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C^k-estimates for the $\overline{\partial}$-equation on concave domains of finite type(*)

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Abstract. — C^k estimates for convex domains of finite type in $\mathbb{C}^n$ are known from [7] for $k = 0$ and from [2] for $k > 0$. We want to show the same result for concave domains of finite type. As in the case of strictly pseudoconvex domain, we fit the method used in the convex case to the concave one by switching z and ζ in the integral kernel of the operator used in the convex case. However the kernel will not have the same behavior on the boundary as in the Diederich-Fischer-Fornæss-Alexandre work. To overcome this problem we have to alter the Diederich-Fornæss support function. Also we have to take care of the so generated residual term in the homotopy formula.

Résumé. — Les estimées C^k pour les domaines convexes de type fini ont été établies dans [7] pour $k = 0$ et dans [2] pour $k > 0$. Nous voulons ici étudier le cas des domaines concaves de type fini. Comme pour le cas strictement pseudoconvexe, nous adaptons les outils utilisés par K. Diederich, B. Fisher et J.E. Fornæss et W. Alexandre en échangeant le rôle des variables dans les noyaux intégraux de leurs opérateurs. Cependant le comportement au bord des nouveaux noyaux n'est plus le même et il faut modifier la fonction de support de K. Diederich et J.E. Fornæss. Elle perdra son holomorphie et générera un terme résiduel dans la formule d'homotopie dont il faudra tenir compte.

1. Introduction

For any convex finite type domain was constructed in [7] a $\overline{\partial}$-solving operator which satisfies the best Hölder estimates. These estimates were generalized in [2] to C^k-estimates for all $k \in \mathbb{N}$. Each article used Cauchy-Fantappiè type integral operators based on the Diederich-Fornæss support function constructed in [6]. In those two articles rose the problem of the

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bad behavior of the normal component of the kernel in the integration variable. In case of Hölder estimates this bad behavior did not matter because the main difficulty was the control of a boundary integral, so the normal component did not play any role. However in [2] was to be controlled an integral over a volume. The problem of the normal component was avoid in two times. First were shown new estimates of the derivatives of the defining function for the domain in the normal direction to the boundary and then the author integrated by parts many times.

In this article we are interested in C^k-estimates on concave finite type domain. We use definitions of C^k-norms used in [11] and show

Theorem 1.1. — *Let $D \subset \mathbb{C}^n$ be a smooth bounded convex domain of finite type m and $q = 1, \ldots, n-2$. There exist a neighborhood $\mathcal{U}$ of bD and a linear operator $T_q : C_{0,q}(\mathcal{U} \setminus D) \to C_{0,q-1}(\mathcal{U} \setminus \overline{D})$ such that for all $k \in \mathbb{N}$ and all $\overline{\partial}$-closed $f \in C^k_{0,q}(\mathcal{U} \setminus D)$ with support included in $\mathcal{U} \setminus D$, we have*

i) $\overline{\partial} T_q f = f$,

ii) $T_q f$ *belongs to* $C^{k+\frac{1}{m}}_{0,q-1}(\mathcal{U} \setminus D)$ *and there exists a constant* $c_k > 0$*, not depending on* f*, such that* $\|T_q f\|_{k+\frac{1}{m},\mathcal{U}\setminus D} \leqslant c_k \|f\|_{k,\mathcal{U}\setminus D}$.

In order to prove theorem 1.1 we try to fit the operator used in [7] for convex domains by exchanging in the integral kernel the integration variables ζ and z. After this manipulation the normal component of the kernel in z has a bad behavior and does not disappear by integrating over the boundary of D and it seems to be impossible to show Hölder estimates for such a kernel.

To overcome the problem we have to alter the Diederich-Fornæss support function. We add some terms to the support function in order to improve the behavior of the normal component of the Cauchy-Fantappiè kernel generated with it. Because the new support function will not be holomorphic, a residual term will appear in the homotopy formula. We will show that the modification is made in such a way that the residual term is extremely regular and $\overline{\partial}$-closed in a neighborhood of $\mathbb{C}^n \setminus D$. Then we solve the $\overline{\partial}$-problem for this residual term and get the operator T_q from the theorem 1.1.

2. Integral operators

We recall the definition of the support function F of [6]. Let D be a bounded convex domain in $\mathbb{C}^n$ of finite type m and r a C^∞ defining function for D. We set $D_\alpha := \{z \in \mathbb{C}^n, r(z) < \alpha\}$, $\alpha \in \mathbb{R}$, and we assume that r is convex on $\mathbb{C}^n$ and that $grad\, r$ is non zero in a bounded neighborhood $\mathcal{V}$ of bD. We fix some ζ in $\mathcal{V}$ and denote by $T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$ the complex tangent space to $bD_{r(\zeta)}$ at ζ and by η_ζ the outer unit normal at ζ to $bD_{r(\zeta)}$. Then we choose an orthonormal basis $w'_1, \dots, w'_n$ such that $w'_1 = \eta_\zeta$ and we set $r_\zeta(\omega) = r(\zeta + \omega_1 w'_1 + \ldots + \omega_n w'_n)$ and

$$F_\zeta(\omega) = \frac{\partial r_\zeta}{\partial \omega_1}(0)\omega_1 + K\left(\frac{\partial r_\zeta}{\partial \omega_1}(0)\omega_1\right)^2 - K'\sum_{j=2}^m \kappa_j M^{2^j} \sum_{\substack{|\beta|=j\\ \beta_1=0}} \frac{1}{\beta!}\frac{\partial^j r_\zeta}{\partial \omega^\beta}(0)\omega^\beta$$

where K, K', M are positive constants $\kappa_j = 1$ when $j \equiv 0\, mod\, 4$, -1 when $j \equiv 2\, mod\, 4$ and 0 otherwise. We also set

$$\begin{aligned} T_\zeta(\omega) = &-K'\sum_{j=2}^m \kappa_j M^{2^j}\frac{j+1}{j}\sum_{\substack{|\beta|=j\\ \beta_1=0}}\frac{1}{\beta!}\frac{\partial^{j+1} r_\zeta}{\partial\overline{\omega}_1\partial\omega^\beta}(0)\overline{\omega_1}\omega^\beta + 2\sum_{i=1}^n \frac{\partial^2 r_\zeta}{\partial\overline{\omega}_1\partial\omega_i}(0)\omega_i\overline{\omega}_1 \\ &-K'\sum_{j=2}^m \kappa_j M^{2^j}\frac{j+2}{2j}\sum_{\substack{|\beta|=j\\ \beta_1=0}}\frac{1}{\beta!}\frac{\partial^{j+2} r_\zeta}{\partial\overline{\omega}_1^2\partial\omega^\beta}(0)\overline{\omega_1}^2\omega^\beta + \frac{3}{2}\sum_{i=1}^n \frac{\partial^3 r_\zeta}{\partial\overline{\omega}_1^2\partial\omega_i}(0)\overline{\omega}_1^2\omega_i, \end{aligned}$$

$$F'_\zeta(\omega) = -K'\sum_{j=2}^m \kappa_j M^{2^j}\sum_{\substack{|\beta|=j\\ \beta_1\neq 0}}\frac{1}{\beta!}\frac{\partial^j r_\zeta}{\partial\omega^\beta}(0)\omega^\beta,$$

$$\tilde{F}_\zeta(\omega) = F_\zeta(\omega) + F'_\zeta(\omega) + T_\zeta(\omega).$$

We should notice that T_ζ and F'_ζ do not depend on the basis $w'_1, \dots, w'_n$ provided $w'_1 = \eta_\zeta$. For $z = \zeta + \omega_{z,1}w'_1 + \ldots + \omega_{z,n}w'_n$ we set

$$\begin{aligned} F(\zeta, z) &:= F_\zeta(\omega_{z,1}, \dots, \omega_{z,n}), \\ T(\zeta, z) &:= T_\zeta(\omega_{z,1}, \dots, \omega_{z,n}), \\ \tilde{F}(\zeta, z) &:= \tilde{F}_\zeta(\omega_{z,1}, \dots, \omega_{z,n}). \end{aligned}$$

The following theorem was shown in [7].

Theorem 2.1. — *There exist a neighborhood $\mathcal{V}$ of bD and positive constants M, K, K', k', c_+, c_- and R such that for all $\zeta \in \mathcal{V}$, all unit vector $v \in T^{\mathbb{C}}_\zeta bD_{r(\zeta)}$ and all $w = (w_1, w_2) \in \mathbb{C}^2$, with $|w| < R$, we have*

$$
\begin{aligned}
&\Re F(\zeta,\zeta+w_1\eta_\zeta+w_2v)\leqslant\\
&\leqslant\ -\left|\frac{\Re w_1}{2}\right|-\frac{K}{2}(\Im w_1)^2-\frac{K'k'}{4}\sum_{j=2}^m\sum_{\alpha+\beta=j}\left|\left.\frac{\partial^j r(\zeta+\lambda v)}{\partial\lambda^\alpha\partial\overline{\lambda}^\beta}\right|_{\lambda=0}\right||w_2|^j\\
&\qquad -c_\pm(r(\zeta)-r(\zeta+w_1\eta_\zeta+w_2v))
\end{aligned}
$$

where $c_\pm=c_+$ *when* $r(\zeta)-r(\zeta+w_1\eta_\zeta+w_2v)>0$ *and* $c_\pm=c_-$ *otherwise.*

With no restriction we assume that $\mathcal{V}=D_\rho\setminus\overline{D_{-\rho}}$, $\rho>0$ sufficiently small.

For $\varepsilon>0$ sufficiently small and $z\in\mathcal{V}$ we denote by $w_1^*,\dots,w_n^*$ an ε-extremal basis at z as defined in [3]. We denote by $\zeta^*=(\zeta_1^*,\dots,\zeta_n^*)$ the ε-extremal coordinates at z of a point ζ and Φ_* the unitary transformation such that $\zeta^*=\Phi_*(\zeta-z)$. We also define the following complex directional level distances $\tau(\zeta,v,\varepsilon):=\sup\{\tau, r(\zeta+\lambda v)-r(\zeta)<\varepsilon \text{ for all}\lambda\in\mathbb{C},|\lambda|<\tau\}$, we write $\tau_i(z,\varepsilon)=\tau(z,w_i^*,\varepsilon)$, $i=1,\dots,n$, and set
$\mathcal{P}_\varepsilon(z):=\{\zeta\in\mathbb{C}^n,|\zeta_i^*|<\tau_i(z,\varepsilon),\ i=1,\dots n\}$ the polydisc of McNeal (see [12, 3, 2, 7]). We have $\tau_i(z,\varepsilon)\gtrsim\varepsilon^{\frac12}$ for all $i\neq 1$ and $\tau_1(z,\varepsilon)\eqsim\varepsilon$ uniformly with respect to z and ε (see proposition 3.1 from [7]). As in [7, 2] we use some kind of poly-annuli defined by

$$\mathcal{P}^0_\varepsilon(z):=\mathcal{P}_\varepsilon(z)\setminus c_1\mathcal{P}_\varepsilon(z)$$

where $c_1>0$ given by the proposition 3.1 of [7] is such that $c_1\mathcal{P}_\varepsilon(z)\subset\mathcal{P}_{\frac{\varepsilon}{2}}(z)$ for all $\varepsilon>0$.

Proposition 2.2. — *For all* $\varepsilon>0$ *sufficiently small, the following inequality holds uniformly for all* $z\in\mathcal{V}$ *all* $\zeta\in\mathcal{P}^0_\varepsilon(z)$

$$|F(z,\zeta)|\gtrsim\varepsilon+c_\pm(r(z)-r(\zeta))$$

with $c_\pm=c_+$ *when* $r(z)-r(\zeta)>0$ *and* $c_\pm=c_-$ *when* $r(z)-r(\zeta)\leqslant 0$.

Proof. — We write $\zeta\in\mathcal{P}^0_\varepsilon(z)$ as $\zeta=z+\lambda\eta_z+\mu v$, v unit vector in $T_z^{\mathbb{C}}bD_{r(z)}$. We have $|\lambda|\gtrsim c_1\varepsilon$ or $|\mu|\gtrsim c_1\tau(z,v,\varepsilon)$. Indeed by proposition 3.1 of [7], we have $\frac{|\mu|}{\tau(z,v,\varepsilon)}\eqsim\sum_{i=2}^n\frac{|\zeta_i^*|}{\tau_i(z,\varepsilon)}$. Therefore when $|\mu|<c_1\tilde{c}\tau(z,v,\varepsilon)$, $\tilde{c}>0$ sufficiently small and not depending on ζ and z, we have $|\zeta_i^*|<c_1\tau_i(z,\varepsilon)$ for $i=2,\dots,n$. So we have $|\lambda|=|\zeta_1^*|\geqslant c_1\tau_1(z,\varepsilon)\gtrsim c_1\varepsilon$.

We first assume $|\mu| < c_1\tilde{c}\tau(z,v,\varepsilon)$ so that $|\lambda| \gtrsim c_1\varepsilon$. We have with the proposition 3.1 (vi) of [7] and the theorem 2.1

$$\begin{aligned}
|F(z,\zeta)| &\gtrsim |\lambda| - K|\lambda|^2 - K'\sum_{j=2}^{m} M^{2^j} \sum_{k+l=j} \frac{|\mu|^j}{k!l!}\left|\left.\frac{\partial^j r(z+\mu v)}{\partial\mu^k\partial\overline{\mu}^l}\right|_{\mu=0}\right| + \\
&\quad + c_\pm(r(z)-r(\zeta)) \\
&\gtrsim \varepsilon(1-\varepsilon-\tilde{c}) + c_\pm(r(z)-r(\zeta)) \\
&\gtrsim \varepsilon + c_\pm(r(z)-r(\zeta))
\end{aligned}$$

provided $\tilde{c}$ and ε are sufficiently small. We now fix such constants $\tilde{c}$ and ε.

When $|\mu| \geqslant c_1\tilde{c}\tau(z,v,\varepsilon)$ we have again with the proposition 3.1 (vi) of [7] and the theorem 2.1 :

$$\begin{aligned}
|F(z,\zeta)| &\gtrsim K'\sum_{j=2}^{m} M^{2^j} \sum_{k+l=j} \frac{|\mu|^j}{k!l!}\left|\left.\frac{\partial^j r(z+\mu v)}{\partial\mu^k\partial\overline{\mu}^l}\right|_{\mu=0}\right| + c_\pm(r(z)-r(\zeta)) \\
&\gtrsim \varepsilon + c_\pm(r(z)-r(\zeta)).
\end{aligned}$$

□

Corollary 2.3. — *For all $\varepsilon > 0$ sufficiently small, the following inequality holds*

$$|\tilde{F}(z,\zeta)| \gtrsim \varepsilon + c_\pm(r(z)-r(\zeta))$$

uniformly for all $z \in \mathcal{V}$ and all $\zeta \in \mathcal{P}^0_\varepsilon(z)$, $c_\pm = c_+$ when $r(z)-r(\zeta) > 0$ and $c_\pm = c_-$ when $r(z)-r(\zeta) \leqslant 0$.

Proof. — We use $w_1^*,\dots,w_n^*$ as a basis for the definition of T_z and F'_z. Using $|\zeta_1^*| < \tau_1(z,\varepsilon) \lesssim \varepsilon$ and $|\zeta_i^*| < \tau_i(z_0,\varepsilon) \lesssim \varepsilon^{\frac{1}{m}}$ for all $\zeta \in \mathcal{P}_\varepsilon(z)$ we get uniformly with respect to z and ζ $|F'_z(\omega_\zeta)| + |T_z(\omega_\zeta)| \lesssim \varepsilon^{1+\frac{1}{m}}$. Therefore when ε is sufficiently small the proposition 2.2 gives the estimate. □

Corollary 2.4. — *There exists a constant $R > 0$ such that for all $(z,\zeta) \in \mathcal{V}\times\mathbb{C}^n$ with $|\zeta - z| < R$ the following inequality holds*

$$|\tilde{F}(z,\zeta)| \gtrsim |\zeta - z|^m + c_\pm(r(z)-r(\zeta))$$

uniformly with respect to ζ and z, $c_\pm = c_+$ when $r(z)-r(\zeta) > 0$ and $c_\pm = c_-$ when $r(z)-r(\zeta) \leqslant 0$.

Proof. — We denote by $\varepsilon_0 > 0$ a constant such that for all $\varepsilon \in]0,\varepsilon_0]$ the corollary 2.3 holds and by $R > 0$ a constant such that $B(z,R) \subset \mathcal{P}_{\varepsilon_0}(z)$

for all $z \in \mathcal{V}$. We fix $\zeta \in B(z,R)$. If $\zeta = z$, the corollary is obvious. Otherwise we have $B(z,R) \subset \mathcal{P}_{\varepsilon_0}(z) \subset \{z\} \cup \bigcup_{i=0}^{+\infty} \mathcal{P}^0_{2^{-i}\varepsilon_0}(z)$ thus there exists $i_0 \in \mathbb{N}$ such that ζ belongs to $\mathcal{P}^0_{2^{-i_0}\varepsilon_0}(z)$. The corollary 2.3 then implies that $|\tilde{F}(z,\zeta)| \gtrsim 2^{-i_0}\varepsilon_0 + c_\pm(r(z) - r(\zeta))$.

Since $\zeta \notin c_1 \mathcal{P}_{2^{-i_0}\varepsilon_0}(z)$, we have $|\zeta - z|^m \lesssim c_1 2^{-i_0}\varepsilon_0$ and so $|\tilde{F}(z,\zeta)| \gtrsim |\zeta - z|^m + c_\pm(r(z) - r(\zeta))$. □

As F, $\tilde{F}$ is a local support function. However we need a global support function to use integral formulas. Therefore we will give a global support function which has locally the same behavior.

We set $\tilde{R} := \frac{1}{4c_-}\left(\frac{R}{4}\right)^m$ and assume $\mathcal{V}$ so small that $r(z) \geqslant -\tilde{R}$ for all $z \in \mathcal{V}$. Now let χ be a C^∞ function such that for all $\zeta \in \mathbb{C}^n$, $0 \leqslant \chi(\zeta) \leqslant 1$, $\chi(\zeta) = 1$ when $|\zeta| \leqslant \frac{R}{2}$ and $\chi(\zeta) = 0$ when $|\zeta| \geqslant R$, R given by the corollary 2.3. We set for all $(\zeta, z) \in \mathbb{C}^n \times \mathcal{V}$

$$\tilde{S}(\zeta,z) \quad = \quad \tilde{F}\left(z, \chi(\zeta - z)\zeta + (1 - \chi(\zeta - z))\left(z + \frac{R}{2|\zeta - z|}(\zeta - z)\right)\right).$$

Proposition 2.5. — *For all $(\zeta, z) \in \mathbb{C}^n \times \mathcal{V}$ we have*

i) $|\tilde{S}(\zeta,z)| \gtrsim 1$ *uniformly with respect to* $\zeta \in \overline{D}_{\tilde{R}}$ *and* $z \in \mathcal{V}$ *with* $|\zeta - z| \geqslant \frac{R}{4}$.

ii) $\tilde{S}(\zeta,z) = \tilde{F}(z,\zeta)$ *when* $|\zeta - z| \leqslant \frac{R}{2}$.

Proof. — (*ii*) follows immediately from the definition of χ. To show (*i*) We set $\tilde{\zeta} = \chi(\zeta - z)\zeta + (1 - \chi(\zeta - z))\left(z + \frac{R}{2|\zeta - z|}(\zeta - z)\right)$.

We have $|\tilde{\zeta} - z| = \frac{R}{2}$ when $|\zeta - z| \geqslant R$ and $\frac{R}{4} \leqslant |\tilde{\zeta} - z| \leqslant R$ when $\frac{R}{4} \leqslant |\zeta - z| \leqslant R$. So the corollary 2.4 gives

$$|\tilde{S}(\zeta,z)| \quad \gtrsim \quad \left(\frac{R}{4}\right)^m + c_\pm(r(z) - r(\tilde{\zeta})). \tag{2.1}$$

Moreover when $|\zeta - z| \geqslant \frac{R}{4}$ we have $r(\tilde{\zeta}) \leqslant \max(r(z), r(\zeta))$. Indeed $\hat{\zeta} := \frac{R}{2|\zeta - z|}\zeta + \left(1 - \frac{R}{2|\zeta - z|}\right)z$ is a point of $[\zeta, z]$ and by convexity we have $r(\hat{\zeta}) \leqslant \max(r(\zeta), r(z))$. Again by convexity we have $r(\tilde{\zeta}) \leqslant \max(r(\zeta), r(\hat{\zeta})) \leqslant \max(r(\zeta), r(z))$. Therefore for $\zeta \in \overline{D}_{\tilde{R}}$ and $z \in \mathcal{V}$ we have $c_\pm(r(z) - r(\tilde{\zeta})) \geqslant -\frac{1}{2}\left(\frac{R}{4}\right)^m$. Now (2.1) implies (*i*). □

We now define the Hefer section and the kernel we use. Let U be an arbitrary unitary matrix and set for $\zeta, \omega \in \mathbb{C}^n$ and $z \in \mathcal{V}$

$$\begin{aligned}
\tilde{\Sigma}(z,\omega) &= \tilde{S}(z+U\omega, z), && (2.2)\\
\Theta(z,\omega) &= T(z+U\omega, z), && (2.3)\\
\tilde{\sigma}_i(z,\omega) &= \int_0^1 \frac{\partial \tilde{\Sigma}}{\partial \omega_i}(z, t\omega)dt, && (2.4)\\
\theta_i(z,\omega) &= \int_0^1 \frac{\partial \Theta}{\partial \omega_i}(z, t\omega)dt, && (2.5)\\
\tilde{Q}(\zeta,z) &= \overline{U}(\tilde{\sigma}_1(z, \overline{U}^t(\zeta-z)), \dots, \tilde{\sigma}_n(z, \overline{U}^t(\zeta-z))), && (2.6)\\
Q(\zeta,z) &= \tilde{Q}(\zeta,z) - \overline{U}(\theta_1(z, \overline{U}^t(\zeta-z)), \dots, \theta_n(z, \overline{U}^t(\zeta-z))). && (2.7)
\end{aligned}$$

One can easily check that $\tilde{Q}$ and Q do not depend on U and satisfies $\tilde{S}(\zeta,z) = \sum_{i=1}^n (\zeta_i - z_i)\tilde{Q}_i(\zeta,z)$. Moreover Q is holomorphic with respect to $\zeta \in B(z, \frac{R}{2})$. Later on we will choose $U := U(z)$ such that $\overline{U}^t \eta_z = (1,0,\dots,0)$. Now we set

$$\begin{aligned}
\tilde{\eta}_1(\zeta,z) &= \sum_{i=1}^n \tilde{Q}_i(\zeta,z)d\zeta_i,\\
\eta_0(\zeta,z) &= \sum_{i=1}^n \overline{\zeta_i - z_i}d\zeta_i,\\
\tilde{\eta}(\zeta,\lambda,z) &= (1-\lambda)\frac{\eta_0(\zeta,z)}{|\zeta-z|^2} + \lambda\frac{\tilde{\eta}_1(\zeta,z)}{\tilde{S}(\zeta,z)}, \text{ for} (\zeta,z) \in \overline{D}\times\mathcal{V} \text{ with } \tilde{S}(\zeta,z) \neq 0,\\
&= \frac{\eta_0(\zeta,z)}{|\zeta-z|^2} \qquad \text{for } (\zeta,z) \in (\mathbb{C}^n - \mathcal{V})\times\mathcal{V},\\
\tilde{\Omega}_{n,q} &= \frac{(-1)^{\frac{q(q-1)}{2}}}{(2i\pi)^n}\binom{n-1}{q}\tilde{\eta}\wedge(\overline{\partial}_{\zeta,\lambda}\tilde{\eta})^{n-q-1}\wedge(\overline{\partial}_z\tilde{\eta})^q \text{ for } n = 0,\dots,n-1
\end{aligned}$$

and $\tilde{\Omega}_{n,-1} = \tilde{\Omega}_{n,n} = 0$.

We also define for $i = 0,1$ $\iota_i : \mathbb{C}^n \times \{i\} \times \mathbb{C}^n \to \mathbb{C}^n \times [0,1] \times \mathbb{C}^n$ the canonical injection and we set $B_{n,q} = \iota_0^*(\tilde{\Omega}_{n,q})$ and $K_{n,q} = \iota_1^*(\tilde{\Omega}_{n,q})$. We finally define $\tilde{T}_q$ and R_q by setting for $f \in C^0_{0,q}(\mathbb{C}^n \setminus D)$ and $z \in \mathcal{V}\setminus D$

$$\begin{aligned}
\tilde{T}_q f(z) &= -\int_{bD\times[0,1]} f(\zeta)\wedge\tilde{\Omega}_{n,q-1}(\zeta,\lambda,z) - \int_{\mathcal{V}\setminus D} f(\zeta)\wedge B_{n,q-1}(\zeta,z),\\
R_q f(z) &= -\int_{bD} f(\zeta)\wedge K_{n,q}(\zeta,z);
\end{aligned}$$

When *suppf*, the support of f, is included in $\mathcal{V}\setminus D$ the Stokes theorem gives (see [15])

$$f = R_q f + \overline{\partial}\tilde{T}_q f + \tilde{T}_{q+1}\overline{\partial} f. \tag{2.8}$$

At the end of section 3 we show that R_q and $\tilde{T}_q$ satisfy the following theorem.

Theorem 2.6. — *i) For all $k \in \mathbb{N}$ and all $\overline{\partial}$-closed $f \in C^0_{0,q}(\mathbb{C}^n \setminus D)$, $1 \leqslant q \leqslant n-2$, $R_q f$ belongs to $C^k_{0,q}(\mathcal{V})$ and $\|R_q f\|_{k,\mathcal{V}} \leqslant \|f\|_{0,bD}$, c_k not depending on f. Moreover $R_q f$ is $\overline{\partial}$-closed on $\mathcal{V}$.*

ii) For all $f \in C^0_{0,q}(\mathbb{C}^n \setminus D)$, $1 \leqslant q \leqslant n-2$, $\tilde{T}_q f$ belongs to $C^{\frac{1}{m}}_{0,q-1}(\overline{\mathcal{V}\setminus D})$ and there exists some constant $c > 0$ not depending on f such that $\|\tilde{T}_q f\|_{\frac{1}{m},\mathcal{V}\setminus D} \leqslant c\|f\|_{0,\mathcal{V}\setminus D}$.

We may not have $\|\tilde{T}_q f\|_{k+\frac{1}{m},\mathcal{V}\setminus D} \leqslant c\|f\|_{k,\mathcal{V}\setminus D}$ for all $\overline{\partial}$-closed $f \in C^k_{0,q}(\mathcal{V}\setminus D)$, $k \in \mathbb{N}^*$. Thus we modify $\tilde{T}_q$ as in [11] or [2]. Even if the method is almost the same, there are some differences because $\tilde{S}$ is not holomorphic.

We set $G = \mathcal{V} \cap \overline{D}$ and E the Seeley type extension operator given by the following lemma (see [11] or [16]).

Lemma 2.7. — *There exists a linear extension operator $E : C^0(\mathcal{V}\setminus D) \to C^0(\mathcal{V})$ such that*

i) for all $u \in C^0(\mathcal{V}\setminus D)$ $Eu|_{\mathcal{V}\setminus D} = u$ and supp $Eu \subset \mathbb{C}^n - (D \setminus \mathcal{V})$,

ii) for all $k \in \mathbb{N}$ and all $u \in C^k(\mathcal{V}\setminus D)$ Eu belongs to $C^k(\mathcal{V})$ and there exists a constant $c_k > 0$ not depending on u such that $\|Eu\|_{k,\mathcal{V}} \leqslant c_k\|u\|_{k,\mathcal{V}\setminus D}$.

We set for $z \in \mathcal{V}\setminus\overline{D}$ and $f \in C^0_{0,q}(\mathcal{V}\setminus D)$

$$\begin{aligned}
\tilde{R}^*_q f(z) &= \int_G Ef(\zeta)\wedge K_{n,q-1}(\zeta,z), \qquad q \geqslant 1,\\
\tilde{M}_q f(z) &= \overline{\partial}_z \int_{G\times[0,1]} Ef(\zeta)\wedge\tilde{\Omega}_{n,q-2}(\zeta,\lambda,z) \qquad \text{for } q > 1,\\
&= 0 \qquad \text{for } q = 1,
\end{aligned}$$

and

$$\tilde{T}^*_q f = \tilde{T}_q f - \tilde{M}_q f - \tilde{R}^*_q f.$$

Since $\tilde{M}_q f$ is trivially $\overline{\partial}$-closed, for all $\overline{\partial}$-closed $f \in C^0_{0,q}(\mathbb{C}^n \setminus D)$ with *supp*$f \subset \mathcal{V}$, (2.8) becomes

$$f \;=\; \overline{\partial}\tilde{T}^*_q f + \overline{\partial}\tilde{R}^*_q f + R_q f. \tag{2.9}$$

Moreover when $f \in C^1_{0,q}(\mathcal{V} \setminus D)$ is $\overline{\partial}$-closed the Stokes theorem gives

$$\int_{G\times[0,1]} \overline{\partial}_{\zeta,\lambda}(Ef \wedge \tilde{\Omega}_{n,q-1}) =$$
$$= \int_{bD\times[0,1]} Ef \wedge \tilde{\Omega}_{n,q-1} - \int_G Ef \wedge B_{n,q-1} + \int_G Ef \wedge K_{n,q-1}.$$

Since $\overline{\partial}_{\zeta,\lambda}\tilde{\Omega}_{n,q-1} = (-1)^{q-1}\overline{\partial}_z\tilde{\Omega}_{n,q-2}$, we get

$$\tilde{T}^*_q f(z) = -\int_{G\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \tilde{\Omega}_{n,q-1}(\zeta,\lambda,z) - \int_{\mathcal{V}} Ef(\zeta) \wedge B_{n,q-1}(\zeta,z). \tag{2.10}$$

Using this expression of $\tilde{T}^*_q$, we will show the following theorem.

THEOREM 2.8. — *i) For $q = 1,\dots n-2$, $k \in \mathbb{N}$ and $f \in C^k_{0,q}(\mathcal{V} \setminus D)$ $\overline{\partial}$-closed with supp$f \subset \mathcal{V} \setminus D$, $\tilde{T}^*_q f$ belongs to $C^{k+\frac{1}{m}}_{0,q-1}(\mathcal{V} \setminus D)$ and there exists a constant $c_k > 0$, not depending on f, such that $\|\tilde{T}^*_q f\|_{k+\frac{1}{m},\mathcal{V}\setminus D} \leqslant c_k\|f\|_{k,\mathcal{V}\setminus D}$.*

ii) For $q = 1,\dots,n-2$ and $f \in C^0_{0,q}(\mathcal{V} \setminus D)$, $\tilde{R}^*_q f$ belongs to $C^k_{0,q}(\mathcal{V} \setminus D)$ for all k and there exists a constant $c_k > 0$ not depending on f such that $\|\tilde{R}^*_q f\|_{k,\mathcal{V}\setminus D} \leqslant c_k\|f\|_{0,\mathcal{V}\setminus D}$.

Proof of theorem 1.1. — Let $\mathcal{U}$ and $\mathcal{W}$ be neighborhoods of bD such that $\overline{\mathcal{U}} \subset \mathcal{W} \subset \overline{\mathcal{W}} \subset \mathcal{V} = D_\rho \setminus \overline{D_{-\rho}}$, $\rho > 0$. There exists an operator $T^*_q : C^0_{0,q}(\overline{\mathcal{W}}) \to C^0_{0,q-1}(\mathcal{U} \setminus \overline{D})$ such that for all $k \in \mathbb{N}$ and all $\overline{\partial}$-closed $f \in C^k_{0,q}(\overline{\mathcal{W}})$ $\overline{\partial}T^*_q f = f$ and $T^*_q f$ belongs to $C^k_{0,q-1}(\overline{\mathcal{U}})$ and satisfies $\|T^*_q f\|_{k,\mathcal{U}} \lesssim \|f\|_{k,\mathcal{W}}$ uniformly with respect to f (see [15]).

We set $T_q = \tilde{T}^*_q + \tilde{R}^*_q + T^*_q R_q$. For all $k \in \mathbb{N}$ and all $\overline{\partial}$-closed $f \in C^k_{0,q}(\mathcal{U}\setminus D)$ with *supp* $f \subset \mathcal{U} \setminus D$, we have $\overline{\partial}T_q f = f$. Moreover the theorem 2.6 (i) and 2.8 imply that $T_q f$ belongs to $C^{k+\frac{1}{m}}_{0,q-1}(\mathcal{U} \setminus D)$ and satisfies $\|T_q f\|_{k+\frac{1}{m},\mathcal{U}\setminus D} \lesssim \|f\|_{k,\mathcal{U}\setminus D}$. □

3. C^0-estimates

As in [7, 9, 2] we use ε-extremal basis to estimate the kernel. We fix $\varepsilon > 0$, $z_0 \in \mathcal{V} \setminus \overline{D}$ and an ε-extremal basis $w_1^*, \dots, w_n^*$ at z_0. We assume that ε is so small that $\left|\frac{\partial r}{\partial w_1^*}(\zeta)\right| \geqslant c$ for all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, c not depending on z_0, ζ and ε. We denote by $(\zeta_1^*, \dots, \zeta_n^*)$ the extremal coordinates of a point $\zeta \in \mathbb{C}^n$ and by Φ_* a unitary matrix such that $\zeta^* = \Phi_*(\zeta - z_0)$. In order to write $\tilde{\Omega}_{n,q}$ in the ε-extremal basis we set $Q^* = \overline{\Phi}_* Q$ and $\tilde{Q}^* = \overline{\Phi}_* \tilde{Q}$. Thus we have

$$\begin{aligned}
\tilde{\eta}_1(\zeta, z) &= \sum_{j=1}^n \tilde{Q}_j^*(\zeta, z) d\zeta_j^*,\\
\overline{\partial}_z \tilde{\eta}_1(\zeta, z) &= \sum_{i,j=1}^n \frac{\partial \tilde{Q}_j^*}{\partial \overline{z}_i^*}(\zeta, z) d\overline{z}_i^* \wedge d\zeta_j^*,\\
\overline{\partial}_\zeta \tilde{\eta}_1(\zeta, z) &= \sum_{i,j=1}^n \frac{\partial \tilde{Q}_j^*}{\partial \overline{\zeta}_i^*}(\zeta, z) d\overline{\zeta}_i^* \wedge d\zeta_j^*.
\end{aligned}$$

We also need a unitary matrix $\Psi(z)$ such that $\Psi(z)\eta_z = (1, 0, \dots, 0)$ for all $z \in \mathcal{P}_\varepsilon(z_0)$. We use the matrix defined and studied in [2]. In the definition of $\tilde{Q}$ (see (2.2), (2.4) and (2.6)) we use $U = \overline{\Psi(z)\Phi_*}^t$ and set $\omega(z, \zeta) = \Psi(z)(\zeta^* - z^*)$. We notice that $\omega(z_0, \zeta) = \zeta^*$ and that for $|\zeta - z| < \frac{R}{2}$

$$\begin{aligned}
Q_1^*(\zeta, z) &= \frac{\partial r}{\partial z_1^*}(z) + K \frac{\partial r}{\partial z_1^*}(z) \sum_{k=1}^n \frac{\partial r}{\partial z_k^*}(z)(\zeta_k^* - z_k^*),\\
Q_j^*(\zeta, z) &= \frac{\partial r}{\partial z_j^*}(z) + K \frac{\partial r}{\partial z_j^*}(z) \sum_{k=1}^n \frac{\partial r}{\partial z_k^*}(z)(\zeta_k^* - z_k^*)\\
&\quad - K' \sum_{k=2}^m \kappa_k M^{2^k} \sum_{|\beta|=k} \frac{\beta_j}{k\beta!} \frac{\partial^k r}{\partial z^{*\beta}}(z) \frac{(\zeta^* - z^*)^\beta}{\zeta_j^* - z_j^*}, \quad j = 2, \dots, n.
\end{aligned}$$

For later use (see section 4) we introduce $\delta_j^* = \frac{\partial}{\partial z_j^*} + \frac{\partial}{\partial \zeta_j^*}$, $j = 1, \dots, n$

Lemma 3.1. — *For $i = 1, \dots, n$, $j = 2, \dots, n$ and $\zeta \in \mathcal{P}_\varepsilon(z_0)$ we have uniformaly with respect to ζ, z_0 and ε*

$$\begin{aligned}
|\omega_i(z_0, \zeta)| &\lesssim \tau_i(z_0, \varepsilon)\\
\left|\frac{\partial \overline{\omega}_1}{\partial z_i^*}(z_0, \zeta)\right| &\lesssim \frac{\varepsilon}{\tau_i'(z_0, \varepsilon)}
\end{aligned}$$

$$\begin{aligned}
|\delta_i^*\omega_j(z_0,\zeta)| + \left|\frac{\partial \omega_j}{\partial \overline{z}_i^*}(z_0,\zeta)\right| &\lesssim \tau_j(z_0,\varepsilon)\\
\left|\frac{\partial \overline{\omega}_1}{\partial \overline{z}_i^*}(z_0,\zeta)\right| &\lesssim \frac{\varepsilon}{\tau_i(z_0,\varepsilon)}\\
\left|\frac{\partial \overline{\omega}_1}{\partial \overline{z}_1^*}(z_0,\zeta)+1\right| &\lesssim \varepsilon^{\frac{1}{2}}
\end{aligned}$$

where $\tau_i'(z_0,\varepsilon)=\tau_i(z_0,\varepsilon)$ *if* $i\neq 1$ *and* $\tau_1'(z_0,\varepsilon)=\varepsilon^{\frac{1}{2}}$*.*

Proof. — We have for all k $\omega_k(z,\zeta)=\sum_{l=1}^n \Psi_{kl}(z)(\zeta_l^*-z_l^*)$. The proposition 4.2 of [2] implies all the inequalities. □

Lemma 3.2. — *For all multiindices β with $|\beta|\geqslant 1$ and $\beta_1=0$ and $j=1,\dots,n$ we have uniformly with respect to z_0 and ε*

$$\begin{aligned}
\left|\frac{\partial^{|\beta|+1} r_{z_0}}{\partial\overline{\omega}_1\partial\omega^\beta}(0)\right| &\lesssim \frac{\varepsilon^{\frac{1}{2}}}{\prod_{k=1}^n \tau_k(z_0,\varepsilon)^{\beta_k}},\\
\left|\frac{\partial}{\partial z_j^*}\frac{\partial^{|\beta|+1} r_{z_0}}{\partial\overline{\omega}_1\partial\omega^\beta}(0)-\frac{\partial^{|\beta|+2} r}{\partial z_j^*\partial\overline{z_1^*}\partial z^{*\beta}}(z_0)\right| &\lesssim \frac{\varepsilon}{\tau_j'(z_0,\varepsilon)\prod_{k=1}^n \tau_k(z_0,\varepsilon)^{\beta_k}},\\
\left|\frac{\partial}{\partial \overline{z}_j^*}\frac{\partial^{|\beta|+1} r_{z_0}}{\partial\overline{\omega}_1\partial\omega^\beta}(0)-\frac{\partial^{|\beta|+2} r}{\partial \overline{z}_j^*\partial\overline{z_1^*}\partial z^{*\beta}}(z_0)\right| &\lesssim \frac{\varepsilon}{\tau_j'(z_0,\varepsilon)\prod_{k=1}^n \tau_k(z_0,\varepsilon)^{\beta_k}}.
\end{aligned}$$

Proof. — Since $r_z(\omega)=r(z+\overline{\Psi(z)}^t\omega)$ and $\Psi(z_0)=Id_{\mathbb{C}^n}$, for all $\alpha_1,\dots,\alpha_l$, $l\geqslant 1$, we have

$$\frac{\partial^{l+1} r_{z_0}}{\partial\overline{\omega}_1\partial\omega_{\alpha_1}\dots\partial\omega_{\alpha_l}}(0)=\frac{\partial^{l+1} r}{\partial\overline{z}_1^*\partial z_{\alpha_1}^*\dots\partial z_{\alpha_l}^*}(z_0).$$

Therefore the first inequality is a straightforward consequence of the corollary 3.4 of [2]. In order to prove the second inequality we compute

$$\begin{aligned}
\frac{\partial}{\partial z_j^*}\frac{\partial^{l+1} r_{z_0}}{\partial\overline{\omega}_1\partial\omega_{\alpha_1}\dots\partial\omega_{\alpha_l}}(0) = &\sum_{\substack{1\leqslant p\leqslant l\\ 1\leqslant s\leqslant n}} \frac{\partial^{l+1} r}{\partial\overline{z}_1^*\partial z_{\alpha_1}^*\dots[\partial z_{\alpha_p}^*]\dots\partial z_{\alpha_l}^*}(z_0)\frac{\partial\overline{\Psi}_{\alpha_p s}}{\partial z_j^*}(z_0)+\\
&+\sum_{1\leqslant s\leqslant n}\frac{\partial^{l+1} r}{\partial\overline{z}_s^*\partial z_{\alpha_1}^*\dots\partial z_{\alpha_l}^*}(z_0)\frac{\partial\Psi_{1s}}{\partial z_j^*}(z_0)+\\
&+\frac{\partial^{l+2} r}{\partial z_j^*\partial\overline{z}_1^*\partial z_{\alpha_1}^*\dots\partial z_{\alpha_l}^*}(z_0).
\end{aligned}\tag{3.1}$$

where the term between $[\cdot]$ is omitted.

The proposition 4.2 of [2] gives $\left|\frac{\partial\overline{\Psi}_{\alpha_k s}}{\partial z_j^*}(z)\right| \lesssim \frac{\varepsilon}{\tau_{\alpha_k}(z_0,\varepsilon)\tau'_j(z_0,\varepsilon)}$ for all s, α_k and j.

The corollary 3.4 of [2] gives $\left|\frac{\partial^{p+1}r}{\partial\overline{z}_s^*\partial z_{\alpha_1}^*\ldots\partial z_{\alpha_p}^*}\right| \lesssim \frac{\varepsilon^{\frac12}}{\prod_{k=1}^p \tau_{\alpha_k}(z_0,\varepsilon)}$ and by proposition 4.2 of [2] we have $\left|\frac{\partial\Psi_{1s}}{\partial z_j^*}(z)\right| \lesssim \frac{\varepsilon^{\frac12}}{\tau'_j(z_0,\varepsilon)}$ from which follows the estimates of the second sum in (3.1). This proves the second inequality. The third can be shown in the same way. □

Proposition 3.3. — *i) For all $\zeta \in B(z_0,\frac{R}{2})$, $i = 2,\ldots,n$ and $j = 1,\ldots,n$ we have*

$$\frac{\partial\tilde{Q}_j^*}{\partial\overline{\zeta}_i^*}(\zeta,z_0) = 0.$$

ii) For all $\zeta \in \mathcal{P}_\varepsilon(z_0)\cap B(z_0,\frac{R}{2})$ and $j = 1,\ldots,n$ holds uniformly

$$\left|\frac{\partial\tilde{Q}_j^*}{\partial\overline{\zeta}_1^*}(\zeta,z_0)\right| \lesssim \frac{\varepsilon^{\frac12}}{\tau_j(z_0,\varepsilon)}.$$

Proof. — We have $\tilde{Q}_j^*(\zeta,z) = Q_j^*(\zeta,z) + \sum_{k=1}^n \Psi_{kj}(z)\theta_k(z,\omega(z,\zeta))$ and

$$\begin{aligned}
\theta_1(z,\omega) &= 0\\
\theta_k(z,\omega) &= -\overline{\omega}_1 K' \sum_{l=2}^m \kappa_l M^{2^l} \sum_{\substack{|\beta|=l\\ \beta_1=0}} \frac{\beta_k}{l\beta!}\frac{\omega^\beta}{\omega_k}\frac{\partial^{l+1}r_z}{\partial\overline{\omega}_1\partial\omega^\beta}(0) + \frac{\partial^2 r_z}{\partial\overline{\omega}_1\partial\omega_k}(0)\overline{\omega}_1\\
&\quad -\overline{\omega}_1^2 K' \sum_{l=2}^m \frac{\kappa_l M^{2^l}}{2} \sum_{\substack{|\beta|=l\\ \beta_1=0}} \frac{\beta_k}{l\beta!}\frac{\omega^\beta}{\omega_k}\frac{\partial^{l+2}r_z}{\partial\overline{\omega}_1^2\partial\omega^\beta}(0) + \frac12\frac{\partial^3 r_z}{\partial\overline{\omega}_1^2\partial\omega_k}(0)\overline{\omega}_1^2.
\end{aligned}$$

$Q^*(\cdot,z_0)$ is holomorphic for $\zeta \in B(z_0,\frac{R}{2})$ and $\Psi(z_0) = Id_{\mathbb{C}^n}$ thus $\frac{\partial\tilde{Q}_j^*}{\partial\overline{\zeta}_i^*}(\zeta,z_0)$ $= \frac{\partial\theta_j(z_0,\omega(z_0,\zeta))}{\partial\overline{\zeta}_i^*}$. On the other hand $\frac{\partial\overline{\omega}_1}{\partial\overline{\zeta}_i^*}(z_0,\zeta) = \Psi_{1i}(z_0)$. Therefore (i) is obvious and (ii) follows from the lemma 3.1 and 3.2. □

Proposition 3.4. — *For all $\zeta \in \mathcal{P}_\varepsilon(z_0)\cap B(z_0,\frac{R}{2})$, $j = 1,\ldots,n$, $i = 2,\ldots,n$ we have uniformly with respect to ζ, z_0 and ε*

$$|\tilde{Q}_j^*(\zeta,z_0)| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)},$$

$$\left|\frac{\partial \tilde{Q}_j^*}{\partial \overline{z}_i^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)},$$
$$\left|\frac{\partial \tilde{Q}_j^*}{\partial \overline{z}_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)}.$$

Proof. — Since $\tau_1(z_0,\varepsilon) \eqsim \varepsilon$ (see proposition 3.1 (v) of [7]), the proposition is obvious when $j = 1$. Therefore we only consider the cases $j = 2,\dots,n$. We use $\tilde{Q}_j^*(\zeta,z) = Q_j^*(\zeta,z) + \sum_{k=1}^n \Psi_{kj}(z)\theta_k(z,\omega(z,\zeta))$. $|\theta_k(z_0,\omega(z_0,\zeta)| \lesssim |\omega_1(z_0,\zeta)| \lesssim \varepsilon$ for all k and the corollary 3.4 of [2] implies that $\left|Q_j^*(\zeta,z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)}$. Therefore the first inequality holds. Because of the proposition 4.2 of [2], to show the others inequalities it remains to estimate $\frac{\partial Q_j^*}{\partial \overline{z}_i^*}(\zeta,z_0) + \left.\frac{\partial \theta_j(z,\omega(z,\zeta))}{\partial \overline{z}_i^*}\right|_{z=z_0}$. The corollary 3.4 of [2] gives $\left|\frac{\partial Q_j^*}{\partial \overline{z}_i^*}(\zeta,z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_i'(z_0,\varepsilon)}$.

The lemma 3.1 then implies $\left|\frac{\partial \tilde{Q}_j^*}{\partial \overline{z}_i^*}(\zeta,z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}$ for $i,j = 2,\dots,n$.

When $i = 1$ we notice that $\frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) = \frac{\partial^{|\beta|+1} r}{\partial \overline{z}_1^* \partial z^{*\beta}}(z_0)$. Therefore lemma 3.1 and 3.2 imply for all multiindices β with $\beta_1 = 0$ and $|\beta| \geqslant 1$ that

$$\left|\beta_j \frac{\omega(z_0,\zeta)^\beta}{\omega_j(z_0,\zeta)}\left(\frac{\partial \overline{\omega}_1}{\partial \overline{z}_1^*}(z_0,\zeta)\frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) + \frac{\partial^{|\beta|} r}{\partial \overline{z}_1^* \partial z^{*\beta}}(z_0)\right)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)}.$$

The inequality $|\overline{\omega}_1(z_0,\zeta)| \lesssim \varepsilon$ then implies that for all β with $\beta_1 = 0$

$$\left|\frac{\partial}{\partial \overline{z}_1^*}\left(\beta_j \frac{\omega(z_0,\zeta)^\beta}{\omega_i(z_0,\zeta)}\overline{\omega}_1(z_0,\zeta)\frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) + \frac{\beta_j \zeta^{*\beta}}{\zeta_j^*}\frac{\partial^{|\beta|} r}{\partial z^{*\beta}}(z_0)\right)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)}. \quad (3.2)$$

Since $|\zeta_1^*| \leqslant \varepsilon$ we have for all multiindices β with $\beta_1 \neq 0$

$$\left|\frac{\partial}{\partial \overline{z}_1^*}\left(\beta_j \frac{\zeta^{*\beta}}{\zeta_j^*}\frac{\partial^{|\beta|} r}{\partial z^{*\beta}}(z_0)\right)\right| \lesssim \varepsilon. \quad (3.3)$$

The corollary 3.4 of [2] gives

$$\left|\frac{\partial^2 r}{\partial \overline{z}_1^* \partial z_j^*}(z_0)\sum_{k=1}^n \frac{\partial r}{\partial z_k^*}(z_0)\zeta_k^* + \frac{\partial r}{\partial z_j^*}(z_0)\sum_{k=1}^n \frac{\partial^2 r}{\partial \overline{z}_1^* \partial z_k^*}(z_0)\zeta_k^*\right| \lesssim \frac{\varepsilon^{\frac{3}{2}}}{\tau_j(z_0,\varepsilon)}. \quad (3.4)$$

All together (3.2), (3.3) and (3.4) hield to $\left|\frac{\partial Q_j^*}{\partial \overline{z}_1^*}(\zeta,z_0) + \left.\frac{\partial \theta_j(z,\omega(z,\zeta))}{\partial \overline{z}_1^*}\right|_{z=z_0}\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)}$ and so $\left|\frac{\partial \tilde{Q}_j^*}{\partial \overline{z}_1^*}(\zeta,z_0)\right| \lesssim \frac{\varepsilon}{\tau_j(z_0,\varepsilon)}$. □

Lemma 3.5. — *For $q = 0, \ldots, n-3$, $(\zeta, z) \in \mathbb{C}^n \times \mathcal{V}$ with $|\zeta - z| < \frac{R}{2}$ we have*

$$K_{n,q}(\zeta, z) = 0.$$

Proof. — We fix $z \in \mathcal{V}$ and we write $K_{n,q}(\zeta, z)$ with respect to an extremal basis at z. We get a sum of the following terms

$$\frac{\tilde{Q}^*_{\nu_0} d\zeta^*_{\nu_0} \wedge \bigwedge_{i=1}^{q} \frac{\partial \tilde{Q}^*_{\nu_i}}{\partial \overline{z}^*_{\mu_i}} d\overline{z}^*_{\mu_i} \wedge d\zeta^*_{\nu_i} \wedge \bigwedge_{i=1+q}^{n-1} \frac{\partial \tilde{Q}^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}} d\overline{\zeta}^*_{\mu_i} \wedge d\zeta^*_{\nu_i}}{\tilde{S}^n}.$$

According to the proposition 3.3, when $n - 3 \geqslant q$, $\bigwedge_{i=1+q}^{n-1} \frac{\partial \tilde{Q}^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}}(\zeta, z) d\overline{\zeta}^*_{\mu_i} \wedge d\zeta^*_{\nu_i} = 0$ for $\zeta \in \mathbb{C}^n$ with $|\zeta - z| < \frac{R}{2}$. Therefore $K_{n,q}(\zeta, z) = 0$ for such ζ. □

Theorem 2.6 (i) follows from lemma 3.5.

Proof of theorem 2.6 (i). — We fix some $\overline{\partial}$-closed form $f \in C^0_{0,q}(\mathbb{C}^n \setminus D)$. We first show the continuity of R_q when $q < n-2$. Let $\tilde{\chi}$ be a C^∞ cut off function such that $\tilde{\chi}(\zeta) = 1$ when $|\zeta| \geqslant \frac{R}{2}$ and $\tilde{\chi}(\zeta) = 0$ when $|\zeta| \leqslant \frac{R}{4}$.

By lemma 3.5 we have $R_q f(z) = \int_{bD} \tilde{\chi}(\zeta - z) f(\zeta) \wedge K_{n,q}(\zeta, z)$ for all $z \in \mathcal{V}$. The proposition 2.5 (i) then implies that $R_q f$ is in $C^\infty_{0,q}(\mathcal{V})$ and for all $k \in \mathbb{N}$ the following inequality holds $\|R_q f\|_{k,\mathcal{V}} \leqslant c_k \|f\|_{0,bD}$, c_k depending only on k.

When $q = n-2$ we show that $R_{n-2} f$ is defined in $\mathcal{V}$ although $K_{n,n-2}(\zeta, z)$ is not defined for $z \in \mathcal{V}$ and $\zeta \in \overline{D}$.

For $z \in \mathcal{V}$ and $\zeta \in D_{\tilde{R}}$ with $|\zeta - z| \geqslant \frac{R}{4}$, $\tilde{S}(\zeta, z) \neq 0$ (see proposition 2.5) so $K_{n,n-3}(\zeta, z) = \iota_1^*(\tilde{\Omega}_{n,n-3})(\zeta, z)$ and $K_{n,n-2}(\zeta, z) = \iota_1^*(\tilde{\Omega}_{n,n-2})(\zeta, z)$ are well defined and we have $\overline{\partial}_\zeta K_{n,n-2}(\zeta, z) = (-1)^n \overline{\partial}_z K_{n,n-3}(\zeta, z)$. On the other hand the lemma 3.5 gives $K_{n,n-3}(\zeta, z) = 0$ for $z \in \mathcal{V}$ and $\zeta \in D_{\tilde{R}}$ such that $|\zeta - z| < \frac{R}{2}$.

Therefore $\overline{\partial}_z K_{n,n-3}$ is $\overline{\partial}_\zeta$-closed for all $z \in \mathcal{V}$ and $\zeta \in D_{\tilde{R}}$. Theorem 2.5 of chapter V of [15] implies the existence of a C^∞-form u defined on $\tilde{\mathcal{U}} \times \mathcal{V}$, $\tilde{\mathcal{U}}$ neighborhood of $\overline{D}$, of bidegree $(n, 1)$ in ζ and $(0, n-2)$ in z such that $\overline{\partial}_\zeta u = (-1)^n \overline{\partial}_z K_{n,n-3}$ on $\tilde{\mathcal{U}} \times \mathcal{V}$.

We fix $z \in \mathcal{V} \setminus D$. We have $\overline{\partial}_\zeta (K_{n,n-2}(\cdot, z) - u(\cdot, z)) = 0$ on $D_{r(z)} \cap \tilde{\mathcal{U}}$. Therefore there exists a C^∞-form v_z of bidegree $(n, 0)$ in ζ and $(0, n-2)$ in z such that $\overline{\partial}_\zeta v_z = K_{n,n-2}(\cdot, z) - u(\cdot, z)$ on a neighborhood of $\overline{D}$. Because

$\overline{\partial} f = 0$ the Stokes theorem implies $\int_{bD} f(\zeta) \wedge (K_{n,n-2}(\zeta, z) - u(\zeta, z)) = (-1)^n \int_{bD} \overline{\partial}_\zeta (f(\zeta) \wedge v_z(\zeta)) = 0$. Thus we have $R_{n-2} f(z) = \int_{bD} f(\zeta) \wedge u(\zeta, z)$. Since u is a C^∞-form on $\tilde{\mathcal{U}} \times \mathcal{V}$, $R_{n-2} f$ belongs to $C^\infty_{0,n-2}(\mathcal{V})$ and for all $k \in \mathbb{N}$ holds $\|R_{n-2} f\|_{k,\mathcal{V}} \leqslant c_k \|f\|_{0,bD}$, $c_k > 0$ depending only on k.

We now show that $R_q f$ is $\overline{\partial}$-closed on $\mathcal{V}$. For $q < n-3$, $K_{n,q}$ and $K_{n,q+1}$ are well defined on $D_{\tilde{R}} \times \mathcal{V}$. We use $\overline{\partial} f = 0$ and Stokes theorem and get for $z \in \mathcal{V}$

$$\begin{aligned}
\overline{\partial}_z R_q f(z) &= \int_{bD} f(\zeta) \wedge \overline{\partial}_z K_{n,q}(\zeta, z) \\
&= \int_{bD} f(\zeta) \wedge (-1)^{q+1} \overline{\partial}_\zeta K_{n,q+1}(\zeta, z) \\
&= -\int_{bD} \overline{\partial}_\zeta (f(\zeta) \wedge K_{n,q+1}(\zeta, z)) = 0.
\end{aligned}$$

For $q = n-3$, $\overline{\partial}_z K_{n,n-3} = (-1)^n \overline{\partial}_\zeta u$ on $\tilde{\mathcal{U}} \times \mathcal{V}$. Therefore we get with the Stokes theorem $\overline{\partial}_z R_{n-3} f(z) = -\int_{bD} \overline{\partial}_\zeta (f(\zeta) \wedge u(\zeta, z)) = 0$ for all $z \in \mathcal{V}$.

For $q = n-2$, $\overline{\partial}_z R_{n-2} f = \int_{bD} f \wedge \overline{\partial}_z u$ on $\mathcal{V}$. We notice that $\overline{\partial}_z u(\cdot, z)$ is $\overline{\partial}_\zeta$-closed on $\mathcal{V}$ because $\overline{\partial}_\zeta \overline{\partial}_z u = -\overline{\partial}_z \overline{\partial}_\zeta u = (-1)^{n+1} \overline{\partial}_z \overline{\partial}_z K_{n,n-3} = 0$. Thus there exists a form $\tilde{v}_z$ of bidegree $(0, n-1)$ in z and $(n, 0)$ in ζ such that $\overline{\partial}_\zeta \tilde{v}_z = \overline{\partial}_z u(\cdot, z)$ on a neighborhood of $\overline{D}$. Therefore for all $z \in \mathcal{V}$ $\overline{\partial}_z R_{n-2} f(z) = \int_{bD} f(\zeta) \wedge \overline{\partial}_\zeta \tilde{v}_z(\zeta) = 0$. □

Proof of the theorem 2.6 (ii). — We denote by $\iota : bD \to \mathbb{C}^n$ the canonical injection of bD into $\mathbb{C}^n$ and first prove the following lemma.

Lemma 3.6. — *Let $\Delta = \frac{\partial}{\partial z_i}$ or $\frac{\partial}{\partial \overline{z}_i}$ be any differentiation of order 1.*

$$\iota^*\left(f(\zeta) \wedge \Delta \frac{\left(\tilde{\eta}_1 \wedge (\overline{\partial}_z \tilde{\eta}_1)^k \wedge (\overline{\partial}_\zeta \tilde{\eta}_1)^{k'} \wedge \eta_0 \wedge (\overline{\partial}_z \eta_0)^{q-k-1} \wedge (\overline{\partial}_\zeta \eta_0)^{n-q-1-k'}\right)(\zeta, z_0)}{\tilde{S}(\zeta, z_0)^{k+k'+1} |\zeta - z_0|^{2(n-k-k'-1)}} \right)$$

can be estimated for $\zeta \in B(z_0, \frac{R}{2}) \cap bD \cap \mathcal{P}^0_\varepsilon(z_0)$ by a sum of terms like

$$\|f\|_{0,bD} \frac{\varepsilon^{-1}}{\prod_{i=0}^{k+\min(1,k')} \tau_{\nu_i}(z_0, \varepsilon) \prod_{\substack{i=1 \\ \mu_i \neq 1}}^{k} \tau_{\mu_i}(z_0, \varepsilon) |\zeta - z_0|^{2(n-k-k'-1)-1}},$$

and for $\zeta \in \mathcal{P}_{|r(z_0)|}(z_0) \cap bD \cap B(z_0, \frac{R}{2})$ by

$$\frac{\|f\|_{0,bD} |r(z_0)|^{-1}}{\prod_{i=0}^{k+\min(1,k')} \tau_{\nu_i}(z_0, |r(z_0)|) \prod_{\substack{i=1 \\ \mu_i \neq 1}}^{k} \tau_{\mu_i}(z_0, |r(z_0)|) |\zeta - z_0|^{2(n-k-k'-1)-1}}.$$

Moreover we always have $\nu_i \neq \nu_j$ and $\mu_j \neq \mu_i$ for $i \neq j$.

Proof. — We show the estimate for $\zeta \in \mathcal{P}^0_\varepsilon(z_0)$, the other case is analogous. We write $f \wedge \Delta\frac{\tilde\eta_1\wedge(\overline\partial_z\tilde\eta_1)^k\wedge(\overline\partial_\zeta\tilde\eta_1)^{k'}\wedge\eta_0\wedge(\overline\partial_z\eta_0)^{q-k-1}\wedge(\overline\partial_\zeta\eta_0)^{n-q-k'-1}}{\tilde S(\zeta,z)^{k+k'+1}|\zeta-z|^{2(n-k-k'-1)}}$ in the ε-extremal basis at z_0. We get a sum of term like

$$\begin{aligned}I^{\mu_1,\dots,\mu_{k+k'}}_{\nu_0,\dots,\nu_{k+k'}} &:= f\wedge\Delta(\eta_0\wedge(\overline\partial_z\eta_0)^{q-k-1}\wedge(\overline\partial_\zeta\eta_0)^{n-q-1-k'}\wedge\\ &\wedge\frac{\tilde Q^*_{\nu_0}d\zeta^*_{\nu_0}\bigwedge_{i=1}^k\frac{\partial\tilde Q^*_{\nu_i}}{\partial\overline z^*_{\mu_i}}d\overline z^*_{\mu_i}\wedge d\zeta^*_{\nu_i}\bigwedge_{i=k+1}^{k+k'}\frac{\partial\tilde Q^*_{\nu_i}}{\partial\overline\zeta^*_{\mu_i}}d\overline\zeta^*_{\mu_i}\wedge d\zeta^*_{\nu_i}}{\tilde S(\zeta,z)^{k+k'+1}|\zeta-z|^{2(n-k-k'-1)}})\end{aligned} \quad (3.5)$$

where necessarily $\nu_i\neq\nu_j$ for $0\leqslant i<j\leqslant k+k'$ and $\mu_i\neq\mu_j$ for $1\leqslant i<j\leqslant k$.

Using the corollary 2.3 and the proposition 2.5 we get $|\tilde S(\zeta,z_0)|\gtrsim\varepsilon$.

For $\zeta\in\mathcal{P}^0_\varepsilon(z_0)$ we have $|\zeta-z_0|\gtrsim\varepsilon$ so $\Delta\frac{\eta_0\wedge(\overline\partial_z\eta_0)^{q-k-1}\wedge(\overline\partial_\zeta\eta_0)^{n-q-1-k'}}{|\zeta-z_0|^{2(n-k-k'-1)}}$ is dominated by $\frac{\varepsilon^{-1}}{|\zeta-z_0|^{2(n-k-k'-1)-1}}$.

We estimate $\tilde Q^*_{\nu_0}$ and $\frac{\partial\tilde Q^*_{\nu_i}}{\partial\overline z^*_{\mu_i}}$ with the proposition 3.4. When one of them is differentiated by Δ, we divide the corresponding estimates by ε. We then estimate bounded functions by unbounded quantities when ε goes to zero. It corresponds to the (worst) case where only $\tilde S$ is differentiated.

Notice that $\frac{\partial\tilde Q^*_i}{\partial\overline\zeta^*_j}(\zeta,z_0)=0$ for $j=2,\dots,n$ (see proposition 3.3). Therefore we only may have in (3.5) $\frac{\partial\tilde Q^*_{\nu_i}}{\partial\overline\zeta^*_1}$ and $\Delta\frac{\partial\tilde Q^*_{\nu_i}}{\partial\overline\zeta^*_{\mu_i}}$. We estimate $\frac{\partial\tilde Q^*_{\nu_i}}{\partial\overline\zeta^*_1}(\zeta,z_0)$ using the proposition 3.3. Since it is always uniformly bounded, we estimate $\Delta\frac{\partial\tilde Q^*_{\nu_i}}{\partial\overline\zeta^*_{\mu_i}}$ by a constant. When all $\mu_i\neq1$ for $i>k$ or $k'=0$, we therefore get

$$|\iota^*(I^{\mu_1,\dots,\mu_{k+k'}}_{\nu_0,\dots,\nu_{k+k'}}(\zeta,z_0))|\lesssim\frac{\|f\|_{0,bD}\varepsilon^{-1}}{\prod_{i=0}^k\tau_{\nu_i}(z_0,\varepsilon)\prod_{\substack{i=1\\ \mu_i\neq1}}^k\tau_{\mu_i}(z_0,\varepsilon)|\zeta-z_0|^{2(n-k-k'-1)-1}}$$

When there exists $i_0>k$ necessarily unique such that $\mu_{i_0}=1$, we should notice that we integrate an $(n,n-1)$ form and that $\iota^*(\bigwedge_{i=1}^n d\zeta^*_i\bigwedge_{\substack{i=1\\ i\neq j}}^n d\overline{\zeta^*_i})=\iota^*\left(\frac{(-1)^{j+1}\frac{\partial r}{\partial\overline{\zeta^*_j}}(\zeta)}{\frac{\partial r}{\partial\overline{\zeta^*_1}}(\zeta)}\bigwedge_{i=1}^n d\zeta^*_i\wedge\bigwedge_{i=2}^n d\overline{\zeta^*_i}\right)$ for $j\neq1$. We have chosen ε such that $\left|\frac{\partial r}{\partial\overline\zeta^*_1}(\zeta)\right|\gtrsim1$ and by proposition 3.1 (vii), (iv) and (v) of [7] we have $\left|\frac{\partial r}{\partial\overline\zeta^*_j}(\zeta)\right|\lesssim\varepsilon^{\frac12}$. Therefore when there exists $i_0>k$ such that $\mu_{i_0}=1$ we have

$$|\iota^*(I^{\mu_1,\dots,\mu_{k+k'}}_{\nu_0,\dots,\nu_{k+k'}}(\zeta,z_0))| \lesssim$$
$$\lesssim \frac{\|f\|_0\varepsilon^{-1}}{\tau_{\nu_{i_0}}(z_0,\varepsilon)\prod_{i=0}^k \tau_{\nu_i}(z_0,\varepsilon)\prod_{\substack{i=1\\ \mu_i\neq 1}}^{k} \tau_{\mu_i}(z_0,\varepsilon)|\zeta-z_0|^{2(n-k-k'-1)-1}}.$$

Maybe after renumbering, this proves the lemma. □

As in [7], estimate given by lemma 3.6 shows the continuity of $\tilde{T}_q$. For $f \in C^0_{0,q}(\mathcal{V}\setminus D)$ we set $\hat{T}_q f = \int_{bD\times[0,1]} f\wedge\tilde{\Omega}_{n,q-1}$. Using the Hardy-Littlewood's lemma, we just have to show for $\Delta = \frac{\partial}{\partial z_j}$ or $\frac{\partial}{\partial \overline{z}_j}$, $j=1,\dots,n$, that the inequality $|\Delta\hat{T}_q f(z)| \lesssim \|f\|_0 d(z,bD)^{\frac{1}{m}-1}$ holds uniformly with respect to f and $z\in\mathcal{V}\setminus\overline{D}$ close enough to bD.

We fix $z_0\in\mathcal{V}\setminus\overline{D}$. We integrate with respect to $\lambda\in[0,1]$ and get as a sum of the following terms

$$J_{k,k'} := f\wedge\Delta\frac{\tilde{\eta}_1\wedge(\overline{\partial}_z\tilde{\eta}_1)^k\wedge(\overline{\partial}_\zeta\tilde{\eta}_1)^{k'}\wedge\eta_0\wedge(\overline{\partial}_z\eta_0)^{q-k-1}\wedge(\overline{\partial}_\zeta\eta_0)^{n-q-k'-1}}{\tilde{S}(\zeta,z)^{k+k'+1}|\zeta-z|^{2(n-k-k'-1)}},$$

$0\leqslant k\leqslant q-1$ and $0\leqslant k'\leqslant n-q-1$.

Since $|\tilde{S}(\zeta,z_0)| \gtrsim 1$ uniformly when $|\zeta-z_0|$ is bounded away from 0 (see proposition 2.5 and corollary 2.4), it suffices to integrate $J_{k,k'}$ over $B(z_0,\frac{R}{2})\cap\mathcal{P}_{\varepsilon_0}(z_0)$.

We use the covering $\mathcal{P}_{\varepsilon_0}(z_0)\subset\mathcal{P}_{r(z_0)}(z_0)\cup\bigcup_{i=0}^{j_0}\mathcal{P}^0_{2^{-i}\varepsilon_0}(z_0)$ where $j_0\in\mathbb{N}$ satisfies $2^{-j_0}\varepsilon_0 \eqsim |r(z_0)|$. Using lemma 3.6, we get as in [7]

$$\left|\int_{bD\cap\mathcal{P}^0_{2^{-i}\varepsilon_0}(z_0)\cap B(z_0,\frac{R}{2})} J_{k,k'}(\zeta,z_0)\right| \lesssim \|f\|_{0,bD}(2^{-i}\varepsilon_0)^{\frac{1}{m}-1}, \quad (3.6)$$

$$\left|\int_{bD\cap\mathcal{P}_{r(z_0)}(z_0)\cap B(z_0,\frac{R}{2})} J_{k,k'}(\zeta,z_0)\right| \lesssim \|f\|_{0,bD}(r(z_0))^{\frac{1}{m}-1}. \quad (3.7)$$

Adding (3.6) for $i=0,\dots,j_0$ and (3.7) and using $2^{-j_0}\varepsilon_0\eqsim|r(z_0)|$, we get $\left|\int_{bD\cap\mathcal{P}_{\varepsilon_0}(z_0)\cap B(z_0,\frac{R}{2})} J_{k,k'}(\zeta,z_0)\right| \lesssim \|f\|_{0,bD}(r(z_0))^{\frac{1}{m}-1}$ uniformly with respect to z_0 and f. This prove the theorem 2.6 (ii). □

4. More kernel estimates

To show (i) of the theorem 2.8 when $k=0$ we just have to prove that $\tilde{M}_q$ and $\tilde{R}_q^*$ satisfy C^0-estimates. For positive k we introduce $\delta_j=\frac{\partial}{\partial z_j}+\frac{\partial}{\partial \zeta_j}$, $j=1,\dots,n$, and set for $f\in C^1_{0,q}(\mathcal{V}\setminus D)$ with $supp f\subset \mathcal{V}\setminus D$ $\tilde{T}'_q f:=\int_{G\times[0,1]}\overline{\partial}_\zeta Ef\wedge\tilde{\Omega}_{n,q-1}$. As in [14] we have

$$\begin{aligned}\frac{\partial \tilde{T}'_q f}{\partial z_j} &=\\ &=\int_G\left(\frac{\partial Ef}{\partial\zeta_j}-E\frac{\partial f}{\partial \zeta_j}\right)\wedge K_{n,q-1}-\int_{\mathcal{V}}\frac{\partial Ef}{\partial\zeta_j}\wedge B_{n,q-1}-\tilde{T}_q^*\left(\frac{\partial f}{\partial\zeta_j}\right)+\\ &\quad+\int_{G\times[0,1]}\overline{\partial}_\zeta Ef\wedge\delta_j\tilde{\Omega}_{n,q-1}+\int_{G\times[0,1]}\left(\frac{\partial Ef}{\partial\zeta_j}-E\frac{\partial f}{\partial\zeta_j}\right)\wedge\overline{\partial}_z\tilde{\Omega}_{n,q-2}.\end{aligned}\tag{4.1}$$

Using this expression we will show that $\frac{\partial\tilde{T}'_q f}{\partial z_j}$ belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ when f belongs to $C^k_{0,q}(\mathcal{V}\setminus D)$, $k\geqslant 1$. Thus it will remain to show that $\frac{\partial\tilde{T}'_q f}{\partial\overline{z}_j}$ belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$. To do this we need estimates of $\frac{\partial\tilde{S}}{\partial\overline{z}^*_j}$, $\frac{\partial^2\tilde{Q}^*_i}{\partial\overline{z}^*_j\partial\overline{z}^*_k},\dots$ We set $\delta^*_j=\frac{\partial}{\partial z^*_j}+\frac{\partial}{\partial\zeta^*_j}$, $j=1,\dots,n$.

Lemma 4.1. — *For all $\zeta\in\mathcal{P}_\varepsilon(z_0)\cap B(z_0,\frac{R}{2})$, $i,j=1,\dots,n$ we have uniformly with respect to ζ, z_0 and ε*

$$|\delta^*_j\tilde{Q}^*_i(\zeta,z_0)|\lesssim\frac{\varepsilon}{\tau'_j(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}.$$

Proof. — We have

$$\delta^*_j\tilde{Q}^*_i(\zeta,z_0)=\delta^*_jQ^*_i(\zeta,z_0)+\sum_{k=1}^n\frac{\partial\Psi_{ki}}{\partial z^*_j}(z_0)\theta_k(z_0,\omega(z_0,\zeta))+\delta^*_j\theta_i(z_0,\omega(z_0,\zeta)).$$

The corollary 3.4 of [2] implies that $|\delta^*_jQ^*_i(\zeta,z_0)|\lesssim\frac{\varepsilon}{\tau_i(z_0,\varepsilon)\tau'_j(z_0,\varepsilon)}$.

Lemma 3.1 gives $|\delta^*_j\overline{\omega}_1(z_0,\zeta)|\lesssim\frac{\varepsilon}{\tau'_j(z_0,\varepsilon)}$. Moreover $|\omega_1(z_0,\zeta)|\lesssim\varepsilon$ so we have $|\delta^*_j\theta_i(z_0,\omega(z_0,\zeta))|\lesssim\frac{\varepsilon}{\tau'_j(z_0,\varepsilon)}$ and $|\theta_k(z_0,\omega(z_0,\zeta))|\lesssim\varepsilon$ for all k. □

Corollary 4.2. — *For all $\zeta\in\mathcal{P}_\varepsilon(z_0)\cap B(z_0,\frac{R}{2})$, $i,j=1,\dots,n$ we have uniformly with respect to ζ, z_0 and ε*

$$\left|\frac{\partial\tilde{S}}{\partial\overline{z}_j}(\zeta,z_0)\right|+|\delta_j\tilde{S}(\zeta,z_0)|\quad\lesssim\quad\varepsilon^{\frac{1}{2}},$$

$$\left|\frac{\partial \tilde{Q}_i^*}{\partial \overline{z}_j}(\zeta, z_0)\right| + |\delta_j \tilde{Q}_i^*(\zeta, z_0)| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0, \varepsilon)}.$$

Proof. — Since $\tau_j'(z_0,\varepsilon) \gtrsim \varepsilon^{\frac{1}{2}}$ for all j lemma 4.1 and proposition 3.4 imply the second estimate. The first inequality follows from the second one because $\tilde{S}(\zeta, z) = \sum_{i=1}^n \tilde{Q}_i^*(\zeta, z)(\zeta_i^* - z_i^*)$. □

Lemma 4.3. — *For $i, j = 1, \dots, n$, $k = 2, \dots, n$ and $\zeta \in \mathcal{P}_\varepsilon(z_0) \cap B(z_0, \frac{R}{2})$ we have uniformly with respect to ζ, z_0 and ε :*

$$\left|\delta_j \frac{\partial \tilde{Q}_i^*}{\partial \overline{\zeta}_k^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0, \varepsilon)}.$$

Proof. — It suffices to show the estimate with δ_j^* instead of δ_j.

$Q_i^*(\cdot, z_0)$ is holomorphic so we have to estimate $\frac{\partial \Psi_{li}}{\partial z_j^*}(z_0) \frac{\partial \theta_i(z_0, \omega(z_0,\zeta))}{\partial \overline{\zeta}_k^*}$ and $\delta_j^* \frac{\partial \theta_i(z, \omega(z,\zeta))}{\partial \overline{\zeta}_k^*}\Big|_{z=z_0}$ $i, j, l = 1, \dots, n$ and $k = 2, \dots, n$.

We have $\frac{\partial \overline{\omega}_1}{\partial \overline{\zeta}_k^*}(z_0, \zeta) = \overline{\Psi}_{1k}(z_0) = 0$ thus $\frac{\partial \theta_i(z_0, \omega(z_0,\zeta))}{\partial \overline{\zeta}_k^*} = 0$ for all $k \neq 1$.

The lemma 3.1 implies that $|\omega_1(z_0, \zeta)| + |\delta_j^* \overline{\omega}_1(z_0, \zeta)| \lesssim \varepsilon^{\frac{1}{2}}$. Moreover the lemma 3.1 and 3.2 give $\left|\frac{\beta_i \omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)} \frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$. Therefore we have $\left|\delta_j^* \frac{\partial \theta_i(z, \omega(z,\zeta))}{\partial \overline{\zeta}_k^*}\Big|_{z=z_0}\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$. □

Lemma 4.4. — *For $i, j = 1, \dots, n$, $k = 2, \dots, n$ and $\zeta \in \mathcal{P}_\varepsilon(z_0) \cap B(z_0, \frac{R}{2})$ we have uniformly with respect to ζ, z_0 and ε :*

$$\left|\delta_j \frac{\partial \tilde{Q}_i^*}{\partial \overline{z}_k^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0, \varepsilon)\tau_k(z_0, \varepsilon)},$$

$$\left|\delta_j \frac{\partial \tilde{Q}_i^*}{\partial \overline{z}_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0, \varepsilon)}.$$

Proof. — It suffices to show the 2 estimates with δ_j^* instead of δ_j. Since $\tau_1(z_0, \varepsilon) \eqsim \varepsilon$ we consider only the cases $i = 2, \dots, n$. We have

$$\delta_j^* \frac{\partial \tilde{Q}_i^*}{\partial \overline{z}_k^*}(\zeta, z) = \sum_{l=2}^{n} \Psi_{li}(z) \delta_j^* \frac{\partial \theta_l(z, \omega(z,\zeta))}{\partial \overline{z}_k^*} + \frac{\partial \Psi_{li}}{\partial z_j^*}(z) \frac{\partial \theta_l(z, \omega(z,\zeta))}{\partial \overline{z}_k^*} +$$
$$+ \sum_{l=2}^{n} \frac{\partial \Psi_{li}}{\partial \overline{z}_k^*}(z) \delta_j^* \theta_l(z, \omega(z,\zeta)) + \frac{\partial^2 \Psi_{li}}{\partial z_j^* \partial \overline{z}_k^*}(z) \theta_l(z, \omega(z,\zeta)) + \delta_j^* \frac{\partial Q_i^*}{\partial \overline{z}_k^*}(\zeta, z).$$

For all l the proposition 4.2 of [2] gives $\left|\frac{\partial \Psi_{li}}{\partial \overline{z}_k^*}(z_0)\right| + \left|\frac{\partial \Psi_{li}}{\partial z_j^*}(z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$ and by lemma 3.1 $|\theta_l(z_0, \omega(z_0,\zeta))| \lesssim \varepsilon$. Thus it remains to estimate $\delta_j^* \frac{\partial Q_i^*}{\partial \overline{z}_k^*}(\zeta, z_0) + \delta_j^* \frac{\partial \theta_i(z,\omega(z,\zeta))}{\partial \overline{z}_k^*}\Big|_{z=z_0}$.

Lemma 3.1 gives $|\omega_1(z_0,\zeta)| \lesssim \varepsilon$, $|\delta_j^* \overline{\omega}_1(z_0,\zeta)| \lesssim \varepsilon^{\frac{1}{2}}$ and $\left|\frac{\partial \overline{\omega}_1}{\partial \overline{z}_k^*}(z_0,\zeta)\right| \lesssim \frac{\varepsilon}{\tau_k(z_0,\varepsilon)}$ for $k \neq 1$. The lemma 3.1 and 3.2 give for all multiindices β with $|\beta| \geqslant 1$ and $\beta_1 = 0$ $\left|\beta_i \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)} \frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$. Thus we have $\left|\delta_j^* \frac{\partial \theta_i(z,\omega(z,\zeta))}{\partial \overline{z}_k^*}\Big|_{z=z_0}\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)\tau_k(z_0,\varepsilon)}$. On the other hand the corollary 3.4 of [2] gives $\left|\delta_j^* \frac{\partial Q_i^*}{\partial \overline{z}_k^*}(\zeta, z)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)\tau_k(z_0,\varepsilon)}$. Thus for all $k \neq 1$ $\left|\delta_j \frac{\partial \tilde{Q}_i^*}{\partial \overline{z}_k^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)\tau_k(z_0,\varepsilon)}$.

To treat the case $k = 1$ we first notice that for all multiindices β with $\beta_1 \neq 0$ $\left|\delta_j^* \frac{\partial}{\partial \overline{z}_1^*}\left(\beta_i \frac{\zeta^{*\beta}}{\zeta_i^*} \frac{\partial^{|\beta|} r}{\partial z^{*\beta}}(z_0)\right)\right| \lesssim \varepsilon$. Moreover the corollary 3.4 of [2] gives $\left|\delta_j^* \frac{\partial}{\partial \overline{z}_1^*}\left(\frac{\partial r}{\partial z_i^*}(z) \sum_{l=1}^n \frac{\partial r}{\partial z_l^*}(z)(\zeta_l^* - z_l^*)\right)\Big|_{z=z_0}\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$.

Thus we must show for all multiindices β with $\beta_1 = 0$ and $|\beta| \geqslant 1$ that $\left|\delta_j^* \frac{\partial}{\partial \overline{z}_1^*}\left(\beta_i \frac{\omega^\beta(z,\zeta)}{\omega_i(z,\zeta)} \overline{\omega}_1(z,\zeta) \frac{\partial^{l+1} r_z}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) + \beta_i \frac{(\zeta^* - z^*)^\beta}{\zeta_i^* - z_i^*} \frac{\partial^{l+2} r}{\partial z^{*\beta}}(z)\right)\Big|_{z=z_0}\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$.

Let β be such that $\beta_1 = 0$ and $|\beta| \geqslant 1$. The lemma 3.1 and 3.2 give

$$\left|\beta_i \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)}\right| \left|\frac{\partial \overline{\omega}_1}{\partial \overline{z}_1^*}(z_0,\zeta) \frac{\partial}{\partial z_j^*} \frac{\partial^{l+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) + \frac{\partial^{l+2} r}{\partial z_j^* \partial \overline{z}_1^* \partial z^{*\beta}}(z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}.$$

By lemma 3.1 we have $\left|\beta_i \delta_j^* \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)}\right| \lesssim \frac{\prod_{l=1}^n \tau_l(z_0,\varepsilon)^{\beta_l}}{\tau_i(z_0,\varepsilon)}$. Since $|\delta_j^* \overline{\omega}_1(z_0,\zeta)| \lesssim \varepsilon^{\frac{1}{2}}$, $|\omega_1(z_0,\zeta)| \lesssim \varepsilon$ (see lemma 3.1) and $\left|\frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\prod_{l=1}^n \tau_l(z_0,\varepsilon)^{\beta_l}}$ (see lemma 3.2) we get

$$\left|\delta_j^* \frac{\partial}{\partial \overline{z}_1^*}\left(\beta_i \frac{\omega^\beta(z,\zeta)}{\omega_i(z,\zeta)}\overline{\omega}_1(z,\zeta)\frac{\partial^{l+1} r_z}{\partial\overline{\omega}_1 \partial\omega^\beta}(0) + \beta_i \frac{(\zeta^* - z^*)^\beta}{\zeta_i^* - z_i^*}\frac{\partial^{l+2} r}{\partial z^{*\beta}}(z)\right)\bigg|_{z=z_0}\right| \lesssim$$
$$\lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}.$$

□

Lemma 4.5. — *For all $\zeta \in \mathcal{P}_\varepsilon(z_0) \cap B(z_0, \frac{R}{2})$, $i,j = 1,\dots,n$ and $k = 2,\dots,n$ we have uniformly with respect to ζ, z_0 and ε*

$$\left|\frac{\partial}{\partial \overline{z}_j}\frac{\partial \tilde{Q}_i^*}{\partial \overline{\zeta}_k^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}.$$

Proof. — As usually it suffices to show the estimate with $\frac{\partial}{\partial \overline{z}_j^*}$ instead of $\frac{\partial}{\partial \overline{z}_j}$ and only for $i = 2,\dots,n$. Since

$$\frac{\partial^2 \tilde{Q}_i^*}{\partial \overline{\zeta}_k^* \partial \overline{z}_j^*}(\zeta, z_0) = \sum_{l=1}^n \frac{\partial \Psi_{li}}{\partial \overline{z}_j^*}(z_0)\frac{\partial \theta_l(z_0,\omega(z_0,\zeta))}{\partial \overline{\zeta}_k^*} + \frac{\partial^2 \theta_i(z,\omega(z,\zeta))}{\partial \overline{z}_j^* \partial \overline{\zeta}_k^*}\bigg|_{z=z_0},$$

by proposition 4.2 of [2] it suffices to estimate $\frac{\partial^2 \theta_i(z,\omega(z,\zeta))}{\partial \overline{z}_j^* \partial \overline{\zeta}_k^*}\Big|_{z=z_0}$ for all k. For all multiindices β with $\beta_1 = 0$ and $|\beta| \geqslant 1$ we have $\left|\beta_i \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)}\frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$. Moreover $\beta_i \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)}\frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0)$ is holomorphic with respect to ζ, $|\omega_1(z_0,\zeta)| \lesssim \varepsilon$ and $\frac{\partial \overline{\omega}_1}{\partial \overline{\zeta}_k^*}(z_0,\zeta) = 0$ for $k \neq 1$. Therefore $\left|\frac{\partial^2 \theta_i(z,\omega(z,\zeta))}{\partial \overline{z}_j^* \partial \overline{\zeta}_k^*}\Big|_{z=z_0}\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$ from which the lemma follows. □

Lemma 4.6. — *For all $\zeta \in \mathcal{P}_\varepsilon(z_0) \cap B(z_0, \frac{R}{2})$, $i,j = 1,\dots,n$ and $k = 2,\dots,n$ we have uniformly with respect to ζ, z_0 and ε*

$$\left|\frac{\partial}{\partial \overline{z}_j}\frac{\partial \tilde{Q}_i^*}{\partial \overline{z}_k^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)\tau_k(z_0,\varepsilon)},$$
$$\left|\frac{\partial}{\partial \overline{z}_j}\frac{\partial \tilde{Q}_i^*}{\partial \overline{z}_1^*}(\zeta, z_0)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}.$$

Proof. — As before it suffices to show the inequality with $\frac{\partial}{\partial \overline{z}_j^*}$ instead of $\frac{\partial}{\partial \overline{z}_j}$. We consider only the cases $i \neq 1$ because $\tau_1(z_0, \varepsilon) \eqsim \varepsilon$. We have

$$\begin{aligned}\frac{\partial^2 \tilde{Q}_i^*}{\partial \overline{z}_k^* \partial \overline{z}_j^*}(\zeta, z_0) = {} & \frac{\partial^2 Q_i^*}{\partial \overline{z}_k^* \partial \overline{z}_j^*}(\zeta, z_0) + \sum_{l=1}^n \frac{\partial \Psi_{li}}{\partial \overline{z}_j^*}(z_0) \left. \frac{\partial \theta_l(z, \omega(z,\zeta))}{\partial \overline{z}_k^*} \right|_{z=z_0} \\ & + \sum_{l=1}^n \frac{\partial \Psi_{li}}{\partial \overline{z}_k^*}(z_0) \left. \frac{\partial \theta_l(z, \omega(z,\zeta))}{\partial \overline{z}_j^*} \right|_{z=z_0} + \left. \frac{\partial^2 \theta_i(z, \omega(z,\zeta))}{\partial \overline{z}_j^* \partial \overline{z}_k^*} \right|_{z=z_0} \\ & + \sum_{l=1}^n \frac{\partial^2 \Psi_{li}}{\partial \overline{z}_k^* \partial \overline{z}_j^*}(z_0) \theta_l(z_0, \omega(z_0, \zeta)).\end{aligned}$$

On one hand for $l = 1, \ldots, n$ $|\theta_l(z_0, \omega(z_0, \zeta))| \lesssim |\omega_1(z_0, \zeta)| \lesssim \varepsilon$. On the other hand the proposition 4.2 of [2] implies for $j, k, l = 1, \ldots, n$ that $\left| \frac{\partial \Psi_{li}}{\partial \overline{z}_j^*}(z_0) \right| + \left| \frac{\partial \Psi_{li}}{\partial \overline{z}_k^*}(z_0) \right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0,\varepsilon)}$. Therefore it suffices to estimate $\left. \frac{\partial^2 \theta_i(z,\omega(z,\zeta))}{\partial \overline{z}_j^* \partial \overline{z}_k^*} \right|_{z=z_0} + \frac{\partial^2 Q_i^*}{\partial \overline{z}_k^* \partial \overline{z}_j^*}(\zeta, z_0)$.

For $k \neq 1$: The corollary 3.4 of [2] implies that $\left| \frac{\partial^2 Q_i^*}{\partial \overline{z}_k^* \partial \overline{z}_j^*}(\zeta, z_0) \right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_k(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}$.

We also have $|\omega_1(z_0, \zeta)| \lesssim \varepsilon$ and $\left| \frac{\partial \overline{\omega}_1}{\partial \overline{z}_k^*}(z_0, \zeta) \right| \lesssim \frac{\varepsilon}{\tau_k(z_0,\varepsilon)}$ (see lemma 3.1). So for all multiindices β we have $\left| \frac{\partial^2}{\partial \overline{z}_k^* \partial \overline{z}_j^*} \left(\overline{\omega}_1^2(z_0, \zeta) \beta_i \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)} \frac{\partial^{|\beta|+2} r_{z_0}}{\partial \overline{\omega}_1^2 \partial \omega^\beta}(0) \right) \right| \lesssim \frac{\varepsilon}{\tau_k(z_0,\varepsilon)}$.

Next on one hand the lemma 3.1 implies for all multiindices β with $\beta_1 = 0$ that $\left| \left(\beta_i \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)} \right) \right| + \left| \frac{\partial}{\partial \overline{z}_k^*} \left(\beta_i \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)} \right) \right| \lesssim \frac{\prod_{l=2}^n \tau_l(z_0,\varepsilon)^{\beta_l}}{\tau_i(z_0,\varepsilon)}$. On the other hand the corollary 3.4 of [2] and the lemma 3.2 imply for all such multiindices β that $\left| \frac{\partial}{\partial \overline{z}_k^*} \frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) \right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_k(z_0,\varepsilon) \prod_{l=2}^n \tau_l(z_0,\varepsilon)^{\beta_l}}$ and $\left| \frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) \right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\prod_{l=2}^n \tau_l(z_0,\varepsilon)^{\beta_l}}$. Therefore we have $\left| \frac{\partial^2}{\partial \overline{z}_j^* \partial \overline{z}_k^*} \left(\beta_i \overline{\omega}_1(z_0, \zeta) \frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)} \frac{\partial^{|\beta|+1} r_{z_0}}{\partial \overline{\omega}_1 \partial \omega^\beta}(0) \right) \right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_k(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}$ for all multiindices β with $|\beta| \geqslant 1$ and $\beta_1 = 0$.

We deduce from these inequalities that $\left| \left. \frac{\partial^2 \theta_i(z,\omega(z,\zeta))}{\partial \overline{z}_j^* \partial \overline{z}_k^*} \right|_{z=z_0} \right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_k(z_0,\varepsilon)\tau_i(z_0,\varepsilon)}$ and the first inequality is shown.

For $k=1$: The lemma 3.1 and 3.2 imply for all multiindices β with $\beta_1=0$ and $|\beta|\geqslant 1$

$$\left|\frac{\partial\overline{\omega}_1}{\partial\overline{z}_1^*}(z_0,\zeta)\frac{\partial}{\partial\overline{z}_j^*}\frac{\partial^{|\beta|+1}r_{z_0}}{\partial\overline{\omega}_1\partial\omega^\beta}(0)+\frac{\partial^{|\beta|+2}r}{\partial\overline{z}_j^*\partial\overline{z}_1^*\partial z^{*\beta}}(z_0)\right|\lesssim\frac{\varepsilon}{\prod_{l=1}^n\tau_l(z_0,\varepsilon)^{\beta_l}\tau_j'(z_0,\varepsilon)}.$$

We also have $|\omega_1(z_0,\zeta)|\lesssim\varepsilon$, $\left|\frac{\partial}{\partial\overline{z}_j^*}\left(\beta_i\frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)}\right)\right|\lesssim\frac{\prod_{l=1}^n\tau_l(z_0,\varepsilon)^{\beta_l}}{\tau_i(z_0,\varepsilon)}$, (see lemma 3.1), $\left|\frac{\partial^{|\beta|+1}r_{z_0}}{\partial\overline{\omega}_1\partial\omega^\beta}(0)\right|\lesssim\frac{\varepsilon^{\frac12}}{\prod_{l=1}^n\tau_l(z_0,\varepsilon)^{\beta_l})}$ (see lemma 3.2) and if $j\neq 1$ $\varepsilon^{\frac12}\gtrsim\left|\frac{\partial\overline{\omega}_1}{\partial\overline{z}_j^*}(z_0,\zeta)\right|$. Thus for all $j\neq 1$ and all β with $\beta_1=0$ and $|\beta|\geqslant 1$

$$\left|\frac{\partial^2}{\partial\overline{z}_j^*\partial\overline{z}_1^*}\left(\beta_i\frac{\omega^\beta(z,\zeta)}{\omega_i(z,\zeta)}\overline{\omega}_1(z,\zeta)\frac{\partial^{|\beta|+1}r_z}{\partial\overline{\omega}_1\partial\omega^\beta}(0)+\right.\right.$$
$$\left.\left.+\ \beta_i\frac{\zeta^{*\beta}}{\zeta_i}\frac{\partial^{|\beta|}r}{\partial z^{*\beta}}(z)+\frac{\beta_i}{2}\overline{\omega}_1^2(z,\zeta)\frac{\omega^\beta(z,\zeta)}{\omega_i(z,\zeta)}\frac{\partial^{|\beta|+2}r_z}{\partial\overline{\omega}_1^2\partial\omega^\beta}(0)\right)\right|_{z=z_0}\right|\lesssim\frac{\varepsilon^{\frac12}}{\tau_i(z_0,\varepsilon)}.$$

If $j=1$ the lemma 3.1 and 3.2 give for all multiindices β with $\beta_1=0$ and $|\beta|\geqslant 1$

$$\left|2\frac{\partial\overline{\omega}_1}{\partial\overline{z}_1^*}(z_0,\zeta)\frac{\partial}{\partial\overline{z}_1^*}\frac{\partial^{|\beta|+1}r_{z_0}}{\partial\overline{\omega}_1\partial\omega^\beta}(0)+\frac{\partial^{|\beta|+2}r}{\partial{\overline{z}_1^*}^2\partial z^{*\beta}}(z_0)+\right.$$
$$\left.+\left(\frac{\partial\overline{\omega}_1}{\partial\overline{z}_1^*}(z_0,\zeta)\right)^2\frac{\partial^{|\beta|+2}r_{z_0}}{\partial\overline{\omega}_1^2\partial\omega^\beta}(0)\right|\lesssim\frac{\varepsilon^{\frac12}}{\prod_{l=1}^n\tau_l(z_0,\varepsilon)^{\beta_l}}.$$

We still have $|\omega_1(z_0,\zeta)|\lesssim\varepsilon$, $\left|\frac{\partial}{\partial\overline{z}_1^*}\left(\beta_i\frac{\omega^\beta(z_0,\zeta)}{\omega_i(z_0,\zeta)}\right)\right|\lesssim\frac{\prod_{l=1}^n\tau_l(z_0,\varepsilon)^{\beta_l}}{\tau_i(z_0,\varepsilon)}$, $\left|\frac{\partial^{|\beta|+1}r_{z_0}}{\partial\overline{\omega}_1\partial\omega^\beta}(0)\right|\lesssim\frac{\varepsilon^{\frac12}}{\prod_{l=1}^n\tau_l(z_0,\varepsilon)^{\beta_l}}$. Therefore we get

$$\left|\frac{\partial^2}{\partial{\overline{z}_1^*}^2}\left(\beta_i\frac{\omega^\beta(z,\zeta)}{\omega_i(z,\zeta)}\overline{\omega}_1(z,\zeta)\frac{\partial^{|\beta|+1}r_z}{\partial\overline{\omega}_1\partial\omega^\beta}(0)+\right.\right.$$
$$\left.\left.+\beta_i\frac{\zeta^{*\beta}}{\zeta_i}\frac{\partial^{|\beta|}r}{\partial z^{*\beta}}(z)+\frac{\beta_i}{2}\overline{\omega}_1^2(z,\zeta)\frac{\omega^\beta(z,\zeta)}{\omega_i(z,\zeta)}\frac{\partial^{|\beta|+2}r_z}{\partial\overline{\omega}_1^2\partial\omega^\beta}(0)\right)\right|_{z=z_0}\right|\lesssim\frac{\varepsilon^{\frac12}}{\tau_i(z_0,\varepsilon)}.$$

We finally get for all i,j

$$\left|\frac{\partial Q_i^*}{\partial\overline{z}_1^*\partial\overline{z}_j^*}(\zeta,z_0)+\left.\frac{\partial\theta_i(z,\omega(z,\zeta))}{\partial\overline{z}_1^*\partial z_j^*}\right|_{z=z_0}\right|\lesssim\frac{\varepsilon^{\frac12}}{\tau_i(z_0,\varepsilon)}$$

which finishes the proof of the second inequality for $k=1$. $\square$

5. Final Estimates

LEMMA 5.1. — *For any differentiation* $\Delta_j = \frac{\partial^j}{\partial z^\alpha \partial \overline{z}^\beta}$ *of order* $j \geqslant 1$, $i = 1, \ldots, n$, $k = 0, \ldots, q-1$, $k' = 0, \ldots, n-q-1$, *and* $\zeta \in \overline{D} \cap \mathcal{P}^0_\varepsilon(z_0)$ *if* $\varepsilon \neq |r(z_0)|$ *or* $\zeta \in \overline{D} \cap \mathcal{P}_\varepsilon(z_0)$ *if* $\varepsilon = |r(z_0)|$,

$$\Delta_j \frac{\left(\tilde{\eta}_1 \wedge (\overline{\partial}_z \tilde{\eta}_1)^k \wedge (\overline{\partial}_\zeta \tilde{\eta}_1)^{k'}\right)(\zeta, z_0)}{\tilde{S}(\zeta, z_0)^{k+k'+1}}$$

can be uniformly estimated by a sum of terms such

$$\frac{\varepsilon^{-j}}{\prod_{l=0}^{k+k'} \tau_{\nu_l}(z_0, \varepsilon) \prod_{\substack{l=1 \\ \mu_l \neq 1}}^{k} \tau_{\mu_l}(z_0, \varepsilon)} \text{ and } \frac{\varepsilon^{-j-\frac{1}{2}}}{\prod_{l=0}^{k+k'} \tau_{\nu_l}(z_0, \varepsilon) \prod_{\substack{l=1 \\ \mu_l \neq 1}}^{k} \tau_{\mu_l}(z_0, \varepsilon)},$$

this last term appearing only for positive k'.

$$\Delta_j \frac{\partial}{\partial \overline{z}_i} \frac{(\tilde{\eta}_1 \wedge (\overline{\partial}_z \tilde{\eta}_1)^k \wedge (\overline{\partial}_\zeta \tilde{\eta}_1)^{k'})(\zeta, z_0)}{\tilde{S}(\zeta, z_0)^{k+k'+1}} \text{ and } \Delta_j \delta_i \frac{(\tilde{\eta}_1 \wedge (\overline{\partial}_z \tilde{\eta}_1)^k \wedge (\overline{\partial}_\zeta \tilde{\eta}_1)^{k'})(\zeta, z_0)}{\tilde{S}(\zeta, z_0)^{k+k'+1}}$$

can be uniformly estimated by a sum of terms such

$$\frac{\varepsilon^{-j-\frac{1}{2}}}{\prod_{l=0}^{k+k'} \tau_{\nu_l}(z_0, \varepsilon) \prod_{\substack{l=1 \\ \mu_l \neq 1}}^{k} \tau_{\mu_l}(z_0, \varepsilon)} \text{ and } \frac{\varepsilon^{-j-1}}{\prod_{l=0}^{k+k'} \tau_{\nu_l}(z_0, \varepsilon) \prod_{\substack{l=1 \\ \mu_l \neq 1}}^{k} \tau_{\mu_l}(z_0, \varepsilon)},$$

this last term appearing only for positive k'.

Moreover we have in each case $\nu_i \neq \nu_j$ *and* $\mu_j \neq \mu_i$ *for* $i \neq j$.

Proof. — We estimate $\Delta_j \frac{\partial}{\partial \overline{z}_i} \frac{(\tilde{\eta}_1 \wedge (\overline{\partial}_z \tilde{\eta}_1)^k \wedge (\overline{\partial}_\zeta \tilde{\eta}_1)^{k'})(\zeta, z_0)}{\tilde{S}(\zeta, z_0)^{k+k'+1}}$ for $\zeta \in \mathcal{P}^0_\varepsilon(z_0) \cap \overline{D}$, the other cases are analogous. We write $\frac{\tilde{\eta}_1 \wedge (\overline{\partial}_z \tilde{\eta}_1)^k \wedge (\overline{\partial}_\zeta \tilde{\eta}_1)^{k'}}{\tilde{S}(\zeta, z)^{k+k'+1}}$ in the ε-extremal basis at z_0. We get a sum of term like

$$\tilde{I}^{\mu_1, \ldots, \mu_{k+k'}}_{\nu_0, \ldots, \nu_{k+k'}}(\zeta, z_0) := \frac{\tilde{Q}^*_{\nu_0} d\zeta^*_{\nu_0} \bigwedge_{i=1}^{k} \frac{\partial \tilde{Q}^*_{\nu_i}}{\partial \overline{z}^*_{\mu_i}} d\overline{z}^*_{\mu_i} \wedge d\zeta^*_{\nu_i} \bigwedge_{i=k+1}^{k+k'} \frac{\partial \tilde{Q}^*_{\nu_i}}{\partial \overline{\zeta}^*_{\mu_i}} d\overline{\zeta}^*_{\mu_i} \wedge d\zeta^*_{\nu_i}}{\tilde{S}(\zeta, z)^{k+k'+1}}$$

where necessarily $\nu_i \neq \nu_j$ for $0 \leqslant i < j \leqslant k + k'$ and $\mu_i \neq \mu_j$ for $1 \leqslant i < j \leqslant k$.

Using the corollary 2.3 and the proposition 2.5 we get $|\tilde{S}(\zeta, z_0)| \gtrsim \varepsilon$.

We estimate $\tilde{Q}^*_i$ and $\frac{\partial \tilde{Q}^*_i}{\partial \overline{z}^*_j}$ using the proposition 3.4. We use $\left|\frac{\partial \tilde{Q}^*_i}{\partial \overline{\zeta}^*_j}(z_0, \zeta)\right| \lesssim \frac{\varepsilon}{\tau_i(z_0, \varepsilon)}$ if $j \neq 1$ and $\left|\frac{\partial \tilde{Q}^*_i}{\partial \overline{\zeta}^*_1}(z_0, \zeta)\right| \lesssim \frac{\varepsilon^{\frac{1}{2}}}{\tau_i(z_0, \varepsilon)}$ (see proposition 3.3). When one of them is differentiated by $\frac{\partial}{\partial \overline{z}_j}$ we divide the corresponding estimate by $\varepsilon^{\frac{1}{2}}$ (see lemma 4.6, 4.5 and corollary 4.2) and we estimate $\frac{\partial \tilde{S}}{\partial \overline{z}_i}$ by $\varepsilon^{\frac{1}{2}}$ (see corollary 4.2). When one of them is differentiated p times by Δ_j we divide the corresponding estimates by ε^p. With those estimates we sometimes estimate bounded functions by unbounded quantities when ε goes to zero.This corresponds to the worst case $k' = 1$, $\mu_{k+1} = 1$ and only $\tilde{S}$ differentiated. This avoids too many cases to distinguish.

When there exists $i_0 > k'$ such that $\mu_{i_0} = 1$ we therefore get

$$|\Delta_j \frac{\partial}{\partial \overline{z}_i} \tilde{I}^{\mu_1,\dots,\mu_{k+k'}}_{\nu_0,\dots,\nu_{k+k'}}(\zeta, z_0)| \lesssim \frac{\varepsilon^{-j-1}}{\prod_{i=0}^{k+k'} \tau_{\nu_i}(z_0,\varepsilon) \prod_{\substack{i=1\\ \mu_i \neq 1}}^{k} \tau_{\mu_i}(z_0,\varepsilon)}.$$

If $k' = 0$ or $\mu_i \neq 1$ for all $i > k$ we get

$$|\Delta_j \frac{\partial}{\partial \overline{z}_i} \tilde{I}^{\mu_1,\dots,\mu_{k+k'}}_{\nu_0,\dots,\nu_{k+k'}}(\zeta, z_0)| \lesssim \frac{\varepsilon^{-j-\frac{1}{2}}}{\prod_{i=0}^{k+k'} \tau_{\nu_i}(z_0,\varepsilon) \prod_{\substack{i=1\\ \mu_i \neq 1}}^{k} \tau_{\mu_i}(z_0,\varepsilon)}.$$

□

We also need the two following obvious lemma.

Lemma 5.2. — *If ε is sufficiently small, then for all $j \in \mathbb{N}$, for all $g \in C^j(\mathcal{V})$, g identically zero on $\mathcal{V} \setminus D$, for all $z_0 \in \mathcal{V} \setminus D$ and for all $\zeta \in \mathcal{P}_\varepsilon(z_0)$, we have, uniformly with respect to z_0, ζ and g,*

$$|g(\zeta)| \lesssim \varepsilon^j \|g\|_{\mathcal{V},j}.$$

As a straightforwards consequence of lemma 3.5 and proposition 2.5 we have

Lemma 5.3. — *For all $g \in C^0_{0,q}(\mathcal{V})$, $q = 1,\dots,n-2$, and all $k \in \mathbb{N}$ $\int_G g(\zeta) \wedge K_{n,q-1}(\zeta,\cdot)$ belongs to $C^k_{0,q-1}(\mathcal{V} \setminus D)$ and there exists a constant c_k not depending on g such that $\left\| \int_G g(\zeta) \wedge K_{n,q-1}(\zeta,\cdot) \right\|_{k,\mathcal{V}\setminus D} \leqslant c_k \|g\|_{0,\mathcal{V}}$.*

Proof of theorem 2.8.— (*ii*) follows immediately from lemma 5.3 and 5.2.

To show (*i*) when $k = 0$ we study $\tilde{M}_q$. We fix some $f \in C^0_{0,q}(\mathcal{V} \setminus D)$ and $z_0 \in \mathcal{V} \setminus \overline{D}$ sufficiently close to bD. For $\zeta \in \mathcal{P}^0_\varepsilon(z_0) \cap G \cap B(z_0, \frac{R}{2})$

we have $|\zeta - z_0| \gtrsim \varepsilon$. Thus for $\Delta = \frac{\partial}{\partial z_j}$ or $\frac{\partial}{\partial \overline{z}_j}$, $j = 1,\ldots,n$ and $\zeta \in \mathcal{P}^0_\varepsilon(z_0) \cap G \cap B(z_0, \frac{R}{2})$, lemma 5.1 and 2.7 give

$$\left|\Delta \int_{\lambda\in[0,1]} Ef(\zeta) \wedge \overline{\partial}_z \tilde{\Omega}_{n,q-2}(\zeta,\lambda,z_0)\right| \lesssim$$

$$\lesssim \sum_{\substack{0\leqslant k\leqslant q-2\\ 0\leqslant k'\leqslant n-q}} \sum_{\substack{\nu_0<\ldots<\nu_{k+k'}\\ \mu_1<\ldots<\mu_k}} \frac{\varepsilon^{-2}\|f\|_{0,\mathcal{V}\setminus D}}{\prod_{i=0}^{k+k'} \tau_{\nu_i}(z_0,\varepsilon) \prod_{\substack{i=1\\ \mu_i\neq 1}}^{k} \tau_{\mu_i}(z_0,\varepsilon) |\zeta - z_0|^{2(n-k-k')-3}} +$$

$$+ \sum_{\substack{0\leqslant k\leqslant q-2\\ 1\leqslant k'\leqslant n-q}} \sum_{\substack{\nu_0<\ldots<\nu_{k+k'}\\ \mu_1<\ldots<\mu_k}} \frac{\varepsilon^{-\frac{3}{2}}\|f\|_{0,\mathcal{V}\setminus D}}{\prod_{i=0}^{k+k'} \tau_{\nu_i}(z_0,\varepsilon) \prod_{\substack{i=1\\ \mu_i\neq 1}}^{k} \tau_{\mu_i}(z_0,\varepsilon) |\zeta - z_0|^{2(n-k-k'-1)}}.$$

This implies that $\left|\int_{(\mathcal{P}^0_\varepsilon(z_0)\cap G\cap B(z_0,\frac{R}{2}))\times[0,1]} Ef(\zeta) \wedge \Delta\overline{\partial}_z \tilde{\Omega}_{n,q-2}(\zeta,\lambda,z_0)\right| \lesssim \|f\|_{0,\mathcal{V}\setminus D}\varepsilon^{\frac{1}{m}-1}$ uniformly with respect to ε, z_0 and f. Using lemma 5.1 and 2.7 we also get $\left|\int_{(\mathcal{P}_{r(z_0)}(z_0)\cap G\cap B(z_0,\frac{R}{2}))\times[0,1]} Ef(\zeta)\wedge\Delta\overline{\partial}_z \tilde{\Omega}_{n,q-2}(\zeta,\lambda,z_0)\right| \lesssim \|f\|_{0,\mathcal{V}\setminus D} r(z_0)^{\frac{1}{m}-1}$. As in the proof of theorem 2.6 (ii) we use the covering $\mathcal{P}_{\varepsilon_0}(z_0) \subset \mathcal{P}_{|r(z_0)|}(z_0) \cup \bigcup_{i=0}^{j_0} \mathcal{P}^0_{2^{-i}\varepsilon_0}(z_0)$ and $2^{-j_0}\varepsilon \eqsim |r(z_0)|$ and we get

$$\left|\Delta \int_{(\mathcal{P}_{\varepsilon_0}(z_0)\cap G\cap B(z_0,\frac{R}{2}))\times[0,1]} Ef(\zeta) \wedge \overline{\partial}_z \tilde{\Omega}_{n,q-2}(\zeta,\lambda,z_0)\right| \lesssim \|f\|_{0,\mathcal{V}\setminus D} |r(z_0)|^{\frac{1}{m}-1}.$$

The Hardy-Littlewood lemma then implies that $\tilde{M}_q f$ belongs to $C^{\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ and satisfies $\|\tilde{M}_q f\|_{\frac{1}{m},\mathcal{V}\setminus D} \lesssim \|f\|_{0,\mathcal{V}\setminus D}$. With theorem 2.6 and theorem 2.8 (ii) this proves that $\tilde{T}^*_q f = \tilde{T}_q f - \tilde{M}_q f - \tilde{R}^*_q f$ belongs to $C^{\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ and satisfies $\|\tilde{T}^*_q f\|_{\frac{1}{m},\mathcal{V}\setminus D} \lesssim \|f\|_{0,\mathcal{V}\setminus D}$, uniformly with respect to f.

To prove the theorem 2.8 (i) for $k > 0$, we assume it is shown for $k-1$. We fix some $\overline{\partial}$-closed $f \in C^k_{0,q}(\mathcal{V}\setminus D)$ with *suppf* $\subset \mathcal{V}\setminus D$. The equality (2.10) shows that it suffices to prove that $\tilde{T}'_q f = \int_{G\times[0,1]} \overline{\partial}_\zeta Ef \wedge \tilde{\Omega}_{n,q-1}$ belongs to $C^{k+\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ and satisfies $\left\|\tilde{T}'_q f\right\|_{k+\frac{1}{m},\mathcal{V}\setminus D} \lesssim \|f\|_{k,\mathcal{V}\setminus D}$ uniformly with respect to f. In order to do this we use (4.1).

As for $\tilde{M}_q$ the lemma 5.1 and 5.2 imply that $\int_{G\times[0,1]} \overline{\partial}_\zeta Ef(\zeta) \wedge \delta_j \tilde{\Omega}_{n,q-1}(\zeta,\lambda,\cdot)$ and $\int_{G\times[0,1]} \left(\frac{\partial Ef}{\partial\zeta_j}(\zeta) - E\frac{\partial f}{\partial\zeta_j}(\zeta)\right) \wedge \overline{\partial}_z \tilde{\Omega}_{n,q-2}(\zeta,\lambda,\cdot)$ belong to $C^{k-1+\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ and have norms uniformly bounded by $\|f\|_{k,\mathcal{V}\setminus D}$.

By induction $\tilde{T}^*_q\left(\frac{\partial f}{\partial \zeta_j}\right) \in C^{k-1+\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ and $\left\|\tilde{T}^*_q\left(\frac{\partial f}{\partial \zeta_j}\right)\right\|_{k-1+\frac{1}{m},\mathcal{V}\setminus D} \lesssim \|f\|_{k,\mathcal{V}\setminus D}$.

Since $\frac{\partial Ef}{\partial \zeta_j}$ has compact support in $\mathcal{V}$ $\int_{\mathcal{V}} \frac{\partial Ef}{\partial \zeta_j}(\zeta)\wedge B_{n,q-1}(\zeta,\cdot)$ belongs to $C^{k-\varepsilon}_{0,q-1}(\mathcal{V}\setminus D)$ and $\left\|\int_{\mathcal{V}} \frac{\partial Ef}{\partial \zeta_j}(\zeta)\wedge B_{n,q-1}(\zeta,\cdot)\right\|_{k-\varepsilon} \lesssim c_{\varepsilon,k}\|f\|_k$ for all $\varepsilon \in]0,1]$.

Lemma 5.2 and 5.3 imply that $\int_G \left(\frac{\partial Ef}{\partial \zeta_j}(\zeta) - E\frac{\partial f}{\partial \zeta_j}(\zeta)\right) \wedge K_{n,q-1}(\zeta,\cdot)$ belongs to $C^k_{0,q-1}(\mathcal{V}\setminus D)$ and satisfies
$\left\|\int_G\left(\frac{\partial Ef}{\partial \zeta_j}(\zeta) - E\frac{\partial f}{\partial \zeta_j}(\zeta)\right) \wedge K_{n,q-1}(\zeta,\cdot)\right\|_{k,\mathcal{V}\setminus D} \lesssim \|f\|_{0,\mathcal{V}\setminus D}$.

Thus equality (4.1) implies for all j that $\frac{\partial \tilde{T}'_q f}{\partial z_j}$ belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ and $\left\|\frac{\partial \tilde{T}'_q f}{\partial z_j}\right\|_{k-1+\frac{1}{m},\mathcal{V}\setminus D} \lesssim \|f\|_{k,\mathcal{V}\setminus D}$ uniformly with respect to f. Morevover the lemma 5.2 and 5.1 imply for all j that $\frac{\partial \tilde{T}'_q f}{\partial \overline{z}_j}$ also belongs to $C^{k-1+\frac{1}{m}}_{0,q-1}(\mathcal{V}\setminus D)$ and $\left\|\frac{\partial \tilde{T}'_q f}{\partial \overline{z}_j}\right\|_{k-1+\frac{1}{m},\mathcal{V}\setminus D} \lesssim \|f\|_{k,\mathcal{V}\setminus D}$ uniformly with respect to f. This achieves the proof of the theorem 2.8 (i). $\square$

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