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Stochastic calculus with respect to fractional Brownian motion[*]

David Nualart [1]

Abstract. — Fractional Brownian motion (fBm) is a centered self-similar Gaussian process with stationary increments, which depends on a parameter $H \in (0, 1)$ called the Hurst index. In this conference we will survey some recent advances in the stochastic calculus with respect to fBm. In the particular case $H = 1/2$, the process is an ordinary Brownian motion, but otherwise it is not a semimartingale and Itô calculus cannot be used. Different approaches have been introduced to construct stochastic integrals with respect to fBm: pathwise techniques, Malliavin calculus, approximation by Riemann sums. We will describe these methods and present the corresponding change of variable formulas. Some applications will be discussed.

Résumé. — Le mouvement brownien fractionnaire (MBF) est un processus gaussien centré auto-similaire à accroissements stationnaires qui dépend d'un paramètre $H \in (0, 1)$, appelé paramètre de Hurst. Dans cette conférence, nous donnerons une synthèse des résultats nouveaux en calcul stochastique par rapport à un MBF. Dans le cas particulier $H = 1/2$, ce processus est le mouvement brownien classique, sinon, ce n'est pas une semi-martingale et on ne peut pas utiliser le calcul d'Itô. Différentes approches ont été utilisées pour construire des intégrales stochastiques par rapport à un MBF : techniques trajectorielles, calcul de Malliavin, approximation par des sommes de Riemann. Nous décrivons ces méthodes et présentons les formules de changement de variables associées. Plusieurs applications seront présentées.

1. Fractional Brownian motion

Fractional Brownian motion is a centered Gaussian process $B = \{B_t, t \geqslant 0\}$ with the covariance function

$$R_H(t, s) = E(B_t B_s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

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The parameter $H \in (0,1)$ is called the Hurst parameter. This process was introduced by Kolmogorov [21] and studied by Mandelbrot and Van Ness in [24], where a stochastic integral representation in terms of a standard Brownian motion was established.

Fractional Brownian motion has the following *self-similar* property: For any constant $a > 0$, the processes $\left\{a^{-H}B_{at}, t \geqslant 0\right\}$ and $\{B_t, t \geqslant 0\}$ have the same distribution.

From (1.1) we can deduce the following expression for the variance of the increment of the process in an interval $[s,t]$:

$$E\left(|B_t - B_s|^2\right) = |t-s|^{2H}. \tag{1.2}$$

This implies that fBm has *stationary increments*. Furthermore, by Kolmogorov's continuity criterion, we deduce that fBm has a version with α-Hölder continuous trajectories, for any $\alpha < H$.

For $H = \frac{1}{2}$, the covariance can be written as $R_{\frac{1}{2}}(t,s) = t \wedge s$, and the process B is an ordinary Brownian motion. In this case the increments of the process in disjoint intervals are independent. However, for $H \neq \frac{1}{2}$, the increments are not independent, and, furthermore, the fBm is not a semimartingale. Let $r(n) := E\left[(B_{t+1} - B_t)(B_{n+1} - B_n)\right]$. Then, $r(n)$ behaves as Cn^{2H-2}, as n tends to infinity (long-memory process). In particular, if $H > \frac{1}{2}$, then $\sum_n |r(n)| = \infty$ (*long-range dependence*) and if $H < \frac{1}{2}$, then, $\sum_n |r(n)| < \infty$ (*short-range dependence*).

The self-similarity and long memory properties make the fractional Brownian motion a suitable input noise in a variety of models. Recently, fBm has been applied in connection with financial time series, hydrology and telecommunications. In order to develop these applications there is a need for a stochastic calculus with respect to the fBm. Nevertheless, fBm is neither a semimartingale nor a Markov process, and new tools are required in order to handle the differentials of fBm and to formulate and solve stochastic differential equations driven by a fBm.

There are essentially two different methods to define stochastic integrals with respect to the fractional Brownian motion:

(i) A path-wise approach that uses the Hölder continuity properties of the sample paths, developed from the works by Ciesielski, Kerkyacharian and Roynette [7] and Zähle [37].

(ii) The stochastic calculus of variations (Malliavin calculus) for the fBm introduced by Decreusefond and Üstünel in [13].

The stochastic calculus with respect to the fBm permits to formulate and solve stochastic differential equations driven by a fBm. The stochastic integral defined using the Malliavin calculus leads to anticipative stochastic differential equations, which are difficult to solve except in some simple cases. In the one-dimensional case, the existence and uniqueness of a solution can be recovered by using the change-of-variable formula and the Doss-Sussmanm method (see [26]). In the multidimensional case, when $H > \frac{1}{2}$, the existence and uniqueness of a solution have been established in several papers (see Lyons [22] and Nualart and Rascanu [28]). For $H \in (\frac{1}{4}, \frac{1}{2})$, Coutin and Qian have obtained in [12] the existence of strong solutions and a Wong-Zakai type approximation limit for multi-dimensional stochastic differential equations driven by a fBm, using the approach of rough path analysis developed by Lyons and Qian in [23]. The large deviations for these equations have been studied by Millet and Sanz-Solé in [25].

The purpose of this talk is to introduce some of the recent advances in the stochastic calculus with respect to the fBm and discuss several applications.

2. Stochastic integration with respect to fractional Brownian motion

We first construct the stochastic integral of deterministic functions.

2.1. Wiener integral with respect to fBm

Fix a time interval $[0,T]$. Consider a fBm $\{B_t, t \in [0,T]\}$ with Hurst parameter $H \in (0,1)$. We denote by $\mathcal{E}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s). \tag{2.1}$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with B. We will denote this isometry by $\varphi \longrightarrow B(\varphi)$, and we would like to interpret $B(\varphi)$ as the Wiener integral of $\varphi \in \mathcal{H}$ with respect to B and to write $B(\varphi) = \int_0^T \varphi dB$. However, we do not know whether the elements of $\mathcal{H}$ can be considered as real-valued functions. This turns out to be true for $H < \frac{1}{2}$, but is false when $H > \frac{1}{2}$ (see Pipiras and Taqqu [30], [31]).

The fBm has the following integral representation:

$$B_t = \int_0^t K_H(t,s) dW_s,$$

where $W = \{W_t, t \geqslant 0\}$ is an ordinary Wiener process and $K_H(t,s)$ is the Volterra kernel given by

$$K_H(t,s) = c_H \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right], \tag{2.2}$$

if $s < t$ and $K_H(t,s) = 0$ if $s \geqslant t$. Here c_H is the normalizing constant

$$c_H = \left[\frac{(2H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)} \right]^{1/2}.$$

The operator $K_H^* : \mathcal{E} \to L^2([0,T])$ defined by

$$\left(K_H^* \mathbf{1}_{[0,t]}\right)(s) = K_H(t,s). \tag{2.3}$$

is a linear isometry that can be extended to the Hilbert space $\mathcal{H}$. In fact, for any $s,t \in [0,T]$ we have, using (2.3) and (2.1),

$$\begin{aligned} \left\langle K_H^* \mathbf{1}_{[0,t]}, K_H^* \mathbf{1}_{[0,s]} \right\rangle_{L^2([0,T])} &= \langle K_H(t,\ \cdot), K_H(s,\ \cdot) \rangle_{L^2([0,T])} \\ &= \int_0^{t \wedge s} K_H(t,u) K_H(s,u) du \\ &= R_H(t,s) = \left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}}. \end{aligned}$$

This operator plays a basic role in the construction of a stochastic calculus with respect to B.

If $H > \frac{1}{2}$, the operator K_H^* can be expressed in terms of *fractional integrals*:

$$\left(K_H^* \varphi\right)(s) = c_H \Gamma(H-\frac{1}{2}) s^{\frac{1}{2}-H} (I_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \varphi(u))(s), \tag{2.4}$$

and $\mathcal{H}$ is the space of distributions f such that $s^{\frac{1}{2}-H} I_{0+}^{H-\frac{1}{2}} (f(u) u^{H-\frac{1}{2}})(s)$ is a square integrable function. In this case, the scalar product in $\mathcal{H}$ has the simpler expression

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr,$$

where $\alpha_H = H(2H-1)$, and $\mathcal{H}$ contains the Banach space $|\mathcal{H}|$ of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi_r| \, |\varphi_u| \, |r-u|^{2H-2} \, dr du < \infty. \tag{2.5}$$

We have the following continuous embeddings (see [31]):

$$L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}.$$

For $H < \frac{1}{2}$, the operator K_H^* can be expressed in terms of *fractional derivatives*:

$$(K_H^* \varphi)(s) = d_H s^{\frac{1}{2}-H} (D_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u))(s), \tag{2.6}$$

where $d_H = c_H \Gamma(H + \frac{1}{2})$. In this case, $\mathcal{H} = I_{T-}^{\frac{1}{2}-H}(L^2)$ (see [13]) and

$$C^\gamma([0,T]) \subset \mathcal{H}$$

if $\gamma > \frac{1}{2} - H$.

As a consequence, we deduce the following *transfer* rule:

$$B(\varphi) = W(K_H^* \varphi), \tag{2.7}$$

for any $\varphi \in \mathcal{H}$.

2.2. Stochastic integrals of random processes

Suppose now that $u = \{u_t, t \in [0,T]\}$ is a random process. By the transfer rule (2.7) we can write

$$\int_0^T u_t dB_t = \int_0^T (K_H^* u)_t \, dW_t. \tag{2.8}$$

However, even if the process u is adapted to the filtration generated by the fBm (which coincides with the filtration generated by W), the process $K_H^* u$ is no longer adapted because the operator K_H^* does not preserves the adaptability. Therefore, in order to define stochastic integrals of random processes with respect to the fBm we need *anticipating integrals.*

In the case of an ordinary Brownian motion, the divergence operator coincides with an extension of Itô's stochastic integral to anticipating processes introduced by Skorohod in [34]. Thus, we could use the Skorohod integral in formula (2.8), and in that case, the integral $\int_0^T u_t dB_t$ coincides with the divergence operator in the Malliavin calculus with respect to the fBm B. The approach of Malliavin calculus to define stochastic integrals with respect to the fBm has been introduced by Decreusefont and Üstünel in [13], and further developed by several authors (Carmona and Coutin [6], Alòs, Mazet and Nualart [3], Alòs and Nualart [4], Alòs, León and Nualart [1], and Hu [18]).

2.2.1. Stochastic calculus of variations with respect to fBm

Let $B = \{B_t, t \in [0,T]\}$ be a fBm with Hurst parameter $H \in (0,1)$. Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \dots, B(\phi_n)), \tag{2.9}$$

where $n \geqslant 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$.

The *derivative operator* D of a smooth and cylindrical random variable F of the form (2.9) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\phi_1), \dots, B(\phi_n))\phi_i.$$

The derivative operator D is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. We denote by $\mathbb{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} = \sqrt{E\left(F^2\right) + E\left(\|DF\|_{\mathcal{H}}^2\right)}.$$

The *divergence operator* δ is the adjoint of the derivative operator. That is, we say that a random variable u in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator, denoted by Dom δ, if

$$|E\left(\langle DF, u\rangle_{\mathcal{H}}\right)| \leqslant c_u \|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$. In this case $\delta(u)$ is defined by the duality relationship

$$E(F\delta(u)) = E\left(\langle DF, u\rangle_{\mathcal{H}}\right), \tag{2.10}$$

for any $F \in \mathbb{D}^{1,2}$.

We have $\mathbb{D}^{1,2}(\mathcal{H}) \subset$Dom δ and for any $u \in \mathbb{D}^{1,2}(\mathcal{H})$

$$E\left(\delta(u)^2\right) = E\left(\|u\|_{\mathcal{H}}^2\right) + E\left(\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}}\right), \tag{2.11}$$

where $(Du)^*$ is the adjoint of (Du) in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$.

2.2.2. The divergence and symmetric integrals in the case $H > \frac{1}{2}$

The following result (see [4]) provides a relationship between the divergence operator and the *symmetric stochastic integral* introduced by Russo and Vallois in [33].

PROPOSITION 2.1. — *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in the space $\mathbb{D}^{1,2}(\mathcal{H})$. Suppose that*

$$E\left(\|u\|_{|\mathcal{H}|}^2 + \|Du\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2\right) < \infty$$

and

$$\int_0^T \int_0^T |D_s u_t| \, |t-s|^{2H-2} \, dsdt < \infty, a.s. \tag{2.12}$$

Then the symmetric integral $\int_0^T u_t dB_t$, defined as the limit in probability as ε tends to zero of

$$(2\varepsilon)^{-1} \int_0^T u_s (B_{(s+\varepsilon)\wedge T} - B_{(s-\varepsilon)\vee 0}) ds,$$

exists and we have

$$\int_0^T u_t dB_t = \delta(u) + \alpha_H \int_0^T \int_0^T D_s u_t \, |t-s|^{2H-2} \, dsdt. \tag{2.13}$$

Remark. — The symmetric integral can be replaced by the forward or backward integrals in the above proposition.

Suppose that $u = \{u_t, t \in [0,T]\}$ is a stochastic process satisfying the conditions of Proposition 2.1. Then, we can define the indefinite integral $\int_0^t u_s dB_s = \int_0^T u_s \mathbf{1}_{[0,t]}(s) dB_s$ and the following decomposition holds

$$\int_0^t u_s dB_s = \delta(u\mathbf{1}_{[0,t]}) + \alpha_H \int_0^t \int_0^T D_r u_s \, |s-r|^{2H-2} \, drds.$$

The second summand in this expression is a process with absolutely continuous paths. The first summand can be estimated using Meyer's inequalities for the divergence operator. For any $p > 1$, we denote by $\mathbb{L}_H^{1,p}$ is the set of processes $u \in \mathbb{D}^{1,p}(\mathcal{H})$ such that

$$\|u\|_{\mathbb{L}_H^{1,p}}^p := E\left(\|u\|_{L^{1/H}([0,T])}^p + \|Du\|_{L^{1/H}\left([0,T]^2\right)}^p\right) < \infty. \tag{2.14}$$

If $u \in \mathbb{L}_H^{1,p}$ with $pH > 1$ and

$$\|u\|_{1,p}^p := \int_0^T |E(u_s)|^p \, ds + \int_0^T E\left(\int_0^T |D_s u_r|^{1/H} \, ds\right)^{pH} dr < \infty$$

then the indefinite divergence integral $X_t = \int_0^t u_s \delta B_s$ has a version with γ-Hölder continuous trajectories and for all $\gamma < H - \frac{1}{p}$ and the following maximal inequality holds

$$E\left(\sup_{t\in[0,T]}\left|\int_0^t u_s \delta B_s\right|^p\right) \leqslant C\,\|u\|_{1,p}\,.$$

2.2.3. Itô's formula for the divergence integral

If F is a function of class C^2, and $H > \frac{1}{2}$, the path-wise Riemann-Stieltjes integral $\int_0^t F'(B_s)dB_s$ exists for each $t \in [0,T]$ by the theory of Young [36]. Moreover the following change of variables formula holds:

$$F(B_t) = F(0) + \int_0^t F'(B_s)dB_s. \tag{2.15}$$

Suppose that F is a function of class $C^2(\mathbb{R})$ such that

$$\max\{|F(x)|, |F'(x)|, |F''(x)|\} \leqslant ce^{\lambda x^2}, \tag{2.16}$$

where c and λ are positive constants such that $\lambda < \frac{1}{4T^{2H}}$. Then, the process $F'(B_t)$ satisfies the conditions of Proposition 2.1. As a consequence, we obtain

$$\begin{aligned}\int_0^t F'(B_s)dB_s &= \int_0^t F'(B_s)\delta B_s + \alpha_H \int_0^t \int_0^s F''(B_s)(s-r)^{2H-2}drds \\ &= \int_0^t F'(B_s)\delta B_s + H\int_0^t F''(B_s)s^{2H-1}ds. \end{aligned}\tag{2.17}$$

Therefore, putting together (2.15) and (2.17) we deduce the following Itô's formula for the divergence process

$$F(B_t) = F(0) + \int_0^t F'(B_s)\delta B_s + H\int_0^t F''(B_s)s^{2H-1}ds. \tag{2.18}$$

The following general version of Itô's formula has been proved in [4]:

Theorem 2.2. — *Let F be a function of class $C^2(\mathbb{R})$. Assume that $u = \{u_t, t \in [0,T]\}$ is a process locally in the space $\mathbb{D}^{2,2}(|\mathcal{H}|)$ such that the indefinite integral $X_t = \int_0^t u_s \delta B_s$ is a.s. continuous. Assume that $\|u\|_2$*

belongs to $\mathcal{H}$. Then for each $t \in [0,T]$ the following formula holds

$$\begin{aligned} F(X_t) &= F(0) + \int_0^t F'(X_s) u_s \delta B_s \\ &+ \alpha_H \int_0^t F''(X_s)\ u_s \left(\int_0^T |s-\sigma|^{2H-2} \left(\int_0^s D_\sigma u_\theta \delta B_\theta \right) d\sigma \right) ds \\ &+ \alpha_H \int_0^t F''(X_s) u_s \left(\int_0^s u_\theta \left(s-\theta\right)^{2H-2} d\theta \right) ds. \end{aligned} \tag{2.19}$$

Remark. — Taking the limit as H converges to $\frac{1}{2}$ in Equation (2.19) we recover the usual Itô's formula for the the Skorohod integral proved by Nualart and Pardoux [27].

The following result on the p-variation of the divergence integral has been obtained by in [17]. Fix $T > 0$ and set $t_i^n := \frac{iT}{n}$, where n is a positive integer and $i = 0, 1, \ldots, n$. Given a stochastic process $X = \{X_t, t \in [0,T]\}$ and $p \geqslant 1$, we set

$$V_n^p(X) := \sum_{i=0}^{n-1} \left| X_{t_{i+1}^n} - X_{t_i^n} \right|^p.$$

THEOREM 2.3. — *Let $\frac{1}{2} < H < 1$ and $u \in \mathbb{L}_H^{1,1/H}$. Set $X_t := \int_0^t u_s \delta B_s$, for each $t \in [0,T]$. Then*

$$V_n^{1/H}(X) n \to \infty \underset{L^1(\Omega)}{\longrightarrow} C_H \int_0^T |u_s|^{1/H} ds, \tag{2.20}$$

where $C_H := E\left(|B_1|^{1/H} \right)$.

2.2.4. Stochastic integration with respect to fBm in the case $H < \frac{1}{2}$

The extension of the previous results to the case $H < \frac{1}{2}$ is not trivial and new difficulties appear. For instance, the forward integral $\int_0^T B_t dB_t$ in the sense of Russo and Vallois does not exists, and one is forced to use symmetric integrals. A counterpart of Proposition 2.1 in the case $H < \frac{1}{2}$ and Itô's formulae 2.18 and 2.19 have been proved in [1] for $\frac{1}{4} < H < \frac{1}{2}$. The reason for the restriction $\frac{1}{4} < H$ is the following. In order to define the divergence integral $\int_0^T F'(B_s)\delta B_s$, we need the process $F'(B_s)$ to belong to $L^2(\Omega; \mathcal{H})$. This is clearly true, provided F satisfies the growth condition

(2.16), because $F'(B_s)$ is Hölder continuous of order $H-\varepsilon > \frac{1}{2} - H$ if $\varepsilon < 2H - \frac{1}{2}$. If $H \leqslant \frac{1}{4}$, one can show (see [9]) that

$$P(B \in \mathcal{H}) = 0,$$

and the space $\mathbb{D}^{1,2}(\mathcal{H})$ is too small to contain processes of the form $F'(B_t)$.

In [9] a new approach is introduced in order to extend the domain of the divergence operator to processes whose trajectories are not necessarily in the space $\mathcal{H}$. The basic tool for this extension of the divergence operator is the adjoint of the operator K_H^* in $L^2(0,T)$ given by

$$\left(K_H^{*,a}\varphi\right)(s) = d_H s^{H-\frac{1}{2}} D_{0+}^{\frac{1}{2}-H}\left(u^{\frac{1}{2}-H}\varphi(u)\right)(s).$$

Set $\mathcal{H}_2 = \left(K_H^*\right)^{-1}\left(K_H^{*,a}\right)^{-1}\left(L^2(0,T)\right)$ and denote by $\mathcal{S}_{\mathcal{H}}$ the space of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{2.21}$$

where $n \geqslant 1$, $f \in C_b^\infty(\mathbb{R}^n)$, and $\phi_i \in \mathcal{H}_2$.

Definition 2.4. — *Let $u = \{u_t, t \in [0,T]\}$ be a measurable process such that*

$$E\left(\int_0^T u_t^2 dt\right) < \infty.$$

We say that $u \in \mathrm{Dom}^*\delta$ if there exists a random variable $\delta(u) \in L^2(\Omega)$ such that for all $F \in \mathcal{S}_{\mathcal{H}}$ we have

$$\int_{\mathbb{R}} E(u_t K_H^{*,a} K_H^* D_t F)dt = E(\delta(u)F).$$

This extended domain of the divergence operator satisfies the following elementary properties:

1. $\mathrm{Dom}\delta \subset \mathrm{Dom}^*\delta$, and δ restricted to $\mathrm{Dom}\delta$ coincides with the divergence operator.

2. If $u \in \mathrm{Dom}^*\delta$ then $E(u)$ belongs to $\mathcal{H}$.

3. If u is a deterministic process, then $u \in \mathrm{Dom}^*\delta$ if and only if $u \in \mathcal{H}$.

This extended domain of the divergence operator leads to the following version of Itô's formula for the divergence process, established by Cheridito and Nualart in [9].

THEOREM 2.5. — *Suppose that F is a function of class $C^2(\mathbb{R})$ satisfying the growth condition (2.16). Then for all $t \in [0,T]$, the process $\{F'(B_s)\mathbf{1}_{[0,t]}(s)\}$ belongs to* $\mathrm{Dom}^*\delta$ *and we have*

$$F(B_t) = F(0) + \int_0^t F'(B_s)\delta B_s + H\int_0^t F''(B_s)s^{2H-1}ds. \tag{2.22}$$

2.2.5. Local time and Tanaka's formula for fBm

Berman proved in [5] that that fractional Brownian motion $B = \{B_t, t \geqslant 0\}$ has a local time l_t^a continuous in $(a,t) \in \mathbb{R} \times [0,\infty)$ which satisfies the occupation formula

$$\int_0^t g(B_s)ds = \int_{\mathbb{R}} g(a)l_t^a da. \tag{2.23}$$

for every continuous and bounded function g on $\mathbb{R}$. Set

$$L_t^a = 2H\int_0^t s^{2H-1}l^a(ds).$$

Then $a \to L_t^a$ is the density of the occupation measure

$$\mu(C) = 2H\int_0^t \mathbf{1}_C(B_s)s^{2H-1}ds,$$

where C is a Borel subset of $\mathbb{R}$. As an extension of the Itô 's formula (2.22), the following result has been proved in [9]:

THEOREM 2.6. — *Let $0 < t < \infty$ and $a \in \mathbb{R}$. Then*

$$\mathbf{1}_{\{B_s>a\}}\mathbf{1}_{[0,t]}(s) \in \mathrm{Dom}^*\delta\,,$$

and

$$(B_t - a)^+ = (-a)^+ + \int_0^t \mathbf{1}_{\{B_s>a\}}\delta B_s + \frac{1}{2}L_t^a\,. \tag{2.24}$$

This result can be considered as a version of Tanaka's formula for the fBm. In [11] it is proved that for $H > \frac{1}{3}$, the process $\mathbf{1}_{\{B_s>a\}}\mathbf{1}_{[0,t]}(s)$ belongs to $\mathrm{Dom}\delta$ and (2.24) holds.

3. Fractional Bessel processes

Let $B = \{(B_t^1, \ldots, B_t^d), t \geqslant 0\}$ be a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$. The fractional Bessel process is defined by $R_t = \sqrt{(B_t^1)^2 + \cdots + (B_t^d)^2}$. If $H > \frac{1}{2}$ and $d \geqslant 2$, as an application of the multidimensional version of the Itô formula (2.18), one obtains (see [17]):

$$R_t = \sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} \delta B_s^i + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds. \tag{3.1}$$

For $d = 1$, Tanaka's formula (2.24) says that (for any $H \in (0,1)$)

$$|B_t| = \int_0^t \operatorname{sign}(B_s) \delta B_s + L_t^0. \tag{3.2}$$

Assume $H > \frac{1}{2}$ and set

$$X_t = \begin{cases} \displaystyle\sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} \delta B_s^i & \text{if} \quad d \geqslant 2 \\ \displaystyle\int_0^t \operatorname{sign}(B_s) \delta B_s & \text{if} \quad d = 1 \end{cases}. \tag{3.3}$$

In the standard Brownian motion case, the process X_t is a one-dimensional Brownian motion, as a consequence of Lévy's characterization theorem. The process X_t is H self-similar and it has the the same $\frac{1}{H}$-finite variation as the fBm. It is then natural to conjecture that X_t is a fBm. Some partial results have been obtained so far:

It has been proved in [19] that X_t is not an $\mathcal{F}_t$ -fractional Brownian motion, where $\mathcal{F}_t$ is the filtration generated by the fBm. Moreover, it is proved in [19] that for $H > 2/3$ it does not have the long-range dependence property and, as a consequence, it is not a fBm. In [14] it is proved that for any Hurst parameter $H \in (0,1)$, $H \neq \frac{1}{2}$, it is not possible for the process X_t defined in (3.3) to be a fBm and to safisfy the equation

$$R_t^2 = 2 \int_0^t R_s \delta X_s + nt^{2H}.$$

4. Vortex filaments based on fBm

The observations of three-dimensional turbulent fluids indicate that the vorticity field of the fluid is concentrated along thin structures called vortex filaments. In his book Chorin [10] suggests probabilistic descriptions of vortex filaments by trajectories of self-avoiding walks on a lattice. Flandoli [15] introduced a model of vortex filaments based on a three-dimensional Brownian motion. A basic problem in these models is the computation of the kynetic energy of a given configuration.

Denote by $u(x)$ the velocity field of the fluid at point $x \in \mathbb{R}^3$, and let $\xi = \mathrm{curl} u$ be the associated vorticity field. The kynetic energy of the field will be

$$\mathbb{H} = \frac{1}{2} \int_{\mathbb{R}^3} |u(x)|^2 dx = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\xi(x) \cdot \xi(y)}{|x-y|} \, dxdy. \tag{4.1}$$

We will assume that the vorticity field is concentrated along a thin tube centered in a curve $\gamma = \{\gamma_t, 0 \leqslant t \leqslant T\}$. Moreover, we will choose a random model and consider this curve as the trajectory of a three-dimensional fractional Brownian motion $B = \{B_t, 0 \leqslant t \leqslant T\}$. This can be formally expressed as

$$\xi(x) = \Gamma \int_{\mathbb{R}^3} \left(\int_0^T \delta(x - y - B_s) \cdot B_s ds \right) \rho(dy), \tag{4.2}$$

where Γ is a parameter called the circuitation, and ρ is a probability measure on $\mathbb{R}^3$ with compact support.

Substituting (4.2) into (4.1) we derive the following formal expression for the kynetic energy:

$$\mathbb{H} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{H}_{xy} \rho(dx) \rho(dy), \tag{4.3}$$

where the so-called interaction energy $\mathbb{H}_{xy}$ is given by the double integral

$$\mathbb{H}_{xy} = \frac{\Gamma^2}{8\pi} \sum_{i=1}^{3} \int_0^T \int_0^T \frac{1}{|x + B_t - y - B_s|} dB_s^i dB_t^i. \tag{4.4}$$

We are interested in the following problems: Is $\mathbb{H}$ a well defined random variable? Does it have moments of all orders and even exponential moments?

In order to give a rigorous meaning to the double integral (4.4) let us introduce the regularization of the function $|\cdot|^{-1}$:

$$\sigma_n = |\cdot|^{-1} * p_{1/n}, \tag{4.5}$$

where $p_{1/n}$ is the Gaussian kernel with variance $\frac{1}{n}$. Then, the smoothed interaction energy

$$\mathbb{H}^n_{xy} = \frac{\Gamma^2}{8\pi} \sum_{i=1}^{3} \int_0^T \left(\int_0^T \sigma_n(x + B_t - y - B_s)\, dB^i_s \right) dB^i_t, \tag{4.6}$$

is well defined, where the integrals are path-wise Riemann-Stieltjes integrals. Set

$$\mathbb{H}^n = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{H}^n_{xy} \rho(dx)\rho(dy). \tag{4.7}$$

The following result has been proved in [29]:

Theorem 4.1. — *Suppose that the measure ρ satisfies*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x - y|^{1-\frac{1}{H}} \rho(dx)\rho(dy) < \infty. \tag{4.8}$$

Let $\mathbb{H}^n_{xy}$ be the smoothed interaction energy defined by (4.6). Then $\mathbb{H}^n$ defined in (4.7) converges, for all $k \geqslant 1$, in $L^k(\Omega)$ to a random variable $\mathbb{H} \geqslant 0$ that we call the energy associated with the vorticity field (4.2).

If $H = \frac{1}{2}$, fBm B is a classical three-dimensional Brownian motion. In this case condition (4.8) would be $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |x - y|^{-1} \rho(dx)\rho(dy) < \infty$, which is the assumption made by Flandoli [15] and Flandoli and Gubinelli [16]. In this last paper, using Fourier approach and Itô's stochastic calculus, the authors show that $Ee^{-\beta\mathbb{H}} < \infty$ for sufficiently small β.

The proof of Theorem 4.1 is based on the stochastic calculus of variations with respect to fBm and the application of Fourier transform.

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