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**When does the  $F$ -signature exists?<sup>(\*)</sup>**IAN M. ABERBACH<sup>(1)</sup>, FLORIAN ENESCU<sup>(2)</sup>


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**ABSTRACT.** — We show that the  $F$ -signature of an  $F$ -finite local ring  $R$  of characteristic  $p > 0$  exists when  $R$  is either the localization of an  $\mathbf{N}$ -graded ring at its irrelevant ideal or  $\mathbf{Q}$ -Gorenstein on its punctured spectrum. This extends results by Huneke, Leuschke, Yao and Singh and proves the existence of the  $F$ -signature in the cases where weak  $F$ -regularity is known to be equivalent to strong  $F$ -regularity.

**RÉSUMÉ.** — Nous prouvons dans cet article l'existence de la  $F$ -signature d'un anneau local  $F$ -fini  $R$ , de caractéristique positive  $p$ , quand  $R$  est la localisation à l'unique idéal homogène maximal d'un anneau  $\mathbf{N}$ -gradué ou quand  $R$  est  $\mathbf{Q}$ -Gorenstein sur son spectre épointé. Ceci généralise les résultats de Huneke, Leuschke, Yao et Singh et prouve l'existence de la  $F$ -signature dans les cas où faible et forte  $F$ -régularité sont équivalentes.

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**1. A sufficient condition for the existence of the  $F$ -signature**

Let  $(R, \mathfrak{m}, k)$  be a reduced, local  $F$ -finite ring of positive characteristic  $p > 0$  and Krull dimension  $d$ . Let

$$R^{1/q} = R^{a_q} \oplus M_q$$

be a direct sum decomposition of  $R^{1/q}$  such that  $M_q$  has no free direct summands. If  $R$  is complete, such a decomposition is unique up to isomorphism. Recent research has focused on the asymptotic growth rate of the numbers  $a_q$  as  $q \rightarrow \infty$ . In particular, the  $F$ -signature (defined below) is studied in [7] and [3], and more generally the Frobenius splitting ratio is studied in [2].

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For a local ring  $(R, \mathfrak{m}, k)$ , we set  $\alpha(R) = \log_p[k_R : k_R^p]$ . It is easy to see that, for an  $\mathfrak{m}$ -primary ideal  $I$  of  $R$ ,  $\lambda(R^{1/q}/IR^{1/q}) = \lambda(R/I^{[q]})/q^{\alpha(R)}$ , where  $\lambda(-)$  represents the length function over  $R$ .

We would like to first define the notion of  $F$ -signature as it appears in [3] and [7].

DEFINITION 1.1. — *The  $F$ -signature of  $R$  is  $s(R) = \lim_{q \rightarrow \infty} \frac{a_q}{q^{d+\alpha(R)}}$ , if it exists.*

The following result, due to Aberbach and Leuschke [3], holds:

THEOREM 1.2. — *Let  $(R, \mathfrak{m}, k)$  be a reduced Noetherian ring of positive characteristic  $p$ . Then  $\liminf_{q \rightarrow \infty} a_q/q^{d+\alpha(R)} > 0$  if and only if  $\limsup_{q \rightarrow \infty} a_q/q^{d+\alpha(R)} > 0$  if and only if  $R$  is strongly  $F$ -regular.*

The question of whether or not, in a strongly  $F$ -regular ring,  $s(R)$  exists, is open. We show in this paper that its existence is closely connected to the question of whether or not weak and strong  $F$ -regularity are equivalent.

Smith and Van den Bergh ([10]) have shown that the  $F$ -signature of  $R$  exists when  $R$  has finite Frobenius representation type (FFRT) type, that is, if only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules occur as direct summands of  $R^{1/q}$  for any  $q = p^e$ . Yao has proven that, under mild conditions, tight closure commutes with localization in a ring of FFRT type, [11]. Moreover, Huneke and Leuschke proved that if  $R$  is also Gorenstein, then the  $F$ -signature exists, [7]. Yao has recently extended this result to rings that are Gorenstein on their punctured spectrum, [12]. Singh has also shown that the  $F$ -signature exists for monomial rings, [9].

Let  $(R, \mathfrak{m})$  be an approximately Gorenstein ring. This means that  $R$  has a sequence of  $\mathfrak{m}$ -primary irreducible ideals  $\{I_t\}_t$  cofinal with the powers of  $\mathfrak{m}$ . By taking a subsequence, we may assume that  $I_t \supset I_{t+1}$ . For each  $t$ , let  $u_t$  be an element of  $R$  which represents a socle element modulo  $I_t$ . Then there is, for each  $t$ , a homomorphism  $R/I_t \hookrightarrow R/I_{t+1}$  such that  $u_t + I_t \mapsto u_{t+1} + I_{t+1}$ . The direct limit of the system will be the injective hull  $E = E_R(R/\mathfrak{m})$  and each  $u_t$  will map to the socle element of  $E$ , which we will denote by  $u$ . Hochster has shown that every excellent, reduced local ring is approximately Gorenstein ([5]).

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Aberbach and Leuschke have shown that, for every  $q$ , there exists  $t_0(q)$ , such that

$$a_q/(q^{d+\alpha(R)}) = \lambda(R/(I_t^{[q]} : u_t^q))/q^d,$$

for all  $t \geq t_0(q)$  (see [3], p. 55).

The situation when  $t_0(q)$  can be chosen independently of  $q$  is of special interest.

DEFINITION 1.3. — *We say that  $R$  satisfies Condition (A), if there exist a sequence of irreducible  $\mathfrak{m}$ -primary ideals  $\{I_t\}$  and a  $t_0$  such that, for all  $t \geq t_0$  and all  $q$*

$$(I_t^{[q]} : u_t^q) = (I_{t_0}^{[q]} : u_{t_0}^q).$$

PROPOSITION 1.4. — *Let  $(R, \mathfrak{m}, k)$  be a local reduced  $F$ -finite ring. If  $R$  satisfies Condition A, then the  $F$ -signature exists.*

*Proof.* — We know that  $R$  is approximately Gorenstein and hence we will use the notation fixed in the paragraph above.

As explained above, Condition A implies that there exists  $t_0$ , independent of  $q$ , such that

$$a_q/(q^{d+\alpha(R)}) = \lambda(R/(I_{t_0}^{[q]} : u_{t_0}^q))/q^d,$$

for all  $q$ .

But  $\lambda(R/(I_{t_0}^{[q]} : u_{t_0}^q)) = \lambda(R/I_{t_0}^{[q]}) - \lambda(R/(I_{t_0} + u_{t_0}R)^{[q]})$ . Dividing by  $q^d$  and taking the limit as  $q \rightarrow \infty$  yields  $s(R) = e_{HK}(I_{t_0}, R) - e_{HK}(I_{t_0} + u_{t_0}R, R)$ .  $\square$

Now we would like to concentrate on another condition, Condition (B), that appeared first in the work of Yao. First we need to introduce some notation.

Assume that  $E$  is the injective hull of the residue field  $k$ . By  $R^{(e)}$  we denote the  $R$ -bialgebra whose underlying abelian group equals  $R$  and the left and right  $R$ -multiplication is given by  $a \cdot r * b = arb^q$ , for  $a, b \in R, r \in R^{(e)}$ .

Let  $k = Ru \rightarrow E$  be the natural inclusion and consider the natural induced map  $\phi_e : R^{(e)} \otimes_R E \rightarrow R^{(e)} \otimes_R (E/k)$ . Then  $a_q/q^{\alpha(R)} = \lambda(\ker(\phi_e))$  (by Aberbach-Enescu, Corollary 2.8 in [2], see also Yao's work [12]).

One can in fact see that

$$\lambda(\ker(\phi_e)) = \lambda(R/(c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R E)) = \lambda(R/\cup_t (I_t^{[q]} : u_t^q)).$$

DEFINITION 1.5. — *We say that  $R$  satisfies Condition (B) if there exists a finite length submodule  $E' \subset E$  such that, if  $\psi_e : R^{(e)} \otimes_R E' \rightarrow R^{(e)} \otimes_R E'/k$ , then  $\lambda(\ker(\phi_e)) = \lambda(\ker(\psi_e))$ , for all  $e$ .*

Yao [12] has shown that Condition (B) implies that the  $F$ -signature of  $R$  exists.

PROPOSITION 1.6. — *Let  $(R, \mathfrak{m}, k)$  be a local reduced  $F$ -finite ring. Then Conditions (A) and (B) are equivalent.*

*Proof.* — Assume that Condition (A) holds. Then one can take  $E' = R/I_{t_0}$  and then Condition (B) follows.

If Condition (B) holds, then take  $t_0$  large enough such that  $E' \subset \text{Im}(R/I_{t_0} \rightarrow E)$ .

As noted above, one can compute the length of the kernel of  $\psi_e$  as the colength of  $\{c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R E'\}$ . Since  $R/I_{t_0}$  injects into  $E$  we see that  $\{c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R E'\}$  is a subset of  $\{c \in R : c \otimes u = 0 \text{ in } R^{(e)} \otimes_R R/I_{t_0}\} = (I_{t_0}^{[q]} : u_{t_0}^q)$ .

Since  $(I_{t_0}^{[q]} : u_{t_0}^q) \subset (I_t^{[q]} : u_t^q)$  for all  $t \geq t_0$ , we see that Condition (B) implies that  $(I_{t_0}^{[q]} : u_{t_0}^q) = (I_t^{[q]} : u_t^q)$  for all  $t \geq t_0$ , which is Condition (A).  $\square$

## 2. N-Graded rings

Let  $(R, \mathfrak{m})$  be a Noetherian  $\mathbf{N}$ -graded ring  $R = \bigoplus_{n \geq 0} R_n$ , where  $R_0 = k$  is an  $F$ -finite field of characteristic  $p > 0$ .

For any graded  $R$ -module  $M$  one can define a natural grading on  $R^{(e)} \otimes M$ : the degree of any tensor monomial  $r \otimes m$  equals  $\deg(r) + q \deg(m)$ .

In what follows we will need the following important Lemma by Lyubeznik and Smith ([8], Theorem 3.2):

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LEMMA 2.1. — *Let  $R$  be an  $\mathbf{N}$ -graded ring and  $M, N$  two graded  $R$ -modules. Then there exists an integer  $t$  depending only on  $R$  such that whenever*

$$M \rightarrow N$$

*is a degree preserving map which is bijective in degrees greater than  $s$ , then the induced map*

$$R^{(e)} \otimes M \rightarrow R^{(e)} \otimes N$$

*is bijective in degrees greater than  $p^e(s + t)$ .*

Let  $E$  be the injective hull of  $R_{\mathfrak{m}}$ . In fact,  $E$  is also the injective hull of  $R/\mathfrak{m}$  over  $R$  and as a result is naturally graded with socle in degree 0. We can write  $E = \bigoplus_{n \leq 0} E_n$ .

Let  $t$  be as in Lemma 2.1, and let  $s \leq -t - 1$ . Obviously the map  $E' = \bigoplus_{s \leq n \leq 0} E_n \rightarrow E = \bigoplus_{n \leq 0} E_n$  is bijective in degrees greater than  $s$ . So by Lemma 2.1, the map  $R^{(e)} \otimes E' \rightarrow R^{(e)} \otimes E$  is bijective in degrees greater than  $p^e(s + t) \leq -p^e$ .

THEOREM 2.2. — *Let  $R$  be an  $\mathbf{N}$ -graded reduced ring over an  $F$ -finite field  $k$  of positive characteristic. Then Condition (B) is satisfied by  $R$  and hence the  $F$ -signature of  $R$  exists.*

*Proof.* — Let  $E$  be the injective hull of  $k = R/\mathfrak{m}$  over  $R_{\mathfrak{m}}$ . As above,  $E = \bigoplus_{n \leq 0} E_n$ , where 0 is the degree of the socle generator  $u$  of  $E$ .

Using the notation introduced above, we will let  $s = -t - 1$  and  $E' = \bigoplus_{s \leq n \leq n_0} E_n \rightarrow E$ . So,  $R^{(e)} \otimes E' \rightarrow R^{(e)} \otimes E$  is bijective in degrees greater than  $-p^e$ . In particular it is bijective in degrees greater or equal to 0.

We have the following exact sequences:

$$0 \rightarrow k = Ru \rightarrow E \rightarrow E/k \rightarrow 0$$

and

$$0 \rightarrow k = Ru \rightarrow E' \rightarrow E'/k \rightarrow 0.$$

After tensoring with  $R^{(e)}$ , we get the exact sequences

$$R^{(e)} \otimes k = R^{(e)} \otimes Ru \rightarrow R^{(e)} \otimes E \xrightarrow{\phi_s} R^{(e)} \otimes E/k \rightarrow 0$$

and

$$R^{(e)} \otimes k = R^{(e)} \otimes Ru \rightarrow R^{(e)} \otimes E' \xrightarrow{\psi_s} R^{(e)} \otimes E'/k \rightarrow 0.$$

One can easily see that  $\ker(\phi_e)$  and  $\ker(\psi_e)$  are the submodules generated by  $1 \otimes u$  in  $R^{(e)} \otimes E$  and  $R^{(e)} \otimes E'$ , respectively.

The degree of  $1 \otimes u$  is  $q \cdot 0 = 0$  and we have noted that the natural map  $R^{(e)} \otimes E' \rightarrow R^{(e)} \otimes E$  is bijective in degrees greater than  $-p^e$ . This shows that  $\ker(\phi_e) \simeq \ker(\psi_e)$  and hence Condition (B) is satisfied.  $\square$

### 3. $\mathbb{Q}$ -Gorenstein rings

We turn now to showing that Condition (A) holds in strongly  $F$ -regular local rings which are  $\mathbb{Q}$ -Gorenstein on the punctured spectrum. Let  $(R, \mathfrak{m}, k)$  be such a ring of dimension  $d$ , and assume that  $R$  has a canonical module (e.g.  $R$  is complete). In this case  $R$  has an unmixed ideal of height 1, say  $J \subseteq R$ , which is a canonical ideal. We may pick an element  $a \in J$  which generates  $J$  at all minimal primes of  $J$ , and then an element  $x_2 \in \mathfrak{m}$  which is a parameter on  $R/J$  such that  $x_2 J \subseteq aR$ . It is easy to see that then  $x^n J^{(n)} \subseteq a^n R$  for all  $n \geq 1$  (where  $J^{(n)}$  is the height one component of  $J^n$ ). The condition that  $R$  is  $\mathbb{Q}$ -Gorenstein on the punctured spectrum implies that there is an integer  $h$  and two sequences of elements  $x_3, \dots, x_d \in \mathfrak{m}$  and  $a_3, \dots, a_d \in J^{(h)}$  such that  $x_i J^{(h)} \subseteq a_i R$  for  $3 \leq i \leq d$ , and  $x_2, \dots, x_d$  is a s.o.p. on  $R/J$ . We may then pick  $x_1 \in J$  such that  $x_1, \dots, x_d$  is an s.o.p. for  $R$ . See [1], section 2.2 for more detail. Then by [1], Lemma 2.2.3 we have that for any  $N \geq 0$  and any  $n \geq 0$ ,

$$(J^{(nh)}, x_2^N, \dots, \widehat{x_i^N}, \dots, x_d^N) : x_i^\infty = (J^{(nh)}, x_2^N, \dots, \widehat{x_i^N}, \dots, x_d^N) : x_i^n. \quad (3.1)$$

**THEOREM 3.1.** — *Let  $(R, \mathfrak{m}, k)$  be an  $F$ -finite strongly  $F$ -regular ring which is  $\mathbb{Q}$ -Gorenstein on the punctured spectrum. Then  $R$  satisfies Condition (A). In particular the  $F$ -signature of  $R$  exists.*

*Proof.* — If  $R$  is not complete, we observe that, since  $R$  is excellent,  $\widehat{R}$  is strongly  $F$ -regular and  $\mathbb{Q}$ -Gorenstein on the punctured spectrum. If  $\{I_t\}$  is a sequence of ideal in  $\widehat{R}$  showing condition (A) in  $\widehat{R}$ , then  $\{I_t \cap R\}$  does so for  $R$ . Thus we will assume that  $R$  is complete.

Let  $J, h$ , and  $x_1, \dots, x_d$  be as discussed above. Let  $I_t = (x_1^{t-1} J, x_2^t, \dots, x_d^t)$ . Since  $x_1^n J \cong J$  as  $R$ -modules, the quotient  $R/x_1^n J$  is Gorenstein. The hypothesis that  $x_2, \dots, x_d$  are parameters on  $R/J$  and  $R/x_1 R$  (hence on  $R/x_1^n J$ ) then shows that  $I_t$  is irreducible (see [4], Proposition 3.3.18). The sequence  $\{I_t\}$  is then a sequence of  $\mathfrak{m}$ -primary irreducible ideals cofinal with

the powers of  $\mathfrak{m}$ . If  $u_1$  represents the socle element of  $I_1$ , then we may take  $u_t = (x_1 \cdots x_d)^{t-1} u_1$  to represent the socle element of  $I_t$ . We will show that  $t_0$  may be taken to be 3.

Suppose that  $c \in I_t^{[q]} : u_t^q$  for some  $q$ . We will show that  $c \in I_3^{[q]} : u_3^q$ . Raising to the  $q'$ th power we have  $c^{q'} u_t^{qq'} = c^{q'} ((x_1 \cdots x_d)^{t-1} u_1)^{qq'} \in I_t^{[qq']}$   $= (x_1^{t-1} J, x_2^t, \dots, x_d^t)^{[qq']}$ . Hence  $c^{q'} ((x_2 \cdots x_d)^{t-1} u_1)^{qq'} \in (x_2^t, \dots, x_d^t)^{[qq']}$  :  $x_1^{(t-1)qq'} + (J, x_2^t, \dots, x_d^t)^{[qq']}$   $= (J, x_2^t, \dots, x_d^t)^{[qq']}$ .

Write  $qq' = n_{q'}h + r_{q'}$  with  $0 \leq r_{q'} < h$ . Repeated application of equation 3.1 (using 1 rather than  $h$  for  $x_2$ ) gives

$$c^{q'} ((x_2 \cdots x_d) u_1)^{qq'} \in (J^{(n_{q'}h)}, x_2^{2qq'}, \dots, x_d^{2qq'}). \quad (3.2)$$

Let  $d \in J^{(h)} \subseteq J^{(r_{q'})}$ . Multiplying by  $x_2^{qq'}$  and using that  $x_2^{qq'} J^{(qq')} \subseteq a^{qq'} R \subseteq J^{[qq']}$  we have  $dc^{q'} ((x_2 \cdots x_d)^2 u_1)^{qq'} \in (J, x_2^3, \dots, x_d^3)^{[qq']}$ . Multiplying by  $x_1^{2qq'}$  shows that  $dc^{q'} u_3^{qq'} = d(cu_3^q)^{q'} \in (I_3^{[q]})^{[q']}$ . Thus  $cu_3^q \in (I_3^{[q]})^* = I_3^{[q]}$ , as desired.  $\square$

## Bibliography

- [1] ABERBACH (I. M.). — Some conditions for the equivalence of weak and strong  $F$ -regularity, *Comm. Alg.* 30, 1635-1651 (2002).
- [2] ABERBACH (I. M.), ENESCU (F.). — The structure of  $F$ -pure rings, *Math. Zeit.*, to appear.
- [3] ABERBACH (I. M.), LEUSCHKE (G.). — The  $F$ -signature and strong  $F$ -regularity, *Math. Res. Lett.* 10, 51-56 (2003).
- [4] BRUNS (W.), HERZOG (J.). — *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge (1993).
- [5] HOCHSTER (M.). — Cyclic purity versus purity in excellent Noetherian rings, *Trans. Amer. Math. Soc.* 231, no. 2, 463-488 (1977).
- [6] HOCHSTER (M.) AND HUNEKE (C.). — Tight closure and strong  $F$ -regularity, *Mémoire no. 38, Soc. Math. France*, 119-133 (1989).
- [7] HUNEKE (C.), LEUSCHKE (G.). — Two theorems about maximal Cohen-Macaulay modules, *Math. Ann.* 324, 391-404 (2002).
- [8] LYUBEZNIK (G.), SMITH (K.E.). — Strong and weak  $F$ -regularity are equivalent for graded rings, *Amer. J. Math.* 121, 1279-1290 (1999).
- [9] SINGH (A.K.). — The  $F$ -signature of an affine semigroup ring, *J. Pure Appl. Algebra*, 196, 313-321 (2005).
- [10] SMITH (K.E.), VAN DEN BERGH (M.). — Simplicity of rings of differential operators in prime characteristic, *Proc. London. Math. Soc.* (3) 75, no. 1, 32-62 (1997).
- [11] YAO (Y.). — Modules with finite  $F$ -representation type, *Jour. London Math. Soc.*, to appear.
- [12] —, Observations on the  $F$ -signature of local rings of characteristic  $p > 0$ , preprint 2003.