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GERD DETHLOFF, TRAN VAN TAN

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Uniqueness problem for meromorphic mappings with truncated multiplicities and few targets(*)

Gerd Dethloff and Tran Van Tan (1)

Abstract. — In this paper, using techniques of value distribution theory, we give a uniqueness theorem for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with truncated multiplicities and "few" targets. We also give a theorem of linear degeneration for such maps with truncated multiplicities and moving targets.

Résumé. — Dans cet article, on donne un théorème d'unicité pour des applications méromorphes de $\mathbb{C}^m$ dans $\mathbb{C}P^n$ avec multiplicités coupées et avec « peu de » cibles. On donne aussi un théorème de dégénération linéaire pour des telles applications avec multiplicités coupées et avec des cibles mobiles. Les preuves utilisent des techniques de la distribution des valeurs.

1. Introduction

The uniqueness problem of meromorphic mappings under a condition on the inverse images of divisors was first studied by R. Nevalinna [8]. He showed that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values then $f \equiv g$. In 1975, H. Fujimoto [3] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. He showed that for two linearly nondegenerate meromorphic mappings f and g of $\mathbb{C}^m$ into $\mathbb{C}P^n$, if they have the same inverse images counted with multiplicities for $(3n+2)$ hyperplanes in general position in $\mathbb{C}P^n$, then $f \equiv g$. Since that time, this problem has been studied intensively by H.Fujimoto ([4], [5] ...), L. Smiley [11], S. Ji [6], M. Ru [10], Z. Tu [12] and others.

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(1) Gerd Dethloff, Tran Van Tan, Université de Bretagne Occidentale, UFR Sciences et Techniques, Département de Mathématiques, 6, avenue Le Gorgeu, BP 452, 29275 Brest Cedex (France).
E-mail: gerd.dethloff@univ-brest.fr

Let f be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. For each hyperplane H we denote by $v_{(f,H)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ such that $v_{(f,H)}(a)$ ($a \in \mathbb{C}^m$) is the intersection multiplicity of the image of f and H at $f(a)$.

Take q hyperplanes $H_1, ..., H_q$ in $\mathbb{C}P^n$ in general position and a positive integer l_0 .

We consider the family $F(\{H_j\}_{j=1}^q, f, l_0)$ of all linearly nondegenerate meromorphic mappings $g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ satisfying the conditions:

(a) $\min\left\{v_{(g,H_j)}, l_0\right\} = \min\left\{v_{(f,H_j)}, l_0\right\}$ for all $j \in \{1, ..., q\}$,

(b) $\dim\left(f^{-1}(H_i) \cap f^{-1}(H_j)\right) \leqslant m-2$, for all $1 \leqslant i < j \leqslant q$, and

(c) $g = f$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

In 1983, L.Smiley showed that:

Theorem A. — ([11]) *If* $q \geqslant 3n+2$ *then* $g_1{=}g_2$ *for any* $g_1, g_2 \in F(\{H_j\}_{j=1}^q, f, 1)$.

For the case $q = 3n{+}1$ in [4],[5],[6] the authors gave the following results:

Theorem B. — ([6]) *Assume that* $q = 3n+1$.*Then for three mappings* $g_1, g_2, g_3 \in F(\{H_j\}_{j=1}^q, f, 1)$, *the map* $g_1{\times}g_2{\times}g_3 : \mathbb{C}^m \longrightarrow \mathbb{C}P^n{\times}\ \mathbb{C}P^n{\times}\ \mathbb{C}P^n$ *is algebraically degenerate, namely,* $\{\ (g_1(z), g_2(z), g_3(z))\ , z \in C^m\}$ *is included in a proper algebraic subset of* $\mathbb{C}P^n{\times}\ \mathbb{C}P^n{\times}\ \mathbb{C}P^n$.

Theorem C. — ([4]) *Assume that* $q = 3n+1$.*Then there are at most two distinct mappings in* $F(\{H_j\}_{j=1}^q, f, 2)$.

Theorem D. — ([5]) *Assume that* $n = 2, q = 7$. *Then there exist some positive integer* l_0 *and a proper algebraic set* V *in the cartesian product of seven copies of the space* $\left(\mathbb{C}P^2\right)^*$ *of all hyperplanes in* $\mathbb{C}P^2$ *such that, for an arbitrary set* $(H_1, ..., H_7) \notin V$ *and two nondegenerate meromorphic mappings* f, g *of* $\mathbb{C}^m$ *into* $\mathbb{C}P^2$ *with* $\min\left\{v_{(g,H_j)}, l_0\right\} = \min\left\{v_{(f,H_j)}, l_0\right\}$ *for all* $j \in \{1, ..., 7\}$, *we have* $f = g$.

In [5], H.Fujimoto also gave some open questions:

+) Does Theorem D remain valid under the assumption that the H_j 's are in general position ?

+) What is a generalization of Theorem D for the case $n \geqslant 3$?

In connection with the above results, it is also an interesting problem to ask whether these results remain valid if the number of hyperplanes is replaced by a smaller one. In this paper, we will try to get some partial answers to this problem. We give a uniqueness theorem for the case $q \geqslant n+I\left(\sqrt{2n(n+1)}\right)+1$ and a theorem of the linear degeneration for the case of $(2n+2)$ moving targets (where we denote $I(x) := \min\{k \in \mathbb{N}_0 : k > x\}$ for a positive constant x).

Let f, a be two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with reduced representations $f = (f_0 : \ldots : f_n)$, $a = (a_0 : \ldots : a_n)$.
Set $(f, a) := a_0 f_0 + \ldots + a_n f_n$. We say that a is "small" with respect to f if $T_a(r) = o(T_f(r))$ as $r \to \infty$. Assuming that $(f, a) \not\equiv 0$, we denote by $v_{(f,a)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ with $v_{(f,a)}(z) = 0$ if $(f, a)(z) \neq 0$ and $v_{(f,a)}(z) = k$ if z is a zero point of (f, a) with multiplicity k.

Let $a_1, \ldots, a_q$ $(q \geqslant n+1)$ be meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with reduced representations $a_j = (a_{j0} : \ldots : a_{jn})$, $j = 1, \ldots, q$. We say that $\{a_j\}_{j=1}^q$ are in general position if for any $1 \leqslant j_0 < \ldots < j_n \leqslant q$, $\det(a_{j_k i}, 0 \leqslant k, i \leqslant n) \not\equiv 0$.

For each $j \in \{1, ..., q\}$, we put $\widetilde{a_j} = (\frac{a_{j0}}{a_{jt_j}} : ... : \frac{a_{jn}}{a_{jt_j}})$ and $(f, \widetilde{a_j}) = f_0 \frac{a_{j0}}{a_{jt_j}} + ... + f_n \frac{a_{jn}}{a_{jt_j}}$ where a_{jt_j} is the first element of $a_{j0}, ..., a_{jn}$ not identically equal to zero. Let $\mathcal{M}$ be the field (over $\mathbb{C}$) of all meromorphic functions on $\mathbb{C}^m$. Denote by $\mathcal{R}\left(\{a_j\}_{j=1}^q\right) \subset \mathcal{M}$ the subfield generated by the set $\{\frac{a_{ji}}{a_{jt_j}}, 0 \leqslant i \leqslant n, 1 \leqslant j \leqslant q\}$ over $\mathbb{C}$. This subfield is independant of the reduced representations $a_j = (a_{j0} : \ldots : a_{jn})$, $j = 1, \ldots, q$, and it is of course also independant of our choice of the a_{jt_j}, because it contains all quotients of the quotients $\frac{a_{ji}}{a_{jt_j}}, i = 0, \ldots, n$.

We say that f is linearly nondegenerate over $\mathcal{R}\left(\{a_j\}_{j=1}^q\right)$ if $f_0, \ldots, f_n$ are linearly independant over $\mathcal{R}\left(\{a_j\}_{j=1}^q\right)$.

Denote by Ψ the Segre embedding of $\mathbb{C}P^n \times \mathbb{C}P^n$ into $\mathbb{C}P^{n^2+2n}$ which is defined by sending the ordered pair $((w_0, ..., w_n), (v_0, ..., v_n))$ to $(..., w_i v_j, ...)$ in lexicographic order.

Let $h : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ be a meromorphic mapping. Let $(h_0 : \cdots : h_{n^2+2n})$ be a representation of $\Psi \circ h$. We say that h is linearly degenerate (with the algebraic structure in $\mathbb{C}P^n \times \mathbb{C}P^n$ given by the Segre embedding) if $h_0, ..., h_{n^2+2n}$ are linearly dependant over $\mathcal{R}\Big(\{a_j\}_{j=1}^q\Big)$.

Our main results are stated as follows: Let n, x, y, p be nonnegative integers. Assume that:

$$2 \leqslant p \leqslant n \ , \ 1 \leqslant y \leqslant 2n \ , \text{ and}$$
$$0 \leqslant x < \min\{2n - y + 1, \frac{(p-1)y}{n+1+y}\}.$$

Let k be an integer or $+\infty$ with $\frac{2n(n+1+y)(3n+p-x)}{(p-1)y-x(n+1+y)} \leqslant k \leqslant +\infty$.

Theorem 1.1. — *Let f, g be two linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q$ be $q := 3n + 1 - x$ hyperplanes in $\mathbb{C}P^n$ in general position.*

Set $E_f^j := \big\{z \in C^m : 0 \leqslant v_{(f,H_j)}(z) \leqslant k\big\}, {}^*E_f^j := \big\{z \in \mathbb{C}^m : 0 < v_{(f,H_j)}(z) \leqslant k\big\}$, and similarily for E_g^j, ${}^*E_g^j$, $j = 1, \ldots, q$.

Assume that :

(a) $\min\{v_{(f,H_j)}, 1\} = \min\{v_{(g,H_j)}, 1\}$ *on* $E_f^j \cap E_g^j$ *for all* $j \in \{n + 2 + y, \ldots, q\}$, *and*

$$\min\{v_{(f,H_j)}, p\} = \min\{v_{(g,H_j)}, p\} \text{ on } E_f^j \cap E_g^j \text{ for all } j \in \{1, \ldots, n+1+y\},$$

(b) $\dim\big({}^*E_f^i \cap {}^*E_f^j\big) \leqslant m - 2$, $\dim\big({}^*E_g^i \cap {}^*E_g^j\big) \leqslant m - 2$ *for all* $1 \leqslant i < j \leqslant q$,

(c) $f = g$ *on* $\bigcup_{j=1}^q \big({}^*E_f^j \cap {}^*E_g^j\big)$.

Then $f = g$.

We state some corollaries of Theorem 1.1:

+) Take $n \geqslant 2, y = 1, p = 2, x = 0$ and $k \geqslant n(n + 2)(6n + 4)$. Then we have:

Corollary 1.2. — *Let f, g be two linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n (n \geqslant 2)$ and $\{H_j\}_{j=1}^{3n+1}$ be hyperplanes in $\mathbb{C}P^n$ in general position.*

Assume that:

(a) $\min\{v_{(f,H_j)},1\} = \min\{v_{(g,H_j)},1\}$ *on* $E^j_f \cap E^j_g$ *for all* $j \in \{n+3,\dots,3n+1\}$*, and*

$\min\{v_{(f,H_j)},2\} = \min\{v_{(g,H_j)},2\}$ *on* $E^j_f \cap E^j_g$ *for all* $j \in \{1,\dots,n+2\}$,

(b) $dim\big({}^*E^i_f \cap {}^*E^j_f\big) \leqslant m-2$, $dim\big({}^*E^i_g \cap {}^*E^j_g\big) \leqslant m-2$ *for all* $1 \leqslant i < j \leqslant 3n+1$,

(c) $f = g$ *on* $\bigcup\limits_{j=1}^{3n+1} \big({}^*E^j_f \cap {}^*E^j_g\big)$.

Then $f = g$.

Corollary 1.2 is an improvement of Theorem C. It is also a kind of generalization of Theorem D to the case where $n \geqslant 2$ and the hyperplanes are in general position.

+) Take $n \geqslant 3, y = n+2, p = 3, x = 1$ and $k = +\infty$. Then we have:

Corollary 1.3. — *Let* f, g *be two linearly nondegenerate meromorphic mappings of* $\mathbb{C}^m$ *into* $\mathbb{C}P^n (n \geqslant 3)$ *and* $\{H_j\}_{j=1}^{3n}$ *be hyperplanes in* $\mathbb{C}P^n$ *in general position.*

Assume that:

(a) $\min\{v_{(f,H_j)},1\} = \min\{v_{(g,H_j)},1\}$ *for all* $j \in \{2n+4,\dots,3n\}$*, and*

$\min\{v_{(f,H_j)},3\} = \min\{v_{(g,H_j)},3\}$ *for all* $j \in \{1,\dots,2n+3\}$,

(b) $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leqslant m-2$ *for all* $1 \leqslant i < j \leqslant 3n$,

(c) $f = g$ on $\bigcup\limits_{j=1}^{3n} f^{-1}(H_j)$.

Then $f = g$.

+) Take $n \geqslant 2, y = I\left(\sqrt{2n(n+1)}\right), p = n, x = 2n - I\left(\sqrt{2n(n+1)}\right)$, $k = +\infty$. Then we have:

Corollary 1.4. — *Let* f, g *be two linearly nondegenerate meromorphic mappings of* $\mathbb{C}^m$ *into* $\mathbb{C}P^n$ $(n \geqslant 2)$ *and* $\{H_j\}_{j=1}^{n+I\left(\sqrt{2n(n+1)}\right)+1}$ *be hyperplanes in* $\mathbb{C}P^n$ *in general position.*

Assume that:

(a) $\min\{v_{(f,H_j)}, n\} = \min\{v_{(g,H_j)}, n\}$ *for all* $j \in \{1, \ldots, n + I\left(\sqrt{2n(n+1)}\right) + 1\}$,

(b) $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leqslant m - 2$ *for all* $1 \leqslant i < j \leqslant n + I\left(\sqrt{2n(n+1)}\right) + 1$,

(c) $f = g$ *on* $\bigcup_{j=1}^{n+I\left(\sqrt{2n(n+1)}\right)+1} f^{-1}(H_j)$.

Then $f = g$.

We finally give a result for moving targets:

THEOREM 1.5.— *Let* $f, g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ $(n \geqslant 2)$ *be two nonconstant meromorphic mappings with reduced representations* $f = (f_0 : \ldots : f_n)$ *and* $g = (g_0 : \ldots : g_n)$.

Let $\{a_j\}_{j=1}^{2n+2}$ *be "small" (with respect to* f *) meromorphic mappings of* $\mathbb{C}^m$ *into* $\mathbb{C}P^n$ *in general position with reduced representations* $a_j = (a_{j0} : \ldots : a_{jn})$, $j = 1, \ldots, 2n+2$. *Suppose that* $(f, a_j) \not\equiv 0$, $(g, a_j) \not\equiv 0$, $j = 1, \ldots, 2n+2$. Take M an integer or $+\infty$ with

$$3n(n+1)\begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix}^2 \left[\begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} - 2\right] \leqslant M \leqslant +\infty.$$

Assume that:

(a) $\min\{v_{(f,a_j)}, M\} = \min\{v_{(g,a_j)}, M\}$ *for all* $j \in \{1, \ldots, 2n+2\}$,

(b) $\dim\{z \in \mathbb{C}^m : (f, a_i)(z) = (f, a_j)(z) = 0\} \leqslant m - 2$ for all $i \neq j, i \in \{1, \ldots, n+4\}, j \in \{1, \ldots, 2n+2\}$,

(c) *There exist* $\gamma_j \in \mathcal{R}\left(\{a_j\}_{j=1}^{2n+2}\right)$ ($j = 1, \ldots, 2n+2$) *such that* $\gamma_j = \frac{a_{j0}f_0 + \ldots + a_{jn}f_n}{a_{j0}g_0 + \ldots + a_{jn}g_n}$ *on* $\left(\bigcup_{i=1}^{n+4}\{z : (f, a_i)(z) = 0\}\right) \setminus \{z : (f, a_j)(z) = 0\}$.

Then the mapping $f \times g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ *is linearly degenerate (with the algebraic structure in* $\mathbb{C}P^n \times \mathbb{C}P^n$ *given by the Segre embedding) over* $\mathcal{R}\left(\{a_j\}_{j=1}^{2n+2}\right)$.

Remark. — The condiction (c) is weaker than the following easier one:

(c') $f = g$ on $\bigcup_{i=1}^{n+4} \{z : (f, a_i)(z) = 0\}$.

We finally remark that we also obtained uniqueness theorems with moving targets (in [1]), and with fixed targets, but not taking into account, at all, truncations from some fixed order on (in [2]). But in both cases the number of targets has to be bigger than in our results above.

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2. Preliminaries

We set $\|z\| = \big(|z_1|^2 + \ldots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define

$B(r) := \{z \in \mathbb{C}^m : |z| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : |z| = r\}$ for all $0 < r \leqslant \infty$.

Define $d^c := \dfrac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$, $\upsilon := (dd^c\|z\|^2)^{m-1}$ and

$$\sigma := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1}.$$

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For every $a \in \mathbb{C}^m$, expanding F as $F = \sum P_i(z-a)$ with homogeneous polynomials P_i of degree i around a, we define

$$v_F(a) := \min\{i : P_i \not\equiv 0\}.$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the map v_φ as follows: for each $z \in \mathbb{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of z such that $\varphi = \dfrac{F}{G}$ on U and $\dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \leqslant m-2$, and then we put $v_\varphi(z) := v_F(z)$.

Set $|v_\varphi| := \overline{\big\{z \in \mathbb{C}^m : v_\varphi(z) \neq 0\big\}}$.

Let k, M be positive integers or $+\infty$.

Set

$$^{\leqslant M}v_{\varphi}^{[k]}(z)=0 \text{ if } \quad v_{\varphi}(z)>M \text{ and } {}^{\leqslant M}v_{\varphi}^{[k]}(z)=\min\{v_{\varphi}(z),k\} \text{ if } \quad v_{\varphi}(z)\leqslant M$$

$$^{>M}v_{\varphi}^{[k]}(z)=0 \text{ if } \quad v_{\varphi}(z)\leqslant M \text{ and } {}^{>M}v_{\varphi}^{[k]}(z)=\min\{v_{\varphi}(z),k\} \text{ if } v_{\varphi}(z)>M.$$

We define

$$^{\leqslant M}N_{\varphi}^{[k]}(r):=\int_1^r \frac{^{\leqslant M}n(t)}{t^{2m-1}}dt$$

and

$$^{>M}N_{\varphi}^{[k]}(r):=\int_1^r \frac{^{>M}n(t)}{t^{2m-1}}dt \qquad (1\leqslant r<+\infty)$$

where,

$$^{\leqslant M}n(t):=\int_{|v_{\varphi}|\cap B(r)} {}^{\leqslant M}v_{\varphi}^{[k]}.\upsilon \text{ for } m\geqslant 2, {}^{\leqslant M}n(t):=\sum_{|z|\leqslant t} {}^{\leqslant M}v_{\varphi}^{[k]}(z) \text{ for } m=1$$

$$^{>M}n(t):=\int_{|v_{\varphi}|\cap B(r)} {}^{>M}v_{\varphi}^{[k]}.\upsilon \text{ for } m\geqslant 2, {}^{>M}n(t):=\sum_{|z|\leqslant t} {}^{>M}v_{\varphi}^{[k]}(z) \text{ for } m=1.$$

Set $N_{\varphi}(r):={}^{\leqslant\infty}N_{\varphi}^{[\infty]}(r)$, $N_{\varphi}^{[k]}(r):={}^{\leqslant\infty}N_{\varphi}^{[k]}(r)$.

We have the following Jensen's formula (see [5], p.177, observe that his definition of $N_{\varphi}(r)$ is a different one than ours):

$$N_{\varphi}(r)-N_{\frac{1}{\varphi}}(r)=\int_{S(r)}\log|\varphi|\sigma-\int_{S(1)}\log|\varphi|\sigma, \quad 1\leqslant r\leqslant\infty.$$

Let $f:\mathbb{C}^m\longrightarrow\mathbb{C}P^n$ be a meromorphic mapping. For arbitrary fixed homogeneous coordinates $(w_0:\ldots:w_n)$ of $\mathbb{C}P^n$, we take a reduced representation $f=(f_0:\ldots:f_n)$ which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z)=(f_0(z):\ldots:f_n(z))$ outside the analytic set $\{f_0=\ldots=f_n=0\}$ of codimension $\geqslant 2$. Set $\|f\|=\big(|f_0|^2+\ldots+|f_n|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T_f(r)=\int_{S(r)}\log\|f\|\sigma-\int_{S(1)}\log\|f\|\sigma, \quad 1\leqslant r<+\infty.$$

For a meromorphic function φ on $\mathbb{C}^m$, the proximity function is defined by

$$m(r,\varphi) := \int\limits_{S(r)} \log^+ |\varphi|\sigma$$

and we have, by the classical First Main Theorem that (see [4], p.135)

$$m(r,\varphi) \leqslant T_\varphi(r) + O(1).$$

Here, the characteristic function $T_\varphi(r)$ of φ is defined as φ can be considered as a meroromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

We state the First and Second Main Theorem of Value Distribution Theory. Let a be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ such that $(f,a) \not\equiv 0$, then for reduced representations $f = (f_0 : \ldots : f_n)$ and $a = (a_0 : \ldots : a_n)$, we have:

First Main Theorem. — (Moving target version, see [12], p.569)

$$N_{(f,a)}(r) \leqslant T_f(r) + T_a(r) + O(1) \quad \textit{for} \quad r \geqslant 1.$$

For a hyperplane $H : a_0w_0 + \ldots + a_nw_n = 0$ in $\mathbb{C}P^n$ with $\operatorname{im} f \not\subseteq H$, we denote $(f,H) = a_0f_0 + \ldots + a_nf_n$, where $(f_0 : \ldots : f_n)$ again is a reduced representation of f.

Second Main Theorem. — (Classical version) *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $H_1, \ldots, H_q$ $(q \geqslant n+1)$ hyperplanes of $\mathbb{C}P^n$ in general position, then*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} N^{[n]}_{(f,H_j)}(r) + o(T_f(r))$$

for all r *except for a set of finite Lebesgue measure.*

3. Proof of Theorem 1.1

First of all, we need the following:

Lemma 3.1. — *Let f,g be two linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$* into *$\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q$ be hyperplanes in $\mathbb{C}P^n$ in general position. Then there exists a dense subset $\mathcal{C} \subset \mathbb{C}^{n+1}\{0\}$ such that for any*

$c = (c_0, ..., c_n) \in \mathcal{C}$, the hyperplane H_c defined by $c_0\omega_0 + ... + c_0\omega_n = 0$ satifies:

$$\begin{aligned} \dim\left(f^{-1}(H_j) \cap f^{-1}(H_c)\right) &\leqslant m-2 \\ \textit{and } \ dim\left(g^{-1}(H_j) \cap g^{-1}(H_c)\right) &\leqslant m-2 \\ \textit{for all } j &\in \{1, ..., q\}. \end{aligned}$$

Proof. — We refer to [6], Lemma 5.1. □

We now begin to prove Theorem 1.1.

Assume that $f \not\equiv g$.

Let j_0 be an arbitrarily fixed index, $j_0 \in \{1, ..., n+1+y\}$. Then there exists a hyperplane H in $\mathbb{C}P^n$ such that:

$$\dim\left(f^{-1}(H_j) \cap f^{-1}(H)\right) \leqslant m-2 \ \ , \ dim\left(g^{-1}(H_j) \cap g^{-1}(H)\right) \leqslant m-2$$

$$\textit{for all } j \in \{1, ..., q\} \textit{ and } \frac{(f, H_{j_0})}{(f, H)} \not\equiv \frac{(g, H_{j_0})}{(g, H)} : \tag{3.1}$$

Indeed, suppose that this assertion does not hold. Then by Lemma 3.1 we have $\frac{(f,H_{j_0})}{(f,H)} \equiv \frac{(g,H_{j_0})}{(g,H)}$ for all hyperplanes H in $\mathbb{C}P^n$. In particular, $\frac{(f,H_{j_0})}{(f,H_{j_i})} \equiv \frac{(g,H_{j_0})}{(g,H_{j_i})}, i \in \{1, ..., n\}$ where $\{j_1, ..., j_n\}$ is an arbitrary subset of $\{1, ..., q\} \setminus \{j_0\}$. After changing the homogeneous coordinates $(w_0 : ... : w_n)$ on $\mathbb{C}P^n$ we may assume that $H_{j_i} : w_i = 0, (i = 0, ..., n)$. Then $\frac{f_0}{f_i} = \frac{g_0}{g_i}$ for all $i \in \{1, ..., n\}$. This means that $f \equiv g$. This is a contradiction. Thus we get (3.1).

Since $\min\{v_{(f,H_{j_0})}, p\} = \min\{v_{(g,H_{j_0})}, p\}$ on $E_f^{j_0} \cap E_g^{j_0}$, $f = g$ on $\bigcup_{j=1}^{q} \left({}^*E_f^j \cap {}^*E_g^j\right)$ and by (3.1) we have that a zero point z_0 of (f, H_{j_0}) with multiplicity $\leqslant k$ is either a zero point of $\frac{(f,H_{j_0})}{(f,H)} - \frac{(g,H_{j_0})}{(g,H)}$ with multiplicity $\geqslant \min\{v_{(f,H_{j_0})}(z_0), p\}$ or a zero point of (g, H_{j_0}) with multiplicity $> k$ (outside an analytic set of codimension $\geqslant 2$). (3.2)

For any $j \in \{1, ..., q\}\{j_0\}$, by the asumptions (a),(c) and by (3.1), we have that a zero point of (f, H_j) with multiplicity $\leqslant k$ is either a zero point of $\frac{(f,H_{j_0})}{(f,H)} - \frac{(g,H_{j_0})}{(g,H)}$ or zero point of (g, H_j) with multiplicity $> k$ (outside an analytic set of codimension $\geqslant 2$). (3.3)

By (3.2) and (3.3), the assumption (b) and by the First Main Theorem we have

$$\begin{aligned} {}^{\leqslant k}N^{[p]}_{(f,H_{j_0})} + \sum_{j=1, j\neq j_0}^{q} {}^{\leqslant k}N^{[1]}_{(f,H_j)}(r) &\leqslant N_{\left(\frac{(f,H_{j_0})}{(f,H)} - \frac{(g,H_{j_0})}{(g,H)}\right)}(r) + {}^{>k}N^{[p]}_{(g,H_{j_0})} \\ &\quad + \sum_{j=1, j\neq j_0}^{q} {}^{>k}N^{[1]}_{(g,H_j)}(r) \end{aligned}$$

$$\begin{aligned} &\leqslant T_{\left(\frac{(f,H_{j_0})}{(f,H)} - \frac{(g,H_{j_0})}{(g,H)}\right)}(r) + \frac{p}{k+1} N_{(g,H_{j_0})}(r) + \frac{1}{k+1} \sum_{j=1, j\neq j_0}^{q} N_{(g,H_j)}(r) + O(1) \\ &\leqslant T_{\frac{(f,H_{j_0})}{(f,H)}}(r) + T_{\frac{(g,H_{j_0})}{(g,H)}}(r) + \frac{p+q-1}{k+1} T_g(r) + O(1) \qquad (3.4) \end{aligned}$$

Since $\dim(f^{-1}(H_{j_0}) \cap f^{-1}(H)) \leqslant m-2$ we have:

$$\begin{aligned} T_{\frac{(f,H_{j_0})}{(f,H)}}(r) &= \int_{S(r)} \log \left(|(f,H_{j_0})|^2 + |(f,H)|^2\right)^{\frac{1}{2}} \sigma \quad + O(1) \\ &\leqslant \int_{S(r)} \log \|f\| \sigma + O(1) = T_f(r) + O(1) \ . \end{aligned}$$

Similarly,

$$T_{\frac{(g,H_{j_0})}{(g,H)}}(r) \quad \leqslant \quad T_g(r) + O(1) \ .$$

So by (3.4) we have

$$\begin{aligned} {}^{\leqslant k}N^{[p]}_{(f,H_{j_0})}(r) + \sum_{j=1, j\neq j_0}^{q} {}^{\leqslant k}N^{[1]}_{(f,H_j)}(r) &\leqslant T_f(r) + T_g(r) \\ &\quad + \frac{p+q-1}{k+1} T_g(r) + O(1). \end{aligned}$$

Similarly,

$$\begin{aligned} {}^{\leqslant k}N^{[p]}_{(g,H_{j_0})}(r) + \sum_{j=1, j\neq j_0}^{q} {}^{\leqslant k}N^{[1]}_{(g,H_j)}(r) &\leqslant T_f(r) + T_g(r) \\ &\quad + \frac{p+q-1}{k+1} T_f(r) + O(1). \end{aligned}$$

Thus,

$${}^{\leqslant k}N^{[p]}_{(f,H_{j_0})}(r) + {}^{\leqslant k}N^{[p]}_{(g,H_{j_0})}(r) + \sum_{j=1, j\neq j_0}^{q} \left({}^{\leqslant k}N^{[1]}_{(f,H_j)}(r) + {}^{\leqslant k}N^{[1]}_{(g,H_j)}(r)\right)$$

$$\leqslant \left(2+\frac{p+q-1}{k+1}\right)(T_f(r)+T_g(r))+O(1).$$

$$\Rightarrow \frac{p}{n}\left({}^{\leqslant k}N^{[n]}_{(f,H_{j_0})}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_{j_0})}(r)\right)$$

$$+\frac{1}{n}\sum_{j=1,j\neq j_0}^{q}\left({}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_j)}(r)\right)$$

$$\leqslant \frac{2(k+1)+(p+q-1)}{k+1}(T_f(r)+T_g(r))+O(1),$$

(note that $p\leqslant n$).

$$\Rightarrow \frac{p-1}{n}\left({}^{\leqslant k}N^{[n]}_{(f,H_{j_0})}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_{j_0})}(r)\right)$$

$$\leqslant \frac{2(k+1)+(p+q-1)}{k}(T_f(r)+T_g(r))$$

$$-\frac{1}{n}\sum_{j=1}^{q}\left({}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_j)}(r)\right)+O(1) \qquad (3.5)$$

By the First and the Second Main Theorem, we have:

$$(q-n-1)T_f(r)\leqslant \sum_{j=1}^{q}N^{[n]}_{(f,H_j)}(r)+o(T_f(r))$$

$$= \frac{k}{k+1}\sum_{j=1}^{q}{}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+\sum_{j=1}^{q}\left(\frac{1}{k+1}\ {}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+{}^{>k}N^{[n]}_{(f,H_j)}(r)\right)$$

$$+o(T_f(r))$$

$$\leqslant \frac{k}{k+1}\sum_{j=1}^{q}{}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+\frac{n}{k+1}\sum_{j=1}^{q}N_{(f,H_j)}(r)+o(T_f(r))$$

$$\leqslant \frac{k}{k+1}\sum_{j=1}^{q}{}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+\frac{nq}{k+1}T_f(r)+o(T_f(r))$$

$$\Rightarrow \sum_{j=1}^{q}{}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)\geqslant \frac{(q-n-1)(k+1)-qn}{k}T_f(r)+o(T_f(r))$$

Similarly,

$$\sum_{j=1}^{q}{}^{\leqslant k}N^{[n]}_{(g,H_j)}(r)\geqslant \frac{(q-n-1)(k+1)-qn}{k}T_g(r)+o(T_g(r))$$

So,

$$\sum_{j=1}^{q}\left({}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_j)}(r)\right)\geqslant\frac{(q-n-1)(k+1)-qn}{k}\left(T_f(r)+T_g(r)\right)$$
$$+o\left(T_f(r)+T_g(r)\right)\qquad(3.6)$$

By (3.5) and (3.6) we have

$$\begin{aligned}&\frac{p-1}{n}\left({}^{\leqslant k}N^{[n]}_{(f,H_{j_0})}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_{j_0})}(r)\right)+o\left(T_f(r)+T_g(r)\right)\\ \leqslant\ &\left(\frac{2(k+1)+(p+q-1)}{k}-\frac{(q-n-1)(k+1)-qn}{nk}\right)\left(T_f(r)+T_g(r)\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow\ &\left({}^{\leqslant k}N^{[n]}_{(f,H_{j_0})}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_{j_0})}(r)\right)+o\left(T_f(r)+T_g(r)\right)\\ \leqslant\ &\frac{(3n+1-q)(k+1)+(2q+p-1)n}{k(p-1)}\left(T_f(r)+T_g(r)\right)\end{aligned}$$

for all $j_0\in\{1,...,n+1+y\}$

So,

$$\begin{aligned}&\sum_{j=1}^{n+1+y}\left({}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+{}^{\leqslant k}N^{[n]}_{(g,H_j)}(r)\right)+o\left(T_f(r)+T_g(r)\right)\\ \leqslant\ &\frac{(n+1+y)\left[(3n+1-q)(k+1)+(2q+p-1)n\right]}{k(p-1)}\left(T_f(r)+T_g(r)\right)\qquad(3.7)\end{aligned}$$

By the First and the Second Main Theorem, we have:

$$yT_f(r)\leqslant\sum_{j=1}^{n+1+y}N^{[n]}_{(f,H_j)}(r)+o(T_f(r))$$

$$\begin{aligned}=\ &\frac{k}{k+1}\sum_{j=1}^{n+1+y}{}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)\\ &+\sum_{j=1}^{n+1+y}\left(\frac{1}{k+1}\,{}^{\leqslant k}N^{[n]}_{(f,H_j)}(r)+{}^{>k}N^{[n]}_{(f,H_j)}(r)\right)+o(T_f(r))\end{aligned}$$

$$\leqslant \quad \frac{k}{k+1}\sum_{j=1}^{n+1+y} {}^{\leqslant k}N^{[n]}_{(f,H_j)}(r) + \frac{n}{k+1}\sum_{j=1}^{n+1+y} N_{(f,H_j)}(r) + o(T_f(r))$$

$$\leqslant \quad \frac{k}{k+1}\sum_{j=1}^{n+1+y} {}^{\leqslant k}N^{[n]}_{(f,H_j)}(r) + \frac{n(n+1+y)}{k+1}T_f(r) + o(T_f(r))$$

$$\Rightarrow \quad \frac{y(k+1)-n(n+1+y)}{k}T_f(r) \leqslant \sum_{j=1}^{n+1+y} {}^{\leqslant k}N^{[n]}_{(f,H_j)}(r) + o(T_f(r)).$$

Similarly,

$$\frac{y(k+1)-n(n+1+y)}{k}T_g(r) \leqslant \sum_{j=1}^{n+1+y} {}^{\leqslant k}N^{[n]}_{(g,H_j)}(r) + o(T_g(r)).$$

So,

$$\frac{y(k+1)-n(n+1+y)}{k}\left(T_f(r)+T_g(r)\right)$$

$$\leqslant \sum_{j=1}^{n+1+y}\Big({}^{\leqslant k}N^{[n]}_{(f,H_j)}(r) + {}^{\leqslant k}N^{[n]}_{(g,H_j)}(r)\Big) + o\left(T_f(r)+T_g(r)\right) \tag{3.8}$$

By (3.7) and (3.8) we have

$$\frac{y(k+1)-n(n+1+y)}{k}\left(T_f(r)+T_g(r)\right) + o\left(T_f(r)+T_g(r)\right)$$
$$\leqslant \quad \frac{(n+1+y)\left[(3n+1-q)(k+1)+(2q+p-1)n\right]}{k(p-1)}\left(T_f(r)+T_g(r)\right)$$

So,

$$(p-1)\left[y(k+1)-n(n+1+y)\right] \leqslant (n+1+y)\left[x(k+1)+(6n+p+1-2x)n\right]$$

$$\Rightarrow \quad k+1 \leqslant \frac{2n(n+1+y)(3n+p-x)}{(p-1)y-x(n+1+y)}$$

(note that $(p-1)y - x(n+1+y) > 0$). This is a contradiction. Thus, we have $f \equiv g$. □

4. Proof of Theorem 1.5

Let $\mathcal{G}$ be a torsion free abelian group and $A = (x_1, ..., x_q)$ be a $q-$tuple of elements x_i in $\mathcal{G}$. Let $1 < s < r \leqslant q$. We say that A has the property $P_{r,s}$ if any r elements $x_{p_1}, ..., x_{p_r}$ in A satisfy the condition that for any subset $I \subset \{p_1, ..., p_r\}$ with $\#I = s$, there exists a subset $J \subset \{p_1, ..., p_r\}$, $J \neq I, \#J = s$ such that $\prod_{i\in I} x_i = \prod_{j\in J} x_j$.

Lemma 4.1. — *If A has the property $P_{r,s}$, then there exists a subset $\{i_1, ..., i_{q-r+2}\} \subset \{1, ..., q\}$ such that $x_{i_1} = ... = x_{i_{q-r+2}}$.*

Proof. — We refer to [3], Lemma 2.6. □

Lemma 4.2. — *Let $f : C^m \longrightarrow CP^n$ be a nonconstant meromorphic mapping and $\{a_i\}_{i=0}^{n}$ be "small" (with respect to f) meromorphic mappings of C^m into CP^n in general position.*

Denote the meromorphic mapping,

$$F = \big(c_0 \cdot (f, \tilde{a}_0) : \cdots : c_n \cdot (f, \tilde{a}_n)\big) : C^m \longrightarrow CP^n$$

where $\{c_i\}_{i=0}^{n}$ are "small" (with respect to f) nonzero meromorphic functions on C^m.

Then,

$$T_F(r) = T_f(r) + o(T_f(r)).$$

Moreover, if

$$\begin{aligned} f &= (f_0 : \cdots : f_n), \\ a_i &= (a_{i0} : \cdots : a_{in}), \\ F &= \Big(\frac{c_0 \cdot (f, \tilde{a}_0)}{h} : \cdots : \frac{c_n \cdot (f, \tilde{a}_n)}{h}\Big) \end{aligned}$$

are reduced representations, where h is a meromorphic function on C^m, then

$$N_h(r) \leqslant o(T_f(r))$$

and

$$N_{\frac{1}{h}}(r) \leqslant o(T_f(r)).$$

Proof. — Set
$$F_i = \frac{c_i \cdot (f, \tilde{a}_i)}{h}, \quad (i = 0, \ldots, n).$$
We have
$$\begin{cases} a_{00} f_0 + \ldots + a_{0n} f_n = \frac{h}{c_0} F_0 a_{0t_0} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n0} f_0 + \ldots + a_{nn} f_n = \frac{h}{c_n} F_n a_{nt_n} \end{cases} \tag{4.1}$$
Since $(F_0 : \cdots : F_n)$ is a reduced representation of F, we have
$$N_{\frac{1}{h}}(r) \leqslant \sum_{i=0}^{n} N_{a_{it_i}}(r) + \sum_{i=0}^{n} N_{\frac{1}{c_i}}(r) = o(T_f(r)).$$
Set
$$P = \begin{pmatrix} a_{00} & \ldots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{n0} & \ldots & a_{nn} \end{pmatrix}$$
and matrices P_i $(i \in \{0, \ldots, n\})$ which are defined from P after changing the $(i+1)^{th}$ column by $\begin{pmatrix} F_0 \dfrac{a_{0t_0}}{c_0} \\ \vdots \\ F_n \dfrac{a_{nt_n}}{c_n} \end{pmatrix}$.

Put $u_i = \det(P_i)$, $u = \det(P)$, then u is a nonzero holomorphic function on C^n and
$$\begin{aligned} N_u(r) &= o(T_f(r)), \\ N_{\frac{1}{u_i}}(r) &\leqslant \sum_{j=0}^{n} N_{c_j}(r) = o(T_f(r)), \quad i = 1, \ldots, n. \end{aligned}$$

By (4.1) we have,
$$\begin{cases} f_0 = \dfrac{h \cdot u_0}{u} \\ \qquad \vdots \\ f_n = \dfrac{h \cdot u_n}{u} \end{cases} \tag{4.2}$$
On the other hand $(f_0 : \cdots : f_n)$ is a reduced representation of f.

Hence,
$$N_h(r) \leqslant N_u(r) + \sum_{i=0}^{n} N_{\frac{1}{u_i}}(r) = o(T_f(r)).$$

We have

$$
\begin{aligned}
T_F(r) &= \int\limits_{S(r)} \log\Big(\sum_{i=0}^{n} |F_i|^2\Big)^{1/2}\sigma + 0(1)\\
&= \int\limits_{S(r)} \log\Big(\sum_{i=0}^{n} \Big|\frac{c_i(f,\tilde{a}_i)}{h}\Big|^2\Big)^{1/2}\sigma + 0(1)\\
&= \int\limits_{S(r)} \log\Big(\sum_{i=0}^{n} |c_i(f,\tilde{a}_i)|^2\Big)^{1/2}\sigma - \int\limits_{S(r)} \log|h|\sigma + 0(1)\\
&\leqslant \int\limits_{S(r)} \log\|f\|\sigma + \int\limits_{S(r)} \log\Big(\sum_{i=0}^{n} |c_i|^2\cdot\|\tilde{a}_i\|^2\Big)^{1/2}\sigma\\
&\qquad -N_h(r) + N_{\frac{1}{h}}(r) + 0(1)\\
&\leqslant T_f(r) + \frac{1}{2}\int\limits_{S(r)} \log^+\Big(\sum_{i=0}^{n}\Big(\Big|c_i\frac{a_{i0}}{a_{it_i}}\Big|^2 + \cdots + \Big|c_i\frac{a_{in}}{a_{it_i}}\Big|^2\Big)\Big)\sigma\\
&\quad + o(T_f(r))\\
&\leqslant T_f(r) + \sum_{i,j=0}^{n} m\Big(r, c_i\frac{a_{ij}}{a_{it_i}}\Big) + o(T_f(r))\\
&= T_f(r) + o(T_f(r)). \qquad (4.3)
\end{aligned}
$$

(4.2) can be written as

$$
\begin{cases}
f_0 = h\cdot\sum\limits_{i=0}^{n} b_{i0}F_i\\
\cdots\quad\cdots\quad\cdots\\
f_n = h\cdot\sum\limits_{i=0}^{n} b_{in}F_i
\end{cases}
$$

where $\{b_{ij}\}_{i,j=0}^{n}$ are "small" (with respect to f) meromorphic functions on C^m.

So,

$$
\begin{aligned}
T_f(r) &= \int\limits_{S(r)} \log\|f\|\sigma + 0(1)\\
&= \int\limits_{S(r)} \log\Big(\sum_{j=0}^{n}\Big|\sum_{i=0}^{n} b_{ij}F_i\Big|^2\Big)^{1/2}\sigma + \int\limits_{S(r)} \log|h|\sigma + 0(1)
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \int\limits_{S(r)} \log\|F\|\sigma + \int\limits_{S(r)} \log\Big(\sum_{i,j}|b_{ij}|^2\Big)^{1/2}\sigma + N_h(r) - N_{\frac{1}{h}}(r) + 0(1)\\
&\leqslant T_F(r) + \int\limits_{S(r)} \log^+\Big(\sum_{i,j}|b_{ij}|^2\Big)^{1/2}\sigma + o(T_f(r))\\
&\leqslant T_F(r) + \sum_{i,j} m(r,b_{ij}) + o(T_f(r))\\
&= T_F(r) + o(T_f(r)) \qquad (4.4)
\end{aligned}$$

By (4.3) and (4.4), we get Lemma 4.2. □

We now begin to prove Theorem 1.5.

The assertion of Theorem 1.5 is trivial if f or g is linearly degenerate over $\mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$. So from now we assume that f and g are linearly nondegenerate over $\mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$.

Define functions

$$h_j := \frac{(a_{j0}f_0 + ... + a_{jn}f_n)}{(a_{j0}g_0 + ... + a_{jn}g_n)}, \quad j \in \{1, \dots, 2n+2\}.$$

For each subset $I \subset \{1, \dots, 2n+2\}$, $\#I = n+1$, set $h_I = \prod_{i\in I} h_i$, $\gamma_I = \prod_{i\in I} \gamma_i$.

Let $\mathcal{M}^*$ be the abelian multiplication group of all nonzero meromorphic functions on $\mathbb{C}^m$. Denote by $\mathcal{H} \subset \mathcal{M}^*$ the set of all $h \in \mathcal{M}^*$ with $h^k \in \mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$ for some positive integer k. It is easy to see that $\mathcal{H}$ is a subgroup of $\mathcal{M}^*$ and the multiplication group $\mathcal{G} := \mathcal{M}^*/\mathcal{H}$ is a torsion free abelian group. We denote by $[h]$ the class in $\mathcal{G}$ containing $h \in \mathcal{M}^*$.

We now prove that:

$$A := ([h_1], ..., [h_{2n+2}]) \text{ has the property } \quad P_{2n+2,n+1}\ . \qquad (4.5)$$

We have

$$\begin{cases} a_{j0}f_0 + \dots + a_{jn}f_n = h_j(a_{j0}g_0 + \dots + a_{jn}g_n) \\ j \in \{1, ..., 2n+2\} \end{cases}$$

$$\Rightarrow \begin{cases} a_{j0}f_0 + \dots + a_{jn}f_n - h_ja_{j0}g_0 - \dots - h_ja_{jn}g_n = 0 \\ 1 \leqslant j \leqslant 2n+2 \end{cases}$$

Therefore,

$$\det(a_{j0},\ldots,a_{jn},h_j a_{j0},\ldots,h_j a_{jn},\ 1\leqslant j\leqslant 2n+2)\equiv 0\ .$$

For each $I=\{i_0,\ldots,i_n\}\subset\{1,\ldots,2n+2\}$, $1\leqslant i_0<\ldots<i_n\leqslant 2n+2$, we define

$$A_I=\frac{(-1)^{\frac{n(n+1)}{2}+i_0+\ldots+i_n}\cdot\det(a_{i_r j},0\leqslant r,j\leqslant n)\cdot\det(a_{i'_s j},0\leqslant s,j\leqslant n)}{a_{j_1 t_{j_1}}\ldots a_{j_{2n+2}t_{j_{2n+2}}}}$$

where $\{i'_0,\ldots,i'_n\}=\{1,\ldots,2n+2\}\setminus\{i_0,\ldots,i_n\}$, $i'_0<\ldots<i'_n$. We have $A_I\in\mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$ and $A_I\not\equiv 0$, since $\{a_j\}_{j=1}^{2n+2}$ are in general position.

Set $L=\{I\subset\{1,\ldots,2n+2\},\#I=n+1\}$, then $\#L=\begin{pmatrix}2n+2\\ n+1\end{pmatrix}$. By the Laplace expansion Theorem, we have

$$\sum_{I\in L}A_I h_I\equiv 0. \tag{4.6}$$

We introduce an equivalence relation on L as follows: $I\backsim J$ if and only if $\frac{h_I}{h_J}\in\mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$.

Set $\{L_1,\ldots,L_s\}=L/\backsim$, ($s\leqslant\begin{pmatrix}2n+2\\ n+1\end{pmatrix}$).

For each $v\in\{1,\ldots,s\}$, choose $I_v\in L_v$ and set

$$\sum_{I\in L_v}A_I h_I=B_v h_{I_v},\quad B_v\in\mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big).$$

Then (4.6) can be written as

$$\sum_{v=1}^{s}B_v h_{I_v}\equiv 0. \tag{4.7}$$

Case 1. — There exists some $B_v\not\equiv 0$. Without loss of generality we may assume that $B_v\not\equiv 0$, for all $v\in\{1,\ldots,l\}, B_v\equiv 0$ for all $v\in\{l+1,\ldots,s\}$, $(1\leqslant l\leqslant s)$.

By (4.7) we have

$$\sum_{v=1}^{l}B_v h_{I_v}\equiv 0. \tag{4.8}$$

Denote by P the set of all positive integers $p \leqslant l$ such that there exist a subset $K_p \subseteq \{1, ..., l\}, \#K_p = p$ and nonzero constants $\{c_i\}_{i \in K_p}$ with $\sum_{i \in K_p} c_i B_i h_{I_i} \equiv 0$. It is clear that $l \in P$ by (4.8). Let t be the smallest integer in $P, (t \leqslant l \leqslant \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix})$.

We may assume that $K_t = \{1, ..., t\}$. Then there exist nonzero constants $c_v, (v = 1, ..., t)$ such that

$$\sum_{v=1}^{t} c_v B_v h_{I_v} \equiv 0. \tag{4.9}$$

Since $\frac{h_{I_i}}{h_{I_j}} \notin \mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$ and $h_{I_i} \not\equiv 0$ for all $1 \leqslant i \neq j \leqslant t$, we have $t \geqslant 3$.

Set $\varphi_1 := (B_1 h_{I_1} : ... : B_{t-1} h_{I_{t-1}})$, $\varphi_2 := (B_2 h_{I_2} : ... : B_t h_{I_t})$, $\varphi_3 := (B_1 h_{I_1} : B_3 h_{I_3} : ... : B_t h_{I_t})$. They are meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^{t-2}$.

Since $t = \min P$, we have that $\varphi_1, \varphi_2, \varphi_3$ are linearly nondegenerate (over $\mathbb{C}$).

Without loss of generality, we may assume that

$$T_{\varphi_1}(r) = \max\{T_{\varphi_1}(r), T_{\varphi_2}(r), T_{\varphi_3}(r)\} \text{ for all } r \in E,$$

where E is a subset of $[1, +\infty)$ with infinite Lesbesgue measure.

Since $t \geqslant 3$ and by the First Main Theorem, we have

$$\begin{aligned} T_{\varphi_1}(r) &\geqslant \frac{1}{3}\left(T_{\varphi_1}(r) + T_{\varphi_2}(r) + T_{\varphi_3}(r)\right) \\ &\geqslant \frac{1}{3}\left(T_{\frac{B_1 h_{I_1}}{B_2 h_{I_2}}}(r) + T_{\frac{B_2 h_{I_2}}{B_3 h_{I_3}}}(r) + T_{\frac{B_3 h_{I_3}}{B_1 h_{I_1}}}(r)\right) \\ &\geqslant \frac{1}{3}\left(T_{\frac{h_{I_1}}{h_{I_2}}}(r) + T_{\frac{h_{I_2}}{h_{I_3}}}(r) + T_{\frac{h_{I_3}}{h_{I_1}}}(r)\right) \\ &\quad -\frac{1}{3}\left(T_{\frac{B_1}{B_2}}(r) + T_{\frac{B_2}{B_3}}(r) + T_{\frac{B_3}{B_1}}(r)\right) \end{aligned}$$

$$\geqslant \frac{1}{3}\left(N_{\frac{h_{I_1}}{h_{I_2}} - \frac{\gamma_{I_1}}{\gamma_{I_2}}}(r) + N_{\frac{h_{I_2}}{h_{I_3}} - \frac{\gamma_{I_2}}{\gamma_{I_3}}}(r) + N_{\frac{h_{I_3}}{h_{I_1}} - \frac{\gamma_{I_3}}{\gamma_{I_1}}}(r)\right) + o(T_f(r)), r \in E \tag{4.10}$$

(note that $\frac{h_{I_i}}{h_{I_j}} \not\equiv \frac{\gamma_{I_i}}{\gamma_{I_j}}$ since $\frac{h_{I_i}}{h_{I_j}} \notin \mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big), 1 \leqslant i \neq j \leqslant 3)$.

Let $(h'_1 : ... : h'_{t-1})$ be a reduced representation of φ_1. Set $h'_t = \frac{B_t h_{I_t} h'_1}{B_1 h_{I_1}}$.

By (4.9) we have

$$\sum_{i=1}^{t} c_i h'_i \equiv 0. \tag{4.11}$$

It is easy to see that a zero of h'_i $(i = 1, ..., t)$ is a zero or a pole of some $B_j h_{I_j}, j \in \{1, \ldots, t\}$.

Thus,

$$\begin{aligned} N^{[1]}_{h'_i}(r) &\leqslant \sum_{j=1}^{t} \Big(N^{[1]}_{B_j h_{I_j}}(r) + N^{[1]}_{\frac{1}{B_j h_{I_j}}}(r) \Big) \\ &\leqslant \sum_{j=1}^{t} \Big(N^{[1]}_{h_{I_j}}(r) + N^{[1]}_{\frac{1}{h_{I_j}}}(r) \Big) + o\,(T_f(r))\,, i \in \{1, \ldots, t\}. \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & \sum_{i=1}^{t} N^{[t-2]}_{h'_i}(r) \leqslant (t-2) \sum_{i=1}^{t} N^{[1]}_{h'_i}(r) \\ \leqslant \quad & t(t-2) \sum_{j=1}^{t} \Big(N^{[1]}_{h_{I_j}}(r) + N^{[1]}_{\frac{1}{h_{I_j}}}(r) \Big) + o\,(T_f(r)) \end{aligned}$$

So, by the Second Main Theorem we have

$$\begin{aligned} T_{\varphi_1}(r) &\leqslant \sum_{i=1}^{t-1} N^{[t-2]}_{h'_i}(r) + N^{[t-2]}_{(c_1 h'_1 + \ldots + c_{t-1} h'_{t-1})}(r) + o\,(T_f(r)) \\ &\overset{(4.11)}{=} \sum_{i=1}^{t} N^{[t-2]}_{h'_i}(r) + o\,(T_f(r)) \end{aligned}$$

$$\leqslant t(t-2) \sum_{j=1}^{t} \Big(N^{[1]}_{h_{I_j}}(r) + N^{[1]}_{\frac{1}{h_{I_j}}}(r) \Big) + o\,(T_f(r)) \tag{4.12}$$

By (4.10) and (4.12) we have

$$N_{\frac{h_{I_1}}{h_{I_2}} - \frac{\gamma_{I_1}}{\gamma_{I_2}}}(r) + N_{\frac{h_{I_2}}{h_{I_3}} - \frac{\gamma_{I_2}}{\gamma_{I_3}}}(r) + N_{\frac{h_{I_3}}{h_{I_1}} - \frac{\gamma_{I_3}}{\gamma_{I_1}}}(r)$$

$$\leqslant 3t(t-2)\sum_{j=1}^{t}\left(N^{[1]}_{h_{I_j}}(r)+N^{[1]}_{\frac{1}{h_{I_j}}}(r)\right)+o\left(T_f(r)\right), r\in E \tag{4.13}$$

Since $\min\{v_{(f,a_i)}, M\} = \min\{v_{(g,a_i)}, M\}$ for $i \in \{1,\dots,2n+2\}$, we have

$$\{z\in\mathbb{C}^m : h_{I_j}(z)=0 \text{ or } h_{I_j}(z)=\infty\}\subset\bigcup_{i\in I_j}\{z\in\mathbb{C}^m : v_{(f,a_i)}(z)>M\ \},$$

$$j=1,...,t.$$

Thus,

$$N^{[1]}_{h_{I_j}}(r)+N^{[1]}_{\frac{1}{h_{I_j}}}(r)\leqslant\sum_{i\in I_j}{}^{>M}N^{[1]}_{(f,a_i)}(r)\leqslant\frac{1}{M+1}\sum_{i\in I_j}N_{(f,a_i)}(r)$$

$$\leqslant\frac{n+1}{M+1}T_f(r)+O(1), j\in\{1,...,t\}$$

(note that $\#I_j = n+1$).

$$\Longrightarrow\sum_{j=1}^{t}\left(N^{[1]}_{h_{I_j}}(r)+N^{[1]}_{\frac{1}{h_{I_j}}}(r)\right)\leqslant\frac{(n+1)t}{M+1}T_f(r)+O(1) \tag{4.14}$$

By (4.13) and (4.14) we have

$$N_{\frac{h_{I_1}}{h_{I_2}}-\frac{\gamma_{I_1}}{\gamma_{I_2}}}(r)+N_{\frac{h_{I_2}}{h_{I_3}}-\frac{\gamma_{I_2}}{\gamma_{I_3}}}(r)+N_{\frac{h_{I_3}}{h_{I_1}}-\frac{\gamma_{I_3}}{\gamma_{I_1}}}(r)$$

$$\leqslant\frac{3(n+1)t^2(t-2)}{M+1}T_f(r)+o(T_f(r)), r\in E \tag{4.15}$$

For each $1\leqslant s<v\leqslant 3$, set $V_{sv}=\{1,...,n+4\}\setminus\left((I_s\cup I_v)\setminus(I_s\cap I_v)\right).$

Since $\dim\{z\in\mathbb{C}^m : (f,a_i)(z)=(f,a_j)(z)=0\}\leqslant m-2$ for all $i\neq j$, $i\in\{1,\dots,n+4\}$, $j\in\{1,\dots,2n+2\}$, and $\gamma_j=h_j$ on $\left(\bigcup_{i=1}^{n+4}\{z:(f,a_i)(z)=0\}\right)\setminus\{z:(f,a_j)(z)=0\}$, we have:

$$N_{\frac{h_{I_1}}{h_{I_2}}-\frac{\gamma_{I_1}}{\gamma_{I_2}}}(r)\geqslant\sum_{i\in V_{12}}N^{[1]}_{(f,a_i)}(r).$$

Indeed, let z_0 be an arbitrary zero point of some $(f,a_i), i\in V_{12}$. By omitting an analytic set of codimension $\geqslant 2$, we may assume that $(f,a_j)(z_0)\neq 0$

for all $j \in \{1, \ldots, 2n+2\} \setminus \{i\}$. In particular, $(f, a_j)(z_0) \neq 0$ for all $j \in (I_1 \cup I_2) \setminus (I_1 \cap I_2)$. So $\gamma_j(z_0) = h_j(z_0)$ for all $j \in (I_1 \cup I_2) \setminus (I_1 \cap I_2)$. Consequently, z_0 is a zero point of $\frac{h_{I_1}}{h_{I_2}} - \frac{\gamma_{I_1}}{\gamma_{I_2}}$. Thus, the above assertion holds.

Similarly,

$$N_{\frac{h_{I_2}}{h_{I_3}} - \frac{\gamma_{I_2}}{\gamma_{I_3}}}(r) \geqslant \sum_{i \in V_{23}} N^{[1]}_{(f,a_i)}(r), \quad N_{\frac{h_{I_3}}{h_{I_1}} - \frac{\gamma_{I_3}}{\gamma_{I_1}}}(r) \geqslant \sum_{i \in V_{13}} N^{[1]}_{(f,a_i)}(r).$$

It is easy to see that: $V_{12} \cup V_{23} \cup V_{13} = \{1, ..., n+4\}$.
Thus,

$$N_{\frac{h_{I_1}}{h_{I_2}} - \frac{\gamma_{I_1}}{\gamma_{I_2}}}(r) + N_{\frac{h_{I_2}}{h_{I_3}} - \frac{\gamma_{I_2}}{\gamma_{I_3}}}(r) + N_{\frac{h_{I_3}}{h_{I_1}} - \frac{\gamma_{I_3}}{\gamma_{I_1}}}(r) \geqslant \sum_{i=1}^{n+4} N^{[1]}_{(f,a_i)}(r) \geqslant \sum_{i=1}^{n+2} N^{[1]}_{(f,a_i)}(r) \tag{4.16}$$

By (4.15) and (4.16) we have

$$\sum_{i=1}^{n+2} N^{[1]}_{(f,a_i)}(r) \leqslant \frac{3(n+1)t^2(t-2)}{(M+1)n} T_f(r) + o(T_f(r)), r \in E \tag{4.17}$$

We now prove that:

$$\frac{1}{n} T_f(r) \leqslant \sum_{i=1}^{n+2} N^{[1]}_{(f,a_i)}(r) + o(T_f(r)). \tag{4.18}$$

Set

$$N_{n+2} := \begin{pmatrix} \frac{a_{10}}{a_{1t_1}} & \cdots & \frac{a_{(n+1)0}}{a_{(n+1)t_{n+1}}} \\ \vdots & \ddots & \vdots \\ \frac{a_{1n}}{a_{1t_1}} & \cdots & \frac{a_{(n+1)n}}{a_{(n+1)t_{n+1}}} \end{pmatrix},$$

and matrices N_i $(i \in \{1, \ldots, n+1\})$ which are defined by N_{n+2} after changing the i^{th} column by $\begin{pmatrix} \frac{a_{(n+2)0}}{a_{(n+2)t_{n+2}}} \\ \vdots \\ \frac{a_{(n+2)n}}{a_{(n+2)t_{n+2}}} \end{pmatrix}$.

Put $c_i = \det(N_i)$, $(i = 1, ..., n+2)$, then $\{c_i\}_{i=1}^{n+2}$ are nonzero meromorphic functions on $\mathbb{C}^m$ and $c_i \in \mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$.

It is easy to see that:

$$\sum_{i=1}^{n+1} c_i(f, \widetilde{a_i}) = c_{n+2}(f, \widetilde{a_{n+2}}). \tag{4.19}$$

Denote by F the meromorphic mapping $(c_1(f, \widetilde{a_1}) : ... : c_{n+1}(f, \widetilde{a_{n+1}})) : \mathbb{C}^m \to \mathbb{C}P^n$.

Since f is linearly nondegenerate over $\mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$ and since $\{a_j\}_{j=1}^{2n+2}$ are in general position, we have that F is linearly nondegenerate (over $\mathbb{C}$).

By Lemma 4.2 we have

$$T_F(r) = T_f(r) + o\left(T_f(r)\right).$$

Let $\Big(\dfrac{c_1(f, \widetilde{a_1})}{h} : ... : \dfrac{c_{n+1}(f, \widetilde{a_{n+1}})}{h}\Big)$ be a reduced representation of F, where h is a meromorphic function on $\mathbb{C}^m$. By Lemma 4.2 we have

$$N_h(r) = o(T_f(r)), N_{\frac{1}{h}}(r) = o(T_f(r)).$$

By the Second Main Theorem, we have:

$$\begin{aligned} T_f(r) + o\left(T_f(r)\right) &= T_F(r) \leqslant \sum_{i=1}^{n+1} N^{[n]}_{\frac{c_i(f,\widetilde{a_i})}{h}}(r) + N^{[n]}_{\sum_{i=1}^{n+1}\frac{c_i(f,\widetilde{a_i})}{h}}(r) + o\left(T_F(r)\right) \\ &\overset{(4.19)}{=} \sum_{i=1}^{n+2} N^{[n]}_{c_i(f,\widetilde{a_i})}(r) + N_{\frac{1}{h}}(r) + o\left(T_F(r)\right) \\ &\leqslant \sum_{i=1}^{n+2} N^{[n]}_{(f,a_i)}(r) + \sum_{i=1}^{n+2} N_{\frac{1}{a_{it_i}}}(r) + \sum_{i=1}^{n+2} N_{c_i}(r) + o(T_F(r)) \\ &\leqslant n\sum_{i=1}^{n+2} N^{[1]}_{(f,a_i)}(r) + o\left(T_f(r)\right). \end{aligned}$$

We get (4.18).

By (4.17) and (4.18) we have :

$$T_f(r) \leqslant \frac{3(n+1)t^2(t-2)}{(M+1)} T_f(r) + o(T_f(r)), r \in E.$$

This contradicts to

$$M \geqslant 3n(n+1)\begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix}^2 \left[\begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} - 2\right] \geqslant 3(n+1)t^2(t-2).$$

Case 2.— $B_v \equiv 0$ for all $v \in \{1, ..., s\}$. Then $\sum_{I \in L_v} A_I h_I \equiv 0$ for all $v \in \{1, ..., s\}$. On the other hand $A_I \not\equiv 0, h_I \not\equiv 0$. Hence, $\# L_v \geqslant 2$ for all $v \in \{1, ..., s\}$.

So, for each $I \in L$, there exists $J \in L, J \neq I$ such that $\frac{h_I}{h_J} \in \mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$. This implies that $\prod_{i \in I} [h_i] = \prod_{i \in J} [h_j]$.

We get (4.5). □

By Lemma 4.1 there exist $j_1, j_2 \in \{1, ..., 2n+2\}$, $j_1 \neq j_2$ such that $[h_{j_1}] = [h_{j_2}]$.

By the definition, we have $\frac{h_{j_1}}{h_{j_2}} \in \mathcal{H}$. This means that $\left(\frac{h_{j_1}}{h_{j_2}}\right)^k \in \mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$ for some positive integer k.

So $\left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k \in \mathcal{R}\Big(\{a_j\}_{j=1}^{2n+2}\Big)$.

Take $\{i_1, ..., i_{n+2}\} \subseteq \{1, ..., n+4\} \setminus \{j_1, j_2\}$.

Similarly to (4.18), we have:

$$\frac{1}{n} T_f(r) \leqslant \sum_{s=1}^{n+2} N^{[1]}_{(f,a_{i_s})}(r) + o(T_f(r)). \tag{4.20}$$

+) If $\left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k - \left(\frac{\gamma_{j_1}}{\gamma_{j_2}}\right)^k \not\equiv 0$, then by the assumptions (b) and (c) we have

$$N_{\left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k - \left(\frac{\gamma_{j_1}}{\gamma_{j_2}}\right)^k}(r) \geqslant \sum_{s=1}^{n+2} N^{[1]}_{(f,a_{i_s})}(r) \tag{4.21}$$

Indeed, let z_0 be an arbitrary zero point of some (f, a_{i_s}), $(1 \leqslant s \leqslant n+2)$. By omitting an analytic set of codimemsion $\geqslant 2$, we may assume that $(f, a_{j_1})(z_0) \neq 0$, $(f, a_{j_2})(z_0) \neq 0$ (note that $j_1, j_2 \neq i_s$). Then $\gamma_{j_1}(z_0) = \frac{(f,a_{j_1})}{(g,a_{j_1})}(z_0)$, $\gamma_{j_2}(z_0) = \frac{(f,a_{j_2})}{(g,a_{j_2})}(z_0)$. Thus z_0 is a zero point of $\left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k - \left(\frac{\gamma_{j_1}}{\gamma_{j_2}}\right)^k$. We get (4.21).

By the First Main Theorem and by (4.20), (4.21) we have:

$$T_{\left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k}(r) + T_{\left(\frac{\gamma_{j_1}}{\gamma_{j_2}}\right)^k}(r) \geqslant N_{\left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k - \left(\frac{\gamma_{j_1}}{\gamma_{j_2}}\right)^k}(r)$$

$$\geqslant \sum_{s=1}^{n+2} N^{[1]}_{(f,a_{i_s})}(r) \geqslant \frac{1}{n} T_f(r) + o(T_f(r)).$$

This is a contradiction, since $\gamma_{j_1}, \gamma_{j_2}, \left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k \in \mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{2n+2}\Big)$.

Thus, $\left(\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})}\right)^k \equiv \left(\frac{\gamma_{j_1}}{\gamma_{j_2}}\right)^k$. So, $\frac{(f,a_{j_1})(g,a_{j_2})}{(g,a_{j_1})(f,a_{j_2})} \equiv \alpha \frac{\gamma_{j_1}}{\gamma_{j_2}}$, where α is a constant. This implies that $f \times g$ is linearly degenerate over $\mathcal{R}\Big(\big\{a_j\big\}_{j=1}^{2n+2}\Big)$.

We have completed proof of Theorem 1.5. □

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