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Global well-posedness for the primitive equations with less regular initial data[*]

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Abstract. — This paper is devoted to the study of the lifespan of the solutions of the primitive equations for less regular initial data. We interpolate the globall well-posedness results for small initial data in $\dot{H}^{\frac{1}{2}}$ given by the Fujita-Kato theorem, and the result from [6] which gives global well-posedness if the Rossby parameter ε is small enough, and for regular initial data (oscillating part in $\dot{H}^{\frac{1}{2}} \cap \dot{H}^1$ and quasigeostrophic part in H^1).

Résumé. — Cet article est consacré à l'étude du temps d'existence des solutions du système des équations primitives pour des données moins régulières. On interpole les résultats d'existence globale à données $\dot{H}^{\frac{1}{2}}$ petites fournis par le théorème de Fujita-Kato, et le résultat de [6] qui donne l'existence globale si le paramètre de Rossby ε est suffisamment petit, et pour des données plus régulières (partie oscillante initiale dans $\dot{H}^{\frac{1}{2}} \cap \dot{H}^1$ et partie quasigéostrophique initiale dans H^1).

1. Introduction

1.1. The primitive equations

The primitive system writes :

$$\begin{cases} \partial_t U_\varepsilon + U_\varepsilon.\nabla U_\varepsilon - LU_\varepsilon + \dfrac{1}{\varepsilon}\mathcal{A}U_\varepsilon = \dfrac{1}{\varepsilon}(-\nabla\Phi_\varepsilon, 0) \\ div\ v_\varepsilon = 0 \\ U_{\varepsilon/t=0} = U_{0,\varepsilon}. \end{cases} \qquad (PE_\varepsilon)$$

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The unknowns are U_ε and Φ_ε. We denote by U_ε a pair $(v_\varepsilon, \theta_\varepsilon)$ where v_ε is a vector field on $\mathbb{R}^3$ (three dimensional velocity), θ_ε a scalar function (the density fluctuation : in the case of the atmosphere it depends on the scalar (potential) temperature and in the case of the ocean it depends on the temperature and the salinity), and Φ_ε the pressure, all of them depending on (t, x). The operator L is defined by

$$LU_\varepsilon \overset{\text{def}}{=} (\nu\Delta v_\varepsilon, \nu'\Delta\theta_\varepsilon),$$

We define :

$$U_\varepsilon.\nabla U_\varepsilon = v_\varepsilon.\nabla U_\varepsilon = \sum_{i=1}^{3} v_\varepsilon^i.\partial_i U_\varepsilon,$$

and the matrix $\mathcal{A}$ by :

$$\mathcal{A} \overset{\text{def}}{=} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & F^{-1} \\ 0 & 0 & -F^{-1} & 0 \end{pmatrix}.$$

The small parameter ε is called here the Rossby number and F is called the Froude number. They are related to the physical Rossby and Froude numbers by the following relations :

$$Ro = \varepsilon, \quad Fr = \varepsilon F.$$

The smaller is ε, the more important are the Coriolis force (induced by the rotation of the earth around its axis) and the vertical stratification of the density.

We refer for example to [6] for the physical meaning of these terms and for a list of physical references.

Definition 1.1. — *If s is a real number, the homogenous (resp. inhomogenous) Sobolev space of order s, which we will denote by $\dot{H}^s$ (resp. H^s), is defined as the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^3)$ whose Fourier transform $\hat{u}$ is locally integrable and has the following property :*

$$\|u\|_{\dot{H}^s}^2 \overset{def}{=} \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty$$

$$(\textit{resp.} \quad \|u\|_{H^s}^2 \overset{def}{=} \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi < \infty).$$

Although the primitive equations have no scaling anymore, we can easily adapt the proofs of the Leray and Fujita-Kato theorems (thanks to the skewsymmetry of matrix $\mathcal{A}$ and the fact that both of these theorems are proved using mainly inner products and energy estimates) to get the following results :

THEOREM 1.2. — *(Leray, 1934, [15]) if the initial data $U_0 \in L^2(\mathbb{R}^3)$, then there exists for all $\varepsilon > 0$ a Leray solution of the system (PE_ε), U_ε, globally defined in time, belonging to $L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ and satisfying the following energy inequality (let $\nu_0 = \min(\nu, \nu') > 0$) :*

$$\forall t \in \mathbb{R}_+, \quad \|U_\varepsilon(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\nu_0 \int_0^t \|\nabla U_\varepsilon(t)\|_{L^2(\mathbb{R}^3)}^2 dt \leqslant \|U_0\|_{L^2(\mathbb{R}^3)}^2.$$

We refer to [5] where we studied the limit of Leray solutions when ε, the Rossby number, goes to zero and introduced the following notations and results in the case of weak solutions : the potential vorticity is defined by

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \partial_1 v_\varepsilon^2 - \partial_2 v_\varepsilon^1 - F\partial_3\theta_\varepsilon.$$

Then from this, we define the orthogonal decomposition of U_ε into its quasigeostrophic part, and its oscillating part :

$$U_{\varepsilon,QG} \stackrel{\text{def}}{=} \begin{pmatrix} -\partial_2 {\Delta_F}^{-1}\Omega_\varepsilon \\ \partial_1 {\Delta_F}^{-1}\Omega_\varepsilon \\ 0 \\ -F\partial_3 {\Delta_F}^{-1}\Omega_\varepsilon \end{pmatrix},$$

with $\Delta_F \stackrel{\text{def}}{=} \partial_1^2 + \partial_2^2 + F^2\partial_3^3$, and :

$$U_{\varepsilon,osc} \stackrel{\text{def}}{=} U_\varepsilon - U_{\varepsilon,QG} = \begin{pmatrix} v_\varepsilon^1 + \partial_2 {\Delta_F}^{-1}\Omega_\varepsilon \\ v_\varepsilon^2 - \partial_1 {\Delta_F}^{-1}\Omega_\varepsilon \\ v_\varepsilon^3 \\ \theta_\varepsilon + F\partial_3 {\Delta_F}^{-1}\Omega_\varepsilon \end{pmatrix}.$$

We have seen in [5] that this decomposition is an orthogonal decomposition and we denoted by $\mathcal{P}$ the orthogonal projector onto the potential vorticity free vector fields (which is built the same way as the orthogonal projector $\mathbb{P}$ on the divergence free vector fields, also called the Leray projector) and $\mathcal{Q} = Id - \mathcal{P}$ the orthogonal projector on the quasigeostrophic vectorfiels. Both of them are homogeneous pseudo differential operators of degree zero.

In [5] we studied the convergence, when ε goes to zero, of the weak Leray solutions towards the quasigeostrophic model (see (1.1) below).

When the initial data is more regular and even if there is no scale invariance for this system we can easily adapt the Fujita and Kato theorem (1964) :

Theorem 1. — *(Fujita and Kato, 1964, [12]) If* $U_0 \in \dot{H}^{\frac{1}{2}}$ *there exist a unique maximal time* $T_\varepsilon^* > 0$*, and a unique solution*

$$U_\varepsilon \in \mathcal{C}([0, T_\varepsilon^*[, \dot{H}^{\frac{1}{2}}) \cap L^2_{loc}([0, T_\varepsilon^*[, \dot{H}^{\frac{3}{2}}).$$

Moreover, if T_ε^* *is finite, then we have*

$$\int_0^{T_\varepsilon^*} \|U_\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} dt = +\infty.$$

Finally there exists a constant c *such that if* $\|U_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \leqslant c\nu_0$ *then* $T_\varepsilon^* = +\infty$*.*

Contrary to the Leray solutions, the solutions are unique but we do not know whether they are global in general. The Fujita-Kato theorem also works on the quasigeostrophic system, and again, it does not say whether the unique solution is global if we do not have a small initial data.

Both of these results are general results directly adaptated from the Navier-Stokes case, without using the special structure of the primitive equations. As we have seen in [5], [6], and [7], when the Rossby number ε goes to zero, the system is stabilized as its solutions go to the solutions of the quasigeostrophic model (We refer to [5], [6], and [7] for the study, in the case of the whole space, and for $F \neq 1$, of the convergence, as ε goes to zero, of the primitive equations solutions to the quasigeostrophic model.), which is closer to the two-dimensionnal Navier-Stokes system than to the three-dimensionnal one :

Theorem 2. — *[6] If* $U_{0,QG} \in H^1(\mathbb{R}^3)$ *then the quasigeostrophic system*

$$\begin{cases} \partial_t U_{QG} - \Gamma U_{QG} + \mathcal{Q}(U_{QG}.\nabla U_{QG}) = 0 \\ U_{QG_{/t=0}} = U_{0,QG}, \end{cases} \tag{1.1}$$

has a unique global solution in $L^\infty(\mathbb{R}_+, H^1) \cap L^2(\mathbb{R}_+, \dot{H}^1 \cap \dot{H}^2)$*, with the following energy estimate for all* $s \in [0,1]$ *:*

$$\forall t \in \mathbb{R}_+, \|\widetilde{U_{QG}}(t)\|^2_{\dot{H}^s} + 2c\nu_0 \int_0^t \|\widetilde{U_{QG}}(t')\|^2_{\dot{H}^{s+1}} dt' \leqslant C(U_{0,QG}), \tag{1.2}$$

where $\nu_0 = \min(\nu, \nu')$*.*

The convergence theorem is the following one :

THEOREM 3. — *([6]) Assume that $U_0 \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $U_{0,QG} \in L^2(\mathbb{R}^3)$. Let us define for $s \in \mathbb{R}$, $\dot{E}^s \stackrel{def}{=} L^\infty(\mathbb{R}_+, \dot{H}^s) \cap L^2(\mathbb{R}_+, \dot{H}^{s+1})$ and let W_ε be a solution of the following linear system :*

$$\begin{cases} \partial_t W_\varepsilon - LW_\varepsilon + \frac{1}{\varepsilon}\mathbb{P}\mathcal{A}W_\varepsilon = -G \\ W_{\varepsilon/t=0} = U_{0,osc} = \mathcal{P}(U_0) \end{cases} \tag{1.3}$$

$$\text{with} \quad G \stackrel{def}{=} \mathbb{P}\mathcal{P}(\widetilde{U_{QG}}.\nabla\widetilde{U_{QG}}) - F(\nu-\nu')\Delta\Delta_F^{-2} \begin{pmatrix} -F\partial_2\partial_3^2 \\ F\partial_1\partial_3^2 \\ 0 \\ (\partial_1^2+\partial_2^2)\partial_3 \end{pmatrix} \widetilde{\Omega_{QG}}. \tag{1.4}$$

Then we have the following results :

- *W_ε exists globally and is unique in the space $\dot{E}^s$ for every $s \in [\frac{1}{2}, 1]$.*
- *Moreover $\|W_\varepsilon\|_{L^2(\mathbb{R}_+, L^\infty)} \to 0$ as $\varepsilon \to 0$.*
- *If we denote by $\gamma_\varepsilon \stackrel{def}{=} U_\varepsilon - \widetilde{U_{QG}} - W_\varepsilon$, then if ε is small enough, $\gamma_\varepsilon \in \dot{E}^s$ and converges to zero in this space $\dot{E}^s$ for every $s \in [\frac{1}{2}, 1]$.*
- *If ε is small enough U_ε is defined for all time in E^s.*

The aim of this paper is to get global existence results on the solutions of the primitive equations but with less initial regularity. The $\dot{H}^{\frac{1}{2}}$ regularity being the minimal regularity as we want to apply at least the Fujita and Kato theorem, we will require in this paper $U_{0,QG} \in H^{\frac{1}{2}+\eta}$, $U_{0,osc} \in \dot{H}^{\frac{1}{2}}$ and we will interpolate between theorem 1 and theorem 3, using the arguments given by [14]. The key point is that we cut the initial data into two parts : the first part being regular enough to apply theorem 3, and the second one being $\dot{H}^{\frac{1}{2}}$ with small initial data, in order to apply theorem 1. We get the following result :

THEOREM 4. — *If the initial data $U_0 = U_{0,QG} + U_{0,osc}$ with $U_{0,QG} \in H^{\frac{1}{2}+\eta}$, $U_{0,osc} \in \dot{H}^{\frac{1}{2}}$ then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \leqslant \varepsilon_0$, (PE_ε) has a unique global solution $U_\varepsilon \in L^\infty(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}_+, \dot{H}^{\frac{3}{2}})$.*

This paper is devoted to the proof of theorem 4 and the structure will be the following : first we will use truncation in order to cut the initial data into two parts. At this point we show how important is the interpolation argument from [14] to get adaptated energy estimates on the quasigeostrophic system. We then prove this interpolation argument and manage to apply theorem 3. Finally we are able to adapt theorem 1 with small initial data, which concludes the proof.

2. Proof of Theorem 4

2.1. Frequency truncation of the initial data

We have seen that the above theorems give two different results concerning the lifespan of the strong solutions of the primitive equations :

- Theorem 1 requires $U_{0,QG} \in \dot{H}^{\frac{1}{2}}$, $U_{0,osc} \in \dot{H}^{\frac{1}{2}}$, and gives local existence of strong solutions, with global lifespan and energy if the initial data are small enough ($\|U_0\|_{\dot{H}^{\frac{1}{2}}} \leqslant c\nu_0$).

- Theorem 3 requires $U_{0,QG} \in H^1$, $U_{0,osc} \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^1$, and gives global strong solutions (and energy and convergence) when the Rossby number ε is small enough.

So the idea is to cut our initial data into two parts : on the first one, which is regular (H^1) and whose norm is large, we will be able to apply Theorem 3, and on the second one, which is $\dot{H}^{\frac{1}{2}}$ with a small norm we will use Theorem 1. We then decompose the initial data in the following way (χ is a $\mathcal{C}^\infty$ truncation function : $\chi(x) \equiv 1$ if $x \in [-1, 1]$ and $\chi(x) \equiv 0$ if $|x| > \frac{3}{2}$ for example) :

$$U_0 = U_{0,QG} + U_{0,osc} = \left(\chi(\frac{|D|}{\lambda})U_{0,QG} + U_{0,osc}\right) + (1 - \chi(\frac{|D|}{\lambda}))U_{0,QG}.$$

So let us begin with the use of Theorem 3 on the first part (low frequencies for the quasigeostrophic part).

2.2. Study of the low frequencies

2.2.1. Global well-posedness

Let us define U_ε^λ solution of the primitive equations :

$$\begin{cases} \partial_t U_\varepsilon^\lambda + U_\varepsilon^\lambda . \nabla U_\varepsilon^\lambda - LU_\varepsilon^\lambda + \frac{1}{\varepsilon}\mathcal{A}U_\varepsilon^\lambda = \frac{1}{\varepsilon}(-\nabla\Phi_\varepsilon^\lambda, 0) \\ \operatorname{div} v_\varepsilon^\lambda = 0 \\ U_{\varepsilon/t=0}^\lambda = \chi(\frac{|D|}{\lambda})U_{0,QG} + U_{0,osc}. \end{cases}$$

Theorem 3 gives then $\varepsilon_0 = \varepsilon_0(\lambda, ...) > 0$ such that $\forall \varepsilon \leqslant \varepsilon_0$, the unique solution U_ε^λ globally exists. Precisely we define U_{QG}^λ and W_ε^λ, solutions of the following systems (we refer to [6] for details concerning the convergence of the strong solutions and its proof) :

$$\begin{cases} \partial_t U_{QG}^\lambda - \Gamma U_{QG}^\lambda + \mathcal{Q}(U_{QG}^\lambda . \nabla U_{QG}^\lambda) = 0 \\ \\ U_{QG}^\lambda{}_{/t=0} = \chi(\frac{|D|}{\lambda}) U_{0,QG}, \end{cases} \tag{2.5}$$

and

$$\begin{cases} \partial_t W_\varepsilon^\lambda - L W_\varepsilon^\lambda + \frac{1}{\varepsilon} \mathbb{P} \mathcal{A} W_\varepsilon^\lambda = -G^\lambda \\ \\ W_\varepsilon^\lambda{}_{/t=0} = U_{0,osc}, \end{cases} \tag{2.6}$$

where $G^\lambda = G^{\lambda,b} + G^{\lambda,l}$,

$$\text{with} \quad G^{\lambda,b} = \mathbb{P}\mathcal{P}(U_{QG}^\lambda . \nabla U_{QG}^\lambda),$$

$$\text{and} \quad G^{\lambda,l} = -F(\nu - \nu')\Delta\Delta_F^{-2} \begin{pmatrix} -F\partial_2\partial_3^2 \\ F\partial_1\partial_3^2 \\ 0 \\ (\partial_1^2 + \partial_2^2)\partial_3 \end{pmatrix} \Omega_{QG}^\lambda. \tag{2.7}$$

Then Theorem 3 gives $\varepsilon_0 = \varepsilon_0(\lambda, ...) > 0$ such that $\forall \varepsilon \leqslant \varepsilon_0$, the difference of the solutions of the primitive and quasigeostrophic systems, to whom we substract rapid oscillations, $\gamma_\varepsilon = U_\varepsilon^\lambda - U_{QG}^\lambda - W_\varepsilon^\lambda$, globally exists in $\dot{E}^{\frac{1}{2}}$ and goes to zero in this space.

The aim is to get estimates in $\dot{E}^{\frac{1}{2}}$ of and then use it in the system satisfied by $V_\varepsilon^\lambda = U_\varepsilon - U_\varepsilon^\lambda$ so we will outline the adaptation of the estimates given in [6] in the proof of Theorem 3.

2.2.2. Estimates for the limit system

First we have to estimate U_{QG}^λ in $\dot{H}^s$: let us recall that in the proof, the fact that the initial data is in $\dot{H}^{\frac{1}{2}}$ allows us to use the Fujita and Kato theorem, which gives a local lifespan $[0, T^{*,\lambda}[$. The fact that, in addition, $\chi(\frac{|D|}{\lambda}) U_{0,QG} \in \dot{H}^1$ allows us to use the regularity propagation theorem that provides enough regularity to the potential vorticity which is in $\mathcal{C}([0, T^{*,\lambda}[, L^2) \cap L^2_{loc}([0, T^{*,\lambda}[, \dot{H}^1)$, and it is sufficient for us to define the scalar product in L^2 of Ω_{QG}^λ by the equation :

$$\partial_t \Omega_{QG}^\lambda - \Gamma \Omega_{QG}^\lambda + U_{QG}^\lambda . \nabla \Omega_{QG}^\lambda = 0,$$

and for all $t < T^*$ ($\nu_0 = \min(\nu, \nu') > 0$) :

$$\|\Omega_{QG}(t)^\lambda\|_{L^2}^2 + 2c\nu_0 \int_0^t \|\nabla \Omega_{QG}^\lambda(\tau)\|_{L^2}^2 d\tau = \|\Omega_{QG}^\lambda(0)\|_{L^2}^2 \leqslant C' \|\chi(\frac{|D|}{\lambda}) U_{0,QG}\|_{\dot{H}^1}^2.$$

Finally, the fact that $\chi(\frac{|D|}{\lambda}) U_{0,QG}$ is in L^2 allows us to get the fact that it is a global weak solution, and the following Leray estimate $\forall t \geqslant 0$, ($\nu_0 = \min(\nu, \nu') > 0$) :

$$\|U_{QG}^\lambda(t)\|_{L^2}^2 + \nu_0 \int_0^t \|\nabla U_{QG}^\lambda(\tau)\|_{L^2}^2 d\tau \leqslant \|\chi(\frac{|D|}{\lambda}) U_{0,QG}\|_{L^2}^2.$$

Then from this, we easily contradict the usual blow-up criterion, and that implies that there is a unique, global solution and for every $s \in [0,1]$:

$$\begin{aligned} \forall t \in \mathbb{R}_+, \|U_{QG}^\lambda(t)\|_{\dot{H}^s}^2 + \nu_0 \int_0^t \|U_{QG}^\lambda(\tau)\|_{\dot{H}^{s+1}}^2 d\tau &\leqslant C \|\chi(\frac{|D|}{\lambda}) U_{0,QG}\|_{H^1}^2 \\ &\leqslant C \lambda^{\frac{1}{2}-\eta} \|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}. \end{aligned} \tag{2.8}$$

2.2.3. Estimates for W_ε^λ and U_ε^λ

We refer to [6] (section 3.2) for the following estimates :

$$\begin{aligned} &\|W_\varepsilon^\lambda(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu_0 \int_0^t \|\nabla W_\varepsilon^\lambda(t')\|_{\dot{H}^{\frac{1}{2}}}^2 e^{\int_{t'}^t \|G^{\lambda,b}(\tau)\|_{\dot{H}^{\frac{1}{2}}} d\tau} dt' \\ &\leqslant \|U_{0,osc}\|_{\dot{H}^{\frac{1}{2}}}^2 e^{\int_0^t \|G^{\lambda,b}(\tau)\|_{\dot{H}^{\frac{1}{2}}} d\tau} + \int_0^t (\|G^{\lambda,b}\|_{\dot{H}^{\frac{1}{2}}} + \frac{1}{\nu_0} \|G^l\|_{\dot{H}^{-\frac{1}{2}}}^2) e^{\int_{t'}^t \|G^{\lambda,b}(\tau)\|_{\dot{H}^{\frac{1}{2}}} d\tau} dt', \end{aligned}$$

and there is no change, in lemma 3.2 from [6], for the estimate on $G^{\lambda,l}$:

$$\int_0^\infty \|G^l\|_{\dot{H}^{-\frac{1}{2}}}^2 dt \leqslant C \|U_{QG}^\lambda\|_{L^2 \dot{H}^{\frac{3}{2}}},$$

contrary to the estimate on $G^{\lambda,b}$, where we could use the argument from [6] :

$$\int_0^\infty \|G^b\|_{\dot{H}^s} dt \leqslant C \begin{cases} \|\widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+,\dot{H}^1)} \|\widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+,\dot{H}^2)} & \text{if} \quad s = \dfrac{1}{2} \\ \|\widetilde{U_{QG}}\|_{L^2(\mathbb{R}_+,\dot{H}^{s+1})}^2 & \text{if} \quad s \in]\dfrac{1}{2}, 1]. \end{cases}$$

But here $s = \frac{1}{2}$ and as we cannot afford to use much regularity, we prefer to do it differently, using product laws in Sobolev spaces :

$$\int_0^\infty \|G^b\|_{\dot{H}^{\frac{1}{2}}} dt \leqslant C \int_0^\infty \|U_{QG}^\lambda . \nabla U_{QG}^\lambda\|_{\dot{H}^{\frac{1}{2}}} \leqslant C \int_0^\infty \|U_{QG}^\lambda\|_{\dot{H}^{\frac{3}{2}-\eta}} . \|\nabla U_{QG}^\lambda\|_{\dot{H}^{\frac{1}{2}+\eta}}.$$

Then

$$\|G^b\|_{L^1\dot{H}^{\frac{1}{2}}} \leqslant \|U_{QG}^\lambda\|_{\dot{H}^{\frac{3}{2}-\eta}}.\|U_{QG}^\lambda\|_{\dot{H}^{\frac{3}{2}+\eta}}$$

So, using the estimate from the previous section, we obtain that :

$$\|W_\varepsilon^\lambda\|_{\dot{E}^{\frac{1}{2}}} \leqslant C(\|U_{0,osc}\|_{\dot{H}^{\frac{1}{2}}} + \lambda^{\frac{1}{2}-\eta}\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}).$$

Finally, as $\gamma_\varepsilon^\lambda$ goes to zero, in particular if ε is small enough, its norm in $\dot{E}^{\frac{1}{2}}$ is less than 1, so we obtain the estimate for U_ε^λ :

$$\|U_\varepsilon^\lambda\|_{\dot{E}^{\frac{1}{2}}} \leqslant C(\|U_{0,osc}\|_{\dot{H}^{\frac{1}{2}}} + \lambda^{\frac{1}{2}-\eta}\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}).$$

But in the following we will use the fact that λ goes to infinity in order to use the results of global well-posedness with small initial data. But in this case the previous estimates explode. We refer to the following section for an explaination of this problem.

2.3. Study of the high frequencies

In the previous section we used Theorem 3 to define U_ε^λ. The Fujita and Kato theorem gives the existence of U_ε in $\mathcal{C}([0,T_\varepsilon^*[,\dot{H}^{\frac{1}{2}})$. Then we can define the difference $V_\varepsilon^\lambda = U_\varepsilon - U_\varepsilon^\lambda$, which satisfies :

$$\begin{cases} \partial_t V_\varepsilon^\lambda + V_\varepsilon^\lambda.\nabla V_\varepsilon^\lambda + V_\varepsilon^\lambda.\nabla U_\varepsilon^\lambda + U_\varepsilon^\lambda.\nabla V_\varepsilon^\lambda - LV_\varepsilon^\lambda + \frac{1}{\varepsilon}\mathcal{A}V_\varepsilon^\lambda = \frac{1}{\varepsilon}(-\nabla\Phi_\varepsilon^\lambda, 0) \\ \\ {V_\varepsilon^\lambda}_{/t=0} = (1-\chi(\frac{|D|}{\lambda}))U_{0,QG}. \end{cases}$$

Then the classical schemes of the proofs of the Leray or Fujita-Kato theorems can be adapted on this system to prove the existence of weak solutions, strong solutions, and global lifespan when the initial data are small. We won't give details in this section, we will only write energy estimates (in the proof, such estimates are proved for regularized approximated solutions and obtained as limits).

First, the inner product in L^2, yields :

$$\frac{1}{2}\frac{d}{dt}\|V_\varepsilon^\lambda\|_{L^2}^2 + \nu_0\|\nabla V_\varepsilon^\lambda\|_{L^2}^2 \leqslant |(V_\varepsilon^\lambda.\nabla U_\varepsilon^\lambda|V_\varepsilon^\lambda)_{L^2}|.$$

Then, classical Sobolev injections ($\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \hookrightarrow L^3(\mathbb{R}^3)$, imply :

$$\frac{1}{2}\frac{d}{dt}\|V_\varepsilon^\lambda\|_{L^2}^2 + \nu_0\|\nabla V_\varepsilon^\lambda\|_{L^2}^2 \leqslant \frac{\nu_0}{2}\|\nabla V_\varepsilon^\lambda\|_{L^2}^2 + \frac{C}{\nu_0}\|V_\varepsilon^\lambda\|_{L^2}^2.\|\nabla U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}.$$

Then a Gronwall estimate implies :

$$\forall t \geqslant 0, \quad \|V_\varepsilon^\lambda(t)\|_{L^2}^2 + \nu_0 \int_0^t \|\nabla V_\varepsilon^\lambda(\tau)\|_{L^2}^2 d\tau$$

$$\leqslant \|(1-\chi(\frac{|D|}{\lambda}))U_{0,QG}\|_{L^2}^2 e^{\frac{2C}{\nu_0}\|\nabla U_\varepsilon^\lambda\|^2_{L^2\dot{H}^{\frac{1}{2}}}}. \tag{2.9}$$

And if we take the inner product in $\dot{H}^{\frac{1}{2}}$, we obtain :

$$\frac{1}{2}\frac{d}{dt}\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu_0 \|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 \leqslant |(V_\varepsilon^\lambda.\nabla V_\varepsilon^\lambda | V_\varepsilon^\lambda)_{\dot{H}^{\frac{1}{2}}}|$$
$$+|(V_\varepsilon^\lambda.\nabla U_\varepsilon^\lambda | V_\varepsilon^\lambda)_{\dot{H}^{\frac{1}{2}}}| + |(U_\varepsilon^\lambda.\nabla U_\varepsilon^\lambda | V_\varepsilon^\lambda)_{\dot{H}^{\frac{1}{2}}}|.$$

Using $|(f|g)_{\dot{H}^{\frac{1}{2}}}| \leqslant C\|f\|_{L^2}\|g\|_{\dot{H}^1}$, the fact that $L^3.L^6 \hookrightarrow L^2$, and linear interpolation arguments, we get :

$$\frac{1}{2}\frac{d}{dt}\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu_0 \|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 \leqslant C\|V_\varepsilon^\lambda\|_{\dot{H}^1}^2 \|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}$$
$$+C\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}\|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}\|\nabla U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}} + C\|U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}}\|\nabla U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^{\frac{1}{2}}\|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^{\frac{3}{2}}.$$

The classical inequality $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$ if $\frac{1}{p}+\frac{1}{q}=1$ gives :

$$\frac{1}{2}\frac{d}{dt}\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu_0 \|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 \leqslant C\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}\|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{\nu_0}{4}\|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 +$$
$$\frac{C}{\nu_0}\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2\|\nabla U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{\nu_0}{4}\|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{C}{\nu_0^3}\|U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2\|\nabla U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2.$$

And then,

$$\frac{d}{dt}\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu_0 \|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 \leqslant 2C\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}\|\nabla V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 +$$
$$\|V_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 \left(\frac{2C}{\nu_0}\|\nabla U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2 + \frac{2C}{\nu_0^3}\|U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2\|\nabla U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2\right).$$

A Gronwall estimate finally gives that for all $t \in [0, T_\varepsilon^*[$:

$$\|V_\varepsilon^\lambda(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_0^t (\nu_0 - 2C\|V_\varepsilon^\lambda(t')\|_{\dot{H}^{\frac{1}{2}}})\|\nabla V_\varepsilon^\lambda(t')\|_{\dot{H}^{\frac{1}{2}}}^2 e^{\int_{t'}^t g(\tau)d\tau} dt' \leqslant$$

$$\|(1-\chi(\frac{|D|}{\lambda}))U_{0,QG}\|_{L^2}^2 e^{\int_0^t g(\tau)d\tau}, \tag{2.10}$$

$$\text{with} \quad g(t) = \|\nabla U_\varepsilon^\lambda(t)\|_{\dot{H}^{\frac{1}{2}}}^2 (\frac{2C}{\nu_0} + \frac{2C}{\nu_0^3}\|U_\varepsilon^\lambda\|_{\dot{H}^{\frac{1}{2}}}^2).$$

In both cases, we obtain a majoration by $\|(1-\chi(\frac{|D|}{\lambda}))U_{0,QG}\|e^{\|U_\varepsilon^\lambda\|_{\dot{E}^{\frac{1}{2}}}}$, that is

$$\|(1-\chi(\frac{|D|}{\lambda}))U_{0,QG}\|e^{\lambda^{1-2\eta}C(\|U_{0,osc}\|_{\dot{H}^{\frac{1}{2}}},\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}})}$$

which does not go to zero when λ goes to infinity which is annoying as we want to adapt a theorem with small initial data. So as everything depends on the estimate of U_{QG}^λ, we will provide, in the following, an estimate whose right-hand member depends on $\|\chi(\frac{|D|}{\lambda}))U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}$ instead of $\|\chi(\frac{|D|}{\lambda}))U_{0,QG}\|_{H^1}$.

2.4. Real interpolation

The aim of this section is to prove the following result :

Lemma 2.1. — *Let $\eta > 0$, $U_{0,QG} \in H^1$, and U_{QG} the unique global solution (we refer to [6]) of the following quasigeostrophic system :*

$$\begin{cases} \partial_t U_{QG} - \Gamma U_{QG} + \mathcal{Q}(U_{QG}.\nabla U_{QG}) = 0 \\ U_{QG}{}_{/t=0} = U_{0,QG}. \end{cases}$$

There exists a constant C such that, for all $t \geqslant 0$,

$$\|U_{QG}(t)\|^2_{H^{\frac{1}{2}+\eta}} + \nu_0 \int_0^t \|\nabla U_{QG}(\tau)\|^2_{H^{\frac{1}{2}+\eta}} d\tau \leqslant C\|U_{0,QG}\|^{2+\frac{1}{\eta}}_{H^{\frac{1}{2}+\eta}}.$$

With this estimate, we will be able to use the results from the previous section on V_ε^λ and prove the global well-posedness as λ is large enough to ensure small initial data.

To prove this lemma, we will use the same Calderon method (see [4]) as Gallagher and Planchon in [14] : thanks to interpolation arguments we will be able to control the norm of a part of the initial data and make it as small as we want, so that we can use the Fujita-Kato theorem.

2.4.1. General interpolation results

In this section we will recall classical real interpolation definitions (we refer for example to [3] for a presentation) and present it in the same way as in [14] together with a useful lemma from this paper :

Definition 2.2. — *Let E_1 and E_2 two Banach spaces. The interpolated space $E = [E_1, E_2]_{\theta,q}$ with $\theta \in [0,1]$ and $q \geqslant 1$ is defined by :*

$$E = [E_1, E_2]_{\theta,q} = \{f \in E_1 + E_2 \quad \textit{such that} \quad \|f\|_E < \infty\},$$

with

$$\|f\|_E = \left(\sum_{j\in\mathbb{Z}} 2^{jq\theta} K(f,j)^q\right)^{\frac{1}{q}},$$

and

$$K(f,j) =^{def} \inf_{f_1+f_2=f} (\|f_1\|_{E_1} + 2^{-j}\|f_2\|_{E_2}) \quad (f_i \in E_i).$$

The following lemma concerns the case when $E_2 \hookrightarrow E \hookrightarrow E_1$. In the following, we will take $E_1 = H^{\frac{1}{2}}$, $E_2 = H^1$ and $E = H^{\frac{1}{2}+\eta}$.

Lemma 2.3. — *There exists a constant $C(\theta, q)$ such that for any integer $j_0 \geqslant 1$ and any function $f \in E$, the following equivalence holds :*

$$\left(\sum_{j\geqslant j_0} 2^{jq\theta} K(f,j)^q\right)^{\frac{1}{q}} \leqslant \|f\|_E \leqslant C(\theta,q) 2^{j_0} \left(\sum_{j\geqslant j_0} 2^{jq\theta} K(f,j)^q\right)^{\frac{1}{q}}.$$

We refer to [14] for the proof of this result (section 4.4).

2.4.2. Decomposition and small data

As said in section 2.2.2, we have different results on the quasigeostrophic system :

- $U_{0,QG} \in L^2$ leads to a global weak solution and a global energy estimate in L^2.
- $U_{0,QG} \in \dot{H}^{\frac{1}{2}}$ leads to a local unique strong solution, global if $\|U_{0,QG}\|_{\dot{H}^{\frac{1}{2}}} \leqslant c\nu_0$ (see Theorem 1) with global energy estimate in $\dot{H}^{\frac{1}{2}}$.
- $U_{0,QG} \in H^1$ leads to a global strong solutions together with a global energy estimate in H^1.

The aim is to decompose the initial data in $H^{\frac{1}{2}+\eta}=[H^{\frac{1}{2}},H^1]_{2\eta,2}$ and solve separatedly a quasigeostrophic system with small data in $H^{\frac{1}{2}}$ (therefore it is small in $\dot{H}^{\frac{1}{2}}$), and a modified quasigeostrophic system with initial data in H^1.

For every $j\in\mathbb{Z}$ let us decompose

$$U_{0,QG}=U_{0,QG}^{1,j}+U_{0,QG}^{2,j},\ \text{with}\quad U_{0,QG}^{1,j}\in E_1=H^{\frac{1}{2}},\quad U_{0,QG}^{2,j}\in E_2=H^1,$$

(actually, as $U_{0,QG}$ is H^1, so does $U_{0,QG}^{1,j}$), and by definition of $K(U_{0,QG},j)$ as an infimum :

$$\|U_{0,QG}^{1,j}\|_{H^{\frac{1}{2}}}+2^{-j}\|U_{0,QG}^{2,j}\|_{H^1}\leqslant\frac{3}{2}K(U_{0,QG},j).$$

As said earlier, we want to define the corresponding solutions, $U_{QG}^{1,j}(t)$ and $U_{QG}(t)^{2,j}$, respectedly given by the Fujita-Kato theorem, or Theorem 1.2. The problem is that we have no information on the smallness of $\|U_{0,QG}^{1,j}\|_{\dot{H}^{\frac{1}{2}}}$.

But, like in [14], using Lemma 2.3, we can write, that, for every $j\geqslant 1$ (with $\theta=2\eta$ and $q=2$) :

$$\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}\geqslant 2^{2j\eta}K(U_{0,QG},j)\geqslant 2^{2j\eta}\frac{2}{3}\left(\|U_{0,QG}^{1,j}\|_{H^{\frac{1}{2}}}+2^{-j}\|U_{0,QG}^{2,j}\|_{H^1}\right).$$

In particular,

$$\|U_{0,QG}^{1,j}\|_{\dot{H}^{\frac{1}{2}}}\leqslant\|U_{0,QG}^{1,j}\|_{H^{\frac{1}{2}}}\leqslant\frac{3}{2}2^{-2j\eta}\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}.$$

So if j_0 is defined such that (c being the constant given by the global lifespan for small initial data result) :

$$\frac{3}{2}2^{-2j_0\eta}\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}=\frac{c\nu_0}{\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}},\tag{2.11}$$

i.-e.

$$j_0=E\Big(-\frac{1}{2\eta}\frac{\log\frac{2c\nu_0}{3\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}}}{\log 2}\Big)+1,$$

and, for all $j\geqslant j_0$, $\|U_{0,QG}^{1,j}\|_{\dot{H}^{\frac{1}{2}}}\leqslant c\nu_0$. This allows us to apply the Fujita-Kato theorem with small initial data and define $U_{QG}^{1,j}$, solution of :

$$\begin{cases}\partial_t U_{QG}^{1,j}-\Gamma U_{QG}^{1,j}+\mathcal{Q}(U_{QG}^{1,j}.\nabla U_{QG}^{1,j})=0\\ U_{QG/t=0}^{1,j}=U_{0,QG}^{1,j},\end{cases}\tag{2.12}$$

with the global energy estimate :

$$\forall t \in \mathbb{R}_+, \|U_{QG}^{1,j}(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \nu_0 \int_0^t \|\nabla U_{QG}^{1,j}(\tau)\|_{\dot{H}^{\frac{1}{2}}}^2 d\tau \leqslant C\|U_{0,QG}^{1,j}\|_{\dot{H}^{\frac{1}{2}}}^2. \quad (2.13)$$

On the other hand, as $U_{0,QG}^{1,j} \in H^{\frac{1}{2}} \hookrightarrow L^2$ the Leray Theorem says that $U_{QG}^{1,j}$ is also a global weak solution, together with the associated energy estimate (in L^2), so we finally obtain :

$$\forall t \in \mathbb{R}_+, \|U_{QG}^{1,j}(t)\|_{H^{\frac{1}{2}}}^2 + \nu_0 \int_0^t \|\nabla U_{QG}^{1,j}(\tau)\|_{H^{\frac{1}{2}}}^2 d\tau \leqslant C\|U_{0,QG}^{1,j}\|_{H^{\frac{1}{2}}}^2 \leqslant C\nu_0^2. \quad (2.14)$$

Remark 2.4. — As $U_{0,QG}^{1,j} = U_{0,QG} - U_{0,QG}^{2,j}$, it is in fact in $\dot{H}^1$ so by the regularity propagation theorem, the solution is more regular and the estimate 2.14 is in fact in H^1.

We now define $U_{QG}^{2,j}(t) = U_{QG}(t) - U_{QG}^{1,j}(t)$, $U_{QG}(t)$ globally given by Theorem 1.2, which satisfies the following system :

$$\begin{cases} \partial_t U_{QG}^{2,j} - \Gamma U_{QG}^{2,j} + \mathcal{Q}(U_{QG}^{2,j}.\nabla U_{QG}^{2,j}) + \mathcal{Q}(U_{QG}^{2,j}.\nabla U_{QG}^{1,j}) + \mathcal{Q}(U_{QG}^{1,j}.\nabla U_{QG}^{2,j}) = 0 \\ U_{QG_{/t=0}}^{2,j} = U_{0,QG}^{2,j}. \end{cases} \quad (2.15)$$

Its potential vorticity satisfying :

$$\partial_t \Omega_{QG}^{2,j} - \Gamma \Omega_{QG}^{2,j} + U_{QG}^{2,j}.\nabla \Omega_{QG}^{2,j} + U_{QG}^{2,j}.\nabla \Omega_{QG}^{1,j} + U_{QG}^{1,j}.\nabla \Omega_{QG}^{2,j} = 0. \quad (2.16)$$

We aim to adapt Theorem 1.1 and get global estimates : if we take the inner product in L^2 of (2.15) by $U_{QG}^{2,j}$:

$$\frac{1}{2}\frac{d}{dt}\|U_{QG}^{2,j}\|_{L^2}^2 + \nu_0\|\nabla U_{QG}^{2,j}\|_{L^2}^2 \leqslant |(U_{QG}^{2,j}.\nabla U_{QG}^{1,j}|U_{QG}^{2,j})_{L^2} \leqslant \|U_{QG}^{2,j}.\nabla U_{QG}^{1,j}\|_{L^2}\|U_{QG}^{2,j}\|_{L^2}.$$

Using the usual product law $L^3.L^6 \hookrightarrow L^2$, we get :

$$\frac{1}{2}\frac{d}{dt}\|U_{QG}^{2,j}\|_{L^2}^2 + \nu_0\|\nabla U_{QG}^{2,j}\|_{L^2}^2 \leqslant \|U_{QG}^{2,j}\|_{\dot{H}^1}\|\nabla U_{QG}^{1,j}\|_{\dot{H}^{\frac{1}{2}}}\|U_{QG}^{2,j}\|_{L^2}.$$

The same Hölder estimate as above gives :

$$\frac{1}{2}\frac{d}{dt}\|U_{QG}^{2,j}\|_{L^2}^2 + \nu_0\|\nabla U_{QG}^{2,j}\|_{L^2}^2 \leqslant \frac{\nu_0}{2}\|\nabla U_{QG}^{2,j}\|_{L^2}^2 + \frac{C}{\nu_0}\|\nabla U_{QG}^{1,j}\|_{\dot{H}^{\frac{1}{2}}}\|U_{QG}^{2,j}\|_{L^2},$$

and then, classical Gronwall estimates give for all $t \geqslant 0$:

$$\|U_{QG}^{2,j}(t)\|_{L^2}^2 + \nu_0 \int_0^t \|\nabla U_{QG}^{2,j}(\tau)\|_{L^2}^2 d\tau \leqslant C\|U_{0,QG}^{2,j}\|_{L^2}^2 e^{\frac{2C}{\nu_0}\|\nabla U_{QG}^{1,j}\|_{L^2\dot{H}^{\frac{1}{2}}}^2}.$$

On the other hand, as in section 2.2.2, the inner product in L^2 of (2.16) by $\Omega_{QG}^{2,j}$ yields (with the same methods as above) :

$$\|U_{QG}^{2,j}(t)\|_{\dot{H}^1}^2+\nu_0\int_0^t \|\nabla U_{QG}^{2,j}(\tau)\|_{\dot{H}^1}^2 d\tau \leqslant C\|U_{0,QG}^{2,j}\|_{\dot{H}^1}^2 e^{\frac{2C}{\nu_0}\|\nabla U_{QG}^{1,j}\|^2_{L^2\dot{H}^{\frac12}}}.$$

Collecting these last two estimates, and using (2.14) we obtain (the new constant C contains ν_0.) :

$$\|U_{QG}^{2,j}(t)\|_{H^1}^2+\nu_0\int_0^t \|\nabla U_{QG}^{2,j}(\tau)\|_{H^1}^2 d\tau \leqslant C\|U_{0,QG}^{2,j}\|_{H^1}^2. \tag{2.17}$$

2.4.3. Application of Lemma 2.3

Since we have decomposed the initial data, we want to use Lemma 2.3 on $U_{QG}^{1,j}$ and $U_{QG}^{2,j}$ in order to estimate U_{QG}.

According to section (2.4) and to the definition K :

$$\|U_{0,QG}\|^2_{H^{\frac12+\eta}} \geqslant \sum_{j\geqslant j_0} 2^{4j\eta} K(U_{0,QG}, j, H^{\frac12}, H^1)^2,$$

After the definition of $U_{QG}^{1,j}$ and $U_{QG}^{2,j}$ we can write that (j_0 fixed in the previous section) :

$$\|U_{0,QG}\|^2_{H^{\frac12+\eta}} \geqslant \frac23 \sum_{j\geqslant j_0} 2^{4j\eta}\left(\|U_{0,QG}^{1,j}\|_{H^{\frac12}}+2^{-j}\|U_{0,QG}^{2,j}\|_{H^1}\right)^2. \tag{2.18}$$

According to the estimates (2.14) and (2.17) we have for all $t\geqslant 0$:

$$\begin{cases} \|U_{0,QG}^{1,j}\|_{H^{\frac12}} \geqslant C\|U_{QG}^{1,j}(t)\|_{H^{\frac12}} \\ \\ \|U_{0,QG}^{2,j}\|_{H^1} \geqslant C\|U_{QG}^{2,j}(t)\|_{H^1}, \end{cases} \tag{2.19}$$

and

$$\begin{cases} \|U_{0,QG}^{1,j}\|_{H^{\frac12}} \geqslant C\sqrt{\nu_0}\|U_{QG}^{1,j}\|_{L^2_tH^{\frac32}} \\ \\ \|U_{0,QG}^{2,j}\|_{H^1} \geqslant C\sqrt{\nu_0}\|U_{QG}^{2,j}\|_{L^2_tH^1}. \end{cases} \tag{2.20}$$

So if we use (2.19) into (2.18), we obtain that :

$$\|U_{0,QG}\|^2_{H^{\frac12+\eta}} \geqslant C \sum_{j\geqslant j_0} 2^{4j\eta}\left(\|U_{QG}^{1,j}(t)\|_{H^{\frac12}}+2^{-j}\|U_{QG}^{2,j}(t)\|_{H^1}\right)^2.$$

And, using $U_{QG}(t) = U_{QG}^{1,j}(t) + U_{QG}^{2,j}(t)$ and the definition of K as an infimum :

$$\|U_{0,QG}\|^2_{H^{\frac{1}{2}+\eta}} \geqslant C \sum_{j\geqslant j_0} 2^{4j\eta} K\left(U_{QG}(t), j, H^{\frac{1}{2}}, H^1\right)^2.$$

Using Lemma 2.3 allows us to go back to $U_{QG}(t)$:

$$\|U_{0,QG}\|^2_{H^{\frac{1}{2}+\eta}} \geqslant C(2^{-j_0}\|U_{QG}(t)\|_{H^{\frac{1}{2}+\eta}})^2.$$

Finally, replacing j_0 (see (2.11)) implies :

$$\forall t \geqslant 0, \|U_{QG}(t)\|^2_{H^{\frac{1}{2}+\eta}} \leqslant C\|U_{0,QG}\|^{2+\frac{1}{\eta}}_{H^{\frac{1}{2}+\eta}}.$$

Similarly if we use (2.20) into (2.18), we obtain that :

$$\|U_{0,QG}\|^2_{H^{\frac{1}{2}+\eta}}$$
$$\geqslant C \sum_{j\geqslant j_0} 2^{4j\eta}\left(\sqrt{\nu_0}\|\nabla U_{QG}^{1,j}\|_{L^2([0,t],H^{\frac{1}{2}})} + 2^{-j}\sqrt{\nu_0}\|\nabla U_{QG}^{2,j}\|_{L^2([0,t],H^1)}\right)^2.$$

Using $\nabla U_{QG} = \nabla U_{QG}^{1,j} + \nabla U_{QG}^{2,j}$ and the definition of K :

$$\|U_{0,QG}\|^2_{H^{\frac{1}{2}+\eta}} \geqslant C\sqrt{\nu_0}\sum_{j\geqslant j_0} 2^{4j\eta} K\left(\nabla U_{QG}, j, L^2([0,t], H^{\frac{1}{2}}), L^2([0,t], H^1)\right)^2.$$

Using the fact that $[L^2_t E_1, L^2_t E_1]_{\theta,2} = L^2_t[E_1, E_1]_{\theta,2}$ (we refer for example to [3], this can be easily proved using the continuous (equivalent) definition of K) and Lemma 2.3, we can write that :

$$\|U_{0,QG}\|^2_{H^{\frac{1}{2}+\eta}} \geqslant C\sqrt{\nu_0}\left(2^{-j_0}\|\nabla U_{QG}\|_{L^2([0,t],H^{\frac{1}{2}+\eta})}\right)^2,$$

and, finally :

$$\nu_0\int_0^t \|\nabla U_{QG}(\tau)\|^2_{H^{\frac{1}{2}+\eta}} d\tau \leqslant C\|U_{0,QG}\|^{2+\frac{1}{\eta}}_{H^{\frac{1}{2}+\eta}}.$$

In particular this implies that $T^* = \infty$ and if we collect the previous results, we obtain that :

$$\forall t \geqslant 0, \quad \|U_{QG}(t)\|^2_{H^{\frac{1}{2}+\eta}} + \nu_0\int_0^t \|\nabla U_{QG}(\tau)\|^2_{H^{\frac{1}{2}+\eta}} d\tau \leqslant C\|U_{0,QG}\|^{2+\frac{1}{\eta}}_{H^{\frac{1}{2}+\eta}}, \tag{2.21}$$

which concludes the proof of Lemma 2.1.

2.5. End of the proof

If we apply Lemma 2.1 to system 2.5, we obtain that for all $t \geqslant 0$:

$$\begin{aligned} \|U_{QG}^{\lambda}(t)\|^2_{H^{\frac{1}{2}+\eta}} + \nu_0 \int_0^t \|\nabla U_{QG}^{\lambda}(\tau)\|^2_{H^{\frac{1}{2}+\eta}} d\tau &\leqslant C\|\chi(\frac{|D|}{\lambda})U_{0,QG}\|^{2+\frac{1}{\eta}}_{H^{\frac{1}{2}+\eta}} \\ &\leqslant C\|U_{0,QG}\|^{2+\frac{1}{\eta}}_{H^{\frac{1}{2}+\eta}}. \end{aligned}$$

And the right-hand member is now a constant (no divergence when λ goes to infinity), so we can go back to 2.10 and now we can estimate the L^1-norm of g where :

$$g(t) = \|\nabla U_\varepsilon^\lambda(t)\|^2_{\dot{H}^{\frac{1}{2}}}(\frac{2C}{\nu_0} + \frac{2C}{\nu_0^3}\|U_\varepsilon^\lambda\|^2_{\dot{H}^{\frac{1}{2}}}),$$

and $\|g\|_{L^1} \leqslant C(\|U_{0,QG}\|_{H^{\frac{1}{2}+\eta}}, \nu_0)$, so we can write :

$$\|V_\varepsilon^\lambda(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \int_0^t (\nu_0 - 2C\|V_\varepsilon^\lambda(t')\|_{\dot{H}^{\frac{1}{2}}})\|\nabla V_\varepsilon^\lambda(t')\|^2_{\dot{H}^{\frac{1}{2}}} e^{\int_{t'}^t g(\tau)d\tau} dt' \leqslant$$

$$\|(1-\chi(\frac{|D|}{\lambda}))U_{0,QG}\|^2_{L^2} e^{\int_0^t g(\tau)d\tau} \leqslant C'\|(1-\chi(\frac{|D|}{\lambda}))U_{0,QG}\|^2_{L^2}. \qquad (2.22)$$

We can now complete the bootstrap argument : if λ is large enough so that :

$$\sqrt{C'}\|(1-\chi(\frac{|D|}{\lambda}))U_{0,QG}\|_{L^2} \leqslant \frac{\nu_0}{8C},$$

and if we define (recall that T_ε^* is the lifespan of V_ε^λ) :

$$T_\varepsilon = \sup\{t \in [0, T_\varepsilon^*[, \text{so that} \quad \forall t' \leqslant t, \|V_\varepsilon^\lambda(t')\|_{\dot{H}^{\frac{1}{2}}} \leqslant \frac{\nu_0}{4C}\}.$$

Then for all $t \leqslant T_\varepsilon$,

$$\|V_\varepsilon^\lambda(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{\nu_0}{2}\int_0^t \|\nabla V_\varepsilon^\lambda(t')\|^2_{\dot{H}^{\frac{1}{2}}} d\tau dt' \leqslant \frac{\nu_0}{8C} < \frac{\nu_0}{4C},$$

then it implies $T_\varepsilon = T_\varepsilon^*$ and we use the blow-up criterion to conclude that $T^{\varepsilon,*} = +\infty$.

Finally, if $\lambda \geqslant \lambda_0$ and $\varepsilon \leqslant \varepsilon(\lambda)$, V_ε is global, and then our solution U_ε is also global, and is an element of $\dot{E}^{\frac{1}{2}}$.

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