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Continuous dependence of the entropy solution of general parabolic equation

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ABSTRACT. — We consider the general parabolic equation :

$$u_t - \Delta b(u) + div\ F(u) = f \text{ in } Q =]0, T[\times \mathbb{R}^N,\ T > 0 \text{ with }\ u_0 \in L^\infty(\mathbb{R}^N),$$

for $a.e\ t \in]0, T[,\ f(t) \in L^\infty(\mathbb{R}^N)$ and $\displaystyle\int_0^T \|f(t)\|_{L^\infty(\mathbb{R}^N)} dt < \infty$.

We prove the continuous dependence of the entropy solution with respect to F, b, f and the initial data u_0 of the associated Cauchy problem.
This type of solution was introduced and studied in [MT3]. We start by recalling the definition of weak solution and entropy solution. By applying an abstract result (Theorem 2.3), we get the continuous dependance of the entropy solution. The contribution of the present work consists of considering the equation in the whole space $\mathbb{R}^n$ instead of a bounded domain and considering a bounded data instead of integrable data.

RÉSUMÉ. — On considère l'équation parabolique générale :

$$u_t - \Delta b(u) + div\ F(u) = f \text{ dans } Q =]0, T[\times \mathbb{R}^N,\ T > 0 \text{ avec } u_0 \in$$
$L^\infty(\mathbb{R}^N),\ f \in L^1_{Loc}(Q)$ pour $p.p\ t \in]0, T[\ f(t) \in L^\infty(\mathbb{R}^N),$

et $\displaystyle\int_0^T \|f(t)\|_{L^\infty(\mathbb{R}^N)} dt < \infty$.

On montre la dépendance continue de la solution entropique du problème de Cauchy associé, par rapport aux données F, b f et la donnée initiale u_0. Ce type de solution a été introdui et étudié dans [MT3].
On commence le travail par un rappel de la définition de la solution faible et entropique ainsi que les résultats importants obtenus dans [MT3]. Ensuite on montre le résultat principal du travail en utilisant le lemme abstrait (Théorème 2.3). La contribution du travail est de traiter le problème dans $\mathbb{R}^N$ ainsi que de considérer des données bornées au lieu des données intégrables utilisées dans la littérature.

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1. Introduction and notations

We consider the Cauchy Problem $(CP) = (CP)(b, F, f, u_0)$:

$$
\begin{cases}
u_t - \Delta b(u) + div\ F(u) = f & \text{in}\ \ Q \\[2mm]
u(0, .) = u_0 & \text{in}\ \ \mathbb{R}^N,
\end{cases}
\tag{CP}
$$

where $F \in C(\mathbb{R}, \mathbb{R}^N)$, $b \in C(\mathbb{R})$, b such that $F(0) = 0$ and $b(0) = 0$.

We denote by $Q =]0, T[\times \mathbb{R}^N$ with $T > 0$, and:

$$
\begin{cases}
1) \ \ u_0 \in L^\infty(\mathbb{R}^N), \\[3mm]
2) \ \ f \in L^1_{Loc}(Q) \ \ \text{for}\ \ a.e \ \ t \in]0, T[\ \ f(t) \in L^\infty(\mathbb{R}^N), \\[3mm]
3) \ \ \displaystyle\int_0^T \|f(t)\|_{L^\infty(\mathbb{R}^N)} dt < \infty.
\end{cases}
\tag{H1}
$$

$(CP)(b, F, f, u_0)$ is a degenerate second order problem. It is known there is no classical solution and the weak solution is not unique. A vast literature exists on the degenerate parabolic equation we consider. In this literature, many results are proved about existence of weak solution, and uniqueness under various additional conditions [AL], [BG], [BT1], [BW], [BT2], [DT] and [YJ].

In our first work [MT3] while studying the existence and the uniqueness of the entropy solution, we introduced sufficient conditions on the continuity modulus of b and F to insure this uniqueness. This type of conditions appears basically in [BK] where the authors gave optimal conditions for uniqueness of entropy solution for the first order problem.

In this paper, we achieve the study of the problem (CP) by giving a result for continuous dependence of the entropy solution. The contribution of this work consists of considering $(CP)(b, F, f, u_0)$ on $\mathbb{R}^n$ instead of a bounded domain. This is done by considering the same type of conditions as in [MT3] and [BK]. Our main result is based on the theory and theorems we have developed and proved in [MT1], [MT2] and [MT3].

In section 2 we recall the definitions of the types of solutions we use (weak solution, entropy solution) and the main result presented in [MT3]. In section 3, we state and prove the main result (Theorem 6).

2. Preliminaries

First of all we introduce the definitions of solution types used below.

DEFINITION 2.1 *(Weak solution of $(CP)(b, F, f, u_0)$).* — *Let u_0 and f be such that $(H1)$ is fulfilled. A weak solution of (CP) is a function $u \in L^\infty(Q)$ such that:*

$$u_t \in L^2((0, T), H_{loc}^{-1}(\mathbb{R}^N)) + L^1((0, T), L^\infty(\mathbb{R}^N)), \qquad (2.1)$$

$$b(u) \in L^2((0, T), H_{loc}^1(\mathbb{R}^N)) \qquad (2.2)$$

$$u_t - \Delta b(u) + div F(u) = f \qquad in \qquad \mathcal{D}'(Q)$$

$$and \quad u(0, x) = u_0 \ in \qquad \mathbb{R}^N.$$

The last condition is taken in the sense that

$$\int_0^T < u_t, \xi > \ dt = - \int_Q u \ \xi_t \ dx \ dt - \int_{\mathbb{R}^N} u_0 \ \xi(0) \ dx, \qquad (2.3)$$

for any $\xi \in L^2((0, T); \mathcal{D}(\mathbb{R}^N)) \cap W^{1,1}((0, T); L^\infty(\mathbb{R}^N))$ so that $\xi(T) = 0$ and $<,>$ represents the duality product between $H^{-1}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$.

It is well known, that we do not have uniqueness of weak solution in general. In order to get uniqueness, we introduce entropy solution following the notion of entropy introduced by S.N. Kruzkhov for conservation law [KA], [C1], [C2] and their references

DEFINITION 2.2 *(Entropy solution of (CP) (b, F, f, u_0)).* — *Let u_0 and f verifying $(H1)$. An entropy solution u of (CP) (b, F, f, u_0) is a weak solution of (CP) such that:*

$$\int_Q H_0(u - s)\{\nabla b(u)\nabla \xi - (F(u) - F(s))\nabla \xi - (u - s)\xi_t - f\xi\} \ dx \ dt$$

$$- \int_{\mathbb{R}^N} (u_0 - s)^+ \xi(0) \ dx \leqslant 0$$

$$(2.4)$$

and

$$\int_Q H_0(s - u)\{\nabla b(u)\nabla \xi - (F(u) - F(s))\nabla \xi - (u - s)\xi_t - f\xi\} \ dx \ dt$$

$$+ \int_{\mathbb{R}^N} (s - u_0)^+ \xi(0) \ dx \geqslant 0$$

$$(2.5)$$

for any $s \in \mathbb{R}$ and $\xi \in \mathcal{D}(Q)$, $\xi \geqslant 0$.

Let recall at first some important results from [MT3]. The first one is the technical Lemma (cf Theorem 17 in [MT3]), it is the main tool in the proof of the uniqueness of the entropy solution of (CP), in fact in this lemma we can see the origin of conditions on the continuity modulus of b and F.

The second result is existence and uniqueness of the entropy solution of (CP) (cf Theorem 14, Theorem 15 and Theorem 16 in [MT3]).

On the other hand, if $u_0 \in L^1(\mathbb{R}^N)$ and $f \in L^1(Q)$, by the semi-group theory, one knows that we have the existence and the uniqueness of the mild solution of (CP) which is an entropy solution, moreover it depends continuously in the data u_0, f, b and F (cf Theorem 9, Theorem 10 in [MT3]) (cf also [C1], [BCP]and their references).

THEOREM 2.3 *(The technical Lemma). — Let ω be non negative function*

$$
\begin{cases}
\lim_{\epsilon \to 0} \dfrac{\omega(\epsilon)}{\epsilon} = +\infty; \quad \lim_{\epsilon \to 0} \omega(\epsilon) = 0, \quad \text{there is } l \text{ such that } 2 < l \leqslant N \\
\liminf_{\epsilon \to 0} \dfrac{\omega^l(\epsilon)}{\epsilon^{l-1}} < +\infty \quad \text{if } N > 2, \text{ and } \quad \liminf_{\epsilon \to 0} \dfrac{\omega^2(\epsilon)}{\epsilon} = 0 \quad \text{if } N = 2,
\end{cases}
\tag{H2}
$$

Let $h \in L^1_{loc}(Q)$ such that $h^+ = max(h,0) \in L^1(Q)$. Let $W_0 \in L^1(\mathbb{R}^N)$, $W \in L^1_{loc}(Q)$, $W \geqslant 0$, $W \in L^\infty(Q)$, and let $\lambda > 0$ so that:

$$
\begin{aligned}
&\int \int_Q W \frac{\partial \xi}{\partial t} + \sum_{j=1}^{l} (W + \epsilon) \frac{\omega(\epsilon)}{\epsilon} [|\frac{\partial^2 \xi}{\partial x_j \partial x_j}| + |\frac{\partial \xi}{\partial x_j}|] \\
&+ \sum_{j=l+1}^{N} \lambda W (|\frac{\partial^2 \xi}{\partial x_j \partial x_j}| + |\frac{\partial \xi}{\partial x_j}|) + h \, \xi \, dx \, dt \geqslant 0,
\end{aligned}
\tag{2.6}
$$

for any $\epsilon > 0$ et $\xi \in \mathcal{D}(Q)$, $\xi \geqslant 0$.

Set also that $(W(t,.) - W_0)^+ \longrightarrow 0$ in $L^1_{loc}(\mathbb{R}^N)$, when $t \longrightarrow 0$ essentially; then $h \in L^1(Q)$, $W \in L^\infty(0,T,L^1(\mathbb{R}^N))$ and

$$
\int_{\mathbb{R}^N} W(\tau,x) \, dx \leqslant \int_{\mathbb{R}^N} W_0(x) \, dx + \int_{Q_\tau} h \, dx \, dt, \quad \text{for a.e. } \tau \in (0,T),
$$

with $Q_\tau =]0,\tau[\times \mathbb{R}^N$.

Remark 1. —

1. Theorem 2.3 is a particular case of Theorem 17 in [MT3] with $\omega_j = \eta_j = \omega$ for $j = 1...l$.

2. Let g be a continuous function, we denote by ω a non negative continuous function which is non-decreasing such that $\omega(0) = 0$ and $|g(x) - g(y)| \leqslant \omega(x - y)$.

   In general we have $\lim\limits_{\epsilon \to 0} \dfrac{\omega(\epsilon)}{\epsilon} = +\infty$. In the particular case where g is Lipschitz continuous, we have $\dfrac{\omega(\epsilon)}{\epsilon} \leqslant \lambda$ where λ is a positive constant.

THEOREM 2.4 *(Existence theorem) (cf. Theorem 14 in [MT3]).* — *For all u_0 and f verifying $(H1)$. There exists an entropy solution of the problem $(CP)(b, F, f, u_o)$.*

Remark 2. — Proceeding if necessary to coordinate change, we can suppose that:

- for $i = 1...l$ the functions F_i are continuous but not Lipschitz continuous.

- For $i = l + 1...N$ the functions F_i are Lipschitz continuous, we denote by λ a constant such that $|F_i(x) - F_i(y)| \leqslant \lambda |x - y|$.

- We may suppose that $l > 2$, we replace if necessary λ by $\tilde{\lambda} = \lambda + \epsilon^{\frac{-1}{N+1}}$.

- In the case where the function b is not Lipschitz continuous, we may suppose $l = N$.

- For $i = 1...l$ we denote by ω the continuity modulus of F_i and b .

THEOREM 2.5 *(Comparison principle).* — *Let (u_{01}, f_1), (u_{02}, f_2) verifying $(H1)$, and let u_1, u_2 be entropy solution with respect to (CP) $(b, F, f_1, u_{01}))$, and (CP) $(b, F, f_2, u_{02}))$. If $(H2)$ holds, then:*

$$\int_{\mathbb{R}^N} (u_1(t) - u_2(t))^+ \, dx \leqslant \int_{\mathbb{R}^N} (u_{01} - u_{02})^+ \, dx + \int_{Q_t} \nu(f_1 - f_2) \, dx \, dt. \quad (2.7)$$

for $\nu \in H(u_1 - u_2)$ a.e so that:

$$\|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^N)} \leqslant \|u_{01} - u_{02}\|_{L^1(\mathbb{R}^N)} + \int_0^t \|f_1 - f_2\|_{L^1(\mathbb{R}^N)} \, ds;$$

which gives uniqueness of entropy solution.

3. The main result

Our main result is:

THEOREM 3.1. — *Let be* $(F_n, b_n, f_n, u_{0,n}) \in \mathcal{C}(\mathbb{R}) \times \mathcal{C}_m(\mathbb{R}) \times L^1_{loc}(Q) \times L^\infty(\mathbb{R})$, *given functions such that:*

$$\begin{cases} F_{in} \to F_i \quad for \quad i=1...N \quad b_n \to b \quad in \quad \mathcal{C}(\mathbb{R}) \\[2ex] (f_n, u_{0,n}) \quad converges \ to \ (f, u_0) \ \ in \ L^1_{loc}(Q) \times L^1_{loc}(\mathbb{R}^N) \\[2ex] \|u_{0,n}\|_{L^\infty(\mathbb{R}^N)} + \int_0^T \|f_n(t)\|_{L^\infty(\mathbb{R}^N)} \ dt \leqslant C, \end{cases} \qquad (3.1)$$

and we suppose that for n large enough we have

$$\begin{cases} |F_{in}(x) - F_{in}(y)| \leqslant \omega(x-y) \quad for \ i = 1...l, \\ |F_{in}(x) - F_{in}(y)| \leqslant \lambda|x-y| \quad for \ i = l+1...N, \\ |b_n(x) - b_n(y)| \leqslant \omega(x-y) \\ and \quad (H2) \quad hold. \end{cases} \qquad (H3)$$

then u_n *converges to* u *in* $\mathcal{C}([0,T], L^1_{loc}(\mathbb{R}^N))$, *where* u_n, u *are respectively the entropy solutions of* $(CP)(F_n, b_n, f_n, u_{0,n})$ *and* $(CP)(F, b, f, u_0)$.

Proof. — Let X be the set of $(\hat{f}, \hat{u}_0) \in L^1(Q) \times L^1(\mathbb{R}^N)$, such that $(H1)$ is satisfied and

$$\|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} + \int_0^T \|\hat{f}(t)\|_{L^\infty(\mathbb{R}^N)} \ dt \leqslant C$$

Let K be a compact in $\mathbb{R}^N$, and $\delta \in \mathbb{R}^+$, there exist $(\hat{f}, \hat{u}_0) \in X$ such that

$$\int_K |u_0 - \hat{u}_0| + \int_0^T \int_K |f - \hat{f}| \ dx \ ds < \frac{\delta}{3}; \qquad (3.2)$$

furthermore, since $(\hat{f}, \hat{u}_0)$ is in X, by the semi group theory in L^1 the Cauchy problem $(CP)(F, b, \hat{f}, \hat{u}_0)$, has a unique mild solution $\hat{u}$ (cf [BCP], [MT2], [MT3]), which depends continuously on the data $(F, b, \hat{f}, \hat{u}_0)$, and one has

$$\sup_{t \in [0,T]} \int_K |u(t) - \hat{u}(t)| \ dx < \frac{\delta}{3}, \qquad (3.3)$$

where u is the unique entropy solution of the Cauchy problem $(CP)(F, b, f, u_0)$.

Set $\hat{u}_n$ the entropy solution of the Cauchy problem $(CP)(F_n, b_n, \hat{f}, \hat{u}_0)$. by the same arguments as before one has

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \int_{\mathbb{R}^N} |\hat{u}_n(t) - \hat{u}(t)|\ dx = 0. \tag{3.4}$$

Moreover

$$\int_K |u_n(t) - u(t)|\ dx \leqslant \int_K |u_n(t) - \hat{u}_n(t)|\ dx$$

$$+ \int_K |\hat{u}_n(t) - \hat{u}(t)|\ dx + \int_K |\hat{u}(t) - u(t)|\ dx. \tag{3.5}$$

By inequality (3.3), (3.4), and (3.5) a sufficient condition to get the result of theorem 7 which can be written as:

$$\lim_{n \to \infty} \sup_{t \in [0,T[} \int_K |u_n(t) - u(t)|\ dx = 0,$$

is to prove that there exist $n_0 \in \mathbb{N}$, such that for $n \geqslant n_0$

$$\sup_{t \in [0,T]} \int_K |u_n(t) - \hat{u}_n(t)|\ dx < \frac{\delta}{3}.$$

By Theorem 16 in [MT3] for all $\xi \in \mathcal{D}(\mathbb{R})$, $\xi \geqslant 0$ et $0 < t \leqslant T$ the "kato" inequality applied to $(CP)(F_n, b_n, f_n, u_{0,n})$ and $(CP)(F_n, b_n, \hat{f}_n, \hat{u}_{0,n})$ leads to:

$$\int_{\mathbb{R}^N} |u_n(t) - \hat{u}_n(t)|\ \xi\ dx \leqslant \int_{\mathbb{R}^N} |u_{0,n} - \hat{u}_{0,n}|\ \xi\ dx$$

$$+ \sum_{j=1}^{N} \int_Q |b_n(u_n) - b_n(\hat{u}_n)| |\frac{\partial^2 \xi}{\partial x_j \partial x_j}|$$

$$+ \sum_{j=1}^{N} \int_Q |F_{in}(u_n) - F_{in}(\hat{u}_n)| |\frac{\partial \xi}{\partial x_j}| \tag{3.6}$$

$$+ \int_{\mathbb{R}} |f_n - \hat{f}|\ \xi\ dx\ ds,$$

In order to use Theorem 2.3 we set

$$W_n(t) = |u_n(t) - \hat{u}_n(t)|; \quad W_{0,n} = |u_{0,n} - \hat{u}_{0,n}|; \quad h_n = |f_n - \hat{f}|.$$

By using the two first inequalities in $(H3)$, one obtain for $n \geqslant n_0$

$$\int_{\mathbb{R}^N} W_n(t) \, \xi \, dx \leqslant \int_{\mathbb{R}^N} W_{0,n} \, \xi \, dx + \int_{\mathbb{R}} h_n \, \xi \, dx \, ds$$

$$+2 \sum_{j=1}^{l} \int_0^t \int_{\mathbb{R}^N} (W_n + \epsilon) \frac{\omega(\epsilon)}{\epsilon} (|\frac{\partial^2 \xi}{\partial x_j \partial x_j}| + |\frac{\partial \xi}{\partial x_j}|) \, dx \, ds$$

$$+2 \sum_{j=l+1}^{N} \int_0^t \int_{\mathbb{R}^N} \lambda W_n (|\frac{\partial^2 \xi}{\partial x_j \partial x_j}| + |\frac{\partial \xi}{\partial x_j}|) \, dx \, ds, \tag{3.7}$$

with $W_{0,n} \to 0$ dans $L^1_{loc}(\mathbb{R}^N)$, $h_n \to 0$ dans $L^1_{loc}(Q)$, and $\|W_n\|_{L^\infty(Q)} \leqslant C$,

By the proof of Theorem 16 of [MT3] one has

$$\frac{1}{2} \sup_{\tau \in (0,T)} ess \int_{\mathbb{R}^N} W_n(\tau) \xi \, dx$$

$$\leqslant \int_{\mathbb{R}^N} W_{0,n} \xi \, dx + \int_0^\tau \int_{\mathbb{R}^N} h_n \xi \, dt \, dx \, + \Lambda(\epsilon) \tag{3.8}$$

where

1) ξ is a particular test function which tends to 1 when ϵ tends to 0.

2) $0 \leqslant \Lambda(\epsilon) \leqslant M \dfrac{\omega^l(\epsilon)}{\epsilon^{1-l} \eta^{l-2}}$, M is a constant.

3) η is a parameter assigned to go to $+\infty$.

From (3.8) one has that for n large enough:

$$\frac{1}{2} \sup_{\tau \in (0,T)} ess \int_K W(\tau) \, \xi \, dx \leqslant \frac{\delta}{3} + \frac{\delta}{3} + \Lambda(\epsilon) \tag{3.9}$$

Then

If $N > 2$ we let η goes to $+\infty$ and ϵ goes to 0 to get the result.

If $N = 2$ then $l = 2$ and $\Lambda(\epsilon) \leqslant M \dfrac{\omega^2(\epsilon)}{\epsilon}$ so we let ϵ going to 0 to get the result.

If $N = 1$ there is no condition on ω.

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