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Orbit Structure of certain \mathbb{R}^2 -actions on solid torus

C. MAQUERA⁽¹⁾, L. F. MARTINS⁽²⁾

ABSTRACT. — In this paper we describe the orbit structure of C^2 -actions of \mathbb{R}^2 on the solid torus $S^1 \times D^2$ having $S^1 \times \{0\}$ and $S^1 \times \partial D^2$ as the only compact orbits, and $S^1 \times \{0\}$ as singular set.

RÉSUMÉ. — Nous décrivons la structure des orbites des actions de class C^2 de \mathbb{R}^2 sur le tore solide $S^1 \times D^2$ ayant uniquement $S^1 \times \{0\}$ et $S^1 \times \partial D^2$ comme orbites compacts, et $S^1 \times \{0\}$ comme ensemble singulier.

1. Introduction

Singular foliations can be defined in different ways and have been studied by several authors (cf. [5], [13], [14]). For a recent account of the theory we refer the reader to [4] and [9]. Singular foliations defined by orbits of an action of a Lie group are in the category of foliations given by Stefan [13] and Sussmann [14], and appear in control theory. The geometric description and characterization of locally free C^2 -actions φ of \mathbb{R}^2 on a compact orientable 3-dimensional manifold N , that is, when all the orbits are of codimension 1 in N , are given in [1], [6] and [10]. To the best of our knowledge, the case when φ is not locally free has not been dealt with previously. The aim of this paper is to initiate this study.

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Recently, Camacho and Scárdua ([4]) considered the effect of the presence of singularities of Morse type in a codimension one oriented smooth foliation defined on a closed, connected and oriented three-manifold N . They showed that if such foliation has more centres than saddles and without saddle connections, then N is diffeomorphic to the three-sphere. (See [12] for a generalization of this result.) So we can ask the following natural question : “What can be said about closed 3-dimensional manifolds supporting an action of \mathbb{R}^2 with only a finite number of singular orbits which are homeomorphic to a circle?” Rosenberg-Roussarie-Weil showed in [10] that every closed 3-manifold that admits a locally free action of \mathbb{R}^2 is a bundle over S^1 with fibre T^2 . For this they proved the following result (Fundamental Lemma) : *Let N be a compact, connected 3-manifold with boundary. If N admits a locally free C^2 -action of \mathbb{R}^2 having the boundary of N as an orbit then N is homeomorphic to $T^2 \times [0, 1]$.*

In order to answer the above question, it may be necessary to have an analogous result to the Fundamental Lemma in [10]. So, it is natural to first investigate the orbit structure of C^2 -actions of \mathbb{R}^2 on the solid torus having only a finite number of singular orbits which are homeomorphic to a circle. In this paper we will restrict ourselves to a family of C^2 -actions φ of \mathbb{R}^2 on the solid torus $N = S^1 \times D^2$ having $\mathcal{O}_0 = S^1 \times \{0\}$ and $\mathcal{O} = S^1 \times \partial D^2$ as the only compact orbits and with singular set $\text{Sing}(\varphi) = S^1 \times \{0\}$. We will denote the set of all the above actions by \mathcal{A} , \mathcal{O}_p the φ -orbit of p , G_p the isotropy group of p , G_0 and G the isotropy groups of \mathcal{O}_0 and \mathcal{O} , respectively (which are isomorphic to $\mathbb{Z} \times \mathbb{R}$ and $\mathbb{Z} \times \mathbb{Z}$, resp.) and F_φ the singular foliation in N induced by φ . The possible φ -orbits in $N \setminus (\mathcal{O} \cup \mathcal{O}_0)$ are homeomorphic to a cylinder or a plane, and we shall say an $S^1 \times \mathbb{R}$ -orbit or an \mathbb{R}^2 -orbit, respectively.

In order to describe the asymptotic behavior of orbits, the topological concept of *limit set* of \mathcal{O}_p (see Definition 2.7), denoted by $\lim \mathcal{O}_p$, is essential. We show in Theorem 2.8 that $\mathcal{O}_0 \subset \lim \mathcal{O}_p$ for each $p \in N \setminus \mathcal{O}$. This result is fundamental for obtaining all other results. For its proof we first show that \mathcal{O} and \mathcal{O}_0 are the only minimal sets of φ . Consequently, the set C of points $p \in N$ such that \mathcal{O}_p is an $S^1 \times \mathbb{R}$ -orbit, is a disjoint union of $C_0 = \{p \in C; G_p \subset G_0^0\}$ and $C_1 = \{p \in C; G_p \cap G_0^0 = \{0\}\}$, where G_0^0 is the connected component of G_0 that contains the origin $(0, 0) \in \mathbb{R}^2$, the neutral element of the group (see Remark 2.9).

In this paper we give a complete geometric description of the orbit structure of $\varphi \in \mathcal{A}$ in the complement of S_Γ (the solid k -tube of φ at \mathcal{O}_0 associated to k -tube Γ , see Definition 3.9). More precisely :

THEOREM A. — *If $\varphi \in \mathcal{A}$, then there exists an φ -invariant neighbourhood U of \mathcal{O} , homeomorphic to $T^2 \times (0, 1]$, such that the frontier $\Gamma = \text{Front}(U)$ of U is a k -tube of φ at \mathcal{O}_0 , for some integer $k \geq 0$, and all the orbits inside $U \setminus \mathcal{O}$ have the same topological type. Furthermore, precisely one of the following cases occurs for each $p \in U \setminus \mathcal{O}$:*

- (1) \mathcal{O}_p is an $S^1 \times \mathbb{R}$ -orbit in C_0 , $C_1 = \emptyset$ and $\lim \mathcal{O}_p = \mathcal{O}_0 \cup \mathcal{O}$. In particular, $U = N \setminus \mathcal{O}_0$,
- (2) \mathcal{O}_p is an $S^1 \times \mathbb{R}$ -orbit in C_1 , $C_0 = \emptyset$ and $\lim \mathcal{O}_p = \Gamma \cup \mathcal{O}$,
- (3) \mathcal{O}_p is an \mathbb{R}^2 -orbit dense in $U \cup \Gamma$, and $\lim \mathcal{O}_p = U \cup \Gamma$.

In case (1) of Theorem A, $k = 0$ and $\Gamma = \mathcal{O}_0 = S_\Gamma$. When $k > 0$, the orbit structure of φ in the interior of S_Γ appears to be very complicated, although in this case the above theorem states that $C_0 = \emptyset$.

For the rest of the paper we take $D^2 = \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 \leq 1\}$, $S^1 = \mathbb{R}/\mathbb{Z}$, and in $N = S^1 \times D^2$, we will consider coordinates (θ, x) where $\theta \in S^1$ and $x = (x_1, x_2) \in D^2$. Given $\varphi \in \mathcal{A}$, the intersections of the φ -orbits with the disks $D_\theta = \{\theta\} \times D^2$, $\theta \in S^1$ (the traces of the φ -orbits on D_θ) clearly yield good information about the geometric behavior of the φ -orbits. With an adaptation of Haefliger's techniques for regular foliations ([7]), we obtain information about these traces. More precisely, we prove in Proposition 3.2 the existence of a closed embedded 2-disk Σ in N with $\partial\Sigma \subset \mathcal{O}$ and in general position with respect to the foliation F_φ . Furthermore, the induced foliation on Σ is given by a vector field $Z_\varphi \in X^2(\Sigma)$. We have thus proved the following theorem :

THEOREM A'. — *If $\varphi \in \mathcal{A}$, then there exists an Z_φ -invariant neighbourhood V of $\partial\Sigma$ in Σ , homeomorphic to $S^1 \times (0, 1]$, such that $\Gamma = \text{Front}(V)$, the frontier of V in Σ , is a k -petal of Z_φ at $\mathcal{O}_0 \cap \Sigma$ for some integer $k \geq 0$, and all orbits inside $V \setminus \partial\Sigma$ have the same topological type. Furthermore, precisely one of the following cases occurs for each $p \in V \setminus \partial\Sigma$:*

- (1) $\mathcal{O}_p(Z_\varphi)$ is periodic and $V = \Sigma \setminus (\mathcal{O}_0 \cap \Sigma)$,
- (2) $\mathcal{O}_p(Z_\varphi)$ is homeomorphic to \mathbb{R} and $\alpha(p) \cup \omega(p) = \partial\Sigma \cup \Gamma$.

The concept of k -petal is analogous to that of k -tube (see Definition 3.8). In case (1) of Theorem A', $k = 0$ and this means that if a trace of one φ -orbit on $V \setminus \partial\Sigma$ is homeomorphic to S^1 , then all the other traces on $\Sigma \setminus (\mathcal{O}_0 \cap \Sigma)$ are also homeomorphic to S^1 (Figure 1 (a)). On the other hand (in case (2)), given $p \in V$, if $\mathcal{O}_p(Z_\varphi)$ is homeomorphic to \mathbb{R} then the

Figures 1 (b) and (c) describe possibilities for the traces of φ on Σ (in (b) we have a 0-petal and in (c) a 3-petal).

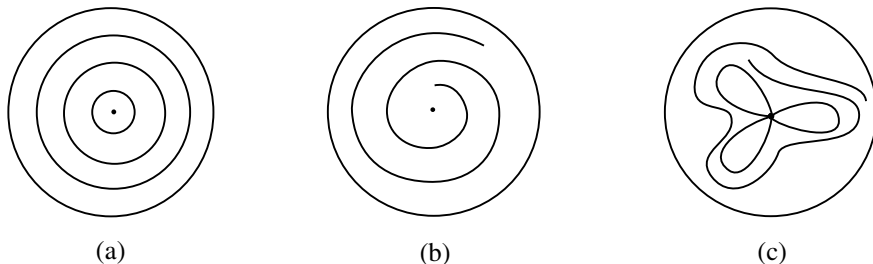


Figure 1. — Possibilities for the traces of the φ -orbits on Σ

Theorems A and A' are related as follows. Theorem A (1) is satisfied if and only if Theorem A' (1) is true, and either (2) or (3) in Theorem A is true if and only if (2) in Theorem A' is true. We shall prove Theorems A and A' simultaneously.

The paper is organized as follows. In the next section we study the orbit structure of actions in \mathcal{A} in neighbourhoods of compact orbits and obtain topological and asymptotic properties of orbits in C_0 and C_1 . In Section 3 we introduce the concept of general position, show the existence of an embedded 2-disk Σ in N which is in general position with respect to the foliation defined by φ , and give the proofs of Theorems A and A'.

The results of our investigation can be used to study 3-manifolds that admit a Heegaard splitting of genus one, since these are obtained by gluing two copies of $S^1 \times D^2$ by a diffeomorphism of $\partial(S^1 \times D^2)$.

This paper is part of the second author's Ph.D. thesis, written under supervision of J. L. Arraut at the University of São Paulo in São Carlos.

2. Properties of actions in \mathcal{A}

In this section we shall first study the orbit structure of $\varphi \in \mathcal{A}$ in neighbourhoods of compact orbits. We shall prove in Proposition 2.1 that, in neighbourhoods of \mathcal{O} and \mathcal{O}_0 , \mathcal{F}_φ is topologically equivalent to one suspension of a vector field (defined below in Section 2.1). This is a fundamental result for the proof of the main theorems. Next, we shall obtain topological and asymptotic properties of φ -orbits (Theorem 2.8 and Proposition 2.10). Finally, in Theorem 2.14, we shall establish information about the generators of cylindrical φ -orbits with relation to \mathcal{O}_0 . Some of the results of this section are proved using infinitesimal generators adapted to φ at \mathcal{O}_0 and

\mathcal{O} . These generators are defined using the concept of “charts adapted to the compact orbits”, which is a fundamental tool in this paper.

For each $w \in \mathbb{R}^2 \setminus \{0\}$, φ induces a C^2 -flow $(\varphi_w^t)_{t \in \mathbb{R}}$ given by $\varphi_w^t(p) = \varphi(tw, p)$ and its corresponding C^1 -vector field X_w is defined by $X_w(p) = D_1\varphi(0, p) \cdot w$. If $\{w_1, w_2\}$ is a basis of \mathbb{R}^2 , then $\{X_{w_1}, X_{w_2}\}$, called a set of *infinitesimal generators* of φ , determines completely the action φ . Moreover, the Lie bracket $[X_{w_1}, X_{w_2}] = 0$.

For the rest of the paper $p_0 = (\theta_0, x_0) \in \mathcal{O}$, $q_0 = (\theta_0, 0) \in \mathcal{O}_0$ and $S_1 = \partial D_{\theta_0}$, where $D_{\theta_0} = \{\theta_0\} \times D^2 \subset N$, for some fixed $\theta_0 \in S^1$.

2.1. The orbit structure in neighbourhoods of \mathcal{O}_0 and \mathcal{O}

Here we will study the orbit structure of $\varphi \in \mathcal{A}$ in neighbourhoods of compact orbits. A classical result in foliation theory states that the leaf structure of a foliation in the neighbourhood of a compact leaf is determined by the holonomy of this leaf (see [3]). We shall determine the holonomy of \mathcal{O} and from this we obtain information about the φ -orbits in a neighbourhood of \mathcal{O} . When we refer to the holonomy of \mathcal{O} we mean the holonomy group of \mathcal{O} as a leaf of \mathcal{F}_φ on $N \setminus \mathcal{O}_0$ (cf. [3], Chapter 4, Section 1, for a definition).

Let S be a smooth compact surface. The set of C^r vector fields on S will be denoted by $\mathfrak{X}^r(S)$, $r \geq 1$. Let $X \in \mathfrak{X}^r(S)$ with a finite number of singularities, all contained in the interior of S when $\partial S \neq \emptyset$. Suppose that $f \in \text{Diff}^r(S)$ preserves the orbits of X . Let M be the manifold obtained from $\mathbb{R} \times S$ by identifying (z, p) with $(z - 1, f(p))$. The suspension of f defines a C^r foliation $\mathcal{F}(X, f)$ of M , which is the image of the foliation of $\mathbb{R} \times S$, whose leaves are $\mathbb{R} \times \mathcal{O}_p(X)$ by the quotient map. The foliation $\mathcal{F}(X, f)$ is called the *suspension of X by f* .

PROPOSITION 2.1. — *If $\varphi \in \mathcal{A}$, then there exist neighbourhoods W_0 of \mathcal{O}_0 , W_1 of \mathcal{O} and, for $i = 0, 1$, a C^2 diffeomorphism $f_i : A_i \rightarrow U_i$ and $Y_i \in \mathfrak{X}^1(A_i)$, where A_0, U_0 and A_1, U_1 are neighbourhoods in D_{θ_0} of q_0 and ∂D_{θ_0} , respectively, such that f_i preserves the orbits of Y_i and $\mathcal{F}_\varphi|_{W_i}$ is topologically equivalent to the suspension of Y_i by f_i .*

The proof of Proposition 2.1 follows from Lemmas 2.2 and 2.4 below, using the concept of “charts adapted to the compact orbits” that we now shall introduce. This concept is a fundamental tool and will be referred to very often in the sequel.

Let $\varphi \in \mathcal{A}$, and let H be a 1-dimensional subspace of \mathbb{R}^2 such that $\mathbb{R}^2 = H \oplus G_0^0$. Let $\{w_1, w_2\}$ be a basis of \mathbb{R}^2 such that w_1 and w_2 generate

the subgroups $G_0 \cap H$ and G_0^0 , respectively. Note that $\{X_i = X_{w_i}; i = 1, 2\}$ is a set of infinitesimal generators of φ such that \mathcal{O}_0 is a periodic orbit of X_1 of period one, and $X_2(q) = 0$ for every $q \in \mathcal{O}_0$. We say that $\{X_1, X_2\}$ is a set of *infinitesimal generators adapted to φ at \mathcal{O}_0* .

Let $I^3(\varepsilon) = \{(\theta, x) \in N; |\theta - \theta_0| < \varepsilon \text{ and } |x| < \varepsilon\}$ and $h : V \rightarrow I^3(\varepsilon)$ be a chart of N at q_0 such that if $(\theta, x_1, x_2) \in I^3(\varepsilon)$, then the vector fields X_i in this chart, $i = 1, 2$, can be written as

$$\begin{aligned} X_1(\theta, x_1, x_2) &= \frac{\partial}{\partial \theta}, \\ X_2(\theta, x_1, x_2) &= a(x_1, x_2) \frac{\partial}{\partial \theta} + b(x_1, x_2) \frac{\partial}{\partial x_1} + c(x_1, x_2) \frac{\partial}{\partial x_2}. \end{aligned} \quad (2.1)$$

The above chart is called *adapted to \mathcal{O}_0 at q_0* . The vector field

$$Y_0(x_1, x_2) = b(x_1, x_2) \frac{\partial}{\partial x_1} + c(x_1, x_2) \frac{\partial}{\partial x_2} \quad (2.2)$$

defined on $A_0(\varepsilon) = \{(\theta_0, x) \in N; |x| < \varepsilon\}$ has only q_0 as singularity.

Note that $\{X_1, Y_0\}$ defines a local \mathbb{R}^2 -action $\widehat{\varphi}$ on $I^3(\varepsilon)$ and $\mathcal{O}_{(\theta, x)}(\widehat{\varphi}) = \mathcal{O}_{(\theta, x)}(h \circ \varphi \circ h^{-1})$ for each $(\theta, x) \in I^3(\varepsilon)$.

Let U_0 be a neighbourhood of q_0 in D_{θ_0} such that the Poincaré diffeomorphism of X_1 at q_0 , $f_0 : A_0(\varepsilon) \rightarrow U_0$, is well defined. Note that f_0 is of class C^2 . For $\varepsilon > 0$ sufficiently small, let $\tau : A_0(\varepsilon) \rightarrow [0, 1 + \varepsilon)$ be the time of the first return map. Let W_0 denote the interior of $\bigcup_{q \in \text{cl}(A_0(\varepsilon))} \{X_1^t(q); 0 \leq t \leq \tau(q)\}$, where $\text{cl}(B)$ denotes the closure in N of a set B .

As an immediate consequence we obtain the following result which is one part of Proposition 2.1 :

LEMMA 2.2. — *There exist $\varepsilon > 0$ and a neighbourhood W_0 of \mathcal{O}_0 such that $f_0 : A_0(\varepsilon) \rightarrow U_0$ preserves the orbits of Y_0 , and $\mathcal{F}_\varphi|_{W_0}$ is topologically equivalent to the suspension of Y_0 by f_0 .*

In order to complete the proof of Proposition 2.1, we shall determine the holonomy of \mathcal{O} , which is stated in Lemma 2.3. First we need to introduce another set of infinitesimal generators.

Suppose now that $\{w_1, w_2\}$ is a basis of \mathbb{R}^2 such that w_1 and w_2 generate the isotropy group G of \mathcal{O} . Write $X_i = X_{w_i}$, $i = 1, 2$. Note that if $q \in \mathcal{O}$, then for $i \in \{1, 2\}$ the orbit of X_i by q is periodic of period one. Without loss of generality, we can assume that for each $\theta \in S^1$, $\{\theta\} \times \partial D^2$

is an orbit of X_1 and for each $x \in \partial D^2$, $S^1 \times \{x\}$ is an orbit of X_2 . We shall say that $\{X_1, X_2\}$ is a set of *infinitesimal generators adapted to φ at \mathcal{O}* .

Now we consider in N another coordinate system (θ, x) where $\theta \in S^1$ and $x \in D^2$ is given in polar coordinates (ϕ, r) . Let S_i , $i = 1, 2$, be the circle orbit of X_i through p_0 , that is, $S_1 = \{\theta_0\} \times \partial D^2$ and $S_2 = S^1 \times \{x_0\}$. For $\varepsilon \in (0, 1)$ let $A_1(\varepsilon) = \{(\theta_0, \phi, r) \in N; r > 1 - \varepsilon\}$ and $A_2(\varepsilon) = \{(\theta, \phi_0, r) \in N; r > 1 - \varepsilon\}$, where $x_0 = (\phi_0, 1)$. For simplicity of notation we write (ϕ, r) and (θ, r) instead of (θ_0, ϕ, r) and (θ, ϕ_0, r) , respectively. Since S_1 (S_2) is transverse to the orbits of X_2 (X_1), there exists $\varepsilon > 0$ such that $A_1(\varepsilon)$ and $A_2(\varepsilon)$ are transverse to the orbits of X_2 and X_1 , respectively. Consequently, $A_1(\varepsilon)$, $A_2(\varepsilon)$ and $J_\varepsilon = A_1(\varepsilon) \cap A_2(\varepsilon)$ are transverse to the orbits of φ . Let $\delta > 0$, $I(\delta) = (-\delta, \delta)$, and for $i = 1, 2$ we consider the C^2 -maps $h_i : A_i(\varepsilon) \times I(\delta) \rightarrow N$ defined by $h_1(q, t) = X_2^t(q)$ and $h_2(q, t) = X_1^t(q)$. There exists $\delta > 0$ such that $h_i|_{A_i(\varepsilon) \times I(\delta)}$ is a diffeomorphism onto its image V_i . Moreover, in the coordinates (h_1^{-1}, V_1) , the infinitesimal generators of φ are of the form

$$\begin{aligned} X_1(\phi, r, t) &= a_1(\phi, r) \frac{\partial}{\partial t} + b_1(\phi, r) \frac{\partial}{\partial \phi} + c_1(\phi, r) \frac{\partial}{\partial r}, \\ X_2(\phi, r, t) &= \frac{\partial}{\partial t}, \end{aligned} \quad (2.3)$$

and in the coordinates (h_2^{-1}, V_2) the infinitesimal generators of φ are of the form

$$\begin{aligned} X_1(\theta, r, t) &= \frac{\partial}{\partial t}, \\ X_2(\theta, r, t) &= a_2(\theta, r) \frac{\partial}{\partial t} + b_2(\theta, r) \frac{\partial}{\partial \theta} + c_2(\theta, r) \frac{\partial}{\partial r}. \end{aligned} \quad (2.4)$$

The maps h_i are called a *cylindrical coordinate system adapted to \mathcal{O} at S_i* , $i = 1, 2$. The vector fields

$$\widehat{X}_1 = b_1(\phi, r) \frac{\partial}{\partial \phi} + c_1(\phi, r) \frac{\partial}{\partial r} \quad \text{and} \quad \widehat{X}_2 = b_2(\theta, r) \frac{\partial}{\partial \theta} + c_2(\theta, r) \frac{\partial}{\partial r} \quad (2.5)$$

define a local flow on $A_1(\varepsilon)$ and $A_2(\varepsilon)$, respectively. Furthermore, $S_i \subset A_i(\varepsilon)$ is an orbit of \widehat{X}_i , $i = 1, 2$.

Note that $p_0 \in J_\varepsilon$ and the map $\alpha_i : [0, 1] \rightarrow A_i(\varepsilon)$ given by $\alpha_i(\tau) = \widehat{X}_i^\tau(p_0)$ is a parametrization of S_i , $i = 1, 2$. Let $P_i : (J_\varepsilon, p_0) \rightarrow (J_\varepsilon, p_0)$ be the Poincaré map of α_i , $i = 1, 2$, and

$$\text{Hol} : \pi_1(\mathcal{O}, p_0) \cong \mathbb{Z}^2 \rightarrow \text{Diff}^2(J_\varepsilon, p_0) \quad (2.6)$$

the holonomy of \mathcal{O} as a leaf of the foliation \mathcal{F}_φ . Then $P_i = \text{Hol}([\alpha_i])$, $i = 1, 2$.

Note also that $\{\widehat{X}_1, X_2\}$ and $\{X_1, \widehat{X}_2\}$ define two local \mathbb{R}^2 -actions $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ on $A_1(\varepsilon) \times I(\delta)$ and $A_2(\varepsilon) \times I(\delta)$, respectively, where \widehat{X}_1 and \widehat{X}_2 are given in Equation 2.5. Moreover,

$$\mathcal{O}_{(\phi,r,t)}(\widehat{\varphi}_1) = \mathcal{O}_{(\phi,r,t)}(h_1 \circ \varphi \circ h_1^{-1}) \text{ and } \mathcal{O}_{(\theta,r,t)}(\widehat{\varphi}_2) = \mathcal{O}_{(\theta,r,t)}(h_2 \circ \varphi \circ h_2^{-1}).$$

The following result is a particular case of Lemma 2.4 in [2]. The condition of C^2 differentiability is necessary.

LEMMA 2.3. — *There exists $\varepsilon \in (0, 1)$ such that for each $i \in \{1, 2\}$ one and only one of the following cases holds :*

- (a) $P_i|_{J_\varepsilon} = \text{id}$; that is, every \widehat{X}_i -orbit near S_i is periodic,
- (b) either $P_i|_{J_\varepsilon}$ or $(P_i|_{J_\varepsilon})^{-1}$ is a topological contraction, i.e. every \widehat{X}_i -orbit near S_i spirals towards S_i .

Furthermore, if P_1 (resp. P_2) satisfies (a), then P_2 (resp. P_1) satisfies (b).

We conclude from the above lemma that, in a neighbourhood of S_1 , the orbits of \widehat{X}_1 are as depicted in one of the figures below, more precisely, they are all homeomorphic to S^1 or all homeomorphic to \mathbb{R} .

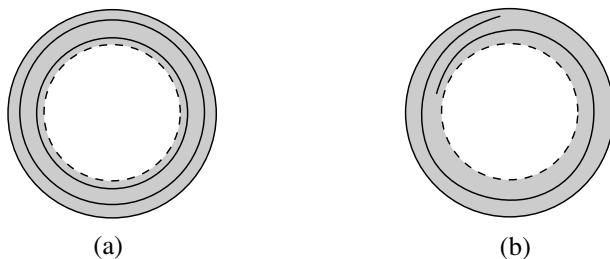


Figure 2. — The possibilities for the orbits of \widehat{X}_1 on a neighbourhood of S_1

We now can conclude the proof of Proposition 2.1 taking $Y_1 = \widehat{X}_1$ in following lemma :

LEMMA 2.4. — *There exist $\varepsilon > 0$, a neighbourhood W_1 of \mathcal{O} , a neighbourhood U_1 in D_{θ_0} of S_1 and a C^2 -diffeomorphism $f_1 : A_1(\varepsilon) \rightarrow U_1$ that preserves the orbits of \widehat{X}_1 such that $\mathcal{F}_\varphi|_{W_1}$ is topologically equivalent to the suspension of \widehat{X}_1 by f_1 .*

Proof. — Since $X_2^1(S_1) = S_1$, there exists $\varepsilon > 0$ such that $X_2^1(A_1(\varepsilon)) \subset V_1$. Consequently, the time of first return map $\tau : A_1(\varepsilon) \rightarrow [0, 1 + \delta)$ is a C^2 map. If $U_1 = \cup_{q \in A_1(\varepsilon)} X_2^{\tau(q)}(q)$, then the C^2 map $f_1 : A_1(\varepsilon) \rightarrow U_1$, defined by $f_1(q) = X_2^{\tau(q)}(q)$, is a diffeomorphism that preserves the orbits of \widehat{X}_1 . Let W_1 be the interior of $\cup_{q \in \text{cl}(A_1(\varepsilon))} \{X_2^t(q); 0 \leq t \leq \tau(q)\}$. By taking a smaller ε if necessary, f_1 and \widehat{X}_1 induce a local diffeomorphism of J_ε , which coincides with P_2 . Therefore, the holonomy of \mathcal{O} as a leaf of the foliation obtained by the suspension of \widehat{X}_1 by f_1 is the same holonomy of \mathcal{O} as a leaf of \mathcal{F}_φ , which is given by Lemma 2.3. Thus, $\mathcal{F}_\varphi|_{W_1}$ is topologically equivalent to the suspension of \widehat{X}_1 by f_1 . \square

Lemma 2.3 yields a natural decomposition of the family \mathcal{A} into a disjoint union :

$$\mathcal{A} = \mathcal{A}_{S^1} \cup \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p,$$

where $\mathcal{A}_{S^1} = \{\varphi \in \mathcal{A}; P_1|_{J_\varepsilon} = \text{id}\}$, $\mathcal{A}_{\mathbb{R}}^c = \{\varphi \in \mathcal{A}; P_2|_{J_\varepsilon} = \text{id}\}$ and $\mathcal{A}_{\mathbb{R}}^p = \{\varphi \in \mathcal{A}; P_1|_{J_\varepsilon}, P_2|_{J_\varepsilon} \neq \text{id}\}$. Note that if $\varphi \in \mathcal{A}_{S^1}$ (resp. $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$) then, for each fixed disk D_θ , the traces of φ -orbits in a neighbourhood of S_1 in D_θ are as in Figure 2 (a) (resp. Figure 2 (b)). So we obtain :

PROPOSITION 2.5. — *If $\varphi \in \mathcal{A}$, then there exists a φ -invariant neighbourhood U of \mathcal{O} such that all φ -orbits inside $U \setminus \mathcal{O}$ have the same topological type and precisely one of the following possibilities occurs for each $p \in U \setminus \mathcal{O}$:*

- (1) \mathcal{O}_p is an $S^1 \times \mathbb{R}$ -orbit, $\text{cl}(\mathcal{O}_p) \setminus \mathcal{O}_p$ has two connected components, with \mathcal{O} being one of them, and $\varphi \in \mathcal{A}_{S^1} \cup \mathcal{A}_{\mathbb{R}}^c$,
- (2) \mathcal{O}_p is an \mathbb{R}^2 -orbit which is dense in U , and $\varphi \in \mathcal{A}_{\mathbb{R}}^p$.

For the proof of this proposition we will need the following result. For the rest of the paper $A_0(\varepsilon)$ and $A_1(\varepsilon)$ will denote the sets given in Lemmas 2.2 and 2.4, respectively.

LEMMA 2.6. — *Let $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$. Then there exists an orientation preserving C^2 diffeomorphism $f : S^1 \rightarrow S^1$ such that for each $m \in \mathbb{N}$ either $\text{Fix}(f^m) = \emptyset$ or $\text{Fix}(f^m) = S^1$.*

Proof. — Since $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$, there exists a C^2 embedding $F : S^1 \rightarrow A_1(\varepsilon)$ such that $S = F(S^1)$ is transverse to \widehat{X}_1 . Then the diffeomorphism f_1 , as in Lemma 2.4, induces a C^2 diffeomorphism $f : S^1 \rightarrow S^1$ defined by $f(q) = F^{-1}(S \cap \mathcal{O}_{f_1 \circ F(q)}(\widehat{X}_1))$. If $q \in S^1$ is a fixed point of f^m , then $f_1^m(F(q)) \in \mathcal{O}_{F(q)}(\widehat{X}_1)$ with $F(q) \neq p_0$. Without loss of generality we can

assume that $F(q) \in J_\varepsilon$. Hence $P_2^m(F(q)) = F(q)$, and it follows from Lemma 2.3 that $P_2^m = \text{id}$, that is, $f^m = \text{id}$. Therefore $f : S^1 \rightarrow S^1$ preserves orientation (otherwise it would have exactly two fixed points, see [8, Exercise 11.2.4]). \square

Proof of Proposition 2.5. — Let U_0 denote the union of φ -orbits by points of $J_\varepsilon \setminus \{p_0\}$. The holonomy of the orbit \mathcal{O} , given in Lemma 2.3, guarantees that the φ -orbits of points in U_0 either are all homeomorphic to $S^1 \times \mathbb{R}$, or all homeomorphic to \mathbb{R}^2 . We take $U = U_0 \cup \mathcal{O}$. If every orbit inside U_0 is homeomorphic to $S^1 \times \mathbb{R}$, then part (1) of the proposition follows from Lemma 2.4. Assume now that every orbit inside U_0 is homeomorphic to \mathbb{R}^2 . In this case $\varphi \in \mathcal{A}_{\mathbb{R}}^p$. Let $f \in \text{Diff}_+^2(S^1)$ be the diffeomorphism given in Lemma 2.6 and $\tau(f)$ its rotation number. We claim that $\tau(f) \in \mathbb{R} \setminus \mathbb{Q}$. Otherwise, if $\tau(f) \in \mathbb{Q}$, then f has at least one periodic point. It follows from Lemma 2.6 that $f^m = \text{id}$ for some $m \in \mathbb{N}$. This implies that $P_2^m = \text{id}$, contradicting the fact that P_2 satisfies (b) of Lemma 2.3. Therefore $\tau(f) \in \mathbb{R} \setminus \mathbb{Q}$. Since f is of class C^2 , by Denjoy's Theorem [8, Theorem 12.1.1] f is topologically conjugate to the rotation in S^1 given by $R_{\tau(f)}(\theta) = \theta + \tau(f)$. Therefore, the set $\{f^n(q)\}_{n \in \mathbb{Z}_+}$ is dense in S^1 for each $q \in S^1$. Consequently, every orbit inside U_0 is dense in U_0 . \square

2.2. Asymptotic properties

We now proceed to study the asymptotic behaviour of φ -orbits. We shall show that \mathcal{O}_0 is in the closure of every φ -orbit in $N \setminus \mathcal{O}$, which is fundamental for the proof of the main results.

DEFINITION 2.7. — The *limit set* of \mathcal{O}_p is the φ -invariant compact set given by $\lim \mathcal{O}_p = \bigcap_{i=1}^\infty \text{cl}(\mathcal{O}_p \setminus K_i)$, where K_i is a compact subset of \mathcal{O}_p , $K_i \subset K_{i+1}$, and $\mathcal{O}_p = \bigcup_{i=1}^\infty K_i$.

It is not difficult to show that $\text{cl}(\mathcal{O}_p) = \mathcal{O}_p \cup \lim \mathcal{O}_p$. The notions of minimal and exceptional minimal sets that we use here are the standard ones (see [3], Chapter 3, Section 4). We now obtain:

THEOREM 2.8. — *If $\varphi \in \mathcal{A}$, then:*

- (i) \mathcal{O} and \mathcal{O}_0 are the only minimal sets of φ ,
- (ii) $\mathcal{O}_0 \subset \lim \mathcal{O}_p$ for each $p \in N \setminus \mathcal{O}$. Consequently $G_p \subset G_0$.

Proof. — (i) Suppose that μ is a minimal set of φ such that $\mu \neq \mathcal{O}$ and $\mu \neq \mathcal{O}_0$. Then μ is also a minimal set of the action φ' of \mathbb{R}^2 on $N' = N \setminus (\mathcal{O} \cup \mathcal{O}_0)$ given by $\varphi' = \varphi|_{\mathbb{R}^2 \times N'}$. Consequently, as φ' has

no exceptional minimal sets (see [11, Theorem 8]), either $\mu = \mathcal{O}_p$ for some $p \in N'$, or $\mu = N'$. If $\mu = \mathcal{O}_p$, then \mathcal{O}_p is a compact orbit of φ , contradicting the fact that $\varphi \in \mathcal{A}$. If $\mu = N'$, then $\text{cl}(\mu)$, the closure of μ in N , contains \mathcal{O} and \mathcal{O}_0 . This contradicts the fact that μ is a minimal set of φ . This completes the proof of (i).

(ii) Suppose that (ii) is not true, i.e. there exists $p \in N \setminus \mathcal{O}$ such that $\mathcal{O}_0 \not\subset \lim \mathcal{O}_p$. Since $\text{cl}(\mathcal{O}_p) = \mathcal{O}_p \cup \lim \mathcal{O}_p$, we have $\mathcal{O}_0 \not\subset \text{cl}(\mathcal{O}_p)$. Therefore, as the φ -invariant compact set $\text{cl}(\mathcal{O}_p)$ contains a minimal set μ which by (i) coincides with \mathcal{O} , we have $\mathcal{O} \subset \text{cl}(\mathcal{O}_p)$. If \mathcal{O}_p is an $S^1 \times \mathbb{R}$ -orbit, then by item (1) of Proposition 2.5, there exists a connected component Δ of $\text{cl}(\mathcal{O}_p) \setminus \mathcal{O}_p$ such that $\mathcal{O} \cap \Delta = \emptyset$. Let $\mu' \subset \Delta$ be a minimal set of φ . By (i), $\mu' = \mathcal{O}_0$, which contradicts the fact that $\Delta \cap \mathcal{O}_0 = \emptyset$. This contradiction shows, in this case, that $\mathcal{O}_0 \subset \text{cl}(\mathcal{O}_p)$. Finally, assume that \mathcal{O}_p is an \mathbb{R}^2 -orbit. Then the neighbourhood U of \mathcal{O} , as in Proposition 2.5(2), satisfies $\text{cl}(\mathcal{O}_p) \cap U = U$, and $\text{Front}(U)$ satisfies $\text{Front}(U) \cap \mathcal{O}_0 = \emptyset = \text{Front}(U) \cap \mathcal{O}$. So, $\text{Front}(U)$ contains a minimal set which is neither \mathcal{O}_0 nor \mathcal{O} , contradicting (i). This concludes the proof of (ii). \square

Remark 2.9. — For each $\varphi \in \mathcal{A}$, let C be the set of points $p \in N$ such that \mathcal{O}_p is an $S^1 \times \mathbb{R}$ -orbit. Since $G_p \subset G_0$, it follows that C is a disjoint union of the sets C_0 and C_1 , with :

$$C_0 = \{p \in C; G_p \subset G_0^0\} \quad \text{and} \quad C_1 = \{p \in C; G_p \cap G_0^0 = \{0\}\}.$$

We shall see in the next subsection that if φ has an $S^1 \times \mathbb{R}$ -orbit in C_0 , then all the other φ -orbits in $N \setminus (\mathcal{O}_0 \cup \mathcal{O})$ are $S^1 \times \mathbb{R}$ -orbits in C_0 , i.e., if $C_0 \neq \emptyset$, then $C_1 = \emptyset$, moreover, $C_0 = N \setminus (\mathcal{O}_0 \cup \mathcal{O})$. We shall also elucidate the orbit structure in C_0 and in C_1 . First, we obtain :

PROPOSITION 2.10. — *If $\varphi \in \mathcal{A}$, then $\lim \mathcal{O}_q = \mathcal{O}_0 \cup \mathcal{O}$ for every $q \in C_0$.*

In the proof of this proposition we shall use the following lemma. Let $\varphi \in \mathcal{A}$, $q \in C_0$ and $\{w_1, w_2\}$ a basis of \mathbb{R}^2 such that the vector fields $\{X_1 = X_{w_1}, X_2 = X_{w_2}\}$ are infinitesimal generators adapted to φ at \mathcal{O}_0 with w_2 a generator of G_q .

LEMMA 2.11. — *Given $p \in \mathcal{O}_0$ and a neighbourhood V_p of p , there exists a neighbourhood $U_p \subset V_p$ of p such that $\mathcal{O}_{q'}(X_2) \subset V_p$ for all $q' \in \mathcal{O}_q \cap U_p$.*

Proof. — Since the orbits of X_2 by points of \mathcal{O}_q are periodic of period 1, it is sufficient to show that there exists a neighbourhood U_p of p such

that $X_2^t(U_p) \subset V_p$, for all $t \in [0, 1]$. Since $X_2^t(p) = p$, $t \in [0, 1]$, then there exists a neighbourhood $U_{p,t}$ of p and an open interval $I_t \subset [0, 1]$ that contains t such that $X_1^s(U_{p,t}) \subset V_p$, for all $s \in I_t$. There exists a finite sub-family $\{I_{t_i}; i = 1, \dots, k\}$ of $\{I_t; t \in [0, 1]\}$ that covers $[0, 1]$. The statement now follows by taking $U_p = \cap_{i=1}^k U_{p,t_i}$. \square

Proof of Proposition 2.10. — Let $q \in C_0$. Since $G_q \subset G_0^0$, we take $\{X_1, X_2\}$ as infinitesimal generators adapted to φ at \mathcal{O}_0 such that w_2 generates G_q , and (h, V_{q_0}) a chart adapted to \mathcal{O}_0 at q_0 . By Lemma 2.11, there exists a neighbourhood U_{q_0} of q_0 such that $\mathcal{O}_{q'}(X_2) \subset V_{q_0}$ for every $q' \in \mathcal{O}_q \cap U_{q_0}$. We first show that $\lim \mathcal{O}_q$ (which in this case is equal to $\text{cl}(\mathcal{O}_q) \setminus \mathcal{O}_q$) has two connected components Δ_0 and Δ with $\mathcal{O}_0 \subset \Delta_0$.

By Lemma 2.2, $\mathcal{F}|_\varphi$ is the suspension, in a neighbourhood of \mathcal{O}_0 , of Y_0 by the Poincaré diffeomorphism $f_0 : A_0(\varepsilon) \rightarrow U_0$ of X_1 at q_0 .

Let $\varepsilon > 0$ such that $A_0(\varepsilon) \subset U_{q_0}$ and $U_0 \subset U_{q_0}$. Since $\mathcal{O}_0 \subset \text{cl}(\mathcal{O}_q)$, there exists $q' \in \mathcal{O}_q \cap A_0(\varepsilon)$ such that $\mathcal{O}_{q'}(X_2) \subset V_{q_0}$ and $\mathcal{O}_{f_0(q')}(X_2) \subset V_{q_0}$. Consequently, the Y_0 -orbits γ_1 and γ_2 by q' and $f_0(q')$, respectively, are periodic, contained in $A_0(\varepsilon)$ and satisfy $f_0(\gamma_1) = \gamma_2$. Furthermore, $\gamma_1 \neq \gamma_2$, otherwise $\mathcal{O}_{q'}$ would be a compact φ -orbit. We can assume that γ_2 is contained in the interior of γ_1 in $A_0(\varepsilon)$. Let $A \subset A_0(\varepsilon)$ be the ring limited by γ_1 and γ_2 and $B \subset \mathcal{O}_q$ the closed cylinder whose boundary is $\gamma_1 \cup \gamma_2$. Then $T = A \cup B$ is an embedded topological torus containing \mathcal{O}_0 in its interior and such that $N \setminus T$ has two connected components N_0 and N_1 , with $\mathcal{O}_0 \subset N_0$, see Figure 3.

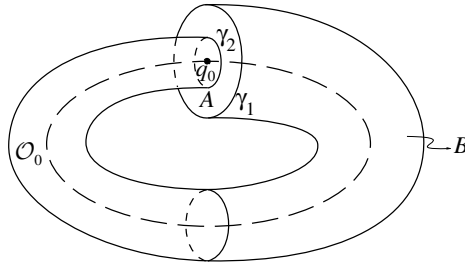


Figure 3

If $\mathcal{O}^+ = \{X_1^t(\gamma_1); t > 0\}$ and $\mathcal{O}^- = \{X_1^t(\gamma_1); t < 0\}$, then $\mathcal{O}_q = \mathcal{O}^- \cup \gamma_1 \cup \mathcal{O}^+$. Since X_1 is transverse to $A_0(\varepsilon) \supset A$, then $\mathcal{O}^+ \subset N_0$ and $\mathcal{O}^- \subset N_1$. Consequently, $\text{cl}(\mathcal{O}_q) \setminus \mathcal{O}_q$ has two connected components, Δ_0 and Δ . Since $\mathcal{O}_0 \subset \text{cl}(\mathcal{O}_q)$, we can assume that $\mathcal{O}_0 \subset \Delta_0$.

Finally, we show that $\Delta = \mathcal{O}$ and $\Delta_0 = \mathcal{O}_0$. We have $\mathcal{O} \subset \Delta$, otherwise φ has a minimal set $\mu \subset \Delta$, with $\mu \neq \mathcal{O}_0$ and \mathcal{O} , contradicting Theorem 2.8(i). It follows then from Proposition 2.5(1) that $\Delta = \mathcal{O}$. We consider a sequence $\{\gamma_k\}_{k \in \mathbb{N}}$, where $\gamma_k = f_0(\gamma_{k-1})$ is a closed orbit of Y_0 . If $\text{int}(\gamma_k)$ denotes the interior of the open 2-disk in $A_0(\varepsilon)$ which γ_k bounds, then $\gamma_k \subset \text{int}(\gamma_{k-1})$. We claim that $\bigcap_{k \in \mathbb{N}} \text{int}(\gamma_k) = \{q_0\}$, otherwise $\bigcap_{k \in \mathbb{N}} \text{int}(\gamma_k)$ is an open 2-disk in $A_0(\varepsilon)$ whose boundary γ is a closed orbit of Y_0 such that $f_0(\gamma) = \gamma$, and thus the φ -orbit that contains γ would be homeomorphic to T^2 . Therefore, $\bigcap_{k \in \mathbb{N}} \text{int}(\gamma_k) = \{q_0\}$. Since φ is given by a suspension in a neighbourhood of \mathcal{O}_0 , it follows that $\Delta_0 = \mathcal{O}_0$, which concludes the proof. \square

Remark 2.12. — Proposition 2.10 is not necessarily true when $q \in C_1$. For example, the closure of an $S^1 \times \mathbb{R}$ -orbit in a k -tube of φ at \mathcal{O}_0 (see Definition 3.9) does not contain \mathcal{O} .

2.3. C_0 versus C_1

Let us now study properties that distinguish the φ -orbits in C_0 from those in C_1 . Firstly we need the following result from [3] (see Chapter 8, Section 3).

LEMMA 2.13. — *Let $\varphi : G \times M \rightarrow M$ be a locally free action of a simply connected Lie group G on M , a C^∞ manifold with $\dim(M) = \dim(G) + 1 \geq 3$. If \mathcal{O} is an orbit of φ and $i : \mathcal{O} \rightarrow M$ is the canonical immersion, then $i_* : \pi_1(\mathcal{O}) \rightarrow \pi_1(M)$ is injective, i.e. if γ is a closed curve in \mathcal{O} homotopic to a constant in M , then γ is homotopic to a constant in \mathcal{O} .*

THEOREM 2.14. — *Suppose that \mathcal{O}_q is an $S^1 \times \mathbb{R}$ -orbit of $\varphi \in \mathcal{A}$ and let $\gamma \subset \mathcal{O}_q$ be a simple closed curve which is not homotopic to a constant in \mathcal{O}_q . Then :*

- (a) $\mathcal{O}_q \subset C_0$ if and only if γ is homotopic to a constant in N ,
- (b) if $\varphi \in \mathcal{A}_{S^1}$, then $C = C_0 = N \setminus (\mathcal{O}_0 \cup \mathcal{O})$,
- (c) $\mathcal{O}_q \subset C_0$ if and only if $\varphi \in \mathcal{A}_{S^1}$,
- (d) if $\mathcal{O}_q \subset C_0$, then γ bounds a closed 2-disk $D \subset N$ such that $\mathcal{O}_0 \cap D \neq \emptyset$,
- (e) $\mathcal{O}_q \subset C_1$ if and only if γ is not homotopic to a constant in N ,
- (f) $\mathcal{O}_q \subset C_1$ if and only if $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$.

In Figure 4 we have examples of $S^1 \times \mathbb{R}$ -orbits. Following the theorem above, the first one is an $S^1 \times \mathbb{R}$ -orbit in C_0 and the two last are in C_1 . Therefore, if \mathcal{O}_q is an φ -orbit whose traces on D_θ in a neighbourhood of S_1 are as in Figure 2 (a), then \mathcal{O}_q is an $S^1 \times \mathbb{R}$ -orbit in C_0 . On the other hand, if its traces are as in Figure 2 (b) and \mathcal{O}_q is an $S^1 \times \mathbb{R}$ -orbit, then \mathcal{O}_q is in C_1 .

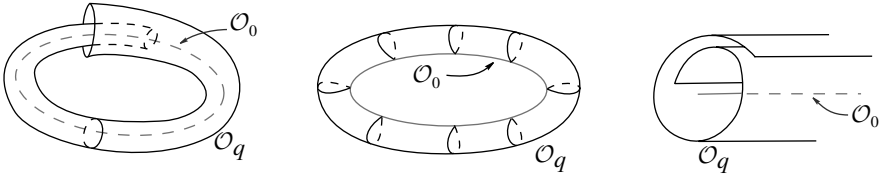


Figure 4. — Examples of $S^1 \times \mathbb{R}$ -orbits in C_0 and C_1

Proof of Theorem 2.14. — (a) Let $\{X_1, X_2\}$ be infinitesimal generators adapted to φ at \mathcal{O}_0 . If $\mathcal{O}_q \subset C_0$, then we take w_2 as the generator of G_q . Let $p \in \mathcal{O}_0$ and V_p a neighbourhood of p . By Theorem 2.8 and Lemma 2.11, there exist a neighbourhood U_p of p and a point $q' \in U_p \cap \mathcal{O}_q$ such that $\mathcal{O}_{q'}(X_2) \subset V_p$. We can take V_p sufficiently small such that $\mathcal{O}_{q'}(X_2)$ bounds a closed 2-disk inside V_p , i.e. $\mathcal{O}_{q'}(X_2)$ is homotopic to a constant in N . Since γ is simple, then $\mathcal{O}_{q'}(X_2)$ and γ (or $-\gamma$) are homotopic in \mathcal{O}_q , and therefore γ is homotopic to a constant in N . Conversely, suppose that γ is null homotopic in N and $\mathcal{O}_q \subset C_1$. Since $\mathbb{R}^2 = H \oplus G_0^0$, we can assume that H is generated by G_q . Let $f_0 : A_0(\varepsilon) \rightarrow U_0$ be the Poincaré diffeomorphism of X_1 at $q_0 \in \mathcal{O}_0$. Let $V(\varepsilon) = S^1 \times D_\varepsilon$ where $D_\varepsilon = \{(x_1, x_2) \in D^2; x_1^2 + x_2^2 < \varepsilon\}$ and $\pi : V(\varepsilon) \rightarrow S^1$ be given by $\pi(\theta, x) = \theta$. There exists $\varepsilon > 0$ such that $\pi^{-1}(\theta)$ is transverse to X_1 , for each $\theta \in S^1$. Since $\mathcal{O}_0 \subset \text{cl}(\mathcal{O}_q)$, there exist $p \in \mathcal{O}_q \cap A_0(\varepsilon)$ and $n \in \mathbb{Z}$ such that $\mathcal{O}_p(X_1) \subset V(\varepsilon)$ is periodic of period n , i.e. $f_0^n(p) = p$. Then $\pi^{-1}(\theta_0) \cap \mathcal{O}_p(X_1)$ has exactly n elements, and consequently every $\pi^{-1}(\theta)$, $\theta \in S^1$, contains exactly n points of $\mathcal{O}_p(X_1)$. In particular, $\pi(\mathcal{O}_p(X_1)) = \mathcal{O}_0$, and thus $\mathcal{O}_p(X_1)$ is not homotopic to a constant in N . This is not possible since $\mathcal{O}_p(X_1)$ is homotopic to γ (or $-\gamma$) in \mathcal{O}_q , which by hypothesis is homotopic to a constant in N . This contradiction shows that $\mathcal{O}_q \subset C_0$ and completes the proof of (a).

(b) If $\varphi \in \mathcal{A}_{S^1}$, by Proposition 2.5 and item (a), there exists a φ -invariant neighbourhood U of \mathcal{O} such that $U \setminus \mathcal{O} \subset C_0$. It follows from Proposition 2.10 that $U = N \setminus \mathcal{O}_0$.

(c) We assume that $q \in C_0$ and $\varphi \notin \mathcal{A}_{S^1}$. By Proposition 2.10, $\mathcal{O} \subset \text{cl}(\mathcal{O}_q)$, and consequently it follows from Proposition 2.5 that $\varphi \in \mathcal{A}_{\mathbb{R}}^c$.

Since $P_2 = \text{id}$, there exists a simple closed curve $\gamma \subset \mathcal{O}_q$ which is neither homotopic to a constant in \mathcal{O}_q , nor homotopic to a constant in N , contradicting item (a). This shows that $\varphi \in \mathcal{A}_{S^1}$. Conversely, if $\varphi \in \mathcal{A}_{S^1}$, it follows from (b), that $\mathcal{O}_q \subset C_0$.

(d) Let $q \in C_0$ and φ' be the action of \mathbb{R}^2 on $N' = N \setminus (\mathcal{O}_0 \cup \mathcal{O})$ defined by $\varphi'(g, p) = \varphi(g, p)$. By item (a), γ bounds a closed 2-disk $D \subset N$. If $\mathcal{O}_0 \cap D = \emptyset$, by Lemma 2.13, γ is homotopic to a constant in \mathcal{O}_q , since \mathcal{O}_q is also a φ' -orbit.

(e)-(f) Since $C = C_0 \cup C_1$ with $C_0 \cap C_1 = \emptyset$, it follows that (e) (resp. (f)) is equivalent to (a) (resp. (c)). \square

3. Proof of the main results

In this section we simultaneously prove Theorems A and A'. We first show the existence of Σ , a 2-disk embedded in N , in general position with respect to \mathcal{F}_φ such that the foliation in Σ induced by φ -orbits is orientable (Proposition 3.2). More precisely, let $\varphi \in \mathcal{A}$ and \mathcal{F}_0 be the restriction of \mathcal{F}_φ to $N \setminus \mathcal{O}_0$. An embedding $g : D^2 \rightarrow N$ with $g(0) \in \mathcal{O}_0$ is said to be in *general position* with respect to \mathcal{F}_φ if g is transverse to \mathcal{F}_φ at $g(0)$ and, for every distinguished map f of \mathcal{F}_0 , the map $(f \circ g)|_{D^2 \setminus \{0\}}$ is locally of Morse type. The submanifold $g(D^2)$ is said to be in *general position* with respect to \mathcal{F}_φ .

Remark 3.1. — If $g : D^2 \rightarrow N$ with $g(0) \in \mathcal{O}_0$ is in general position with respect to \mathcal{F}_φ , then g induces a foliation \mathcal{F}^* in $g(D^2)$ whose leaves are the connected components of the intersection of the leaves of \mathcal{F}_φ with $g(D^2)$. Furthermore, \mathcal{F}^* has a finite number of singularities that are centres or saddles, except maybe for $g(0)$. The singularities of \mathcal{F}^* are the points where $g(D^2)$ is tangent to a leaf of \mathcal{F}_φ .

PROPOSITION 3.2. — *If $\varphi \in \mathcal{A}$, then there exists Σ , a closed 2-disk embedded in N , in general position with respect to \mathcal{F}_φ , such that $\partial\Sigma \subset \mathcal{O}$ and the foliation \mathcal{F}^* in Σ induced by \mathcal{F}_φ is given by a vector field $Z_\varphi \in \mathfrak{X}^2(\Sigma)$.*

Proof. — Let $j : D^2 \rightarrow N$ be the inclusion such that $j(D^2) = D_{\theta_0} = \{\theta_0\} \times D^2$, for some $\theta_0 \in S^1$. Given $\varepsilon > 0$ and an integer $r \geq 2$, with an adaptation of Haefliger's techniques [7], we obtain a C^∞ -embedding $g : D^2 \rightarrow N$ in general position with respect to \mathcal{F}_φ such that g is ε -close to j in the C^r -topology and coincides with j in neighbourhoods of ∂D^2 and 0.

The foliation \mathcal{F}^* , in a neighbourhood of $q_0 = (\theta_0, 0)$, is given by the vector field Y_0 , as in the Equation 2.2. Thus, by Remark 3.1, \mathcal{F}^* is C^2 locally orientable, and consequently by [3], Chapter 6, Section 4, \mathcal{F}^* is C^2 orientable. Set $\Sigma = g(D^2)$. It follows that the foliation \mathcal{F}^* is given by a vector field $Z_\varphi \in \mathfrak{X}^2(\Sigma)$. \square

Remark 3.3. — There exists $\varepsilon > 0$ such that $A_0(\varepsilon)$ and $A_1(\varepsilon)$ are contained in Σ . We can take Z_φ such that $Z_\varphi|_{A_0(\varepsilon)} = Y_0$ and $Z_\varphi|_{A_1(\varepsilon)} = \widehat{X}_1$, where Y_0 and \widehat{X}_1 are the vector fields given in Equations (2.2) and (2.5), respectively. Furthermore, by a small isotopy of Σ in a neighbourhood of each singularity different from q_0 , we may assume that no two singularities of Z_φ in $\Sigma \setminus \{q_0\}$ are on the same leaf of \mathcal{F}_φ . Thus we can assume that there is no connection between two different saddles of Z_φ in $\Sigma \setminus \{q_0\}$.

As an immediate consequence of Theorem 2.14(a), we have :

COROLLARY 3.4. — *If $Z_\varphi \in \mathfrak{X}^2(\Sigma)$ is a vector field induced by $\varphi \in \mathcal{A}$ and γ is a closed orbit of Z_φ such that the interior of the open 2-disk in Σ which γ bounds contains q_0 , then the φ -orbit that contains γ is an $S^1 \times \mathbb{R}$ -orbit that is contained in C_0 .*

For the proof of Theorems A and A' we need a result that shows that the vector fields as shown in Figure 5 are not induced by the action of any $\varphi \in \mathcal{A}$.

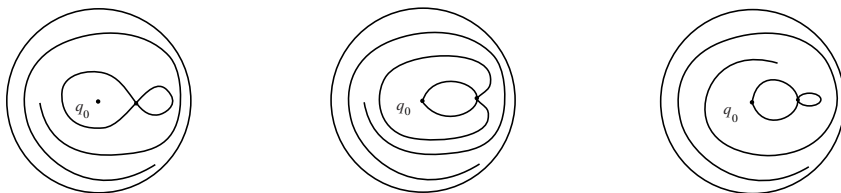


Figure 5. — Impossible configurations for induced vector fields by $\varphi \in \mathcal{A}$

A *limit cycle* Γ of Z_φ is a non-empty limit set $\alpha(\gamma)$ or $\omega(\gamma)$, of some orbit γ of Z_φ such that $\gamma \cap \Gamma = \emptyset$, and which does not consist of a singular point. By the Poincaré-Bendixson Theorem, if $q_0 \notin \Gamma$, then Γ is either a periodic orbit or a graph of Z_φ . In the last case, by taking Z_φ as in Remark 3.3, Γ is the union of a saddle p with one or two self-connections of p . Since Γ is connected, if $\Gamma \neq \partial\Sigma$, then $\Sigma \setminus \Gamma$ has at least two connected components, a neighbourhood of $\partial\Sigma$ being one of them.

PROPOSITION 3.5. — *Let $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$ and $\Gamma \neq S_1$ be a limit cycle of Z_φ . Then $q_0 \in \Gamma$ and $\Gamma \setminus \text{Sing}(Z_\varphi)$ has a finite number of connected components, each one of them contains q_0 in its closure.*

Proof. — We start by showing that $q_0 \in \Gamma$. Let \mathcal{O}_Γ be the φ -orbit containing Γ , and $q \in \Gamma$ a regular point of Z_φ . Since Γ is a limit cycle, the holonomy transformation of \mathcal{O}_Γ at q associated to Γ is not trivial. Consequently, Γ is not homotopic to a constant in \mathcal{O}_Γ . If $q_0 \notin \Gamma$, we claim that $q_0 \in R(\Gamma)$, where $R(\Gamma)$ is the union of the connected components of $\Sigma \setminus \Gamma$ that do not contain $\partial\Sigma$. Indeed, if $q_0 \notin R(\Gamma)$, then Γ is homotopic to a constant in $\Sigma \setminus \{q_0\}$, and hence, homotopic to a constant in $N \setminus (\mathcal{O}_0 \cup \mathcal{O})$. Consequently, by Lemma 2.13, Γ is null homotopic in \mathcal{O}_Γ . But this contradicts the fact that Γ is a limit cycle of Z_φ . Therefore, $q_0 \in R(\Gamma)$. By taking Z_φ as in Remark 3.3, Γ is either a simple closed curve or the union of two simple closed curves Γ_1 and Γ_2 with $\Gamma_1 \cap \Gamma_2 = \{q\} = \text{Sing}(Z_\varphi) \cap \Gamma$. Since $q_0 \in R(\Gamma)$, it follows that either Γ or Γ_i , for some $i = 1, 2$, is not homotopic to a constant in \mathcal{O}_Γ . However these are homotopic to a constant in N . Thus, \mathcal{O}_Γ is an $S^1 \times \mathbb{R}$ -orbit, and by Theorem 2.14(a), $\mathcal{O}_\Gamma \subset \mathcal{C}_0$. Hence, by item (c) of Theorem 2.14, $\varphi \in \mathcal{A}_{S^1}$. This contradiction proves that $q_0 \in \Gamma$.

Finally, by the Poincaré-Bendixson Theorem, $\Gamma \setminus \text{Sing}(Z_\varphi)$ has a finite number of connected components. Let $s \subset \Gamma$ be a separatrix of $p \in \Gamma \cap \text{Sing}(Z_\varphi)$. We show that $\text{cl}(s) = s \cup \{p, q_0\}$. If this is not the case, since Z_φ has no connection between two different saddles in $\Sigma \setminus \{q_0\}$, then $\gamma = s \cup \{p\}$ is a simple closed curve which bounds a closed 2-disk $D \subset N$ such that $q_0 \notin \text{int}(D)$. Then, γ is homotopic to a constant in $N \setminus (\mathcal{O}_0 \cup \mathcal{O})$, and by Lemma 2.13, γ is homotopic to a constant in \mathcal{O}_Γ . Again, this contradicts the fact that $\Gamma \supset \gamma$ is a limit cycle, and completes the proof. \square

Remark 3.6. — Note that if $Z_\varphi \in \mathfrak{X}^2(\Sigma)$ has no limit cycle, then q_0 is the only singularity of Z_φ .

We also need the following lemma and some definitions. Let $\{w_1, w_2\}$ be a basis of \mathbb{R}^2 such that the vector fields $\{X_1 = X_{w_1}, X_2 = X_{w_2}\}$ are infinitesimal generators adapted to φ at \mathcal{O} .

LEMMA 3.7. — *If $\varphi \in \mathcal{A}_{S^1} \cup \mathcal{A}_{\mathbb{R}}^c$, then there exist a neighbourhood V of \mathcal{O} and, for each $i \in \{1, 2\}$, a C^2 function $u_i : V \rightarrow \mathbb{R}^2$ such that $u_i(p_0) = w_i$ and for each $q \in V$, $u_i(q)$ generates G_q , $i = 1, 2$.*

Proof. — Let $I^2(\varepsilon) = \{(x_1, x_2) \in \mathbb{R}^2; |x_1|, |x_2| < \varepsilon\}$ and $h : V_{p_0} \rightarrow I^2(\varepsilon) \times (-\varepsilon, 0]$ be a chart at p_0 with $h(p_0) = 0$, such that, if (x_1, x_2, x_3) are the coordinates of $I^2(\varepsilon) \times (-\varepsilon, 0]$, then $h_*X_k = \partial/\partial x_k$ for $k = 1, 2$. Let

$D_k = D_k(\varepsilon) = \{(x_1, x_2, x_3) \in I^2(\varepsilon) \times (-\varepsilon, 0]; x_k = 0\}$ and $\Sigma_k = \Sigma_k(\varepsilon) = h^{-1}(D_k)$ for $k = 1, 2$. The function $\tau_k : V_{p_0} \rightarrow (-\varepsilon, \varepsilon)$ given by $\tau_k(q) = -x_k(q)$, where $h(q) = (x_1(q), x_2(q), x_3(q))$, is such that $X_k^{\tau_k(q)}(q) \in \Sigma_k$, for $k = 1, 2$. Since $X_k^1(p_0) = p_0$, $k = 1, 2$, it follows that there exists $0 < \delta < \varepsilon$ such that $X_k^1(\Sigma_k(\delta)) \subset V_{p_0}$, $k = 1, 2$. Let $\Sigma_{p_0} = \Sigma_{p_0}(\delta) = \Sigma_1(\delta) \cap \Sigma_2(\delta)$. Then Σ_{p_0} is a transverse section to \mathcal{O} at p_0 . Consider the functions $u_i : \Sigma_{p_0} \rightarrow \mathbb{R}^2$ given by

$$\begin{aligned} u_1(q) &= (1 + \tau_1(X_1^1(q)))w_1 + \tau_2(X_1^1(q))w_2, \text{ if } i = 1, \\ u_2(q) &= \tau_1(X_2^1(q))w_1 + (1 + \tau_2(X_2^1(q)))w_2, \text{ if } i = 2. \end{aligned} \tag{3.7}$$

It can be shown that every orbit of $X_{u_i(q)}$ inside \mathcal{O}_q , $q \in \Sigma_{p_0}$, is periodic of period one and $u_i(p_0) = w_i$, $i = 1, 2$. We can therefore extend the functions u_i to the open set $V = \cup_{q \in \Sigma_{p_0}} (\mathcal{O}_q \cap V_{p_0})$ by defining $u_i(q) = u_i(\Sigma_{p_0} \cap \mathcal{O}_q)$. \square

DEFINITION 3.8. — Let Z be a vector field on \mathbb{R}^2 , p a singularity of Z , $k \geq 0$ an integer and $\Gamma = \cup_{i=0}^k \gamma_i$, where $\gamma_0 = p$ and γ_i , $i = 1, \dots, k$, is a regular orbit of Z . We then say that Γ is a k -petal of Z at p if $\text{cl}(\gamma_i) \setminus \gamma_i = \{p\}$ and $\text{cl}(\gamma_i)$ is the frontier of an open 2-disk D_i such that $D_i \cap D_j = \emptyset$ for $j = 1, \dots, k$ with $j \neq i$.

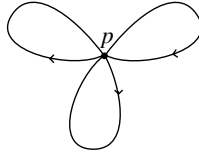


Figure 6. — A 3-petal at p

DEFINITION 3.9. — Let $\varphi \in \mathcal{A}$, $k \geq 0$ an integer and $\Gamma = \cup_{i=0}^k \mathcal{O}_i$, where \mathcal{O}_i , $i = 1, \dots, k$, is an $S^1 \times \mathbb{R}$ -orbit. One says that Γ is a k -tube of φ at \mathcal{O}_0 if $\text{cl}(\mathcal{O}_i) \setminus \mathcal{O}_i = \mathcal{O}_0$ and $\text{cl}(\mathcal{O}_i)$ is the frontier of an open solid torus T_i such that $T_i \cap T_j = \emptyset$ for $j = 1, \dots, k$ with $j \neq i$. The set $S_\Gamma = \cup_{i=1}^k \text{cl}(T_i)$ is said to be a *solid k -tube* of φ at \mathcal{O}_0 associated to k -tube Γ .

Remark 3.10. — The second figure of Figure 4 gives an example of a 1-tube. Note that every $S^1 \times \mathbb{R}$ -orbit in a k -tube is contained in C_1 .

3.1. Proof of Theorems A and A'

We consider the following two cases separately. Let $\{w_1, w_2\}$ be a basis of \mathbb{R}^2 such that the vector fields $\{X_1 = X_{w_1}, X_2 = X_{w_2}\}$ are infinitesimal generators adapted to φ at \mathcal{O} .

(i) **The case when** $\varphi \in \mathcal{A}_{S^1}$

We shall show that there exists a closed 2-disk Σ embedding in N transverse to \mathcal{F}_φ such that the induced vector field Z_φ satisfies Theorem A'(1). Theorem A(1) follows then by Corollary 3.4 and Proposition 2.10.

By Theorem 2.14, $\mathcal{O}_q \subset \mathcal{C}_0$ for each $q \in N \setminus (\mathcal{O}_0 \cup \mathcal{O})$. Let U be the φ -invariant neighbourhood of \mathcal{O} as in Proposition 2.5 and assume it contains the neighbourhood V given in Lemma 3.7. Since $\mathcal{O} \subset \text{cl}(\mathcal{O}_q)$, $q \in U$, then $G_q \subset G$. Consequently, it follows by Lemma 3.7 that for each $q \in U \setminus \mathcal{O}$, G_q is generated by w_1 . Let $A = \cup_{q \in J} \mathcal{O}_q(X_1)$, where J is a segment in J_ε with end points p_0 and p_1 . It is not hard to show that A is homeomorphic to $S^1 \times [0, 1]$ and transverse to X_2 , and consequently A is transverse to \mathcal{F}_φ . Given a neighbourhood V_0 of q_0 , by Lemma 2.11, there exists a neighbourhood U_0 of q_0 such that $\mathcal{O}_q(X_1) \subset V_0$, for each $q \in U_0 \cap \mathcal{O}_{p_1}$. Since $\mathcal{O}_0 \subset \text{cl}(\mathcal{O}_{p_1})$, there exists $t \in \mathbb{R}$ such that $X_2^t(p_1) \in U_0$ and, consequently, $\mathcal{O}_{X_2^t(p_1)}(X_1) = X_2^t(\mathcal{O}_{p_1}(X_1)) \subset V_0$. The ring $A_2 = X_2^t(A)$ is the union of closed orbits of X_1 , transverse to \mathcal{F}_φ , with $\partial A_2 = \mathcal{O}_{X_2^t(p_0)}(X_1) \cup \mathcal{O}_{X_2^t(p_1)}(X_1)$. We consider V_0 as the domain of a chart adapted to φ at q_0 . Then there exists a closed 2-disk $D \subset V_0$ transverse to \mathcal{F}_φ such that $\partial D = \mathcal{O}_{X_2^t(p_1)}(X_1)$. Let $X \in \mathfrak{X}^2(D)$ be the vector field whose orbits are the connected components of the intersection of D with the φ -orbits. We claim that every orbit of X inside $D \setminus \mathcal{O}_0$ is periodic. Otherwise, there exists $q \in D$ such that $\omega_X(q) = \gamma$ (or $\alpha_X(q) = \gamma$), where $\gamma \subset D$ is a closed orbit of X and $q \notin \gamma$. Then \mathcal{O}_γ , the φ -orbit containing γ , is contained in $\text{cl}(\mathcal{O}_q)$ and is different from \mathcal{O}_0 and \mathcal{O} . But this contradicts Proposition 2.10. Therefore all the orbits of X in $D \setminus \mathcal{O}_0$ are periodic. Let $\tilde{D} = A_2 \cup D$, $f : D^2 \rightarrow \tilde{D} \subset N$ a homeomorphism, and D_0 be a neighbourhood of $f^{-1}(\partial D)$ in D^2 . We take a C^∞ embedding $g : D^2 \rightarrow N$, arbitrarily close to f in the C^0 -topology, such that $g|_{(D^2 \setminus D_0)} = f|_{(D^2 \setminus D_0)}$ and transverse to \mathcal{F}_φ . Then, the foliation \mathcal{F}^* in $\Sigma = g(D^2)$, induced by \mathcal{F}_φ , is given by a vector field $Z_\varphi \in \mathfrak{X}^2(\Sigma)$ whose orbits in $\Sigma \setminus (\mathcal{O}_0 \cap \Sigma)$ are all periodic.

(ii) **The case when** $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$

In this case we take Z_φ as in Remark 3.3. Let V be the union of Z_φ -orbits by points of $A_1(\varepsilon)$. Then V is homeomorphic to $S^1 \times (0, 1]$ and every orbit of Z_φ inside $V \setminus S_1$ is homeomorphic to \mathbb{R} . Hence, the frontier Γ of V in Σ , either coincides with $\{q_0\}$ or is a limit cycle of Z_φ .

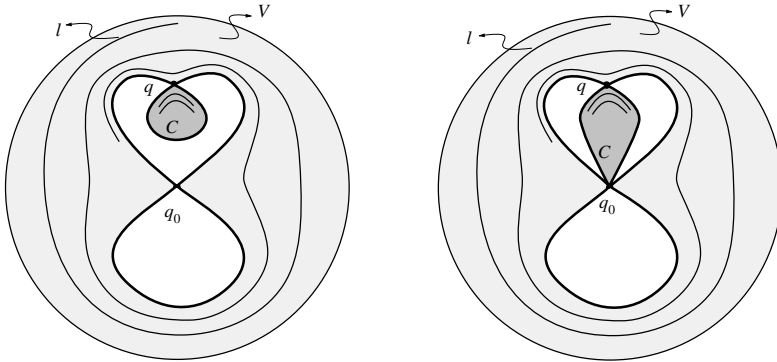


Figure 7. — The φ -orbit that contain l intersects the region D

We suppose that Γ is a limit cycle of Z_φ . By Lemma 2.4, there exist a neighbourhood W_1 of \mathcal{O} , a neighbourhood U_1 of S_1 in D_{θ_0} , and a C^2 diffeomorphism $f_1 : A_1(\varepsilon) \rightarrow U_1$ that preserves orbits of \hat{X}_1 , such that $\mathcal{F}_\varphi|_{W_1}$ is the suspension of $\hat{X}_1 = Z_\varphi|_{A_1(\varepsilon)}$ by f_1 . Then, as V is invariant by Z_φ , we obtain an extension $F_1 : V \rightarrow V$ of f_1 , that is given by $F_1(q) = Z_\varphi^t(f_1(Z_\varphi^{-t}(q)))$, where $t \in \mathbb{R}$ is such that $Z_\varphi^{-t}(q) \in S$ with $S \subset A_1(\varepsilon)$ a circle transverse to Z_φ . Note that, by definition, F_1 preserves the orbits of Z_φ , and thus $\mathcal{F}_\varphi|_U$ is the suspension of $Z_\varphi|_V$ by F_1 , where U is as in Proposition 2.5. Therefore, $V = U \cap \Sigma$. We claim that Γ is a k -petal, i.e. $\Gamma \cap \text{Sing}(Z_\varphi) = \{q_0\}$. Suppose that there exists $q \in \Gamma \cap \text{Sing}(Z_\varphi)$ with $q \neq q_0$. Since Γ is a limit cycle, q is necessarily a saddle. Hence, as Σ is in general position with respect to \mathcal{F}_φ , every φ -orbit by points in V has points in $\Sigma \setminus \text{cl}(V)$, see Figure 7. But this contradicts the fact that $V = U \cap \Sigma$. This contradiction and Proposition 3.5 show that Γ is a k -petal of Z_φ at q_0 , and consequently $\text{Front}(U)$ is a k -tube of φ at \mathcal{O}_0 . Theorem A (2) and (3) then follow from Theorem 2.14. \square

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