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Equisingular generic discriminants and Whitney conditions[(∗)]

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Abstract. — The purpose of this article is to show that *the Whitney conditions* are satisfied for complex analytic families of normal surface singularities for which the *generic discriminants* are *equisingular*. According to J. Briançon and J. P. Speder the constancy of the topological type of a family of surface singularities does not imply Whitney conditions in general. We will see here that for a family of *minimal normal surface singularities* these two equisingularity conditions are equivalent.

Résumé. — L'objet de cet article est de montrer que les conditions de Whitney sont satisfaites pour les familles analytiques complexes de singularités de surfaces normales à discriminants génériques équisinguliers. D'après J. Briançon et J. P. Speder, la constance du type topologique d'une famille de singularités de surfaces n'implique pas en general les conditions de Whitney. Nous verrons ici que pour une famille de singularités minimales de surfaces normales ces deux conditions d'équisingularité sont équivalentes.

Introduction

The principal tools we will use are the theory of polar varieties as developed by B. Teissier and Lê Dũng Tráng (cf. [14], [19]). In [19], B. Teissier gives numerical conditions for a stratification to be Whitney. Our situation is to consider a family of normal surface singularities $f : (X, 0) \longrightarrow (\mathbb{D}, 0)$ with *equisingular generic discriminants* and to show that *Whitney conditions* are satisfied. We refer to section three for more details. Let us recall that by Thom-Mather's isotopy lemma the Whitney conditions imply the constancy of the topological type [15]. Let us also note that different

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surface singularities with equisingular generic discriminants are not topologically equivalent in general (cf. Example 4.4). The question resolved in this article was suggested by Lê Dũng Tráng. The answer is certainly well known by the specialists. We did not find any proof anywhere. However we refer to [7] (cf. Theorem 4.1 and Remark 4.4) for families of complex surface singularities $X \hookrightarrow \mathbb{C}^3 \times \mathbb{D}$. It is shown in [7] that for such families the constancy of some numerical invariants is equivalent to the "generic equisingularity "in Zariski's sense (cf. [7]) and that generic equisingularity is stronger than Whitney conditions along the singular locus. We will use the class of *minimal surface singularities* for some applications. These singularities were studied by M. Spivakovsky [18], and recently by R. Bondil [3], [4], (see also [1]). R. Bondil gives in [3] the algebraic structure of the generic discriminants of minimal surface singularities (see also [4], [1]). It turns out that *equisingular minimal surface singularities* (according to Definition 3.7) have equisingular generic discriminants.
In the first section we review the concept of polar varieties (we refer to [19] for more details) and give the main result of B. Teissier in [19] on the numerical characterization of the Whitney conditions. *The generic discriminants* of normal surface singularities will be defined in section 2. The sections 3 and 4 are the principal part of this article. We refer to [6], [20] for the concept of equisingularity of reduced plane curves.

1. Local polar varieties and Whitney conditions

The notion of polar varieties was developed as a mean of studying the singularities of analytic varieties.
Let us consider an analytic morphism $f : (X, 0) \longrightarrow (Y, 0)$ of reduced complex analytic spaces such that the fibers of f are smooth of dimension $d = \dim X - \dim Y$ on the complement of a closed nowhere dense analytic subset F of X. Here we suppose that Y is a smooth space. We can embed $(X, 0)$ in $Y \times \mathbb{C}^N$ so that the following diagram commutes.

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (Y,0) \times (\mathbb{C}^N, 0) \\ f \downarrow & \swarrow & \\ (Y,0) & & \end{array}$$

The relative k-th polar variety, $0 \leqslant k \leqslant d$ is obtained by choosing a generic projection $p : X \longrightarrow Y \times \mathbb{C}^{d-k+1}$ (with kernel D_{d-k+1}), calculating the critical set of the restriction of p to $f^{-1}(s) \setminus (f^{-1}(s) \cap F)$ for all s and taking the closure of this. This set is denoted $p_k(f, D_{d-k+1})$ or sometimes

$p_k(X, f)$ when p is understood. B. Teissier showed in [19] that for a generic projection $p : X \longrightarrow Y \times \mathbb{C}^{d-k+1}$ the polar variety $p_k(X, D_{d-k+1})$ is a closed analytic subset of X, purely of codimension k in X if it is not empty. The key invariant of $p_k(f, D_{d-k+1})$ is its multiplicity which we denote by $m_0(p_k(f, D_{d-k+1}))$ or $m_k(X, f)$ if D_{d-k+1} is understood. If f is the constant map, we denote this multiplicity by $m_k(X)$. For a generic projection this turns out to be independent of D_{d-k+1} and is in fact an analytic invariant (we refer to [19] for more details). B. Teissier uses these invariants to give a necessary and sufficient conditions so that a stratification be Whitney.
The key result that we will use is ([19], page 470):

Theorem 1.1. — *Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be a morphism with a section σ, whose fibers are purely of dimension d and reduced. Set $Y = \sigma(\mathbb{C})$ and suppose we have a local $\mathbb{C}$-embedding*

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathbb{C},0)\times(\mathbb{C}^N,0) \\ f \big\downarrow \big\uparrow \sigma & \swarrow & \\ (\mathbb{C},0) & & \end{array}$$

such that $Y = \mathbb{C} \times \{0\}$.
The following conditions are equivalent (here X^o stands for the smooth locus of the family X).
i) The hyperplane $K = \{0\} \times \mathbb{C}^N$ is transverse to all limiting tangent planes at X^o and the map $Y \longrightarrow N^{d+1}$ defined by:
$y \longrightarrow (m_0(X, f)_y, m_1(X, f)_y, \ldots, m_d(X, f)_y)$ is constant on Y in a neighborhood of 0.
ii) The pair of strata (X^o, Y) satisfies the Whitney conditions at 0.

2. Generic discriminants of normal surface singularities

Let $(S, 0)$ be a germ of normal complex surface singularity and take a representative S embedded in $(\mathbb{C}^N, 0)$. For any $(N-2)$-dimensional subspace D in $\mathbb{C}^N$, we consider the linear projection $\mathbb{C}^N \longrightarrow \mathbb{C}^2$ with kernel D and denote by $p_D : (S, 0) \longrightarrow (\mathbb{C}^2, 0)$ the restriction of this projection to $(S, 0)$.
Considering a small representative S of the germ $(S, 0)$ and restricting ourselves to the $(N-2)$-dimensional subspaces D such that p_D is finite, we define as in [19] the *polar curve* $C(D)$ of the projection p_D as the closure in S of the critical locus of the restriction of p_D to $S \setminus \{0\}$. It is a reduced analytic curve. It is shown in [19] that it makes sense to say that for an open

dense subset of the grassmannian of $(N-2)$-dimensional linear subspaces of $\mathbb{C}^N$ the polar curves $C(D)$ are equisingular in term of strong simultaneous resolution (cf. [6] for this notion). It also turns out that this equisingularity class depends only on the analytic type of the germ $(S,0)$ (cf. [19], page 430).

The discriminant of the finite projection p_D is (the germ at 0 of) the reduced analytic curve of $(\mathbb{C}^2,0)$, image of the polar curve $C(D)$.

We can state the following result ([6], Proposition VI.2, [19], page 352, 462).

Theorem 2.1. — *There is an open dense subset $\mathcal{W}$ of the grassmannian of $(N-2)$-linear subspaces of $\mathbb{C}^N$ such that the discriminants Δ_{p_D}, $D \in \mathcal{W}$ obtained are equisingular (germs of) plane curves.*

We refer to [20], [6] for the concept of equisingularity of reduced plane curves. As explained in [6] the equisingularity class of the discriminant Δ_{p_D}, $D \in \mathcal{W}$ is uniquely defined by the saturation ring $\widetilde{\mathcal{O}}_{C(D),0}$ of the polar curve $C(D)$. Also note that the equisingularity class of the discriminant Δ_{p_D}, $D \in \mathcal{W}$ depends only on the analytic type of the germ $(S,0)$. We will denote by $\Delta_{S,0}$ the equisingularity class of the discriminant of a generic projection p_D and call it *the generic discriminant* of the normal surface singularity $(S,0)$. Note that for a generic projection the polar curve and the discriminant have the same multiplicity at 0 (cf. [19]).

3. Whitney conditions for a family of normal surface singularities

Our situation is the following: let $f : (X,0) \longrightarrow (\mathbb{D},0)$ be an analytic morphism of reduced complex spaces with a section σ, where $\mathbb{D}$ is a small complex disk centered at 0. We suppose that for all $t \in \mathbb{D}$, X_t is a normal surface with isolated singularity at $\sigma(t)$, i.e. $X_t \setminus \sigma(t)$ is smooth. We can consider a $\mathbb{D}$-installation (i.e., an embedding $(X,0) \hookrightarrow (\mathbb{D},0)\times(\mathbb{C}^N,0)$ such that the following diagram commutes).

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathbb{D},0)\times(\mathbb{C}^N,0) \\ f\big\downarrow\big\uparrow\sigma & \swarrow & \\ (\mathbb{D},0) & & \end{array}$$

We suppose $\sigma(\mathbb{D}) = \mathbb{D}\times\{0\}$.

Theorem 3.1. — *Assume that the generic discriminants $\Delta_{X_t,\sigma(t)}$, $t \in \mathbb{D}$ of the normal surface singularities $(X_t,\sigma(t))$, $t \in \mathbb{D}$ are equisingular. Then* *we can find a projection $\pi : X \longrightarrow \mathbb{D} \times \mathbb{C}^2$ compatible with f such that for all $t \in \mathbb{D}$ the discriminant $(\Delta_t, \sigma(t))$ of the restriction $\pi_t : X_t \longrightarrow \{t\} \times \mathbb{C}^2$ of π to X_t is equisingular to the generic discriminant $\Delta_{X_t,\sigma(t)}$.*

Proof. — Let us consider a projection $\pi : X \longrightarrow \mathbb{D} \times \mathbb{C}^2$ which is generic for both (X, f) and the special fiber $X_0 := f^{-1}(0)$ (i.e., $\pi : X \longrightarrow \mathbb{D} \times \mathbb{C}^2$ gives the generic relative polar variety (cf. section 1) and $\pi_0 : X_0 \longrightarrow \{0\} \times \mathbb{C}^2$ gives the generic discriminant of the singularity $(X_0, 0)$). We denote by Δ the "divisorial" part of the Fitting discriminant of the projection $\pi : X \longrightarrow \mathbb{D} \times \mathbb{C}^2$ (we refer to [5] for this notion). The discriminant $(\Delta, 0)$ is a reduced hypersurface in $(\mathbb{D} \times \mathbb{C}^2, 0)$. The discriminant of the restriction π_t of π to X_t is $\Delta_t = \Delta \cap (\{t\} \times \mathbb{C}^2)$. Since Δ is Cohen-Macaulay the induced morphism $h : \Delta \longrightarrow \mathbb{D}$ is flat so that the generic fibers Δ_t are reduced (Δ_0 is reduced by hypothesis).
Let us show that the projections π_t are generic, i.e. Δ_t is the generic discriminant of $(X_t, \sigma(t))$.

Since the Milnor number (cf. [6], [16]) is upper semi-continuous we have:

$$\mu(\Delta_0, 0) \geqslant \mu(\Delta_t, \sigma(t)). \tag{3.1}$$

This inequality cannot be strict. Indeed, suppose that the following inequality holds for some t:

$$\mu(\Delta_0, 0) > \mu(\Delta_t, \sigma(t)). \tag{3.2}$$

There is a family of projections $P = (p_l)_{l \in \mathbb{D}}$, $P : \mathbb{D} \times X_t \longrightarrow \mathbb{D} \times \mathbb{C}^2$ such that the projections p_l, $l \in \mathbb{D} \setminus \{0\}$ are generic and the discriminant of $p_0 : \{0\} \times X_t \longrightarrow \{0\} \times \mathbb{C}^2$ is equisingular to $(\Delta_t, \sigma(t))$. We have $\mu(\Delta_{X_t,\sigma(t)}) = \mu(\Delta_{p_l})$ for $l \neq 0$. Since the Milnor number is upper semi-continuous we have

$$\mu(\Delta_{p_0}) \geqslant \mu(\Delta_{p_l}). \tag{3.3}$$

It follows that

$$\mu(\Delta_0, 0) > \mu(\Delta_t, \sigma(t)) \geqslant \mu(\Delta_{p_l}) = \mu(\Delta_{X_t,\sigma(t)}).$$

This is contrary to the fact that $\mu(\Delta_0, 0) = \mu(\Delta_{X_t,\sigma(t)})$ since $(\Delta_0, 0)$ is equisingular to $\Delta_{X_t,\sigma(t)}$. So the morphism $h : \Delta \longrightarrow \mathbb{D}$ defines a family of reduced planes curves with constant Milnor number. It is well known that such plane curves are equisingular [9]. □

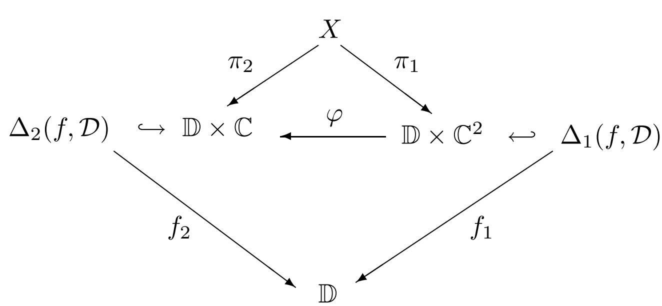
Remark 3.2. — It follows from the previous theorem (cf. Theorem 3.1) that the multiplicities $m_0(X_t, \sigma(t))$, $t \in \mathbb{D}$ of the singularities $(X_t, \sigma(t))$, $t \in \mathbb{D}$ are the same.

In fact the projections $\pi_t : X_t \longrightarrow \{t\} \times \mathbb{C}^2$ are generic so that the degree of each projection π_t (i.e., the number of elements in a generic fiber of π_t) is equal to the multiplicity $m_0(X_t, \sigma(t))$ of the singularity $(X_t, \sigma(t))$. The degree of the projection π is the number of elements in a generic fiber $\pi^{-1}(y)$ where $y \in (\mathbb{D} \times \mathbb{C}^2) \setminus \Delta$. Taking y in $(\{t\} \times \mathbb{C}^2) \setminus \Delta$, we have $Card\, \pi^{-1}(y) = deg\, \pi_t$ and taking $y \in \{0\} \times \mathbb{C}^2 \setminus \Delta$ gives $Card\, \pi^{-1}(y) = deg\, \pi_0$. It follows that $m_0(X_t, \sigma(0)) = deg\, \pi_0 = deg\, \pi = deg\, \pi_t = m_0(X_t, \sigma(t))$.

Let us take a family of normal surface singularities $X \hookrightarrow \mathbb{D} \times \mathbb{C}^N$ as above. We denote $T = \mathbb{D} \times \{0\}$. Note that if the pair of strata (X^o, T) satisfies Whitney conditions at 0 then one can easily show that for any k, $0 \leqslant k \leqslant 2$ the k th absolute local polar variety and the k-th relative polar variety are the same (subspace of X). It also turns out that the generic absolute polar variety $p_2(X, f)$ in X is empty if the pair of strata (X^o, T) satisfies the Whitney conditions (cf. [19], chapter V, Proposition 1.2.1). In our situation we can state (cf. [1]).

Proposition 3.3. — *Suppose that there is a generic projection $\pi : X \longrightarrow \mathbb{D}\times\mathbb{C}^2$ such that the discriminants of the projections $\pi_t : X_t \longrightarrow \{t\}\times\mathbb{C}^2$, $t \in \mathbb{D}$ are equisingular. Then the relative generic polar variety $p_2(X, f)$ is empty.*

Proof. — Let us consider a general flag $\mathcal{D}$: $D_2 \subset D_1 \subset \mathbb{C}^N$, where D_2 (resp. D_1) is a $(N-2)$-subspace (resp. $(N-1)$-subspace) of $\mathbb{C}^N$. We have the following diagram



where $\pi_1 (= \pi)$ is the projection with kernel D_2 and π_2 is the projection with kernel D_1 ($\pi_2 = \varphi \circ \pi_1$) and $\Delta_i(f, \mathcal{D})$ is the discriminant of the projection π_i, $i = 1, 2$ and $f_i : \Delta_i(f, \mathcal{D}) \longrightarrow \mathbb{D}$ is the morphism induced by f. By transitivity (cf. [11], Corollary 4.3.12) we have:

$$\Delta_1(f_1, \mathcal{D}_1) = \Delta_2(f, \mathcal{D}) \tag{3.4}$$

(where $\mathcal{D}_1$ denote the flag induced by $\mathcal{D}$).

The morphism f_1 defines also the family $(\Delta_t)_{t\in\mathbb{D}}$ of equisingular discriminants. Since the Milnor number is constant along $\mathbb{D}\times\{0\}$ it follows that the singular locus of the discriminant $\Delta = (\Delta_t)_{t\in\mathbb{D}}$ is $\mathbb{D}\times\{0\}$ and the pair of strata $(\Delta^0, \mathbb{D}\times\{0\})$ is a "good stratification"(in the sense of [13]). Then the polar variety $p_1(f_1,\mathcal{D}_1) = \emptyset$ is necessarily empty.
The space $\Delta_1(f_1,\mathcal{D}_1)$ is the image of $p_1(f_1,\mathcal{D}_1)$. By the equality (3.4) it follows that $\Delta_2(f,\mathcal{D}) = \emptyset$ and since $\Delta_2(f,\mathcal{D})$ is the image of $p_2(f,\mathcal{D})$ we have $p_2(f,\mathcal{D}) = \emptyset$. □

The following result is by T. Gaffney (cf. [10], Theorem 5.10).

THEOREM 3.4. — *Suppose $X \subset \mathbb{D}\times\mathbb{C}^N \longrightarrow \mathbb{D}$ is a 1-parameter family of d-dimensional reduced complex analytic spaces with isolated singularity at $(t,0)$. Suppose $Sing(X) = T$ and $p_d(X,f) = \emptyset$. We also assume that for any fixed k, $0 \leqslant k \leqslant d-1$, the polar multiplicities $m_k(X_t)_0$ are the same. Then $m(p_k(X,f))_{(t,0)} = m_k(X_t)_0$.*

COROLLARY 3.5. — *Suppose $X \subset \mathbb{D}\times\mathbb{C}^N \longrightarrow \mathbb{D}$ is a 1-parameter family of d-dimensional reduced complex analytic spaces with isolated singularity at $(t,0)$. Suppose $Sing(X) = T$ and $p_d(X,f) = \emptyset$. Then the pair of strata (X^o, T) is Whitney equisingular at 0 if and only if the multiplicities $m_k(X_t)_0$ are constant on T, $0 \leqslant k \leqslant d-1$.*

Proof. — The hyperplane $K := \{0\}\times\mathbb{C}^N$ is transverse to all limiting tangent planes to X at 0 (cf. [10], Theorem 5.2). The conclusion comes from Theorem 1.1 and the above theorem. □

We can state the main result.

THEOREM 3.6. — *Let $(X,0) \hookrightarrow \mathbb{D}\times\mathbb{C}^N$ be a family of normal surface singularities as above,*

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathbb{D},0)\times(\mathbb{C}^N,0) \\ f\big\downarrow\big\uparrow\sigma & \swarrow & \\ (\mathbb{D},0) & & \end{array}$$

$(T := \sigma(\mathbb{D}))$.
Suppose the generic discriminants $\Delta_{X_t,\sigma(t)}$, $t \in \mathbb{D}$ of the normal surface singularities $(X_t,\sigma(t))$ are equisingular. Then the pair of strata (X^o,T) satisfies Whitney conditions.

Proof. — By Theorem 3.1 and Proposition 3.3 we have $p_2(X, f) = \emptyset$. The hyperplane $K := \{0\} \times \mathbb{C}^N$ is transverse to all limiting tangent planes to X at 0 (cf. [10], Theorem 5.2). By Remark 3.2 the normal surface singularities $(X_t, \sigma(t))$, $t \in \mathbb{D}$ have the same multiplicity. The conclusion comes from Corollary 3.5. □

Let us consider a germ of a normal complex surface singularity $(S, 0)$ and take a representative S embedded in $(\mathbb{C}^N, 0)$. The *link* Σ of the singularity is the intersection of S with a small $(2N-1)$-sphere centered at the origin. The link Σ is a compact oriented 3-manifold. The local topology of the pair $(S, \mathbb{C}^N)$ is given by the topological cone over (Σ, S^{2N-1}) (in particular S is a topological manifold at the origin iff Σ is homeomorphic to the 3-sphere). The diffeomorphism class of the link Σ does not depend on the embedding $(S, 0) \subset (\mathbb{C}^N, 0)$ (we refer to [16] for more details).
It is well known that the usual way to see the topology of a normal surface singularity $(S, 0)$ is via a "good" resolution $\pi : (X, E) \longrightarrow (S, 0)$. Namely X is smooth, π is proper and maps $X \setminus E$ isomorphically onto $S \setminus \{0\}$ and $E = \pi^{-1}(0)$ is a divisor consisting of smooth projective curves E_i, intersecting transversally (there is in fact a minimal good resolution in an obvious sense). The weighted resolution dual graph Γ is then associated to E in the usual way: each irreducible component E_i of E gives a vertex. Intersection points give edges of the graph and each vertex is weighted by the degree of the normal bundle of the corresponding irreducible curve. It is well known that the link Σ can be reconstructed from Γ so that the topological type of a normal surface singularity is given by the weighted dual graph of its minimal good resolution (cf. [17]).

Definition 3.7. — *Two normal surface singularities will be said to be* equisingular *if the dual graphs of their minimal good resolutions.*

4. Application to minimal surface singularities

The class of minimal normal surface singularities can be defined as the subclass of rational surface singularities with reduced tangent cone. The reader can find in [12] the definition of minimal singularities in any dimension. Minimal normal surfaces singularities are exactly the rational surface singularities with reduced fundamental cycle (in the terminology of [2]). By using a result of Spivakovsky in [18] R. Bondil gives in [3] the equisingularity type of the generic discriminants of minimal surface singularities (see also [3], [1]). It turns out that equisingular minimal surface singularities (according to the definition 3.7) have equisingular generic discriminants (cf. [3], [4], [1]). We can then state:

Corollary 4.1. — *Let* $(X,0) \hookrightarrow \mathbb{D} \times \mathbb{C}^N$ *be a family of equisingular minimal surface singularities*

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathbb{D},0) \times (\mathbb{C}^N,0) \\ f \downarrow \uparrow \sigma & \swarrow & \\ (\mathbb{D},0) & & \end{array}$$

($T =: \sigma(\mathbb{D})$). Then the pair of strata (X^o, T) satisfies Whitney conditions.

Proof. — The generic discriminants $\Delta_{X_t,\sigma(t)}$ of a family $(X_t, \sigma(t))_{t\in\mathbb{D}}$ of equisingular minimal surface singularities are equisingular (cf. [3], [4], [1]). The conclusion comes from Theorem 3.6. □

Remark 4.2. — Whitney conditions for a family of normal surface singularities imply the constancy of the topological type of that family [15]. According to J. Briançon and J. P. Speder the constancy of the topological type of a family of surface singularities does not imply Whitney conditions in general [8].
Let us recall an important result of W. Neumann which says that the topological type of the link of a normal surface singularity determines the graph of the minimal good resolution (with the exceptions of cyclic quotient singularities and "cusp"singularities where orientation must be taken into account) [17].

Corollary 4.3. — *Let* $(X,0) \hookrightarrow \mathbb{D}\times\mathbb{C}^N$ *be a family of minimal surface singularities (which does not contain cyclic quotients)*

$$\begin{array}{ccc} (X,0) & \hookrightarrow & (\mathbb{D},0) \times (\mathbb{C}^N,0) \\ f \downarrow \uparrow \sigma & \swarrow & \\ (\mathbb{D},0) & & \end{array}$$

($T =: \sigma(\mathbb{D})$). The pair of strata (X, T) satisfies Whitney conditions if and only if the family is equisingular (i.e, the minimal dual graphs are the same).

Note that different surface singularities with equisingular generic discriminants are not equisingular in general (cf. [1]).

Example 4.4. — Let Γ_1 and Γ_2 be the following minimal graphs (i.e., the dual graphs of the minimal resolutions of minimal surface singularities).

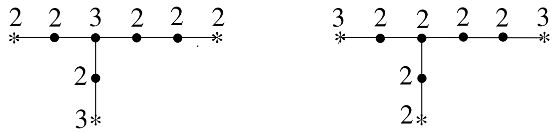


Figure 1. — Minimal graphs Γ_1 and Γ_2

The generic discriminant of the minimal singularities with minimal graph Γ_1, Γ_2 is $\Delta = \delta_2 \cup A_5 \cup A_6$, where δ_2 denotes two distinct lines and A_5 (resp. A_6) denotes a plane curve analytically isomorphic to the curve $x^2 + y^6 = 0$ (resp. $x^2 + y^7 = 0$). The contact between A_5 and A_6 (i.e., the number of blow-up points necessary to separate A_5 and A_6) is 3. The contact between δ_2 and A_5 (resp. A_6) is one.

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