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Real analytic manifolds in $\mathbb{C}^n$ with parabolic complex tangents along a submanifold of codimension one

Real analytic manifolds in $\mathbb{C}^n$ with parabolic complex tangents along a submanifold of codimension one

Patrick Ahern and Xianghong Gong[1]

Résumé. — Nous classifions les sous-variétés réelles analytiques de dimension n dans $\mathbb{C}^n$, qui ont un ensemble de points de tangence complexe paraboliques de dimension réelle $n-1$. Ces sous variétés sont toutes équivalentes via biholomorphisme formel. Nous montrons que les classes d'équivalence sous changement de variables par biholomorphisme local (convergent) forment un 'espace de modules' de dimension infinie. Nous montrons aussi qu'il existe une sous-variété M de dimension n dans $\mathbb{C}^n$, dont les images par les biholomorphismes $(z_1, \ldots, z_n) \mapsto (rz_1, \ldots, rz_{n-1}, r^2 z_n)$, $r > 1$, ne sont pas équivalentes à M via biholomorphisme local préservant le volume.

Abstract. — We will classify n-dimensional real submanifolds in $\mathbb{C}^n$ which have a set of parabolic complex tangents of real dimension $n-1$. All such submanifolds are equivalent under formal biholomorphisms. We will show that the equivalence classes under convergent local biholomorphisms form a moduli space of infinite dimension. We will also show that there exists an n-dimensional submanifold M in $\mathbb{C}^n$ such that its images under biholomorphisms $(z_1, \ldots, z_n) \mapsto (rz_1, \ldots, rz_{n-1}, r^2 z_n)$, $r > 1$, are not equivalent to M via any local volume-preserving holomorphic map.

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1. Introduction

In [1], E. Bishop introduced the local study of a real analytic manifold M of real dimension n in $\mathbb{C}^n$ with $0 \in M$. If M were totally real at 0 then M would be locally biholomorphic to $\mathbb{R}^n$ so he assumed that M has a complex tangent at 0. After some non-degeneracy assumptions he was able to assign a number $\gamma \in [0, \infty]$ to $(M, 0)$ in such a way that γ is a holomorphic invariant, i.e. if M_1 and M_2 are as described above and there is a biholomorphic map F defined near 0 with $F(M_1) = M_2$ and $F(0) = 0$ then $(M_1, 0)$ and $(M_2, 0)$ have the same invariant. Bishop was interested in polynomial hulls. He was able to show that if M was as above and if $0 \leqslant \gamma < 1/2$ (called the elliptic case) then there are holomorphic mappings f defined on the closed unit disc into $\mathbb{C}^n$, taking the boundary of the disc into M but $f(0) \notin M$, thereby showing that M has a non trivial polynomial hull.

In [8], Moser and Webster returned to the class of manifolds considered by Bishop. For $0 < \gamma < 1/2$ they found normal forms. As is often the case in these matters, there is a formal biholomorphic map that takes the manifold to its normal form and then one asks if there is a convergent biholomorphic map that takes the manifold to its normal form. They showed that there is such a convergent map in the case just cited. (By the way, the existence of holomorphic discs for the normal forms is more or less evident in the elliptic case so this gives another way to look at Bishop's result.) In the hyperbolic case ($\gamma > 1/2$) and for a countable set E, they had normal forms for $\gamma \in (1/2, \infty) \setminus E$ but they also found algebraic real surfaces M which are not equivalent to their normal forms by any convergent biholomorphic mappings. They do this as follows: the normal forms all lie in a real linear subspace of codimension one, and they then show that there are manifolds with a hyperbolic complex tangent that can not be so holomorphically embedded. At this point it can be asked if a manifolds with a hyperbolic complex tangent is already contained in real codimension one subspace can it be mapped to its normal form by a convergent mapping? In [3] it is shown that the answer to this question is *no*.

We want to mention that the Moser-Webster normal form excludes the case $\gamma = 0$. Very recently Huang and Yin [6] have constructed formal normal forms of infinitely many invariants for this case. They also showed that two real analytic surfaces of the same formal normal form are holomorphically equivalent.

In this paper we consider the *parabolic* case, $\gamma = 1/2$. In the elliptic and hyperbolic cases the set of all points with a complex tangent exactly has real dimension $n - 2$. However in the parabolic case there is the possibility that the set of complex tangents can have dimension $n - 1$. For example when

$n = 2$ the quadratic surface $z_2 = (z_1 + \overline{z}_1)^2$ has the set $z_1 + \overline{z}_1 = 0 = z_2$ as its set of complex tangents. (Note that when $n = 2$ the complex tangent is isolated in the elliptic and hyperbolic cases.) In this paper we will let $\mathcal{M}$ denote the set of parabolic manifolds M whose set of complex tangents is a real hypersurface of M. It is also natural to consider a subclass of $\mathcal{M}$. Let $\omega = dz_1 \wedge \cdots \wedge dz_n$ and let $\mathcal{M}_\omega$ be the set of M in $\mathcal{M}$ such that $\mathrm{Re}\,\omega|_M = 0$. For $\mathcal{M}_\omega$, we consider equivalence under unimodular maps, that is ones that preserve ω. We show that there is a quadratic surface Q (an n dimensional version of the surface defined above) such that if $M \in \mathcal{M}$ then M is formally equivalent to Q and hence any two manifolds in $\mathcal{M}$ are formally equivalent. Now there is a method called *functional moduli* that can be used in some cases to show that, in a situation where the formal theory shows that everything is equivalent, exactly the opposite is true. In one variable it was discovered independently by Écalle and Voronin and published in 1981; see [2] and [9]. Here is an example: let $\mathcal{A}$ be the set of all germs of holomorphic functions $f(z) = z + z^2 + z^3 + \cdots$ where the dots mean higher order terms and we assume the series has positive radius of convergence. The formal theory says that for $f \in \mathcal{A}$ there is a formal series g such that $g \circ f \circ g^{-1} = p$ where $p(z) = z + z^2 + z^3$ (no dots!) and hence any two germs in $\mathcal{A}$ are formally equivalent. The theory of functional moduli says that given $f \in \mathcal{A}$ there is a way to associate to f a functional modulus which consists of a pair of holomorphic functions of period one, one defined on an upper half plane and the other defined on a lower half plane. There is also an equivalence relation on the set of moduli (which is transparent). Then there are two theorems, one says that two germs are equivalent if and only if their moduli are equivalent and the other says that given any potential modulus there is a germ in $\mathcal{A}$ that has that modulus. Since the equivalence at the level of moduli is transparent it is very easy to construct non equivalent moduli and hence we can see that if we pick two moduli *at random* they are not equivalent and so if we pick two germs in $\mathcal{A}$ *at random* they are not equivalent. Such a theory is useful in that it shows us the big picture but it is not a useful way to decide if two concretely given germs are equivalent because the correspondence between germ and modulus is not *constructive*, in either direction. Voronin [10] has developed a theory of functional moduli for certain rather special germs of mappings defined in a neighborhood of the origin in $\mathbb{C}^n$ taking 0 to 0. The theory for $n > 1$ is in some ways quite different from $n = 1$; in particular, the equivalence relation at the level of moduli is not nearly so transparent. We will apply Voronin theory to show that even though the formal theory shows that all manifolds in $\mathcal{M}$ are equivalent the convergent theory is quite the opposite.

How do we get from an element of $\mathcal{M}$, a manifold, to one of Voronin's germs? We use the pair of Moser-Webster involutions [8] associated to M

in $\mathcal{M}$. (These involutions will be described in section 3.) In our case the composition of these involutions is a special case of one of Voronin's germs. As usual in applying functional moduli theory we must identify which germs we are using and then we must identify which moduli correspond to these germs and finally we must show that the set of moduli is non-trivial, i.e. that there is a large class of mutually inequivalent moduli. In the one dimensional Écalle-Voronin case this last step is actually trivial. In more than one dimension it usually is not, due to the lack of *transparency* of the equivalence relation at the level of moduli. However the theory still works as is is supposed to: construction a large class of inequivalent moduli is still possible and it is the only known to prove the existence of a large class of inequivalent germs which are formally equivalent. Sections 7 and 8 are devoted to showing the non-triviality of the set of moduli for both the case of $\mathcal{M}$ and $\mathcal{M}_\omega$.

Finally we consider the relation between holomorphic and unimodular equivalence for elements of $\mathcal{M}$. For $n = 2$ Webster [11] showed that if $M \in \mathcal{M}_\omega$ and if there is a convergent holomorphic map taking M to the quadric $z_2 = (z_1 + \overline{z}_1)^2$ then M and the quadric are equivalent by convergent unimodular map. Also in [4] for $n = 2$ and in the hyperbolic case it is shown that if M is convergently equivalent to its Moser-Webster normal form then it is equivalent to a normal form by a convergent unimodular mapping. In contrast we will use our methods to show that if for $r > 0$ we define $L_r(z_1, \cdots, z_n) = (rz_1, \cdots, rz_{n-1}, r^2 z_n)$ then there is $M \in \mathcal{M}_\omega$ such that for $r \neq 1$ $L_r M$ is not equivalent to M by any unimodular convergent map, but all $L_r M$ are equivalent to the quadric under unimodular formal maps. (Of course our M is formally but not holomorphically equivalent to the parabolic quadric.) Therefore in general there is no result that says that convergent holomorphic equivalence plus formal unimodular equivalence implies convergent unimodular equivalence.

2. Statements of main results and organization of the paper

The construction of moduli spaces uses the pair of Moser-Webster involutions [8] that characterizes real n-manifolds M in $\mathbb{C}^n$ with a non-vanishing Bishop invariant and Voronin's classification of local biholomorphisms $f(z) = (z_1, z_2 + z_1, z_3, \ldots, z_n) + O(2)$ that have constant eigenvalue 1 of multiplicity n along a complex hypersurface of fixed points [10].

Let M be a real analytic n-dimensional submanifold in $\mathbb{C}^n$. Assume that M has a non-degenerate complex tangent at 0 and the set of complex tangents of M is a real hypersurface C of M. We will see that in suitable local holomorphic coordinates z_1, $'z \stackrel{\text{def}}{=\!=} (z_2, \ldots, z_{n-1}) = {'x} + i\,{'y}$ and z_n, C is

the linear space $z_n = x_1 = {}'y = 0$ and the real n-manifold has the form

$$M \colon z_n = (z_1 + \overline{z}_1)^2 p(z_1, \overline{z}_1, {}'x), \quad {}'y = (z_1 + \overline{z}_1) q(z_1, \overline{z}_1, {}'x), \qquad (2.1)$$

where p, with $p(0) = 1$, is a convergent power series in $z_1, \overline{z}_1, {}'x$, and q, with $q(0) = 0$, is a vector of real-valued convergent power series.

The main purpose of this paper is to describe a complete set of equivalence classes for each of the following two problems: classify the above mentioned real analytic submanifolds M by local change of holomorphic coordinates, and classify the M satisfying the additional condition $\mathrm{Re}\,\omega|_M = 0$, under *unimodular* holomorphic maps, i.e. the ones preserving $\omega = dz_1 \wedge \cdots \wedge dz_n$.

We now describe which elements of Voronin's moduli space are relevant to the above-mentioned two classification problems.

Moduli space without volume-form. — Let $x, y \in \mathbb{C}$ and $\zeta = (\zeta_2, \ldots, \zeta_{n-1}) \in \mathbb{C}^{n-2}$. A power series $h(x, y, \zeta) = \sum_{k \geqslant 0} h_j(y, \zeta) x^j$ is called *semi-formal* in x, if all h_j are holomorphic in (y, ζ) on some fixed neighborhood W of the origin of $\mathbb{C}^{n-1}$. Define semi-formal maps analogously.

We say that $S = V \times W$ is a sectorial domain, if V is a sector of the form $V_{\alpha, \beta, \epsilon} = \{x \in \mathbb{C} \colon \arg x \in (\alpha, \beta), 0 < |x| < \epsilon\}$ and W is a neighborhood of the origin in $\mathbb{C}^{n-1}$, where $\beta - \alpha$ is called *aperture* of the domain. A semi-formal power series $G = \sum_{k=0}^{\infty} G_k(y, \zeta) x^k$ is called an *asymptotic expansion* of a holomorphic function g on $V \times W$, denoted by $g \sim G$ on $V \times W$, if there is a possibly smaller neighborhood $\widetilde{W}$ of $0 \in \mathbb{C}^{n-1}$ such that for each fixed N

$$\lim_{V \ni x \to 0} |x|^{-N} \Big| g(x, y, \zeta) - \sum_{k=0}^{N} G_k(y, \zeta) x^k \Big| = 0$$

holds uniformly for $(y, \zeta) \in \widetilde{W}$. Analogously, we say that a semi-formal map Φ is asymptotic to a holomorphic map H on $V \times W$, if each component of Φ is asymptotic to the corresponding component of H.

Put

$$\hat{\tau}_1 \colon (x, y, \zeta) \to (-x, y + 2x, \zeta), \quad \rho \colon (x, y, \zeta) \to (\overline{x}, -\overline{y}, \overline{\zeta}),$$
$$\hat{\tau}_2 = \rho \hat{\tau}_1 \rho \colon (x, y, \zeta) \to (-x, y - 2x, \zeta),$$
$$\hat{\sigma} = \hat{\tau}_2 \hat{\tau}_1 \colon (x, y, \zeta) \to (x, y + 4x, \zeta).$$

Let $V_1 = V_{\epsilon, \delta} = V_{-\epsilon, \frac{\pi}{2} + \epsilon, \delta}$. Put $V_j = \sqrt{-1}^{1-j} V_1$, $\Delta_\delta = \{t \in \mathbb{C} \colon |t| < \delta\}$. Assume that $0 < \epsilon < \frac{\pi}{4}$. In particular, $S_{j\,j+1} = (V_j \cap V_{j+1}) \times \Delta_\delta^{n-1}$ are

disjoint for $j = 1, 2, 3, 4$. Let $\mathcal{H}$ be the set of $H = \{H_{1\,2}, H_{2\,3}, H_{3\,4}, H_{4\,1}\}$ satisfying the following: $H_{j\,j+1}$ is defined on $S_{j\,j+1}$ and

$$H_{1\,2}^{-1} = \rho H_{1\,2} \rho, \ H_{4\,1}^{-1} = \rho H_{2\,3} \rho, \ H_{3\,4} = \hat{\tau}_j H_{1\,2} \hat{\tau}_j,$$
$$H_{j\,j+1} \sim \mathrm{id}, \quad \text{on } S_{j\,j+1}$$

in which the positive numbers ϵ and δ depend on H. Note that we define $H_{4\,5} = H_{4\,1}$, $H_{5\,6} = H_{1\,2}$, etc. Throughout the paper, that an identity holds on a sectorial domain such as $V_{\alpha,\beta,\epsilon} \times W$ means that it holds on $V_{\alpha+\delta,\beta-\delta,\epsilon'} \times \Delta_{\epsilon'}^{n-1}$ for any $\delta > 0$ and some ϵ' dependent of δ. This is justified by Lemma 10.3.

We say that $H, \widetilde{H}$ are *equivalent* and write $\widetilde{H} \sim H$, if there exist a semiformal map Ψ and biholomorphic maps $G_j = G_{j+4}$, defined on $S_j' \equiv i^{1-j} V_{\epsilon',\delta'} \times \Delta_{\delta'}^{n-1}$ (for some positive ϵ', δ') or on S_{j+2}' and satisfying

$$\widetilde{H}_{j\,j+1} = G_j^{-1} H_{j\,j+1} G_{j+1}, \ j = 1, \ldots, 4; \quad \text{or}$$
$$\widetilde{H}_{j+2\,j+3} = G_j^{-1} H_{j\,j+1} G_{j+1}, \ j = 1, \ldots, 4; \tag{2.2}$$

$$G_2 = \rho G_1 \rho, \quad G_4 = \rho G_3 \rho, \quad G_{j+2} = \hat{\tau}_k G_j \hat{\tau}_k; \tag{2.3}$$

$$G_j \sim \Psi, \quad \text{on } S_j' \text{ or on } S_{j+2}'; \tag{2.4}$$

$$\Psi \colon (x, y, \zeta) \to (a(x, \zeta)x, ya(x, \zeta) + b(x, \zeta), c(x, \zeta)), \tag{2.5}$$

where a, b, c are semi-formal in x, $a(0) \neq 0, b(0) = 0, c(0) = 0$, and $\zeta \to c(0, \zeta)$ is biholomorphic. Note that $\Psi = \rho \Psi \rho = \hat{\tau}_j \Psi \hat{\tau}_j$. In particular, $a(0)$ is real.

Moduli space with volume-form.. — Let $\mathcal{H}_{\hat{\omega}}$ be the set of $H \in \mathcal{H}$ satisfying the additional condition

$$H_{j\,j+1}^* \hat{\omega} = \hat{\omega}, \quad \hat{\omega} = x\,dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}.$$

For $H, \widetilde{H} \in \mathcal{H}_{\hat{\omega}}$, we denote $\widetilde{H} \sim H$, if there are G_j, Ψ satisfying (2.2)-(2.5) and $G_j^* \hat{\omega} = \hat{\omega}$. Note that $\Psi^* \hat{\omega} = \hat{\omega}$. Denote by $\mathcal{H}/\sim$ and $\mathcal{H}_{\hat{\omega}}/\sim$ the corresponding sets of equivalence classes.

Recall that $\mathcal{M}$ is the set of real analytic n-manifolds M in $\mathbb{C}^n$, of which complex tangents form a germ of real analytic set of dimension $n - 1$ at the origin, while the origin is a parabolic complex tangent of M. Denote by $\mathcal{M}_\omega$ the set of $M \in \mathcal{M}$ satisfying $\mathrm{Re}\,\omega|_M = 0$. Denote by $\mathcal{M}/\sim$ the set of holomorphic equivalence classes in $\mathcal{M}$, and by $\mathcal{M}_\omega/\sim$ the set of equivalence classes in M_ω under unimodular holomorphic maps.

The following theorem solves the two classification problems mentioned early in this section.

THEOREM 2.1. — *Each $M \in \mathcal{M}$ is formally biholomorphic to*

$$Q: z_n = (z_1 + \bar{z}_1)^2, \quad \mathrm{Im}z_2 = \cdots = \mathrm{Im}z_{n-1} = 0$$

and each $M \in \mathcal{M}_\omega$ is equivalent to Q under some formal unimodular holomorphic map. There are one-to-one correspondence between $\mathcal{M}/\sim$ and $\mathcal{H}/\sim$ and one-to-one correspondence between $\mathcal{M}_\omega/\sim$ and $\mathcal{H}_{\hat{\omega}}/\sim$; moreover, $\mathcal{H}/\sim$ and $\mathcal{H}_{\hat{\omega}}/\sim$ are of infinite dimension.

Our moduli spaces, as moduli spaces given by Voronin [10], are not explicit. However, they are useful to obtain results which are not achieved by other approaches. For example, using our moduli spaces we obtain

THEOREM 2.2. — *Let $n \geqslant 2$ and let L_r be the dilation $z_j \to rz_j$ $(1 \leqslant j < n), z_n \to r^2 z_n$. There exists a germ M of real analytic n-submanifold in $\mathbb{C}^n$ at the origin such that $L_r M$ is not equivalent to M under any unimodular holomorphic map if r is a positive number with $r \neq 1$, while all $L_r M$ $(r > 0)$ are equivalent to $Q: z_n = 2z_1\bar{z}_1 + z_1^2 + \bar{z}_1^2, y_j = 0$ $(1 < j < n)$ under unimodular formal maps.*

The paper is organized as follows.

In section 3, we will obtain some preliminary normalization for a real analytic n-manifold M by flattening its set of complex tangent points of dimension $n - 1$, from which a preliminary holomorphic normalization for the pair of Moser-Webster involutions follows.

In section 4 one can find normal forms for pairs of holomorphic linear involutions τ_1, τ_2 in general and for special pairs $\tau_1, \tau_2 = \rho\tau_1\rho$ intertwined by an anti-holomorphic linear involution ρ, under the assumptions that $\sigma = \tau_2\tau_1$ is not diagonalizable and fixes a hyperplane pointwise. The linear involutions τ_1, τ_2, ρ discussed in section 4 are more general than those arising from real manifolds with complex tangents.

In section 5 we will discuss the semi-formal normalization of pairs of involutions whose linear parts are classified in section 4. In section 6, we will identify the two classification problems formulated at the beginning of this section with the problem on classifying pairs of involutions $\tau_1, \tau_2 = \rho\tau_1\rho$ under holomorphic maps.

Section 6 also contains some results on classifying real n-manifolds M with a parabolic complex tangent under unimodular holomorphic maps and we will also obtain a formal normal form showing infinitely many invariants. However, the results in this direction are not complete, and the main

difficulties arise from the volume form ω which is invariant under τ_1 but not τ_2.

Sections 7 and 8 are devoted to the construction of moduli spaces stated in Theorem 2.1, by adapting Voronin's moduli space [10]. The non-triviality of the moduli spaces are proved in the two sections too.

Theorem 2.2 is proved in section 9. The reader could read the proof of Theorem 2.2 *first*, since it also outlines the construction from the involutions of real manifolds to the moduli spaces.

In Appendix A (section 10), for the convenience of the reader we will give a proof for a fundamental theorem of Voronin [10]. Voronin's proof is for two dimensional case, which can be easily adapted to higher dimensional case.

We would like to conclude the section with the following two open problems.

A) Classify all real analytic n-manifolds M in $\mathbb{C}^n$ which have a parabolic complex tangent at the origin. Here the set of complex tangent points has dimension less than $n-1$.

B) Classify all real analytic n-manifolds M in $\mathbb{C}^n$ having a parabolic complex tangent under unimodular holomorphic maps. Here $\mathrm{Re}\, dz_1 \wedge \cdots \wedge dz_n|_M \not\equiv 0$. The problem remains open even if the set of complex tangent points of M has dimension $n-1$.

3. Complex tangent points and a pair of Moser-Webster involutions

We will recall the pair of Moser-Webster involutions [8].

Let $M \subset \mathbb{C}^n$ be an n-dimensional real analytic submanifold containing the origin, given by $R_1(z,\overline{z}) = \ldots = R_n(z,\overline{z}) = 0$ with $dR_1 \wedge \cdots \wedge dR_n \neq 0$, where $R_j(z,\overline{z})$ are convergent power series and real-valued, i.e. $\overline{R_j(z,\overline{z})} \stackrel{\text{def}}{=} \overline{R_j(\overline{z},z)} = R_j(\overline{z},z)$. The complexification $M^c \subset \mathbb{C}^n \times \mathbb{C}^n$ of M is defined by $R_1(z,w) = \ldots = R_n(z,w) = 0$. Then M becomes a totally real and real analytic submanifold of M^c via the embedding $z \to (z,\overline{z})$, and $dz_1 \wedge \ldots \wedge dz_n|_M$ extends uniquely to a holomorphic n-form ω on M^c, and uniquely to an anti-holomorphic n-form $\overline{\omega}_2 = d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_n$.

We say that $p \in M$ is a *complex tangent* of M, if $T_pM \cap iT_pM \neq \{0\}$, i.e. if $dz_1 \wedge \ldots \wedge dz_n|_M$ vanishes at p since M has dimension n. We assume that $0 \in M$ is a complex tangent point so $T_0M \cap iT_0M$ is a complex space

of positive dimension. We also assume that it has the smallest positive dimension so it is a complex line. Hence by a change of coordinates we have $T_0 M = \mathbb{C} \times \mathbb{R}^{n-2} \times 0$. Then M is given by

$$
\begin{aligned}
z_n \;&=\; a z_1 \overline{z}_1 + b z_1^2 + c \overline{z}_1^2 + \sum_{1 < \alpha < n} (c_\alpha z_1 x_\alpha + d_\alpha \overline{z}_1 x_\alpha) \\
&\quad + \sum_{1 < \alpha, \beta < n} e_{\alpha\beta} x_\alpha x_\beta + O(|(z_1, {}'x)|^3), \\
{}'y \;&=\; O(|(z_1, {}'x)|^2).
\end{aligned}
$$

Recall that ${}'\zeta = {}'x + i\,{}'y = (z_2, \ldots, z_{n-1})$. Put $\omega = dz_1 \wedge \ldots \wedge dz_n$. Then $\omega|_M = A \, d\overline{z}_1 \wedge dz_1 \wedge dx_2 \wedge \ldots \wedge dx_{n-1}$ for $A = (-1)^{n-1}(a z_1 + 2c\overline{z}_1 + \sum_{\alpha=2}^{n-1} d_\alpha x_\alpha + O(2))$. We assume that M has a *non-degenerate* complex tangent at the origin, i.e. that $\mathcal{D}_{z_1} A$ or $\mathcal{D}_{\overline{z}_1} A$ does not vanish; equivalently $|a| + |c| \neq 0$. By a quadratic change of coordinates, we may achieve that

$$
M : z_n = a z_1 \overline{z}_1 + b z_1^2 + c \overline{z}_1^2 + O(3), \quad y_\alpha = O(2), \qquad a \geqslant 0, c \geqslant 0
$$

where $\gamma = c/a \in [0, \infty]$ is the Bishop invariant [1]. The complex tangent point $0 \in M$ is said to be *elliptic* if $\gamma < 1/2$, or *parabolic* if $\gamma = 1/2$, or *hyperbolic* if $\gamma > 1/2$. Put $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$. When $\gamma \neq 0$, $\pi_j \colon M^c \to \mathbb{C}^n$ are two-to-one branched coverings, and the covering transformations form a pair of holomorphic involutions $\tau_j \colon M^c \to M^c$ satisfying $\tau_2 = \rho\tau_1\rho$ for $\rho \colon (z, w) \to (\overline{w}, \overline{z})$. When $\gamma \neq 1/2$, the set of complex tangent points of M is real submanifold of M of dimension $n - 2$, and τ_1, τ_2 fix a complex submanifold of M^c of dimension $n - 2$ when $\gamma \neq 0, 1/2$.

Let us compute τ_1, τ_2 for our special case, and find local coordinates such that M is given by (2.1).

We assume that C, the set of complex tangent points of M, is a real submanifold of dimension $n - 1$. So we are dealing with parabolic complex tangents. $\omega|_{M^c}$ vanishes on the complexification C^c of C in M^c, and we will see soon that C^c is precisely the zero set of $\omega|_{M^c}$. First, $a = 2c$. By a quadratic change of coordinates, we may achieve that

$$
M : z_n = (z_1 + \overline{z}_1)^2 + O(3), \quad {}'y = O(2). \tag{3.1}
$$

Now $\omega|_M = (-1)^{n-1}(4x_1 + O(2)) d\overline{z}_1 \wedge dz_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}$. Since $\dim C = n - 1$, we see that C is a totally real submanifold parameterized by

$$
\begin{aligned}
z_1 = a(y_1, {}'x) + i y_1, \quad {}'z = {}'x + i b(y_1, {}'x), \quad z_n = c(y_1, {}'x), \\
a(y_1, {}'x) = O(2), \quad b(y_1, {}'x) = O(2), \quad c(y_1, {}'x) = O(3).
\end{aligned}
$$

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Put

$$F(z_1, {'z}, z_n) = \big(a(-iz_1, {'z}) + z_1, {'z} + ib(-iz_1, {'z}), z_n + c(-iz_1, {'z})\big).$$

$F^{-1}(M) \equiv M$ is still of the form (3.1) and C is flattened to the linear space $x_1 = z_n = y_\alpha = 0$. Now we can write

$$M \subset \mathbb{C}^n : \begin{cases} z_n = (z_1 + \overline{z}_1)^2 + (z_1 + \overline{z}_1)p_0(z_1, \overline{z}_1, {'x}), & p_0 = O(2), \\ {'y} = (z_1 + \overline{z}_1)q(z_1, \overline{z}_1, {'x}), & q(0) = 0. \end{cases}$$

Looking at the zero set of $dz_1 \wedge \ldots \wedge dz_n|_M$, we see that x_1 divides $p_0(x_1, y_1, {'x})$. Thus M is given by

$$M \subset \mathbb{C}^n : z_n = (z_1 + \overline{z}_1)^2 p(z_1, \overline{z}_1, {'x}), \quad {'y} = (z_1 + \overline{z}_1)q(z_1, \overline{z}_1, {'x})$$

with $p(0) = 1, q(0) = 0$, and $\overline{q}(z_1, \overline{z}_1, {'x}) = q(\overline{z}_1, z_1, {'x})$. We have derived (2.1).

The complexification of M is

$$M^c \subset \mathbb{C}^n \times \mathbb{C}^n : \begin{cases} z_n = (z_1 + w_1)^2 p(z_1, w_1, \frac{'z + 'w}{2}), \\ w_n = (z_1 + w_1)^2 \overline{p}(w_1, z_1, \frac{'z + 'w}{2}), \\ {'z} - {'w} = 2i(z_1 + w_1)q(z_1, w_1, \frac{'z + 'w}{2}). \end{cases}$$

One can see that the set $\widetilde{C}(\subset M^c) : z_1 + w_1 = 0$ is fixed pointwise by τ_1, τ_2 and is invariant under ρ. On M^c, introduce coordinates

$$x \overset{\text{def}}{=\!=} z_1 + w_1, \quad y \overset{\text{def}}{=\!=} z_1 - w_1, \quad \zeta \overset{\text{def}}{=\!=} (z_2 + w_2, \ldots, z_{n-1} + w_{n-1}).$$

Then $\rho|_{M^c} : (z, w) \to (\overline{w}, \overline{z})$ becomes

$$\rho : (x, y, \zeta) \to (\overline{x}, -\overline{y}, \overline{\zeta}).$$

We also have

$$\tau_1 : x' = -x + xa(x, y, \zeta), \ y' = y + 2x + xb(x, y, \zeta), \ \zeta' = \zeta + xc(x, y, \zeta).$$

with $a(0) = b(0) = c(0) = 0$. Note that $\tau_1^2 = \text{id}$ implies $a(x, y, \zeta) = x\widetilde{a}(x, y, \zeta)$.

Conversely, assume that a smooth holomorphic hypersurface $\widetilde{C}$ is fixed pointwise by both τ_1, τ_2 of M, which means that $\rho(C)$ is fixed pointwise by τ_j too. We may assume that M is given by (3.1). By linearizing τ_1, one sees that $\widetilde{C}$ is the unique smooth holomorphic hypersurface fixed by τ_1 pointwise and

hence $\rho(\widetilde{C}) = \widetilde{C}$, and that $\widetilde{C}$ is tangent to $z_1 + w_1 = 0$. Put $\zeta_\alpha = (z_\alpha + w_\alpha)/2$, so z_1, w_1, ζ form coordinates of M^c and ρ becomes $(z_1, w_1, \zeta) \to (\overline{w}_1, \overline{z}_1, \overline{\zeta})$. $\widetilde{C}$ is the zero set of $R = (R_1 + iR_2)(z_1, w_1, \zeta) = z_1 + w_1 + O(2)$, where $R_j(z_1, w_1, \zeta)$ are real when $\zeta_2, \ldots, \zeta_{n-1}$ are real and $w_1 = \overline{z}_1$. Now $\rho(\widetilde{C}) = \widetilde{C}$ implies that $(R_1 - iR_2)(z_1, w_1, \zeta) = \overline{R \circ \rho(z_1, w_1, \zeta)}$ vanishes on $\widetilde{C}$, i.e., that $(R_1 - iR_2)(z_1, w_1, \zeta)$ is a multiple of $R(z_1, w_1, \zeta)$; equivalently, R_2 is a multiple of R_1. Then

$$\omega|_{M^c} = uR_1(z_1, w_1, \zeta)dz_1 \wedge dw_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_{n-1}$$

with $u(0) \neq 0$. Thus $\omega|_M$ vanishes when $R_1(z_1, \overline{z}_1, \zeta) = 0$, which defines a smooth hypersurface in M.

In summary, we proved that the complex tangent points of M form a codimension one set in M, if and only if τ_1, τ_2 fix the same complex hypersurface pointwise. In particular, when M has a non-degenerate complex tangent at 0 and Re $dz_1 \wedge \cdots \wedge dz_n|_M = 0$, the zero set of $dz_1 \wedge \cdots \wedge dz_n|_M = i$Im $dz_1 \wedge \cdots \wedge dz_n|_M$ has codimension 1 in M.

Remark. — When the dimension of the set C of complex tangent points is less than $n-1$, the zero set $\widetilde{C}$ of ω_{M^c} is still a smooth complex hypersurface in M^c, τ_1 fixes $\widetilde{C}$ pointwise, and τ_2 fixes $\rho(\widetilde{C})$ pointwise. However, $\widetilde{C} \cap \rho(\widetilde{C})$ is a complex analytic variety of pure dimension $n - 2$. The map $z \to (z, \overline{z})$ identifies the set of complex tangent of M with a real analytic subset of $\widetilde{C}$.

Before we state the next result, we need the notion of invariant smooth formal holomorphic hypersurfaces. By a smooth formal holomorphic hypersurface passing through $0 \in \mathbb{C}^n$, we mean an equation $u = 0$, where u is a formal power series in $z \in \mathbb{C}^n$ with $u(0) = 0$ and $du(0) \neq 0$. Two such equations $u = 0$ and $\widetilde{u} = 0$ are considered to be the same if $\widetilde{u} = vu$ for some formal power series v. The formal hypersurface $u = 0$ is invariant under a formal biholomorphic map F if $F(0) = 0$ and $u \circ F = au$ for some formal power series a; we say that the hypersurface is fixed by F *pointwise*, if $F(T(t)) = T(t)$ for some (and hence for all) formal holomorphic map $t = (t_1, \ldots, t_{n-1}) \to T(t)$ satisfying $u \circ T = 0$, $T(0) = 0$, and rank$T'(0) = n - 1$.

PROPOSITION 3.1. — *Let $M \subset \mathbb{C}^n$ be a real analytic submanifold with a parabolic complex tangent at the origin. Let $\{\tau_1, \tau_2, \rho\}$ be the Moser-Webster involutions on M^c. Then the germ of the set C of complex tangent points of M at the origin is of real dimension $n - 1$, if and only if τ_1, τ_2 fix the same formal smooth hypersurface pointwise, in which case under suitable holomorphic coordinates on M^c we have*

$$\tau_j \colon \begin{cases} x' = -x + xa_j(x,y,\zeta), \\ y' = y + (-1)^{j-1}2x + xb_j(x,y,\zeta), & b_j(0) = 0, \\ \zeta' = \zeta + xc_j(x,y,\zeta), & c_j(0) = 0, \quad j = 1,2, \end{cases}$$

$$\tau_2 = \rho\tau_1\rho, \quad \rho(x,y,\zeta) = (\overline{x}, -\overline{y}, \overline{\zeta}).$$

Proof. — τ_1 is holomorphically equivalent to $(x,y,\zeta) \to (-x,y,\zeta)$ and the latter fixes pointwise a unique smooth formal hypersurface containing 0, which is actually given by $x = 0$. Hence the unique smooth formal holomorphic hypersurface fixed by τ_1 pointwise is actually given by a holomorphic function. The same argument works for τ_2 also. So both τ_j fix the same smooth holomorphic hypersurface pointwise. From the argument given before the previous remark, we know that the set of complex tangent points of M has dimension $n - 1$. $\quad\square$

Before we normalize τ_1, τ_2, ρ under semi-formal maps, we will deal with the linear involutions first in next section by considering a more general situation. The semi-formal normalization for the involutions is given in section 5.

4. Normal forms of a pair of linear involutions

We will find two normal forms: one for pairs of linear involutions τ_1, τ_2 on $\mathbb{C}^n$ of which the indicator $\sigma = \tau_2\tau_1$ is not diagonalizable and the set of fixed points of σ is a hyperplane, and the other for distinct linear involutions τ_1, τ_2 such that $\tau_2 = \rho\tau_1\rho$ for some anti-holomorphic linear involution ρ and the set of fixed points of σ is a hyperplane. We will show that the latter σ is not diagonalizable either.

By assumption, the set of fixed points of σ is a hyperplane, and that σ is not diagonalizable. So in suitable linear coordinates $\sigma = \hat{\sigma}\colon x' = x, y' = y + 4x, \zeta' = \zeta$ with $\zeta \in \mathbb{C}^{n-2}$.

We want to further normalize τ_2, τ_1, while σ remains unchanged. Note that $x = 0$ is the set of fixed points of σ. Hence τ_1 preserves $x = 0$ and we can write

$$\tau_1 \colon \begin{cases} x' = -\delta_0 x, \quad \delta_0 = \pm 1, \\ y' = \delta y + b_2 x + c_2\zeta, \\ \zeta' = \epsilon\zeta + b_3 x + c_3 y, \end{cases}$$

where c_2^T, c_3, b_3 are column vectors and ϵ is a matrix. Since $\tau_1\sigma = \sigma^{-1}\tau_1$, the ζ-components say $c_3 = 0$ and y-components say $\delta_0 = \delta$. Since $\tau_1^2 = \mathrm{id}$

then

$$c_2(\delta I + \epsilon) = 0, \quad \epsilon^2 = I, \quad (\epsilon - \delta I)b_3 = 0.$$

Obviously, $\zeta \to \epsilon\zeta$ is an involution. By choosing a linear transformation $(x, y, \zeta) \to (x, y, S\zeta)$ we may assume that ϵ is a diagonal matrix with diagonal elements $\epsilon_j = \pm 1$. Since $(\epsilon - \delta I)b_3 = 0$, we can choose $\widetilde{b}$ such that $(\delta I + \epsilon)\widetilde{b} + b_3 = 0$. Putting $S(x, y, \zeta) = (x, y, \zeta + \widetilde{b}x)$, we obtain $b_3 = 0$ for $S^{-1}\tau_1 S$. Note that $c_2(\delta I + \epsilon) = 0$ implies that $c_{2j} = 0$ when $\epsilon_j = \delta$. Let $\varphi_2(x, y, \zeta) = (x, y + (1 - \frac{\delta b_2}{2})x - \frac{\delta c_2}{2}\zeta, \zeta)$. We obtain $b_2 = 2\delta$ and $c_2 = 0$ for $\varphi_2^{-1}\tau_1\varphi_2$. In other words, by a possible permutation of ζ coordinates and by possible splitting ζ into two sets of variables, denoted by (ζ, w) by an abuse of notation, the pair $\{\tau_1, \tau_2\}$ is normalized to

$$\hat{\tau}_1 \colon \begin{cases} x' = -\delta x, \\ y' = \delta y + 2\delta x, \\ \zeta' = \zeta, \\ w' = -w, \end{cases} \qquad \hat{\tau}_2 \colon \begin{cases} x' = -\delta x, \\ y' = \delta y - 2\delta x, \\ \zeta' = \zeta, \\ w' = -w. \end{cases} \tag{4.1}$$

Assume now that τ_1, τ_2 are linear involutions and that $\tau_2 = \rho\tau_1\rho$ for some anti-holomorphic linear involution ρ. We also assume that the set of fixed points of $\sigma = \tau_2\tau_1$ is a hyperplane. Then σ is not diagonalizable. Otherwise, in suitable linear coordinates we have $\sigma \colon z \to (\epsilon z_1, z_2, \ldots, z_n)$. Note that $\det \sigma = \det \tau_1 \det \tau_2 = \pm 1$ and $\sigma \neq \mathrm{id}$. Hence $\epsilon = -1$. Now $z_1 = 0$ is preserved by τ_1, τ_2, ρ. One may assume that the first component of ρ is $z_1 \to \overline{z}_1$. The first component of τ_1 is $z_1 \to \pm z_1$, since $\tau_1^2 = \mathrm{id}$. So the first component of σ is the identity, which is a contradiction.

We will normalize ρ, while $\tau_1 = \hat{\tau}_1, \tau_2 = \hat{\tau}_2$ remain unchanged at each step of coordinate changes.

Consider the case $\delta = 1$ first.

Since $\sigma^{-1} = \rho\sigma\rho = \tau_1\sigma\tau_1$, then ρ, τ_1 preserve the set of fixed points of σ defined by $x = 0$. By a change of coordinates $(x, y, \zeta, w) \to (cx, cy, \zeta, w)$ we may assume that the first component of $\rho(x, y, \zeta, w)$ is $\overline{x}$. Restricting to $x = 0$, we have $\tau_1 = \rho\tau_1\rho$. Hence ρ preserves the eigen-spaces $x = y = \zeta = 0$ and $x = w = 0$ of $\tau_1|_{x=0}$. Thus we can write

$$\rho \colon \begin{cases} x' = \overline{x}, \\ y' = \mu\overline{y} + p\overline{x} + q\overline{\zeta}, \\ \zeta' = s\overline{\zeta} + s'\overline{x} + \widetilde{s}\overline{y}, \\ w' = t\overline{w} + t'\overline{x}. \end{cases}$$

Since $t\bar{t} = \mathrm{id}$, by a linear change of coordinates of the w-space, we may assume that $t = \mathrm{id}$. Thus $\rho^2 = \mathrm{id}$ implies that t' is pure-imaginary. Let $S(x, y, \zeta, w) = (x, y, \zeta, w + \frac{1}{2}t'x)$. Then S preserves τ_j, and for $\rho \equiv S^{-1}\rho S$ we get $t' = 0$. The y and ζ components of $\rho^2 = \mathrm{id}$ produce

$$\mu\bar{\mu} + q\overline{\tilde{s}} = 1, \tag{4.2}$$

$$\mu\bar{p} + p + q\bar{s}' = 0, \tag{4.3}$$

$$\mu\bar{q} + q\bar{s} = 0, \tag{4.4}$$

$$s\bar{s} + \tilde{s}\,\bar{q} = \mathrm{id}, \tag{4.5}$$

$$s\bar{s}' + s' + \tilde{s}\,\bar{p} = 0, \tag{4.6}$$

$$s\overline{\tilde{s}} + \tilde{s}\,\bar{\mu} = 0. \tag{4.7}$$

From $\rho\tau_2 = \tau_1\rho$, we get

$$p = -\mu - 1, \tag{4.8}$$

$$s' = -\tilde{s}. \tag{4.9}$$

From (4.9), (4.2)-(4.3) we get $\mu\bar{\mu} + \mu\bar{p} + p = 1$, and combining with (4.8) yields $\mu = -1$ and $p = 0$. Now (4.9) and (4.6)-(4.7) imply $s' = \tilde{s} = 0$. So (4.5) becomes $s\bar{s} = \mathrm{id}$. We get $s = \mathrm{id}$ by a change of ζ-coordinates alone. Now (4.4) says $q = \bar{q}$. Put $\varphi(x, y, \zeta, w) = (x, y + \frac{1}{2}q\zeta, \zeta, w)$. Then $\varphi^{-1}\rho\varphi$ becomes

$$\rho\colon x' = \bar{x}, \quad y' = -\bar{y}, \quad \zeta' = \bar{\zeta}, \quad w' = \bar{w}. \tag{4.10}$$

Consider now the case $\delta = -1$, by a reduction to $\delta = 1$. Assume that $\tilde{\tau}_1, \tilde{\tau}_2 = \rho_0\tilde{\tau}_1\rho_0$ are given by (4.1) with $\delta = -1$, and that ρ_0 is a linear anti-holomorphic involution. Put $L(x, y, \zeta, w) = (x, y, w, \zeta)$, and put $L = \mathrm{id}$ when one of ζ, w is absent. Then $\hat{\tau}_j = -L\tilde{\tau}_j L$ are given by (4.1) with $\delta = 1$. Still $\hat{\tau}_2 = L\rho_0 L\hat{\tau}_1 L\rho_0 L$. By the above argument there is a linear map K such that $KL\rho_0 LK^{-1} = \rho = L\rho L$ is given by (4.10) and $K\hat{\tau}_j K^{-1}$ is still given by (4.1) with $\delta = 1$. Then $LKL\tilde{\tau}_j LK^{-1}L = \tilde{\tau}_j$ is given by (4.1) with $\delta = -1$ and $LKL\rho^0(LKL)^{-1} = \rho$.

In summary, we proved

PROPOSITION 4.1. — *Let τ_1, τ_2 be two linear involutions on $\mathbb{C}^n$. Assume that the set of fixed points of $\sigma = \tau_2\tau_1$ is a hyperplane. Then we have*

(i) if σ is not diagonalizable, there is a linear transformation φ such that $\varphi\tau_j\varphi^{-1}$ are given by

$$\hat{\tau}_1: \begin{cases} x' = -\delta x, \\ y' = \delta y + 2\delta x, \\ \zeta' = \epsilon\zeta, \end{cases} \qquad \hat{\tau}_2: \begin{cases} x' = -\delta x, \\ y' = \delta y - 2\delta x, \\ \zeta' = \epsilon\zeta, \end{cases}$$

where $\delta = \pm 1$ and $\epsilon = \mathrm{diag}(\epsilon_2, \ldots, \epsilon_{n-1})$ is a diagonal matrix with diagonal elements $\epsilon_j = \pm 1$;

(ii) if $\tau_2 = \rho^0\tau_1\rho^0$ for some linear anti-holomorphic involution ρ^0, then φ can be chosen to satisfy (i) and $\varphi\rho^0\varphi^{-1} = \rho: (x, y, \zeta) \to (\overline{x}, -\overline{y}, \overline{\zeta})$.

Remark. — $\delta = 1$ if and only if the above $\hat{\tau}_1, \hat{\tau}_2$ have the same set of fixed points. The same conclusion holds for τ_1, τ_2 below.

COROLLARY 4.2. — *Let τ_1^0, τ_2^0 be a pair of holomorphic involutions on $\mathbb{C}^n$. Assume that the set of fixed points of $\sigma = \tau_2^0\tau_1^0$ is a smooth hypersurface, and that $\sigma'(0)$ is not diagonalizable. There exists a biholomorphic map φ such that $\varphi\tau_j^0\varphi^{-1}$ is given by*

$$\tau_j: \begin{cases} x' = -\delta x + xa_j(x, y, \zeta), & a_j(0) = 0, \\ y' = \delta y + (-1)^{j-1}2\delta x + xb_j(x, y, \zeta), & b_j(0) = 0, \\ \zeta' = \epsilon\zeta + xc_j(x, y, \zeta), & c(0) = 0, \quad j = 1, 2, \end{cases} \tag{4.11}$$

where $\delta = \pm 1$ and $\epsilon = \mathrm{diag}(\epsilon_2, \ldots, \epsilon_{n-1})$ with $\epsilon_j = \pm 1$. If $\tau_2^0 = \rho\tau_1^0\rho$ for some anti-holomorphic involution ρ^0, we can choose φ to satisfy $\varphi\rho^0\varphi^{-1}(x, y, \zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$ additionally.

Proof. — We first choose a linear map φ_0 such that the linear parts of $\widetilde{\tau}_j = \varphi_0\tau_j^0\varphi_0^{-1}$ are given by (4.1). Choose a biholomorphic map φ_1 with $\varphi_1'(0) = \mathrm{id}$, sending the set of fixed points of $\widetilde{\tau}_2\widetilde{\tau}_1$ into $x = 0$. Then $\tau_j^* = \varphi_1\widetilde{\tau}_j\varphi_1^{-1}$ preserves $x = 0$ and is tangent to $\widetilde{\tau}_j$. Now $\varphi_2 = (\mathrm{id} + \widetilde{\tau}_1'(0)\tau_1^*)/2$ preserves $x = 0$ and hence linearizes $\tau_1^*|_{x=0} = \tau_2^*|_{x=0}$ into $\widetilde{\tau}_1'(0)|_{x=0} = \widetilde{\tau}_2'(0)|_{x=0}$. Thus $\varphi_2\tau_j^*\varphi_2^{-1}$ are given by (4.11).

Assume now that $\tau_2^0 = \rho^0\tau_1^0\rho^0$. Choose linear coordinates such that τ_j^0 is tangent to (4.11) and ρ^0 is tangent to ρ. The above argument shows that there is a biholomorphic map tangent to the identity, sending τ_j^0 into (4.11). So ρ^0 is still tangent to ρ, and we may assume that τ_j^0 are given by (4.11). Let $\psi = (\mathrm{id} + \rho\rho^0)/2$. Since ρ^0 preserves $x = 0$, the set of fixed point of $\tau_2^0\tau_1^0$, then ψ preserves $x = 0$. Restricted on $x = 0$ we have $\tau_j^0 = \tau_j$, so the restrictions

are linear. Hence $\psi\tau_j^0 = (\tau_j^0 + \rho\tau_{j+1}^0\rho_0)/2 = (\tau_j^{0\prime}(0) + \rho\tau_{j+1}^{0\prime}(0)\rho_0)/2 = (\tau_j^{0\prime}(0) + \tau_j^{0\prime}(0)\rho\rho_0)/2 = \tau_j'(0)(\mathrm{id} + \rho\rho^0)/2 = \tau_j\psi$. Thus $\psi\tau_j^0\psi^{-1}$ are still of the form (4.11), while $\psi\rho^0\psi^{-1} = \rho$. $\qquad\square$

Remark 4.3. — 1) Let τ_1, τ_2 be given by (4.11). Computing the Jacobian matrix shows that the eigenvalues of $\tau_2\tau_1$ at its fixed point $(0, y, \zeta)$ are 1 and

$$[1 - \delta a_2(0, \delta y, \epsilon\zeta)][1 - \delta a_1(0, y, \zeta)].$$

If $\delta = 1$ and $\epsilon = \mathrm{id}$, a direct computation from the first component of $\tau_j^2 = \mathrm{id}$ shows that $a_j(0, y, \zeta) = 0$. This turns out to be crucial in the Voronin theory. Note that for the real analytic manifolds with a parabolic tangent we *do* have $\delta = 1$ and $\epsilon = \mathrm{id}$.

2) If $\delta \neq 1$ or $\epsilon \neq \mathrm{id}$ the above non-trivial eigenvalue may not be constant. In particular $\tau_2\tau_1$ is not formally linearizable. To see an example, let $\phi(x, y, \zeta) = (xb(y, \zeta), y, \zeta)$, where b is a holomorphic function vanishing nowhere. Let $\tau_1 = \phi\hat{\tau}_1\phi^{-1}$ and $\tau_2 = \rho\tau_1\rho$. The non-trivial eigenvalue is

$$\frac{\overline{b}(-y, \zeta)b(\delta y, \epsilon\zeta)}{\overline{b}(-\delta y, \epsilon\zeta)b(y, \zeta)},$$

which is not constant in general.

5. Semi-formal normalization

We will normalize pairs of involutions whose indicators fix a smooth hypersurface pointwise. The arguments follow proofs in [11] and [10]. We will give an averaging argument, which is also useful to construct Voronin's module functions of other linear symmetries.

We first normalize the composition $\tau_2\tau_1$ by semi-formal transformations. The proof of next result is in [10], [11], when $n = 2$. We include the proof for $n \geqslant 2$ for the convenience of the reader.

PROPOSITION 5.1. — *Let τ_1, τ_2 be holomorphic involutions given by (4.11). Assume further that $\tau_2\tau_1$ has constant eigenvalue 1 along its set of fixed points, if $\delta \neq 1$ or $\epsilon \neq \mathrm{id}$. There exists a unique semi-formal map*

$$\Phi\colon \begin{cases} x' = x + xu(x, y, \zeta), \\ y' = y + v(x, y, \zeta), \\ \zeta' = \zeta + w(x, y, \zeta), \end{cases} \tag{5.1}$$

$$u(x, 0, \zeta) = v(x, 0, \zeta) = w(x, 0, \zeta) = 0, \tag{5.2}$$

$$(v(x, y, \zeta), w(x, y, \zeta)) = O(|(x, y, \zeta)|^2) \tag{5.3}$$

such that

$$\Phi^{-1} \tau_2 \tau_1 \Phi = \hat{\sigma} \colon x' = x, \quad y' = y + 4x, \quad \zeta' = \zeta.$$

If $\tau_2 = \rho \tau_1 \rho$ for $\rho \colon (x, y, \zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$, the unique Φ satisfies $\Phi = \rho \Phi \rho$.

Proof. — As we remarked at the end of last section, our assumptions mean that $\tau_2 \tau_1$ has constant eigenvalue 1 at its fixed points (given by $x = 0$). Therefore, we can write

$$\sigma = \tau_2 \tau_1 \colon \begin{cases} x' = x + x^2 p(x, y, \zeta), \\ y' = y + 4x + xq(x, y, \zeta), \quad q(0) = 0, \\ \zeta' = \zeta + xr(x, y, \zeta), \qquad r(0) = 0. \end{cases}$$

For $\Phi^{-1} \sigma \Phi = \hat{\sigma}$ we need

$$\begin{aligned} u(x, y + 4x, \zeta) - u(x, y, \zeta) &= x(1 + u(x, y, \zeta))^2 p\Phi(x, y, \zeta), \\ v(x, y + 4x, \zeta) - v(x, y, \zeta) &= x(4u(x, y, \zeta) + \\ &\qquad (1 + u(x, y, \zeta))q\Phi(x, y, \zeta)), \\ w(x, y + 4x, \zeta) - w(x, y, \zeta) &= x(1 + u(x, y, \zeta))r\Phi(x, y, \zeta). \end{aligned} \tag{5.4}$$

Let u_k, v_k and $w_{\alpha k}$ be coefficients of expansions of $u(x, y, \zeta)$, $v(x, y, \zeta)$ and $w_\alpha(x, y, \zeta)$ in x variable, respectively. Comparing the coefficients of x^1 in 5.4, we obtain

$$4 \frac{\partial u_0}{\partial y} = (1 + u_0)^2 p_0(y + v_0, \zeta + w_0),$$

$$4 \frac{\partial v_0}{\partial y} = 4u_0 + (1 + u_0)q_0(y + v_0, \zeta + w_0),$$

$$4 \frac{\partial w_0}{\partial y} = (1 + u_0)r_0(y + v_0, \zeta + w_0).$$

Let u_0, v_0, w_0 be the unique set of solutions satisfying $u_0(0, \zeta) = v_0(0, \zeta) = w_0(0, \zeta) = 0$. For $k > 0$ comparing coefficients of x^{k+1} in (5.4) yields

$$4 \begin{pmatrix} \frac{\partial u_k}{\partial y} \\ \frac{\partial v_k}{\partial y} \\ \frac{\partial w_k}{\partial y} \end{pmatrix} = \begin{pmatrix} 2p_0 & p_{0y} & p_{0\zeta} \\ 4 + q_0 & q_{0y} & q_{0\zeta} \\ r_0 & r_{0y} & r_{0\zeta} \end{pmatrix} (y + v_0, \zeta + w_0) \cdot \begin{pmatrix} u_k \\ v_k \\ w_k \end{pmatrix} + \cdots,$$

where the omitted terms are polynomials in $u_j(y, \zeta)$, $v_j(y, \zeta)$, $w_j(y, \zeta)$ $(j < k)$ and partial derivatives of p_j, q_j, r_j of order less than k evaluated at $(y + v_0(y, \zeta), \zeta + w_0(y, \zeta))$. One readily sees that there are unique solutions u_k, v_k, w_k that are holomorphic on Δ_δ^{n-1} and vanish for $y = 0$.

Since $\rho\sigma\rho = \sigma^{-1}$, then $\rho\Phi\rho$ linearizes σ. We have $\rho\Phi\rho(x, y, \zeta) = (x + x\overline{u}(x, -y, \zeta), y - \overline{v}(x, -y, \zeta), \zeta + \overline{w}(x, -y, \zeta))$. By the uniqueness of the solutions u_k, v_k, w_k under the conditions $u(x, 0, \zeta) = v(x, 0, \zeta) = w(x, 0, \zeta) = 0$, we get $\rho\Phi\rho = \Phi$.   □

Remark. — The above proof is valid if σ, not necessary a composition of a pair of involutions, has the form

$$(x, y, \zeta) \rightarrow (x + x^2 p(x, y, \zeta), y + 4x + xq(x, y, \zeta), \zeta + xr(x, y, \zeta))$$

with $q(0) = 0 = r(0)$.

Having normalized the composition $\tau_2\tau_1$, we now normalize the actual pair $\tau_1, \tau_2 = \rho\tau_1\rho$.

COROLLARY 5.2. — *Let τ_1, τ_2 and Φ be as in 5.1. There is a semi-formal map*

$$\Psi\colon \begin{cases} x' = x + xu(x, \zeta), & u(0) = 0 \\ y' = y + yu(x, \zeta) + v(x, \zeta), & v = O(2) \\ \zeta' = \zeta + w(x, \zeta), & w = O(2) \end{cases}$$

such that $\Psi^{-1}\Phi^{-1}\tau_j\Phi\Psi$ are equal to

$$\hat{\tau}_j\colon (x, y, \zeta) \rightarrow (-\delta x, \delta y + (-1)^{j-1}2\delta x, \epsilon\zeta), \quad j = 1, 2.$$

If $\tau_2 = \rho\tau_1\rho$ for $\rho(x, y, \zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$, the $\Phi\Psi$ commutes with ρ.

Proof. — We prove it by an averaging argument.

Let Φ be as in Proposition 5.1. Then Φ preserves $x = 0$ and $\Phi = \mathrm{id} + O(2)$. Put $\widetilde{\tau}_j = \Phi^{-1}\tau_j\Phi = \hat{\tau}_j + O(2)$. Note that $\widetilde{\tau}_2\widetilde{\tau}_1 = \hat{\tau}_2\hat{\tau}_1$ implies that $\Psi^{-1} \stackrel{\text{def}}{=} (\mathrm{id} + \hat{\tau}_1\widetilde{\tau}_1)/2 = (\mathrm{id} + \hat{\tau}_2\widetilde{\tau}_2)/2$. Since $\hat{\tau}_j$ are linear, we have $\hat{\tau}_j\Psi^{-1}\widetilde{\tau}_j = \Psi^{-1}$.

We now assume that $\tau_2 = \rho\tau_1\rho$ with $\rho(x, y, \zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$. Let $\Phi = \rho\Phi\rho$ be as in Proposition 5.1. We have $\rho\widetilde{\tau}_1 = \widetilde{\tau}_2\rho$. Let Ψ be as above. Then $\rho\Psi^{-1} = (\rho + \rho\hat{\tau}_1\widetilde{\tau}_1)/2 = (\rho + \hat{\tau}_2\widetilde{\tau}_2\rho)/2 = \Psi^{-1}\rho$.

To show that the above $\Psi = \mathrm{id} + O(2)$ has the desired form, we recall that Φ, τ_j preserve $x = 0$. This shows that $\widetilde{\tau}_j$ and hence Ψ preserves $x = 0$ too. Now $\Psi\hat{\sigma} = \hat{\sigma}\Psi$ implies that the x, ζ components of Ψ are independent of y. One can also verify that the y-component of Ψ has the above form.   □

It is a fundamental theorem of Voronin that there exists a biholomorphic map H on a sectorial domain such that H is asymptotic to Φ and $H\tau_2\tau_1H^{-1}$ is linear (see section 10). Moreover, it is easy to normalize the pair τ_1, τ_2 on

sector when $\tau_2\tau_1$ is already linear. See Proposition 7.1 in section 7. For the moment, we will realize Ψ as an asymptotic expansion of a biholomorphic map H on sectors by the Borel-Ritt theorem. We will require that the holomorphic map preserves $\hat{\sigma}$ for later purpose.

Let Ψ be as in Corollary 5.2. So

$$\Psi(x,y,\zeta) = (x + xu(x,\zeta), y + yu(x,\zeta) + v(x,\zeta), \zeta + w(x,\zeta)),$$

where the formal power series expansions of u, v, w in x have coefficients $u_k(\zeta)$, $v_k(\zeta)$, $w_k(\zeta)$ which are holomorphic and bounded in a neighborhood W of $0 \in \mathbb{C}^{n-2}$. Moreover $u = O(1), v = O(2)$ and $w = O(2)$. Let V be any bounded sectorial domain with opening less than 2π. Fix a square root $\sqrt{x}$ on V and choose μ with $|\mu| = 1$ so that $e^{\mu/\sqrt{x}}$ tends to zero as $x \to 0$ in $\overline{V}$. Then $|1 - e^{c\mu/\sqrt{x}}| \leqslant c/|\sqrt{x}|$ for $c > 0$. Let c_k be a positive sequence such that $c_k \sup |u_k(\zeta)| < 1$. Then $\widetilde{u} \sim u$ on $V \times W$. Construct $\widetilde{v}, \widetilde{w}$ analogously such that $\widetilde{v} \sim v$ and $\widetilde{w} \sim w$ on $V \times W$. Put

$$H\colon \begin{cases} x' = x + x\widetilde{u}(x,\zeta), \\ y' = y + y\widetilde{u}(x,\zeta) + \widetilde{v}(x,\zeta), \\ \zeta' = \zeta + \widetilde{w}(x,\zeta). \end{cases}$$

Then the semi-formal map Ψ in Corollary 5.2 is asymptotic to H and $H^{-1}\hat{\sigma}H = \hat{\sigma}$. We get the following.

PROPOSITION 5.3. — *Let Ψ be the semi-formal map given by Corollary 5.2. Let $0 < \beta - \alpha < 2\pi$, and $\epsilon > 0$ be small. There exists a biholomorphic map $H = \hat{\sigma}H\hat{\sigma}^{-1}$ such that H is asymptotic to Ψ on $V_{\alpha,\beta,\epsilon} \times \Delta_\epsilon^{n-1}$.*

Next proposition gives a uniqueness condition on the semi-formal map $\Phi\Psi$. However, Corollary 5.2 suffices for our further discussions.

PROPOSITION 5.4. — *Let τ_1, τ_2 be holomorphic involutions given by (4.11). Assume further that $\tau_2\tau_1$ has constant eigenvalue 1 along its set of fixed points, if $\delta \neq 1$ or $\epsilon \neq$ id. There is a unique semi-formal map of the form*

$$\Phi\colon \begin{cases} x' = x + xu(x,y,\zeta), \\ y' = y + v(x,y,\zeta), \\ \zeta' = \zeta + w(x,y,\zeta), \end{cases}$$

$$u(0,0,\epsilon\zeta) = -u(0,0,\zeta), \quad v(0,0,\zeta) = w(0,0,\zeta) = 0, \qquad (5.5)$$

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$$u(x,0,\zeta) = -u(-\delta x,0,\epsilon\zeta), \quad v(-\delta x,0,\epsilon\zeta) = -\delta v(x,0,\zeta),$$
$$w(-\delta x,0,\epsilon\zeta) = -\epsilon w(x,0,\zeta) \tag{5.6}$$

so that

$$\Phi\tau_j\Phi^{-1} = \hat{\tau}_j \colon (x,y,\zeta) \to (-\delta x, \delta y + (-1)^{j-1}2\delta x, \epsilon\zeta), \quad j = 1,2.$$

If $\tau_2 = \rho\tau_1\rho$ for $\rho(x,y,\zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$, the unique Φ satisfies $\Phi = \rho\Phi\rho$.

Proof. — We will adapt a proof in [11] to the semi-formal case. By Proposition 5.1, there is Φ_0 such that $\Phi_0^{-1}\tau_2\tau_1\Phi_0 = \hat{\sigma}$. Note that the normalizing condition on Φ_0, given in Proposition 5.1, is equivalent to that Φ_0 is tangent to the identify, preserves $x = 0$, and is the identity when restricted to $y = 0$. In particular the inverse of Φ_0 satisfies the normalizing condition also. The new normalizing conditions (5.5)-(5.6) is about the map $(x,\zeta) \to \Phi(x,0,\zeta)$. Consequently, Φ satisfies the new normalizing condition if and only if $\Phi\Phi_0^{-1}$ satisfies the same normalizing conditions. Therefore, we may assume that $\tau_2\tau_1 = \hat{\sigma}$. Then Φ must have the form $x' = xp(x,\zeta), y' = yp(x,\zeta) + q(x,\zeta), \zeta' = r(x,\zeta)$.

Write $\tau_j = \hat{\tau}_j + H_j$. Since $\tau_2\tau_1 = \hat{\sigma} = \hat{\tau}_2\hat{\tau}_1$ and $\hat{\tau}_j$ are linear, then

$$\hat{\tau}_2 \circ H_1 + H_2 \circ \tau_1 = 0 = \hat{\tau}_1 \circ H_2 + H_1 \circ \tau_2.$$

Hence $\hat{\sigma}H_1\hat{\sigma} = H_1$, i.e.,

$$H_1(x,y,\zeta) = (xa(x,\zeta), -ya(x,\zeta) + xb(x,\zeta), xc(x,\zeta)).$$

First we want to find a sequence of maps $\phi_k(x,y,\zeta) = (x(1+p_k(\zeta)), y(1+p_k(\zeta)), \zeta)$, where $p_k(\zeta)$ is a homogeneous polynomial of degree k in ζ, such that the x-component of $T_{k+1} = \phi_k T_k \phi_k^{-1}$, denoted by $-\delta x + xa_{k+1}(\zeta) + O(|x|^2)$, satisfies $a_{k+1}(\zeta) = O(|\zeta|^{k+1})$. Moreover p_k are uniquely determined, if $\phi_k \circ \cdots \circ \phi_1(x,y,\zeta) = (x(1 + c_k(\zeta)), y(1 + c_k(\zeta)), \zeta)$ satisfies $c_k(\epsilon\zeta) = -c_k(\zeta) + O(|\zeta|^{k+1})$. We prove by induction. The assertion is trivial for $k = 0$ (with $p_0 \equiv 0$), since the linear part of T_0 is tangent to $\hat{\tau}_1$. Assume that $p_{k-1}(\zeta), k \geqslant 1$, has been uniquely determined. Now the x-component of $T_k^2 = \text{id}$ implies that $(1 - \delta a_k(\zeta))(1 - \delta a_k(\epsilon\zeta)) = 1$, in particular, $a_k(\epsilon\zeta) = -a_k(\zeta) + O(|\zeta|^{k+1})$. The x-component of $\phi_k T_k \phi_k^{-1}$ is $-\delta x + x[\delta p_k(\zeta) - \delta p_k(\epsilon\zeta) + a_k(\zeta) + O(|\zeta|^{k+1})] + O(|x|^2)$. On the other hand $c_{k+1}(\zeta) = c_k(\zeta) + p_k(\zeta) + O(|\zeta|^{k+1})$. Thus we need to solve the equations

$$p_k(\epsilon\zeta) + c_k(\epsilon\zeta) = -p_k(\zeta) - c_k(\zeta) + O(|\zeta|^{k+1}),$$
$$p_k(\zeta) - p_k(\epsilon\zeta) + \delta a_k(\zeta) = O(|\zeta|^{k+1}),$$

which admits a unique solution $p_k(\zeta)$ because $a_k(\zeta) + a_k(\epsilon\zeta) = O(|\zeta|^{k+1})$.

To show the convergence of formal map

$$\Phi_0 = \lim_{k\to\infty} \phi_k \circ \cdots \circ \phi_1 \colon (x, y, \zeta) \to (x(1 + p(\zeta)), y(1 + p(\zeta)), \zeta),$$

we look at the x-components of $\Phi_0 T_0 \Phi_0^{-1} = T_\infty = \lim_{k\to\infty} T_k$. The latter is $\delta x + O(|x|^2)$, and we get

$$\frac{1 + p(\epsilon\zeta)}{1 + p(\zeta)}(1 - \delta a(0, \zeta)) = 1.$$

Hence $p(\epsilon\zeta) - p(\zeta) = \delta a(0, \zeta) + \delta p(\epsilon\zeta)a(0, \zeta)$. Since $p(\zeta) = \sum p_\alpha \zeta^\alpha$ satisfies $p_\alpha = 0$ for $\epsilon^\alpha = 1$, then $|p_\alpha| \leqslant p_\alpha^*$ if $\sum p_\alpha^* \zeta^\alpha = p^*$ satisfies $2p^* = a_0^* + p^* a_0^*$ with $a^*(\zeta) = \sum |a_{0,\alpha}| \zeta^\alpha$. This shows that Φ_0 is convergent.

Set $T_0 = \Phi_0 \tau_1 \Phi_0^{-1}$. Next, we want to find a sequence of semi-formal maps

$$\Phi_k \colon \begin{cases} x' = x + u_k(\zeta)x^{k+1}, \\ y' = y + yu_k(\zeta)x^k + v_k(\zeta)x^k, \\ \zeta' = \zeta + w_k(\zeta)x^k \end{cases}$$

such that $T_k = \Phi_k T_{k-1} \Phi_k^{-1}$ have the form

$$T_k \colon \begin{cases} x' = -\delta x + x^{k+2} A_k(x, \zeta), \\ y' = \delta y + 2\delta x - yx^{k+1} A_k(x, \zeta) + x^{k+1} B_k(x, \zeta), \\ \zeta' = \epsilon\zeta + x^{k+1} C_k(x, \zeta) \end{cases} \tag{5.7}$$

and $\widetilde{\Phi}_k = \Phi_k \cdots \Phi_2 \Phi_1 \Phi_0$ have the form

$$\widetilde{\Phi}_k \colon \begin{cases} x' = x + x\widetilde{u}_k(x, \zeta), \\ y' = y + y\widetilde{u}_k(x, \zeta) + \widetilde{v}_k(x, \zeta), \\ \zeta' = \zeta + \widetilde{w}_k(x, \zeta), \end{cases}$$

$$\widetilde{u}_{k,0,L} = 0, \quad \text{if } \epsilon^L = 1; \quad \widetilde{v}_{k,0,L} = \widetilde{w}_{k,0,L} = 0, \tag{5.8}$$

$$\begin{cases} \widetilde{u}_{k,j,L} = 0, & \text{if } (-\delta)^j \epsilon^L = 1 \text{ and } 1 \leqslant j \leqslant k, \\ \widetilde{v}_{k,j,L} = 0, & \text{if } (-\delta)^{j-1} \epsilon^L = -1 \text{ and } 1 \leqslant j \leqslant k, \\ \widetilde{w}_{k,\alpha,j,L} = 0, & \text{if } (-\delta)^j \epsilon^L = \epsilon_\alpha \text{ and } 1 \leqslant j \leqslant k, \end{cases} \tag{5.9}$$

where $\epsilon^L = \epsilon_1^{l_2} \cdots \epsilon_{n-1}^{l_{n-1}}$ for $L = (l_2, \ldots, l_{n-1})$.

To achieve (5.7) and (5.9), we apply induction on k. Assume that we have chosen Φ_{k-1}. We need to find Φ_k such that (5.7) and (5.9) hold. We have

$$\Phi_k^{-1}: \begin{cases} x' = x - u_k(\zeta)x^{k+1} + O(|x|^{k+2}), \\ y' = y - yu_k(\zeta)x^k - v_k(\zeta)x^k + O(|x|^{k+1}), \\ \zeta' = \zeta - w_k(\zeta)x^k + O(|x|^{k+1}). \end{cases}$$

$T_k = \Phi_k T_{k-1} \Phi_k^{-1}$ has the form

$$\begin{cases} x' = -\delta x + x^{k+1}(A_{k-1}(x,\zeta) + \delta u_k(\zeta) + (-\delta)^{k+1}u_k(\epsilon\zeta)) \\ \quad + O(|x|^{k+2}), \\ y' = \delta y + 2\delta x - yx^k(A_{k-1}(x,\zeta) + \delta u_k(\zeta) + (-\delta)^{k+1}u_k(\epsilon\zeta)) \\ \quad + x^k(B_{k-1}(x,\zeta) - \delta v_k(\zeta) + (-\delta)^k v_k(\epsilon\zeta)) + O(|x|^{k+1}), \\ \zeta' = \epsilon\zeta + x^k(C_{k-1}(x,\zeta) - \epsilon w_k(\zeta) + (-\delta)^k w_k(\epsilon\zeta)) + O(|x|^{k+1}), \end{cases}$$

which yields (5.7), provided

$$\begin{cases} A_{k-1}(0,\zeta) + \delta u_k(\zeta) + (-\delta)^{k+1}u_k(\epsilon\zeta) = 0, \\ B_{k-1}(0,\zeta) - \delta v_k(\zeta) + (-\delta)^k v_k(\epsilon\zeta) = 0, \\ C_{k-1}(0,\zeta) - \epsilon w_k(\zeta) + (-\delta)^k w_k(\epsilon\zeta) = 0. \end{cases}$$

The above equations are solvable, since $T_{k-1}^2 = \mathrm{id}$ implies that

$$A_{k-1}(0,\zeta) = -(-\delta)^k A_{k-1}(0,\epsilon\zeta), \quad B_{k-1}(0,\zeta) = (-\delta)^{k-1} B_{k-1}(0,\epsilon\zeta),$$
$$\epsilon C_{k-1}(0,\zeta) = -(-\delta)^k C_{k-1}(0,\epsilon\zeta).$$

We also have

$$\widetilde{\Phi}_k = \Phi_k \widetilde{\Phi}_{k-1}: \begin{cases} x' = x + x\widetilde{u}_k(x,\zeta), \\ y' = y + y\widetilde{u}_k(x,\zeta) + \widetilde{v}_k(x,\zeta), \\ \zeta' = \zeta + \widetilde{w}_k(x,\zeta), \end{cases}$$

$$\widetilde{u}_k(x,\zeta) = \widetilde{u}_{k-1}(x,\zeta) + u_k(\zeta)x^k + O(|x|^{k+1}),$$
$$\widetilde{v}_k(x,\zeta) = \widetilde{v}_{k-1}(x,\zeta) + v_k(\zeta)x^k + O(|x|^{k+1}),$$
$$\widetilde{w}_k(x,\zeta) = \widetilde{w}_{k-1}(x,\zeta) + w_k(\zeta)x^k + O(|x|^{k+1}).$$

In particular
$$\widetilde{u}_k(0,\zeta) = \widetilde{u}_{k-1,k}(0,\zeta) + u_k(\zeta),$$
$$\widetilde{v}_k(0,\zeta) = \widetilde{v}_{k-1,k}(0,\zeta) + v_k(\zeta),$$
$$\widetilde{w}_k(0,\zeta) = \widetilde{w}_{k-1,k}(0,\zeta) + w_k(\zeta).$$

This shows that there exist unique u_k, v_k, w_k such that (5.7) and (5.9) hold. More specifically, the solution is given by

$$u_k(\zeta) = - \sum_{(-\delta)^k \epsilon^L \neq 1} \frac{A_{k-1,0,L}}{\delta + (-\delta)^{k+1}\epsilon^L} \zeta^L - \sum_{(-\delta)^k \epsilon^L = 1} \widetilde{u}_{k-1,k,L}\zeta^L,$$

$$v_k(\zeta) = \sum_{(-\delta)^{k-1}\epsilon^L \neq -1} \frac{B_{k-1,0,L}}{\delta - (-\delta)^k\epsilon^L} \zeta^L - \sum_{(-\delta)^{k-1}\epsilon^L = -1} \widetilde{v}_{k-1,k,L}\zeta^L,$$

$$w_{k,\alpha}(\zeta) = \sum_{\epsilon_\alpha \neq (-\delta)^k\epsilon^L} \frac{C_{k-1,\alpha,0,L}}{\epsilon_\alpha - (-\delta)^k\epsilon^L} \zeta^L - \sum_{\epsilon_\alpha = (-\delta)^k\epsilon^L} \widetilde{w}_{k-1,\alpha,k,L}\zeta^L.$$

The above formulae also say that if $A_{k-1}(0,\zeta)$, $B_{k-1}(0,\zeta)$, $C_{k-1}(0,\zeta)$, $u_{k-1}(\zeta)$, $v_{k-1}(\zeta)$ and $w_{k-1}(\zeta)$ are holomorphic on Δ_r^{n-2}, then $u_k(\zeta)$, $v_k(\zeta)$, $w_k(\zeta)$ are holomorphic on the same polydisc.

We now assume that $\tau_2 = \rho\tau_1\rho$ with $\rho(x,y,\zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$. Here we take $\hat{\tau}_1(x,y,\zeta) = (-\delta x, \delta y + 2\delta x, \epsilon\zeta)$. Then $\hat{\sigma} = \hat{\tau}_2\hat{\tau}_1$ and $\hat{\tau}_2 = \rho\hat{\tau}_1\rho$. Note that $\rho\Phi\rho$ still normalizes τ_j and satisfies the normalizing condition. By the uniqueness of Φ we obtain $\rho\Phi\rho = \rho$. $\qquad\square$

We should remark that the normalized maps, i.e, the maps Φ satisfying (5.5)-(5.6), do not form a (pseudo)group, even when $\delta = 1$ and $\epsilon = \mathrm{id}$.

6. Realization of pairs of involutions and holomorphic n-forms

Let $\tau_1, \tau_2 = \rho\tau_1\rho$ be the pair of involutions generated by a real analytic n-submanifold $M \subset \mathbb{C}^n$ with a parabolic complex tangent at 0. Recall that τ_1 is defined on $M^c \subset \mathbb{C}^n \times \mathbb{C}^n \ni (z,w)$ and preserves the holomorphic n-form $\omega = dz_1 \wedge \cdots \wedge dz_n$, while τ_2 preserves the holomorphic n-form $\omega_2 = dw_1 \wedge \cdots \wedge dw_n = \overline{\rho^*\omega}$. When M satisfies $\mathrm{Re}\,\omega|_M = 0$, we have $\omega = -\omega_2$ and hence both τ_1 and τ_2 preserve ω.

Let $\mathcal{L}$ be the set of real analytic n dimensional submanifolds $M \subset \mathbb{C}^n$ which have non-degenerate (parabolic) complex tangent point at 0 and satisfy $\mathrm{Re}\,dz_1 \wedge \cdots \wedge dz_n|_M = 0$. Note that we have proved in section 3 that the set of complex tangent points of M has real dimension $n-1$. Write $M \sim \widetilde{M}$ if they are equivalent by a biholomorphic map f preserving $dz_1 \wedge \cdots \wedge dz_n$.

Next, we adapt the Moser-Webster involutions to the classification of $\mathcal{L}$.

Consider the set $\mathcal{L}^*$ of the following data $\{\tau_1, \tau_2, \rho, \omega\}$: (i) ρ is an antiholomorphic involution, and $\tau_1, \tau_2 = \rho\tau_1\rho$ are a pair of holomorphic involutions on $\mathbb{C}^n$ fixing the same smooth holomorphic hypersurface N pointwise.

(ii) $(\tau_2\tau_1)'(0) \neq$ id. (iii) $\omega = Adw_1 \wedge dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-1}$ is a holomorphic n form on $\mathbb{C}^n$ vanishing on N to first order (i.e. $A = 0$ on N and $dA \neq 0$) and $\tau_j^*\omega = \omega = -\overline{\rho^*\omega}$, where $z_1, w_1, z_2, \ldots, z_{n-1}$ are holomorphic coordinates of $\mathbb{C}^n$.

Write $\{\tau_1, \tau_2, \rho, \omega\} \sim \{\widetilde{\tau}_1, \widetilde{\tau}_2, \widetilde{\rho}, \widetilde{\omega}\}$ if there is a biholomorphic map f satisfying $\widetilde{\tau}_j = f\tau_j f^{-1}, \widetilde{\rho} = f\rho f^{-1}$, and $f^*\widetilde{\omega} = \omega$.

Note that by Proposition 4.1 the linear parts of τ_1, τ_2, ρ are equivalent to

$$\hat{\tau}_1(x, y, \zeta) = (-x, y + 2x, \zeta), \quad \hat{\tau}_2 = \rho\hat{\tau}_1\rho, \quad \rho(x, y, \zeta) = (\overline{x}, -\overline{y}, \overline{\zeta}). \quad (6.1)$$

(δ in Proposition 4.1 equals 1, since the above (i) (ii) imply that $\tau_1'(0), \tau_2'(0)$ have the same set of fixed points which is a hyperplane.) We may of course assume that the ρ is of the above form by linearizing ρ first.

PROPOSITION 6.1. — *Let $M, \widetilde{M} \in \mathcal{L}$. Then $M \sim \widetilde{M}$ if and only if the corresponding $\{\tau_1, \tau_2, \rho, \omega\}$ are equivalent. Each $\{\tau_1, \tau_2, \rho, \omega\} \in \mathcal{L}^*$ is equivalent to one arising from some $M \in \mathcal{L}$.*

Proof. — It is clear that if $M, \widetilde{M}$ are equivalent by f preserving ω then the restriction of $(z, w) \to (f(z), \overline{f}(w))$ to M^c transforms the involutions τ_j, ρ and ω of M^c to those of $\widetilde{M}^c$. Conversely, if F transforms the involutions τ_j, ρ and n-form ω of M^c to those of $\widetilde{M}^c$. Let π_1 be the projection $(z, w) \to z$. Then $f = \pi_1 F \pi_1^{-1}$ is well-defined and $F(z, w) = (f(z), \overline{f}(w))$ on M^c. Since F transforms M^c into $\widetilde{M}^c$ then f transforms M into $\widetilde{M}$. Obviously f is a biholomorphism since F is, and f preserves $dz_1 \wedge \cdots \wedge dz_n$.

We need to show the realization. Assume that $\tau_1, \tau_2 = \rho\tau_1\rho$, with $(\tau_2\tau_1)'(0) \neq$ id, are holomorphic involutions on $\mathbb{C}^n$ and that the common fixed point set of τ_j is a smooth hypersurface N. We also assume that $\omega = a(x, y, \zeta)dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$ with $a|_N = 0$ and $da \neq 0$ is a holomorphic n-form satisfying $\tau_j^*\omega = \omega$ and $\overline{\rho^*\omega} = -\omega$. We shall find an n-dimensional real analytic manifold M in $\mathbb{C}^n$, and a biholomorphic map $\varphi: \mathbb{C}^n \to M^c$ such that $\varphi^{-1}\rho\varphi = \rho_0$ is the restriction of $(z, w) \to (\overline{w}, \overline{z})$ on M^c, and $\varphi^{-1}\tau_j\varphi$ are the involutions on M^c generated by M, and $\varphi^*\omega = dz_1 \wedge \ldots \wedge dz_n|_{M^c}$.

By Proposition 4.1, we may also assume that the linear parts of τ_1, τ_2, ρ are given by (6.1). By averaging, we may assume that $\rho(x, y, \zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$. Let $\xi_1 = \frac{x+y}{2}, \eta_1 = \frac{x-y}{2}$. Then $\rho(\xi_1, \eta_1, \zeta) = (\overline{\eta}_1, \overline{\xi}_1, \overline{\zeta})$ and $\tau_1(\xi_1, \eta_1, \zeta) = (\xi_1, -\eta_1 - 2\xi_1, \zeta) + O(2)$. Let

$$p_1 = \frac{1}{2}(\xi_1 + \xi_1 \circ \tau_1(\xi_1, \eta_1, \zeta)) = \xi_1 + O(2),$$

$$p_\alpha = \frac{1}{2}(\zeta_\alpha + \zeta_\alpha \circ \tau_1(\xi_1, \eta_1, \zeta)) = \zeta_\alpha + O(2),$$

$$p_n = \eta_1 \circ \tau_1(\xi_1, \eta_1, \zeta) \cdot \eta_1 = -2\xi_1\eta_1 - \eta_1^2 + O(3).$$

By linearizing τ_1 alone, one can see that any holomorphic function that is invariant under τ_1 is a holomorphic function in $p_1, \ldots, p_{n-1}, p_n$. Furthermore, $\tau_1^*\omega = \omega$ implies that $\omega = dp_1 \wedge \cdots \wedge dp_{n-1} \wedge dA(p_1, \cdots, p_{n-1}, p_n)$ with $A(p_1, \ldots, p_{n-1}, 0) = 0$. Define $M \subset \mathbb{C}^n$ by equations

$$z_j = p_j(\xi_1, \overline{\xi}_1, t), \ 1 \leqslant j < n, \quad z_n = A(p_1, \ldots, p_{n-1}, p_n)(\xi_1, \overline{\xi}_1, t)$$

with $\xi_1 \in \mathbb{C}$, $t \in \mathbb{R}^{n-2}$. Since $\frac{\partial A}{\partial p_n}(0) \neq 0$, looking at the leading terms we see that M is smooth and of dimension n. The complexification M^c is then parameterized by

$$z_j = p_j(\xi_1, \eta_1, T), \ 1 \leqslant j < n, \quad z_n = A(p_1, \ldots, p_{n-1}, p_n)(\xi_1, \eta_1, T),$$

$$w_j = \overline{p}_j(\eta_1, \xi_1, T), \ 1 \leqslant j < n, \quad w_n = \overline{A}(\overline{p}_1, \ldots, \overline{p}_{n-1}, \overline{p}_n)(\eta_1, \xi_1, T)$$

with $(\xi_1, \eta_1) \in \mathbb{C}^2, T \in \mathbb{C}^{n-2}$.

Obviously, M has a parabolic complex tangent at the origin, by looking at the above expansions of p_j and by using $A = 0$ for $p_n = 0$ and $\frac{\partial A}{\partial p_n} \neq 0$. Since $p_1, \cdots, p_{n-1}, p_n$ are invariant by τ_1, then τ_1 is the unique non-trivial branched covering transformation of $\pi \colon M^c \to \mathbb{C}^n$. Thus $\tau_1, \tau_2 = \rho\tau_1\rho$ are the involution associated to M.

It is clear that $\omega = dp_1 \wedge \cdots \wedge dp_{n-1} \wedge dA(p_1, \cdots, p_{n-1}, p_n)$ is the complexification of $dz_1 \wedge \cdots \wedge dz_n|_M$ in M^c.

Since $\overline{\rho^*\omega} = -\omega$ and $\rho|_M = \mathrm{id}$, it follows that $\mathrm{Re}\, dz_1 \wedge \cdots \wedge dz_n|_M = 0$. Note that the condition $\rho\omega = -\overline{\omega}$ (and $\tau_2^*\omega = \omega$) is used only at the last step. $\quad\square$

We now consider the set $\mathcal{V}^*$ of the following data $\{\tau_1, \tau_2, \rho, \omega\}$, where $\tau_1, \tau_2 = \rho\tau_1\rho$, $\tau_1'(0) \neq \tau_2'(0)$, are holomorphic involutions on $\mathbb{C}^n$, ρ is an anti-holomorphic involution, and $\omega = Adx \wedge dy \wedge d\zeta_2 \cdots \wedge d\zeta_{n-1}$ is a holomorphic n-form on $\mathbb{C}^n$. Moreover, τ_1, τ_2 fix the same smooth holomorphic hypersurface N pointwise, A vanishes to order 1 along N and $\tau_1^*\omega = \omega$. Note that the latter implies that $\tau_2^*\overline{\rho^*\omega} = \overline{\rho^*\omega}$, which is, however, not a constant multiple of ω in general.

Write $\{\tau_1, \tau_2, \rho, \omega\} \sim \{\widetilde{\tau}_1, \widetilde{\tau}_2, \widetilde{\rho}, \widetilde{\omega}\}$ if there is a biholomorphic map f satisfying $\widetilde{\tau}_j = f\tau_j f^{-1}, \widetilde{\rho} = f\rho f^{-1}$, and $f^*\widetilde{\omega} = \omega$.

Let $\mathcal{V}$ be the set of real analytic n dimensional submanifolds $M \subset \mathbb{C}^n$ which have non-degenerate (parabolic) complex tangent points on a smooth

hypersurface in M. Write $M \sim \widetilde{M}$ if they are equivalent by a biholomorphic map f preserving $dz_1 \wedge \cdots \wedge dz_n$.

Dropping the last paragraph in the proof of the above proposition, we have

PROPOSITION 6.2. — *Let $M, \widetilde{M} \in \mathcal{V}$. Then $M \sim \widetilde{M}$ if and only if the corresponding $\{\tau_1, \tau_2, \rho, \omega\}$ are equivalent. Each $\{\tau_1, \tau_2, \rho, \omega\} \in \mathcal{V}^*$ is equivalent to one arising from some $M \in \mathcal{V}$.*

The above classification problem is of course interests in its own right. We are unable to solve it in general in this paper. Nevertheless, we have the following formal normal form, when $n = 2$ and ω is invariant under τ_1 and τ_2.

PROPOSITION 6.3. — *Let $\tau_1, \tau_2 = \rho_0 \tau_1 \rho_0$ be two holomorphic involutions with ρ_0 an anti-holomorphic involution on $\mathbb{C}^2$. Let $\omega_0 = x\alpha(x,y)dx \wedge dy$ with $\alpha(0) \neq 0$ be a holomorphic 2-form. Assume that τ_1, τ_2 fix $x = 0$ pointwise, $\tau_1'(0) \neq \tau_2'(0)$ and $\omega_0 = \tau_j^* \omega_0$ for $j = 1, 2$. There is a semi-formal transformation φ satisfying*

$$\hat{\tau}_j = \varphi^{-1}\tau_j\varphi \colon (x,y) \to (-x, y + (-1)^{j-1}2x),$$

$$\rho = \varphi^{-1}\rho_0\varphi \colon (x,y) \to (\overline{x}, -\overline{y}),$$

$$\omega = \varphi^*\omega_0 = e^{i\theta(x^2)}xdx \wedge dy,$$

in which $\theta(x) = \overline{\theta}(x)$ with $\theta(0) \in [0, \pi)$ is unique.

Proof. — By Proposition 5.1, we can choose a semi-formal map φ_0, which transforms τ_j, ρ_0 into the above $\hat{\tau}_j, \rho$ and transforms ω_0 into ω_1. Then any semi-formal map φ_1 preserving $\hat{\tau}_j, \rho$ must have the form

$$\varphi_1 \colon (x,y) \to (xa(x), ya(x) + b(x)),$$

$$a(x) = \overline{a}(x) \neq 0, \quad b(x) = -\overline{b}(x).$$

Since $\hat{\sigma}^*\omega_1 = \omega_1$ then $\omega_1 = \tilde{r}(x)e^{i\tilde{\theta}(x)}xdx \wedge dy$. Since $\hat{\tau}_1^*\omega_1 = \omega_1$, then $\tilde{r}, \tilde{\theta}$ are even formal power series. Write

$$\omega_1 = r(x^2)e^{i\theta(x^2)}xdx \wedge dy,$$

$$r(x) = \overline{r}(x) \neq 0, \quad \theta(x) = \overline{\theta}(x), \quad \theta(0) \in [0, \pi).$$

The identity $\varphi_1^*\omega = \omega_1$ is equivalent to

$$xa^2(x)a'(x) + a^3(x) = r(x^2).$$

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Note that if $a(x) = a_0 + \sum_{j<k/2} a_{2j} x^{2j} + a_k x^k + O(k+1)$ and k is odd, then the coefficient of x^k of $xa^2(x)a'(x) + a^3(x)$ is $(k+3)a_0^2 a_k$, which must be zero. Hence a must be even in x. For $a(x) = [A(x^2)]^{1/3}$, the above equation becomes $\frac{2}{3}xA'(x) + A(x) = r(x)$, which admits a unique solution A.

After ω_1 is normalized to ω, the maps that preserve the form of ω are given by $(x, y) \to (x, y + b(x))$ and they preserve ω. This shows the uniqueness of θ. $\square$

We can also prove the following by an averaging argument.

PROPOSITION 6.4. — *Let $\tau_1, \tau_2, \rho_0, \omega_0$ be as in Proposition 6.3. If $\tau_2\tau_1$ is holomorphically linearizable, there exists a holomorphic map normalizing τ_1, τ_2, ρ_0 and ω_0 simultaneously.*

Proof. — We normalize τ_1, τ_2, ρ_0 first by averaging. We may assume that $\tau_2\tau_1$ is the linear map $\hat{\sigma}$. By a linear transformation, we may assume by Proposition 4.1 that τ_j, ρ_0 are tangent to $\hat{\tau}_j, \rho$ respectively (for $n = 2$). Then ρ_0 reverses $\hat{\sigma}$. Define $g_1 = (\mathrm{id} + \rho\rho_0)/2$. Then $g_1\hat{\sigma} = \hat{\sigma}g_1$ and $g_1\rho_0 = \rho g_1$. So for $\tilde{\tau}_j = g_1\tau_j g_1^{-1}$, we still have that $\tilde{\tau}_2\tilde{\tau}_1 = \hat{\sigma}$ and $\tilde{\tau}_1$ reverses $\hat{\sigma}$. Put $g_2 = (\mathrm{id} + \hat{\tau}_1\tilde{\tau}_1)/2$. Then $\hat{\tau}_1 g_2 = g_2\tilde{\tau}_1$ and $\hat{\sigma}g_2 = g_2\hat{\sigma}$, and hence $\hat{\tau}_2 g_2 = g_2\tilde{\tau}_2$. Also $\rho g_2 = (\rho + \hat{\tau}_2\tilde{\tau}_2\rho)/2 = g_2\rho$. We have therefore linearized τ_1, τ_2 and ρ_0 by a convergent map. From now on we assume that τ_1, τ_2 and ρ_0 are linear.

We now look at the holomorphic n-form $\omega_0 = \omega_1$. One readily sees that the a in the proof of Proposition 6.3 is convergent, when r is convergent. Using the a, we normalize the holomorphic n-form. $\square$

The above proposition, when θ is constant, is due to Webster[11]. We should mention that the above proposition does not mean that the equivalence class of $\tau_1, \tau_2, \rho_0, \omega_0$ is determined by the holomorphic equivalence class of the *indicator* $\tau_2\tau_1$, as shown by Theorem 2.2.

We conclude this section by recalling a result of Moser-Webster, which is needed to classifying real analytic manifolds that are formally equivalent to the quadric.

Let $\mathcal{C}$ be the set of real analytic n-manifolds M in $\mathbb{C}^n$ such that M has a parabolic complex tangent at 0 and its set of complex tangents is a hypersurface in M. Let $\mathcal{C}/\sim$ be the set of holomorphic equivalence classes.

Let $\mathcal{C}^*$ be the set of $\{\tau_1, \tau_2, \rho\}$ satisfying the following: ρ is an anti-holomorphic involution and $\tau_1, \tau_2 = \rho\tau_1\rho$ are holomorphic involutions on $\mathbb{C}^n$. τ_1 and τ_2 fix pointwise the same hypersurface, and $(\tau_2\tau_1)'(0) \neq \mathrm{id}$. We say that $\{\tau_1, \tau_2, \rho\}, \{\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\rho}\} \in \mathcal{C}^*$ are equivalent if there is a biholomorphic

map g such that $g\tau_j g^{-1} = \widetilde{\tau}_j$ and $g\rho g^{-1} = \widetilde{\rho}$. Denote by $\mathcal{C}^*/\sim$ the set of equivalence classes.

We now recall the following result of Moser and Webster[8]:

PROPOSITION 6.5. — *There is a one-to-one correspondence between $\mathcal{C}/\sim$ and $\mathcal{C}^*/\sim$.*

7. Moduli spaces without volume-form - first half of Theorem 2.1

We will first recall the moduli space of Voronin [10] on holomorphic mappings that have

$$\sigma: x' = x + x^2 a(x,y,\zeta), y' = y + 4x + xb(x,y,\zeta), \zeta' = \zeta + xc(x,y,\zeta) \quad (7.1)$$

with $\zeta \in \mathbb{C}^{n-2}, b(0) = c_j(0) = 0$. This moduli space will be adapted to our classification problems.

Semi-formal maps and asymptotic expansions. Recall from the introduction that a power series $h(x,y,\zeta) = \sum_{k\geqslant 0} h_j(y,\zeta)x^j$ is called semi-formal in x, if all h_j are holomorphic in $(y,\zeta) \in \mathbb{C} \times \mathbb{C}^{n-2}$ on some fixed neighborhood W of the origin. Given two semi-formal maps F, G from $\mathbb{C}^n$ to itself, the composition $F \circ G$ is a well-defined semi-formal map, when G preserves $x = 0$.

Recall that $S = V \times W$ is a sectorial domain, if V is a sector of the form $V_{\alpha,\beta,\epsilon} = \{x : \arg x \in (\alpha,\beta), 0 < |x| < \epsilon\}$ and W is a neighborhood of the origin in $\mathbb{C}^{n-1}$. A semi-formal function G is called an asymptotic expansion of a holomorphic function g on $V \times W$, denoted by $g \sim G$ on $V \times W$, if $G(x,y,\zeta) = \sum_{k=0}^{\infty} G_k(y,\zeta)x^k$ with all G_k being holomorphic on some neighborhood $\widetilde{W}$ of the origin and if

$$\lim_{V \ni x \to 0} |x|^{-N} \left| g(x,y,\zeta) - \sum_{k=0}^{N} G_k(y,\zeta)x^k \right| = 0$$

uniformly on $\widetilde{W}$ for each N. Analogously, we say that a semi-formal map Φ is asymptotic to a holomorphic map H on $V \times W$, if each component of Φ is asymptotic to the corresponding component of H.

Semi-formal or sectorial normalizations. A semi-formal map $\Phi(x,y,\zeta) = (x + xu(x,y,\zeta), y + v(x,y,\zeta), \zeta + w(x,y,\zeta))$ is *normalized* if it satisfies the *normalizing condition*

$$u(x,0,\zeta) = v(x,0,\zeta) = w(x,0,\zeta) = 0.$$

The normalized semi-formal maps form a (pseudo)group. By Proposition 5.1 (and its remark) there is a unique normalized semi-formal map Φ such that

$$\hat{\sigma} = \Phi^{-1}\sigma\Phi \colon x' = x, \quad y' = y + 4x, \quad \zeta' = \zeta.$$

Note that a semi-formal map Ψ commutes with $\hat{\sigma}$, if and only if

$$\Psi \colon x' = x\widetilde{a}(x,\zeta), \quad y' = y\widetilde{a}(x,\zeta) + b(x,\zeta), \quad \zeta' = c(x,\zeta).$$

Let σ be given by (7.1), and Φ be the above unique normalized semi-formal map satisfying $\Phi^{-1}\sigma\Phi = \hat{\sigma}$. Voronin [10] shows that for any $\alpha < \beta < \alpha + \pi$ if $r > 0$ is sufficiently small there is a biholomorphic map H, defined on $S = \{x \colon \alpha < \arg x < \beta, |x| < r\} \times \Delta_r^{n-1}$ such that $H^{-1}\sigma H = \hat{\sigma}$, while $H \sim \Phi$ on S.

Voronin [10] proved the result on H_1 for $n = 2$. The same proof can be modified easily for $n > 2$ (see section 10).

We need to choose the position of sectors so that no further condition is imposed on the individual sectorial transformation H even in the presence of the reality condition $\rho g = g\rho$, which is required to normalize $\sigma = \tau_2\tau_1$. We should emphasize that such an arrangement may not be necessary, for the Voronin normalizing maps H_j can always be chosen to preserve ρ, by a possible averaging when a sector is invariant under ρ. Indeed, if σ *commutes* with anti-holomorphic involution ρ and V_1 is invariant under ρ, Voronin [10] constructed directly an H_1 that commutes with ρ. We are in the case that σ is *reversed* by ρ and it is not clear that such an H_1 can be obtained directly from Voronin's construction.

PROPOSITION 7.1. — *Let* $S_1 = \{x \colon -\epsilon < \arg x < \pi/2 + \epsilon, |x| < r\} \times \Delta_r^{n-1}$, *and* $S_j = i^{1-j}S_1$. *Let* τ_1, τ_2 *be holomorphic involutions given by Proposition 5.1, and let* $\sigma = \tau_2\tau_1$. *Let* $\mu = \delta\epsilon_2\cdots\epsilon_{n-1}$, *where* ϵ_j *are as in (4.11). For each* $0 < \epsilon < \pi/4$ *there exist* $r > 0$ *and holomorphic maps* H_j *on* $S_j \cup \hat{\sigma}(S_j)$ *such that* H_j *admit the same asymptotic expansion* Φ *of semi-formal map on* $S_j \cup \hat{\sigma}(S_j)$, $\Phi'(0) = \mathrm{id}$ *and* Φ *preserves* $x = 0$. *Moreover, we have*

(i) If $\delta = 1$, *then* $H_{k+2}^{-1}\tau_j H_k = \hat{\tau}_j$; *if* $\delta = -1$, *then* $H_k^{-1}\tau_j H_k = \hat{\tau}_j$.

(ii) If $\tau_2 = \rho\tau_1\rho$ *additionally for* $\rho(x,y,\zeta) = (\overline{x}, -\overline{y}, \overline{\zeta})$, *one can choose* H_j *satisfying (i) and* $\rho H_1\rho = H_2$, $\rho H_3\rho = H_4$.

(iii) If $\tau_j^*\omega = \mu\omega, j = 1, 2$ *additionally and* $\omega = a(x,y,\zeta)xdx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$ *is a holomorphic n-form with* $a(0) = 1$, *one can choose* H_j *satisfying (i) and* $H_j^*\omega = \hat{\omega} = xdx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$.

(iv) If τ_j, ρ, ω are as in (i)-(iii) and $\overline{\rho}^*\omega = -\omega$, one can choose H_j satisfying (i)-(iii).

Proof. — The arguments for cases $\delta = 1$ and $\delta = -1$ are different. The case $\delta = 1$ relies on Proposition 5.3. The case $\delta = -1$ requires only an averaging argument.

(i) By Proposition 5.1, there is a semi-formal map $\Phi_0 = \mathrm{id} + O(2)$, preserving $x = 0$ and satisfying $\Phi_0^{-1}\sigma\Phi_0 = \hat{\sigma}$. By a theorem of Voronin, there is a biholomorphic map $H_1^* \sim \Phi_0$ on some $V_1 \times W$ such that $H_1^{*-1}\sigma H_1^* = \hat{\sigma}$ on $V_1 \times W$. Next, we will find additional changes of coordinates, which are composed with H_1^*. It is understood that the composed maps will be defined on sectorial domains by shrinking the aperture of V_1 slightly and choose a smaller radius. This is justified by Lemma 10.3, since we will use changes of coordinates on sectorial domains which preserve $x = 0$ and admit asymptotic expansions that are tangent to the identity.

Consider the case $\delta = 1$ first. By Corollary 5.2, there is a semi-formal map $\Psi = \mathrm{id} + (2)$, preserving $x = 0$, such that

$$\Psi^{-1}\hat{\sigma}\Psi = \hat{\sigma}, \quad \Psi^{-1}\Phi_0^{-1}\tau_j\Phi_0\Psi = \hat{\tau}_j.$$

By Proposition 5.3, there is a biholomorphic map $H_1^{**} \sim \Psi$ on $V_1 \times \Delta_r^{n-1}$ such that $H_1^{**-1}\hat{\sigma}H_1^{**} = \hat{\sigma}$. Put $H_1 = H_1^*H_1^{**}$. Construct H_2 analogously on $V_2 \times W$. Put $H_3 = \tau_1 H_1 \hat{\tau}_1 \sim \Phi_0\Psi \equiv \Phi$ and $H_4 = \tau_1 H_2 \hat{\tau}_1 \sim \Phi$. We have

$$H_{k+2}^{-1}\tau_j H_k = \hat{\tau}_j, \quad H_k \sim \Phi_0\Psi \equiv \Phi = \mathrm{id} + O(2).$$

For the case $\delta = -1$, we simply use averaging. Construct H_2^*, H_3^*, H_4^* analogously as in (i) for H_1^*. Define $\widetilde{\tau}_{1,k} = H_k^{*-1}\tau_1 H_k^*$. Put

$$H_k^{**-1} = (\mathrm{id} + \hat{\tau}_1\widetilde{\tau}_{1,k})/2 \sim (\mathrm{id} + \hat{\tau}_1\Phi_0^{-1}\tau_1\Phi_0)/2 \stackrel{\mathrm{def}}{=\!=} \Psi^{-1}.$$

In particular, the asymptotic expansions of H_k^{**} are independent of k. It is clear that $\hat{\tau}_1 H_k^{**-1}\widetilde{\tau}_{1,k} = H_k^{**-1}$. Since $H_k^{*-1}\tau_2\tau_1 H_k^* = \hat{\tau}_2\hat{\tau}_1$, we also have $H_k^{**-1} = (\mathrm{id} + \hat{\tau}_2\widetilde{\tau}_{2,k})/2$ and hence $\hat{\tau}_2 H_k^{**-1}\widetilde{\tau}_{2,k} = H_k^{**-1}$. This shows that $H_k = H_k^*H_k^{**}$ linearizes τ_1, τ_2 on a sector.

(ii) If $\tau_2 = \rho\tau_1\rho$, we should construct H_1 first for $\delta = \pm 1$ and then take $H_2 = \rho H_1\rho$. When $\delta = 1$, put $H_3 = \tau_1 H_1\hat{\tau}_1$ and $H_4 = \rho H_3\rho$. When $\delta = -1$, construct H_3 analogous to H_1 and put $H_4 = \rho H_3\rho$.

(iii) Next we assume that $\tau_j^*\omega = \mu\omega$. We may assume that H_j are already constructed as in (i). Put

$$\theta_j = H_j^*\omega = a_j(x, y, \zeta)x\, dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}.$$

Since $H_j \sim \Phi$ and $\Phi'(0) = \mathrm{id}$ then $a_j(x, y, \zeta) \sim A(x, y, \zeta)$ and $A(0) = 1$. Put

$$b_j(x, y, \zeta) = \left\{ \int_0^1 3s^2 a_j(xs, sy, \zeta) \, ds \right\}^{1/3} = \frac{1}{x} \left\{ \int_0^x 3\xi^2 a_j(\xi, \xi y/x, \zeta) \, d\xi \right\}^{1/3},$$

$$h_j \colon (x, y, \zeta) \to (xb_j(x, y, \zeta), yb_j(x, y, \zeta), \zeta),$$

where $b_j(0) = 1$. Then

$$\begin{aligned}
h_j^* \hat{\omega} &= \frac{1}{2} d(xb_j(x, y, \zeta))^2 \wedge d(yb_j(x, y, \zeta)) \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1} \\
&= \frac{1}{3} d(xb_j(x, y, \zeta))^3 \wedge d(y/x) \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1} = \theta_j,
\end{aligned}$$

$$b_j(x, y, \zeta) \sim \left\{ \int_0^1 3s^2 A(xs, sy, \zeta) \, ds \right\}^{1/3} = 1 + O(1).$$

Assume $\delta = -1$ first. Since $\hat{\tau}_j^* \theta_j = \mu \theta_j$, then $a_k \hat{\tau}_j = a_k$. Then h_k commute with $\hat{\tau}_j$. Assume that $\delta = 1$. Then $\hat{\tau}_j^* \theta_k = \mu \theta_{k+2}$, $a_k \hat{\tau}_j = a_{k+2}$, and $h_{k+2} \hat{\tau}_j = \hat{\tau}_j h_k$. Thus $H_j h_j^{-1}$ are the required maps.

(iv) Assume that H_j are already constructed as in (ii). Let h_j be as above. When $\rho^* \omega = -\overline{\omega}$, we have $a_1 \rho(x, y, \zeta) = \overline{a_2(x, y, \zeta)}$ and $a_3 \rho(x, y, \zeta) = \overline{a_4(x, y, \zeta)}$. So $h_2 = \rho h_1 \rho$ and $h_4 = \rho h_3 \rho$. Therefore $H_j h_j^{-1}$ are the required maps. $\square$

Having constructed H_j by Proposition 7.1, we are ready to construct two moduli spaces. This will finish the proof of Theorem 2.1.

Moduli space of real manifolds with parabolic complex tangents on a codimension one submanifold. We will first construct the moduli space for real analytic n-manifolds in $\mathbb{C}^n$ which have parabolic complex tangents along an $n - 1$ dimensional submanifold. We will then show the moduli space is infinite dimensional.

By Proposition 6.5, it suffices to construct the moduli space for $\mathcal{C}^*$. Recall that $\{\tau_1, \tau_2, \rho\}$ is in $\mathcal{C}^*$, if τ_1, τ_2 fix the same hypersurface and $(\tau_2 \tau_1)'(0) \neq \mathrm{id}$. Also, $\{\tau_1, \tau_2, \rho\}, \{\widetilde{\tau}_1, \widetilde{\tau}_2, \widetilde{\rho}\} \in \mathcal{C}^*$ are equivalent if there is a biholomorphic map g such that $g\tau_j g^{-1} = \widetilde{\tau}_j$ and $g\rho g^{-1} = \widetilde{\rho}$.

Take $\{\tau_1^0, \tau_2^0, \rho^0\} \in \mathcal{C}^*$. By Corollary 4.2, there exists a biholomorphic map φ such that $\tau_j = \varphi \tau_j^0 \varphi^{-1}$ fix $x = 0$ pointwise and are tangent to $\hat{\tau}_j \colon (x, y, \zeta) \to (-x, y + (-1)^{j-1} 2x, \zeta)$ and such that $\rho = \varphi \rho^0 \varphi^{-1}$ is given by $(x, y, \zeta) \to (\overline{x}, -\overline{y}, \overline{\zeta})$. By Proposition 7.1 for each $\epsilon \in (0, \pi/4)$ there exists $r > 0$ such that for

$$S_j = S_j(\epsilon, r) = i^{1-j} \{ x \colon -\epsilon < \arg x < \pi/2 + \epsilon, |x| < r \} \times \Delta_r^{n-1} \qquad (7.2)$$

there are H_j defined on $S_j \cup \hat{\sigma}(S_j)$ and a semi-formal map Φ satisfying

$$H_{j+2}^{-1}\tau_k H_j = \hat{\tau}_k, \quad \rho H_1 \rho = H_2, \quad \rho H_3 \rho = H_4 \tag{7.3}$$

$$H_j \sim \Phi = \mathrm{id} + O(2), \quad \text{on } S_j \cup \hat{\sigma}(S_j), \quad j = 1, \dots, 4, \tag{7.4}$$

and moreover Φ preserves $x = 0$. Put

$$H_{j\,j+1} = H_j^{-1} H_{j+1}. \tag{7.5}$$

We have

$$H_{1\,2}^{-1} = \rho H_{1\,2} \rho, \; H_{4\,1}^{-1} = \rho H_{2\,3} \rho, \; H_{k+2\,k+3} = \hat{\tau}_j H_{k\,k+1} \hat{\tau}_j, \tag{7.6}$$

$$H_{j\,j+1} \sim \mathrm{id}, \quad \text{on } S_j \cup \hat{\sigma}(S_j). \tag{7.7}$$

We shall call $\{H_{j\,j+1}\}$ a *moduli function* of $\{\tau_1^0, \tau_2^0, \rho^0\}$, if $H_{j\,j+1}$ satisfies (7.3)-(7.7). Denote by $\mathcal{H}$ the set of moduli functions $\{H_{j\,j+1}\}$ satisfying (7.6)-(7.7) for some positive ϵ and δ

Remark. — As remarked in section 2, our biholomorphic maps $H_{j\,j+1}$ are defined on sectorial domains and admit asymptotic expansions. Therefore, we will have good controls on the domains and ranges of inverse or composition maps. One can see this by applying Lemma 10.3. Sometimes we need to shrink a sectorial domain, but this is done in terms the radius of the sctorial domain. The aperture of the sectorial domain is only shrunk slightly. In particular, $H_{j\,j+1} = H_j^{-1} H_{j+1}$ is defined on $S_j \cap S_{j+1} \cup \hat{\sigma}(S_j \cap S_{j+1})$ which is non-empty.

Of course, $H_{j,j+1}$ depend on the choices of initial coordinate map φ and H_j. Let us first determine how moduli functions change for different φ and H_j. Assume that $\widetilde{\varphi}$ is another choice such that $\widetilde{\tau}_j = \widetilde{\varphi} \tau_j^0 \widetilde{\varphi}^{-1}$ fix $x = 0$ pointwise and are tangent to $\hat{\tau}_j$ and such that $\rho = \widetilde{\varphi} \rho^0 \widetilde{\varphi}^{-1}$. Assume that $\widetilde{\Phi}$ and $\widetilde{H}_j$ satisfy

$$\widetilde{H}_{j+2}^{-1} \widetilde{\tau}_k \widetilde{H}_j = \hat{\tau}_k, \quad k = 1, 2, \quad \widetilde{H}_j \sim \widetilde{\Phi}, \quad j = 1, \dots, 4;$$

$$\rho \widetilde{H}_1 \rho = \widetilde{H}_2, \quad \rho \widetilde{H}_3 \rho = \widetilde{H}_4.$$

Recall that H_j and $\tilde{H}_j$ are defined on domains of the form (7.2). Choose small ϵ, r such that they are both defined on $S_j(\epsilon, r)$. For a possibly smaller r, we can put $g = \varphi \widetilde{\varphi}^{-1}$. Then $\widetilde{\tau}_j = g^{-1} \tau_j g$ with $g\rho = \rho g$ and the first component of $g(x, y, \zeta)$ is $\mu x(1 + O(1))$ with $\mu \neq 0$ a real number. If $\mu > 0$, put $G_j = H_j^{-1} g \widetilde{H}_j$, which is defined on S_j when ϵ and r are sufficiently small;

if $\mu < 0$, put $G_j = H_j^{-1} g \widetilde{H}_{j-2}$. Put $\Psi = \Phi^{-1} g \widetilde{\Phi}$. Then $\widetilde{H}_j = g^{-1} H_j G_j$ for $\mu > 0$ and $\widetilde{H}_{j-2} = g^{-1} H_j G_j$ for $\mu < 0$. Also

$$G_j^{-1} H_{j\,j+1} G_{j+1} = \begin{cases} \widetilde{H}_{j\,j+1}, & c > 0, \\ \widetilde{H}_{j+2\,j+3}, & c < 0; \end{cases} \tag{7.8}$$

$$G_2 = \rho G_1 \rho, \quad G_4 = \rho G_3 \rho, \quad G_{k+2} = \hat{\tau}_j G_k \hat{\tau}_j; \tag{7.9}$$

$$G_j \sim \Psi = \rho \Psi \rho = \hat{\tau}_1 \Psi \hat{\tau}_1, \text{ on } S_j \text{ or on } S_{j-2}. \tag{7.10}$$

Since $\Psi'(0) = g'(0)$ is biholomorphic and Ψ commutes with $\hat{\sigma}$ then

$$\Psi \colon (x, y, \zeta) \to (a(x, \zeta)x, ya(x, \zeta) + b(x, \zeta), c(x, \zeta)), \tag{7.11}$$

where a, b, c are semi-formal in x, $a(0) \neq 0, b(0) = 0, c(0) = 0$, and $\zeta \to c(0, \zeta)$ is biholomorphic. Note that $a(0) = \mu$. The above asymptotic expansion implies that

$$G_j \colon \quad \text{sign}(\mu)\, i^{1-j} V_{-\epsilon, \pi/2+\epsilon, (1-\epsilon)\delta} \times \Delta_{(1-\epsilon)\delta}^{n-1} \to$$

$$i^{1-j} V_{-2\epsilon, \pi/2+2\epsilon, c_0(1+\epsilon)\delta} \times \Delta_{c_0(1+\epsilon)\delta}^{n-1},$$

where c_0 is a positive number independent of ϵ, δ. Conversely, assume that there are semi-formal map Ψ and biholomorphisms G_j such that $\{\tau_1, \tau_2, \rho\}$, $\{\widetilde{\tau}_1, \widetilde{\tau}_2, \rho\}$ have moduli functions $H_{j,j+1}$ and $\widetilde{H}_{j,j+1}$ satisfying (7.8)-(7.11). If the first case in (7.8) occurs then $g = H_j G_j \widetilde{H}_j^{-1}, j = 1, \ldots, 4$ agree on the overlap. Hence g is well-defined on $\Delta_{\delta'}^n \cap \mathbb{C}^* \times \mathbb{C}^{n-1}$, if δ' is sufficiently small, and g extends to a holomorphic map defined near the origin. On a sectorial domain, we know that $g \sim \Phi \Psi \widetilde{\Phi}^{-1}$. Thus $g'(0) = \Psi'(0)$, which implies that g is a biholomorphic map. Now $g^{-1} \tau_1 g = \widetilde{H}_3 G_3^{-1} H_3^{-1} \tau_1 H_1 G_1 \widetilde{H}_1^{-1} = \widetilde{\tau}_1$ and $g^{-1} \rho g = \widetilde{H}_2 G_2^{-1} H_2^{-1} \rho H_1 G_1 \widetilde{H}_1^{-1} = \rho$. Hence $g^{-1} \tau_2 g = \widetilde{\tau}_2$. If the second case in (7.8) occurs, define $g = H_j G_j \widetilde{H}_{j+2}^{-1}$. Then $g^{-1} \tau_1 g = \widetilde{\tau}_1$ and $g^{-1} \rho g = \rho$.

Conversely, assume that we are given moduli functions $H_{j\,j+1}$ satisfying (7.6)-(7.7), where $H_{j\,j+1}$ are defined on $S_{j\,j+1}(\epsilon_0, r_0) = S_j(\epsilon_0, r_0) \cap S_{j+1}(\epsilon_0, r_0)$ and $S_j(\epsilon_0, r_0)$ with $0 < \epsilon_0 < \pi/4$ are given by (7.2). Following Malgrange [7], we shall construct the corresponding pairs $\tau_1, \tau_2 = \rho \tau_1 \rho$ as follows.

Let $A_1 = \{x \colon |\arg x - \pi/2| < 2\epsilon, |x| < 2r\} \times \Delta_{2r}^{n-1}$, $B_1 = \{x \colon \epsilon < \arg x < \pi/2 - \epsilon, |x| < r\} \times \Delta_r^{n-1}$. Put $A_j = i^{1-j} A_1$ and $B_j = i^{1-j} B_1$. Choose small and positive ϵ, r so that $H_{j\,j+1}$ are asymptotic to identity on A_{j+1} and so that the first component $h_{j\,j+1}$ of $H_{j\,j+1}$ satisfies $4/5 < |\frac{1}{x} h_{j\,j+1}(x, y, \zeta)| < 5/4$ and $|\arg\{\frac{1}{x} h_{j\,j+1}(x, y, \zeta)\}| < \epsilon/4$ on A_j. We may assume the first

component of $H_{j\,j+1}^{-1}$ satisfies the same estimates. Then $H_{j\,j+1}$ is a biholomorphic map from A_{j+1} onto $\widetilde{C}_j = H_{j\,j+1}(A_{j+1})$. Let $S_j = A_j \cup B_j \cup \widetilde{C}_j$. Let X_0 be the disjoint union $\sqcup_{j=1}^4 S_j$. We identify $p \in A_{j+1}$ with $H_{j\,j+1}(p) \in \widetilde{C}_j$, which defines an equivalence relation on X_0 since $\widetilde{C}_j$ does not intersect A_k for $k \neq j, j+1, j-2$. Let X be the quotient space of X_0 by the equivalence relation, and $\pi\colon X_0 \to X$ be the projection. So $U \subset X$ is open if and only if $\pi^{-1}(U) \cap S_j$ are open for all j; in particular, if V is open in S_j then $\pi^{-1}(\pi(V)) = V \cup H_{j-1\,j}(V \cap A_j) \cup H_{j\,j+1}^{-1}(V \cap \widetilde{C}_j)$ is open and hence $\pi(V)$ is open. We need to show that X is Hausdorff. Let p, q be in X_0 with $\pi(p) \neq \pi(q)$. If p, q are in the same S_j, take disjoint open sets $U_p \ni p, U_q \ni q$ in S_j. Since $H_{j\,j+1}$ is one-to-one then $\pi(U_p), \pi(U_q)$ are also disjoint open sets. If p is in S_j and q is in S_k for $k \neq j, j-1, j+1$, then $\pi(S_j), \pi(S_k)$ separate p and q. Finally it remains to check the case that $p \in S_j$ and $q \in S_{j+1}$. If $q \in A_{j+1}$, then p and $H_{j\,j+1}(q)$ are both in S_j, which is reduced to a previous case. The same argument applies if $p \in \widetilde{C}_j$. Assume now that p is in $S_j \setminus \widetilde{C}_j$ and q is in $S_{j+1} \setminus A_{j+1}$. In particular $|\arg(p/q)| < \epsilon/2$ does not hold. Choose open sets $U_p \ni p$ and $U_q \ni q$ such that $|\arg(\widetilde{p}/H_{j\,j+1}(\widetilde{q}))| < \epsilon/4$ does not hold for $\widetilde{p} \in U_p$ and $\widetilde{q} \in U_q \cap A_{j+1}$. Therefore, $\pi(U_p) \cap \pi(U_q)$ is empty and X is Hausdorff.

Now X is a complex manifold with the coordinates $\pi_j^{-1} = (x_j, y_j, \zeta_j)$ defined on $\pi(S_j)$ and with value in $S_j \subset \mathbb{C}^n$, and we also have its inverse $\pi_j\colon S_j \hookrightarrow X_0 \overset{\pi}{\to} X$. Note that $H_{j\,j+1} = \pi_j^{-1}\pi_{j+1}$ on A_{j+1}. On $\pi(X_0/4)$, we define

$$\widetilde{\tau}_k\colon (x_j, y_j, \zeta_j) \to (x_{j+2}, y_{j+2}, \zeta_{j+2}) = (-x_j, y_j + (-1)^{k-1}2x_j, \zeta_j),$$
$$\widetilde{\rho}\colon \begin{cases} (x_1, y_1, \zeta_1) \to (x_2, y_2, \zeta_2) = (\overline{x}_1, -\overline{y}_1, \overline{\zeta}_1), \\ (x_3, y_3, \zeta_3) \to (x_4, y_4, \zeta_4) = (\overline{x}_3, -\overline{y}_3, \overline{\zeta}_3). \end{cases}$$

Take smooth non-negative functions $\chi_j(x, y, \zeta) \equiv \chi_j(x/|x|)$ such that χ_j equals 1 for $\arg x \in (\epsilon/2 + (1-j)\pi/2, -\epsilon/2 + (2-j)\pi/2)$ and 0 for $\arg x \notin (-\epsilon/2 + (1-j)\pi/2, \epsilon/2 + (2-j)\pi/2)$, and such that $\chi_1 + \cdots + \chi_4 = 1$. Define

$$K(p) = \sum_{j=1}^4 \chi_j(x_j(p))(x_j(p), y_j(p), \zeta_j(p)).$$

Since $H_{j\,j+1} \sim \mathrm{id}$ then $K(X) = D \cap (\mathbb{C}^* \times \mathbb{C}^{n-1})$, where D is an open neighborhood of the origin in $\mathbb{C}^n$. Note that K is a biholomorphism when the complex structure on $K(X)$ is defined by $K_{j*}\mathcal{D}_x, K_{j*}\mathcal{D}_y, K_{j*}\mathcal{D}_\zeta$ and $K_j \circ \pi_j^{-1} = K$ on $\pi(S_j)$. Note that $\pi_k^{-1}(p) = H_{kj}\pi_j^{-1}(p)$ when $\chi_k(\pi_k^{-1}(p))\chi_j(\pi_j^{-1}(p)) \neq 0$. Thus

$$K_j(t) = \sum_{k=1}^{4} \chi_k(H_{k\,j}(t)) H_{k\,j}(t) \sim \sum_{k=1}^{4} \chi_k(t) H_{k\,j}(t)$$

$$\sim \sum_{k=1}^{4} \chi_k(t) t = t, \quad t = \pi_j^{-1}(p) \in S_j.$$

Hence the complex structure extends to D and agrees with the standard one along $x = 0$ to infinitely order. By the Newlander-Nirenberg theorem, there is a diffeomorphism $\psi \colon \tilde{D} \to \Omega \subset \mathbb{C}^n$ with $\psi(0) = 0$ such that ψK is biholomorphic. Since $\psi(\{x = 0\} \cap \tilde{D})$ is a holomorphic hypersurface in (the standard Euclidean space) $\mathbb{C}^n$, by a holomorphic change of coordinates (and by shrinking $\tilde{D}$ if necessary), one may assume that ψ preserves $x = 0$. Now the inverse ψ^{-1}, expanded as a formal power series in $x, \overline{x}$, is a formal power series in x only and has coefficients holomorphic in y, ζ in a fixed domain. Using a finite order Taylor expansion of ψ^{-1} (in x) if necessary, one may also assume that $\psi(x, y, \zeta) = (x, y, \zeta) + O(|x|^2)$. On $\Omega \cap (\mathbb{C}^* \times \mathbb{C}^{n-1})$ define $\tau_j^0 = \psi K \tilde{\tau}_j K^{-1} \psi^{-1}$ and $\rho^0 = \psi K \tilde{\rho} K^{-1} \psi^{-1}$. Again, since $H_{j,j+1} \sim \mathrm{id}$ then τ_j^0, ρ^0 extends to Ω with $\tau_j^0(x, y, \zeta) = \hat{\tau}_j(x, y, \zeta) + O(|x|^2)$ and $\rho^0(x, y, \zeta) = \rho(x, y, \zeta) + O(|x|^2)$. We need to show that $\{\tau_j^0, \rho^0\}$ is the required realization. Let $\varphi_0 = (\mathrm{id} + \rho \rho^0)/2$. Then φ^0 is tangent to the identity and fixes $x = 0$ pointwise, and $\rho \varphi^0 = \varphi^0 \rho^0$. Take $H_j = \varphi^0 \psi K_j = \varphi^0 \psi K \pi_j$. Then $H_j^{-1} H_{j+1} = H_{j,j+1}$. On S_j we have

$$H_j(t) = \varphi^0 \psi K_j(t) \sim \varphi^0 \hat{\psi}(t) \equiv \Phi(t) = \mathrm{id} + O(2), \quad t = (x_j, y_j, \zeta_j),$$

where $\hat{\psi}(x, y, \zeta)$ is the Taylor series expansion of $\psi(x, y, \zeta)$ in $x, \overline{x}$. Since φ^0 and ψ preserve $x = 0$, Φ preserves $x = 0$ too. As mentioned above, $\hat{\psi}(x, y, \zeta)$ is actually a formal (holomorphic) power series in x whose coefficients are holomorphic in y, ζ in a fixed domain. This finishes the proof of the realization.

To deal with mappings defined on a sectorial domain $S = V \times \Delta_r^{n-1}$ that commute with $\hat{\sigma}(x, y, \zeta) = (x, y + 4x, \zeta)$, it is convenient to consider the quotient space $S/\hat{\sigma}$ by the projection $(x, t, \zeta) = \pi(x, y, \zeta) = (x, e^{\frac{\pi i y}{2x}}, \zeta)$. More specifically, if H commutes with $\hat{\sigma}$ then it has the form $H(x, y, \zeta) = (x a(x, y, \zeta), y a(x, y, \zeta) + b(x, y, \zeta), c(x, y, \zeta))$ with $a\hat{\sigma} = a$, $b\hat{\sigma} = b$ and $c\hat{\sigma} = c$, which yields a mapping in (x, t, ζ)-space defined for $x \in V$, $e^{-\frac{\pi r}{2|x|}} < |t| < e^{\frac{\pi r}{2|x|}}$ and $\zeta \in \Delta_r^{n-2}$, given by

$$\tilde{H} \colon x' = x\tilde{a}(x, t, \zeta), \quad t' = t\lambda(x, t, \zeta), \quad \zeta' = \tilde{c}(x, t, \zeta),$$

$$\tilde{a} = a\pi^{-1}, \quad \tilde{c} = c\pi^{-1}, \quad \pi^{-1}(x, t, \zeta) = (x, \frac{x \log t}{\pi i}, \zeta),$$

$$\lambda = e^d, \quad d(x,t,\zeta) = \frac{\pi i b \pi^{-1}(x,t,\zeta)}{2xa\pi^{-1}(x,t,\zeta)}.$$

When H is asymptotic to the identity on the sectorial domain $V \times \Delta_r^{n-1}$, such as a mapping H_{jj+1} in $\{H_{jj+1}\}$, we have $|a(x,y,\zeta) - 1| < c_0|x|$ and $|y(a(x,y,\zeta) - 1) + b(x,y,\zeta)| < c_0|x|^2$ for $x \in V \cap \Delta_\delta$ and $(y,\zeta) \in \Delta_\epsilon^{n-1}$, which implies that

$$|d(x,t,\zeta)| \leqslant \pi c_0 |x| + \pi c_0 |y| < \pi$$

for $|x|, |y|, |\zeta|$ sufficiently small. Hence $\tilde{H}$ determines H uniquely. We will also consider mappings G, such as a mapping G_j appearing in the equivalence relation of moduli space. The G is defined on a sectorial domain $V \times \Delta_r^{n-1}$, commutes with $\hat{\sigma}$ and admits an asymptotic expansion $\Psi(x,y,\zeta) = (xA(x,y,\zeta), yA(x,y,\zeta) + B(x,y,\zeta))$ with $A(0) \neq 0, B(0) = 0 = C(0)$. Note that the semi-formal map Ψ still commutes with $\hat{\sigma}$, so $A\hat{\sigma} = A$ and $B\sigma = B$. However, G is not uniquely determined by $\tilde{G}$; $\tilde{G} = \tilde{G}'$ if and only if

$$A' = A, \quad C' = C, \quad B'(x,y,\zeta) = B(x,y,\zeta) + 4kxA(x,y,\zeta), \quad k \in \mathbb{Z},$$

i.e. $G' = \hat{\sigma}^k G$. Therefore, the asymptotic expansion of G determines k; in particular, the equivalence class of $\{H_{jj+1}\}$ is determined by its equivalence class in the (x,t,ζ)-space.

In (x,t,ζ)-space, define $\hat{\tau}_1(x,t,\zeta) = \hat{\tau}_2(x,t,\zeta) = (-x, -t^{-1}, \zeta)$, and $\rho(x,t,\zeta) = (\overline{x}, \overline{t}, \overline{\zeta})$. Then moduli functions $H_{jj+1}, j = 1, \ldots, 4$ will still satisfy the conditions (7.6) (with the new $\hat{\tau}_1$ and ρ). The G_j in the equivalence relation still satisfy (7.8)-(7.10). Moreover, the asymptotic expansion of G_j becomes $\Psi = (xa(x,\zeta), t\lambda(x,\zeta), c(x,\zeta))$. On the (x,t,ζ)-space the moduli functions H_{jj+1} and mappings G_j are required to satisfy asymptotic expansion conditions, and by definition the asymptotic expansion conditions mean the conditions described in the (x,y,ζ)-space.

Next we want to show the non-triviality of the moduli space.

Define

$$H_{12}(x,y,\zeta) = (x, y + r(x,t), \zeta), \tag{7.12}$$

$$H_{34}(x,y,\zeta) = (x, y + r(-x, -t^{-1}), \zeta), \quad H_{23} = H_{41} = \text{id}, \tag{7.13}$$

$$r(x,t) = \frac{2x}{\pi i} \log \frac{1 + (1 - t^{-1})c_p(x)}{1 - (1 - t)c_p(x)}, \tag{7.14}$$

$$t = e^{\frac{i\pi y}{2x}}, \quad c_p(x) = e^{-1/x}p(x), \quad p(x) = p(-x), \quad p(0) \neq 0, \tag{7.15}$$

where $p(x)$ is holomorphic near the origin. Put $H_{23} = H_{41} = \text{id}$. We have $H_{34} = \hat{\tau}_1 H_{12} \hat{\tau}_1$ and $\hat{\sigma} H_{jj+1} \hat{\sigma}^{-1} = H_{jj+1} \sim \text{id}$ when $|y| < \delta < 1/2$. Note that

$$H_{1\,2}^{-1}(x, y, \zeta) = (x, y + \widetilde{r}(x, t), \zeta),$$

$$\widetilde{r}(x, t) = \frac{2x}{\pi i} \log \frac{1 - (1 - t^{-1})c_p(x)}{1 + (1 - t)c_p(x)}.$$

Thus $\rho H_{1\,2}\rho = H_{1\,2}^{-1}$ holds, if and only if $p(x) = -\overline{p}(x)$.

By a result of Voronin (Lemma 16 in [10]), if $\widetilde{H}_{j,j+1}$ are of the form (7.12)-(7.15) with p replaced by $\widetilde{p}$, and if $\widetilde{H}_{j,j+1}, H_{j,j+1}$ are equivalent under mapping $G_j \sim \Psi$ satisfying (7.8) and (2.5), then $\widetilde{p} = p$.

The above argument shows the moduli space is infinitely dimensional when $n = 2$. The higher dimension case can be obtained by trivial extensions as follows.

PROPOSITION 7.2. — *Let M_1, M_2 be two real analytic surfaces in $\mathbb{C}^2$ with a non-degenerate complex tangent at the origin. Define $M_j^* = M_j \times \mathbb{R}^{n-2} \subset \mathbb{C}^2 \times \mathbb{C}^{n-2}$. Then M_1, M_2 are holomorphically equivalent, if and only if M_1^*, M_2^* are holomorphically equivalent.*

Proof. — By two local changes of holomorphic coordinates in $\mathbb{C}^2$ one may assume that M_j are given by $z_2 = a_j z_1^2 + b_j z_1 \overline{z}_1 + c_j \overline{z}_1^2 + E_j(z_1, \overline{z}_1)$ with $E_j = O(3)$ and $|b_j| + |c_j| \neq 0$. Assume that $f = (f_1, \ldots, f_n) \colon \mathbb{C}^n \to \mathbb{C}^n$ is a local biholomorphism such that $f(M_1^*) = M_2^*$. Since $T_0 M_j^* \cap i T_0 M_j^*$ is spanned by z_1-axis, then $f'(0)$ preserves the z_1-axis. So $f_1(z_1, 0) = \mu z_1 + O(2)$ with $\mu \neq 0$. Since $f(M_1^*) = M_2^*$, we have $f_2(z) = a_2 f_1^2(z) + b_2 f_1(z)\overline{f_1(z)} + c_2 \overline{f_1^2(z)} + E_2(f_1(z), \overline{f_1(z)})$ for $z \in M_1^*$. Let $g_j(z_1, z_2) = f_j(z_1, z_2, 0)$ for $j = 1, 2$. We get

$$g_2(w) = a_2 g_1^2(w) + b_2 g_1(w)\overline{g_1(w)} + c_2 \overline{g_1^2(w)} + E_2(g_1(w), \overline{g_1(w)}) \quad (7.16)$$
$$w_2 = a_1 w_1^2 + b_1 w_1 \overline{w}_1 + c_1 \overline{w}_1^2 + E_1(w_1, \overline{w}_1).$$

Write $g_2(w_1, w_2) = \alpha w_1 + \beta w_2 + O(2)$. Comparing coefficients of $w_1 \overline{w}_1$ and $\overline{w}_1^2$ in (7.16), we see that $\beta b_1 = b_2 \mu\overline{\mu}$ and $\beta c_1 = c_2 \overline{\mu}^2$. Since $\mu \neq 0$ and $|b_2| + |c_2| \neq 0$, then $\beta \neq 0$. This shows that $(z_1, z_2) \to g(z_1, z_2)$ is a biholomorphism sending M_1 into M_2. $\qquad\square$

8. Moduli spaces with the volume-form - Second half of Theorem 2.1

The second half of the proof of Theorem 2.1 involves a volume-form.

We will now construct the moduli space for real analytic n-manifolds in $\mathbb{C}^n$ on which $\mathrm{Re}\, dz_1 \wedge \cdots \wedge dz_n$ vanishes and which have parabolic complex tangents along an $n - 1$ dimensional submanifold. We will then show

the moduli space is infinite dimensional. This will finish the proof of Theorem 2.1.

Along the lines of the second half of the proof of Theorem 2.1, we will also prepare some results for Theorem 2.2.

The proof is very similar to the part in the previous section. We will be brief and emphasize the needed changes.

By Proposition 6.1, it suffices to construct the moduli space for $\mathcal{L}^*$. Recall that $\{\tau_1, \tau_2, \rho, \omega\}$ is in $\mathcal{L}^*$, if ρ is an anti-holomorphic involution, and $\tau_1, \tau_2 = \rho\tau_1\rho$ are a pair of holomorphic involutions on $\mathbb{C}^n$ fixing a smooth holomorphic hypersurface N pointwise. The set of fixed points of $\sigma'(0)$ is a hyperplane. $\omega = Ad\overline{z}_1 \wedge dz_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-1}$ is a holomorphic n-form on $\mathbb{C}^n$ vanishing on N to first order (i.e. $A = 0$ on N and $dA \neq 0$) and $\tau_j^*\omega = \omega = -\overline{\rho^*\omega}$.

Take $\{\tau_1^0, \tau_2^0, \rho^0, \omega^0\} \in \mathcal{L}^*$.

By Corollary 4.2, there exists a biholomorphic map φ such that $\tau_j = \varphi\tau_j^0\varphi^{-1}$ fix $x = 0$ pointwise and are tangent to $\hat{\tau}_j\colon (x, y, \zeta) \to (-x, y + (-1)^{j-1}2x, \zeta)$ and such that $\rho = \varphi\rho^0\varphi^{-1}$ is given by $(x, y, \zeta) \to (\overline{x}, -\overline{y}, \overline{\zeta})$. Write $\omega = \varphi^*\omega^0 = A(x, y, \zeta)x\,dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$. Since $\rho^*\omega = -\overline{\omega}$, then $A(0)$ is real. By a change of coordinates $(x, y, \zeta) \to (cx, cy, \zeta)$ with $c \in \mathbb{R}$, we may assume that $A(0) = 1$. By Proposition 7.1 there exist $V_1 = V_{\epsilon,\delta} = \{x \in \mathbb{C}\colon 0 < |x| < \delta, \arg x \in (-\epsilon, \pi/2 + \epsilon)\}$ with $0 < \epsilon < \pi/4$, $V_j = i^{1-j}V_1$, and H_j defined on $V_j \times \Delta_\delta^{n-1}$ $(\delta > 0)$ and semi-formal map Φ satisfying

$$H_{j+2}^{-1}\tau_k H_j = \hat{\tau}_k, \quad \rho H_1 \rho = H_2, \quad \rho H_3 \rho = H_4 \tag{8.1}$$

$$H_j \sim \Phi = \mathrm{id} + O(2), \text{ on } V_j \times \Delta_\delta^{n-1}, \quad j = 1, \ldots, 4, \tag{8.2}$$

where Φ preserves $x = 0$. Also

$$H_j^*\omega = \hat{\omega}, \quad \hat{\omega} = x\,dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}. \tag{8.3}$$

Put $H_{j,j+1} = H_j^{-1}H_{j+1}$. We have

$$H_{1\,2}^{-1} = \rho H_{1\,2}\rho, \; H_{4\,1}^{-1} = \rho H_{2\,3}\rho, \; H_{k+2\,k+3} = \hat{\tau}_j H_{k\,k+1}\hat{\tau}_j, \tag{8.4}$$

$$H_{j\,j+1} \sim \mathrm{id}, \quad \text{on } i^{1-j}V_{-\epsilon,\epsilon,\delta} \times \Delta_\delta^{n-1} \tag{8.5}$$

for possibly smaller $\epsilon, \delta > 0$. Also

$$H_{j\,j+1}^*\hat{\omega} = \hat{\omega}. \tag{8.6}$$

We shall call $\{H_{j,j+1}\}$ a *moduli function* of $\{\tau_1, \tau_2, \rho, \omega\}$, if $H_{j\,j+1} = H_j^{-1} H_{j+1}$ satisfy (8.1)-(8.6). Denote by $\mathcal{H}$ the set of moduli functions $\{H_{j\,j+1}\}$ satisfying (8.4)-(8.6) for some positive ϵ and δ.

Let us determine how moduli functions change for different φ and H_j. Assume that $\widetilde{\varphi}$ is another choice such that $\widetilde{\tau}_j = \widetilde{\varphi}\tau_j^0\widetilde{\varphi}^{-1}$ fix $x = 0$ pointwise and are tangent to $\hat{\tau}_j$ and such that $\rho = \widetilde{\varphi}\rho^0\widetilde{\varphi}^{-1}$ and $\widetilde{\omega} = \varphi^*\omega^0 = \widetilde{A}(x, y, \zeta)x dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$, $\widetilde{A}(0) = 1$. Assume that $\widetilde{\Phi}$ and $\widetilde{H}_j$ satisfy $\widetilde{H}_{j+2}^{-1}\widetilde{\tau}_k\widetilde{H}_j = \hat{\tau}_k, \widetilde{H}_j \sim \widetilde{\Phi}, \rho\widetilde{H}_1\rho = \widetilde{H}_2, \rho\widetilde{H}_3\rho = \widetilde{H}_4$, and $\widetilde{H}_j^*\widehat{\omega} = \omega$. Then $\widetilde{\tau}_j = g^{-1}\tau_j g$ with $g\rho = \rho g$ and the first component of $g(x, y, \zeta)$ is $cx(1 + O(|x|))$ with c being real and non-zero. If $c > 0$, put $G_j = H_j^{-1}g\widetilde{H}_j$; if $c < 0$, put $G_j = H_j^{-1}g\widetilde{H}_{j+2}$. Put $\Psi = \Phi^{-1}g\widetilde{\Phi}$. Then $\widetilde{H}_j = g^{-1}H_j G_j$ for $c > 0$ and $\widetilde{H}_{j+2} = g^{-1}H_j G_j$ for $c < 0$. So

$$G_j^{-1}H_{j\,j+1}G_{j+1} = \begin{cases} \widetilde{H}_{j\,j+1}, & c > 0, \\ \widetilde{H}_{j+2\,j+3}, & c < 0; \end{cases} \qquad (8.7)$$

$$G_2 = \rho G_1\rho, \quad G_4 = \rho G_3\rho, \quad G_{k+2} = \hat{\tau}_j G_k\hat{\tau}_j, \quad G_j^*\widehat{\omega} = \widehat{\omega}; \qquad (8.8)$$

$$G_j \sim \Psi, \text{ on } V_j \times \Delta_\delta^{n-1} \text{ or on } V_{j-2} \times \Delta_\delta^{n-1}, \quad \det\Psi'(0) \neq 0. \qquad (8.9)$$

Note that $G_j^*\widehat{\omega} = \widehat{\omega}$ implies that $\Psi^*\widehat{\omega} = \widehat{\omega}$. In particular, Ψ preserves $x = 0$. Conversely, assume that there are G_j such that $\{\tau_1, \tau_2, \rho\}, \{\widetilde{\tau}_1, \widetilde{\tau}_2, \rho\}$ have moduli functions $H_{j,j+1}$ and $\widetilde{H}_{j,j+1}$ satisfying (8.7)-(8.9). If the first case in (8.7) occurs then $H_j G_j\widetilde{H}_j^{-1}, j = 1, \ldots, 4$ agree on the overlap, which extend to a biholomorphic map g defined near the origin. As before $g^{-1}\tau_1 g = \widetilde{\tau}_1$ and $g^{-1}\rho g = \rho$. Hence $g^{-1}\tau_2 g = \widetilde{\tau}_2$. If the second case in (8.7) occurs, define $g = H_j G_j\widetilde{H}_{j+2}^{-1}$. Then $g^{-1}\tau_1 g = \widetilde{\tau}_1$ and $g^{-1}\rho g = \rho$. In both cases, we have $g^*\omega = \widetilde{\omega}$.

Conversely, assume that we are given moduli functions $H_{j\,j+1}$ satisfying (8.4)-(8.6). We already constructed $\tau_1^0, \tau_2^0 = \rho^0\tau_1^0\rho^0$ which realize $H_{j\,j+1}$: Recall that $H_{j\,j+1} = \pi_j^{-1}\pi_{j+1}$ are the transition functions on X. Hence $H_{j\,j+1}^*\widehat{\omega} = \widehat{\omega}$ implies that there is a well-defined holomorphic n-form $\widetilde{\omega}$ on X such that $\pi_j^*\widetilde{\omega} = \widehat{\omega}$. Let $\omega^0 = (\psi K)^{-1*}\widetilde{\omega}$. From $\omega^0 = (\psi K\pi_j)^{-1*}\widehat{\omega}$, one sees that ω^0 extends to a holomorphic n-form vanishing precisely on $x = 0$ to first order. Recall that $x = 0$ is also the set of fixed points of τ_j^0. Since $\hat{\tau}_j^*\widehat{\omega} = \widehat{\omega} = -\overline{\rho^*\widehat{\omega}}$, then $\tau_j^*\omega^0 = \omega^0 = -\overline{\rho^*\omega}$. Therefore, $\{\tau_j^0, \rho^0, \omega^0\}$ is a realization of $H_{j\,j+1}$.

We need to show the moduli space is infinite-dimensional.

Case $n = 2$. It is convenient to choose coordinates such that the mappings commuting with $\hat{\sigma}$ corresponds to maps without any restriction in new coordinates. We will use coordinates (x, t) with

$$t = e^{\frac{i\pi y}{2x}}.$$

Each map H, defined on $V_j \times R$, that commute with $\hat{\sigma}$ gives arise a map (still denoted by H) defined on $V_j \times \hat{R}$, where R is a domain defined by $e^{-\delta/|x|} < |t| < e^{\delta/|x|}$ for some constant $\delta > 0$ (see section 7 on equivalence relations in (x, y, ζ) and (x, t, ζ) spaces). Although τ_j and ρ do not commute with $\hat{\sigma}$, we define $\hat{\tau}_1 = \hat{\tau}_2$, ρ and the n-form $\hat{\omega}$ as follows

$$\hat{\tau}_1(x, t) = (-x, -1/t), \quad \rho(x, t) = (\overline{x}, \overline{t}), \quad \hat{\omega} = dx^3 \wedge d \log t.$$

We will define $H_{1\,2} = \rho K \rho K^{-1}$. Using the local generating function $x^3 \log \hat{t} + \hat{t}x^3 p(x^3)e^{-\frac{1}{x^3}}$ with a holomorphic function $p(x)$ vanishing at 0, we want to define $(\hat{x}, \hat{t}) = K(x, t)$ by the identity

$$\log t \, dx^3 + \hat{x}^3 \, d \log \hat{t} = d\{x^3 \log \hat{t} + \hat{t}x^3 p(x^3)e^{-1/x^3}\}.$$

So K (and hence $H_{1\,2}$) preserves $dx^3 \wedge d \log t$, if K defines a biholomorphic map. Thus we want to find where K and K^{-1} are defined and estimate them for later purpose.

We first rewrite the above identity as

$$\hat{x} = x(1 + \hat{t}p(x^3)e^{-\frac{1}{x^3}})^{1/3}, \quad p(0) = 0, \tag{8.10}$$

$$\hat{t} = te^{-\widetilde{tp}(x^3)e^{-\frac{1}{x^3}}}, \quad \widetilde{p}(x) = xp'(x) + p(x) + \frac{p(x)}{x}. \tag{8.11}$$

We need to check that K and K^{-1} are defined on $\{(x, y): 0 < |x| < r, |\arg x| < \pi/9, e^{-\frac{\delta}{|x|}} < |t| < e^{\frac{\delta}{|x|}}\}$ for some positive constants δ and r, when r, δ are sufficiently small.

Let us start with equation (8.11). By the contraction map theorem, for some small $r > 0$ the equation $T = e^{-wT}$ admits a unique solution $T = T(w)$ which is holomorphic in w for $|w| < r$, by requiring $|T| < 4$. Note that

$$T = T(w) = 1 - w + O(w^2), \tag{8.12}$$

$$|T(w) - 1| = |e^{-wT(w)} - 1| < \frac{4|wT(w)|}{1 - 4|wT(w)|} < 1/2.$$

Hence (8.11) admits a unique solution $\hat{t} = tT(t\widetilde{p}(x^3)e^{-\frac{1}{x^3}})$ with $|\frac{\hat{t}}{t} - 1| < 1/2$. Substituting $tT(t\widetilde{p}(x^3)e^{-\frac{1}{x^3}})$ for $\hat{t}$ in (8.10), we see that K is defined on a desired domain.

From (8.11)-(8.12) we get

$$\hat{t} = t(1 - \widetilde{p}(x^3)e^{-1/x^3}t + O((te^{-1/x^3})^2)), \quad |\arg \hat{x}| < \frac{\pi}{9}$$

for $|t| < e^{\frac{\delta}{|x|}}$, where $O((te^{-1/x^3})^2)$ has absolute value bounded by $c|te^{-1/x^3}|^2$ and its Laurent series (and hence Taylor series) expansion in t has no t^k terms for $k < 2$. Now (8.10) says that

$$\hat{x} = x\bigl(1 + \frac{1}{3}p(x^3)e^{-1/x^3}t + O((te^{-1/x^3})^2)\bigr).$$

To find where K^{-1} is defined, we start with (8.10). Replace $x^3, \hat{x}^3$ by $x, \hat{x}$ respectively first and then set $x(1 + u) = \hat{x}$. We are led to a simpler equation

$$u = \hat{t}p\bigl(\frac{\hat{x}}{1 + u}\bigr)e^{-\frac{1+u}{\hat{x}}}. \tag{8.13}$$

First, a contraction argument shows that for $|\hat{t}| < e^{\frac{\delta}{|\hat{x}|}}$, $|\arg \hat{x}| < \pi/3$ and $0 < |\hat{x}| < r$ with small r, there is a unique solution $u = u(\hat{x}, \hat{t})$ that is holomorphic in $\hat{x}, \hat{t}$ and satisfies $|u| < |e^{-\frac{1}{2\hat{x}}}|$. Substituting $\frac{\hat{x}}{1+u(\hat{x},\hat{t})}$ for x in (8.11), we see that K^{-1} is defined on a desired domain.

To estimate K^{-1}, we use (8.13) and get $|u| \leqslant |\hat{t}||e^{-\frac{1}{2\hat{x}}}|$ and

$$|u(\hat{x}, \hat{t}) - \hat{t}p(\hat{x})e^{-\frac{1}{\hat{x}}}| \leqslant |\hat{t}(p(\frac{\hat{x}}{1 + u}) - p(\hat{x}))e^{-\frac{1}{\hat{x}}}| + |\hat{t}p(\frac{\hat{x}}{1 + u})e^{-\frac{1}{\hat{x}}}(e^{-\frac{u}{\hat{x}}} - 1)|.$$

Thus

$$u(\hat{x}, \hat{t}) = \hat{t}p(\hat{x})e^{-\frac{1}{\hat{x}}} + O(|\hat{t}e^{-\frac{1}{2\hat{x}}}|^2). \tag{8.14}$$

Returning to the original equation (8.10), by $p(0) = 0$ we get $x = \hat{x}(1 + u(\hat{x}^3, x\hat{t}))^{-1/3}$. Hence

$$x = \hat{x}\bigl(1 - \frac{1}{3}\hat{t}p(\hat{x}^3)e^{-1/\hat{x}^3} + O(|\hat{t}e^{-\frac{1}{2\hat{x}^3}}|^2)\bigr). \tag{8.15}$$

Solve (8.11) for t by substituting (8.15) for x. To get an estimate for the expansion of t, note that

$$e^{-(1+u(\hat{x}^3,\hat{t}))^3/\hat{x}^3} = e^{-1/\hat{x}^3}(1 + O(\frac{u(\hat{x}^3, \hat{t})}{\hat{x}^3})) = e^{-1/\hat{x}^3}(1 + O(|\hat{t}e^{-1/\hat{x}^3}|))$$

by (8.14) and $p(0) = 0$. Combining the above, (8.11) and (8.15), we get

$$t = \hat{t}\bigl(1 + \widetilde{p}(\hat{x}^3)\hat{t}e^{-1/\hat{x}^3} + O(|\hat{t}^2e^{-1/\hat{x}^3}|)\bigr).$$

Recall that K preserves $\hat{\omega} = dx^3 \wedge d\log t$. From the above computation we obtain expression

$$H_{1\,2} = \rho K \rho K^{-1} : \begin{cases} \hat{x} = x\big(1 - \frac{1}{3}t(p-\overline{p})(x^3)e^{-1/x^3} + O(t^2 e^{-1/x^3})\big), \\ \hat{t} = t\big(1 + t(\widetilde{p} - \overline{\widetilde{p}})(x^3)e^{-1/x^3} + O(t^2 e^{-1/x^3})\big). \end{cases}$$

Define $H_{41} = H_{23} = \mathrm{id}$, $H_{34} = \hat{\tau}_1 H_{1\,2} \hat{\tau}_1$. Let $H_{j\,j+1}^*$ be of the same form with p replaced by p^*. Assume that $\tau_2^* \tau_1^*$ and $\tau_2 \tau_1$ are holomorphically equivalent. Then H^* and H are equivalent by G_j. By [10], we have $G_1(x,y) = G_4(x,y) = (a(x), a(x)y/x + b(x))$. Put $a(x) = x\alpha(x)$. By assumptions, G_j preserves $\hat{\omega}$ and admits an asymptotic expansion. Hence $x^3 \alpha^3(x) = \int_0^x \frac{d[x^3 \alpha^3(x)]}{dx}\, dx = x^3$, i.e. $\alpha^3 = 1$. Since G_j commutes with ρ, then $a(x) = x$. In (x,t)-space, we get $G_j(x,t) = (x, t\lambda_j(x))$ with $\lambda_1 = \lambda_4$ and $\lambda_2 = \lambda_3$. Recall that G_j have the same asymptotic expansion. Hence $\lambda_j(x)$ are asymptotic to the same formal power series $\Lambda(x)$.

In (x,t)-space, we have $G_1^{-1} H_{1\,2} G_2 = H_{1\,2}^*$. The x-component of $G_1^{-1} H_{1\,2} G_2$ is

$$x\big(1 - \frac{1}{3}t(p-\overline{p})(x^3)\lambda_2(x)e^{-1/x^3} + O(t^2 e^{-1/x^3})\big).$$

On $(V_1 \cap V_2) \times \{t \in \mathbb{C} : 1 - \delta < |t| < 1 + \delta\}$ the coefficient of t^1 of the Laurent series expansion of the x-component of $G_1^{-1} H_{1\,2} G_2 = H_{1\,2}^*$ gives us

$$(p - \overline{p})(x^3)\lambda_2(x) = (p^* - \overline{p}^*)(x^3). \tag{8.16}$$

In particular, λ_2 is meromorphic near the origin (assuming $p - \overline{p} \not\equiv 0$). Since λ_2 admits the asymptotic expansion Λ, then $\lambda_2 = \Lambda$ is holomorphic near the origin and we must have $(p - \overline{p})(x^3)\Lambda(x) = (p^* - \overline{p}^*)(x^3)$ as formal power series in x. Since $\hat{\tau}_1 G_4 \hat{\tau}_1 = G_2 = \rho G_1 \rho$, we have $\overline{\Lambda(\overline{x})} = \Lambda(x) = \Lambda(-x)^{-1}$. Hence $\Lambda(iy)\overline{\Lambda(iy)} = 1$ as formal power series in the real variable y. So

$$(p - \overline{p})(-iy^3)(p - \overline{p})(iy^3) = (p^* - \overline{p}^*)(-iy^3)(p^* - \overline{p}^*)(iy^3)$$

as holomorphic functions in $y \in \mathbb{C}$.

We now consider the family of holomorphic functions $p \not\equiv 0$ satisfying

$$p(-\zeta) = p(\zeta) = -\overline{p}(\zeta), \quad p(0) = 0.$$

If p and p^* are in the above family and if the corresponding moduli functions are equivalent, then $p^* = \pm p$. In particular, the above result shows that the moduli space is of infinite dimension. This finishes the proof of Theorem 2.1 for $n = 2$.

For the proof of Theorem 2.2, we need to show that for the above $H_{j,\,j+1}$ with

$$p(x) - \overline{p}(x) \not\equiv 0, \quad p(0) = 0$$

and for distinct positive numbers r, the moduli functions $\{D_r^{-1}H_{j\,j+1}D_r\}$ are not equivalent under G_j preserving $\hat{\omega}$, where

$$D_r(x,t,\zeta) = (rx, t, r\zeta) \quad (D_r(x,t) = (rx,t)).$$

This is quite easy to see. First we know that $G_j(x,t) = (xa_j(x), t\lambda_j(x))$ with $G_1 = G_4, G_2 = G_3$. Since G_j preserve $dx^3 \wedge d\log t$, then $a_j = 1$. The x component of $D_r^{-1}H_{1\,2}D_r$ is given by

$$x\big(1 - \frac{1}{3}t(p - \overline{p})(r^3 x^3)e^{-\frac{1}{r^3 x^3}}\big) + O(t^2 e^{-\frac{1}{r^3 x^3}}).$$

The x component of $D_r^{-1}H_{3\,4}D_r$ is given by

$$x\big(1 + \frac{1}{3}t^{-1}(p - \overline{p})(r^3 x^3)e^{-\frac{1}{r^3 x^3}}\big) + O(e^{\frac{1}{r^3 x^3}}).$$

Since $G_j(x,t) = (x, t\lambda_j(x))$ and $\lambda_j \sim \Lambda$, the asymptotic expansion alone shows that $\{D_{r_1}^{-1}H_{j\,j+1}D_{r_1}\}$ and $\{D_{r_2}^{-1}H_{j\,j+1}D_{r_2}\}$ are not equivalent, as long as $p(x) - \overline{p}(x) \not\equiv 0$ and r_1, r_2 are distinct positive numbers.

For later purpose we remark that the only G_j, in the (x,t)-space, that preserves $dx^3 \wedge d\log t$ and $H_{j\,j+1}$ is the identity, by (8.16). In the (x,y)-space, the $\{G_j\}$ that preserve $H_{j\,j+1}$ must be $G_j = \hat{\sigma}^k$, where k is the same for all j. Since $\hat{\tau}_j$ reverses $\hat{\sigma}$ then $G_{j+2} = \hat{\tau}_j G_j \hat{\tau}_j$ imply that $k = 0$, i.e. $G_j = \mathrm{id}$ in the (x,y)-space too.

Case $n > 2$. Put $'\zeta = (\zeta_3, \ldots, \zeta_{n-1})$ for $n > 3$ and $\zeta = (\zeta_2, \ldots, \zeta_{n-1})$. Recall that

$$\hat{\tau}_1(x,t,\zeta) = (-x, -1/t, \zeta), \quad \rho(x,t,\zeta) = (\overline{x}, \overline{t}, \overline{\zeta}),$$
$$\hat{\omega} = dx^3 \wedge d\log t \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}.$$

Let K be a map of the form

$$\begin{cases} \hat{x} = x, \\ \hat{t} = t + p(x)e^{-1/x} \stackrel{\text{def}}{=} tq_1(x,t), \\ \hat{\zeta}_2 = \zeta_2 q_1(x,t), \quad \hat{'\zeta} = '\zeta \end{cases}$$

with

$$p(x) = \sum_{k=1}^{\infty} \frac{2ic_k}{(k-1)!(k^2 x^2 + 1)}, \quad c_{2k} = 1, \ 1 < c_{2k+1} < 2. \qquad (8.17)$$

Note that $p(x)$ is meromorphic on $\mathbb{C}^*$ with simple poles at $\frac{1}{ki}$, and that its residues at $\frac{1}{ki}$ decay rapidly as $|k| \to \infty$. One can verify that K preserves $dx^3 \wedge d \log t \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$, and that the s-th iterate of K is given by

$$K^s \colon \hat{x} = x, \ \hat{t} = t + sp(x)e^{-1/x} \overset{\text{def}}{=} tq_s(x,t), \ \hat{\zeta}_2 = \zeta_2 q_s(x,t), \ '\hat{\zeta} = {}'\zeta.$$

Put $H_{1\,2} = \rho K \rho K^{-1}$. Since $\overline{p(\overline{x})} = -p(x)$, we get $\rho K \rho = K^{-1}$ and $H_{1\,2} = K^{-2}$. Define $H_{4\,1} = H_{2\,3} = \text{id}$ and $H_{3\,4} = \hat{\tau}_1 H_{1\,2} \hat{\tau}_1$. Let $\{H^*_{j\,j+1}\}$ be another set of moduli functions, defined as above with p being replaced by p^*. We still assume that p^* has the same form (8.17) with c_k being replaced by $c_k^* \in (1,2)$. Denote the corresponding q_s by q_s^*.

We will prove that if p and p^* are distinct, then $\{H_{j\,j+1}\}$ and $\{H^*_{j\,j+1}\}$ are not equivalent by G_j satisfying $G_2 = \rho G_1 \rho$ and $G_3 = \hat{\tau}_1 G_1 \hat{\tau}_1 = \rho G_4 \rho$, without assuming $G_j^* \hat{\omega} = \hat{\omega}$. In other words, the corresponding M, M^*, satisfying $\text{Re } dz_1 \wedge \cdots \wedge dz_n|_M = \text{Re } dz_1 \wedge \cdots \wedge dz_n|_{M^*} = 0$, are not even holomorphically equivalent.

Assume that $G_1 H^*_{1\,2} = H_{1\,2} G_2$. We know that $G_1 = G_4$ and $G_2 = G_3$ with $G_j \colon x' = xa_j(x,\zeta), t' = t\lambda_j(x,\zeta), \zeta' = c_j(x,\zeta)$. The x, t components of $G_1 H^*_{1\,2}(x,t,\zeta)$ are

$$x' = xa_1(x, \zeta_2 q^*_{-2}(x,t), {}'\zeta), \quad t' = tq^*_{-2}(x,t)\lambda_1(x, \zeta_2 q^*_{-2}(x,t), {}'\zeta).$$

The x, t components of $H_{1\,2} G_2(x,t,\zeta)$ are

$$x' = xa_2(x,\zeta), \quad t' = t\lambda_2(x,\zeta)q_{-2}(xa_2(x,\zeta), t\lambda_2(x,\zeta)).$$

Set $\zeta = 0$ and equate the two x and t components respectively. We get

$$a_1(x,0) = a_2(x,0),$$

$$q^*_{-2}(x,t)\lambda_1(x,0) = \lambda_2(x,0)q_{-2}(xa_2(x,0), t\lambda_2(x,0)). \qquad (8.18)$$

Both sides of the second identity are polynomials in t^{-1} and their coefficients say that $\lambda_2(x,0) = \lambda_1(x,0)$ on $V_1 \cap V_2$. Since $G_j \hat{\tau}_1 = \hat{\tau}_1 G_{j+2}$ then $\lambda_j(-x,\zeta) = \lambda_{j+2}(x,\zeta)^{-1}$ and $a_j(-x,\zeta) = a_{j+2}(x,\zeta)$. Hence $\lambda_2(x,0) = \lambda_1(x,0)$ and $a_1(x,0) = a_2(x,0)$ on $V_3 \cap V_4$. Thus we can define $a(x) = a_j(x,0)$ and $\lambda(x) = \lambda_j(x,0)$, which are holomorphic near $x = 0$, with $\lambda(0) \neq 0 \neq a(0)$. Now (8.18) becomes

$$p^*(x) = \lambda(x)^{-1}p(xa(x))e^{\frac{1}{x} - \frac{1}{xa(x)}}. \qquad (8.19)$$

The last identity holds on a small sector in the x plane, and hence in a punctured neighborhood of the origin. For $0 < x < 1$ and for some positive constant c_0, we have $c_0 < |p(x)| < 1/c_0$. So (8.19) implies that

$a(0) = 1$. All meromorphic functions p given by (8.17) have a simple pole at $\pm\frac{i}{k}, k = 1, 2, \ldots$, with residue $\pm\frac{c_k}{k!}$. Now (8.19) says that if an integer k is sufficiently large, then $\frac{i}{k}a(\frac{i}{k}) = \frac{i}{k'}$ for some integer $k' > 0$. Since $x \to xa(x)$ is biholomorphic near the origin and $a(0) > 0$, then $k/c_2 < k' < c_2 k$ for some constant c_2. Therefore, the residue $b_{k'}$ of the right-hand side of (8.19) at i/k' satisfying $1/(c_3 k'!) < |b_{k'}| < c_3/k'!$. The residue b'_k of the left-hand side of (8.19) at i/k satisfies $1/k! \leqslant |b'_k| \leqslant 2/k!$. We conclude that $k' = k$. Therefore $a(i/k) = 1$, $\lambda(\frac{i}{2k}) = 1$ and $p^* = p$.

Assume now that $G_1 H^*_{1\,2} = H_{3\,4} G_2$. We have

$$H_{3\,4}(x, t, \zeta) = \hat{\tau}_1 H_{1\,2}(-x, -t^{-1}, \zeta)$$
$$= \hat{\tau}_1(-x, -t^{-1}q_{-2}(-x, -t^{-1}), \zeta_2 q_{-2}(-x, -t^{-1}), {}'\zeta)$$
$$= (x, t q_{-2}(-x, -t^{-1})^{-1}, \zeta_2 q_{-2}(-x, -t^{-1}), {}'\zeta).$$

The t-component of $H_{3\,4} G_2(x, t, \zeta)$ is

$$t' = t\lambda_2(x, \zeta) q_{-2}(-xa_2(x, \zeta), -(t\lambda_2(x, \zeta))^{-1})^{-1}.$$

Again, the t-components of both sides of $G_1 H^*_{1\,2}(x, t, 0) = H_{3\,4} G_2(x, t, 0)$ say that

$$q^*_{-2}(x, t)\lambda_1(x, 0) = \lambda_2(x, 0) q_{-2}(-xa_2(x, 0), -(t\lambda_2(x, 0))^{-1})^{-1},$$

which never holds since the left-hand side is a polynomial in t^{-1} of degree 1 while the right-hand side is not.

Case $n > 2$ - another family. Let $\zeta = (\zeta_2, \ldots, \zeta_{n-1})$ and ${}'\zeta = (\zeta_3, \ldots, \zeta_{n-1})$. The above family cannot be used for the proof of Theorem 2.2 for $n > 2$, since for $\{D_r^{-1} H_{j\,j+1} D_r\}$, where $D_r(x, t, \zeta) = (rx, t, r\zeta)$, are obviously equivalent for all $r > 0$. We need another construction. Put $v(\zeta) = \zeta_2^2 + \zeta_2$ for $n = 3$, and put

$$v(\zeta) = \zeta_2^n + \zeta_2^{n-1} + \zeta_2 + \sum_{j=3}^{n-1} \zeta_j \zeta_2^{j-1}$$

for $n > 3$. Using the equation

$$\log t d\zeta_2 + \hat{\zeta}_2 d\log\hat{t} = d_{\log\hat{t}, \zeta_2}\{\zeta_2\log\hat{t} + \hat{t}v(\zeta)p(x)e^{-1/x}\},$$

we define a map $(\hat{x}, \hat{t}, \hat{\zeta}) = K(x, t, \zeta)$, preserving $dx^3 \wedge d\log t \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$, by

$$\begin{cases} \hat{x} = x, \quad {}'\hat{\zeta} = {}'\zeta, \\ \hat{t} = te^{-\hat{t}v_{\zeta_2}(\zeta)p(x)e^{-1/x}}, \\ \hat{\zeta}_2 = \zeta_2 + \hat{t}v(\zeta)p(x)e^{-1/x}, \end{cases}$$

where p still has the form (8.17). As above K, K^{-1} are defined on the desired domains. We write components of K and K^{-1} as Laurent series expansion in t and get

$$K: \begin{cases} \hat{t} = t - t^2 v_{\zeta_2}(\zeta) p(x) e^{-1/x} + O(t^3), \\ \hat{\zeta}_2 = \zeta_2 + t v(\zeta) p(x) e^{-1/x} + O(t^2), \\ \hat{x} = x, \quad \hat{'\zeta} = {'\zeta}, \end{cases}$$

where by $O(t^2)$ we mean a Laurent series expansion in t $(e^{-\frac{\delta}{|x|}} < |t| < e^{\frac{\delta}{|x|}})$ containing no terms t^k for $k = 1, 0, -1, \ldots$. Also

$$K^{-1}: \begin{cases} t' = t + t^2 v_{\zeta_2}(\zeta) p(x) e^{-1/x} + O(t^3), \\ \zeta_2' = \zeta_2 - t v(\zeta) p(x) e^{-1/x} + O(t^2), \\ x' = x, \quad {'\zeta}' = {'\zeta}. \end{cases}$$

Define $H_{12} = \rho K \rho K^{-1}$. Recall that $p = -\overline{p}$. We get

$$H_{12}: \begin{cases} t' = t + 2t^2 v_{\zeta_2}(\zeta) p(x) e^{-1/x} + O(t^3), \\ \zeta_2' = \zeta_2 - 2t v(\zeta) p(x) e^{-1/x} + O(t^2), \\ x' = x, \quad {'\zeta}' = {'\zeta}. \end{cases}$$

Put $H_{41} = H_{23} = \mathrm{id}$, and $H_{34} = \hat{\tau}_1 H_{12} \hat{\tau}_1$.

For $r > 0$ put $H^*_{j\,j+1} = D_r^{-1} H_{j\,j+1} D_r$ with $D_r(x, t, \zeta) = (rx, t, r\zeta)$. We get

$$H^*_{12}: \begin{cases} t' = t + 2t^2 v_{\zeta_2}(r\zeta) p(rx) e^{-\frac{1}{rx}} + O(t^3), \\ \zeta_2' = \zeta_2 - 2r^{-1} t v(r\zeta) p(rx) e^{-\frac{1}{rx}} + O(t^2), \\ x' = x, \quad {'\zeta}' = {'\zeta}. \end{cases}$$

Assume that H and H^* are equivalent by some G_j satisfying $G_j^* \hat{\omega} = \hat{\omega}$ and other conditions. So we have $G_1 = G_4$, $G_3 = \hat{\tau}_1 G_1 \hat{\tau}_1 = \rho G_4 \rho$ and $G_j: x' = x a_j(x, \zeta), t' = t \lambda_j(x, \zeta), \zeta' = c_j(x, \zeta)$. Also $d(x a_j(x, \zeta))^3 \wedge d c_{j2}(x, \zeta) \wedge \cdots \wedge d c_{j\,n-1}(x, \zeta) = dx^3 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$.

Assume that $G_1 H_{12} = H^*_{12} G_2$. The x-component of $G_1 H_{12} = H^*_{12} G_2$ says that $a_1 \circ H_{12}(x, t, \zeta) = a_2(x, \zeta)$ on $(V_1 \cap V_2) \times \{t: e^{-\frac{\delta}{|x|}} < |t| < e^{-\frac{\delta}{|x|}}\} \times \Delta_\delta^{n-2}$. In particular $a_1(x, \zeta)$ is independent of ζ_2 since $p(x) v(\zeta) \not\equiv 0$, and $a_1(x, \zeta) = a_2(x, \zeta)$ on $(V_1 \cap V_2) \times \Delta_\delta^{n-2}$ and hence on $(V_3 \cap V_4) \times \Delta_\delta^{n-2}$ because $\hat{\tau}_1 G_j = G_{j+2} \hat{\tau}_1$. Now $a(x, {'\zeta}) \overset{\mathrm{def}}{=\!=} a_j(x, \zeta)$ is holomorphic near $0 \in \mathbb{C}^{n-1}$. The t-component of $G_1 H_{12} = H^*_{12} G_2$ says that

$$(1 + 2tv_{\zeta_2}(\zeta)p(x)e^{-1/x})\lambda_1(x, \zeta_2 - 2tv(\zeta)p(x)e^{-1/x}, {}'\zeta)$$
$$= \lambda_2(x, \zeta)\{1 + 2t\lambda_2(x, \zeta)v_{\zeta_2}(rc_2(x, \zeta))p(rxa(x, {}'\zeta))e^{-\frac{1}{rxa(x, {}'\zeta)}}\} + O(t^2). \tag{8.20}$$

Comparing the constant terms in the expansion in t, we get $\lambda_1(x, \zeta) = \lambda_2(x, \zeta)$ on $(V_1 \cap V_2) \times \Delta_\delta^{n-2}$ and hence on $(V_3 \cap V_4) \times \Delta_\delta^{n-2}$ by $\hat\tau_1 G_j = G_{j+2}\hat\tau_1$. Write $\lambda_j = \lambda$. The coefficients of t in (8.20) say that

$$(v_{\zeta_2}(\zeta)\lambda(x, \zeta) - \lambda_{\zeta_2}(x, \zeta)v(\zeta))p(x)$$
$$= \lambda(x, \zeta)^2 v_{\zeta_2}(rc_2(x, \zeta))p(rxa(x, {}'\zeta))e^{\frac{1}{x} - \frac{1}{rxa(x, {}'\zeta)}}. \tag{8.21}$$

Recall that by the definition of G_j, $\lambda(x, \zeta)$ admits an asymptotic expansion vanishes nowhere. As $x > 0$ tends to zero, the left hand side of (8.21), by setting $\zeta = 0$, is $\lambda(0, 0) \neq 0$ since $v_{\zeta_2}(0) \neq 0 = v(0)$, while the right-hand side, after removing $e^{\frac{1}{x} - \frac{1}{rxa(x, {}'\zeta)}}$, admits an asymptotic expansion which is not identically zero too. Note that $a(x, 0)$ is real-valued when x is real. Hence $ra(0) = 1$. Note that the identity (8.21) holds for $\operatorname{Im} x < 0$ and small $|x|$. The location of the poles of $p(x)$ indicates that $a(x, {}'\zeta)$ is independent of ${}'\zeta$. Set ${}'\zeta = 0$ and let $x \to 0^+$. When j is large and ζ is small, we have $|a(0)|/2 < |a(x_j, {}'\zeta)| < 2|a(0)|$. As before the magnitudes of the residues on both sides of (8.21) indicate that

$$ra(x, {}'\zeta) = 1.$$

The ζ-component of $G_1 H_{1\,2} = H_{1\,2}^* G_2$ says that on $(V_1 \cap V_2) \times \Delta_\delta^{n-2}$

$$c_{1\,2} \circ H_{1\,2}(x, t, \zeta) = c_{2\,2}(x, \zeta) - 2r^{-1}t\lambda(x, \zeta)v(rc_2(x, \zeta))p(x)e^{-\frac{1}{x}} + O(t^2),$$
$$c_{1\,j} \circ H_{1\,2}(x, t, \zeta) = c_{2\,j}(x, \zeta), \quad 2 < j < n. \tag{8.22}$$

The last identity implies that $c_{1\,j}(x, \zeta)$ are independent of ζ_2 and $c_{1\,j} = c_{2\,j}$ for $j > 2$. In the first identity above, the constant terms of the Laurent series expansion in t say that $c_{1\,2} = c_{2\,2}$. We obtain $c_1 = c_2$ on $(V_1 \cap V_2) \times \Delta_\delta^{n-2}$ and hence on $(V_3 \cap V_4) \times \Delta_\delta^{n-2}$ by $\hat\tau_1 G_j = G_{j+2}\hat\tau_1$, and we can write $c_j = c$. Put $c = (c_2, \ldots, c_{n-1})$ by abuse of notation. From $d(r^{-1}x)^3 \wedge dc_2(x, \zeta) \wedge \cdots \wedge dc_{n-1}(x, \zeta) = dx^3 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$, we see that for ${}'\zeta = (\zeta_3, \ldots, \zeta_{n-1})$

$$c_2(x, \zeta) = \alpha(x, {}'\zeta)z_2 + \beta(x, {}'\zeta), \quad c_j(\zeta) = c_j({}'\zeta) \; (j = 3, \ldots, n-1), \tag{8.23}$$

$$r^3\alpha^{-1} = \det \frac{\partial(c_3, \ldots, c_{n-1})}{\partial(\zeta_3, \ldots, \zeta_{n-1})}, \tag{8.24}$$

in which the right-hand side of (8.24) is 1 when $n = 3$. The coefficients of t in (8.22) say that

$$c_{2\zeta_2}(x, \zeta)v(\zeta) = r^{-1}\lambda(x, \zeta)v(rc(x, \zeta)). \tag{8.25}$$

Looking at the vanishing order of both sides, we see that $c_{2\zeta_2}(0) \neq 0$.

So far we have not used the particular form of $v(\zeta)$, except that $v_{\zeta_2} \neq 0 = v(0)$.

Assume $n > 3$ first. Now we use the definition of $v(\zeta)$. Comparing the Weierstrass polynomials in ζ_2 of both sides of (8.25) and recalling that $c_3, \ldots, c_{n-1}$ are independent of ζ_2, we obtain

$$(r\alpha\zeta_2 + r\beta)^n + (r\alpha\zeta_2 + r\beta)^{n-1} + r\alpha\zeta_2 + r\beta + \sum_{j=3}^{n-1} rc_j(x,\zeta)(r\alpha\zeta_2 + r\beta)^{j-1}$$
$$= (r\alpha)^n(\zeta_2^n + \zeta_2^{n-1} + \zeta_3\zeta_2 + \cdots + \zeta_{n-1}\zeta_2^{n-2} + \zeta_2).$$
$$(8.26)$$

Since $\beta(0) = 0$ and $c_j(0) = 0$, comparing the coefficients of $\zeta_2^0, \zeta_2^1, \ldots$ yields $\beta = 0$ and

$$r\alpha(x, {'\zeta}) = 1, \quad rc_j(x,\zeta) = \zeta_j, \ j > 2.$$

Now (8.24) implies that $r = 1 = \alpha$ since $r > 0$. And (8.25) implies $\lambda = 1$. When $n = 3$, it is straightforward that $r = \alpha = \lambda = 1$ and $\beta = 0$.

Assume that $G_1 H_{12} = H^*_{34} G_2$ for

$$H^*_{34} = \hat{\tau}_1 H^*_{12} \hat{\tau}_1 : \begin{cases} t' = t + 2v_{\zeta_2}(r\zeta)p(-rx)e^{\frac{1}{rx}} + O(t^{-1}), \\ \zeta'_2 = \zeta_2 + 2r^{-1}t^{-1}v(r\zeta)p(-rx)e^{\frac{1}{rx}} + O(t^{-2}), \\ x' = x, \quad {'\zeta}' = {'\zeta}, \end{cases}$$

where by $O(t^{-k})$ with $k > 0$ we mean a Laurent series expansion in t (for $e^{-\frac{\delta}{|x|}} < |t| < e^{\frac{\delta}{|x|}}$) containing no terms t^l for $l = -k+1, -k+2, \ldots$. Since G_j is given by $x' = xa_j(x,\zeta), t' = t\lambda_j(x,\zeta), \zeta' = c_j(x,\zeta))$, it is clear that the ζ_2-component of $H^*_{34} G_2(x,t,\zeta)$ has a non-zero coefficient for t^{-1}. But the ζ_2-component of $G_1 H_{12}(x,t,\zeta)$ has zero coefficient for t^{-1}. Therefore $G_1 H_{12} = H^*_{34} G_2$ never occurs.

We just proved that $\{D_r^{-1} H_{j\,j+1} D_r\}$ is not equivalent to $\{H_{j\,j+1}\}$ by G_j preserving $\hat{\omega}$, if $r > 0$ and $r \neq 1$.

We have finished the proof of Theorem 2.1.

One can also conclude that different positive r values correspond to different equivalent classes too. For if $D_{r_1}^{-1} H_{j\,j+1} D_{r_1}$ and $D_{r_2}^{-1} H_{j\,j+1} D_{r_2}$ are equivalent by G_j. Then $H_{j\,j+1}$ and $D_{r_2 r_1^{-1}}^{-1} H_{j\,j+1} D_{r_2 r_1^{-1}}$ are equivalent by $D_{r_1} G_j D_{r_1}^{-1}$.

Remark 8.1. — The above proof also shows that in the (x, t, ζ)-space $\{H_{j\,j+1}\}$ is equivalent to itself by $G_j = $ id only. In the (x, y, ζ)-space,

this means that $G_j = \hat{\sigma}^k$. Now the reality condition on $\{G_j\}$ implies that $G_j = \mathrm{id}$.

9. Proof of Theorem 2.2

Put $'x = (x_2, \ldots, x_{n-1})$, $\zeta = (\zeta_2, \ldots, \zeta_{n-1})$, $\omega_0 = dz_1 \wedge \cdots \wedge dz_n$ and $\hat{\omega} = xdx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$. Consider a real analytic manifold

$$M: \begin{cases} z_n = (z_1 + \overline{z}_1)^2 + E(z_1, \overline{z}_1, 'x), & E(z_1, \overline{z}_1, 'x) = O(3), \\ y_\alpha = F_\alpha(z_1, \overline{z}_1, 'x), & F_\alpha(z_1, \overline{z}_1, 'x) = O(2), \ 1 < \alpha < n \end{cases}$$

with $\mathrm{Re}\, \omega_0|_M = 0$.

Consider the linear map

$$f = L_r^{-1}: z \to (r^{-1}z_1, \cdots, r^{-1}z_{n-1}, r^{-2}z_n), \quad r > 0.$$

Then $\widetilde{M} = f(M)$ is given by

$$\widetilde{M}: \begin{cases} z_n = (z_1 + \overline{z}_1)^2 + \widetilde{E}(z_1, \overline{z}_1), & E(z_1, \overline{z}_1, 'x) = O(3), \\ y_\alpha = \widetilde{F}_\alpha(z_1, \overline{z}_1, 'x), & \widetilde{F}_\alpha(z_1, \overline{z}_1, 'x) = O(2), \ 1 < \alpha < n \end{cases}$$

with $\mathrm{Re}\,\omega_0|_{\widetilde{M}} = 0$. The complexification of f is

$$F: (z, w) \to (f(z), \overline{f}(w)).$$

Let $M^c \subset \mathbb{C}^n \times \mathbb{C}^n$ be the complexification of M. Let τ_1^0, τ_2^0 be the branched-covering transformations of projections M^c to z and w spaces, respectively. Define $\tilde{\tau}_1^0, \tilde{\tau}_2^0$ analogously for $\widetilde{M}^c$.

Then $F\tau_j^0 = \tilde{\tau}_j^0 F$ and $F\rho^0 = \rho^0 F$.

Now we apply Proposition 7.1. The first part of requirements on moduli functions is the existence of a biholomorphism $\varphi: \mathbb{C}^n \to M^c$, $\varphi(0) = (0,0)$ satisfying

$$\varphi^{-1}\tau_j^0\varphi = \tau_j: (x, y, \zeta) \to (-x, y + (-1)^{j-1}2x, \zeta) + x \cdot O(1),$$

$$\omega \overset{\text{def}}{=\!=} \varphi^*\omega_0|_{M^c} = A(x, y, \zeta)\hat{\omega}, \quad A(0) = 1.$$

The second part of requirements on moduli function $\{H_{j\,j+1}\}$ of M is the followings

$$H_{j+2}^{-1}\tau_k H_j = \hat{\tau}_k: (x, y, \zeta) \to (-x, y + (-1)^{j-1}2x, \zeta),$$

$$\rho H_1 \rho = H_2, \quad \rho H_3 \rho = H_4, \quad \rho: (x, y, \zeta) \to (\overline{x}, -\overline{y}, \zeta),$$

$$H_j^* \omega = \hat{\omega} = x\, dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1},$$

$$H_j \sim \Phi: (x, y, \zeta) \to (xa(x, y, \zeta), y + b(x, y, \zeta), 'z + c(x, y, \zeta)) \text{ on } V_j \times \Delta_r^{n-1}$$

$$H_{j\,j+1} = H_j^{-1} H_{j+1}, \quad (V_j \cap V_{j+1}) \times \Delta_r^{n-1},$$

where $V_j = i^{1-j}\{x: -\epsilon < \arg x < \pi/2 + \epsilon, 0 < |x| < r\}$, $\Delta_r = \{y \in \mathbb{C}: |y| < r\}$, and $a(x, y, \zeta)$ with $a(0) = 1$ and $(b(x, y, \zeta), c(x, y, \zeta)) = O(2)$, are formal power series whose coefficients are holomorphic in y, ζ in a neighborhood of the origin.

Next, we assume that a set of moduli function $H_{j\,j+1}$ of M has been given. Then we want to find moduli functions for $\widetilde{M}$, by using those of M.

For a positive number r put

$$D_r: (x, y, \zeta) \to (rx, ry, r\zeta),$$

$$\widetilde{\varphi} = F\varphi D_r: \mathbb{C}^n \to M_r^c, \quad \widetilde{\varphi}(0) = (0, 0).$$

Then we have

$$
\begin{aligned}
\widetilde{\varphi}^{-1} \widetilde{\tau}_k^0 \widetilde{\varphi} &= (F\varphi D_r)^{-1} F \tau_k^0 F^{-1} (F\varphi D_r) = D_r^{-1} \tau_k D_r \\
&\stackrel{\text{def}}{=} \widetilde{\tau}_k: (x, y, \zeta) \to (-x, y + (-1)^{j-1} 2x, \zeta) + x \cdot O(1), \\
\widetilde{\varphi}^{-1} \rho^0 \widetilde{\varphi} &= (F\varphi D_r)^{-1} F \rho^0 F^{-1} (F\varphi D_r) = \rho, \quad r \in \mathbb{R}.
\end{aligned}
$$

Also

$$
\begin{aligned}
\widetilde{\omega} &\stackrel{\text{def}}{=} \widetilde{\varphi}^* dz_1 \wedge \cdots \wedge dz_n |_{\widetilde{M}^c} = (F\varphi D_r)^* dz_1 \wedge \cdots \wedge dz_n |_{\widetilde{M}^c} \\
&= \frac{1}{r^{n+1}} D_r^* \varphi^* dz_1 \wedge \cdots \wedge dz_n |_{M^c} \\
&= \frac{1}{r^{n+1}} D_r^* \omega = A(rx, ry, r\zeta) x\, dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1} \\
&\stackrel{\text{def}}{=} \widetilde{A}(x, y, \zeta) x\, dx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}, \quad \widetilde{A}(0) = 1.
\end{aligned}
\tag{9.1}
$$

Remark. — The equation (9.1), and hence $\widetilde{A}(0) = 1$, is the only place where we used the non-isotropic dilation $f = L_r^{-1}$. All other computations remain true for any biholomorphism f.

We still need to find a set of moduli functions for $\widetilde{M}$. Put $\widetilde{H}_j = D_r^{-1} H_j D_r$. We have

$$\widetilde{H}_{j\,j+1} \stackrel{\text{def}}{=} \widetilde{H}_{j+1} \widetilde{H}_j = D_r^{-1} H_{j\,j+1} D_r,$$

$$\widetilde{H}_{j+2}\widetilde{\tau}_k\widetilde{H}_j = D_r^{-1}H_{j+2}^{-1}\tau_k H_j D_r = \hat{\tau}_k,$$

$$\rho\widetilde{H}_1\rho = \rho D_r^{-1}H_1 D_r\rho = \widetilde{H}_2, \ \ \rho\widetilde{H}_3\rho = \widetilde{H}_4,$$

$$\widetilde{H}_j^*\widetilde{\omega} = (D_r^{-1}H_j D_r)^*(\frac{1}{r^{n+1}}D_r^*\omega) = \hat{\omega},$$

$$\widetilde{H}_j \sim D_r^{-1}\Phi D_r \colon x \to xa(rx, ry, r\zeta), \quad y \to y + \frac{1}{r}b(rx, ry, r\zeta),$$

$$\zeta \to \frac{1}{r}c(rx, ry, r\zeta).$$

Therefore $D_r^{-1}H_{j\,j+1}D_r$ form a set of moduli functions of $\widetilde{M}$. The theorem is proved by choosing $H_{j\,j+1}$ such that $D_r^{-1}H_{j\,j+1}D_r$ and $H_{j\,j+1}$ are not equivalent if $r > 0$ and $r \neq 1$. The existence of such an $\{H_{j\,j+1}\}$ has been constructed in previous two sections.

This finishes the proof of Theorem 2.2.

Given a germ of real manifold M in $\mathbb{C}^n$ at the origin, denote by $\mathrm{Aut}_{vol}(M)$ the germs of holomorphic maps φ on $\mathbb{C}^n$ such that $\varphi(0) = 0$, $\varphi(M) = M$, and $\varphi^*dz_1 \wedge \cdots \wedge dz_n = dz_1 \wedge \cdots \wedge dz_n$. The proof of Theorem 2.1 can be modified to yield some real analytic manifolds M with a parabolic complex tangent of which $\mathrm{Aut}_{vol}(M)$ is finite.

PROPOSITION 9.1. — *Let $n \geqslant 2$, and let $k = 1$ for $n = 2$ and $k = 2^j$ for some integer $0 \leqslant j \leqslant n-2$. There exists a real analytic manifold M which is equivalent to $z_n = (z_1 + \overline{z}_1)^2, \mathrm{Im}z_2 = \cdots = \mathrm{Im}z_{n-1} = 0$ under some formal unimodular holomorphic map such that $\mathrm{Aut}_{vol}(M)$ has exactly k elements.*

Proof. — Let us recall the correspondence between M and its moduli functions. For a real analytic manifold M which has parabolic complex tangents along a hypersurface in M. We have a totally real and real analytic embedding $\Delta\colon M \to M^c \subset \mathbb{C}^{2n}$. Two branched coverings from M^c to $\mathbb{C}^n$ yields two involutions τ_j. There are holomorphic maps H_j defined on sectorial domains such that $H_{j+2}^{-1}\tau_k H_j = \hat{\tau}_k$ and $H_j^*dz_1 \wedge \cdots \wedge dz_n|_{M^c} = xdx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$. We have $H_{j\,j+1} = H_j^{-1}H_{j+1}$. Thus if a unimodular holomorphic map φ preserves M then $G_j = H_j^{-1}\varphi H_j$ or $G_j = H_{j+2}^{-1}\varphi H_j$ preserves the moduli functions $H_{j\,j+1}$ and $xdx \wedge dy \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_{n-1}$. Clearly, distinct maps φ correspond to distinct sets of $\{G_j\}$.

Now the proof of Theorem 2.2 yields examples of M with $|\mathrm{Aut}_{vol}(M)| = 1$.

Let $n > 2$ and $k = 2^j$ for some integer $1 \leqslant j \leqslant n - 1$. Let p be defined by (8.17). For $j = 1$ put $v(\zeta) = \zeta_2^{2n-1} + \zeta_2^{2n-3} + \zeta_2 + \sum_{k=3}^{n-1} \zeta_k \zeta_2^{2k-3}$. For $1 < j \leqslant n - 2$ put

$$v(\zeta) = \zeta_2^{2n-1} + \zeta_2^{2n-3} + \zeta_2 + \sum_{k=3}^{j+1} \zeta_k^2 \zeta_2^{2k-3} + \sum_{k=j+2}^{n-1} \zeta_k \zeta_2^{2k-3}.$$

If $G_j H_{j\,j+1} G_{j+1}^{-1} = H_{j\,j+1}$, then (8.26), which is obtained from (8.25), is just $v(c(x,\zeta)) = \alpha^{2n-1}(x,\zeta)v(\zeta)$. By (8.23) we obtain $\beta = 0$ and

$$\alpha^2 = 1, \quad c_3^2 = \cdots = c_{j+1}^2 = 1 = c_{j+2} = \cdots = c_{n-1}.$$

One can also show that $G_j H_{j,j+1} G_{j+1}^{-1} = H_{j+2\,j+3}$ is impossible. Therefore, in the (x, y, ζ)-space, $\{H_{j\,j+1}\}$ is preserved by $G_j(x,y,\zeta) = (\alpha x, \alpha y, c_2\zeta_2, \ldots, c_{n-1}\zeta_{n-1})$ and $\hat{\sigma}^m G_j$. For the latter we must have $m = 0$, since $G_{j+2} = \hat{\tau}_1 G_j \hat{\tau}_1$. This shows that $\mathrm{Aut}_{vol}(M)$ has exactly 2^j elements. $\square$

10. Appendix A – Normalization on sectorial domains

We now recall the following fundamental theorem of Varonin[10].

We change the notation slightly. Let $z = (z_3, \cdots, z_n)$ and

$$\hat{f}(x, y, z) = (x, y + x, z), \quad \hat{f}^*(x, y) = (x, y + x).$$

THEOREM 10.1. — *Let (x, y, z) be the coordinates of $\mathbb{C}^n$. Let f be a holomorphic map on $\mathbb{C}^n$ of the form*

$$(x, y, z) \rightarrow (x + x^2 p(x, y, z), y + x + xq(x, y, z), z + xs(x, y, z))$$

where $q(0) = 0 = s(0)$. Let $\alpha < \beta < \alpha + \pi$. There exist r depending on α, β, and a holomorphic map B defined on $\{x: \alpha < \arg x < \beta, |x| < r\} \times \Delta_r^{n-1}$ such that on $\{x: \alpha < \arg x < \beta, |x| < r\} \times \Delta_r^{n-1}$, $B^{-1}fB = \hat{f}$ and B admits the asymptotic expansion Φ which preserves $x = 0$ and satisfies $\Phi'(0) = \mathrm{id}$ and $\Phi|_{y=0} = \mathrm{id}$.

Proof. — Voronin gave a proof for $n = 2$. For the convenience of the reader only, we modify it for $n > 2$.

Applying $(x, y, z) \rightarrow (ax, ay, z)$, we may assume that the sector is

$$V_{\epsilon,\alpha} \subset \mathbb{C}: |\arg x| < \frac{\pi}{2} - \alpha, \quad 0 < |x| < \epsilon \sin 2\alpha.$$

In the y-plane consider the rhombus $\hat{V}_{\epsilon,\alpha}$ with vertices

$$0, \quad A = \epsilon e^{i(\frac{\pi}{2} - \alpha)}, \quad \overline{A}, \quad B = 2\epsilon \sin \alpha.$$

$\hat{V}_{\epsilon,\alpha}$ is the smallest rhombus that contains the sector $V_{\epsilon,\alpha}$ and has the origin as one of vertices. All sides of $\hat{V}_{\epsilon,\alpha}$ have length ϵ. Let $R_{\epsilon,\alpha}$ be the smallest rhombus centered at the origin and has four sides parallel to sides of $\hat{V}_{\epsilon,\alpha}$. The vertices of $R_{\epsilon,\alpha}$ are

$$B, \quad -B, \quad C = i2\epsilon \cos \alpha, \quad -C.$$

We also need to consider rhombuses which are contained in $R_{\epsilon,\alpha}$. We start with rhombus

$$R_{(1-\theta)\epsilon,(1+\frac{\theta}{\pi})\alpha} \subset \mathbb{C}, \quad 0 < \theta < 1/2, \quad (1 + \frac{\theta}{\pi})\alpha < \frac{\pi}{2}.$$

Its vertices are

$$B' = 2(1-\theta)\epsilon \sin(1 + \frac{\theta}{\pi})\alpha, \quad -B', \quad C' = i2(1-\theta)\epsilon \cos(1 + \frac{\theta}{\pi})\alpha, \quad -C'.$$

Let us first verify that $R_{(1-\theta)\epsilon,(1+\frac{\theta}{\pi})\alpha} \subset R_{\epsilon,\alpha}$. Indeed, using $\sin x \geqslant \frac{2}{\pi}x$ for $0 \leqslant x \leqslant \pi/2$ we obtain

$$\frac{1}{2\epsilon}(B - B') = \sin \alpha - (1-\theta)\sin(1 + \frac{\theta}{\pi})\alpha \geqslant \theta \frac{2\alpha}{\pi} - (1-\theta)\frac{\theta\alpha}{\pi} \geqslant \frac{\theta\alpha}{\pi}.$$

We also have

$$
\begin{aligned}
\frac{1}{2\epsilon i}(C - C') &= \cos \alpha - (1-\theta)\cos(1 + \frac{\theta}{\pi})\alpha \\
&\geqslant \theta \cos \alpha + (1-\theta)\sin \alpha \sin \frac{\theta\alpha}{\pi} \\
&\geqslant \min\{\frac{\theta}{\sqrt{2}}, \frac{\theta}{4\sqrt{2\pi}}\} = \frac{\theta}{4\sqrt{2\pi}}
\end{aligned}
$$

Note that $\frac{1}{2\epsilon}(B - B') \leqslant \theta\alpha$ and $\frac{1}{2\epsilon i}(C - C') \leqslant 2\theta$. Denote $[B,C]$ the line segment connecting B to C. It is so oriented if needed. The boundary $\Gamma = \Gamma_{\epsilon,\alpha}$ of $R_{\epsilon,\alpha}$ is the union of

$$\Gamma^+ = [C, -B] \cup [-B, -C], \quad \Gamma^- = [-C, B] \cup [B, C].$$

We will orient Γ^+, Γ^- counterclockwise as above when an orientation is needed. We have $\mathrm{dist}(\Gamma_{\epsilon,\alpha}, \Gamma_{(1-\theta)\epsilon,(1+\frac{\theta}{\pi})\alpha}) = \mathrm{dist}([B,C],[B',C'])$ and by $\cos \alpha > \cos \frac{\pi}{2(1+\frac{\theta}{\pi})} = \sin \frac{\theta}{2(1+\frac{\theta}{\pi})} \geqslant \frac{\theta}{2\pi}$ we get

$$\text{dist}([B, C], [B', C']) = \min((B - B')\cos\alpha, |C - C'|\sin\alpha) \leqslant 4\epsilon\theta\alpha,$$

$$\text{dist}([B, C], [B', C']) \geqslant 2\epsilon \min\left(\frac{\theta\alpha\cos\alpha}{\pi}, \frac{\theta\alpha}{2\sqrt{2}\pi^2}\right) \geqslant \frac{\epsilon\theta^2\alpha}{\pi^2}. \qquad (10.1)$$

For $x \in V_{(1-\theta)\epsilon, (1+\frac{\theta}{\pi})\alpha}$, we have $\text{dist}(x, \mathbb{C} \setminus V_{\epsilon,\alpha}) \geqslant \min(|x|\sin\frac{\theta\alpha}{\pi}, \theta\epsilon)$. Hence

$$\text{dist}(x, \mathbb{C} \setminus V_{\epsilon,\alpha}) \geqslant \alpha\theta|x|/8, \quad x \in V_{(1-\theta)\epsilon, (1+\frac{\theta}{\pi}\alpha)}. \qquad (10.2)$$

Let φ be a holomorphic function on $\overline{S}$ with

$$S = S_{\epsilon,\alpha} = V_{\epsilon,\alpha} \times R_{\epsilon,\alpha} \times \Delta_\epsilon^{n-2}.$$

Let $y_0 = -C$ and

$$\varphi_+(x, y, z) = \int_{\Gamma+} \frac{\varphi(x, t, z)}{2\pi i}\left[\frac{1}{t - y} - \frac{1}{y_0 - y}\right]dt,$$

$$\varphi_-(x, y, z) = \int_{\Gamma-} \frac{\varphi(x, t, z)}{2\pi i}\left[\frac{1}{t - y} - \frac{1}{y_0 - y}\right]dt.$$

Then on S we have $\varphi = \varphi_+ + \varphi_-$. Let

$$h_+(x, y, z) = \sum_{k=0}^{\infty} \varphi_+(x, y + kx, z), \quad h_-(x, y, z) = \sum_{k=-1}^{-\infty} \varphi_-(x, y + kx, z).$$

Put $\tilde{h} = h_+ + h_-$ and $h(x, y, z) = \tilde{h}(x, y, z) - \tilde{h}(x, 0, z)$. Note that h_+, h_- are defined on $U = S \cup \hat{f}(S)$, and that on S

$$h(x, y, z) - h(x, y + x, z) = \varphi(x, y, z), \quad h(x, 0, z) = 0.$$

We can write

$$h_\pm(x, y, z) = \int_{\Gamma\pm} \varphi(x, t, z)E_\pm(t, x, y, y_0)dt$$

with $E(t, x, y, y_0) = \frac{1}{2\pi ix}G_\pm\left(\frac{y_0-t}{x}, \frac{y-t}{x}\right)$ for

$$G_+(a, b) = \sum_{k=0}^{\infty}\left(\frac{1}{k + a} - \frac{1}{k + b}\right), \quad G_-(a, b) = \sum_{k=-1}^{-\infty}\left(\frac{1}{k + a} - \frac{1}{k + b}\right).$$

If $|\arg a| < \pi - \gamma$, $0 < \gamma < \pi/2$ and $k \geqslant 0$, then $|a + k| \geqslant |a|\sin\gamma \geqslant 2|a|\gamma/\pi$. In particular, $|a + k| \geqslant k\gamma/\pi$ for $k \geqslant |a|/2$. Assume that $0 < |a| \leqslant |b|$, and

$|\arg a|, |\arg b|$ are less than $\pi - \gamma$. Then $\frac{1}{|b-a|}|G_+(a,b)|$ is less than the sum of $\frac{1}{|a||b|}$ and 4 partial sums:

$$\sum_{1 \leqslant k \leqslant |a|/2} \frac{1}{|a||b|\sin^2\gamma} < \frac{c_0'}{|b|\gamma^2}, \qquad \sum_{|a|/2 < k \leqslant 2|a|} \frac{\pi}{|b|k\gamma\sin\gamma} < \frac{c_0'}{|b|\gamma^2},$$

$$\sum_{2|a| < k \leqslant 2|b|} \frac{2}{k|b|\sin\gamma} < \frac{c_0'}{|a|\gamma}, \qquad \sum_{k > 2|b|} \frac{4}{k^2} < \frac{c_0'}{|b|}.$$

Thus for $0 < \gamma < \pi/2$

$$\frac{1}{|b-a|}|G_+(a,b)| \leqslant \frac{c_0''}{|a||b|} + \frac{c_0''}{\gamma^2\min(|a|,|b|)}, \quad |\arg a| < \pi-\gamma, |\arg b| < \pi-\gamma.$$

Recall that $\hat{f}^*(x,y) = (x, y+x)$ and denote $S_j^* = V_{(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha} \times R_{(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha}$. When $t \in \Gamma_{\epsilon,\delta}^+$ and $(x,y) \in \overline{S}_1^*$, we have $|\arg(y-t)| < |\arg(-B+C)| = \frac{\pi}{2} + \alpha$ and $|\arg x| < \frac{\pi}{2} - (1+\frac{\theta}{\pi})\alpha$. So

$$\left|\arg\frac{y-t}{x}\right| \leqslant |\arg(y-t)| + |\arg x| \leqslant \pi - \frac{\theta\alpha}{\pi}.$$

Also $\left|\frac{y-t}{x}\right| \geqslant \frac{\theta^2\alpha\epsilon}{\pi^2|x|}$ by (10.1) and $\left|\frac{y-t}{x} - \frac{y_0-t}{x}\right| \leqslant \frac{2\epsilon}{|x|}$. One gets

$$|E_\pm(t,x,y,y_0)| \leqslant \frac{c_0}{4\theta^4\alpha^3|x|}, \quad (x,y) \in S_1^* \cup \hat{f}^*(S_1^*), \quad t \in \Gamma_{\epsilon,\alpha}^\pm.$$

(The estimate for E_- on $\hat{f}^* S_1^*$ is analogous to that of E_+ on S_1.) Fix $z \in \Delta_\epsilon^{n-2}$. We have

$$|h_\pm(x,y,z)| \leqslant \frac{c_0\epsilon}{|x|\theta^4\alpha^3} \sup_{y\in\Gamma_{\epsilon,\alpha}} |\varphi(x,y,z)|, \quad (x,y) \in S_1^* \cup \hat{f}^*(S_1^*).$$

The above is for a solution h to $h - h\hat{f} = \varphi$ when φ is a function. We want to apply the solution to the mapping case.

We need a lemma on mappings defined on sectorial domains.

For a mapping $\psi = (\psi_1, \ldots, \psi_n)$ defined on $S_{\epsilon,\alpha}$ it is convenient to use norm

$$[\psi]_{N,\epsilon,\alpha} = \sup_{S_{\epsilon,\alpha}}\{|\frac{\psi_1}{x^{N+1}}|, |\frac{\psi_2}{x^N}|, \ldots |\frac{\psi_n}{x^N}|\}.$$

Recall $S_{\epsilon,\alpha} = V_{\epsilon,\alpha} \times R_{\epsilon,\alpha} \times \Delta_\epsilon^{n-2}$. In brief, put $V_j = V_{(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha}$, $S_j = S_{(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha}$, and $[\psi]_{N,j} = [\psi]_{N,(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha}$, etc. For a matrix $A = (a_{i,j})_{n\times n}$, we denote $|A| = \max_i \sum_j |a_{i,j}|$.

LEMMA 10.2. — *Let k, N be positive integers. Let $0 < \epsilon < 1/4$ and $0 < \theta < \frac{1}{2k}$, and $0 < (1 + \frac{k\theta}{\pi})\alpha < \frac{\pi}{2}$. Let $H = \mathrm{id} + h \colon S_{\epsilon,\alpha} \to \mathbb{C}^n$ be holomorphic. There exist constant $c_1 > 1, c_{k,n} > 1$, independent of N, h, θ, ϵ, such that if*

$$[h]_{N,\epsilon,\alpha} \leqslant c_{k,n}^{-1}\theta^2\alpha$$

then H is injective on S_1, and H, $H^{-1} = \mathrm{id} + \tilde{h}$ satisfy

$$H \colon S_j \to S_{j-1}, \quad j = 1, 2, \ldots, k, \tag{10.3}$$

$$[\tilde{h}]_{N,2} \leqslant 2[h]_{N,\epsilon,\alpha}, \tag{10.4}$$

$$H^{-1} \colon S_{j+1} \to S_j, \quad j = 1, 2, \ldots, k. \tag{10.5}$$

Moreover, $HH^{-1} = \mathrm{id}$ on S_2 and $H^{-1}H = \mathrm{id}$ on S_3. In particular, $H(S_j) \supset S_{j+1}$ and $H^{-1}(S_{j+1}) \supset S_{j+2}$ for $j = 1, \ldots, k$.

In the above lemma if we assume additionally that $[h\hat{f}]_{N,\epsilon,\alpha} \leqslant c_{k,n}^{-1}\theta^2\alpha$, then H is injective on $S_1 \cup \hat{f}S_1$ by choosing a possibly large constant $c_{k,n}$. (See the proof of the lemma.) Consequently, H^{-1} is well-defined on $S_{j+1} \cup \hat{f}S_{j+1}$ (and map it into $S_j \cup \hat{f}S_j$) for $j = 1, \ldots, k$.

Let us postpone the proof of the lemma and continue the proof of the theorem. For the rest of proof of the theorem, all constants $\tilde{c}_1, c_2, c_2', \ldots$ may depend only on n but not on other quantities and they are all larger than 1.

Write $H = \mathrm{id} + h$ with $h = (h_1, \ldots, h_n)$, $f = \hat{f} + \varphi$ with $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then $HfH^{-1} = \hat{f}$ becomes $\varphi + hf = \hat{f}h$. Voronin solved the equation through a sequence of approximations. The linearized equation is $\varphi + h\hat{f} = \hat{f}h$, and in components it becomes

$$
\begin{aligned}
h_1(x, y, z) - h_1(x, y + x, z) &= \varphi_1(x, y, z), \\
h_2(x, y, z) - h_2(x, y + x, z) &= \varphi_2(x, y, z) - h_1(x, y, z), \\
h_j(x, y, z) - h_j(x, y + x, z) &= \varphi_j(x, y, z), \quad j = 3, \ldots, n.
\end{aligned}
$$

Solve the first and third equations with estimates on $S^*_{(1-\frac{\theta}{2})\epsilon,(1+\frac{\theta}{2\pi})\alpha}$ and then the second equation on $S^*_{(1-\theta)\epsilon,(1+\frac{\theta}{\pi})\alpha}$. For $(x, y) \in S^*_{(1-\theta)\epsilon,(1+\frac{\theta}{\pi})\alpha} \cup \hat{f}^*(S^*_{(1-\theta)\epsilon,(1+\frac{\theta}{\pi})\alpha})$ we have

$$
\begin{aligned}
|h_j(x, y, z)| &\leqslant \frac{\tilde{c}_1\epsilon}{|x|\theta^4\alpha^3} \sup_{\Gamma_{\epsilon,\alpha}} |\varphi_j(x, y, z)|, \quad j \neq 2, \\
|h_2(x, y, z)| &\leqslant \frac{\tilde{c}_1\epsilon}{|x|\theta^4\alpha^3} \sup_{\Gamma_{\epsilon,\alpha}} |\varphi_2(x, y, z)| + \frac{\tilde{c}_1^2\epsilon^2}{|x|^2\theta^8\alpha^6} \sup_{\Gamma_{\epsilon,\alpha}} |\varphi_1(x, y, z)|.
\end{aligned}
$$

The above estimates imply

$$[h]_{N-1,1} \leqslant \frac{c_2\epsilon}{\theta^8\alpha^6}[\varphi]_{N,\epsilon,\alpha}, \quad [h \circ \hat{f}]_{N-1,1} \leqslant \frac{c_2\epsilon}{\theta^8\alpha^6}[\varphi]_{N,\epsilon,\alpha}. \tag{10.6}$$

Write $HfH^{-1} = \hat{f} + \tilde{\varphi}$. We need to estimate $\tilde{\varphi}$. Roughly, we want to bound the norm $[\tilde{\varphi}]$ (on a shrunk sectorial domain) by $[\varphi]^2$. We will apply Lemma 10.2 several times. We need to keep track the domains and ranges of mappings. We now take $k = 20$ in Lemma 10.2, which will suffice our purpose. (So we assume that $0 < \theta < \frac{1}{40}$ and $(1 + \frac{20\theta}{\pi})\alpha < \frac{\pi}{2}$.) And denote $c_{20,n} = c_*$.

Assume that

$$[\varphi] \overset{\text{def}}{=\!=} [\varphi]_{N,\epsilon,\alpha} \leqslant \frac{\theta^{10}\alpha^7}{2c_*c_2}. \tag{10.7}$$

We then have $[\hat{f}^{-1}\varphi]_{N,0} \leqslant 2[\varphi] \leqslant \frac{\theta^{10}\alpha^7}{c_*c_2}$. Hence $f(S_j) \subset \hat{f}(S_{j-1})$, $f^{-1}(S_{j+1}) \subset \hat{f}^{-1}(S_j)$ for $j = 1, \ldots, 20$.

Both $[h]_{N-1,1}$ and $[h\hat{f}]_{N-1,1}$ are less than $\frac{\theta^2\alpha}{2c_*}$. We have $[\hat{f}^{-1}h\hat{f}]_{N-1,1} \leqslant \frac{\theta^2\alpha}{c_*}$. To apply Lemma 10.2 to $\hat{f}^{-1}Hf = \text{id} + \hat{f}^{-1}h\hat{f}$, in which ϵ, α are replaced by $(1-\theta)\epsilon, (1 + \frac{\theta}{\pi})\alpha$ we let

$$S'_j = S_{(1-j\theta)(1-\theta)\epsilon, (1+\frac{j\theta}{\pi})(1+\frac{\theta}{\pi})\alpha}.$$

We obtain $H(\hat{f}(S'_1)) \subset \hat{f}(S'_0) = \hat{f}(S_1)$. Since $S_{j+2} \subset S'_j \subset S_j$ for $1 \leqslant j \leqslant 20$, then $H(\hat{f}(S_3)) \subset \hat{f}(S_1)$. Applying Lemma 10.2 to $\hat{f}^{-1}f$, we get $f(S_4) \subset \hat{f}(S_3)$. Applying Lemma 10.2 to H, we obtain $H^{-1}(S'_5) \subset S'_4 \subset S_4$.

Therefore, $\tilde{f} = HfH^{-1}$ maps S'_5 into $\hat{f}(S_1)$.

Note that
$$H(S'_j) \subset S'_{j-1}, \quad H(\hat{f}S'_j) \subset \hat{f}S'_{j-1},$$
$$H^{-1}(S'_{j+1}) \subset S'_j, \quad H^{-1}(\hat{f}S'_{j+1}) \subset \hat{f}S'_j, \quad j = 1, \ldots, 20.$$
Also $HH^{-1} = \text{id}$ on $S'_2 \cup \hat{f}S'_2$ and $H^{-1}H = \text{id}$ on $S'_3 \cup \hat{f}S'_3$.

The domains $\hat{f}(S_j)$ are not product domains, which cause difficulties in estimating derivatives. So we will pull all maps back on S_j. Set $g = \hat{f}^{-1}f = \text{id} + \psi$. We have $[\psi] \overset{\text{def}}{=\!=} [\psi]_{N,\epsilon,\alpha} \leqslant 2[\varphi]$. Recall that $\tilde{\varphi} = HfH^{-1} - \hat{f}$ is defined on S'_5. From the linearized equation $\varphi = \hat{f}h - h\hat{f}$, which holds on $S_1(\supset H^{-1}S'_5)$, we get $\tilde{\varphi}H = hf - h\hat{f}$ on $H^{-1}S'_5$. The latter is actually defined on S_4, for which we first estimate. We write $h\hat{f} = p$. So

$$hf - h\hat{f} = (p_1g - p_1, p_2g - p_2, \ldots, p_ng - p_n).$$

Write $p = (x^N p'_1, x^{N-1} p'_2, \ldots, x^{N-1} p'_n)$ and $\psi = (x^{N+1}\psi'_1, x^N \psi'_2, \ldots, x^N \psi'_n)$. Note that $g(S_4) = \hat{f}^{-1} f(S_4) \subset S_3$. Next, we estimate the derivatives of p_j. For $0 \leqslant t \leqslant 1$, put $x_t = x + tx^{N+1}\psi'_1(x,y,z), y_t = y + tx^N \psi'_2(x,y,z)$ and $z_t = z + tx^N(\psi'_3, \ldots, \psi'_n)(x,y,z)$ we have

$$\frac{\partial p_1}{\partial x}(x_t, y_t, z_t) = N x_t^{N-1} p'_1(x_t, y_t, z_t) + x_t^N \frac{\partial p'_1}{\partial x}(x_t, y_t, z_t).$$

Since $|x| < \epsilon < 1/2$ and $|\psi'_1| < 1/2$ by (10.7) we have

$$|(x + tx^{N+1}\psi'_1)^{N-1}| \leqslant |x|^{N-1} e^{(N-1)|x|^N ||\psi'_1|} < 2|x|^{N-1}.$$

Note that $(x_t, y_t, z_t) = tg(x,y,z) + (1-t)(x,y,z)$ and it is in the convex domain S_3; consequently

$$|p'_1(x_t, y_t, z_t)| \leqslant \sup_{S_3} |p'_1| \leqslant [p]_{N-1,1} = [h\hat{f}]_{N-1,1}.$$

Also $|tx^{N+1}\psi'_1(x,y,z)| < |x|[\varphi] < |x|/2$. By (10.2) we have $\text{dist}(x_t, \mathbb{C}\setminus V_1) \geqslant \theta\alpha|x_t|/8$ and hence $|\frac{\partial p'_1}{\partial x}(x_t, y_t, z_t)| \leqslant \frac{8}{\theta\alpha|x_t|} \sup_{S_1}|p'_1| \leqslant \frac{c_3}{\theta^9\alpha^7|x|}[\varphi]$, by (10.6). Also

$$|\frac{\partial p'_1}{\partial y}(x_t, y_t, z_t)| \leqslant \frac{c'_3}{\theta^2\alpha\epsilon} \sup_{S_1}|p'_1| \leqslant \frac{c_2 c'_3}{\theta^{10}\alpha^7}[\varphi], \quad |\frac{\partial p'_1}{\partial z_j}(x_t, y_t, z_t)| \leqslant \frac{c_2 c'_3}{\theta^9\alpha^7}[\varphi].$$

Recall that on S_4 we have $|\psi'_j| \leqslant [\psi] \leqslant 2[\varphi]$. By using a line integral, we conclude

$$|(p_1 g - p_1)(x,y,z)| = |p_1(x + x^{N+1}\psi'_1, y + x^N\psi'_2, \ldots, z_n + x^N\psi'_n) - p_1(x,y,z)|$$

$$\leqslant \frac{c_4 N}{\theta^{10}\alpha^7}|x|^{2N}[\varphi]^2_{N,\epsilon,\alpha}, \quad (x,y,z) \in S_4.$$

Analogously, we obtain

$$|(p_j g - p_j)(x,y,z)| = |p_j(x + x^{N+1}\psi'_1, y + x^N\psi'_2, \ldots, z_n + x^N\psi'_n) - p_j(x,y,z)|$$

$$\leqslant \frac{c'_4 N}{\theta^{10}\alpha^7}|x|^{2N-1}[\varphi]^2_{N,\epsilon,\alpha}, \quad (x,y,z) \in S_4, \ 1 < j \leqslant n.$$

Hence for $\tilde{f} = HfH^{-1} = id + \tilde{\varphi}$ on S'_5, we have

$$\frac{|\tilde{\varphi}_1|}{|x \circ H^{-1}|^{2N}} \leqslant \frac{c_5 N}{\theta^{10}\alpha^7}[\varphi]^2_{N,\epsilon,\alpha}, \quad \frac{|\tilde{\varphi}_j|}{|x \circ H^{-1}|^{2N-1}} \leqslant \frac{c_5 N}{\theta^{10}\alpha^7}[\varphi]^2_{N,\epsilon,\alpha}, \ j > 1.$$

By Lemma 10.2 we have $[\tilde{h}]_{N-1,6} \leqslant 2[h]_{N-1,1}$, and by (10.6) and (10.7) we get $|x_1 \circ H^{-1}|^k = |x + x^N \tilde{h}_1(x,y,z)|^k \geqslant |x|^k(1 - k\epsilon^{N-1}[\tilde{h}]_{N-1,6}) \geqslant |x|^k/2$ for $k = 2N, 2N-1$. Hence

$$[\tilde{\varphi}]_{2N-1,(1-5\theta)(1-\theta)\epsilon,(1+\frac{5\theta}{\pi})(1+\frac{\theta}{\pi})\alpha} \leqslant \frac{c_6 N}{\theta^{10}\alpha^7}[\varphi]^2_{N,\epsilon,\alpha}.$$

We now linearize f by a sequence of mappings. We change notations. If we are given $f_k = \hat{f} + \varphi_k$ on $S_k = S_{\epsilon_k, \alpha_k}$ we can find $H_{k+1} = \mathrm{id} + h_{k+1}$ on $S_{k+1} = S_{\epsilon_{k+1}, \alpha_{k+1}}$ with $\epsilon_{k+1} = (1 - 5\theta_k)(1 - \theta_k)\epsilon_k$ and $\alpha_{k+1} = (1 + \frac{5\theta_k}{\pi})(1 + \frac{\theta_k}{\pi})\alpha_k$, such that $f_{k+1} = H_{k+1} f_k H_{k+1}^{-1} = \hat{f} + \varphi_{k+1}$ is defined on S_{k+1} and

$$
\begin{aligned}
[h_{k+1}] &\stackrel{\mathrm{def}}{=} \max\Big\{[h_{k+1}\hat{f}]_{N_k-1, (1-\theta_k)\epsilon_k, (1+\frac{\theta_k}{\pi})\alpha_k}, [h_{k+1}]_{N_k-1, (1-\theta_k)\epsilon_k, (1+\frac{\theta_k}{\pi})\alpha_k}\Big\} \\
&\leqslant \frac{c_2 \epsilon_k}{\theta_k^8 \alpha_k^6}[\varphi_k], \\
[\varphi_{k+1}] &\stackrel{\mathrm{def}}{=} [\varphi_{k+1}]_{N_{k+1}, \epsilon_{k+1}, \alpha_{k+1}} \leqslant \frac{c_7 N_k}{\theta_k^{10} \alpha_k^7}[\varphi_k]^2
\end{aligned}
\tag{10.8}
$$

for $N_{k+1} = 2N_k - 1$, provided

$$
[\varphi_k] = [\varphi_k]_{N_k, \epsilon_k, \alpha_k} \leqslant \frac{\theta_k^{10} \alpha_k^7}{2 c_2 c_*} \stackrel{\mathrm{def}}{=} 8n(k+1)^2 c_3 b_k. \tag{10.9}
$$

Note that by Lemma 10.2 and (10.9) we have

$$
f_k(S_{k,j}) \subset \hat{f}(S_{k,j-1}), \quad j = 2, \ldots, 20
$$

for $S_{k,j} = S_{(1-j\theta_k)\epsilon_k, (1+\frac{j\theta_k}{\pi})\alpha_k}$. Set $S'_{k,j} \stackrel{\mathrm{def}}{=} S_{(1-j\theta_k)(1-\theta_k)\epsilon_k, (1+\frac{j\theta_k}{\pi})(1+\frac{\theta_k}{\pi})\alpha_k}$. For $H_{k+1} = \mathrm{id} + h_{k+1}$, using (10.8) and Lemma 10.2, we obtain

$$
H_{k+1}(S'_{k,j}) \subset S'_{k,j-1}, \quad H_{k+1}^{-1}(S'_{k,j+1}) \subset S'_{k,j}, \quad j = 1, \ldots, 20,
$$

$$
H_{k+1}(\hat{f}S'_j) \subset \hat{f}S'_{j-1}, \quad H_{k+1}^{-1}(\hat{f}S'_{j+1}) \subset \hat{f}S'_j, \quad j = 1, \ldots, 20. \tag{10.10}
$$

Also $H_{k+1} H_{k+1}^{-1} = \mathrm{id}$ on $S'_{k,2} \cup \hat{f}S'_{k,2}$ and $H_{k+1}^{-1} H_{k+1} = \mathrm{id}$ on $S'_{k,3} \cup \hat{f}S'_{k,3}$.

Take $N_0 = 3$. So $N_k = 2^{k+1} + 1$. We are given $\alpha_0 \in (0, \frac{\pi}{2})$. Take $\theta_k = \frac{\beta}{2^k}$, and fix $\beta > 0$ such that $\theta_0 < \frac{1}{40}$ and $\alpha_\infty = \lim_{k \to \infty} \alpha_k < \pi/2$. Set $\epsilon_\infty = \lim_{k \to \infty} \epsilon_k$ and $S_\infty = S_{\epsilon_\infty, a_\infty}$. It remains to choose an $\epsilon_0 \in (0, 1/4)$. Let $a_0 = [\varphi_0]$, which depends only on ϵ_0 (and f), and let

$$
a_{k+1} = \frac{c_7 N_k}{\theta_k^{10} \alpha_k^7} a_k^2.
$$

We may assume that $a_0 \neq 0$. Then $\frac{a_{k+1}}{a_k} \leqslant 2^{10}(\frac{a_k}{a_{k-1}})^2$ for $k \geqslant 1$. With b_k being defined in (10.9), $\frac{b_{k+1}}{b_k} \geqslant \frac{1}{4}(\frac{\theta_{k+1}}{\theta_k})^{10} = \frac{1}{2^{12}}$. By the semi-formal theory, we can find a holomorphic map

$$
B_0(x, y, z) = (x + xB'_0, y + B''_0, z + B'''_0), \quad B_0 = \mathrm{id} + O(2)
$$

such that $f \equiv B_0 f B_0^{-1} = \hat{f} + O(|x|^4)$. We choose $\epsilon_0 \in (0, 1/4)$ such that $[\varphi_0] \leqslant b_0$ (in particular (10.9) holds for $k = 0$) and

$$\frac{a_1}{a_0} = \frac{c_7 N_0}{\theta_0^{10} \alpha_0^7} [\varphi_0] < \frac{1}{2^{12}}.$$

For induction, we assume that $a_k \leqslant b_k$ and $\frac{a_{k+1}}{a_k} < \frac{1}{2^{12}}$. Then $a_{k+1} \leqslant 2^{12} (\frac{a_k}{a_{k-1}})^2 a_k \leqslant \frac{1}{2^{12}} a_k \leqslant \frac{1}{2^{12}} b_k \leqslant b_{k+1}$ and $\frac{a_{k+2}}{a_{k+1}} \leqslant 2^{12} (\frac{a_{k+1}}{a_k})^2 < \frac{1}{2^{12}}$. Therefore $a_k \leqslant b_k$ for all k. We have $[\varphi_1] \leqslant a_1 \leqslant b_1$. So we can find H_1. By induction, we can show that the sequence H_k is well-defined for all k and $[\varphi_k] \leqslant a_k \leqslant b_k$ for all k.

Next, we want to find a domain on which $A_k = H_k H_{k-1} \cdots H_1$ is defined. Recall that $\theta_k = \frac{\beta}{2^k}$. For $i = 14$, we have

$$\begin{aligned} &H_{k+1}(S_{(1-i\theta_k)(1-\theta_k)\epsilon_k, (1+\frac{i\theta_k}{\pi})(1+\frac{\theta_k}{\pi})\alpha_k}) \\ &\subset S_{(1-(i-1)\theta_k)(1-\theta_k)\epsilon_k, (1+\frac{(i-1)\theta_k}{\pi})(1+\frac{\theta_k}{\pi})\alpha_k} \quad \text{(by 10.3)} \\ &\subset S_{(1-i\theta_{k+1})(1-\theta_{k+1})\epsilon_{k+1}, (1+\frac{i\theta_{k+1}}{\pi})(1+\frac{\theta_{k+1}}{\pi})\alpha_{k+1}} \overset{\text{def}}{=\!=} S'_{k+1,i}. \end{aligned}$$

(For the last inclusion to hold, we might need to choose a smaller β, and hence a smaller ϵ_0. However, the inclusion remains true regardless the choice of ϵ_0. We will also adjust β a few times.) Hence $A_k(S'_{0,14}) \subset S'_{k,14}$. Also $A_k(\hat{f} S'_{0,14}) \subset \hat{f} S'_{k,14}$. From $H_k f_{k-1} H_k^{-1} = f_k$ we want to conclude that $A_k f_k = f_0 A_k$ holds on $S'_{0,17}$. The statement is trivial for $k = 1$. Assume it holds for $k = m$. First, we have $f_0(S'_{0,17}) \subset f_0(S_{0,17}) \subset \hat{f}(S_{0,16}) \subset \hat{f}(S'_{0,14})$, where the last inclusion is obtained by choosing a possibly smaller β. Thus on $S'_{0,17}$, $A_{m+1} f_0$ is well-defined, and it equals $H_{m+1}(A_m f_0) = H_{m+1} f_m A_m$. By definition $H_{m+1} f_m = f_{m+1} H_{m+1}$ holds on $H_{m+1}^{-1}(S_{m+1})$. Since $A_m(S'_{0,17}) \subset A_m(S'_{0,14}) \subset S'_{m,14} \subset H_{m+1}^{-1}(S'_{m,5}) = H_{m+1}^{-1}(S_{m+1})$, then $H_{m+1} f_m A_m = f_{m+1} H_{m+1} A_m$ holds on $S'_{0,17}$. This shows that $A_k f_0 = f_k A_k$ on $S'_{0,17}$.

Next, we want to show that A_k converges to a holomorphic map A_∞ on $S'_{0,14} \cup \hat{f} S'_{0,14}$. Write $h_k(x, y, z) = (x + x^{N_k} h'_k(x, y, z), y + x^{N_k-1} h''_k(x, y, z), z + x^{N_k-1} h'''_k(x, y, z))$, where $h''_k = (h_{k,2} \ldots, h_{k,n})$, and $A_k(x, y, z) = (x + x^{N_1} A'_k(x, y, z), y + x^{N_1-1} A''_k(x, y, z), z + x^{N_1-1} A'''_k(x, y, z))$. We have $A'_1 = h'_1$, $A''_1 = h''_1$, $A'''_1 = h'''_1$. Hence on $S'_{0,14} \cup \hat{f} S'_{0,14}$ we have

$$\max\{|A'_1|, |A''_1|, |A'''_1|\} \leqslant [h_1] < 1.$$

Assume that on $S'_{0,14} \cup \hat{f} S'_{0,14}$ we have $|A'_k| \leqslant 1 + \cdots + \frac{1}{(k-1)^2}$. Then

$$
A_{k+1} \begin{cases}
x' = x + x^{N_1} A'_k + (x + x^{N_1} A'_k)^{N_{k+1}} h'_{k+1}(A_k), \\
y' = y + x^{N_1 - 1} A''_k + (x + x^{N_1} A'_k)^{N_{k+1} - 1} h''_{k+1}(A_k), \\
z' = z + x^{N_1 - 1} A'''_k + (x + x^{N_1} A'_k)^{N_{k+1} - 1} h'''_{k+1}(A_k).
\end{cases}
$$

Since $|x| < 1/4$ then

$$
\begin{aligned}
|A'_{k+1} - A'_k| \quad &= |x|^{N_{k+1} - N_1} |(1 + x^{N_1 - 1} A'_k)^{N_{k+1}} h'_{k+1}(A_k)| \\
&\leqslant |x|^{N_{k+1} - N_1} (1 + |x|^{N_1 - 1}|A'_k|)^{N_{k+1}} |h'_{k+1}(A_k)| \\
&\leqslant 4^{N_k} |x|^{N_k} |h'_{k+1}(A_k)| \leqslant \frac{1}{k^2}, \quad \text{by (10.8)-(10.9).}
\end{aligned}
$$

By the same argument, we obtain $|A''_{k+1} - A''_k| < \frac{1}{k^2}$ and $|A'''_{k+1} - A'''_k| < \frac{1}{k^2}$. Therefore, A_k converges to $A_\infty = (x + x^{N_0} H', y + x^{N_0 - 1} H'', z + x^{N_0 - 1} H''')$ uniformly on $S'_{0,14} \cup \hat{f} S'_{0,14}$. Note that the sup norm of (H', H'', H''') on $S'_{0,14} \cup \hat{f} S'_{0,14}$ is less than 3. By definition, $H_{k+1} f_k H_{k+1}^{-1} = f_{k+1}$ on S_{k+1}. Hence $H_{k+1} f_k = f_{k+1} H_{k+1}$ on $H_{k+1}^{-1} S_{k+1}$. Since $H_{k+1}^{-1} S_{k+1} \subset S_k$ then $H_{k+1} H_k f_{k-1} H_k^{-1} = f_{k+1} H_{k+1}$ still holds on $H_{k+1}^{-1} S_{k+1}$. Now $H_{k+1} H_k f_{k-1} = f_{k+1} H_{k+1} H_k$ holds on $H_k^{-1} H_{k+1}^{-1} S_{k+1}$. In general, we have $A_{k+1} f_0 = f_{k+1} A_{k+1}$ on $H_1^{-1} \cdots H_{k+1}^{-1} S_{k+1}$. Using

$$
\begin{aligned}
H_{k+1}^{-1} (S_{(1-j\theta_k)(1-\theta_k)} \epsilon_k, (1 + \tfrac{j\theta_k}{\pi})(1 + \tfrac{\theta_k}{\pi}) \alpha_k) \\
\supset S_{(1-(j+1)\theta_k)(1-\theta_k)\epsilon_k, (1 + \frac{(j+1)\theta_k}{\pi})(1 + \frac{\theta_k}{\pi}) \alpha_k}
\end{aligned}
$$

and by a computation as above, we can verify that $H_1^{-1} \cdots H_{k+1}^{-1} S_{k+1} \supset S'_{0,13}$. This shows that $A_{k+1} f_0 = f_{k+1} A_{k+1}$ on $S'_{0,13}$. Taking limits, we get $A_\infty f_0 = \hat{f} A_\infty$ on $S'_{0,17}$.

We want to show that A_∞ admits an asymptotic expansion. By Lemma 8 in [10] and by $H_j|_{y=0} = \text{id}$, we know that each H_j admits an asymptotic expansion. As above we can verify that $\tilde{H}_k = \lim_{j \to \infty} H_j \cdots H_{k+1} = (x + x^{N_k} C', y + x^{N_k - 1} C'', z + x^{N_k - 1} C''')$ satisfies $\max\{|C'_k|, |C''_k|, |C'''_k|\} < 3$ on $S'_{k-1,14}$. In particular $|\tilde{H}_k(x, y, z) - (x, y, z)| < 2|x|^{N_k - 1}$. One sees that A_∞ admits an asymptotic expansion. Finally, we can set $B = A_\infty^{-1}$, defined on $(S'_{0,14} \cup \hat{f} S'_{0,14}) \cap \Delta_r^n$ for some $r > 0$. Choosing a smaller r if necessary, we conclude that $B^{-1} f B = \hat{f}$ holds on $S'_{0,18} \cap \Delta_r^n$. The asymptotic expansion $\tilde{\Phi}$ of B must be Φ since $\tilde{\Phi} = \text{id} + O(2)$, $\tilde{\Phi}|_{y=0} = \text{id}$, and $\tilde{\Phi}$ normalizes f. $\square$

Proof of Lemma 10.2. — Recall that $H = \text{id} + h$. For $h = (x^{N+1} h'_1, x^N h'_2, \ldots, x^N h'_n)$, we define $[h]_{N,\epsilon,\alpha} = \sup_{S_{\epsilon,\alpha}} \{|h'_j(x, y, z)|\}$. Let $r = [h]_{N,\epsilon,\alpha}$

$< c_{k,n}^{-1}\theta^2\alpha$ with $c_{k,n} = c > 1$ to be determined. Recall that $1 \leqslant k < \frac{1}{2\theta}$, $0 < \epsilon < 1/4$ and $(1+\frac{k\theta}{\pi})\alpha < \pi/2$. Fix $1 \leqslant j \leqslant k$. Denote $S_j = S_{(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha}$, $R_j = R_{(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha}$, etc. Note that

$$R_{(1-j\theta)\epsilon,(1+\frac{j\theta}{\pi})\alpha} \subset R_{(1-\frac{\theta}{2})(1-(j-1)\theta)\epsilon,(1+\frac{\theta}{2\pi})(1+\frac{(j-1)\theta}{\pi})\alpha}$$

since $\frac{(j-1)\theta}{\pi} < 1$. By (10.1) the distance between boundaries of R_j and R_{j-1}, denoted by $\mathrm{dist}(bR_{j-1}, bR_j)$, is larger than $(\frac{\theta}{2})^2(1 - (j-1)\theta)\epsilon(1 + \frac{(j-1)\theta}{\pi})\alpha$ $> \frac{\theta^2\alpha\epsilon}{c_1}$ (for $c_1 = 16$).

Fix $(x, y, z) \in S_j$. We want to show that if $\max\{|w_1|, \ldots, |w_n|\} < \frac{\theta^2\alpha}{\pi^2 c_1}$ then $(x', y', z') = (x + x^{N+1}w_1, y + x^N w_2, \ldots, z_n + x^N w_n) \in S_{j-1}$. (i) We have $y' = y + x^N w_2 \in R_{j-1}$, since $|x^N w_2| < \epsilon\frac{\theta^2\alpha}{\pi^2 c_1} < \mathrm{dist}(bR_{j-1}, bR_j)$. (ii) We have $x' = x + x^{N+1}w_1 \in V_{j-1}$. First, since $|\arg(1+\xi)| \leqslant \frac{\pi}{2}|\xi|$ for $|\xi| < 1$ then $|\arg(1 + x^N w_1)| \leqslant \frac{\pi}{2}|w_1| < \frac{\theta\alpha}{\pi}$. Also $|x^{N+1}w_1| \leqslant \epsilon\theta$. (iii) We have $z'_j = z_j + x^N w_j \in \Delta_{(1-(j-1)\theta)\epsilon}$, since $|x^N h'_j(x,y,z)| \leqslant \epsilon\theta$. By (i)-(iii), (10.3) holds when $c > \pi^2 c_1$.

To show that H is injective, we need to estimate the derivatives of h. For $x \in V_1$, the disc centered at x with radius $\theta|x|\alpha/8$ ($\leqslant \min\{|x|\sin\frac{\alpha\theta}{\pi}, \theta\epsilon\}$) is contained in V_0. Hence for $(x, y, z) \in S_1$, we have

$$\left|\frac{\partial(x^{N+1}h_1(x,y,z))}{\partial x}\right| \leqslant \frac{8|2x|^{N+1}r}{\theta|x|\alpha} \leqslant \frac{16|x|^N r}{\theta\alpha} < \frac{1}{n}$$

if $c > 16n$. For $c > 2nc_1$ we can also obtain $|\frac{\partial(x^N h_j(x,y,z))}{\partial y}| < \frac{2c_1|x|^{N-1}}{\theta^2\alpha}r$ $< 1/n$ and $|\frac{\partial(x^N h_j(x,y,z))}{\partial z_k}| < \frac{2}{\theta}|x|^{N-1}r < 1/n$ on S_1, from which we conclude that H is injective on S_1.

We need to find $H^{-1} = \mathrm{id} + \tilde{h}$. Let $\tilde{h}_1(x,y,z) = x^{N+1}u_1(x,y,z)$ and $\tilde{h}_j(x,y,z) = x^N u_j(x,y,z)$ for $j > 1$. For $HH^{-1} = \mathrm{id}$ on S_2, we need

$$u_1 = -(1 + x^N u_1)^{N+1}h'_1(x + x^{N+1}u_1, y + x^N u_2, \ldots, z_n + x^N u_n),$$

$$u_j = -(1 + x^N u_1)^N h'_j(x + x^{N+1}u_1, y + x^N u_2, \ldots, z_n + x^N u_n)$$

for $j = 2, \ldots, n$. Rewriting (10.4) in sup norm, we want to get

$$|u|_{(1-2\theta)\epsilon,(1+\frac{2\theta}{\pi})\alpha} \leqslant 2r.$$

Fix $(x, y, z) \in S_{(1-2\theta)\epsilon,(1+\frac{2\theta}{\pi})\alpha}$. Write (10.11) as $u = Tu$. By (i)-(iii), T is defined on Δ_{2r}^n, assuming $2r \leqslant \frac{\epsilon\theta^2\alpha}{\pi^2 c_1}$. We can also have

$$2(N + 1)\epsilon^N r < 4r < \ln 2.$$

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By $(1 + 2|x|^N r)^{N+1} \leqslant e^{2(N+1)|x|^N r} < 2$, we get $T(\Delta_{2r}^n) \subset \Delta_{2r}^n$.

We also need T to be a contraction. By (i)-(iii) we see that $(x', y', z') = (x + x^{N+1}u_1, y + x^N u_2, \ldots, z_n + x^N u_n)$ is in S_1 for $u \in \Delta_{2r}^n$ (and $(x, y, z) \in S_2$). Hence $|\frac{\partial h'_j}{\partial x}(x', y', z')| \leqslant \frac{8 \sup_{S_0} |h'_j|}{\theta |x| \alpha}$, $|\frac{\partial h'_j}{\partial y}(x', y', z')| \leqslant \frac{c_1 \sup_{S_0} |h'_j|}{\epsilon \theta^2 \alpha}$, and $|\frac{\partial h'_j}{\partial z_k}(x', y', z')| \leqslant \frac{c_1 \sup_{S_0} |h'_j|}{\epsilon \theta}$. Now we get

$$|\frac{\partial T_j u}{\partial u_1}| \leqslant (N+1)e^{2N\epsilon^N r}|x|^N r + e^{2(N+1)\epsilon^N r}\frac{8|x|^N r}{\theta \alpha},$$
$$|\frac{\partial T_j u}{\partial u_k}| \leqslant 2Nre^{2(N+1)\epsilon^N r}\frac{c_1 |x|^{N-1}}{\theta^2 \alpha}, \quad k > 1.$$

Obviously, $T \colon \Delta_{2r}^n \to \Delta_{2r}^n$ is a contraction map, if $r \leqslant \frac{\theta^2 \alpha}{c}$ and c is sufficiently large. We have shown that H^{-1} is defined on S_2 and satisfies (10.4). Then (10.5) follows from (10.3) (by applying it to H^{-1} and by choosing a possibly larger $c_{n,k}$).

We have $HH^{-1} = $ id on S_2 and $HH^{-1}H = H$ on S_3. Since H is injective on S_1, we get $H^{-1}H = $ id on S_3. Finally, we have $S_j = HH^{-1}(S_j) \subset H(S_{j-1})$ and $S_{j+1} = H^{-1}H(S_{j+1}) \subset H^{-1}(S_j)$ for $j = 2, \ldots, k$.

Assume now that $[h\hat{f}]_{N,\epsilon,\alpha} \leqslant c_{k,n}^{-1}\theta^2\alpha$ also holds. We know that H is injective on S_1 and on $\hat{f}S_1$. We want to show that it is also injective on the union of $S_2 \cup \hat{f}S_2$. Assume that distinct $(x, y, z) \in S_2$ and $(x', y', z') \in \hat{f}S_2$ satisfy $H(x, y, z) = H(x', y', z')$. Then (x', y', z') is not in S'_1. Note that $S_2 \cap \hat{f}S_2$ contains $S_2 \cap \Delta_{\epsilon/4}^n$. Then $(x, y, z), (x', y', z')$ are not in $\Delta_{\epsilon/4}^n$. Hence $|(x', y', z') - (x, y, z)| > c\epsilon\theta\alpha^2$ for some constant c. Now $|H(x', y', z') - H(x, y, z)| \geqslant c\epsilon\theta\alpha^2 - 2\epsilon \sum |h'_j(x', y', z')| + |h'_j(x, y, z)| > 0$, by choosing a larger $c_{k,n}$. This shows that H is injective on $S_2 \cup \hat{f}S_2$. We can obtain the injectivity on $S_1 \cup \hat{f}S_1$, by applying the result to the case where θ is replaced with $\theta/2$. $\square$

Lemma 10.2 is mainely used in the proof of Theorem 10.1. We also have the following lemma, which has been used throughout the paper.

LEMMA 10.3. — *Let $0 < \alpha < \pi$. Let H be a holomorphic mapping defined on $W_{\alpha,r} = \{x \colon |\arg w| < \pi - \alpha\} \cap \Delta_{r_0}^n$. Assume that H admits an asymptotic expansion Φ of semi-formal map. Assume that Φ preserves $x = 0$, i.e. the x-component of Φ has the form $xa(x, y, z)$. Suppose that $\Phi'(0)$ is biholomorphic. Let $a(0) = |a(0)|\mu$. Let $0 < \epsilon_1 < \epsilon_2 < \pi - \alpha$. There exist r_1, r_2 with $0 < r_2 < r_1 < r_0$ such that $H \colon W_{\alpha+\epsilon_1, r_1} \to \widetilde{W}$ is biholomorphic and $\widetilde{W} \supset \mu W_{\alpha+\epsilon_2, r_2}$.*

Proof. — Write $\Phi(x,y,z) = \sum_{k=0}^{\infty}(x^{k+1}A_k(y,z), x^k B_k(y,z), x^k C_k(y,z))$. Let $\Phi_1(x,y,z) = \sum_{k=0}^{3}(x^{k+1}A_k(y,z), x^k B_k(y,z), x^k C_k(y,z))$. Then Φ_1 is biholomorphic near the origin of $\mathbb{C}^n$. Writing $\tilde{H}(x,y,z) = (xa(x,y,z), b(x,y,z), c(x,y,z))$, one can obtain the conclusions directly from Lemma 10.2 by considering the map

$$(x,y,z) \rightarrow (x(a(tx^4, ty, tz))^{\frac{1}{4}}, t^{-1}b(tx^4, ty, tz), t^{-1}c(tx^4, ty, tz))$$

for some small positive t. $\quad\square$

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