epsilon > 0$, such that if (2.10.5) and (2.10.8) hold, then there exists a 3-dimensional minimal cone $Y(y,t)$ of type $\mathbb{Y}$, centered at $y$, such that

\[
d_{y,t}(E,Y(y,t)) \leq \epsilon_3 \text{ for } 0 < t \leq 2^5r. \quad (2.10.10)
\]

We note as above, for $y \in B(0,r)$ and $t \leq 2^5r$, $Y(y,t)$ the cone of type $\mathbb{Y}$ that satisfies (2.10.10).

**Case 2, $y$ is a $P$ point.**

Let $d = \text{dist}(y, E_Y) > 0$. Take a point $u \in E_Y$ such that $d(y,u) = d$. Since $z \in B(0,r)$ and $0 \in E_Y$, we have $d \leq d(0,y) \leq r$. We take the cone $Y(u, 2d)$ as in (2.10.10), then

\[
d_{u,2d}(E,Y(u,2d)) \leq \epsilon_3. \quad (2.10.11)
\]

Call $L$ the spine of $Y(u, 2d)$, then $L$ is a 2-dimensional plane passing through $u$. We want to show that

\[
\text{dist}(y,L) \geq d/2. \quad (2.10.12)
\]
Indeed, if (2.10.12) fails, then there exists \( u' \in L \) such that \( d(y, u') = \text{dist}(y, L) < d/2 \). So \( d(u', u) \leq d(u', y) + d(y, u) \leq 3d/2 \). As a consequence, \( B(u', d/2) \subset B(u, 2d) \). We have next

\[
d_{u',d/2}(E,Y(u,2d)) \leq 4d_{u,2d}(E,Y(u,2d)) \leq 4\varepsilon_3. \tag{2.10.13}
\]

By Proposition 2.7, we can choose \( \varepsilon_3 > 0 \) such that if (2.10.13) holds, then there is a point \( u_1 \in E \cap B(u', d/1000) \), which is not of type \( P \). Next, \( d(y, u_1) \leq d(y, u') + d(u', u_1) \leq d/2 + d/1000 < 3d/4 \) and since \( y \in B(0, r) \), \( u' \in B(0, r + 3d/4) \subset B(0, 4r) \). As \( u' \) is not a \( P \)-point, we have that \( u' \in E_Y \). So we can find a point \( u' \in E_Y \) for which \( d(y, u') < d \), a contradiction. We have then (2.10.12).

Since \( B(y, d/2) \subset B(u, 2d) \), we have

\[
d_{y,d/2}(E,Y(u,2d)) \leq 4d_{u,2d}(E,Y(u,2d)) \leq 4\varepsilon_3. \tag{2.10.14}
\]

By [3, 16.43], for each \( \varepsilon_4 > 0 \), we can find \( \varepsilon_3 > 0 \) such that if (2.10.14) holds, then

\[
H^3(E \cap B(y, d/4)) \leq H^3(Y(u, 2d) \cap B(y, (1 + \varepsilon_4)d/4) + \epsilon_4d^3. \tag{2.10.15}
\]

Now as \( \text{dist}(y, L) \geq d/2 \), we see that \( Y(u, 2d) \) coincide with a 3-dimensional plane in the ball \( B(y, (1 + \varepsilon_4)d/4) \). So \( H^3(Y(u, 2d) \cap B(y, (1 + \varepsilon_4)d/4) \leq d_P((1 + \varepsilon_4)d/4)^3 \), together with (2.10.15), we obtain

\[
\theta_E(y, d/4) \leq d_P + C\varepsilon_4. \tag{2.10.16}
\]

By the proof of Proposition 2.6, we have that for each \( \varepsilon_5 > 0 \), we can find \( \varepsilon_4 > 0 \) such that for each \( t \leq d/8 \), there exists a plane \( P(y, t) \) of dimension 3 passing by \( y \), such that

\[
d_{y,t}(E, P(y, t)) \leq \epsilon_5. \tag{2.10.17}
\]

For the case \( d/8 \leq t \leq r \), we take the cone \( Y(u, t + d) \) as in 2.10.10 which is possible since \( t + d < 8r \). Since \( B(y, t) \subset B(u, t + d) \), we have

\[
d_{y,t}(E,Y(u,t+d)) \leq \frac{t+d}{t}d_{u,t+d}(E,Y(u,t+d)) \leq 10\varepsilon_3. \tag{2.10.18}
\]

From (2.10.10), (2.10.17) and (2.10.18) we conclude that, for each \( y \in E \cap B(0, r) \) and \( t \leq r \), there exists a 3-dimensional minimal cone \( Z(y,t) \) of type \( P \) or \( Y \), such that \( d_{y,t}(E, Z(y,t)) \leq \varepsilon_6 \), where \( \varepsilon_6 = \max\{\varepsilon_5, 10\varepsilon_3\} \). By [6,2,2], we conclude that for each \( \alpha > 0 \), we can find \( \varepsilon > 0 \) such that if (2.10.1) and (2.10.2) hold, then \( E \) is Bi-Hölder equivalent to a 3-dimensional minimal
cone of type $Y$, centered at 0 in the ball $B(x, r)$, with Hölder exponent $1 + \alpha$. □

Now we see that Theorem 1 is a consequence of Theorem 2.10, since $\theta_E(x) = d_Y$ which lies between $d_Y$ and $d_Y + \epsilon$ for any $\epsilon > 0$. Next, for each $\epsilon > 0$, since $\lim_{r \to 0} \theta_E(x, r) = \theta_E(x)$, so we can find $r > 0$ such that $\theta_E(x, 2^{11}r) \leq \theta_E(x) + \epsilon = d_Y + \epsilon$. We conclude that $E$ is Bi-Hölder equivalent to a cone of type $\mathbb{Y}$ in the ball $B(x, r)$.

**Corollary 2.11.** For each $\alpha > 0$, we can find $\epsilon > 0$ such that the following holds. Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$, $x \in E$, $r$ be a radius $> 0$ and $Y$ be a 3-dimensional minimal cone of type $\mathbb{Y}$, centered at $x$ such that

$$d_{x,2^{14}r}(E,Y) \leq \epsilon.$$  

Then $E$ is Bi-Hölder equivalent to $Y$ in the ball $B(x, r)$, with Hölder exponent $1 + \alpha$.

**Proof.** By Proposition 2.7, we can find $\epsilon$ small enough such that there exists a point $y \in B(x, r/1000)$ which is not of type $\mathbb{P}$. So $\theta_E(y) \geq d_Y$. Since $B(y, 2^{12}r) \subset B(x, 2^{13}r)$, we have

$$d_{y,2^{13}r}(E,Y) \leq 2d_{x,2^{14}r}(E,Y) \leq 2\epsilon.$$  

By [3, 16.43], for each $\epsilon_1 > 0$, we can find $\epsilon > 0$ such that if (2.11.2) holds, then

$$H^3(E \cap B(y, 2^{12}r)) \leq H^3(Y \cap B(y, (1 + \epsilon_1)2^{12}r)) + \epsilon_1 r^3,$$

which implies that

$$\theta_E(y, 2^{12}r) \leq d_Y + C\epsilon_1.$$  

Now (2.11.4) together with the fact that $\theta_E(y) \geq d_Y$ are the conditions in the hypothesis of Theorem 2.10 with the couple $(x, 2r)$. Following the proof of the theorem, for each $\epsilon_2 > 0$, we can find $\epsilon_1 > 0$ such that for each $z \in B(y, 2r)$ and for each $t \leq 2r$, there is a 3-dimensional minimal cone $Z(z, t)$ of type $\mathbb{P}$ or $\mathbb{Y}$ such that $d_{z,t}(Z(z, t), E) \leq \epsilon_2$. Since $B(x, r) \subset B(y, 2r)$, the above holds for any $z \in B(x, r)$ and $t \leq r$. Now since $d_{x,r}(E,Y) \leq 2^{14}\epsilon \leq \epsilon_2$, we can apply [DDT,2.2] to conclude that for each $\alpha > 0$, we can find $\epsilon > 0$ such that if (2.11.1) holds, then $E$ is Hölder equivalent to $Y$ in $B(x, r)$, with Hölder exponent $1 + \alpha$. □

By construction of the Bi-Hölder function in [6], we see that if $E$ is Bi-Hölder equivalent to a $Y$ of type $\mathbb{Y}$ in $B(x, r)$ by a function $f$, then $f$ is a bijection of the spine of $Y$ in $B(x, r/2)$ to the points of type non-$\mathbb{P}$ of $E$ in a neighborhood of $x$. We have the remark.
Remark 2.12.— Let $E$ be a 3-dimensional minimal set in $\mathbb{R}^4$, $x \in E$ and $r > 0$. Suppose that $E$ is Bi-Hölder equivalent to a 3-dimensional minimal cone $Y$ of type $\mathcal{Y}$ and centered at $x$ in the ball $B(x, r)$. Note $E_Y$ the set of the points of type non-$\mathcal{Y}$ of $E$ in $B(x, r)$ and $L$ the spine of $Y$. Then

$$E_Y \cap B(x, r/8) \subset f(L \cap B(x, r/4)) \subset E_Y \cap B(x, r/2).$$  \tag{2.12.1}$$

3. Existence of a point of type non-$\mathbb{P}$ and non-$\mathcal{Y}$ for a Mumford-Shah minimal set in $\mathbb{R}^4$ which is near a $\mathbb{T}$

Let us restate Theorem 2.

**Theorem 2.**— There exists an absolute constant $\epsilon > 0$ such that the following holds. Let $E$ be an MS-minimal set in $\mathbb{R}^4$, $r > 0$ be a radius and $T$ be a 3-dimensional minimal cone of type $\mathbb{T}$ centered at the origin such that

$$d_{0,r}(E, T) \leq \epsilon.$$  \tag{2.1}$$

Then in the ball $B(0, r)$, there is a point which is neither of type $\mathbb{P}$ nor $\mathcal{Y}$ of $E$.

We will prove Theorem 2 by contradiction. By homothety, we may assume that $r = 2^{10}$. Suppose that (2.1) fails, that is

there are only points of type $\mathbb{P}$ and $\mathcal{Y}$ in $E \cap B(0, 2^{10})$. \tag{2.2}$$

We fix a coordinate $(x_1, x_2, x_3, x_4)$ of $\mathbb{R}^4$. Without loss of generality, we suppose that $T$ is of the form $T = T' \times l$, where $T'$ is a 2-dimensional minimal cone of type $\mathbb{T}$ which belong to a 3-dimensional plane $P$ of equation $P = \{x_1, x_2, x_3, x_4\}$; $x_4 = 0$ and $l$ the line of equation $x_1 = x_2 = x_3 = 0$. We call $l$ the spine of $T$, which is also the set of $\mathbb{T}$-points of $T$. Let $l_1, l_2, l_3, l_4$ be the four axes of $T'$; then $L_i = l_i \times l, i = 1, ..., 4$ are the 2-faces of $T$. We see that $\bigcup_{i=1}^4 L_i \setminus l$ is the set of $\mathcal{Y}$-points of $T$. Finally, let $F_j, 1 \leq j \leq 6$ the faces of $T'$ in $P$. Then $F_j \times l, 1 \leq j \leq 6$ are the 3-faces of $T$ and $\bigcup_{j=1}^6 F_j$ minus the set of $\mathcal{Y}$-points and the set of $\mathbb{T}$-points of $T$ is the set of $\mathbb{P}$-points of $T$. The proof of Theorem 2 requires several lemmas. We begin with a lemma about the connected components of $\overline{B}(0, 2) \setminus E$.

**Lemma 3.1.**— Let $a_i, 1 \leq i \leq 4$ be the four points in $\partial B(0, 2^9) \cap P$ whose distances to $T'$ are maximal. Set $V_i, 1 \leq i \leq 4$ the connected component of $\overline{B}(0, 2^{10}) \setminus E$ which contains $a_i$. Then we have $V_i \neq V_j$ for $1 \leq i \neq j \leq 4$.

**Proof.**— Suppose that the lemma fails. Then there are $i \neq j$ such that $V_i = V_j$. Without loss of generality, we may assume that $V_1 = V_2 = V$. Now
the point $a = (a_1 + a_2)/2$ belongs to a 3-face $P_{12}$ of $T$ and $T$ coincide with
$P_{12}$ in $B(a, 2^8)$.

Since $d_{0,2^{-10}}(E,T) \leq \epsilon$, we have
\[ d_{a,2^8}(E,T) = d_{a,2^8}(E,P_{12}) \leq 4\epsilon. \] (3.1.1)

By Proposition 2.6, for a constant $\tau$ very small, say, $10^{-25}$, we can find $\epsilon > 0$
such that $E$ is Bi-H"older equivalent to $P_{12}$ in the ball $B(a, 2^3)$, with H"older
exponent $1+\tau$. We note $f$ this H"older function; then $f$ is a homeomorphism and
\[ E \cap B(a,4) \subset f(P_{12} \cap B(a,8)) \subset E \cap B(a,16), \] (3.1.2)
and
\[ |f(x)-x| \leq \tau \text{ for } x \in B(a,16). \] (3.1.3)

We want to show that
\[ \text{if } z \in \partial B(a,4) \setminus E, \text{ then } z \in V. \] (3.1.4)

Indeed, set $z' = f^{-1}(z)$, then $z' \in B(a,8)$ and as $z \notin E$, we have $z' \notin P_{12}$. Now the 3-plane $P_{12}$ separate $\mathbb{R}^4$ into two half-spaces $H_1$ and $H_2$ which contain $a_1$ and $a_2$, respectively. Let $z_1 \in H_1$ and $z_2 \in H_2$ be two points in
$\partial B(a,4)$ whose distances to $P_{12}$ are maximal. We see that $a$ is the mid-point
of the segment $[z_1, z_2]$ and this segment is orthogonal to $P_{12}$. Since $z_1$ and $z_2$ lie in two different half-spaces of $\mathbb{R}^4$ separated by $P_{12}$, one of the two
segment $[z', z_1]$ and $[z', z_2]$ doesn’t meet $P_{12}$. We suppose that is the case
of $[z', z_1]$; then the curve $\gamma = f([z', z_1])$ doesn’t meet $E$.

Next, it is clear that $\text{dist}(u, T) \geq 2$ for $u \in [a_1, f(z_1)]$ as $|f(z_1) - z_1| \leq \tau$.
Since $d_{0,2^{-10}}(E,T) \leq \epsilon$, the segment $[a_1, f(z_1)]$ doesn’t meet $E$. Now the
curve $\gamma'$ which goes first from $a_1$ to $f(z_1)$ by the segment $[a_1, f(z_1)]$ and
then from $f(z_1)$ to $f(z') = z$ by the curve $\gamma$ is a curve in $B(0,2^9)$ which
joint $a_1$ to $z$ and doesn’t meet $E$. We deduce that $z \in V_1 = V$, which is
(3.1.4).

Now we want to obtain a contradiction. We will construct an MS-
competitor $F$ for $E$ whose Hausdorff measure in $B(0, 2^{10})$ is smaller than
that of $E$ in the same ball. We set
\[ F = E \setminus B(a,4). \] (3.1.5)
It is clear that $F \setminus \overline{B}(0,2^{10}) = E \setminus \overline{B}(0,2^{10})$. We want to show that $F$ is an
MS-competitor for $E$. For this, we suppose that $x_1, x_2 \in \mathbb{R}^4 \setminus (\overline{B}(0,2^{10}) \cup E)$
such that $x_1, x_2$ are separated by $E$. We want to show that they are also
separated by $F$. 

- 483 -
We proceed by contradiction. Suppose that there is a curve \( \Gamma \subset \mathbb{R}^4 \) connecting \( x_1 \) and \( x_2 \) which doesn’t meet \( F \).

\( \text{(3.1.6)} \)

Now if \( \Gamma \cap \overline{B}(a, 4) = \emptyset \), then \( \Gamma \) doesn’t meet \( E \). Next, as \( F = E \setminus B(a, 4) \), we have that \( x_1, x_2 \) are not separated by \( E \), a contradiction. So we must have that \( \Gamma \) meets \( \overline{B}(a, 4) \). Let \( x_1' \) be the first point at which \( \Gamma \) meets \( \overline{B}(a, 4) \) and \( x_2' \) be the last point at which \( \Gamma \) meets \( \overline{B}(a, 4) \). Then it is clear that \( x_1', x_2' \in \partial B(a, 4) \). We note \( \Gamma_1 \) the sub-curve of \( \Gamma \) from \( x_1 \) to \( x_1' \) and \( \Gamma_2 \) the sub-curve of \( \Gamma \) from \( x_2' \) to \( x_2 \). Since \( \Gamma_1 \) and \( \Gamma_2 \) belong to the same connected component of \( F \) and \( \Gamma_1, \Gamma_2 \) don’t meet \( B(a, 4) \) and \( F = E \setminus B(a, 4) \), we deduce that \( \Gamma_1 \) and \( \Gamma_2 \) belong to the same connected component of \( \mathbb{R}^4 \setminus E \).

In addition, since \( x_1', x_2' \in \partial B(a, 4) \setminus E \), so by (3.1.4), they both belong to \( V \) and then we can connect \( x_1' \) and \( x_2' \) by a curve \( \Gamma_3 \) which doesn’t meet \( E \).

Now the curve \( \Gamma_4 \) which is the union of \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) is a curve that connects \( x_1 \) and \( x_2 \) and doesn’t meet \( E \). This is a contradiction, as we suppose that \( x_1 \) and \( x_2 \) are separated by \( E \).

Now since \( \text{dist}(a, E) \leq 2^{10} \epsilon \), there is a point \( a' \in E \) such that \( d(a, a') \leq 2^{10} \epsilon \) and by consequence \( B(a', 2) \subset B(a, 4) \). Next

\[
H^3(F \cap B(0, 2^{10})) = H^3(E \cap B(0, 2^{10}) \setminus B(a, 4)) \\
\leq H^3(E \cap B(0, 2^{10}) \setminus B(a', 2)) \\
= H^3(E \cap B(0, 2^{10})) - H^3(E \cap B(a', 2)) \\
\leq H^3(E \cap B(0, 2^{10})) - C2^3 < H^3(E \cap B(0, 2^{10})).
\]

(3.1.7)

Where the last line is obtained from the fact that \( E \) is Alhfors-regular (see [7]). Now (3.1.7) contradicts the hypothesis that \( E \) is MS-minimal, we thus obtain the lemma. \( \Box \)

If \( x \) is a point of type \( \mathbb{P} \) or \( \mathbb{Y} \) of \( E \), then by Proposition 2.6 and Theorem 1, for \( \tau = 10^{-25} \), for example, we can find a radius \( r > 0 \) and a Bi-Hölder mapping \( \psi_x : B(x, 2r) \to \mathbb{R}^4 \), and a 3-dimensional minimal cone \( Y \) of type \( \mathbb{P} \) or \( \mathbb{Y} \), respectively, centered at \( x \), such that

\[
|\psi_x(z) - z| \leq \tau r \text{ for } z \in B(x, 2r)
\]

(2)

\[
E \cap B(x, r) \subset \psi_x(Y \cap B(x, 3r/2)) \subset E \cap B(x, 2r).
\]

(3)
By (2.2), there are only points of type $\mathbb{P}$ or $\mathbb{Y}$ of $E \cap \overline{B}(0, 2^{10})$. We set then

$$E_\mathbb{Y} \text{ the set of } \mathbb{Y}\text{-points of } E \cap \overline{B}(0, 2^{10}).$$  

(4)

It is clear that $E_\mathbb{Y}$ is closed by the proof of Theorem 2.10. If $x \in E_\mathbb{Y} \cap B(0, 2^{10})$, then there exists $r_x > 0$ such that $B(x, r_x) \subset B(0, 2^{10})$ and a minimal cone $Y_x$ of type $\mathbb{Y}$, centered at $x$, and a Hölder mapping $\psi_x : B(x, 2r_x) \rightarrow \mathbb{R}^4$ such that (2) and (3) hold for $\psi_x$ and $Y_x$. Let $L_x$ be the spine of $Y_x$, then $L_x$ is a 2-plane passing through $x$. By Remark 2.12, there is a neighborhood $U_x$ of $x$ such that

$$E_\mathbb{Y} \cap U_x = \psi_x(B(x, r_x) \cap L_x).$$  

(5)

Now we take four points $d_i, 1 \leq i \leq 4$ such that $0$ is the mid-point of the segments $[a_i, d_i], 1 \leq i \leq 4$, here $a_i$ is as in Lemma 3.1. It is clear that $d_i \in T' \subset T$. In addition, $d_i \in L_i, 1 \leq i \leq 4$, where $L_i$ are described just after the second statement of Theorem 2. Next, for $1 \leq i \leq 4$, we have $d_{i,4}(E, T) \leq 2^8d_{0,2^{10}}(E, T) \leq 2^8 \epsilon$. But in the ball $B(d_i, 2)$, $T$ coincide with a cone $Y_i$ of type $\mathbb{Y}$ whose spine is $L_i$. So $d_{i,4}(E, Y_i) \leq 2^8 \epsilon$. By Corollary 2.11, for $\tau = 10^{-25}$, we can find $\epsilon > 0$ such that $E$ is Bi-Hölder equivalent to $Y_i$ in the ball $B(d_i, 2)$, with Hölder exponent $1 + \tau$. Call $\psi_i$ this Hölder mapping, then by Remark 2.12

$$E_\mathbb{Y} \cap B(d_i, 1) \subset \psi_i(L_i \cap B(d_i, 3/2)) \subset E_\mathbb{Y} \cap B(d_i, 2)$$  

(6)

and

$$|\psi_i(z) - z| \leq \tau \text{ for } z \in B(d_i, 2).$$  

(7)

Setting

$$b_i = \psi_i(d_i), 1 \leq i \leq 4.$$  

(8)

By (7), we have $d(d_i, b_i) \leq \tau$. We want to prove the following lemma.

**Lemma 3.2.** — The point $b_i \in E_\mathbb{Y}$ can be connected to another point $b_i \in E_\mathbb{Y}, i \neq 1$ by a curve $\gamma \subset E_\mathbb{Y} \cap B(0, 3 \cdot 2^8)$.

**Proof.** — Recall that $\psi_i, b_i, d_i$ are the same as (6),(7),(8) above. In addition, for each $x \in E_\mathbb{Y} \cap B(0, 2^{10})$, there are a radius $r_x$ and a Bi-Hölder mapping $\psi_x$, a minimal cone $Y_x$ of type $\mathbb{Y}$, centered at $x$ such that (2),(3), and (5) hold.

We proceed by contradiction. We denote by $E_\mathbb{Y}$ the connected component of $E_\mathbb{Y} \cap B(0, 2^{10})$ which contains $b_1$. Since in each ball $B(b_i, 2)$, $E_\mathbb{Y}$ is Hölder equivalent to a 2-plane, by (6), we deduce that each $z \in E_\mathbb{Y} \cap B(b_i, 1)$
Tien Duc Luu

can be connected to \( b_i \) by a curve in \( E_Y \). So if the lemma fails, that is \( E_Y^1 \)
doesn’t contain any \( b_i, i \neq 1 \), we must have
\[
E_1^1 \cap B(b_i, 1) = \emptyset \text{ for } i \neq 1. \tag{3.2.1}
\]
Recall next that \( T = T' \times l \), where \( T' \) is a 2-dimensional minimal cone of
type \( T \) in the 3-plane \( P \) of equation \( x_4 = 0 \) and \( l \) is the line of equation
\( x_1 = x_2 = x_3 = 0 \).

Now we construct a family of functions \( f_t, 0 \leq t \leq 1 \) from \( \mathbb{R}^4 \) to \( \mathbb{R}^2 \) by
the formula
\[
f_t(x) = (x_4, |x - td_2|^2 - ((1 - t)2^9)^2), \tag{3.2.2}
\]
where \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \) and \( 0 \leq t \leq 1 \). If \( x \in E_Y^1 \), then
\[
|f_1(x)| \geq |x - d_2| \geq 1/2, \tag{3.2.3}
\]
by (3.2.1) and the fact that \( |d_2 - b_2| \leq \tau \). We will construct a finite number
of functions to go from \( f_0 \) to \( f_1 \). First, let \( K = E_Y^1 \cap \overline{B}(0, 3 \cdot 2^8) \). Then for
each \( z \in K \), there is a radius \( r_z \) such that \( E_Y^1 \) is Bi-Hölder equivalent to a
2-plane \( P_z \), with Hölder exponent \( 1 + \tau \). Since \( K \) is compact, we can cover
\( K \) by a finite number of balls \( B(z_i, r_z), 1 \leq i \leq N \). Finally, we choose \( \eta > 0 \)
which is smaller than \( \frac{1}{10} \min \{r_{z_i}\}, 1 \leq i \leq N \).

Next, let \( \{x_i\}, 1 \leq i \leq l \) be a maximal collection of points in \( K \) such that
\(|x_i - x_j| \geq \eta \) for \( i \neq j \). Set \( \tilde{\varphi}_j \) a bump function with support in \( B(x_j, 2\eta) \)
and such that \( \tilde{\varphi}_j(x) = 1 \) for \( x \in \overline{B}(x_j, \eta) \) and \( 0 \leq \tilde{\varphi}_j(x) \leq 1 \) everywhere.
We note that \( \sum_j \tilde{\varphi}_j(x) \geq 1 \) for \( x \in E_Y^1 \cap \overline{B}(0, 3 \cdot 2^8) \) since \( x \) must lie in
one of the ball \( B(x_j, \eta) \) by the maximality of the family \( \{x_i\} \). Set \( \tilde{\varphi}_0 \) a \( C^\infty \)
function in \( \mathbb{R}^4 \) such that \( \tilde{\varphi}_0(x) = 0 \) for \( |x| \leq 3 \cdot 2^8 - \eta \) and \( \tilde{\varphi}_0(x) = 1 \) for
\(|x| > 3 \cdot 2^8 \) and \( 0 \leq \tilde{\varphi}_0(x) \leq 1 \) everywhere. We have then \( \sum_{j=0}^l \tilde{\varphi}_j(x) \geq 1 \)
on \( E_Y^1 \) and we set
\[
\varphi_j(x) = \tilde{\varphi}_j(x) \left\{ \sum_{j=0}^l \tilde{\varphi}_j(x) \right\}^{-1} \text{ for } x \in E_Y^1 \text{ and } 0 \leq j \leq l. \tag{3.2.4}
\]
The functions \( \varphi_j, 0 \leq j \leq l \) have the following properties.
\[
\varphi_j \text{ has support in } B(x_j, 2\eta) \text{ for } j \geq 1, \tag{3.2.5}
\]
\[
\sum_{j=0}^l \varphi_j(x) = 1 \text{ for } x \in E_Y^1, \tag{3.2.6}
\]
\[
\sum_{j=1}^l \varphi_j(x) = 1 \text{ for } x \in E_Y^1 \cap B(0, 3 \cdot 2^8 - \eta),
\]
On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

since $\varphi_0(x) = 0$ on $B(0, 3 \cdot 2^8 - \eta)$. Our first approximation is a sequence of functions given by

$$g_k = f_0 + \sum_{0 < j < k} \varphi_j(f_1 - f_0),$$

(3.2.7)

with $0 \leq k \leq l$. Then $g_0 = f_0$ and

$$g_l(x) = f_1(x) \text{ for } x \in E \cap B(0, 3 \cdot 2^8 - \eta).$$

(3.2.8)

We note that for $k \geq 1$

$$g_k(x) - g_{k-1}(x) = \varphi_k(x)(f_1(x) - f_0(x)) \text{ is supported in } B(x_k, 2\eta).$$

(3.2.9)

We compute the number of solutions in $E_Y^1$ of the equations $g_k(x) = 0$. We will modify $f_0$ and the $g_k$ such that they have only a finite number of zeroes. We modify first $f_0$.

**Sub-Lemma 3.2.1.** — There exists a continuous function $h_0$ on $E_Y^1$ such that

$$|h_0(x) - f_0(x)| \leq 10^{-6} \text{ for } x \in E_Y^1,$$

(3.2.9)

$h_0$ has exactly one zero $b_1$ in $E_Y^1$, and $b_1$ is a simple, non-degenerate zero of $h_0$.

Here, we say that $\xi \in E_Y^1$ is a non-degenerate, simple zero of a continuous function $h$ on $E_Y^1$ if $h(\xi) = 0$ and there is a ball $B(\xi, \rho)$ and a Bi-Hölder function $\gamma$ with Hölder exponent $1 + \tau$ which maps $E_Y^1 \cap B(\xi, \rho)$ to an open set $V$ of a 2-plane, such that $h \circ \gamma^{-1}$ is of class $C^1$ on $V$ and the differential $D(h \circ \gamma^{-1})$ at the point $\gamma(\xi)$ is of rank 2.

**Proof.** — We modify $f_0$ in a neighborhood of $d_1$. We have already our Bi-Hölder homeomorphism $\psi_1$ which satisfies (6),(7) and (8). Next, since $E_Y^1$ is the connected component of $E_Y$ which contains $b_1$, we have

$$E_Y \cap B(d_1, 1) = E_Y^1 \cap B(d_1, 1),$$

thus

$$E_Y^1 \cap B(d_1, 1/3) \subset \psi_1(B(L_1 \cap B(d_1, 1/2))) \subset E_Y^1 \cap B(d_1, 1),$$

(3.2.10)

here $L_1$ is the 2-face of $T$ that contains $d_1$, which is Bi-Hölder equivalent to $E_Y^1$ in the ball $B(d_1, 1)$.

Set $h_0 = f_0$ outside the ball $B(d_1, 1/2)$. In $B(d_1, 1/4)$, we set $h_0 = f_0 \circ \psi_1$. In the region between the two balls $R = \overline{B(d_1, 1/2)} \setminus B(d_1, 1/4)$, we set

$$h_0(x) = \alpha(x)f_0(x) + (1 - \alpha(x))f_0 \circ \psi^{-1}(x),$$

(3.2.11)
where \( \alpha(x) = 4|x - d_1| - 1 \). We have then
\[
|h_0(x) - f_0(x)| \leq |f_0(x) - f_0 \circ \psi^{-1}_1(x)| \leq C\tau \text{ for } x \in B(d_1, 1/2) \text{ since } |\psi_1(x) - x| \leq \tau \text{ and the differential of } f_0 \text{ is bounded in this ball. We have then (3.2.9)}.
\]

Since \( f_0(x) = (x_4, |x|^2 - 4^9) \), so \( |f_0(x)| \geq 1/500 \) for \( x \in E_Y^1 \setminus B(d_1, 10^{-2}) \).

By consequence, all the zeroes of \( h_0 \) must lie in the ball \( B(d_1, 1/4) \).

We verify next that \( h_0 \) has exactly one zero in \( B(d_1, 1/4) \), which is simple and non-degenerate. Set \( \gamma_1(x) = \psi_1^{-1}(x) \) for \( x \in E_Y^1 \cap B(d_1, 1/4) \). Then \( \gamma_1 \) is a homeomorphism from \( E_Y^1 \cap B(d_1, 1/4) \) onto its image, which is an open set in \( L_1 \).

Since \( h_0 = f_0 \circ \psi_1^{-1} = f_0 \circ \gamma_1 \) on \( E_Y^1 \cap B(d_1, 1/4) \), we have that \( h_0(\xi) = 0 \) for \( \xi \in E_Y^1 \cap B(d_1, 1/4) \) if and only if \( \gamma_1(\xi) \) is a zero of \( f_0(\xi) = (x_4, |x|^2 - 4^9) \) in \( L_1 \cap B(d_1, 1/2) \), which can only be \( d_1 \). The verification that \( Df_0 \) is of maximal rank at \( d_1 \) is clear. The sub-lemma follows.

We need another sub-lemma which allows us to go from \( h_{k-1} \) to \( h_k \).

**Sub-lemma 3.2.2.** — We can find continuous functions \( \theta_k, 1 \leq k \leq l \), such that
\[
\theta_k \text{ is supported in } B(x_k, 3\eta), \quad (3.2.12)
\]
and
\[
||\theta_k||_{\infty} \leq 2^{-k}10^{-6}, \quad (3.2.13)
\]
and if we set
\[
h_k = h_{k-1} + \varphi_k(f_1 - f_0) + \theta_k, \quad (3.2.14)
\]
for \( 1 \leq k \leq l \), then
\[
(3.2.15)
\]
each \( h_k \) has a finite number of zeroes in \( E_Y^1 \), which are all simple and non-degenerate.

**Proof.** — We will construct \( h_k \) by induction. For \( k = 0 \), the function \( h_0 \) satisfy clearly (3.2.15). Let \( k \geq 1 \), and we suppose that we have already constructed \( h_{k-1} \) such that (3.2.15) holds.

We note that \( h_{k-1} + \varphi_k(f_1 - f_0) \) coincide with \( h_{k-1} \) outside the ball \( B(x_k, 2\eta) \), by (3.2.5). We take a thin annulus
\[
A = \overline{B}(x_k, \rho_2) \setminus B(x_k, \rho_1), 2\eta < \rho_1 < \rho_2 < 3\eta, \quad (3.2.16)
\]
which doesn’t meet the finite set of zeroes of \( h_{k-1} \). Recall that there is a Bi-Hölder function \( \psi_k : B(x_k, 20\eta) \to \mathbb{R}^4 \) and a 2-plane \( P_k \) passing through
that $|\psi_k(x) - x| \leq 10\eta\tau$ for $x \in B(x_k, 20\eta)$ and

$$E^1_Y \cap B(x_k, 19\eta) \subset \psi_k(P_k \cap B(x_k, 20\eta)) \subset E^1_Y. \quad (3.2.17)$$

We choose $\theta_k$ such that $\theta_k$ is supported in $B(x_k, \rho_2)$ and $||\theta_k||_\infty < \min\{2^k10^{-6}, \inf_{x \in A} |h_{k-1}(x)|\}$, of course $\inf_{x \in A} |h_{k-1}(x)| > 0$ since $A$ doesn’t meet the set of zeroes of $h_{k-1}$. Then $h_k = h_{k-1}$ outside the ball $B(x_k, \rho_2)$.

We will control $h_k$ in the ball $B(x_k, \rho_1)$. Set $\gamma(x) = \psi_k^{-1}(x)$ for $x \in E^1_Y \cap B(x_k, \rho_1)$. By (3.2.17) and since $\psi_k$ is Bi-Hölder on $B(x_k, 20\eta)$, $\gamma$ is a Bi-Hölder homeomorphism from $E^1_Y \cap B(x_k, \rho_1)$ onto an open set $V$ of the 2-plane $P_k$.

By the density of $C^1$ function in the space of bounded continuous functions on $V$ with the sup norm, we can choose $\theta_k$ with the above properties and such that

$$h_k \circ \theta_k$$

is of class $C^1$ on $V$. \quad (3.2.18)

We can also add a very small constant $w \in \mathbb{R}^2$ to $\theta_k$ on $E^1_Y \cap B(x_k, \rho_1)$, and then interpolate continuously on $A$. We verify that for almost every choice of $w,$

$$h_k$$

has a finite number of zeroes in $E^1_Y \cap B(x_k, \rho_1)$. \quad (3.2.19)

For this, we set $Z_y = \{z \in V; h_k \circ \psi_k(z) = y\}$. By (3.2.18), we can apply the co-area formula ([9, 3.2.22]) for $h_k \circ \psi_k$ on $V$, and we obtain

$$\int_V J(z) dH^2(z) = \int_{y \in \mathbb{R}^2} H^0(Z_y) dH^2(y), \quad (3.2.20)$$

here, $J(z)$ denote the Jacobian of $h_k \circ \psi_k$ at $z$, which is clearly bounded. We deduce that $Z_y$ is finite for almost-every $y \in \mathbb{R}^2$. If we choose $w$ such that $Z_w$ is finite and then add $-w$ to $\theta_k$ in $E^1_Y \cap B(x_k, \rho_1)$, then the new $Z_0$ will be finite, and we have (3.2.19).

We consider now the rank of the differential. By Sard’s theorem, the set of critical values of $h_k \circ \psi_k$ has measure 0 in $\mathbb{R}^2$. So if we choose $w \in \mathbb{R}^2$ which is not a critical value, and add $-w$ to $\theta_k$ in $E^1_Y \cap B(x_k, \rho_1)$, then the differential of the new function $h_k \circ \psi_k$ at each zero of $h_k \circ \psi_k$ is of rank 2.

So we take $w$ very small with the above properties, and add $-w$ to $\theta_k$ in $B(x_k, \rho_1)$; next, we interpolate in the region $A$, we obtain a function $h_k$ having a finite number of zeroes in $E^1_Y \cap B(x_k, \rho_1)$ which are all simple and non-degenerate. The sub-lemma follows.

Now let $N(k)$ be the number of zeroes of $h_k$ in $E^1_Y$. Then $N(0) = 1$ since the only zero of $h_0$ in $E^1_Y$ is $b_1$. Let us check that for the last index $l,$
\[ N(l) = 0. \] First we have
\[ h_l - h_0 = \sum_{1 \leq k \leq l} (h_k - h_{k-1}) = \sum_{1 \leq k \leq l} \varphi_k(f_1 - f_0) + \sum_{1 \leq k \leq l} \theta_k. \]

If \( x \in E^l_Y \cap B(0, 3 \cdot 2^8 - \eta) \), then \( \sum_{1 \leq k \leq l} \varphi_k(x) = 1 \), thus
\[ h_l(x) = h_0(x) + f_1(x) - f_0(x) + \sum_{1 \leq k \leq l} \theta_k(x) \]
so that
\[
|h_l(x)| \geq |f_1(x)| - |h_0(x) - f_0(x)| - \sum_{1 \leq k \leq l} |\theta_k(x)| \\
\geq 1/4 - 10^{-6} - \sum_{1 \leq k \leq l} 2^{-k} 10^{-6} > 0
\]
by (3.2.3), (3.2.6) and (3.2.13).

If \( x \in E^l_Y \cap B(0, 2^{10}) \setminus B(0, 3 \cdot 2^8 - \eta) \), then \( \sum_{1 \leq k \leq l} \varphi_k(x) = 1 - \varphi_0(x) \), so
\[ h_l(x) = h_0(x) + (1 - \varphi_0(x))(f_1(x) - f_0(x)) + \sum_{1 \leq k \leq l} \theta_k(x) \]
which implies
\[
|h_l(x) - f_0(x) - (1 - \varphi_0(x))(f_1(x) - f_0(x))| \\
\leq |h_0(x) - f_0(x)| + \sum_{1 \leq k \leq l} |\theta_k(x)| \leq 2 \cdot 10^{-6}.
\]

But the second coordinate of \( f_0(x) + (1 - \varphi_0(x))(f_1(x) - f_0(x)) \) is
\[
|x|^2 - 4^9 + (1 - \varphi_0(x))(|x - d_2|^2 - |x|^2 + 4^9) \\
= \varphi_0(x)(|x|^2 - 4^9) + (1 - \varphi_0(x))|x - d_2|^2 \geq 1/4,
\]
by (3.2.2) and because \( |x| \geq 3 \cdot 2^8 - \eta \). Thus \( h_l(x) \neq 0 \) in this case also. We deduce that \( h_l \) has no zero in \( E^l_Y \), and \( N(l) = 0 \).

**Sub-lemma 3.2.3.** — \( N(k) - N(k - 1) \) is even for \( 1 \leq k \leq l \).

**Proof.** — We observe that \( h_{k-1} \) don’t vanish on \( A \), where \( A \) is the annulus defined in (3.2.16), and we took \( ||\theta_k||_{\infty} \) very small so that \( h_k \) does not vanish on \( A \) as well. Next, by definition of \( \varphi_k, \varphi_0 = 0 \) on \( A \). Setting
\[ m_t(x) = h_{k-1}(x) + t[h_k(x) - h_{k-1}(x)] = h_{k-1}(x) + \theta_k(x), \quad (3.2.21) \]
On some properties of three-dimensional minimal sets in $\mathbb{R}^4$

for $x \in E_Y^1 \cap B(x_k, \rho_2)$ and $0 \leq t \leq 1$. Then $m_0 = h_{k-1}$ and $m_1 = h_k$ on $E_Y^1 \cap B(x_k, \rho_2)$. Since $m_t(x) = h_{k-1}(x) + t\theta(x)$ for $x \in E_Y^1 \cap A$ and $0 \leq t \leq 1$, so $m_t(x) \neq 0$ if we take $\theta$ small enough. Let $\beta_k > 0$ such that $|m_t(x)| \geq \beta_k$ for $x \in E_Y^1 \cap A$. Set $S_\infty = \mathbb{R}^2 \cup \{\infty\}$, so that $S_\infty$ can be stereographically identified with a sphere of dimension 2, we define $\pi : \mathbb{R}^2 \rightarrow S_\infty$ by

$$\pi(x) = \infty \text{ if } |x| \geq \beta_k \text{ and } \pi(x) = \frac{x}{\beta_k - |x|} \text{ otherwise.} \quad (3.2.22)$$

Next, we set

$$p_t(x) = \pi(m_t(x)) \text{ for } x \in E_Y^1 \cap B(x_k, \rho_2) \text{ and } 0 \leq t \leq 1. \quad (3.2.23)$$

Then $p_t(x)$ is a continuous function of $x$ and $t$, which takes values in $S_\infty$. By the definition of $\beta_k$,

$$p_t(x) = \infty \text{ for } x \in E_Y^1 \cap A \text{ and } 0 \leq t \leq 1. \quad (3.2.24)$$

We want to replace the domain $E_Y^1 \cap B(x_k, \rho_2)$ by an open set in a 2-plane $P_k$. We keep our Bi-Hölder function $\psi_k$ as above, which maps an open set $V$ of a 2-plane $P_k$ onto $E_Y^1 \cap B(x_k, \rho_2)$ and its inverse $\gamma$ which is also Bi-Hölder and maps $E_Y^1 \cap B(x_k, \rho_2)$ onto $V$. For $0 \leq t \leq 1$, we set

$$q_t(x) = p_t(\psi_k(x)) \text{ for } x \in V \text{ and } q_t(x) = \infty \text{ for } x \in P_k \setminus V. \quad (3.2.25)$$

We check that $q_t$ is continuous in $P_k \times [0, 1]$. It is continuous in $V \times [0, 1]$, since $p_t$ is continuous in $[E_Y^1 \cap B(x_k, \rho_2)] \times [0, 1]$. It is also continuous in $[P_k \setminus V] \times [0, 1]$, because it is $\infty$ here. Now if $x \in \partial V$, then $\psi_k(x) \in E_Y^1 \cap \partial B(x_k, \rho_2)$, so there is a neighborhood of $\psi_k(x)$ in $\overline{B(x_k, \rho_2)}$ which is contained in $A$, and we have $p_t(\psi_k) = \infty$ on this neighborhood, so $q_t = \infty$ near $x$.

We set $q_t(\infty) = \infty$, so $q_t$ is well defined on $S' = P_k \cup \{\infty\}$ and it is clear that each $q_t$ is continuous for $0 \leq t \leq 1$.

Now since $q_0$ and $q_1$ are two continuous functions from the 2-sphere $S'$ to the 2-sphere $S_\infty$, we can compute their degrees. First, as $q_0$ and $q_1$ are homotopic, they have the same degrees. We compute the degree of $q_0$, for example. Let

$$q_0^{-1}(\{0\}) = \{y_1, y_2, \ldots, y_m\}, \quad (3.2.26)$$

the set of zeroes of $q_0$. This is a finite set since $q_t$ has only finite number of zeroes for $t \leq 1$. Since each zero of $q_0$ is simple and non-degenerate, for each $1 \leq k \leq m$, there exists a neighborhood $W_k$ of $y_k$ such that

$q_0$ is a homeomorphism from $W_k$ to $q_0(W_k)$, \quad (3.2.27)
and
\[ W_k \cap W_l = \emptyset \text{ if } k \neq l. \quad (3.2.28) \]
So the degree of \( q_0 \) is computed as follows. We begin by 0, next, for \( 1 \leq k \leq m \), if \( q_0 \) preserve the orientation of \( W_k \), we add 1, if \( q_0 \) doesn’t preserve the orientation of \( W_k \), we add -1. Then it is clear that
\[ d(q_0) \text{ is of the same parity as } m. \quad (3.2.29) \]
Here \( d(q) \) denote the degree of the function \( q \). By the same arguments, we have
\[ d(q_1) \text{ is of the same parity as the number of zeroes of } q_1. \quad (3.2.30) \]
But \( d(q_0) = d(q_1) \) as above, we obtain

the number of zeroes of \( q_0 \) is of the same parity as the number of zeroes of \( q_1 \). \( (3.2.31) \)

We want to prove next that the number of zeroes of \( h_{k-1} \) is of the same parity as the number of zeroes of \( h_k \). Since \( h_{k-1} = h_k \) outside the ball \( B(x_k, \rho_2) \) and they both don’t vanish on \( E^1_Y \cap A \), we need only to consider their number of zeroes in \( E^1_Y \cap B(x_k, \rho_1) \). We verify that

the number of zeroes of \( h_{k-1+s} \) in \( E^1_Y \cap B(x_k, \rho_1) \) is equal to the number of zeroes of \( q_s \) in \( S' \) for \( s = 0, 1 \). \( (3.2.32) \)

We verify for \( s = 0 \). If \( q_0(x) = 0 \), then \( x \in V \) (otherwise \( q_0(x) = \infty \)), so \( q_0(x) = p_0(\psi_k(x)) \) and then \( p_0(\psi_k(x)) = 0 \). Since \( m_0(\psi_k(x)) = 0 \), we have \( h_{k-1}(\psi_k(x)) = 0 \). Because \( x \in V \), we have \( \psi_k(x) \in B(x_k, \rho_1) \). So if \( q_0(x) = 0 \), then \( \psi_k(x) \in B(x_k, \rho_1) \) and is a zero of \( h_{k-1} \).

Conversely, if \( y \in B(x_k, \rho_1) \) is such that \( h_{k-1}(y) = 0 \), then \( p_0(y) = 0 \) and then there exists \( y' \in V \) such that \( \psi_k(y') = y \) because \( \psi_k \) is a homeomorphism from \( V \) to \( B(x_k, \rho_1) \). Now \( q_0(y') = p_0(\psi_k(y')) = 0 \) and thus \( y' \) is a zero of \( q_0 \).

So we have \( (3.2.32) \) for \( s = 0 \). The case \( s = 1 \) is the same, and we have then \( (3.2.32) \). By \( (3.2.31) \), we obtain that the number of zeroes of \( h_{k-1} \) is of the same parity as the number of zeroes of \( h_k \), which means that \( N(k) - N(k - 1) \) is even. The sub-lemma follows.

Now by sub-lemma 3.2.3, we know that \( N(0) - N(1) \) is even, but it is 1, so we obtain a contradiction, and we finish the proof of Lemma 3.2. \( \square \)
3.3. Proof of Theorem 2

Let \(U(y), y \in E_Y \cap B(0, 3 \cdot 2^8)\) be the set of connected components \(V\) of \(B(0, 2^{10}) \setminus E\) such that \(y \in V\). Since for each \(y \in E_Y\), there is a neighborhood \(W\) of \(y\) on which \(E\) is Bi-Hölder equivalent to a \(Y\), we see that \(U(y)\) is locally constant. By Lemma 3.2, we can connect \(b_1\) to another point \(b_i, i \neq 1\), by a curve in \(E_Y\), and we can suppose that \(i = 2\). Because \(b_1, b_2 \in E_Y\) and \(U(y)\) is locally constant on \(E_Y\), we have \(U(b_1) = U(b_2)\). By Lemma 3.1, and the fact that \(E\) is Bi-Hölder equivalent to a \(Y\) near each point of type \(Y\), we have

\[
\{V_2, V_3, V_4\} = U(b_1)
\]

and

\[
\{V_1, V_3, V_4\} = U(b_2),
\]

where \(V_i, 1 \leq i \leq 4\) is as in Lemma 3.1. So we see that \(U(b_1) \neq U(b_2)\), which is a contradiction. We finish the proof of Theorem 2. \(\Box\)

Bibliography


