Quantum expanders and growth of group representations

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ABSTRACT. — Let π be a finite dimensional unitary representation of a group \( G \) with a generating symmetric \( n \)-element set \( S \subset G \). Fix \( \varepsilon > 0 \). Assume that the spectrum of \( |S|^{-1} \sum_{s \in S} \pi(s) \otimes \pi(s) \) is included in \([-1, 1 - \varepsilon]\) (so there is a spectral gap \( \geq \varepsilon \)). Let \( r'_N(\pi) \) be the number of distinct irreducible representations of dimension \( \leq N \) that appear in \( \pi \). Then let \( R'_{n,\varepsilon}(N) = \sup r'_N(\pi) \) where the supremum runs over all \( \pi \) with \( n, \varepsilon \) fixed. We prove that there are positive constants \( \delta_\varepsilon \) and \( c_\varepsilon \) such that, for all sufficiently large integer \( n \) (i.e. \( n \geq n_0 \) with \( n_0 \) depending on \( \varepsilon \)) and for all \( N \geq 1 \), we have \( \exp \delta_\varepsilon nN^2 \leq R'_{n,\varepsilon}(N) \leq \exp c_\varepsilon nN^2 \). The same bounds hold if, in \( r'_N(\pi) \), we count only the number of distinct irreducible representations of dimension exactly \( = N \).

RÉSUMÉ. — Soit \( \pi \) une représentation unitaire de dimension finie d’un groupe \( G \) munie d’un ensemble générateur symétrique \( S \subset G \) à \( n \)-éléments. Fixons \( \varepsilon > 0 \) et supposons que le spectre de \( |S|^{-1} \sum_{s \in S} \pi(s) \otimes \pi(s) \) est inclus dans \([-1, 1 - \varepsilon]\) (il y a donc un trou spectral \( \geq \varepsilon \)). Soit \( r'_N(\pi) \) le nombre de représentations irréductibles distinctes de dimension \( \leq N \) qui apparaissent dans la décomposition de \( \pi \). Soit alors \( R'_{n,\varepsilon}(N) = \sup r'_N(\pi) \) où le sup court sur toutes les \( \pi \) possibles avec \( n, \varepsilon \) fixés. Nous démontrons l’existence de constantes positives \( \delta_\varepsilon \) et \( c_\varepsilon \) telles que, pour tout entier \( n \) suffisamment grand (i.e. \( n \geq n_0 \) ou \( n_0 \) peut dépendre de \( \varepsilon \)) et pour tout \( N \geq 1 \), on a \( \exp \delta_\varepsilon nN^2 \leq R'_{n,\varepsilon}(N) \leq \exp c_\varepsilon nN^2 \). Les mêmes bornes sont valables si, dans \( r'_N(\pi) \), on compte seulement le nombre de représentations irréductibles distinctes de dimension exactement \( = N \).

1. Introduction

We wish to formulate and answer a natural extension of a question raised explicitly by Wigderson in several lectures (see e.g. [23, p. 59]) and also

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implicitly in [18]. Although the variant that we answer seems to be much easier, it may shed some light on the original question. Wigderson’s question concerns the growth of the number $r_N(G)$ of distinct irreducible representations of dimension $\leq N$ that may appear on a finite group $G$ when the order of $G$ is arbitrarily large and all that one knows is that $G$ admits a generating set $S$ of $n$ elements for which the Cayley graph forms an expander with a fixed spectral gap $\varepsilon > 0$. The problem is to find the best bound of the form $r_N(G) \leq R(N)$ with $R(N)$ independent of the order of $G$ (but depending on $n, \varepsilon$). We consider a more general framework: the finite group $G$ is replaced by a finite dimensional representation $\pi$ (playing the role of the regular representation $\lambda_G$ for finite groups) such that the representation $\pi \otimes \bar{\pi}$ admits a spectral gap, meaning that the trivial representation is isolated with a gap $\geq \varepsilon$ from the other irreducible components of $\pi \otimes \bar{\pi}$. When $\pi = \lambda_G$ we recover the previous notion of spectral gap. Let then $r'_N(\pi)$ be the number of distinct irreducible representations of dimension $\leq N$ appearing in $\pi$ (note that $r_N(G) = r'_N(\lambda_G)$), and let $R'(N)$ denote the least upper bound $r'_N(\pi) \leq R'(N)$ when the only restriction on $\pi$ is that $n, \varepsilon$ remain fixed (but the dimension of $\pi$ is arbitrary). We observe that the previously known bound for $R(N)$ namely $R(N) = e^{O(nN^2)}$ is also valid for $R'(N)$ and also that $R(N) \leq R'(N)$. Our main result, which follows from the metric entropy estimate for quantum expanders in [20], is that this bound for $R'(N)$ is sharp: there is $\delta > 0$ such that for all $n$ large enough (i.e. $\forall n \geq n_0(\varepsilon)$) we have $R'(N) \geq e^{\delta nN^2}$ for all $N$.

The term “quantum expander” was coined in [2, 3, 8] to which we refer for background (see also [7, 9]).

2. Main result

Let $G$ be any group with a finite generating set $S \subseteq G$ with $|S| = n$. For any unitary representation $\pi : G \to H_\pi$ we set

$$\lambda(\pi, S) = n^{-1} \sup \{ \Re \left( \sum_{s \in S} \pi(s) \xi, \xi \right) \mid \xi \in H_\pi^{\text{inv}}, \|\xi\|_{H_\pi} = 1 \}.$$ 

where $H_\pi^{\text{inv}} \subseteq H_\pi$ denotes the subspace of all $\pi$-invariant vectors. When $S$ is symmetric, $\sum_{s \in S} \pi(s)$ being selfadjoint, the real part sign $\Re$ can be omitted. We then set

$$\varepsilon(\pi, S) = 1 - \lambda(\pi, S).$$

It will be useful to record here the elementary observation that if $\pi$ is unitarily equivalent to the direct sum $\oplus_{i \in I} \pi_i$ of a family of unitary representations,
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then \( \lambda(\pi, S) = \sup_{i \in I} \lambda(\pi_i, S) \) and hence

\[
\varepsilon(\pi, S) = \inf_{i \in I} \varepsilon(\pi_i, S).
\] (2.1)

In particular, if \( \pi_1 \) is contained in \( \pi_2 \), then \( \varepsilon(\pi_1, S) \geq \varepsilon(\pi_2, S) \).

We denote

\[
\varepsilon(G, S) = \inf \{ \varepsilon(\pi, S) \}
\]

where the infimum runs over all unitary representations \( \pi : G \to H_\pi \). Thus the condition

\[
\varepsilon(G, S) > 0
\]

means that \( G \) has Kazhdan’s “property (T)”, (or in other words is a Kazhdan-group), see [1] for more background.

We start by the following result somewhat implicitly due to S. Wassermann [22] and explicitly proved in detail in [6].

**Proposition 2.1** ([22, 6]). — For any \( \varepsilon > 0 \) there is a constant \( c_\varepsilon \) such that for any \( n \), any group \( G \) and any \( S \subset G \) with \( |S| = n \) such that \( \varepsilon(G, S) \geq \varepsilon \), the number \( r_N(G) \) of distinct irreducible unitary representations \( \sigma : G \to B(H_\sigma) \) with \( \dim(H_\sigma) \leq N \) is majorized as follows:

\[
r_N(G) \leq \exp(c_\varepsilon nN^2).
\] (2.2)

Of course, here distinct means up to unitary equivalence.

**Remark 2.2.** — Note that it suffices to prove a bound of the same form for the number of distinct irreducible unitary representations \( \sigma : G \to B(H_\sigma) \) with \( \dim(H_\sigma) = N \). Indeed, if the latter number is denoted by \( s_N(G) \), we have \( r_N(G) = \sum_{d=1}^{N} s_d(G) \), so that it suffices to have a bound of the form \( s_d(G) \leq \exp(c_d' nd^2) \) to obtain (2.2). See [14, 15] for some examples of estimates of the growth of \( r_N(G) \).

We note that it was originally proved by Wang [21] that for any Kazhdan-group \( G \) this number \( r_N(G) \) is finite for any \( N \). There is an indication of proof of (2.2) in [22], and detailed proofs appear in [6] (see also [18]). We will prove a simple extension of this bound below.

Recall that a sequence \( (G_k, S_k) \) of finite groups equipped with generating sets \( S_k \subset G_k \) such that

\[
\sup_k |S_k| < \infty, \quad |G_k| \to \infty \quad \text{and} \quad \inf_k \varepsilon(G_k, S_k) > 0
\]

is called an expander or an expanding family. This corresponds to the usual notion among Cayley graphs to which we restrict the entire discussion. Let \( \hat{G} \) denote as usual the (finite) set of all irreducible unitary representations of a finite group \( G \) (up to unitary equivalence). We note in passing that it
is well known (and this also can be derived from Proposition 2.1) that any expander satisfies
\[
\lim_{k \to \infty} \max \{ \dim(H_\sigma) \mid \sigma \in \hat{G}_k \} = \infty. \tag{2.3}
\]

We refer the reader to the surveys [10, 17] for more information on expanders.

The question raised by Wigderson in this context can be formulated as follows:

Let
\[
R_{n,\varepsilon}(N) = \sup \{ r_N(G) \}
\]
where the supremum runs over all finite groups \( G \) admitting a subset \( S \) with \( |S| = n \) such that \( \varepsilon(G, S) \geq \varepsilon \). Actually the question is just as interesting for arbitrary (Kazhdan) groups \( G \), but it is more natural to restrict to finite groups, because there are infinite Kazhdan groups without any (nontrivial) finite dimensional representations.

Moreover, since, for a finite group \( G \), all representations are weakly contained in the left regular representation \( \lambda_G \), we have clearly by (2.1)
\[
\varepsilon(G, S) = \varepsilon(\lambda_G, S). \tag{2.4}
\]

By (2.2), we have
\[
R_{n,\varepsilon}(N) \leq \exp(c_\varepsilon n N^2). \tag{2.5}
\]
and a fortiori simply \( R_{n,\varepsilon}(N) = \exp O(N^2) \). Wigderson asked whether this upper bound can be improved. More explicitly, what is the precise order of growth of \( \log R_{n,\varepsilon}(N) \) when \( N \to \infty \). Does it grow like \( N \) rather than like \( N^2 \)? The motivation for this question can be summarized like this: In [18, Th. 1.4] an exponential bound \( \exp O(N) \) is proved for a special class of groups \( G \) (namely monomial groups), admitting a fixed spectral gap with generating sets of very slowly growing size (but not bounded) and it is asked whether the same exponential bound holds in general for \( R_{n,\varepsilon}(N) \). Moreover, in a remark following the proof of [18, Th. 1.4], Meshulam and Wigderson observe that for any prime number \( p > 2 \), there is a group \( G_p \) with a generating set of (unbounded) size \( \log p \) admitting a fixed spectral gap and such that \( r_p(G) \approx 2^p/p \).

**Remark 2.3.** — By classical results, originating in the works of Kazhdan and Margulis (see e.g. [16] or [17, Cor. 2.4]), for any fixed \( m \geq 3 \), the family \( \{ SL_m(\mathbb{Z}_p) \mid p \text{ prime} \} \) is an expander, so that we have (for suitable \( \ell, \delta \))
\[
R_{\ell,\delta}(N) \geq \sup_p r_N(SL_m(\mathbb{Z}_p)).
\]
Similarly, let \( G_k \) denote the symmetric group of all permutations of a \( k \) element set. Kassabov [11] proved that the family \( \{ G_k \mid k \geq 1 \} \) forms an
expanding family with respect to subsets $S_k \subset G_k$ of a fixed size $\ell$ and a fixed spectral gap $\delta > 0$. Thus we find a lower bound
\[ R_{\ell,\delta}(N) \geq \sup_k r_N(G_k). \]
Quite remarkably, it is proved in [13] that the family itself of all non-commutative finite simple groups forms an expander (for some suitable $n, \varepsilon$).

Remark 2.4. — However, it seems the resulting lower bounds are still far from being exponential in $N$. Actually, in many important cases (see e.g. [4]), the proof that certain finite groups $G$ give rise to expanders uses the fact that the smallest dimension of a (non-trivial) irreducible representation on $G$ is $\geq c|G|^a$ for some $a > 0$. Then since $|G| = \sum_{\pi \in \hat{G}} \dim(\pi)^2$ the cardinal of $\hat{G}$ is bounded above by $|G|^{1-2a}/c^2$. Therefore, for any $N \geq c|G|^a$ we have $r_N(G) \leq |G|^{1-2a}/c^2 \leq c'N^{(1/a)-2}$, so that the resulting growth implied for $R_{n,\varepsilon}(N)$ is at most polynomial in $N$. (I am grateful to N. Ozawa for drawing my attention to this point).

Nevertheless, we have:

Remark 2.5 (Communicated by Martin Kassabov). — For suitable $n, \varepsilon$ the numbers $R_{n,\varepsilon}(N)$ grow faster than any power of $N$. In fact, we will prove the

Claim. — There is an expanding family of Cayley graphs $(G_k)$ of groups generated by 3 elements with a positive spectral gap $\varepsilon$ and such that for $N_k = 2^{3k} - 2$, $G_k$ admits $2^{k^2}$ distinct irreducible representations of dimension $N_k$.

From this claim follows that $R_{3,\varepsilon}(N_k) \geq 2^{k^2} \geq 2^{(\log(N_k))^2}$, say for all $k$ large enough, and hence
\[ R_{n,\varepsilon}(N) \geq 2^{(\log(N))^2} \text{ for infinitely many } N \text{'s.} \]

To prove the claim we use the ideas from [12]. Let $\mathcal{R}_k$ denote the (finite) ring $M_k(F_2)$ of $k \times k$ matrices with entries in the field with 2 elements. It is known that the cartesian product $\Pi_k = \mathcal{R}_k^{2k^2}$ of $|\mathcal{R}_k| = 2^{k^2}$ copies of $\mathcal{R}_k$ is generated by 3 elements. Indeed, $\mathcal{R}_k$ itself is generated as a ring by two elements, e.g. $a = e_{12}$ and the shift $b = e_{12} + e_{23} + \cdots + e_{k-1k} + e_{k1}$, then $\Pi_k$ is generated as a ring by $\{A, B, C\}$ where $A$ (resp. $B$) is the element with all coordinates equal to $a$ (resp. $b$), and $C$ is such that its coordinates are in one to one correspondence with the elements of $\mathcal{R}_k$. To check this, let $R \subset \Pi_k$ be the ring generated by $\{A, B, C\}$. Note, by the choice of $C$, the following easy observation: for any coordinate $i$, there is $x \in R$ such that $x_i = 0$ but $x_j \neq 0$ for all $j \neq i$. For any subset $I$ of the index set let $p_I : R \to \mathcal{R}_k^I$ be the coordinate projection. One can then prove by induction
on $m = |I|$ that $p_I(R) = R_k^I$ for all $I$. Indeed, assume the fact established for $m - 1$. For any $I$ with $|I| = m$ we pick $i \in I$ and we consider the set $I = \{y \in R_k^I \mid (0, y) \in p_I(R)\}$. By the induction hypothesis, $I$ is an ideal in $R_k^I$, but, since $R_k$ is simple, the above observation implies that $I = R_k^I$, and since $a, b$ generate $R_k$ we have $p_{\{i\}}(R) = R_k$, so we obtain $p_I(R) = R_k^I$.

This implies that the free associative ring $\mathbb{Z}\langle x, y, z \rangle$ (in 3 non-commutative variables) can be mapped onto the product $\Pi_k$. Consider now the group $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ generated by the elementary matrices in $GL_3(\mathbb{Z}\langle x, y, z \rangle)$. This is a noncommutative universal lattice in the terminology of [5, 12].

First observe that $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ is generated by 3 elements. Indeed, let $\alpha, \beta$ generate $SL_3(\mathbb{Z})$. Then $\alpha, \beta, \gamma$ will generate $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ where

$$\gamma = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, by [5, Th. 1.1] $EL_3(\mathbb{Z}\langle x, y, z \rangle)$ has Kazhdan’s property T. It follows that the groups

$$G_k = EL_3(\Pi_k)$$

have expanding generating sets with 3 elements. But it turns out that $G_k$ can be identified with the product

$$SL_{3k}(F_2)^{2k^2}.$$

Indeed, firstly one easily checks the natural isomorphism $EL_3(R_k^{2k^2}) \simeq EL_3(R_k)^{2k^2}$, secondly it is well known that, since $F_2$ is a field, $EL_n(F_2) = SL_n(F_2)$ for any $n$, and hence (taking $n = 3k$) we have a natural isomorphism $EL_3(R_k) = SL_{3k}(F_2)$; this yields the identification $G_k = SL_{3k}(F_2)^{2k^2}$.

To conclude, we will use the fact that $SL_{3k}(F_2)$ admits a nontrivial irreducible representation $\pi$ with dimension $N_k = 2^{3k} - 2$. (Just consider its action by permutation on the projective space, which has $2^{3k} - 1$ elements; the action is transitive and doubly transitive, therefore the associated Koopman representation $\pi$ is irreducible and of dimension $2^{3k} - 2$). This immediately produces $2^{k^2}$ distinct irreducible representations of dimension $N_k$ on $SL_{3k}(F_2)^{2k^2}$. Indeed, it is an elementary fact that if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$ is a product group, and if $\pi_1, \ldots, \pi_m$ are arbitrary nontrivial irreducible representations on the factor groups $\Gamma_1, \ldots, \Gamma_m$, then the representations $\tilde{\pi}_j$ defined on $\Gamma$ by $\tilde{\pi}_j(g) = \pi_j(g_j)$ are distinct (meaning not unitarily equivalent), irreducible on $\Gamma$ and $\dim(\tilde{\pi}_j) = \dim(\pi_j)$ for any $j$. So taking all $\Gamma_j$’s equal to $SL_{3k}(F_2)$, with $\pi_j = \pi$ and $m = 2k^2$, we obtain the announced claim.
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In any case, the problem of finding the correct behaviour of \( \log R_{n,\varepsilon}(N) \) (or of \( R_{n,\varepsilon}(N) \) itself) when \( N \to \infty \) appears to be still wide open.

In this paper we consider a modified version of this question involving “quantum expanders” and show that for this (much easier) modified version, \( N^2 \) is the correct order of growth.

The term “quantum expander” was introduced in [8] and [2, 3], independently, to designate a sort of non-commutative, or matricial, analogue of expanders, as follows.

Fix an integer \( n \). Consider an \( n \)-tuple of \( N \times N \) unitary matrices, say \( u = (u_j) \in U(N)^n \). We view each of them \( u_j \) as a linear operator on the \( N \)-dimensional Hilbert space \( H \). Then \( u_j \otimes \bar{u}_j \) is naturally viewed as a linear operator on the (Hilbert space sense) tensor product \( H \otimes \bar{H} \). Using the (canonical) identification \( H^* \simeq \bar{H} \), the tensor product \( H \otimes \bar{H} \) can be isometrically identified with the space of linear operators from \( H \) to \( H \) equipped with the Hilbert–Schmidt norm denoted by \( \| \cdot \|_2 \) (sometimes called the Frobenius norm in the present finite dimensional context). Then, the identity operator \( \text{Id}_H : H \to H \) defines a distinguished element of \( H \otimes \bar{H} \) that we denote by \( I \).

We set \( \lambda(u) = n^{-1} \sup \left\{ \Re \left( \sum_1^n u_j \otimes \bar{u}_j \right) \xi, \xi \right\} \left| \xi \in H \otimes \bar{H}, \xi \perp I, \|\xi\|_{H \otimes \bar{H}} = 1 \right. \), and \( \varepsilon(u) = 1 - \lambda(u) \).

In other words, with the preceding identifications, the condition \( \varepsilon(u) \geq \varepsilon \) means that for any \( x \in M_N \) with \( \text{tr}(x) = 0 \) we have
\[
\Re \sum_1^n \text{tr}(u_j x u_j^* x^*) \leq (1 - \varepsilon) \|x\|_2,
\]
where \( \|x\|_2 = (\text{tr}(x^* x))^{1/2} \). When \( T = \sum_1^n u_j \otimes \bar{u}_j \) is self adjoint (in particular when the set \( \{u_1, \ldots, u_n\} \) is self adjoint) the real part \( \Re \) can be omitted in the two preceding lines.

In group theoretic language, if \( \pi : F_n \to U(N) \) is the group representation on the free group \( F_n \), equipped with a set of \( n \) free generators \( S = \{g_1, \ldots, g_n\} \), such that \( \pi(g_j) = u_j \ (1 \leq j \leq n) \), then we have \( \varepsilon(u) = \varepsilon(\pi \otimes \pi, S) \).

**Definition 2.6.** — A sequence \( \{u(k) \mid k \in \mathbb{N}\} \) with each \( u(k) \in U(N_k)^n \) such that \( N_k \to \infty \) (with \( n \) remaining fixed) and \( \inf_k \{\varepsilon(u(k))\} > 0 \) is called a quantum expander. We say that \( n \) is its degree and \( \inf_k \{\varepsilon(u(k))\} > 0 \) its spectral gap.
Remark 2.7. — The existence of quantum expanders can be deduced as follows from that of expanders. Recalling (2.4), assume given a finite group $G$ and $S \subset G$ as before such that $\epsilon(G, S) = \epsilon(\lambda_G, S) \geq \epsilon > 0$. Recall that each $\sigma \in \hat{G}$ is contained in $\lambda_G$. Let $\pi \in \hat{G}$. Since any representation on $G$ without invariant vectors, being a direct sum of non-trivial irreps, is weakly contained in $\lambda_G$, the representation $\rho = \pi \otimes \overline{\pi}$ restricted to $H^\text{inv}_\rho$ is weakly contained in the non-trivial part of $\lambda_G$. In particular, we have by (2.1)

$$\lambda(\rho, S) \leq \lambda(\lambda_G, S).$$

Therefore, we have

$$\epsilon(\pi \otimes \overline{\pi}, S) \geq \epsilon(\lambda_G, S) \geq \epsilon.$$

Thus if we are given an expander $(G_k, S_k)$ as above, say with $S_k = \{s_1(k), \ldots, s_n(k)\}$, we can choose by (2.3) $\sigma_k \in \hat{G}_k$ such that $\dim(H_{\sigma_k}) \to \infty$, and if we set $u_j(k) = \sigma_k(s_j(k))$ $(1 \leq j \leq n)$, then $u(k) = \{u_1(k), \ldots, u_n(k)\}$ forms a quantum expander.

The next statement is a simple generalization of Proposition 2.1

**Proposition 2.8.** — For any $0 < \varepsilon < 1$ there is a constant $c'_\varepsilon > 0$ for which the following holds. Let $G$ be any group and let $\pi : G \to B(H)$ be any unitary representation on a finite dimensional Hilbert space $H$. Let us assume that there is an $n$-element subset $S \subset G$ and $\varepsilon > 0$ such that

$$\epsilon(\pi \otimes \overline{\pi}, S) \geq \varepsilon.$$

In other words, $\pi$ satisfies the following spectral gap condition:

$$\lambda(\pi \otimes \overline{\pi}, S) \leq 1 - \varepsilon \quad (2.6)$$

Let $\pi = \bigoplus_{t \in T} \pi_t$ be the decomposition into distinct irreducibles (where each $\pi_t$ has multiplicity $d_t \geq 1$), then

$$|\{t \in T \mid \dim(\pi_t) \leq N\}| \leq \exp c'_\varepsilon nN^2. \quad (2.7)$$

**Proof.** — Let $\sigma = \bigoplus_{t \in T} \pi_t$ be the direct sum where each component is included with multiplicity equal to 1. We may clearly view $\sigma$ as a subrepresentation of $\pi$, acting on a subspace $K \subset H$ so that the orthogonal projection $Q : H \to K$ is intertwining, i.e. satisfies $Q\pi = \sigma Q$. Then we also have $(Q \otimes \overline{Q})(\pi \otimes \overline{\pi}) = (\sigma \otimes \overline{\sigma})(Q \otimes \overline{Q})$, from which it is easy to derive that if we denote $V_\pi = H^\text{inv}_\pi$, we have $(Q \otimes \overline{Q})V_\pi = V_\sigma$ and $(Q \otimes \overline{Q})V_\pi^\perp = V_\sigma^\perp$. This implies

$$\lambda(\sigma \otimes \overline{\sigma}, S) \leq \lambda(\pi \otimes \overline{\pi}, S) \leq 1 - \varepsilon.$$

Thus, replacing $\pi$ by $\sigma$, we may as well assume that the multiplicities $d_t$ are all equal to 1.

Let $H = \bigoplus_{t \in T} \chi_t$ denote the decomposition corresponding to $\pi = \bigoplus_{t \in T} \pi_t$. We have $\pi \otimes \overline{\pi} = \bigoplus_{t, r \in T} \pi_t \otimes \overline{\pi_r}$, with associated decomposition $H \otimes H = \bigoplus_{t, r \in T} \chi_t \otimes \chi_r$.
\[ \bigoplus_{t,r \in T} H_t \otimes \overline{H}_r. \] From this follows that the subspace \( V_\pi \subset H \otimes \tilde{H} \) of \( \pi \otimes \overline{\pi} \)-invariant vectors is equal to \( \bigoplus_{t,r \in T} V_{t,r} \) where \( V_{t,r} \subset H_t \otimes \overline{H}_r \) is the subspace of invariant vectors of \( \pi_t \otimes \overline{\pi}_r \). Since for any \( t \neq r \in T \), \( \pi_t \not\cong \pi_r \), by Schur’s lemma \( V_{t,r} = \{0\} \), and hence \( V_\pi \subset \bigoplus_{t \in T} V_{t,t} \). In particular, this shows that
\[ \forall t \neq r \in T \quad H_t \otimes \overline{H}_r \subset V_{\pi}^\perp. \]

Let \( T' = \{ t \in T \mid \dim(\pi_t) = N \} \). It suffices to show an estimate of the form
\[ |T'| \leq \exp c_\varepsilon n N^2. \tag{2.8} \]

Let \( \mathcal{H} \) be the Hilbert space obtained by equipping \( M_N^\pi \) with the norm
\[ \|x\|_{\mathcal{H}}^2 = N^{-1} n^{-1} \sum_1^n \text{tr}(x_j^* x_j). \]

Let \( S = \{ s_1, \ldots, s_n \} \). For any \( t \in T' \) we define \( x(t) \in M_N^\pi \) by
\[ x(t)_j = \pi_t(s_j) \quad 1 \leq j \leq n. \]

Note that, by our normalization, \( \|x(t)\|_{\mathcal{H}} = 1 \) for any \( t \in T' \). Moreover, since for any \( t \neq r \in T \pi_t \not\cong \pi_r \), by Schur’s lemma the representation \( \pi_t \otimes \overline{\pi}_r \) has no invariant vector, and hence lies inside \( (\pi \otimes \overline{\pi})|_{V_\pi^\perp} \). Therefore, by (2.1)
\[ \lambda(\pi_t \otimes \overline{\pi}_r, S) \leq \lambda(\pi \otimes \overline{\pi}, S), \]
and hence for any unit vector \( \xi \in H_{\pi_t} \otimes \overline{H}_{\pi_r} \) we have
\[ n^{-1} \Re \left( \sum_{s \in S} (\pi_t \otimes \overline{\pi}_r)(\xi, \xi) \right) \leq 1 - \varepsilon. \]

In particular, if \( t \neq r \in T' \), we may realize \( \pi_t, \pi_r \) as representations on the same \( N \)-dimensional space, so that taking \( \xi = N^{-1/2} I \) we find
\[ \Re \langle x(t), x(r) \rangle_{\mathcal{H}} = (nN)^{-1} \Re \left( \sum_{s \in S} \text{tr}(\pi_t(s)^* \pi_r(s)) \right) \leq 1 - \varepsilon, \]
which implies
\[ \|x(t) - x(r)\|_{\mathcal{H}} \geq \sqrt{2\varepsilon}. \]

Thus we have \( |T'| \) points in the unit sphere of \( \mathcal{H} \) that are \( \sqrt{2\varepsilon} \)-separated. Since \( \dim(\mathcal{H}) = n N^2 \), (2.8) follows immediately by a well known elementary volume argument (see e.g. [19, p. 57]). \( \square \)

**Remark 2.9.** — To derive Proposition 2.1 from the preceding statement, consider, in the situation of Proposition 2.1, a finite set \( \{ \sigma_t \mid t \in T \} \) of distinct finite dimensional irreducible representations of \( G \), let \( \pi \) be their direct sum and let \( \rho = \pi \otimes \overline{\pi} \). By the assumption in Proposition 2.1, we know \( \varepsilon(\rho, S) \geq \varepsilon \), and hence (2.7) implies \( |T| \leq \exp c_\varepsilon n N^2 \). Applying this to \( \pi = \lambda_G \), this shows that Proposition 2.8 contains Proposition 2.1.
For any finite dimensional unitary representation $\pi : G \to B(H)$ on an arbitrary group, let us denote by $r_N'(\pi)$ the number of distinct irreducible representations appearing in the decomposition of $\pi$ of dimension at most $N$. Let then

$$R_{n,\varepsilon}'(N) = \sup r_N'(\pi)$$

where the sup runs over all $\pi$’s and $G$’s admitting an $n$-element generating set $S \subset G$ such that

$$\varepsilon(\pi \otimes \overline{\pi}, S) \geq \varepsilon.$$

Note that $r_N'(\lambda_G) = r_N(G)$ and hence

$$R_{n,\varepsilon}(N) \leq R_{n,\varepsilon}'(N).$$

With this notation (2.7) means that

$$R_{n,\varepsilon}'(N) \leq \exp c'_\varepsilon nN^2.$$

While it seems very difficult to give a good lower bound for $R_{n,\varepsilon}(N)$, we can answer the analogous question for $R_{n,\varepsilon}'(N)$: Indeed, the main result of [20] (see [20, Th. 1.3]), which follows, implies the desired lower bound when reformulated in terms of representations.

**Theorem 2.10 ([20]).** — For each $0 < \varepsilon < 1$, there is a constant $\beta_\varepsilon > 0$ such that and for all sufficiently large integer $n$ (i.e. $n \geq n_0$ with $n_0$ depending on $\varepsilon$) and for all $N \geq 1$, there is a subset $T \subset U(N)^n$ with

$$|T| \geq \exp \beta_\varepsilon nN^2$$

such that

$$\forall u \neq v \in T \quad \left\| \sum_{j=1}^{n} u_j \otimes \overline{v}_j \right\| \leq n(1 - \varepsilon) \quad (\text{we call these “$\varepsilon$-separated”}),$$

and $\varepsilon(u) \geq \varepsilon$ for all $u \in T$ (we call these “$\varepsilon$-quantum expanders”). More precisely, for all $u \in T$ we have

$$\left\| \left( \sum_{j=1}^{n} u_j \otimes \overline{u}_j \right) |I^\perp \right\| \leq n(1 - \varepsilon).$$

**Theorem 2.11.** — The estimate in Proposition 2.8 is best possible in the sense that for any $0 < \varepsilon < 1$ there is a constant $\beta_\varepsilon > 0$ such that for any $n$ large enough (i.e. $n \geq n_0(\varepsilon)$), for any $N \geq 1$ there is a group $G$ and a finite dimensional representation $\pi$ on $G$ satisfying (2.6) and admitting a decomposition $\pi = \oplus_{t \in T} \pi_t$, with distinct irreducibles $\pi_t$ each with multiplicity 1 (or any specified value $\geq 1$) and acting on an $N$-dimensional space, with

$$|T| \geq \exp \beta_\varepsilon nN^2.$$

**Proof.** — Fix $N > 1$. Let $T \subset U(N)^n$ be the subset appearing in Theorem 2.10, i.e. $T$ is such that $|T| \geq \exp \beta_\varepsilon nN^2$ and $\forall t \neq r \in T$ we have

$$\left\| \sum_{j} t_j \otimes \overline{r}_j \right\| \leq n(1 - \varepsilon). \quad (2.9)$$
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and also

$$\| (\sum t_j \otimes \bar{t}_j)_{[I]} \| \leq n(1 - \varepsilon).$$

(2.10)

Let $s_j = \oplus_{t \in T} t_j \in U(m)$ with $m = |T|N$, and let $G \subset U(m)$ be the subgroup generated by $S = \{s_1, \cdots, s_n\}$. Note that $\pi(G) \subset \oplus_{t \in T} M_N$. Let $\pi : G \to U(m)$ be the inclusion map viewed as a representation on $G$. Let $P_t : \oplus_{t \in T} M_N \to M_N$ be the $*$-homomorphism corresponding to the projection onto the coordinate of index $t$. For any $t \in T$, let $\pi_t : G \to U(N)$ be the representation defined by $\pi_t = P_t(\pi)$. Then, by definition, we have $\pi = \oplus_{t \in T} \pi_t$. By the spectral gap condition (2.10) the commutant of $\pi_t(S)$ (which is but the commutant of $\{t_1, \cdots, t_n\}$) is reduced to the scalars, so $\pi_t$ is irreducible, and by (2.9) for any $t \neq r \in T$ the representations $\pi_t$ and $\pi_r$ are not unitarily equivalent. □

**Remark 2.12.** — In particular, this means that $\forall n \geq n_0(\varepsilon)$ and $\forall N$

$$R_{n, \varepsilon}'(N) \geq \exp \beta \varepsilon nN^2.$$

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Bibliography


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