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Hölder regularity of two-dimensional almost-minimal sets in $\mathbb{R}^n$ (*)

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Résumé. — On donne une démonstration différente et sans doute plus élémentaire d'une bonne partie du résultat de régularité de Jean Taylor sur les ensembles presque-minimaux d'Almgren. On en profite pour donner des précisions sur les ensembles presque minimaux, généraliser une partie du théorème de Taylor aux ensembles presque minimaux de dimension 2 dans $\mathbb{R}^n$, et donner la caractérisation attendue des ensembles fermés E de dimension 2 dans $\mathbb{R}^3$ qui sont minimaux, au sens où $H^2(E \setminus F) \leqslant H^2(F \setminus E)$ pour tout fermé F tel qu'il existe une partie bornée B telle que $F = E$ hors de B et F sépare les points de $\mathbb{R}^3 \setminus B$ qui sont séparés par E.

Abstract. — We give a different and probably more elementary proof of a good part of Jean Taylor's regularity theorem for Almgren almost-minimal sets of dimension 2 in $\mathbb{R}^3$. We use this opportunity to settle some details about almost-minimal sets, extend a part of Taylor's result to almost-minimal sets of dimension 2 in $\mathbb{R}^n$, and give the expected characterization of the closed sets E of dimension 2 in $\mathbb{R}^3$ that are minimal, in the sense that $H^2(E \setminus F) \leqslant H^2(F \setminus E)$ for every closed set F such that there is a bounded set B so that $F = E$ out of B and F separates points of $\mathbb{R}^3 \setminus B$ that E separates.

1. Introduction

The core of this paper is a slightly different, probably more elementary, and very detailed proof of a good part of Jean Taylor's regularity theorem [Tay] for Almgren almost-minimal sets of dimension 2 in $\mathbb{R}^3$.

Recall that these sets give a very good description of soap films and bubbles, and that Taylor's result says that they are locally C^1-equivalent to minimal cones (i.e., planes or cones of type $\mathbb{Y}$ or $\mathbb{T}$, as described near Figures 1 and 2 and easily seen in soap films). We shall only prove the biHölder equivalence in this text; the C^1-equivalence needs a little bit of extra work, which is left for a next paper [D4].

The proof will extend to 2-dimensional sets in $\mathbb{R}^n$, and will also allow us to show that every MS-minimal set of dimension 2 in $\mathbb{R}^3$ (where the definition of competitors uses the separation constraint in Definition 1.4 below) is a minimal cone. The issue shows up naturally in the study of the Mumford-Shah functional (what are the global Mumford-Shah minimizers (u, K) in $\mathbb{R}^3$ for which u is locally constant), to the point that the author felt compelled to announce Theorem 1.9 in [D2].

The local biHölder equivalence to minimal cones can be useful as a first step to C^1 equivalence, as in [D4], but could also give enough geometric control on the almost-minimal sets to allow one, perhaps, to prove existence results for functionals with a dominant area term under suitable topological constraints.

One of the main initial motivations for this text was to help the author understand Jean Taylor's paper, and possibly help some readers too. Part of the reason for this interest was the potential applications of the technical lemmas to the regularity of minimal segmentations for the Mumford-Shah functional in dimension 3, and indeed A. Lemenant [Le] recently obtained a C^1-regularity result for minimal segmentations in $\mathbb{R}^3$ that look enough like a minimal cone, using this paper and [D4].

Since this was the triggering reason for this paper, let us define MS-minimal sets first. We give the definition for $(n-1)$-dimensional sets in $\mathbb{R}^n$, but for the theorem we shall restrict to $n = 3$.

DEFINITION 1.1. — *Let E be a closed set in $\mathbb{R}^n$. A MS-competitor for E is a closed set $F \subset R^n$ such that we can find $R > 0$ such that*

$$F \setminus B(0, R) = E \setminus B(0, R) \tag{1.2}$$

and

$$\begin{array}{c} F \text{ separates } y \text{ from } z \text{ whenever} \\ y, z \in \mathbb{R}^n \setminus (B(0, R) \cup E) \text{ are separated by } E. \end{array} \tag{1.3}$$

By "F separates y from z", we just mean that y and z lie in different connected components of $\mathbb{R}^n \setminus F$. Note that if (1.2) and (1.3) hold for R, they also hold for every $R' > R$. Here MS stands for Mumford-Shah; indeed the separation condition (1.3) is the same one as in the definition of global minimizers for the Mumford-Shah functional in $\mathbb{R}^n$. We shall discuss this a little more in Section 18, but let us mention here that the MS-minimal sets defined below are exactly the sets E such that the pair $(0, E)$ is a global minimal Mumford-Shah segmentation in $\mathbb{R}^n$. Also see [D2].

DEFINITION 1.4. — *A MS-minimal set in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that*

$$H^{n-1}(E \setminus F) \leqslant H^{n-1}(F \setminus E) \tag{1.5}$$

for every MS-competitor F for E.

Here H^{n-1} denotes the $(n-1)$-dimensional Hausdorff measure. Notice that if E is MS-minimal, then $H^{n-1}(E \cap B(0, R)) < +\infty$ for $R > 0$, because we can compare with $F = [E \setminus B(0, R)] \cup \partial B(0, R)$. We shall also use a weaker notion of minimality, based on a more restrictive notion of competitors. The notion is the same as for Almgren's restricted sets [Al2].

DEFINITION 1.6. — *Let E be a closed set in $\mathbb{R}^n$ and $d \leqslant n - 1$ be an integer. An Al-competitor for E is a closed set $F \subset R^n$ that can be written as $F = \varphi(E)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz mapping such that $W_\varphi = \{x \in \mathbb{R}^n ; \varphi(x) \neq x\}$ is bounded.*

An Al-minimal set of dimension d in $\mathbb{R}^n$ is a closed set $E \subset \mathbb{R}^n$ such that $H^d(E \cap B(0, R)) < +\infty$ for every $R > 0$ and

$$H^d(E \setminus F) \leqslant H^d(F \setminus E) \tag{1.7}$$

for every Al-competitor F for E.

Here it makes sense to define Al-minimal sets of all dimensions $d < n$; this is not the case for MS-minimal sets because of the special role played by separation. Also, the reader may be surprised because Definition 1.6 seems simpler than if we used the standard definition of Almgren restricted sets; we shall check in Section 4 that the two definitions coincide.

Remark 1.8. — Let us check that Al-competitors for E are automatically MS-competitors for E, and hence MS-minimal sets are also Al-minimal. Let F be an Al-competitor for E, let φ and W_φ be as in Definition 1.6, and choose R so that $W_\varphi \cup \varphi(W_\varphi) \subset B(0, R)$. Obviously (1.2) holds. For (1.3),

let y, $z \in \mathbb{R}^n \setminus (B(0, R) \cup E)$ be separated by E. Consider the continuous deformation given by $\varphi_t(x) = (1 - t)x + t\varphi(x)$ for $x \in \mathbb{R}^n$ and $0 \leqslant t \leqslant 1$. Notice that none of the sets $E_t = \varphi_t(E)$ ever contain y or z, because all the modifications occur in $B(0, R)$. Then $F = \varphi_1(E) = E_1$ separates y from z, because $E = E_0$ does. See for instance 4.3 in Chapter XVII of [Du], on page 360. Actually, that result is only stated when E is a compact set of $\mathbb{R}^n$. To accommodate this minor difficulty, we should instead assume that F does not separate y from z, pick a path $\gamma \subset \mathbb{R}^n \setminus F$ from y to z, choose $S > R$ such that $\gamma \subset B(0, S)$, and notice that $E' = [E \cap B(0, S)] \cup \partial B(0, S)$ is compact and separates y from z, but its image $\varphi_1(E') = [F \cap B(0, S)] \cup \partial B(0, S)$ does not (because of γ). From the ensuing contradiction we would deduce that F is a MS-competitor for E, as needed.

Here is a list of MS-minimal sets in $\mathbb{R}^3$. First, the empty set, and planes, are MS-minimal. Next let P be a plane, pick a point $O \in P$, and let S be the union of three half lines in P that start from O and make $120°$ angles with each other at O. The product of S with a line perpendicular to P is MS-minimal. We shall call such a set a cone of type $\mathbb{Y}$; thus cones of type $\mathbb{Y}$ are made up of three half planes with a common boundary L, and that meet along the line L with $120°$ angles. See Figure 1.
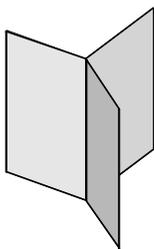

Figure 1. — A cone of type $\mathbb{Y}$
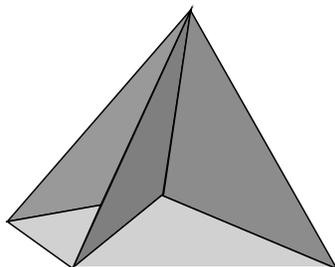
Figure 2. — A set of type $\mathbb{T}$

For the last example, take a regular tetrahedron R, call O its center, and denote by T the cone centered at O over the union of the six vertices of R. See Figure 2. Thus T is composed of six angular faces that meet at O, and are bounded by four half-lines that leave from O with equal (and maximal) angles. We shall call such a set a cone of type $\mathbb{T}$. See K. Brakke's home page http://www.susqu.edu/brakke/ for nicer pictures of T and how Y and T arise as tangent objects to minimal sets.

The verification that cones of type $\mathbb{Y}$ and $\mathbb{T}$ are MS-minimal can be done with a clever integration by parts; see [LM]. It turns out that modulo sets of measure 0, the list is complete.

THEOREM 1.9 . — *If E is an MS-minimal set in $\mathbb{R}^3$, then there is a set E^* such that $E^* \subset E$, $H^2(E \setminus E^*) = 0$, and E^* is the empty set, a plane, or a cone of type $\mathbb{Y}$ or $\mathbb{T}$.*

The fact that we may have to remove a set of vanishing H^2-measure is natural: it is easy to see that the union of a minimal set with a closed set of vanishing H^2-measure is still minimal.

As we shall see, Theorem 1.9 is a rather simple consequence of the proof of Jean Taylor's regularity theorem, but to the author's surprise, it seems that this was not stated in the literature yet. An analogue of this result for size-minimizing currents is stated in Section 4 of [Mo2]; the proof sketched there has the same sort of ingredients as in the present paper, but does not seem to me to apply directly.

The situation for Al-minimal sets is a little less clear; although it seem quite probable that Theorem 1.9 extends to Al-minimal sets in $\mathbb{R}^3$, the author does not know how to prove this without an additional assumption, for instance on the separating properties of E. See Proposition 18.29 and Remark 18.43.

As was hinted before, Theorem 1.9 is much easier to deduce from a proof of Jean Taylor's regularity theorem than from its mere statement; we shall take this as a first excuse to give, in full details, a slightly different proof of that regularity result.

Let us be a little more specific about this. Her theorem concerns almost-minimal sets of dimension 2 in $\mathbb{R}^3$; we shall give very precise definitions of almost-minimal sets, but for the moment let us just say that they are like the minimal sets of Definition 1.6, except that we may add a small term to the right-hand side of (1.7), and also localize the definition. Recall that J. Taylor proves that if E is a reduced almost-minimal set of dimension 2

in $\mathbb{R}^3$, with a sufficiently small gauge function (a function that controls the small term that we add to the right-hand side of (1.7)), then every point of E has a neighborhood where E is C^1-equivalent to a plane or a cone of type $\mathbb{Y}$ or $\mathbb{T}$. [See Definition 2.12 for "reduced sets".]

We shall cut this regularity result in two, and first prove a mere biHölder equivalence to a plane, a $\mathbb{Y}$, or a $\mathbb{T}$, under slightly weaker assumptions on the gauge function. See Theorem 16.1 for a precise statement, and observe that in our definition the biHölder equivalence comes from a local biHölder mapping of the ambient space, so the equivalence also concerns the way E is embedded in $\mathbb{R}^n$. This is a little simpler to prove than the C^1 equivalence, and in particular does not require epiperimetry estimates. This is also the only part of the argument that is needed for Theorem 1.9, and possibly for other results as well.

The full C^1 equivalence (or even $C^{1+\alpha}$ if the gauge function is small enough) seems to need a more precise argument, which uses epiperimetry or some comparison with harmonic functions. We intend to give in [D4] a special argument for this, which uses a little bit of the local biHölder equivalence through separation properties.

Theorem 16.1. also applies to the two-dimensional almost-minimal sets in $\mathbb{R}^n$, $n > 3$. In this case, we don't know the precise list of minimal cones, and by minimal cone of type $\mathbb{T}$ we shall then mean a (reduced) minimal cone T which is neither empty, nor a plane, nor a minimal cone of type $\mathbb{Y}$ as above. As we shall see, these are also the minimal cones whose density at the origin is larger than $3\pi/2$. See Section 14 for a rapid description of these cones. Observe that we do not exclude the possibility that E may have two significantly different tangent cones of type $\mathbb{T}$ at x, and then E will be locally biHölder equivalent to either cone.

The extension of the full Taylor theorem, with the C^1 (and in fact slightly better) equivalence, will be addressed in [D4], but we shall only get it under assumptions on the tangent cones to E. So it could be that when $n > 3$ the existence of epiperimetric inequalities and the local C^1 equivalence of E to a minimal cone near x depends on the type of tangent cones that E has at x. The author does not even know whether E could have an interesting one-parameter family of different tangent cones at x, which would of course exclude the C^1 equivalence if they are not isometric to each other.

We are only interested here in the local regularity of Almgren almost-minimal sets, as defined above, but many other notions exist, corresponding to other approaches to minimal sets and surfaces. The fact that the Lipschitz mappings φ in Definitions 1.6 and 2.7 are not required to be injective is

important. For instance, it prevents the union of two planes from being minimal, and is also the reason why our minimal sets have singularities even in low dimensions. The present definition has the advantage of giving a very good description of soap films and bubbles [Al], [Tay].

Even in the context of soap films and bubbles, there are a few different ways to formalize things, and in particular to state and solve existence problems. Many of them involve currents, in particular because, thanks to the work of Federer, De Giorgi, and many others, currents provide very powerful tools for proving existence results. In such contexts, Almgren almost-minimal sets correspond to size-minimizing currents. In particular, unless the author is mistaken, the support of a size-minimizing integral current away from the boundary constraints is almost-minimal, so J. Talor's result applies to it. The converse is less clear. See for instance [Mo4], [Bo], [R], [SS], [Tam] for different approaches.

Incidentally, the author [D3] dreams of stating and proving existence results in the context of deformations and sets (as opposed to currents), but so far only very simple cases are known [Fv]. For this purpose, getting more precise results on the local (and hopefully boundary) regularity of almost-minimal sets may be useful.

Our proof of biHölder equivalence relies on a few constructions by hand (for instance, to get the monotonicity of density), a result of stability of minimal and almost-minimal sets under Hausdorff limits [D1], and an extension of Reifenberg's topological disk theorem to the case when E is a two-dimensional set in $\mathbb{R}^n$ that stays close to planes or cones of type $\mathbb{Y}$ or $\mathbb{T}$ at all scales and locations [DDT]. In particular, we never use currents or epiperimetry arguments. Even so, we wish to stress the fact that many of our arguments look like existing ones, mostly from [Tay], but also from other sources, such as [Mo2].

Many of the ingredients in the proofs can be generalized, but the author does not know exactly how far the generalizations may lead. Because of this, we shall try to keep a reasonably large amount of generality in the intermediate statements, especially (but not always) when this is not too costly. For the same sort of reasons, the author decided to go ahead and prove in the first sections that various definitions of almost- and quasi-minimality for sets are equivalent, or imply others. Hopefully this will help some readers, and the other ones will not find it too hard to skip.

The plan for the rest of this paper is as follows. Part A is mostly concerned in the definitions, regularity, and limiting properties of almost-minimal sets.

Section 2 contains the definitions and main regularity properties for a slight generalization of Almgren's restricted sets (we shall call them generalized quasiminimal sets); the small amount of new material will only be used to allow us to simplify our notions of almost-minimality (in Section 4), and state some results (like Proposition 5.24 and Theorem 16.1) with a little more generality.

In Section 3 we remind the reader that for sequences of reduced quasiminimal sets with uniform constants, limits are also quasiminimal with the same constants, and we have lower- and upper semicontinuity results for the Hausdorff measure. This stays true for the generalized quasiminimal sets of Section 2. See Lemmas 3.3 and 3.12 in particular.

Section 4 contains various definitions of almost-minimal sets, and the equivalence between the two main ones. Just like Section 2, this one is not really needed in view of Theorems 1.9 and 16.1, for instance.

In Part B we prove the almost monotonicity of the density function for almost-minimal sets with a small enough gauge function, and results of approximation by a minimal cone in annuli where the density is almost constant.

The main result of Section 5 is the fact that if E is a minimal set of dimension d, the density $r^{-d}H^d(E \cap B(x,r))$ is a nondecreasing function of r (Proposition 5.16). The very standard argument is essentially a comparison of $E \cap B(x,r)$ with the cone over $E \cap \partial B(x,r)$. Also see Propositions 5.24 and 5.30 for generalizations to almost-minimal sets.

In Section 6 we show that if E is a reduced minimal set of dimension 2 in $B(0,b) \subset \mathbb{R}^3$ and the density $r^{-d}H^d(E \cap B(0,r))$ is constant on (a,b), then E coincides with a reduced minimal cone on the annulus $\{z \, ; \, a < |z| < b\}$. The proof is surprisingly painful, in the sense that the author did not manage to avoid using the minimality of E again. See Proposition 6.2.

Section 7 contains a variant of Proposition 6.2 for almost-minimal sets and very small variations of density, obtained from it by a compactness argument. See Proposition 7.1 for the initial statement in a cone, Proposition 7.24 for the variant in a ball which is used most of the time, and Proposition 7.31 for the fact that if the density of E at x exists, the tangent objects to E at x are minimal cones with the same density.

Part C contains a structural description of the reduced minimal cones of dimension 2 in $\mathbb{R}^n$. We show that for such a cone E, $K = E \cap \partial B(0,1)$ is composed of a finite number of arcs of great circles, that may only meet by

sets of three at their ends, and with 120° angles. See Proposition 14.1 for a more precise description (but observe that we shall not give a complete list of cones).

We start Section 8 with the simple fact that if the product set $E \times \mathbb{R}^m$ is a reduced Al-minimal set of dimension $d + m$ in $\mathbb{R}^{n+m}$, then E is a d-dimensional Al-minimal set in $\mathbb{R}^n$. See Proposition 8.3. We don't know about, nor need the converse (see Remark 8.22). Proposition 8.3 applies to the blow-up limits of a minimal cone at a point of the unit sphere, and we can use this to prove that if E is a reduced minimal cone of dimension d and $\varepsilon > 0$ is small, then $K = E \cap \partial B(0, 1)$ is ε-close (in normalized Hausdorff distance) to a minimal set of dimension $d - 1$ in every small enough ball centered on K.

This would lead to some information on E, but since we want something more precise, we start a different approach and try to study the local regularity of $K = E \cap \partial B(0, 1)$ with our hands. In section 9 we show that K is a "weakly almost-minimal set" (where "weakly" comes from the fact that we have to pay some price when we start from competitors for K and construct a competitor for E). See Definition 9.1 and Proposition 9.4.

Then we turn more specifically to one-dimensional sets. In Section 10 we show that the nonempty reduced minimal sets of dimension 1 in $\mathbb{R}^n$ are the lines and the propellers Y. See Theorem 10.1. The proof is also used in Section 11 to show that weakly almost-minimal set are often close (in Hausdorff distance) to lines or propellers (Proposition 11.2).

In Section 12 we state a simpler, one-dimensional version of the Reifenberg-type theorem that will be stated in Section 15. This is enough to show that when $K = E \cap \partial B$ as above, K is essentially composed of C^1 arcs. See Proposition 12.6 for a first statement, and Proposition 12.27 for the C^1 version.

In Section 13 we compare the cone over an arc of Lipschitz curve in $\partial B(0, 1)$ with the graph of a harmonic function f; this is a small computation where we use the fact that the main term in the area functional is the energy $\int |\nabla f|^2$, and whose conclusion is that the curve cannot yield a minimal cone unless it is an arc of great circle.

We apply all this in Section 14 and find the description of the two-dimensional minimal cones of Proposition 14.1.

Part D studies the local regularity of the almost-minimal sets. We start in Section 15 with the description of a result from [DDT], where a slight

generalization of Reifenberg's topological disk theorem is proved. This is the result that will be used in Section 16 to produce parameterizations of E when we know that is is well approximated by two-dimensional minimal cones in small balls.

Section 16 contains the main local regularity results for almost-minimal sets of dimension 2 in $\mathbb{R}^n$. Theorem 16.1 is the main statement (the promised generalization of J. Taylor's theorem), but some intermediate results may have independent interest, such as Lemmas 16.19 and 16.48 (which gives sufficient conditions for E to be biHölder equivalent to a plane in a given ball B), Proposition 16.24 (which proves the existence of a point of density $3\pi/2$ in B), Lemmas 16.25 and 16.51 (sufficient conditions for E to be biHölder equivalent in B to a cone of type $\mathbb{Y}$), and Lemma 16.56 and Proposition 18.1 (for the biHölder equivalence to a cone of type $\mathbb{T}$). Proposition 16.24 is proved in Section 17, with a topological degree argument.

Theorem 1.9 is proved in Section 18, with another topological argument. Also see Proposition 18.29 for a version for Al-minimal sets of dimension 2 in $\mathbb{R}^3$, but with an extra separation assumption.

In Section 19 we describe a set which looks like cone of type $\mathbb{T}$ at large scales, but has no point of type T; thus there is no trivial way to make the proof of Theorem 1.9 work for Al-minimal sets. In the very short Section 20 we compute the density of the set T of Figure 1.2.

A. ALMOST-MINIMAL SETS

In this first part we shall give a few different definitions of almost-minimal sets of dimension d in $\mathbb{R}^n$, check that many definitions are equivalent, discuss some of the main properties of these sets (including local Ahlfors-regularity and local uniform rectifiability), and state convergence theorems that will be used in the rest of this paper.

The equivalence between various definitions of almost-minimality is not vital for the rest of the paper, but it will be more convenient to use simple definitions like Definition 1.6, and good to know that they are equivalent to more classical ones, in particular those that come more naturally from

Almgren's definitions in [Al2]. The equivalence is not particularly subtle or original, but some argument is needed, and the author decided that this is a good opportunity to settle the issue as explicitly as possible.

It appears that for the proof of equivalence, it is useful to introduce a class of sets which is slightly larger than the class of restricted sets introduced by F. Almgren, and then check that the main regularity properties of restricted sets still hold in this class. This is what we do in Section 2.

In Section 3, we use these properties to show that the Hausdorff measure has good semicontinuity properties when we restrict to bounded sequences of generalized Almgren quasiminimal sets. In other words, the Hausdorff measure of these sets goes to the limit well, as in Lemma 3.3 and Lemma 3.12 below, and we shall be able to apply this to our various types of almost-minimal sets.

The various definitions of almost-minimality, and the equivalence of the main ones, are given in Section 4. We also use the results of Section 3 to show that limits of almost-minimal sets are almost-minimal; see Lemma 4.7. We try to arrange this text so that the impatient reader can refer to this part only occasionally, for a definition or a believable statement.

2. Generalized Almgren quasiminimal sets

The main results of this paper apply not only to minimal sets, but to almost-minimal sets of various types. We shall give in Section 4 a few definitions of almost-minimal sets and clarify the relations between the various classes, but before we do this we need to talk about Almgren restricted sets (we'll call them quasiminimal), and even generalize the notion slightly.

The standard properties described in this section will be used in Section 4 to prove the equivalence between different notions of almost-minimality, and then also in later sections, but only applied to more restricted classes of almost-minimal sets.

A good part of this is not needed for the main results of the paper, the main point of it is merely to try to give some statements in the best possible context. It should be possible for the reader to skip Sections 2 and 4 almost entirely if she is just interested in minimal sets or happy to use a stronger form of almost-minimality.

So we want to discuss minor variations over the notion of "restricted set" of Almgren [Al2]. For the general setting we are given an open set $U \subset \mathbb{R}^n$, a quasiminimality constant $M \geqslant 1$, and a diameter $\delta \in (0, +\infty]$. Now E is

a closed subset of U, and we want to assume that

$$H^d(E \cap B) < +\infty \ \text{ for every compact ball } B \subset U. \tag{2.1}$$

As in Definition 1.6, we want to compare E with competitors $F = \varphi_1(E)$, where φ_t, $0 \leqslant t \leqslant 1$, is a one-parameter family of continuous functions $\varphi_t : U \to U$ with the following properties:

$$\varphi_0(x) = x \ \text{ for } x \in U, \tag{2.2}$$

$$\text{the function } (t, x) \to \varphi_t(x), \text{ from } [0,1] \times U \text{ to } U, \text{ is continuous}, \tag{2.3}$$

$$\varphi_1 \text{ is Lipschitz}, \tag{2.4}$$

and, if we set

$$W_t = \{x \in U; \varphi_t(x) \neq x\} \ \text{ and then } \widehat{W} = \bigcup_{t \in [0,1]} \Big[W_t \cup \varphi_t(W_t) \Big], \tag{2.5}$$

then

$$\widehat{W} \text{ is relatively compact in } U \ \text{ and } \ \text{diam}(\widehat{W}) < \delta. \tag{2.6}$$

DEFINITION 2.7. — *We say that the set E is an (M, δ)-quasiminimal set in U when E is closed in U, (2.1) holds, and*

$$H^d(E \cap W_1) \leqslant M H^d(\varphi_1(E \cap W_1)) \tag{2.8}$$

for every φ such that (2.2)-(2.6) hold.

Remarks 2.9. — a) This definition is almost the same thing as Almgren's definition of a restricted set. The only difference is that Almgren does not require (2.2) or (2.3). That is, he only considers mappings $\varphi_1 : U \to U$ and does not care whether they come from deformations or not. Our definition of quasiminimal sets is slightly less restrictive (because we add constraints on the mappings φ_1), but this is all right, because the regularity results are the same. In fact, in most regularity arguments for quasiminimal sets, one uses functions φ_1 such that $W_1 \cup \varphi_1(W_1)$ is contained in a compact ball B contained in U and with diameter at most δ. Then φ_1 automatically comes from a one-parameter family of deformations as above, namely $\varphi_t(x) = (1-t)x + t\varphi_1(x)$, for which $\widehat{W}$ is contained in B by convexity.

b) We require φ_1 to be Lipschitz (in (2.4)) only to make sure that (2.8) makes sense, but we do not require any Lipschitz bound for φ_1.

c) It is important that we do not require φ_1 to be injective. That is, we want to allow competitors obtained from E by gluing pieces together, or making entire regions collapse to one point.

d) We can take $\delta = +\infty$ in Definition 2.7. Our definition of *Al*-minimality in Definition 1.6 corresponds to $U = \mathbb{R}^n$, $\delta = +\infty$, and $M = 1$. In addition, the accounting in (1.7) is not the same as in (2.8), but we shall see soon that in this case they are equivalent. In general δ and U give more flexibility in the definition; they may allow us to localize or account for boundary conditions. For instance, catenoids are $(1, \delta)$-quasiminimal sets in $\mathbb{R}^3$, but only for δ small enough.

The quasiminimal sets of Definition 2.7 are not yet general enough for us, so we introduce an even larger class. This extension will be needed only because we want to work with different notions of almost-minimal sets; the uninterested reader may skip.

DEFINITION 2.10. — *Let U and M be as above, and let $\delta_0 \in (0, +\infty]$ and $h \in [0, 1)$ be given. We shall denote by $GAQ(M, \delta_0, U, h)$ the set of closed subsets E of U such that (2.1) holds and*

$$H^d(E \cap W_1) \leqslant M H^d(\varphi_1(E \cap W_1)) + h\delta^d \qquad (2.11)$$

for every $\delta \leqslant \delta_0$ and every φ such that (2.2)-(2.6) hold.

Elements of $GAQ(M, \delta_0, U, h)$ will be called "generalized Almgren quasi-minimal sets". The only difference with the quasiminimal sets from Definition 2.7 is that we added the error term $h\delta^d$ in the right-hand side of (2.11). So $h = 0$ corresponds to the previous situation. We should always think that h is small, because in all the regularity results below, h has to be small enough, depending on n, d, and M. Also notice that the only situations where we expect a real difference between (2.11) and (2.8) (possibly with a larger M) is when $H^d(E \cap W_1)$ and $H^d(\varphi_1(E \cap W_1))$ are very small, i.e., when φ_1 only moves very few points. The main point of the discussion below is that these situations can be avoided when we prove regularity theorems.

We like quasiminimal sets because they have lots of good regularity properties. We want to say that these properties still hold for generalized quasiminimal sets (always assuming that h is small enough), and that the long proofs given elsewhere still work. So, for the rest of this section, we shall try to describe the regularity results that we want to use, and at the same time say why the proof for quasiminimal sets can be generalized rather easily.

Notice that if E is a quasiminimal set and Z is any closed set such that $H^d(Z) = 0$, then $E \cup Z$ is also a quasiminimal set, with the same constants. For instance, in the situation of (2.8), we have that $H^d((E \cup Z) \cap W_1) = H^d(E \cap W_1) \leqslant M H^d(\varphi_1(E \cap W_1)) \leqslant M H^d(\varphi_1((E \cup Z) \cap W_1))$. This is why we introduce the (hopefully simpler) set E^* below, and the notion of "reduced set".

DEFINITION 2.12. — *For each closed subset E of U, denote by*

$$E^* = \big\{ x \in E \,;\, H^d(E \cap B(x,r)) > 0 \text{ for every } r > 0 \big\}$$

the closed support (in U) of the restriction of H^d to E. We say that E is <u>reduced</u> when $E^* = E$.

It is easy to see that

$$H^d(E \setminus E^*) = 0; \tag{2.13}$$

indeed, we can cover $E \setminus E^*$ by countably many balls B_j such that $H^d(E \cap B_j) = 0$ (first reduce to $E \cap K$, where K is compact and does not meet E^*, and then use the definition of E^* and the compactness of $E \cap K$).

Remark 2.14. — If E is quasiminimal, then E^* is quasiminimal with the same constants. This is very easy, because when the φ_t satisfy (2.2)-(2.6), $H^d(\varphi_1(E \setminus E^*)) = H^d(E \setminus E^*) = 0$ by (2.13) and because φ_1 is Lipschitz, so the conditions (2.8) and (2.11) are the same for E^* as for E. Because of this, it is enough to study reduced quasiminimal sets. The same remark will apply to the various types of almost-minimal sets introduced in Section 4.

We start our list of regularity properties with the basic but useful local Ahlfors-regularity of E^*.

LEMMA 2.15. — *For each $M \geqslant 1$ there exist constants $h > 0$ and $C \geqslant 1$, that depend only on n, d, and M, such that if $E \in GAQ(M, \delta, U, h)$, $x \in E^*$, and $0 < r < \delta$ are also such that $B(x, 2r) \subset U$, then*

$$C^{-1} r^d \leqslant H^d(E \cap B(x,r)) \leqslant C r^d. \tag{2.16}$$

See Proposition 4.1 in [DS] for standard quasiminimal sets, but note that this is only a reasonably minor modification of a result in [Al2]. Now we should say a few words about why the proof of (2.16) in [DS] extends to generalized quasiminimal sets. For the upper bound, there is no serious problem; the idea of the proof is to suppose that for some cube Q, $\operatorname{diam}(Q)^{-d} H^d(E \cap Q)$ is very large, and then compare E with a set obtained from E by a Federer-Fleming projection on d-dimensional skeletons

of cubes inside Q. The numbers at stake in the argument are much larger than $\operatorname{diam}(Q)^d$, and the additional small error $h \operatorname{diam}(Q)^d$ does not really change the estimates. More specifically, the first place where quasiminimality is used (in [DS]) is (4.5), and we should add an extra $Ch \operatorname{diam}(Q)^d$ in the right-hand side there. [Let us point out to the reader that would wish to check things in [DS], that our standard quasiminimal set is called S there, and the quasiminimality constant M is called k.] The computations can stay the same, up to (4.9) and (4.11) where we should also add an extra $Ch \operatorname{diam}(Q)^d$ in the right-hand side. Even with $h = 1$, this term is easily eaten up by the other term $C2^{j(n-d)} \operatorname{diam}(Q)^d$ in the right-hand side of (4.9) and (4.11) (because $j \geqslant 1$); the rest of the proof is the same.

The proof of the lower bound in (2.16) is probably the place where we should be the most careful, since the mass of E nearby could be very small and $h\delta^d$ could be the main term in (2.10).

In the argument of [DS] (which begins a little below (4.21) there), we start with a cube Q, assume that $\operatorname{diam}(Q)^{-d}H^d(E \cap Q)$ is very small, and then we construct a competitor $F = \varphi(E)$ (with the constraints (2.2)-(2.6) with any $\delta > \operatorname{diam}(Q)$). The cube Q divides into a thin annular region $A_j(Q)$ near the boundary ∂Q, and the main part $H_j(Q) = Q \setminus A_j(Q)$. Here j is an integer parameter, and the width of $A_j(Q)$ is roughly $2^{-j} \operatorname{diam}(Q)$. In the region $H_j(Q)$, we do a Federer-Fleming projection on skeletons of cubes of size $2^{-j} \operatorname{diam}(Q)$, and it turns out that if $\operatorname{diam}(Q)^{-d}H^d(E \cap Q)$ is small enough, depending on j, the projection contains no full face of dimension d. This allows us to continue the Federer-Fleming projection one more step, up to $(d-1)$-dimensional faces. As a result, our final mapping φ is such that $H^d(\varphi(E \cap H_j(Q))) = 0$. In the intermediate region $A_j(Q)$, we construct a mapping that matches the desired values on the two sides of $\partial A_j(Q)$. We do this so that $H^d(\varphi(E \cap A_j(Q))) \leqslant CH^d(E \cap A_j(Q))$. The comparison argument in [DS] yields $H^d(E \cap H_j(Q)) \leqslant CMH^d(E \cap A_j(Q))$, where M is the quasiminimality constant. See (4.23)-(4.25) in [DS], and the sentence that follows. Here it should be

$$H^d(E \cap H_j(Q)) \leqslant CMH^d(E \cap A_j(Q)) + Ch \operatorname{diam}(Q)^d, \qquad (2.17)$$

because of the extra term in (2.10). It is important that the constant C does not depend on j, so that we can still take j as large as we wish; the only price to pay is that we have to assume that $\operatorname{diam}(Q)^{-d}H^d(E \cap Q)$ is accordingly small at the beginning of the argument, if we want our story about no full face in $H_j(Q)$, and then (2.17), to hold.

The way we can deduce the lower bound in (2.16) from this is slightly different from what was done in [DS]. First we observe that it is enough to

prove this lower bound for $x \in E$ such that

$$\limsup_{r \to 0} \left[r^{-d} H^d(E \cap B(x, r)) \right] \geqslant c_0, \tag{2.18}$$

where c_0 is some positive dimensional constant. Indeed, if c_0 is chosen small enough, (2.18) holds for almost-every $x \in E$ (see for instance [Ma], Theorem 6.2 on p.89, or even [D2], p.113). Then every point $x \in E^*$ is a limit of points $x_k \in E$ such that (2.18) holds, and we easily get the desired lower bound on $H^d(E \cap B(x, r))$ by using the lower bound that we can get for a smaller ball centered at some x_k.

Suppose that (2.18) holds, but not the lower bound in (2.16). Then we can find $r > 0$ such that $r < \delta$, $B(x, 2r) \subset U$, but $H^d(E \cap B(x, r)) < \alpha r^d$, where α is the inverse of C in (2.16), and can be chosen as small as we want. Set $\eta = 2^{-d-3}$. If α is sufficiently small compared to c_0, we can pick a new radius $r_1 < r$ such that

$$H^d(E \cap B(x, r_1)) \leqslant \alpha r_1^d \tag{2.19}$$

but

$$H^d(E \cap B(x, \eta r_1)) \geqslant \alpha(\eta r_1)^d. \tag{2.20}$$

Indeed, we can try successively all the radii $\eta^k r$, $k \geqslant 0$; if none of them works, we can easily prove by induction that $H^d(E \cap B(x, \eta^k r)) < \alpha(\eta^k r)^d$ for every $k > 0$, but if α is small enough this gives a very small lower bound on $t^{-d} H^d(E \cap B(x, t))$ for every $t \leqslant r$, which contradicts (2.18).

Choose r_1 as in (2.19) and (2.20). Now we can pick a cube Q centered at x, such that

$$B(x, 2\eta r_1) \subset Q \subset B(x, r_1/2) \tag{2.21}$$

and

$$H^d(E \cap A_j(Q)) \leqslant C2^{-j} H^d(E \cap B(x, r_1)) \leqslant C2^{-j} \alpha r_1^d. \tag{2.22}$$

Indeed we can easily find $C^{-1} 2^j$ cubes Q such that (2.21) holds, and for which the $A_j(Q)$ are disjoint, so (2.22) is just obtained by picking the one for which $H^d(E \cap A_j(Q))$ is the smallest. On the other hand,

$$H^d(E \cap H_j(Q)) \geqslant H^d(B(x, \eta r_1)) \geqslant \alpha \eta^d r_1^d \tag{2.23}$$

by (2.21) and (2.20), so (2.22) says that

$$H^d(E \cap A_j(Q)) \leqslant C2^{-j} H^d(E \cap H_j(Q)). \tag{2.24}$$

Recall that we can take j as large as we want. If we do so, and then take α small enough (to make sure that (2.17) applies to Q), the term

$CMH^d(E \cap A_j(Q))$ in (2.17) is smaller than $\frac{1}{2}H^d(E \cap H_j(Q))$, and so (2.17) says that $H^d(E \cap H_j(Q)) \leqslant 2Ch \operatorname{diam}(Q)^d$. Recall that $\operatorname{diam}(Q)$ and r_1 are comparable, by (2.21). So if h is now chosen small enough, we get the desired contradiction with (2.23). This completes our proof of Lemma 2.15. $\square$

The next important property of quasiminimal sets is the local uniform rectifiability of E^*. Let us not give a precise statement here, and refer to Theorem 2.11 in [DS] instead. We claim that this extends to $E \in GAQ(M, \delta, U, h)$, provided again that we take h small enough, depending on n, d, and M.

The proof in [DS] is rather long and complicated, but fortunately a lot of it consists in manipulations on Ahlfors-regular sets, and there are only a limited number of places where the quasiminimality of E is used.

In Chapter 5, it is used to get a Lipschitz mapping from E to $\mathbb{R}^d$, with big image. Quasiminimality is used in (5.7), but not in (5.8)-(5.10) (that is, only through Ahlfors-regularity), so the effect of an additional $Ch \operatorname{diam}(Q)^d$ in the right-hand side of (5.7) is just an extra $Ch \operatorname{diam}(Q)^d$ in (5.11). If h is small enough, this is easily eaten up by the $C^{-1} \operatorname{diam}(Q)^d$ in the left-hand side, and we can continue the argument with no further modification.

In Chapter 6 we show that if $E \subset U \subset \mathbb{R}^n$ is a quasiminimal set and $g : U \to R^l$ is Lipschitz, the new set $\widehat{S} = \{(x, y)\, ; \, x \in S \text{ and } y = g(x)\} \subset \mathbb{R}^{n+l}$ is a quasiminimal set too. Quasiminimality is used in (6.10), and our extra $Ch \operatorname{diam}(Q)^d$ gets multiplied by a constant C (that depend also on the Lipschitz constant for g) in (6.12) and (6.13). Then we add things that do not depend on quasiminimality, and eventually we get an extra $Ch \operatorname{diam}(Q)^d$ in (6.16). That is, $\widehat{S} \in GAQ(\widehat{M}, \delta, \widehat{U}, Ch)$, where $\widehat{U} = U \times R^l$, $\widehat{M}$ is a constant that depends on M, n, d and the Lipschitz constant for g, and C also depends on these constants.

Chapter 7 only talks about general Ahlfors-regular sets, and Chapter 8 talks about how to deduce the local uniform rectifiability of E^* from a main lemma, but only uses results from previous chapters, with no comparison argument. We can also relax through most of Chapter 9, because the first sections concern a deformation lemma that has nothing to do with quasiminimal sets. Even Section 9.2, whose goal is to apply the deformation lemma to quasiminimal sets, starts with a general discussion with dyadic cubes and Ahlfors-regularity. Quasiminimality itself appears only on page 75, where a deformation ϕ is constructed. Thus ϕ in [DS] plays the role of φ_1 in (2.2)-(2.6), and W in [DS] is the same as W_1 here. Recall also that S and k there play the roles of E and M here.

By (9.92), $W \cup \phi(W) \subset B'$, where B' is a ball of radius $C_3 N + nN + C$ that is compactly contained in U (by (9.56)). In this argument, N is a very large integer, to be chosen near the end, and we shall see soon that for the present discussion we do not really need to know who the large constant C_3 is, because our small constant h can be chosen last. We can make sure that our set $\widehat{W}$ from (2.5) is also contained in B', because B' is convex (see Remark 2.9.a above).

The quasiminimality of E is used in (9.93), and we should add an extra $h \operatorname{diam}(B') \leqslant h(C + C_3) N^d$ to the right-hand side, because of the additional term in (2.11). Then there are a few inclusions and identities, and (9.93) is not used until (9.103), where we should also add ChN^d to the right-hand side (let us drop the dependence on C_3 and other constants). Thus (9.103) should be replaced with

$$H^d(E \cap W) \leqslant CMN^{d-1} + ChN^d. \tag{2.25}$$

The computations after (9.103) can stay the same; they just use previous results to show that $H^d(E \cap W) \geqslant C^{-1} N^d$ (see (9.105) and two lines above it). The desired contradiction is then derived by choosing N large enough and, in our case, h small. This takes care of local uniform rectifiability.

We are also interested in Chapter 10 of [DS], where one shows that E^* also has big projections locally. Quasiminimality shows up in (10.22), which should be replaced with

$$H^d(E \cap W) \leqslant MH^d(\phi(E \cap W)) + Chr_1^d. \tag{2.26}$$

But since we see (independently) in (10.25) that $H^d(E \cap W) \geqslant C^{-1} r_1^d$, the extra term in (2.26) can be eaten up by the term in the left-hand side, and then one can proceed as in [DS].

This completes our discussion of the extension of results from [DS], and in particular of Theorem 2.11 there, to generalized quasiminimal sets in $GAQ(M, \delta, U, h)$. But we are not finished yet, because now we want to extend results from [D1], that concern limits of sequences of quasiminimal sets.

3. Limits of quasiminimal sets

We want to show that the Hausdorff measure goes to the limit well along bounded sequences of generalized quasiminimal sets; this will be used in the next section to prove the equivalence of various definitions, and also in later sections, although we shall only apply it to almost-minimal sets then.

Let $U \subset \mathbb{R}^n$, $M \geqslant 1$, $\delta > 0$ and $h \geqslant 0$ be given, and let $\{E_k\}$ be a sequence in $GAQ(M, \delta, U, h)$. We shall assume that the E_k are reduced, i.e, that $E_k^* = E_k$ (otherwise, we could always replace E_k with E_k^*, which is a generalized quasiminimal set with the same constants by Remark 2.14).

We shall also assume that $\{E_k\}$ converges to a closed set E in U. By this we mean that for each compact set $H \subset U$,

$$\lim_{k \to +\infty} d_H(E, E_k) = 0, \qquad (3.1)$$

where the local variant d_H of the Hausdorff distance is defined by

$$d_H(E, F) = \sup\{\operatorname{dist}(x, F)\,;\, x \in E \cap H\} + \sup\{\operatorname{dist}(x, E)\,;\, x \in F \cap H\}. \qquad (3.2)$$

We use the convention that $\sup\{\operatorname{dist}(x, F)\,;\, x \in E \cap H\} = 0$ when $E \cap H$ is empty. By the way, if we did not assume the E_k to be reduced, they could converge to essentially anything, because we could not keep track of the $E_k \setminus E_k^*$. Indeed it is easy to find sequences of finite sets that converge to any given closed set E.

LEMMA 3.3. — *Let U, M, δ, h, $\{E_k\}$, and E be as before, and assume that h is small enough, depending on n, d, and M. Then E is reduced,*

$$H^d(E \cap V) \leqslant \liminf_{k \to +\infty} H^d(E_k \cap V) \text{ for every open set } V \subset U, \qquad (3.4)$$

and

$$E \in GAQ(M, \delta, U, h) \text{ (i.e., with the same constants).} \qquad (3.5)$$

When $h = 0$, this is the main result of [D1]. Here we just have to say why the proof of [D1] goes through when h is small. As we shall see soon, E is locally Ahlfors-regular, so it is reduced. For (3.4) (i.e., Theorem 3.4 in [D1]), there is nothing to do: the proof just consists in putting together results from [DS] that we already discussed in Section 2. So we may now discuss the proof of (3.5) (Theorem 4.1 in [D1]).

The verification that E is locally Ahlfors-regular, with big pieces of Lipschitz graphs, does not need to be modified (it is again a mechanical consequence of the results of Section 2). The same comment applies to the remarks about rectifiability and the existence of tangent planes, which leads us near (4.13) in [D1].

Then we come to the verification of quasiminimality itself. Let us remind the reader of how it goes. We are given a deformation $f : U \to U$, which can

be written as $f = \varphi_1$ for a family of mappings φ_t that satisfy (2.2)-(2.6), where (2.6) holds for some $\delta' \leqslant \delta$, and we want to prove (2.11), i.e., that

$$H^d(E \cap W_f) \leqslant M H^d(f(E \cap W_f)) + h(\delta')^d, \qquad (3.6)$$

where we set $W_f = \{x \in U; f(x) \neq x\}$. We know that the sets E_k have the same property, but we cannot take limits directly, because $H^d(f(E \cap W_f))$ could be much smaller than the $H^d(f(E_k \cap W_f))$. A typical unfriendly situation where this may happen would be when f is not injective on E and sends two different pieces of E to the same set, while it is still injective on each of the E_k.

So we replace f with a different deformation h which does not have the same defect. That is, we try to force h to merge points systematically near the places where we know that it merges points of E. This is easier to do at places where E is close to a tangent plane P and f is close to a linear function A, because there we can compose f with the orthogonal projection onto $A(P)$. A good part of the argument of [D1] consists in doing this in many places, gluing the different functions that we construct, and checking that the places where we do the gluing or where we cannot control things do not matter too much, because the part of E that lives there has small measure.

Fortunately we can keep the construction of h as it is in [D1]; we just have to say how the final estimates go. Things start being dangerous near (4.94), when the function h is finally defined and we introduce its variant h_1. Quasiminimality is used in (4.95), with the function h_1 and the set E_k. So we add the usual $Ch(\delta')^d$ to the right-hand side of (4.95), and get that

$$H^d(E_k \cap W_{h_1}) \leqslant M H^d(h_1(E_k \cap W_{h_1})) + h(\delta')^d. \qquad (3.7)$$

Then we quietly follow the computations, up to (4.107) which gives an upper bound on $H^d(h_1(E_k \cap W_{h_1}))$. We pick the various constants as explained after (4.107), so as to get the equivalent but simpler (4.108), and then prove that

$$H^d(h_1(E_k \cap W_{h_1})) \leqslant H^d(f(E \cap W_f)) + 2\eta_1, \qquad (3.8)$$

where η_1 is as small as we want. In [D1], (3.8) follows from (4.108) (our upper bound for $H^d(h_1(E_k \cap W_{h_1}))$) and a lower bound (4.109) for $H^d(f(E \cap W_f))$. This bound itself is the direct consequence of Lemma 4.111.

Unfortunately, we need to say a little more about the proof of Lemma 4.111, because it is very badly written. The author claims at the beginning of the proof that E is a quasiminimal set, and this is a rather stupid thing to say, because this is more or less what we try to prove. So the reader is

asked not to pay attention to any of the remarks about quasiminimality, and follow quietly the construction of the sets S_1 (just the intersection of E with a ball B_1), then $S_2 = \{(x, f(x)) \, ; \, x \in S_1\} \subset \mathbb{R}^{2n}$, and then S_3, which is the image of S_2 by some simple affine bijection φ of $\mathbb{R}^{2n}$, whose goal is just to make some tangent plane horizontal. Then we prove (4.113), which says that S_3 stays very close to a horizontal d-plane in some ball. This part only uses the geometry of the construction, so it is all right.

Now replace E with E_k; this gives a new set $S_{3,k}$, and this set is a quasi-minimal set, by the same argument that was falsely applied to E in [D1]; this involves constructions from [DS], in particular Chapter 5, which also apply to generalized quasiminimal sets, by Section 2. In addition, E_k also satisfies the distance estimate (4.113) for k large, because the E_k converge to E. So we can apply Lemma 2.18 in [D1] (or Chapter 10 of [DS]) to get that for k large, the projection of $S_{3,k}$ on some d-plane contains some fixed ball. Then this is also true of S_3, by compactness, and we can conclude as was claimed in [D1].

So (3.8) holds, and hence

$$H^d(E_k \cap W_{h_1}) \leqslant M H^d(f(E \cap W_f)) + 2M\eta_1 + h(\delta')^d \qquad (3.9)$$

by (3.7). We also know that

$$H^d(E \cap W_{h_1}) \leqslant H^d(E_k \cap W_{h_1}) + \eta_1 \qquad (3.10)$$

for k large, by (3.4). Recall from (4.94) in [D1] and the line below it that $W_{h_1} \subset W_f$, but we may choose the function φ in (4.94) so that the difference is as close to ∂W_f as we want. As a result, we can make $H^d(E \cap W_f \setminus W_{h_1})$ (incidentally, another typo in [D1]) as small as we want. When we combine this with (3.10) and (3.9), and then let small constants like η_1 tend to 0, we get that

$$H^d(E \cap W_f) \leqslant M H^d(f(E \cap W_f)) + h(\delta')^d. \qquad (3.11)$$

This is just the same as (2.11), and this completes our proof of Lemma 3.3. $\square$

We shall also need an upper semicontinuity result for H^d, which unfortunately the author did not care to write down in [D1].

LEMMA 3.12. — *Let* $U, M, \delta, h, \{E_k\}$, *and* E *be as in Lemma 3.3. In particular, assume that* h *is small enough, depending on* d *and* M. *Then*

$$\limsup_{k \to +\infty} H^d(E_k \cap H) \leqslant (1 + Ch) M H^d(E \cap H) \text{ for every compact set } H \subset U.$$

$$(3.13)$$

Here C *depends only on* d *and* M.

This will be a consequence of the rectifiability of E. Since E is rectifiable, E has a d-dimensional "approximate tangent plane" P_x at x for H^d-almost every $x \in E$. See for instance [Ma], Theorem 15.19 on page 212. Since in addition E is locally Ahlfors-regular (i.e., (2.16) holds), P_x is a true tangent plane at x. See for instance [D2], Exercise 41.21 on page 277. In addition,

$$\lim_{r \to 0} r^{-d} H^d(E \cap B(x,r)) = c_d \qquad (3.14)$$

for H^d-almost every $x \in E$, where c_d is the H^d-measure of the unit ball in $\mathbb{R}^d$. See again [Ma], Theorem 16.2 on page 222. Call E' the set of points $x \in E$ such that E has a tangent plane P_x at x and (3.14) holds; thus almost-every point of E lies in E'.

Let $\varepsilon > 0$ be small, and denote by $\mathcal{B}$ the set of balls $B = B(x,r)$ such that $x \in E' \cap H$, $2r < \delta$,

$$B(x, 2r) \subset \widehat{H}, \qquad (3.15)$$

where $\widehat{H}$ is some compact subset of U that we choose so that $H \subset \operatorname{int}(\widehat{H})$,

$$H^d(E \cap \partial B) = 0, \qquad (3.16)$$

$$\operatorname{dist}(y, P_x) \leqslant \varepsilon r \quad \text{for } y \in E \cap B(x, 2r), \qquad (3.17)$$

and

$$|r^{-d} H^d(E \cap B(x,r)) - c_d| \leqslant \varepsilon. \qquad (3.18)$$

Notice that for $x \in E' \cap H$, $B(x,r) \in \mathcal{B}$ for almost-every small enough r: (3.17) and (3.18) come from the definition of E', and the set of small r such that (3.16) fails is at most countable, because $H^d(E)$ is locally finite. Thus the balls B, $B \in \mathcal{B}$, form what is called a Vitali covering of $E' \cap H$. Let us check that there is an at most countable collection $\{B_i\}_{i \in I}$ in $\mathcal{B}$ such that the B_i, $i \in I$, are disjoint and almost cover $E' \cap H$ in the sense that

$$H^d\big([E' \cap H] \setminus \bigcup_{i \in I} B_i\big) = 0. \qquad (3.19)$$

Let us define successive finite collections $\{B_i\}_{i \in I_k}$. Start with $I_0 = \emptyset$. If I_k was already defined, set $H_k = [E' \cap H] \setminus \bigcup_{i \in I_k} \overline{B}_i$. For each $x \in H_k$, there are lots of balls $B \in \mathcal{B}$ centered on x, and such that $5B$ does not meet the B_i, $i \in I_k$. This gives a covering of H_k by balls of $\mathcal{B}$; by the standard 5-covering lemma (see the first pages of [St]), we can find an at most countable subcollection B_i, $i \in J_k$, such that the B_i, $i \in J_k$, are disjoint, and the $5B_i$, $i \in J_k$, cover H_k. Then

$$\sum_{i \in J_k} H^d(E \cap B_i) \geqslant C^{-1} \sum_{i \in J_k} H^d(E \cap 5B_i) \geqslant C^{-1} H^d(H_k), \qquad (3.20)$$

by (2.16) or (3.18) and because the $5B_i$ cover H_k. Notice that

$$\sum_{i \in J_k} H^d(E \cap B_i) \leqslant H^d(E \cap \widehat{H}) < +\infty \qquad (3.21)$$

because the B_i are disjoint and contained in $\widehat{H}$ (by (3.15)), $\widehat{H}$ can be covered by a finite collection of compact balls in U, and by (2.1). So we can pick a finite collection $J'_k \subset J_k$, so that

$$\sum_{i \in J'_k} H^d(E \cap B_i) \geqslant \frac{1}{2} \sum_{i \in J_k} H^d(E \cap B_i) \geqslant (2C)^{-1} H^d(H_k), \qquad (3.22)$$

and then set $I_{k+1} = I_k \cup J'_k$. This gives a definition of I_k by induction, and then we take $I = \cup_k I_k$. The B_i, $i \in I$, are disjoint by construction. Then

$$\sum_k \sum_{i \in J'_k} H^d(E \cap B_i) \leqslant H^d(E \cap \widehat{H}) < +\infty, \qquad (3.23)$$

by the proof (3.21). In particular, the series in k converges, and (3.22) says that $H^d(H_k)$ tends to 0. But

$$H^d\Big([E' \cap H] \backslash \bigcup_{i \in I} B_i\Big) \leqslant H^d\Big([E' \cap H] \backslash \bigcup_{i \in I_k} B_i\Big) = H^d\Big([E' \cap H] \backslash \bigcup_{i \in I_k} \overline{B}_i\Big) = H^d(H_k),$$
$$(3.24)$$

by (3.16). This proves (3.19).

Let us cover $[E \cap H] \backslash \bigcup_{i \in I} B_i$ by a collection of open balls B_j, $j \in J$, such that $\mathrm{diam}(B_j) < \delta$, $2B_j \subset U$ for each j, and

$$\sum_{j \in J} r_j^d \leqslant \varepsilon, \qquad (3.25)$$

where r_j denotes the radius of B_j. Now the balls B_i, $i \in I \cup J$, cover $E \cap H$, and by compactness we can find finite sets $I' \subset I$ and $J' \subset J$ such that $E \cap H \subset \bigcup_{i \in I' \cup J'} B_i$. Since the E_k converges to E, we also get that

$$E_k \cap H \subset \bigcup_{i \in I' \cup J'} B_i \qquad (3.26)$$

for k large enough, hence

$$H^d(E_k \cap H) \leqslant \sum_{i \in I' \cup J'} H^d(E_k \cap B_i) \leqslant \sum_{i \in I'} H^d(E_k \cap B_i) + C\varepsilon, \qquad (3.27)$$

by (2.16) and (3.25). Next pick $i \in I'$, and write $B_i = B(x, r)$. Notice that for k large,

$$\mathrm{dist}(y, P_x) \leqslant 2\varepsilon r \quad \text{for } y \in E_k \cap B(x, 3r/2), \qquad (3.28)$$

by (3.17) and because the E_k converge to E. Also recall that we only have a finite set I' of indices i, so for k large, (3.28) holds for all of them at the same time. In addition, (3.28) also holds if we replace P_x with any plane P parallel to P_x, and close enough to it.

We want to use the quasiminimality of E_k, so we define a Lipschitz deformation φ. Take

$$\varphi(y) = y \ \ \text{for} \ \ y \in [\mathbb{R}^n \setminus B(x,r)] \cup \{y \in \mathbb{R}^n \,;\, \text{dist}(y,P) \geqslant 3\varepsilon r\}, \quad (3.29)$$

then call π the orthogonal projection onto P, and set

$$\varphi(y) \ \ = \pi(y) \text{ on the inside region } \ D = \big\{y \in B(x,(1-\varepsilon)r) \,; \atop 0 \leqslant \text{dist}(y,P) \leqslant 2\varepsilon r\big\}. \quad (3.30)$$

On the remaining annular region $A = \{y \in B(x,r) \,;\, \text{dist}(y,P) < 3\varepsilon r\} \setminus D$ we choose φ so that $\varphi(A) \subset \overline{B}(x,r)$ and φ is 10-Lipschitz. For instance, take $\varphi(x) = \theta(y)y + (1-\theta(y))\pi(y)$ for $y \in A$, with $\theta(y) = \text{Min}(1, (2\varepsilon r)^{-1}\,\text{dist}(y,D))$. [We added a factor 2 to make sure that $\varphi(A) \subset \overline{B}(x,r)$ if P is close enough to P_x, even if it does not exactly go through x.]

Set $\varphi_t(y) = t\varphi(y) + (1-t)y$ for $0 \leqslant t \leqslant 1$; this family obviously satisfies (2.2)-(2.6), with $\widehat{W} \subset \overline{B}_i$. Recall that $\text{diam}(B_i) = 2r < \delta$ by definition of $\mathcal{B}$. So we may apply Definition 2.10, and (2.11) holds (with any $\delta' \in (2r, \delta)$). That is,

$$H^d(E_k \cap W_1) \leqslant M H^d(\varphi_1(E_k \cap W_1)) + (2r)^d h. \quad (3.31)$$

Set $B' = B(x, (1-\varepsilon)r)$ and notice that if $y \in E_k \cap B'$, then $\text{dist}(y,P) \leqslant 2\varepsilon r$ by (3.28), so $\varphi(y) = \pi(y)$ by (3.30). Thus $\varphi(y) = y$ if and only if $y \in P$, and

$$E_k \cap W_1 \cap B' = \big\{y \in E_k \cap B' \,;\, 0 < \text{dist}(y,P) \leqslant 2\varepsilon r\big\} = E_k \cap B' \setminus P. \ (3.32)$$

There are at most countably many planes P parallel to P_x and for which $H^d(E_k \cap B(x,r) \cap P) > 0$, so we may choose P such that this is not the case. Then

$$\begin{aligned}
H^d(E_k \cap B') = H^d(E_k \cap W_1 \cap B') &\leqslant M H^d(\varphi(E_k \cap W_1)) + (2r)^d h \\
&\leqslant M H^d(\varphi(E_k \cap B')) + M H^d(\varphi(E_k \cap B(x,r) \setminus B')) + (2r)^d h \\
&\leqslant M H^d(\varphi(E_k \cap B')) + 10^d M H^d(E_k \cap B(x,r) \setminus B') + (2r)^d h
\end{aligned}$$
$$(3.33)$$

by (3.32) and (3.31), because $\varphi_1 = \varphi$, and because $E_k \cap W_1 \subset E_k \cap B(x,r) \subset (E_k \cap B') \cup (E_k \cap B(x,r) \setminus B')$ and φ is 10-Lipschitz. Next let us check that

$$H^d(E_k \cap B(x,r) \setminus B') \leqslant C\varepsilon r^d, \quad (3.34)$$

where C may depend on the quasiminimality constants, but not on ε. We can cover $P_x \cap B(x,r) \setminus B'$ by $C\varepsilon^{-d+1}$ balls $B(z_j, \varepsilon r)$ centered on $P_x \cap B(x,r) \setminus B'$. By (3.28), the $B(z_j, 3\varepsilon r)$ cover $E_k \cap B(x,r) \setminus B'$. Notice also that the $B(z_j, 10\varepsilon r)$ are contained in U, by (3.15). Then $H^d(E_k \cap B(z_j, 3\varepsilon r)) \leqslant C\varepsilon^d r^d$ by (2.16), because the E_k all lie in the same $GAQ(M, \delta, U, h)$, and if h is small enough; (3.34) follows. Now

$$
\begin{aligned}
H^d(E_k \cap B(x,r)) &= H^d(E_k \cap B') + H^d(E_k \cap B(x,r) \setminus B') \\
&\leqslant MH^d(\varphi(E_k \cap B')) + CH^d(E_k \cap B(x,r) \setminus B') + (2r)^d h \\
&\leqslant MH^d(\varphi(E_k \cap B')) + C\varepsilon r^d + (2r)^d h
\end{aligned}
\tag{3.35}
$$

by (3.33) and (3.34). Recall from (3.28) and (3.30) that $\varphi(y) = \pi(y)$ for $y \in E_k \cap B'$, so $\varphi(E_k \cap B') \subset P \cap B(x,r)$ and

$$
\begin{aligned}
H^d(E_k \cap B(x,r)) &\leqslant MH^d(P \cap B(x,r)) + C\varepsilon r^d + (2r)^d h \\
&\leqslant Mc_d r^d + C\varepsilon r^d + (2r)^d h,
\end{aligned}
\tag{3.36}
$$

where c_d still denotes the H^d-measure of the unit ball in $\mathbb{R}^d$. On the other hand, (3.18) says that

$$
H^d(E \cap B(x,r)) \geqslant c_d r^d - \varepsilon r^d.
\tag{3.37}
$$

Incidentally, we could also have obtained something like (3.37) directly from (3.17) and the facts that $x \in E$ and E is a reduced quasiminimal sets as in (3.5), by the same proof as in Lemma 10.10 in [DS]. But (3.18) is not that hard to obtain anyway.

Recall that $B_i = B(x,r)$; when we put (3.36) and (3.37) together, we get that

$$
H^d(E_k \cap B_i) \leqslant MH^d(E \cap B_i) + C\varepsilon r^d + (2r)^d h \leqslant (1 + C\varepsilon + Ch)MH^d(E \cap B_i).
\tag{3.38}
$$

Recall that the B_i, $i \in I'$, are disjoint, so

$$
\begin{aligned}
H^d(E_k \cap H) &\leqslant \sum_{i \in I' \cup J'} H^d(E_k \cap B_i) \leqslant \sum_{i \in I'} H^d(E_k \cap B_i) + C\varepsilon \\
&\leqslant (1 + C\varepsilon + Ch)M \sum_{i \in I'} H^d(E \cap B_i) + C\varepsilon \\
&\leqslant (1 + C\varepsilon + Ch)MH^d\big(E \cap (\cup_{i \in I'} B_i)\big) + C\varepsilon
\end{aligned}
\tag{3.39}
$$

by (3.26), (2.16) and (3.25), and then (3.38). [We can apply (2.16) with uniform bounds, by (3.14).]

Let V be any open neighborhood of $E \cap H$; we can choose $\mathcal{B}$ above so that all the B_i are contained in V (just add an extra smallness condition on r near (3.14), or equivalently make $\widehat{H}$ smaller). Then

$$H^d(E_k \cap H) \leqslant (1 + C\varepsilon + Ch)MH^d(E \cap V) + C\varepsilon \qquad (3.40)$$

for k large, by (3.39). Since ε is as small as we want,

$$\limsup_{k \to +\infty} H^d(E_k \cap H) \leqslant (1 + Ch)MH^d(E \cap V). \qquad (3.41)$$

Finally, $H^d(E \cap V)$ is as close to $H^d(E \cap H)$ as we want, because V is any open neighborhood of $E \cap H$ and $H^d(E \cap V) < +\infty$ for some V. So

$$\limsup_{k \to +\infty} H^d(E_k \cap H) \leqslant (1 + Ch)MH^d(E \cap H), \qquad (3.42)$$

and this completes our proof of Lemma 3.12. $\qquad \square$

4. Almost-minimal sets

For us, almost-minimal sets will be sets that look more and more like minimal sets when you look at them at smaller scales. There are a few different possible definitions, even in the same context as in Sections 2 and 3 where we only consider Almgren competitors. We shall give three of them. The first one (with A_+) is the closest to our definition of quasiminimal sets (Definition 2.7), and it has the advantage that we essentially do not need this section or many details of the previous two to use it. So the reader that would not be interested in subtleties about classes of almost-minimal sets is encouraged to have a look at Definition 4.1 and skip the rest.

The other two notions (with A and A') are weaker, and turn out to be equivalent to each other (see Proposition 4.10). For A, which is very close to our notion of generalized quasiminimal sets, we have a nice convergence result (Lemma 4.7). The definition for A' is a little bit simpler, and this is the definition that we used for Al-minimal sets (in Definition 1.6), so the equivalence between the two will make some of our statements a little simpler.

For all these definitions (with A_+, A, and A'), we shall measure closeness to minimality with a nondecreasing function $h : (0, +\infty) \to [0, +\infty]$, such that $\lim_{\delta \to 0} h(\delta) = 0$. We shall call such a function a <u>gauge function</u>.

We start with a first notion which is reasonably close to Definition 2.7.

DEFINITION 4.1. — *Let E be a closed subset of U. We say that E is an A_+-almost-minimal set (of dimension d) in U, with gauge function h, if (2.1) holds and, for each $\delta > 0$ and each family $\{\varphi_t\}_{0 \leqslant t \leqslant 1}$ of deformations such that (2.2)-(2.6) hold,*

$$H^d(E \cap W_1) \leqslant (1 + h(\delta)) \, H^d(\varphi_1(E \cap W_1)), \qquad (4.2)$$

where $W_1 = \{x \in U; \varphi_1(x) \neq x\}$, as in (2.5).

In other words, we require that for each $\delta > 0$, E be an $(1 + h(\delta), \delta)$-quasiminimal set in U (as in Definition 2.7). The main difference is that $M(\delta) = 1 + h(\delta)$ tends to 1 when δ tends to 0. Observe that we allowed the possibility that $h(\delta) = +\infty$ for $\delta \geqslant \delta_0$, to make it plain that in some cases we do not have any information coming from large scales. This is the same as saying that we only consider almost-minimality at scales $\delta \leqslant \delta_0$.

Definition 4.1 looks nice, but we shall prefer to use slightly weaker versions where the error term is allowed not to depend on the set W_1 where $\varphi_1(x) \neq x$, but only on δ. The next definition is slightly simpler, but still uses the same sort of accounting method as in Definitions 2.7 and 4.1.

DEFINITION 4.3. — *We say that the closed set E in U is an A-almost-minimal set in U, with gauge function h, if (2.1) holds and, for each $\delta > 0$ and each family $\{\varphi_t\}_{0 \leqslant t \leqslant 1}$ such that (2.2)-(2.6) hold,*

$$H^d(E \cap W_1) \leqslant H^d(\varphi_1(E \cap W_1)) + h(\delta)\delta^d. \qquad (4.4)$$

Again, $W_1 = \{x \in U; \varphi_1(x) \neq x\}$ as in (2.5).

Remark 4.5. — So we just replaced the error term $h(\delta)H^d(\varphi_1(E \cap W_1))$ in (4.2) with the simpler $h(\delta)\delta^d$. Let us check that A_+-almost-minimal set are automatically A-almost-minimal sets, at least in any smaller open set $U_\tau = \{x \in U \, ; \, \overline{B}(x, \tau) \subset U\}$.

Choose $\delta < \tau$ such that $h(\delta) < +\infty$, and let $\{\varphi_t\}$ satisfy (2.2)-(2.6). Observe that $\operatorname{diam}(W_1) \leqslant \delta$, by (2.6), and that $\operatorname{dist}(W_1, \mathbb{R}^n \setminus U) \geqslant \delta$, because $W_1 \subset U_\tau$ (by (2.6)) and we took $\delta < \tau$. Then $H^d(E \cap W_1) \leqslant C\delta^d$, by the local Ahlfors-regularity of E^* (see Lemma 2.15, but we are still in the standard quasiminimal case). Here C depends on $M = 1 + h(\delta)$, but this is all right.

If $H^d(E \cap W_1) \leqslant H^d(\varphi_1(E \cap W_1))$, then (4.4) holds stupidly. Otherwise,

$$\begin{aligned} H^d(E \cap W_1) \; &\leqslant (1 + h(\delta))H^d(\varphi_1(E \cap W_1)) \\ &\leqslant H^d(\varphi_1(E \cap W_1)) + h(\delta)H^d(E \cap W_1) \qquad (4.6) \\ &\leqslant H^d(\varphi_1(E \cap W_1)) + Ch(\delta)\delta^d, \end{aligned}$$

by (4.2). This is (4.4), with $\widetilde{h}(\delta) = Ch(\delta)$.

The author does not know for sure that Definition 4.1 is strictly stronger than Definition 4.3, but it also seems difficult to prove that they are equivalent. This is why we spent some time dealing with generalized quasiminimal sets in the previous sections. Notice that E is an A-almost-minimal set in U, with gauge function h, if and only if it lies in the class $GAQ(1, \delta, U, h(\delta))$ of Definition 2.10 for every $\delta > 0$. If δ is small, $h(\delta)$ is as small as we want, and then we can apply all the results of Sections 2 and 3. In particular, we have the following consequence of Lemma 3.3.

LEMMA 4.7. — *Let $\{E_k\}$ be a sequence of A-almost-minimal sets in U, with the same gauge function h. Assume that each E_k is reduced (i.e., that $E_k^* = E_k$), and that the sequence $\{E_k\}$ converges to the closed set E in U (i.e., (3.1) holds for each compact $H \subset U$). Then E is a reduced A-almost-minimal set in U, with the same gauge function h.*

To be fair, since Lemma 3.3 needs h to be small enough, it seems that we should replace $h(\delta)$ with $+\infty$ for δ large, but in fact we do not even need to do this. We can apply Lemma 3.3 with δ small, and this gives (3.4) and all the good local properties of E and the E_k, which we can then use as in the proof of Lemma 3.3 to give the desired result even when $h(\delta)$ is large. But anyway it makes no real difference, because we shall not be able to use the conclusion when $h(\delta)$ is large. $\square$

Let us now give a variant of Definition 4.3 where the accounting is done more simply, as in Definitions 1.4 and 1.6.

DEFINITION 4.8. — *Let E be a closed subset of U such that (2.1) holds. We say that E is an A'-almost-minimal set in U, with gauge function h, if for each $\delta > 0$ and each family $\{\varphi_t\}_{0 \leqslant t \leqslant 1}$ such that (2.2)-(2.6) hold,*

$$H^d(E \setminus F) \leqslant H^d(F \setminus E) + h(\delta)\delta^d, \tag{4.9}$$

were we set $F = \varphi_1(E)$.

When E and F coincide out of some compact ball $B \subset U$, (4.9) simply says that $H^d(E \cap B) \leqslant H^d(F \cap B) + h(\delta)\delta^d$; we wrote (4.9) the way we did just to avoid mentioning the existence of such a B. Notice that the Al-minimal sets of Definition 1.6 are the same as A'-almost-minimal sets in $\mathbb{R}^n$, with gauge function $h = 0$. Definition 4.8 looks simpler, but is in fact equivalent to Definition 4.3.

PROPOSITION 4.10. — *Let E be a closed subset of U such that (2.1) holds. Then E is an A-almost-minimal set in U with gauge function h if and only if E is an A'-almost-minimal set in U with (the same) gauge function h.*

First suppose that E is an A'-almost-minimal set, and let $\{\varphi_t\}$ satisfy (2.2)-(2.6). We know (4.9) and we want to check (4.4).

Set $E_1 = E \cap W_1$, $E_2 = E \setminus E_1$, and $F = \varphi_1(E)$. Then

$$H^d(E \cap W_1) = H^d(E_1) = H^d(E_1 \cap F) + H^d(E_1 \setminus F) \qquad (4.11)$$

trivially. Now $E_2 \subset F$ because $\varphi_1(x) = x$ on E_2, so $E_1 \setminus F = E \setminus F$. Then

$$H^d(E \cap W_1) = H^d(E_1 \cap F) + H^d(E \setminus F) \leqslant H^d(E_1 \cap F) + H^d(F \setminus E) + h(\delta)\delta^d, \qquad (4.12)$$

by (4.11) and (4.9). Observe that $F = \varphi_1(E) = \varphi_1(E_1) \cup \varphi_1(E_2)$ and $\varphi_1(E_2) = E_2 \subset E$ because $\varphi_1(x) = x$ on E_2. Then $E_1 \cap F = E_1 \cap [\varphi_1(E_1) \cup \varphi_1(E_2)] = E_1 \cap \varphi_1(E_1)$ because $E_1 \cap \varphi_1(E_2) = E_1 \cap E_2 = \emptyset$. Similarly, $F \setminus E = \varphi_1(E_1) \setminus E$ because $\varphi_1(E_2) \subset E$. Thus (4.12) says that

$$
\begin{aligned}
H^d(E \cap W_1) &\leqslant H^d(E_1 \cap \varphi_1(E_1)) + H^d(\varphi_1(E_1) \setminus E) + h(\delta)\delta^d \\
&\leqslant H^d(E_1 \cap \varphi_1(E_1)) + H^d(\varphi_1(E_1) \setminus E_1) + h(\delta)\delta^d \\
&= H^d(\varphi_1(E_1)) + h(\delta)\delta^d = H^d(\varphi_1(E \cap W_1)) + h(\delta)\delta^d,
\end{aligned}
$$
$$(4.13)$$

as needed for (4.4). This proves our first inclusion.

Now let us assume that E is an A-almost-minimal set, and prove that it is an A'-almost-minimal set with the same gauge function. Again let $\{\varphi_t\}$ satisfy (2.2)-(2.6); we would like to show that (4.9) holds. We cannot deduce this so easily from (4.4), because if we are too unlucky, $\varphi_1(E \cap W_1)$ may have a big part in $E \setminus W_1$, which is accounted for in (4.4) (and helps it hold) but not in (4.9). This may happen if W_1 is somewhat smaller than its image; if this is the case, we shall need to modify the φ_t a bit, to make W_1 larger before we apply (4.4).

Set $\varphi = \varphi_1$ to save notation. We want to replace φ with functions φ_k defined by

$$\varphi_k(x) = \varphi(x) + \varepsilon_k \psi(\varphi(x))\, v, \qquad (4.14)$$

where v is a given unit vector in $\mathbb{R}^n$, $\{\varepsilon_k\}$ is a sequence that tends to 0, and $\psi : U \to \mathbb{R}$ is a correctly chosen smooth function with compact support in U. Recall that there is a relatively compact set $\widehat{W} \subset U$, defined by (2.5) and that satisfies (2.6). Let S and S' be compact subsets of U such that

$$\widehat{W} \subset S \subset \operatorname{int}(S') \subset S' \subset U \quad \text{and} \quad \operatorname{diam}(S') < \delta; \qquad (4.15)$$

other conditions may show up later. We choose ψ so that

$$0 \leqslant \psi(x) \leqslant 1 \text{ everywhere}, \qquad (4.16)$$

$$\psi(x) = 1 \text{ for } x \in S, \text{ and } \psi(x) = 0 \text{ for } x \in U \setminus S'. \qquad (4.17)$$

Since we want to apply (4.4) to φ_k, we need to check that it is the endpoint of an acceptable family $\varphi_{k,t}$, $0 \leqslant t \leqslant 1$. Set

$$\varphi_{k,t}(x) = \varphi_t(x) + t\varepsilon_k \psi(\varphi_t(x))v. \qquad (4.18)$$

The conditions (2.2)-(2.4) clearly hold for this family, so we just have to check (2.6). Set $W_{k,t} = \{x \in U \,;\, \varphi_{k,t}(x) \neq x\}$. Then $W_{k,t} \subset S'$, because if $x \notin S'$, (4.15) says that $\varphi_t(x) = x$, and then (4.17) and (4.18) say that $\varphi_{k,t}(x) = x$. Let us also check that

$$\varphi_{k,t}(W_{k,t}) \subset S'_k =: \{y \,;\, \operatorname{dist}(y, S') \leqslant \varepsilon_k\}. \qquad (4.19)$$

Notice that

$$|\varphi_{k,t} - \varphi_t(x)| \leqslant \varepsilon_k \text{ for } x \in U, \qquad (4.20)$$

by (4.18) and (4.16). If $x \in W_t$, then $\varphi_t(x) \in \varphi_t(W_t) \subset S$ by (4.15), and so $\varphi_{t,k}(x) \in S'_k$ by (4.20). If $x \in W_{k,t} \setminus W_t$, then $\varphi_t(x) = x \in W_{k,t} \subset S'$, and then $\varphi_{k,t}(x) \in S'_k$, by (4.20) again. This proves (4.19).

We are now ready to prove (2.6) for the $\varphi_{k,t}$. Set $\widehat{W}_k = \bigcup_t [W_{k,t} \cup \varphi_{k,t}(W_{k,t})]$. We just proved that $\widehat{W}_k \subset S'_k$, and it is obvious from (4.15) and the compactness of S' that for k large, S'_k is a compact subset of U with $\operatorname{diam}(S'_k) < \delta$. This proves (2.6). So we can apply (4.4), and we get that

$$H^d(E \cap W_{k,1}) \leqslant H^d(\varphi_k(E \cap W_{k,1})) + h(\delta)\delta^d. \qquad (4.21)$$

Now we shall cut our sets into pieces to control the overlaps and eventually get something that looks like (4.9). What comes out of the small annular region $A = S' \setminus S$ will be estimated brutally, and later thrown out to the trash, so we shall concentrate on $E_1 = E \cap S$ and $E_2 = E \setminus S'$.

First observe that $\varphi_k(x) = x$ for $x \in E_2$, by (4.15) and (4.17). Also, $\varphi(E_1) \subset S$ because if $\varphi(x) \neq x$ then $x \in W_1$ and $\varphi(x) \in S$ by (4.15). Consequently, $\varphi_k(E_1) \subset S'$ for k large enough, by (4.20). In particular, $\varphi_k(E_1)$ does not meet $\varphi_k(E_2) = E_2$. Observe also that $W_{k,1}$ does not meet $E_2 = E \setminus S'$. Thus

$$\begin{aligned}
H^d(\varphi_k(E \cap W_{k,1})) &= H^d(\varphi_k(E \cap W_{k,1} \cap S')) \\
&\leqslant H^d(\varphi_k(E_1 \cap W_{k,1})) + H^d(\varphi_k(E \cap A)) \qquad (4.22) \\
&= H^d(\varphi(E_1 \cap W_{k,1})) + H^d(\varphi_k(E \cap A)),
\end{aligned}$$

where the last equality comes from the fact that $\varphi_k - \varphi$ is constant on E_1, by (4.17) and because we just said that $\varphi(E_1) \subset S$.

It is not necessarily true that $E_1 \subset W_{k,1}$, but the difference is small: we claim that

$$E_1 \setminus W_{k,1} \subset Z_k =: \{x \in E_1 \,;\, 0 < |\varphi(x) - x| \leqslant \varepsilon_k\}. \qquad (4.23)$$

Indeed let $x \in E_1$ be given, and assume that it does not lie in the set on the right-hand side. If $\varphi(x) = x$, then $\varphi_k(x) \neq x$ because $\varphi(x) \in S$ and (4.17) says that $\psi(\varphi(x)) = 1$. And if $|\varphi(x) - x| > \varepsilon_k$, (4.20) says that $\varphi_k(x) \neq x$ too. So $x \in W_{k,1}$ in both cases; (4.23) follows. Now

$$\begin{aligned}
H^d(E_1) &= H^d(E_1 \cap W_{k,1}) + H^d(E_1 \setminus W_{k,1}) \\
&\leqslant H^d(\varphi_k(E \cap W_{k,1})) + h(\delta)\delta^d + H^d(Z_k) \\
&\leqslant H^d(\varphi(E_1 \cap W_{k,1})) + H^d(\varphi_k(E \cap A)) + h(\delta)\delta^d + H^d(Z_k) \\
&\leqslant H^d(\varphi(E_1)) + H^d(\varphi_k(E \cap A)) + h(\delta)\delta^d + H^d(Z_k)
\end{aligned}$$
$$(4.24)$$

by (4.21) and (4.23), and then (4.22). To get closer to (4.9), observe that

$$H^d(E \setminus \varphi(E)) = H^d(E \cap S' \setminus \varphi(E)) = H^d(E \cap S') - H^d(E \cap S' \cap \varphi(E)) \quad (4.25)$$

because $E_2 = E \setminus S'$ is contained in $\varphi(E)$. Similarly,

$$H^d(\varphi(E) \setminus E) \geqslant H^d(\varphi(E) \cap S' \setminus E) = H^d(\varphi(E) \cap S') - H^d(E \cap S' \cap \varphi(E)). \qquad (4.26)$$

Set $\Delta = H^d(E \setminus F) - H^d(F \setminus E) = H^d(E \setminus \varphi(E)) - H^d(\varphi(E) \setminus E)$. This is the quantity that we want to majorize, and we just proved that

$$\Delta \leqslant H^d(E \cap S') - H^d(\varphi(E) \cap S'). \qquad (4.27)$$

Recall that $\varphi(E_1) \subset S$ (see four lines above (4.22)), so $\varphi(E) \cap S'$ contains $\varphi(E_1)$ and (4.27) implies that

$$\Delta \leqslant H^d(E \cap S') - H^d(\varphi(E_1)) = H^d(E_1) + H^d(E \cap A) - H^d(\varphi(E_1)), \quad (4.28)$$

because $E \cap S' = (E \cap S) \cup (E \cap S' \setminus S) = E_1 \cup (E \cap A)$ by definitions. Then (4.24) yields

$$\Delta \leqslant H^d(E \cap A) + H^d(\varphi_k(E \cap A)) + h(\delta)\delta^d + H^d(Z_k). \qquad (4.29)$$

Notice that $\varphi_k = h_k \circ \varphi$, where h_k is Lipschitz, with a Lipschitz constant that stays below 2 when k is large enough. Then for k large, $H^d(\varphi_k(E \cap A)) \leqslant C_\varphi H^d(E \cap A)$, with a constant C_φ that depends on φ but not k or ψ. On the other hand, when ε_k tends to 0, the set Z_k in (4.23) decreases to the empty set, and so $\lim_{k \to +\infty} H^d(Z_k) = 0$. Thus (4.29) yields

$$\Delta \leqslant (1 + C_\varphi) H^d(E \cap A) + h(\delta)\delta^d. \qquad (4.30)$$

Recall that $A = S' \setminus S$, where we only required that S and S' be compact and satisfy (4.15). We can do this so that $H^d(E \cap A)$ is as small as we want, and so $\Delta \leqslant h(\delta)\delta^d$, as needed for (4.9). Thus E is an A'-almost-minimal set, with the same gauge function h as for its A-almost-minimality. This completes our proof of Proposition 4.10. $\quad\square$

B. MONOTONICITY OF DENSITY AND APPROXIMATION BY CONES

5. Density is nearly nondecreasing

In this section we show that if E is a minimal set of dimension d in $\mathbb{R}^n$, then, for each choice of $x \in \mathbb{R}^n$, $r^{-d}H^d(E \cap B(x,r))$ is a nondecreasing function of r. We also give a local version of this, and generalizations to almost-minimal sets with sufficiently small gauge functions. See Propositions 5.16, 5.24, and 5.30 below. Incidentally, all this is supposed to be very classical, and the general idea will be to compare E with a cone. We do not do this directly because the cone is not exactly a competitor, and to avoid (probably minor) problems of definitions or regularity. Instead we first prove an integrated version.

LEMMA 5.1. — *Let $U \subset \mathbb{R}^n$ be open, and let E be a reduced A- or A'-almost-minimal set in U, with gauge function h. Let $x \in E$ and $0 < \rho_1 < \rho_2$ be such that $B(x, \rho_2) \subset U$. Then*

$$H^d(E \cap B(x, \rho_1)) \leqslant h(2\rho_2)(2\rho_2)^d$$
$$+ \frac{1}{\rho_1 - \rho_2} \int_{E \cap B(x,\rho_2) \setminus B(x,\rho_1)} \frac{r(y)\cos\theta(y)}{d} \, dH^d(y), \qquad (5.2)$$

where we set $r(y) = |y - x|$, and $\theta(y) \in [0, \pi/2]$ denotes the (smallest) angle between the radius $[x, y]$ and the tangent d-plane to E at y.

See Definition 4.3 or 4.8 for almost-minimal sets. Also recall that E is rectifiable, by Theorem 2.11 in [DS] (generalized as in Section 2). The rectifiability of E gives an approximate tangent plane to E at y for H^d-almost every $y \in E$, which is enough to define $\theta(y)$. But here E is locally Ahlfors-regular too, so the approximate tangent plane is even a true tangent d-plane (for instance by Exercise 41.21 on page 277 of [D2]).

To prove the lemma, we may assume that $x = 0$. Let $0 < r_1 < r_2$ be such that $\overline{B}(0, r_2) \subset U$; we want to compare E with the competitor $F = \phi(E)$, where ϕ is the radial function defined by $\phi(r\xi) = \varphi(r)\,\xi$ for $r \geqslant 0$ and $\xi \in \partial B(0, 1)$, and where φ is the piecewise linear function such that (as suggested by Figure 5.1)

$$\varphi(r) = 0 \text{ for } 0 \leqslant r \leqslant r_1, \quad \varphi(r) = \tfrac{r_2(r-r_1)}{r_2-r_1} \text{ for } r_1 \leqslant r \leqslant r_2,$$
$$\text{and } \varphi(r) = r \text{ for } r \geqslant r_2 \,. \tag{5.3}$$

Note that the one-parameter family ϕ_t defined by $\phi_t(y) = (1-t)y + t\phi(y)$ satisfies (2.2)-(2.6) with any $\delta > 2r_2$. Here $W_1 = B(0,r_2) \setminus \{0\}$, and (4.4) says that

$$H^d(E \cap B(0,r_2)) \leqslant H^d(\phi(E \cap B(0,r_2))) + h(\delta)\delta^d. \tag{5.4}$$

Notice that (4.9) yields exactly the same thing, after adding $H^d(E \cap \phi(E) \cap B(0,r_2))$, so we don't need to worry about which definition of almost-minimality we take. We shall remember that (5.4) holds for all $\delta > 2r_2$.

Next we need to evaluate $H^d(\phi(E \cap B(0,r_2))$. Set $A = B(0,r_2) \setminus B(0,r_1)$. We just need to evaluate $H^d(\phi(E \cap A))$, because $E \cap B(0,r_1)$ is mapped to the origin.

We shall use the rectifiability of E. Let $y \in E$ be given, and assume that E has a tangent d-plane at y. Denote by P this d-plane and by P' the vector d-plane parallel to P. Also set $r(y) = |y|$ and $\widetilde{y} = y/r(y)$.

We want to compute how the differential $D\phi(y)$ acts on vectors. In the radial direction, we just use the fact that for $t > 0$ such that $t\widetilde{y} \in A$, $\phi(t\widetilde{y}) = \varphi(t)\widetilde{y} = \dfrac{r_2(t-r_1)}{r_2-r_1}\,\widetilde{y}$, by (5.3), so $D\phi(y) \cdot \widetilde{y} = [r_2/(r_2-r_1)]\,\widetilde{y}$. For vectors w orthogonal to $\widetilde{y}$, (5.3) yields $D\phi(y) \cdot w = [\varphi(r(y))/r(y)]w$.
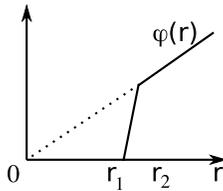

Figure 3
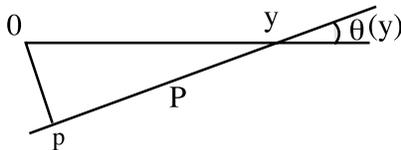
Figure 4

If P is orthogonal to $[0, y]$ at y, $D\phi(y)$ acts on P' by multiplication by $\varphi(r(y))/r(y)$. Otherwise, denote by p the orthogonal projection on P of the origin, and set $v = \frac{(y-p)}{|y-p|} \in P'$. [See Figure 4] By definition, $\theta(y)$ is the angle of $[0, y]$ with v, so $v = \cos\theta(y)\,\widetilde{y} + \sin\theta(y)\,w$ for some unit vector w orthogonal to $\widetilde{y}$. Then $D\phi(y)\cdot v = [r_2/(r_2 - r_1)]\cos\theta(y)\,\widetilde{y} + [\varphi(r(y))/r(y)]\sin\theta(y)\,w$ and $|D\phi(y)\cdot v| = \beta(y)$, where we set

$$\beta(y) = \left\{ [r_2/(r_2 - r_1)]^2 \cos^2\theta(y) + [\varphi(r(y))/r(y)]^2 \sin^2\theta(y) \right\}^{1/2}. \quad (5.5)$$

We are ready to apply the area formula to compute $H^d(\phi(E \cap A))$ (in other words, we cover $E \cap A$ with disjoint measurable subsets of C^1 surfaces, plus a set of vanishing H^d measure, and then compute the Hausdorff measure of images on each C^1 piece with the derivative of ϕ; we don't need to worry about overlaps, since ϕ is bijective on A). We get that

$$H^d(\phi(E \cap B(0, r_2))) = H^d(\phi(E \cap A)) = \int_{E \cap A} [\varphi(r(y))/r(y)]^{d-1}\beta(y)dH^d(y), \quad (5.6)$$

where the extra $[\varphi(r(y))/r(y)]^{d-1}$ comes from the directions of P' that are orthogonal to v. When P is orthogonal to $[0, y]$ at y, (5.6) also gives the right factor $[\varphi(r(y))/r(y)]^d$, because $\theta(y) = \pi/2$. We plug this back into (5.4) and get that

$$H^d(E \cap B(0, r_2)) \leqslant h(\delta)\delta^d + \int_{E \cap A} [\varphi(r(y))/r(y)]^{d-1}\beta(y)dH^d(y). \quad (5.7)$$

It is tempting to let $r_2 - r_1$ tend to 0 now, but we decided to take an average before we do this, and this is what will lead to (5.2). So we fix $0 < \rho_1 < \rho_2$ such that $B(0, \rho_2) \subset U$, pick a small $\eta > 0$, and for each $t \in (\rho_1, \rho_2)$, we apply (5.7) with $r_2 = t$, $r_1 = t - \eta$, and $\delta = 2\rho_2$. [Recall that we can take any $\delta > 2r_2$ in (5.4) and (5.7).] Then we average over t and get that

$$H^d(E \cap B(0, \rho_1 - \eta)) \leqslant h(2\rho_2)(2\rho_2)^d + \frac{1}{\rho_2 - \rho_1}\, I(\eta) \quad (5.8)$$

where

$$I(\eta) = \int_{t \in (\rho_1, \rho_2)} \int_{y \in E\,;\, t-\eta \leqslant r(y) \leqslant t} [\varphi_t(r(y))/r(y)]^{d-1}\beta_t(y)dH^d(y)dt, \quad (5.9)$$

with obvious notations concerning φ_t and β_t. By Fubini,

$$I(\eta) = \int_{y \in E\,;\, \rho_1 - \eta < r(y) < \rho_2} g_\eta(y)\, dH^d(y), \quad (5.10)$$

with

$$g_\eta(y) = \int_{t \in (\rho_1, \rho_2) \cap [r(y), r(y)+\eta]} [\varphi_t(r(y))/r(y)]^{d-1} \beta_t(y) dt. \qquad (5.11)$$

First observe that here $r_2/(r_2 - r_1) = t/\eta$. Then (5.5) yields

$$\begin{aligned} \beta_t(y) &= \left\{ (t/\eta)^2 \cos^2 \theta(y) + [\varphi_t(r(y))/r(y)]^2 \sin^2 \theta(y) \right\}^{1/2} \\ &\leqslant \left\{ (t/\eta)^2 \cos^2 \theta(y) + 1 \right\}^{1/2} \leqslant (t/\eta) \cos \theta(y) + 1 \end{aligned} \qquad (5.12)$$

because $\varphi_t(r(y)) \leqslant r(y)$ and $a^2 + b^2 \leqslant (a+b)^2$. This yields

$$\beta_t(y) \leqslant \eta^{-1} t \cos \theta(y) + 1 \leqslant \eta^{-1}[(r(y)+\eta)\cos\theta(y) + \eta] \qquad (5.13)$$

on the domain of integration in (5.11). Also

$$\frac{\varphi_t(r(y))}{r(y)} = \frac{r_2}{r(y)} \frac{r(y) - r_1}{r_2 - r_1} = \frac{t}{r(y)} \frac{r(y) - t + \eta}{\eta} \leqslant \eta^{-1} \frac{r(y)+\eta}{r(y)} \left(r(y) - t + \eta \right), \qquad (5.14)$$

by (5.3) and because $t \leqslant r(y) + \eta$ on the domain of integration, so

$$\begin{aligned} g_\eta(y) &\leqslant \eta^{-d} \left[(r(y)+\eta) \cos\theta(y) + \eta \right] \left(\tfrac{r(y)+\eta}{r(y)} \right)^{d-1} \\ &\qquad \int_{t \in [r(y), r(y)+\eta]} (r(y) - t + \eta)^{d-1} dt \\ &= \frac{1}{d} \left[(r(y)+\eta)\cos\theta(y) + \eta \right] \left(\tfrac{r(y)+\eta}{r(y)} \right)^{d-1} \leqslant \frac{1}{d} r(y) \cos\theta(y) + C\eta, \end{aligned}$$

where C is allowed to depend on many things, but not on η. We return to (5.10) and get that $I(\eta) \leqslant \dfrac{1}{d} \displaystyle\int_{y \in E \,;\, \rho_1 - \eta < r(y) < \rho_2} [r(y)\cos\theta(y) + C\eta] \, dH^d(y)$. Then (5.8) becomes

$$\begin{aligned} H^d(E \cap \; B(0, \rho_1 - \eta)) &\leqslant h(2\rho_2)(2\rho_2)^d \\ &+ \frac{1}{\rho_2 - \rho_1} \frac{1}{d} \int_{y \in E \,;\, \rho_1 - \eta < r(y) < \rho_2} [r(y)\cos\theta(y) + C\eta] \, dH^d(y). \end{aligned} \qquad (5.15)$$

Then we let η tend to 0 and get (5.2). This completes our proof of Lemma 5.1.
□

Remark. — The deformation ϕ that we used in the proof of Lemma 5.1 is not a bijection, but we could have used the homeomorphism $\phi_t(y) = (1-t)y + t\phi(y)$, with $t < 1$ very close to 1, and then go to the limit and get (5.2) anyway. So Lemma 5.1, and Propositions 5.16 and 5.30 below, also hold when E lies in the larger class of almost-minimal sets where the

deformations in (2.2)-(2.6) are also required to be homeomorphisms, at least if we assume that E is rectifiable. I do not think that Proposition 5.24 below extends to this context (we shall also use local Ahlfors-regularity there).

We want to use Lemma 5.1 to get monotonicity results for $r^{-d}H^d(E \cap B(x, r))$. We start with the easier case of minimal sets.

PROPOSITION 5.16. — *Let E be a minimal set in U. This means that E is almost-minimal in U, with gauge function $h = 0$ (and Definitions 4.1, 4.3, and 4.8 are all equivalent in this case). Let $x \in U$ and $R > 0$ be such that $B(x, R) \subset U$. Then $\theta(x, r) = r^{-d}H^d(E \cap B(x, r))$ is a nondecreasing function of r on $(0, R]$.*

Let $B(x, R)$ be as in the statement, and define a locally finite Borel measure μ on $[0, R)$ by

$$\mu(A) = H^d(E \cap \pi^{-1}(A)) \text{ for every Borel set } A \subset [0, R), \qquad (5.17)$$

where π is the radial projection defined by $\pi(y) = |x - y|$. Then decompose μ into its absolutely continuous part μ_a and its singular part μ_s, and denote by $a(r)$ the density of μ_a with respect to the Lebesgue measure. The Lebesgue density theorem allows us to compute $a(r)$ at almost-every point $r \in [0, R)$ as a limit of densities, i.e.,

$$a(r) = \lim_{t \to 0^+} \frac{1}{t} \mu([r - t, r)) = \lim_{t \to 0^+} \frac{1}{t} H^d\big(E \cap B(x, r) \setminus B(x, r - t)\big). \quad (5.18)$$

Similarly $a(r) = \lim_{t \to 0^+} \frac{1}{t} H^d(E \cap B(x, r + t) \setminus B(x, r))$ for almost-every r. Consequently, if we set

$$l(r) = H^d(E \cap B(x, r)) \text{ for } 0 \leqslant r < R, \qquad (5.19)$$

then $l(r)$ is differentiable almost-everywhere on $(0, R)$, and $l'(r) = a(r)$ almost-everywhere. Let us only assume for the moment that E is almost-minimal, and apply Lemma 5.1 with $\rho_2 = r \in (0, R)$ and $\rho_1 = r - t$. We can even drop $\cos \theta(y)$ and get that

$$l(r - t) \leqslant (2r)^d h(2r) + \frac{r}{td} H^d(E \cap B(x, r) \setminus B(x, r - t)). \qquad (5.20)$$

Then we let t tend to 0, use (5.18), and get that

$$l(r) \leqslant (2r)^d h(2r) + \frac{r}{d} a(r) = (2r)^d h(2r) + \frac{r}{d} l'(r) \text{ for almost-every } r \in (0, R). \qquad (5.21)$$

Here E is even a minimal set, so (5.21) says that $l(r) \leqslant \frac{r}{d} l'(r)$. Recall that $l(r) = \int_{[0,r)} d\mu$; we claim that for $0 < a < b < R$,

$$\theta(x, b) - \theta(x, a) = \int_{[a,b)} t^{-d} d\mu(t) - d \int_{[a,b)} t^{-d-1} l(t) dt. \qquad (5.22)$$

Set $g(t) = t^{-d}$; thus $\theta(x,r) = g(r)l(r)$, and we just want to show that the derivative of the product gl is $gd\mu + g'l$. Let $0 < a < b < R$ be given, and apply Fubini's theorem to $I = \int_{a \leqslant s \leqslant t < b} g'(s)ds d\mu(t)$. One computation gives $I = \int_{[a,b)} [g(t) - g(a)]d\mu(t)$, and the other one yields $I = \int_{[a,b)} g'(s)[l(b) - l(s)]ds$. Then

$$\int_{[a,b)} g(t)d\mu(t) + \int_{[a,b)} g'(s)l(s)ds = \left[I + g(a) \int_{[a,b)} d\mu(t) \right]$$
$$+ \left[-I + l(b) \int_{[a,b)} g'(s)ds \right] \quad (5.23)$$
$$= g(a)(l(b) - l(a)) + l(b)(g(b) - g(a)) = g(b)l(b) - g(a)l(a).$$

This is just (5.22); note that (5.23) would work with any smooth function g.

We may now conclude. Recall that $t^{-d}d\mu(t) \geqslant t^{-d}l'(t)dt \geqslant d\,t^{-d-1}l(t)dt$ because $d\mu(t) \geqslant d\mu_a(t) = l'(t)dt$ and by (5.21) with $h(2r) = 0$, so (5.22) says that $\theta(x, \cdot)$ is nondecreasing, and Proposition 5.16 follows. $\qquad\square$

PROPOSITION 5.24. — *There is a constant $\lambda > 1$, that depends only on n and d, such that the following holds. Suppose that E is an A- or A'-almost-minimal set in U, with gauge function h. Let $x \in E^* \cap U$ and $R > 0$ be such that $B(x,R) \subset U$, and also assume that $h(R)$ is small enough (depending only on n and d) and that $\int_0^R h(2r)\frac{dr}{r} < +\infty$. Then*

$$\theta(x,r)\, e^{\lambda A(r)} \text{ is a nondecreasing function of } r \in (0, R), \quad (5.25)$$

where we set $A(r) = \displaystyle\int_0^r h(2t)\frac{dt}{r}$ for $0 < r < R$.

Notice that here we had to take $x \in E^* \cap U$ to get the conclusion (Recall from Definition 2.12 that E^* is the closed support of $H^d_{|E}$). Since $\lim_{r \to 0} A(r) = 0$, $e^{\lambda A(r)}$ tends to 1 and can be seen as a mild penalty that measures a potential defect in monotonicity.

We start the proof as in Proposition 5.16, but we multiply $l(t)$ with $g(t) = t^{-d}e^{\lambda A(t)}$. By (5.23), the distribution derivative of $g(t)l(t)$ is $g(t)d\mu(t) + g'(t)l(t)dt$. Also, $d\mu(t) \geqslant d\mu_a(t) = a(t)dt = l'(t)dt$, so

$$\theta(x,b)e^{\lambda A(b)} - \theta(x,a)e^{\lambda A(a)} \geqslant \int_a^b [g(t)l'(t) + g'(t)l(t)]dt, \quad (5.26)$$

as in (5.22). So it is enough to check that $g(t)l'(t) + g'(t)l(t) \geqslant 0$ almost-everywhere. But $g'(t) = -\frac{d}{t}g(t) + \lambda A'(t)g(t) = -\frac{d}{t}g(t) + \lambda\frac{h(2t)}{t}g(t)$, so

$$g(t)l'(t) + g'(t)l(t) = g(t)\left[l'(t) - \frac{d}{t}l(t) + \lambda\frac{h(2t)}{t}l(t)\right]. \quad (5.27)$$

Now (5.21) says that $l'(t) \geqslant \frac{d}{t}\left[l(t) - (2t)^d h(2t)\right]$, so it is enough to check that $(2t)^d \leqslant \lambda l(t)$. This comes from Lemma 2.15; we just have to take $\lambda \geqslant 2^d C$, where C is as in (2.16). This is the reason why we have to take $x \in E^*$ and assume that $h(R)$ is small enough. $\quad\square$

The case of A_+-almost-minimal sets is a little easier, because we may replace the error term $h(\delta)\delta^d$ in (5.4) with $h(\delta)H^d(\phi(E \cap B(0, r_2))$. [We use (4.2) instead of (4.4).] When we follow the computations, we see that we can replace (5.2) with

$$H^d(E \cap B(x, \rho_1)) \leqslant \frac{1 + h(2\rho_2)}{\rho_1 - \rho_2} \int_{E \cap B(x,\rho_2)\setminus B(x,\rho_1)} \frac{r(y)\cos\theta(y)}{d}\,dH^d(y). \tag{5.28}$$

Then (5.21) may be replaced with

$$l(r) \leqslant [1 + h(2r)]\frac{r}{d}\,l'(r) \quad \text{for almost-every } r \in (0, R), \tag{5.29}$$

and (5.27) allows us to show that $\theta(x, r)e^{\lambda A(r)}$ is nondecreasing as soon as $[1 + h(2t)]^{-1}\frac{d}{t} - \frac{d}{t} + \lambda\frac{h(2t)}{t} \geqslant 0$, i.e., when $\lambda \geqslant d\{1 - [1 + h(2t)]^{-1}\}$. This proves the following intermediate statement for A_+-almost-minimal sets, where the interest is that we no longer need to take a center in E^*.

PROPOSITION 5.30. — *For each $\lambda > 0$ we can find $\tau > 0$, that depends only on n, d, and λ, with the following property. Let E be an A_+-almost-minimal set in U, with gauge function h. Let $x \in U$ and $R > 0$ be such that $B(x, R) \subset U$, and suppose that $h(2R) \leqslant \tau$ and $\int_0^R h(2r)\frac{dr}{r} < +\infty$. Set $A(r) = \int_0^r h(2r)\frac{dt}{t}$ for $0 < r < R$. Then*

$$\theta(x, r)e^{\lambda A(r)} \text{ is a nondecreasing function of } r \in (0, R). \tag{5.31}$$

Remark 5.32. — It is tempting to try to prove Propositions 5.16, 5.24, and 5.30 without using the rectifiability of E. It would be enough to prove that

$$H^d(E \cap B(x, \rho_1)) \leqslant h(2\rho_2)(2\rho_2)^d + \frac{1}{\rho_1 - \rho_2}\int_{E \cap B(x,\rho_2)\setminus B(x,\rho_1)} \frac{r(y)}{d}\,dH^d(y) \tag{5.33}$$

(or the corresponding variant in the case of Proposition 5.30) instead of (5.2) (i.e., without the cosine), and then continue as before. To do so, we would need to change the argument near (5.5), when we compute the effect of $D\phi$ on the tangent plane. We observed that $D\phi(y).v = [r_2/(r_2 - r_1)]v$ in the radial direction and $D\phi(y).v = [\varphi(r(y))/r(y)]v$ in the other direction, so we expect that ϕ will at most multiply the Hausdorff measure of E by

$[r_2/(r_2-r_1)][\varphi(r(y))/r(y)]^{d-1}$ at y, with the effect of replacing $\beta(y)$ in (5.5) with $[r_2/(r_2-r_1)]$ and eventually leading to (5.33). But the author is not aware of a generalization to unrectifiable sets of the obvious estimate on Hausdorff measure of affine images (of rectifiable sets) that would allow to conclude here. We could also try to replace the Hausdorff measure with a more friendly variant. Anyway, this may not be so useful, because the cosine that shows up in Lemma 5.1 will really be used in the next section.

6. Minimal sets with constant density

In addition to the monotonicity of density proved in the last section, it will be useful to know that when E is a minimal set and

$$\theta(r) = r^{-2}H^d(B(0,r)) \qquad (6.1)$$

is constant on some interval (a,b), then E^* coincides on $B(0,b) \setminus \overline{B}(0,a)$ with a minimal cone (i.e., a minimal set which is a cone) centered at the origin.

THEOREM 6.2. — *Let E be a reduced minimal set of dimension d in $B(0,b) \subset \mathbb{R}^n$. This means that E is a reduced almost-minimal in $B(0,b)$, as in Definition 4.3 or 4.8 (also see Definition 2.12), with the gauge function $h = 0$. Assume that we can find $a \in [0,b)$ and a constant $\theta_0 \geqslant 0$ such that*

$$\theta(r) = \theta_0 \text{ for } a < r < b. \qquad (6.3)$$

Then there is a reduced minimal cone $\mathcal{C}$ centered at the origin such that

$$E \cap [B(0,b) \setminus \overline{B}(0,a)] = \mathcal{C} \cap [B(0,b) \setminus \overline{B}(0,a)]. \qquad (6.4)$$

Recall that our two definitions of minimality are equivalent. We shall not need to know this, because it will be just as simple to prove the theorem for both notions. Of course Theorem 6.2 also applies if E is minimal in a domain that contains $B(0,b)$.

Observe also that if we did not assume that E is reduced, we would still get that $E^* \cap [B(0,b) \setminus \overline{B}(0,a)] = \mathcal{C} \cap [B(0,b) \setminus \overline{B}(0,a)]$ (instead of (6.4)); this is easy, because E^* is a reduced minimal set, with the same density, by Remark 2.14 and (2.13), so we can apply Theorem 6.2 to it.

Let E be as in the theorem. We first look at Proposition 5.16 to see if we can derive extra information because (6.3) holds. Set $A = B(0,b) \setminus \overline{B}(0,a) = \pi^{-1}((a,b))$, where π denotes the radial projection defined by $\pi(y) = |y|$. We want to prove that

$$\cos \theta(y) = 1 \text{ for } H^d\text{-almost every } y \in E \cap A, \qquad (6.5)$$

where $\cos \theta(y)$ is as in Lemma 5.1.

Let ν be the finite Borel measure on (a, b) defined by

$$\nu(S) = \int_{E \cap \pi^{-1}(S)} \cos \theta(y) dH^2(y) \qquad (6.6)$$

for $S \subset (a, b)$. Notice that $\nu \leqslant \mu$, where μ is as in (5.17) and corresponds to replacing $\cos \theta(y)$ with 1. By (6.3), $\mu([\alpha, \beta)) = \theta_0(\beta - \alpha)$ for $a < \alpha < \beta < b$, so $d\mu = \theta_0 dx$ on (a, b). Then ν is absolutely continuous too. Let f denote its density (with respect to the Lebesgue measure). Notice that $f(r) = \lim_{t \to 0+} \frac{1}{t} \nu([r - t, r))$ almost-everywhere, as in (5.18). We apply Lemma 5.1 with $\rho_2 = r \in (a, b)$ and $\rho_1 = r - t$, as we did for (5.20), and we get that

$$l(r - t) \leqslant \frac{r}{dt} H^d(E \cap B(x, r) \setminus B(x, r - t)) \qquad (6.7)$$

instead of (5.20) (just because $h = 0$ here). Then we let t tend to 0 and get that $l(r) \leqslant \frac{r}{d} f(r)$ (instead of (5.21)). We also have that $f(r) \leqslant l'(r)$, because $l'(r)$ is the density of μ and $\nu \leqslant \mu$. In addition, $l(r) = r^d \theta(r) = r^d \theta_0$ by (6.1) and (6.3), so $l'(r) = dr^{d-1} \theta_0 = dl(r)/r$. Altogether

$$l(r) \leqslant \frac{r}{d} f(r) \leqslant \frac{r}{d} l'(r) = l(r) \quad \text{almost-everywhere on } (a, b), \qquad (6.8)$$

and in particular $f(r) = l'(r)$ almost-everywhere on (a, b). Recall that f is the density of the absolutely continuous measure ν, so

$$\nu((a, b)) = \int_a^b f(r) dr = \int_a^b l'(r) dr = \mu((a, b)), \qquad (6.9)$$

and (6.5) follows by comparison.

In geometrical terms, (6.5) just says that for H^d-almost-every point $y \in E \cap A$, the tangent plane to E at y goes through the origin. This looks like pretty strong evidence that E coincides with a cone in A, but apparently we need some nontrivial amount of additional work to get the result. In particular, we shall use the minimality of E again.

Denote by π_S the radial projection onto the unit sphere. Thus $\pi_S(y) = y/|y|$. We claim that

$$H^d(\pi_S(E \cap A)) = 0. \qquad (6.10)$$

This is an easy consequence of (6.5). Indeed recall from Section 2 that E is both rectifiable and locally Ahlfors-regular, so by Exercise 41.21 on page 277 of [D2]) it has a true tangent d-plane at almost every point. [We don't really need to know this here, but this will make the discussion simpler.]

By rectifiability, we can cover $E \cap A$, up to a negligible set N such that $H^d(N) = 0$, by a countable collection of d-dimensional surfaces Γ_i of class

C^1. For almost-every point y of $E \cap A \cap \Gamma_i$, the tangent d-plane to E at y is the same as the tangent d-plane to Γ_i at y, because we may assume that y is a point of positive upper density of $E \cap \Gamma_i$), and then (6.5) says that this d-plane goes through the origin. So y is a critical point of π_S on Γ_i, and Sard's theorem (applied to π_S on Γ_i) says that $H^d(\pi_S(E \cap A \cap \Gamma_i)) = 0$. Since $H^d(\pi_S(N)) = 0$ because π_S is Lipschitz on A, we just need to sum over i to get (6.10).

PROPOSITION 6.11. — *Let $y \in E \cap A$ be such that E has a tangent d-plane P at y which goes through 0. Then E contains the radial line segment $L = A \cap \pi_S^{-1}(\pi_S(y))$ through y.*

Let us first check that Proposition 6.11 implies that

$$E \text{ coincides on } A \text{ with a cone;} \tag{6.12}$$

we shall only see later that this cone is a reduced minimal set.

Call E_1 the set of points $y \in E \cap A$ that satisfy the conditions of Proposition 6.11; by the discussion above (the proof of (6.10)), almost-every point of $E \cap A$ lies in E_1. In particular, E_1 is dense in $E \cap A$. Call $\mathcal{C}$ the cone over E_1; then $E \cap A \subset \overline{E_1} \subset \overline{\mathcal{C}}$. Conversely, $\overline{\mathcal{C}} \cap A \subset E$, by Proposition 6.11 and because E is closed. So (6.12) follows from Proposition 6.11.

The proof of Proposition 6.11 will take some time. Let $y \in E^* \cap A$, P, and L be as in the proposition, and suppose in addition that L is not contained in E. We want to construct a deformation of E and eventually get a contradiction with its minimality.

We need lots of <u>notation</u> before we start. Since L is not contained in E, we can pick $x \in L \setminus E$. Let ε_0 be small, to be chosen near the end. Then pick a small radius r_y such that

$$\text{dist}(z, P) \leqslant \varepsilon_0 |z - y| \quad \text{for } z \in E \cap 2B_y, \tag{6.13}$$

where we set $B_y = B(y, r_y)$ and $2B_y = B(y, 2r_y)$. Notice that (6.13) holds for r_y small enough, because P is tangent to E at y. We choose $r_y < b/100$ and so small that $2B_y$ is contained in A and does not contain x.

Now let r_x be a small radius, smaller than r_y and such that if $B_x = B(x, r_x)$, then

$$2B_x \subset A \setminus [E \cup 2B_y]. \tag{6.14}$$

We shall also need to take r_x small enough, depending on r_y, for the final estimate to work.

Without loss of generality, we may assume that coordinates were chosen so that L is contained in the positive first axis, and P is the horizontal d-plane where $x_{d+1} = \cdots = x_n = 0$. We know that $L \subset P$, by (6.5), so this is coherent. We shall identify $\mathbb{R}^n$ with $L' \times P' \times Q$, where L' is the line that contains L and the first axis, P' is the orthogonal complement of L' in P (so P' corresponds to $x_1 = x_{d+1} = \cdots = x_n = 0$), and Q is the orthogonal of P in $\mathbb{R}^n$. Thus Q is $(n-d)$-dimensional.

We shall often use this identification to denote points of $\mathbb{R}^n$ by $z = (z_1, z_2, z_3)$, with $z_1 \in L'$, $z_2 \in P'$, and $z_3 \in Q$. Also, we identify L' with $\mathbb{R}$, so there is an order on L', and we assume that $0 < x < y$ in L'; the other case would be treaded similarly.

So we want to construct a deformation f and use it to reach a contradiction. We start with a first deformation f_1 that tries to send to P points that lie near L and roughly between x and y. The effect of this map will be to free from $E \setminus P$ a small tunnel near L. This map f_1 will be composed with other maps f_j, $j \geqslant 2$, that will be defined later; the main goal is to make most of $E \cap B_y$ disappear, by retracting it on a boundary; the tunnel will be used as a clean path that leads from x to y.

LEMMA 6.15. — We can find $\xi \in P' \cap B(0, r_x/100)$ and $\rho \in (r_x/2, 2r_x/3)$ such that $E \cap A$ does not meet the cone over $S_{\xi,\rho} = x + \xi + [Q \cap \partial B(0, \rho)]$.

Thus $S_{\xi,\rho}$ is a $(n-d-1)$-dimensional sphere centered at $x + \xi$. When $d = n - 1$, $S_{\xi,\rho}$ is just composed of two points that lie on different sides of P. See Figure 5 for two special cases when $n = 3$.

To prove Lemma 6.15, pick any small constant $\tau > 0$ and set $Z_\tau = \big\{(z_1, z_2, z_3) \in \pi_S(A \cap E)\,;\, z_1 \geqslant \tau\big\}$, where π_S still denotes the radial projection on the unit sphere. We know from (6.10) that $H^d(Z_\tau) = 0$. Then define $\psi_\tau : Z_\tau \to P' \times \mathbb{R}$, by $\psi_\tau(z_1, z_2, z_3) = (xz_2/z_1, x|z_3|/z_1)$. Notice that ψ_τ is Lipschitz (with a bound that depends on τ, but this does not matter) so $H^d(\psi_\tau(Z_\tau)) = 0$.

Recall that $P' \times \mathbb{R}$ is d-dimensional, so we can pick $\xi \in P' \cap B(x, r_x/100)$ and $\rho \in (r_x/2, 2r_x/3)$ such that (ξ, ρ) lies in no $\psi_\tau(Z_\tau)$. Let us check that the cone over $S_{\xi,\rho}$ does not meet $E \cap A$. Otherwise, we can find $z = (z_1, z_2, z_3) \in E \cap A$ and $\lambda \geqslant 0$ such that $\lambda z \in S_{\xi,\rho}$. This last forces $\lambda z_1 = x > 0$ (we identify x with its first component), so $z_1 > 0$ and $\pi_S(z)$ lies in Z_τ for τ small enough. Next $\psi_\tau(\pi_S(z)) = \psi_\tau(z/|z|) = (xz_2/z_1, x|z_3|/z_1) = (\lambda z_2, \lambda|z_3|)$ because $\lambda z_1 = x$. But $\lambda z \in S_{\xi,\rho}$, so $\lambda z_2 = \xi$ and $\lambda|z_3| = \rho$, hence $\psi_\tau(\pi_S(z)) = (\xi, \rho)$, which is impossible by choice of ξ and ρ. So the cone over $S_{\xi,\rho}$ does not meet $E \cap A$; this proves Lemma 6.15. $\square$
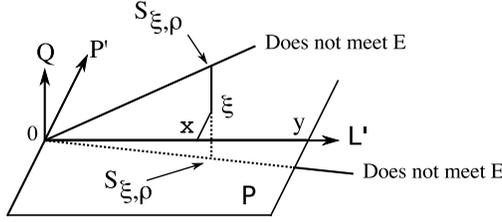
Figure 5. — $n = 3$, $d = 2$
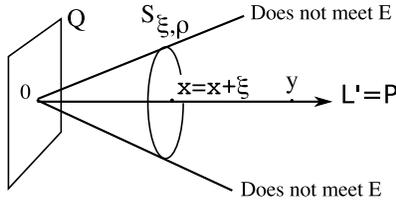
Figure 6. — $n = 3$, $d = 1$

Choose a pair (ξ, ρ) as in Lemma 6.15. For $a \leqslant s \leqslant t \leqslant b$, set

$$V(s, t) = \left\{ z \in \mathbb{R}^n \, ; \, s \leqslant z_1 \leqslant t \right\}. \tag{6.16}$$

With this notation, $S_{\xi,\rho}$ is contained in the vertical hyperplane $V(x, x)$. Pick a_1 and b_1 on the first axis, with $a < a_1 < b_1 < b$, so that

$$\overline{B}_x \cup \overline{B}_y \subset V(a_1, b_1). \tag{6.17}$$

Set $S_{\xi,\rho}^\varepsilon = \left\{ z \in V(x, x) \, ; \, \mathrm{dist}(z, S_{\xi,\rho} \leqslant 2\varepsilon \right\}$ for $\varepsilon > 0$. Also call $\mathcal{C}_\varepsilon$ the cone over $S_{\xi,\rho}^\varepsilon$. If r_x and ε are small enough, $\mathcal{C}_\varepsilon \cap V(a_1, b_1)$ is a compact subset of A. Then, by compactness and because the cone over $S_{\xi,\rho}$ does not meet $E \cap A$, we can choose ε so small that

$$\mathcal{C}_\varepsilon \cap V(a_1, b_1) \text{ does not meet } E. \tag{6.18}$$

Of course ε may be very small, so we shall need to be careful and avoid letting our final estimates depend on ε. Also choose ε smaller than $r_x/100$ to avoid possible geometric complications.

Our first map f_1 will act inside vertical hyperplanes $V(s, s)$, and we first want to describe how it works in $V(x, x)$. The reader may refer to Figure 7, which is a picture in $V(x, x)$. The dimensions of the picture correspond

to $n = 3$ and $d = 2$; in general, $V(x,x)$ is $n-1$ dimensional, P' is $(d-1)$-dimensional, and Q is $(n-d)$-dimensional, but the picture is simple in P', and things will be invariant under rotations in $x + Q$, so Figure 7 still gives a fair idea of what happens.

Set $J = P' \cap \overline{B}(x + \xi, r_x/10)$ and $S' = x + \xi + Q \cap \partial B(0, \rho + \varepsilon)$ (a sphere concentric with $S_{\xi,\rho}$, but just a bit larger), and then denote by T the convex hull of $J \cup S'$. Thus

$$T = \big\{ (x, z_2, z_3) \,;\, |z_2 - \xi| \leqslant (\rho + \varepsilon - |z_3|)\, r_x/(10\rho + 10\varepsilon) \big\}. \qquad (6.19)$$

Set $\widetilde{f}(z) = z$ in $V(x,x) \setminus T$. For $z \in T \setminus \mathcal{C}_\varepsilon$, write $z = (x, z_2, z_3)$, call $z^* = (x, \xi, \rho z_3/|z_3|)$ the closest point of $S_{\xi,\rho}$, and then let $\widetilde{f}(z)$ be the radial projection of z onto $P' \cup \partial T$, with origin z^*. That is, $\widetilde{f}(z)$ lies on $P' \cup \partial T$, z lies on the segment $[z^*, \widetilde{f}(z)]$, and the open segment does not meet $P' \cup \partial T$. Obviously $\widetilde{f}$ is continuous across ∂T.

The reader may be worried about the Lipschitz-continuity of $\widetilde{f}$ inside $T \setminus \mathcal{C}_\varepsilon$, so let us say a few words about that. Let us first check that

$$|z_3| \leqslant \rho - \varepsilon \quad \text{when} \quad (x, z_2, z_3) \in T \setminus \mathcal{C}_\varepsilon, \qquad (6.20)$$

which is already almost obvious from Figure 7. Otherwise, $|z_2 - \xi| \leqslant (\rho + \varepsilon - |z_3|)r_x/(10\rho + 10\varepsilon) \leqslant 2\varepsilon r_x/10\rho \leqslant 4\varepsilon/10$, by (6.19) and because $\rho \geqslant r_x/2$. Recall that $z^* = (x, \xi, \rho z_3/|z_3|)$, so $|z - z^*|^2 = |z_2 - \xi|^2 + (\rho - |z_3|)^2 \leqslant 16\varepsilon^2/100 + (\rho - |z_3|)^2 \leqslant 16\varepsilon^2/100 + \varepsilon^2 < 4\varepsilon^2$. This is impossible because $z \notin \mathcal{C}_\varepsilon$; (6.20) follows.
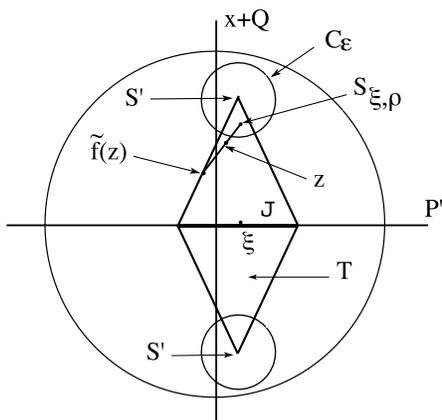

Figure 7. — A picture in $V(x,x)$

We may now compute the radial projection of z onto P', again with origin z^*. This point should be of the form $z' = z^* + \lambda(z - z^*)$, with $\lambda \geqslant 1$. Since the last coordinate $z_3^* + \lambda(z_3 - z_3^*)$ vanishes and $z_3^* = \rho z_3 / |z_3|$, we get that $\lambda = \rho/(\rho - |z_3|)$, which is indeed defined by (6.20) and larger than 1. In addition, λ and z' are Lipschitz functions of $z \in T \setminus \mathcal{C}_\varepsilon$. Now the radial projection of z onto ∂T, again with origin z^*, is also Lipschitz on $T \setminus \mathcal{C}_\varepsilon$. And so is $\widetilde{f}(z)$, because the infimum of two Lipschitz functions λ is Lipschitz. Since $\widetilde{f}$ is continuous across ∂T, $\widetilde{f}$ is Lipschitz on $V(x, x) \setminus \mathcal{C}_\varepsilon$.

Extend $\widetilde{f}$ to the whole $V(x, x)$, so that it is Lipschitz and $\widetilde{f}(T) \subset T$. [If this last condition is not satisfied, we can always compose the restriction of $\widetilde{f}$ to T with a Lipschitz retraction onto T.] Finally extend $\widetilde{f}$ to $V(a_1, b_1)$, by homogeneity: set

$$\widetilde{f}(z_1, z_2, z_3) = \frac{z_1}{x} \, \widetilde{f}(x, z_2 x / z_1, z_3 x / z_1). \tag{6.21}$$

Notice that $\widetilde{f}$ moves the points inside vertical planes $V(s, s)$.

We are now ready to define f_1. Set

$$x_0 = x - \frac{r_x}{5} \quad \text{and} \quad y_0 = y + \frac{r_y}{5}, \tag{6.22}$$

and then let ψ be the continuous cut-off function such that

$$\psi(t) = 0 \text{ out of } (x_0, y_0), \ \psi(t) = 1 \text{ on } [x, y], \quad \text{and}$$
$$\psi \text{ is affine on the two remaining intervals.} \tag{6.23}$$

Finally set

$$f_1(z) = \psi(z_1)\widetilde{f}(z) + (1 - \psi(z_1)) z \quad \text{for } z \in \mathbb{R}^n, \tag{6.24}$$

where z_1 still denotes the first coordinate of z. Notice that $z \in V(a_1, b_1)$ when $\psi(z_1) \neq 0$, by (6.17).

Let us record a few properties of f_1 before we compose with a second deformation. As promised, f_1 only moves the points inside vertical planes $V(s, s)$. Notice also that

$$f_1(z) = z \quad \text{out of } \ \widehat{T} \cap V(x_0, y_0), \tag{6.25}$$

where $\widehat{T}$ denotes the cone over T, and that

$$f_1\big(\widehat{T} \cap V(x_0, y_0)\big) \subset \widehat{T} \cap V(x_0, y_0). \tag{6.26}$$

We also want to see where the set $E_1 = f_1(E)$ lies. Let us check that

$$E_1 \cap \widehat{T} \cap V(x, y) \subset \big[\partial\widehat{T} \cup \widehat{J}\big] \cap V(x, y), \tag{6.27}$$

where $\widehat{J}$ is the cone over J (also notice that $\partial \widehat{T}$ is the cone over the boundary of T in $V(x,x)$). Indeed, let $w \in E_1 \cap \widehat{T} \cap V(x,y)$ be given, and let $z \in E$ be such that $f(z) = w$. By (6.25), $z \in \widehat{T}$. Also, $z \in V(x,y)$ because f_1 moves points vertically. By (6.23) and (6.24), $w = \widetilde{f}(z)$. By (6.21), $\widetilde{f}(z) = \frac{z_1}{x}\widetilde{f}(xz/z_1)$. By (6.18), z and xz/z_1 lie out of C_ε, and then $\widetilde{f}(xz/z_1)$ is the projection on $\partial T \cup J$ of some point z^*; (6.27) follows.

Near x, things are also easy. Indeed $\widehat{T} \cap V(x_0, x) \subset B_x$ by construction, and B_x does not meet E by (6.14), so

$$E_1 \cap \widehat{T} \cap V(x_0, x) = \emptyset \tag{6.28}$$

(again recall that f_1 only moves points on planes $V(s,s)$, and inside $\widehat{T}$).

Near y, we want to use the fact that

$$\operatorname{dist}(z, P) \leqslant \varepsilon_0 |z - y| \tag{6.29}$$

for $z \in E \cap 2B_y$ (by (6.13)) to show that

$$\operatorname{dist}(w, P) \leqslant 2\varepsilon_0 |w - y| \text{ for } w \in E_1 \cap \overline{B}_y \tag{6.30}$$

Let $w \in E_1 \cap \overline{B}_y$ be given, and let $z \in E$ be such that $w = f_1(z)$. Notice that $z \in V(y - r_y, y + r_y)$ because f_1 moves points vertically. If $f_1(z) = z$, then $\operatorname{dist}(w, P) = \operatorname{dist}(z, P) \leqslant \varepsilon_0 |w - y|$ by (6.29). Otherwise, $z \in \widehat{T}$ by (6.25). If r_x is small enough (depending on x, y, and r_y, but this is all right), $\widehat{T} \cap V(y - r_y, y + r_y) \subset 2B_y$, so $z \in 2B_y$ and (6.29) holds.

Set $z' = xz/z_1$, and let us first worry about $\widetilde{f}(z')$. Recall from (6.18) that z and z' lie out of C_ε; then $z' \in T \setminus C_\varepsilon$, and (6.20) says that $|z_3'| \leqslant \rho - \varepsilon$. The point z^* used to compute $\widetilde{f}(z')$ lies "above" z', and so

$$\operatorname{dist}(\widetilde{f}(z'), P) = |\widetilde{f}(z')_3| \leqslant |z_3'| = \operatorname{dist}(z', P). \tag{6.31}$$

Let us also check that

$$|\widetilde{f}(z') - z'| \leqslant 6 \operatorname{dist}(z', P). \tag{6.32}$$

If $\operatorname{dist}(z', P) \geqslant \rho/2$, then (6.32) holds because the diameter of T is less than 3ρ (recall that $\rho \geqslant r_x/2 \geqslant 5 \operatorname{diam}(J)/2$). Otherwise, set $z^* = (x, \xi, \rho z_3'/|z_3'|)$, as in the definition of $\widetilde{f}(z')$, and observe that $\widetilde{f}(z') - z^* = \lambda(z' - z^*)$, for some $\lambda \geqslant 1$. In addition, $\lambda \leqslant \rho/(\rho - |z_3'|)$, which corresponds to the radial projection of z' onto P; see the computation above (6.21). Then

$$|\widetilde{f}(z') - z'| \leqslant (\lambda - 1)|z^* - z'| \leqslant \frac{|z_3'|}{\rho - |z_3'|}|z^* - z'| \leqslant 6|z_3'| = 6 \operatorname{dist}(z', P) \tag{6.33}$$

because $\rho - |z_3'| \geqslant \rho/2$ and $|z^* - z'| \leqslant \operatorname{diam}(T) \leqslant 3\rho$. So (6.32) holds in all cases.

By homogeneity, (6.32) yields $|\widetilde{f}(z) - z| \leqslant 6 \operatorname{dist}(z, P)$. Then

$$|f_1(z) - z| \leqslant 6 \operatorname{dist}(z, P) \leqslant 6\varepsilon_0 |z - y|, \qquad (6.34)$$

because $f_1(z)$ lies on the segment $[z, \widetilde{f}(z)]$, and then because (6.29) holds for z. Similarly, $\operatorname{dist}(\widetilde{f}(z), P) \leqslant \operatorname{dist}(z, P)$ by (6.31) and homogeneity, and then

$$\operatorname{dist}(f_1(z), P) \leqslant \operatorname{dist}(z, P) \qquad (6.35)$$

because $f_1(z)$ lies on $[z, \widetilde{f}(z)]$.

Next $|f_1(z) - y| \geqslant |z - y| - |f_1(z) - z| \geqslant (1 - 6\varepsilon_0)|z - y|$, by (6.34). Therefore

$$\operatorname{dist}(f_1(z), P) \leqslant \operatorname{dist}(z, P) \leqslant \varepsilon_0 |z - y| \leqslant 2\varepsilon_0 |f_1(z) - y| \qquad (6.36)$$

by (6.35) and (6.29). Recall that $w = f_1(z)$; hence (6.36) proves (6.30).

To summarize, $E_1 \cap \widehat{T} \cap V(x, y)$ is contained in the union of the flat floor $\widehat{J}$ and the triangular tunnel $\partial \widehat{T}$, by (6.27). In addition, there is no piece of E_1 at the end of $\widehat{T}$ near x (by (6.28)), and near y we know that E_1 stays very close to P (by (6.30)), so the end of the tunnel near y is not completely blocked by E_1 either.

We are now ready to construct a second deformation f_2, to be composed with f_1. Its goal is to project $E_1 \cap B_y$ onto P. We take

$$f_2(z) = z \quad \text{for } z \in \mathbb{R}^n \setminus B_y, \qquad (6.37)$$

and inside B_y we want f_2 to look like a radial projection onto $P \cup \partial B_y$. It will act a little like f_1 above, but we won't need to move points in vertical planes. On the other hand, we want to keep the invariance under rotations in Q.

Set $S_y = (y/x)S_{\xi, \rho} = y + (y/x)\xi + Q \cap \partial B(0, \rho_1)$, where we set $\rho_1 = y\rho/x$. The precise choice of sphere will be useful later. Also set $S_{y,+} = \{z \in \mathbb{R}^n \, ; \, \operatorname{dist}(z, S_y) \leqslant \rho_1/100\}$. Notice that S_y and $S_{y,+}$ both lie well inside B_y, because $\rho \leqslant r_x$ and we take r_x much smaller than r_y. See Figure 8.
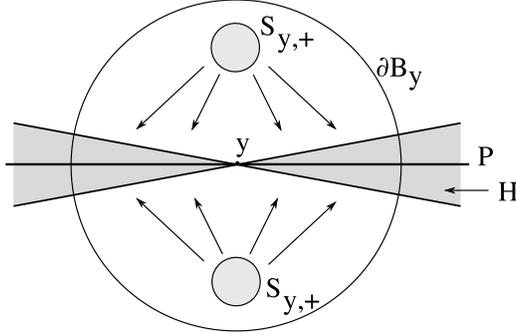
Figure 8

For $z \in B_y \setminus S_{y,+}$, denote by $z^\sharp$ the closest point of S_y. Thus $z^\sharp = (y, (y/x)\xi, \rho_1 z_3/|z_3|)$. Then let $f_2(z)$ be the radial projection of z onto $\partial B_y \cup P$, centered at $z^\sharp$. Let us check that f_2 is Lipschitz on $B_y \setminus S_{y,+}$. Let $\tau > 0$ be very small. In the region where $|z_3| \leqslant \rho_1 - \tau$, the radial projection of z onto P is well defined, and Lipschitz (we even checked below (6.20) that it is equal to $z^\sharp + \lambda(z - z^\sharp)$, with $\lambda = \rho_1(\rho_1 - |z_3|)^{-1}$. Then f_2 is Lipschitz on this region. In the region where $|z_3| > \rho_1 - \tau$, and if τ is small enough, the projection of z onto P is either not defined, or else lies out of B_y. Then $f_2(z)$ is the projection onto ∂B_y, and it is Lipschitz on the region. Notice also that $f_2(z) = z$ on ∂B_y, so it is continuous across ∂B_y and Lipschitz on $\mathbb{R}^n \setminus S_{y,+}$.

We now extend f_2 to the whole $\mathbb{R}^n$, so that it is Lipschitz (probably with a very bad constant) and maps $\overline{B}_y$ to itself (otherwise, compose its restriction to $\overline{B}_y$ with a Lipschitz retraction onto $\overline{B}_y$).

We need to know where the set $E_2 = f_2(E_1)$ lies. Notice that

$$E_2 \setminus \overline{B}_y = E_1 \setminus \overline{B}_y \quad \text{and} \quad E_2 \cap \overline{B}_y = f_2(E_1 \cap \overline{B}_y) \tag{6.38}$$

by (6.37) and because $f_2(\overline{B}_y) \subset \overline{B}_y$. Set

$$H = \left\{ z \in \mathbb{R}^n ; \, \mathrm{dist}(z, P) \leqslant 2\varepsilon_0 |z - y| \right\}; \tag{6.39}$$

thus (6.30) says that $E_1 \cap \overline{B}_y$ is contained in H. It will be good to know that

$$E_1 \cap \overline{B}_y \subset \overline{B}_y \setminus S_{y,+}, \tag{6.40}$$

and for this it is enough to check that $S_{y,+}$ does not meet H. Let $z = (z_1, z_2, z_3) \in S_{y,+}$ be given; then $||z_3| - \rho_1| \leqslant \mathrm{dist}(z, S_y) \leqslant \rho_1/100$, so $\mathrm{dist}(z, P) = |z_3| \geqslant 99\rho_1/100$. On the other hand, $|z - y| \leqslant \mathrm{dist}(z, S_y) + \rho_1$

$+ |(y/x)\xi - y| \leqslant \rho_1/100 + \rho_1 + yr_x/100x \leqslant \rho_1/100 + \rho_1 + y\rho/50x \leqslant 2\rho_1$ (see the statement of Lemma 6.15 and recall that $\rho_1 = y\rho/x$). So z lies out of H; (6.40) follows.

Next we check that

$$E_2 \cap \overline{B}_y \subset P \cup [\partial B_y \cap H]. \tag{6.41}$$

Let $w \in E_2 \cap \overline{B}_y$ be given, and let $z \in E_1$ be such that $w = f_2(z)$. By (6.37), $z \in \overline{B}_y$. By (6.40), z lies out of $S_{y,+}$, so w is the radial projection of z onto $P \cap \partial B_y$, with origin $z^\sharp$ as above. In particular, $w \in P \cup \partial B_y$, and we just have to check that $w \in H$ if $w \notin P$.

First assume that $z, z^\sharp$, and y are not collinear, and let V denote the plane that contains them. Also assume that $P \cap V$ is a line; then $H \cap V$ is a thin double cone centered at y that contains $P \cap V$. Now $z^\sharp$ lies out of H (because $z^\sharp \in S_y \subset S_{y,+}$), and $z \in H$ by (6.30). So the half line $[z^\sharp, z)$ starts out of H, then gets in H before it hits z (or at the same time), and it does not leave H before it meets P. This last may not happen, but if it happens, it happens after $[z^\sharp, z)$ hits z (the open line segment $(z^\sharp, z)$ does not meet P by definition of $z^\sharp$). In both cases, $w = f_2(z)$ lies in H.

If $P \cap V$ is not a line, it is reduced to $\{y\}$ (because $z^\sharp$ lies out ou P). Then the line through z and $z^\sharp$ does not meet P (recall that $z, z^\sharp$, and y are not collinear). By definition of $z^\sharp = (y, (y/x)\xi, \rho_1 z_3/|z_3|)$, this can only happen if $|z_3| = \rho_1$. In this case, $\text{dist}(w, P) = \text{dist}(z, P) \leqslant 2\varepsilon_0|z - y| \leqslant 2\varepsilon_0 r_y$, because $z \in H \cap \overline{B}_y$. In addition, $w \in \partial B_y$ (because it does not lie in P), hence $|w - y| = r_y$ and $w \in H$.

Finally suppose that $z, z^\sharp$, and y are collinear. In this case z lies between $z^\sharp$ and y (because otherwise z would not lie in H; already $z^\sharp$ does not), and $w = y$ (because $y \in P$ and $[z^\sharp, y]$ does not meet ∂B_y). This completes our proof of (6.41).

Let us also check that E_2 does not block our access to the tube $\widehat{T}$, i.e., that

$$E_2 \cap \overline{B}_y \cap V(x, y) \text{ does not meet } \text{int}(\widehat{T}) \setminus P. \tag{6.42}$$

Let $w \in E_2 \cap \overline{B}_y \cap V(x, y)$ be given, and suppose it lies in $\text{int}(\widehat{T}) \setminus P$. Let $z \in E_1$ be such that $f_2(z) = w$. Then $z \in \overline{B}_y$, because otherwise $w = z$ by (6.37), and (6.27) says that this is impossible. In particular, w is obtained from z by the usual radial projection process with $z^\sharp$. Then $z \notin P$, because otherwise $w = z$, and we assumed that $w \notin P$.

Since $w \in V(x, y)$ and $z^\sharp \in V(y, y)$, z lies in $V(x, y)$. Then (6.27) says that z lies out of $\text{int}(\widehat{T})$. Call w', z', and z^* the radial projections of $w, z,,$

and $z^\sharp$ on $V(x,x)$. Thus $w' \in \text{int}(T)$, $z' \notin \text{int}(T)$, and $z^* = (x, \xi, \rho z_3'/|z_3'|)$ lies in $\text{int}(T)$. This is impossible, because T is convex (see near (6.19)) and z' lies on $[z^*, w]$. So (6.42) holds.

Let us summarize the situation now. Set

$$Z = [\widehat{T} \cap V(x_0, y)] \cup \overline{B}_y. \tag{6.43}$$

Recall from (6.25), the fact that $\widehat{T} \cap V(y, y_0) \subset B_y$ if r_x is small enough (see (6.22)), and (6.37) that

$$f_2(z) = f_1(z) = z \quad \text{for } z \in \mathbb{R}^n \setminus Z. \tag{6.44}$$

Let us also check that

$$f_2 \circ f_1(Z) \subset Z. \tag{6.45}$$

First take $z \in \widehat{T} \cap V(x_0, y)$, $f_1(z) \in \widehat{T} \cap V(x_0, y)$, by (6.26) and because f_1 moves points vertically. If $f_1(z)$ lies in $\overline{B}_y$, $f_2 \circ f_1(z) \in \overline{B}_y \subset Z$ because $f_2(\overline{B}_y) \subset \overline{B}_y$. Otherwise, $f_2 \circ f_1(z) = f_1(z) \in \widehat{T} \cap V(x_0, y) \subset Z$, by (6.37). Next suppose that $z \in \widehat{T} \cap V(y, y_0)$; then $f_1(z) \in \widehat{T} \cap V(y, y_0) \subset B_y$ by (6.26), because f_1 moves points vertically, and because r_x is very small. Then $f_2 \circ f_1(z) \in \overline{B}_y \subset Z$, as before. Finally, if $z \in \overline{B}_y \setminus [\widehat{T} \cap V(x_0, y_0)]$, $f_1(z) = z$ by (6.25), and $f_2 \circ f_1(z) = f_2(z) \in \overline{B}_y \subset Z$; (6.45) follows.

We now verify that

$$E_2 \cap Z \subset P \cup [\partial\widehat{T} \cap V(x, y) \setminus B_y] \cup [\partial B_y \cap H \setminus (\text{int}(\widehat{T}) \cap V(x, y))]. \tag{6.46}$$

Indeed there is no part in $V(x_0, x)$ because of (6.28) and (6.38). The part of $E_2 \cap Z \setminus P$ that lies in $V(x, y) \setminus \overline{B}_y$ lies in $\partial\widehat{T}$, by (6.27) and (6.38). The rest lies in $\overline{B}_y$, hence in $\partial B_y \cap H$ by (6.41), and out of $\text{int}(\widehat{T}) \cap V(x, y)$ by (6.42).

This is somewhat better than before; we now have a nice floor $P \cap Z$, and we managed to surround its interior with a tunnel that leads from x to y and a big cavity $B_y \setminus P$.

Our third mapping f_3 will contract most of the floor $P \cap Z$ onto its boundary; this will be good, because it will make a big part of the floor vanish. More precisely, set $F = P \cap Z \cap V(x, b)$. See Figure 9. Thus F is the union of $\widehat{J} \cap V(x, y)$ and $P \cap \overline{B}_y$, so it looks like a long d-dimensional cylinder attached to a much larger d-ball. [Recall that $\widehat{J}$ is the cone over the $(d-1)$-ball $J = P' \cap \overline{B}(x + \xi, r_x/10)$; see above (6.19).] There is a bilipschitz mapping ψ from F to the cylinder $C = [x, y] \times J \times \{0\}$, and which fixes the points of the end $F \cap V(x, x) = \{x\} \times J \times \{0\}$.

To contract C to a piece of its boundary, we use the mapping φ which to $z \in C$ associates the radial projection of z onto the box $\big[[x,y] \times \partial J \times \{0\}\big] \cup \big[\{y\} \times J \times \{0\}\big]$ (open at one side) with center $(x_0, \xi, 0)$. [We don't care about the precise value; the point is that $x_0 < x$, so φ is Lipschitz. The contraction that we use on F is $\psi^{-1} \circ \varphi \circ \psi$. It is Lipschitz on F, even though with a very large constant. We want to set

$$f_3(z) = \psi^{-1} \circ \varphi \circ \psi(z) \ \text{ for } z \in F \tag{6.47}$$

and

$$f_3(z) = z \ \text{ for } z \in \overline{\mathbb{R}^n \setminus Z}, \tag{6.48}$$

so let us check that this defines a Lipschitz function f_3 on $F \cup (\overline{\mathbb{R}^n \setminus Z})$. The main point is that $f_3(z) = z$ on the part of F that touches $\overline{\mathbb{R}^n \setminus Z}$; the rest of the verification is straightforward. Indeed, we want to check that $|f_3(z_2) - f_3(z_1)| \leqslant C|z_2 - z_1|$ when z_1 and z_2 lie in $F \cup (\overline{\mathbb{R}^n \setminus Z})$. This is trivial if z_1 and z_2 both lie in F or in $\overline{\mathbb{R}^n \setminus Z}$, so we may assume that $z_1 \in F$ and $z_2 \in \overline{\mathbb{R}^n \setminus Z}$. But, due to the simple geometry of Z, we can find $z_3 \in F \cap \partial Z$ such that $|z_3 - z_1| \leqslant C|z_2 - z_1|$, and then $|f_3(z_2) - f_3(z_1)| \leqslant |f_3(z_2) - f_3(z_3)| + |f_3(z_3) - f_3(z_1)| \leqslant |z_3 - z_2| + C|z_3 - z_1| \leqslant |z_3 - z_1| + |z_1 - z_2| + C|z_3 - z_1| \leqslant C'|z_2 - z_1|$, as needed.
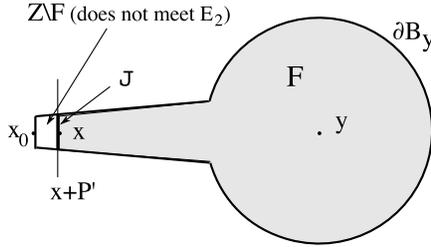
Figure 9. — A picture in P

Extend f_3 to the rest of Z, so that

$$f_3 \text{ is Lipschitz on } \mathbb{R}^n \text{ and } f_3(Z) \subset Z \tag{6.49}$$

(the existence of a Lipschitz extension is a well-known general fact, and we can arrange that $f_3(Z) \subset Z$ by composing the restriction of f_3 to Z with a Lipschitz retraction onto Z (such a retraction exists because there is a bilipschitz mapping of $\mathbb{R}^n$ that maps Z to a cylinder).

Set $E_3 = f_3(E_2)$, and let us check that

$$E_3 \cap Z \subset Z_1 \cup Z_2, \tag{6.50}$$

where

$$Z_1 = V(x,y) \cap \partial \widehat{T} \setminus B_y \quad \text{and} \quad Z_2 = \partial B_y \cap H. \qquad (6.51)$$

Let $w \in E_3 \cap Z$ and let $z \in E_2$ be such that $w = f_3(z)$. Notice that $z \in Z$ because $f(z) = z$ for $z \notin Z$. Next, z lies out of $V(x_0,x)$, because $E_2 \cap V(x_0,x) = E_1 \cap V(x_0,x)$ by (6.38), and $E_1 \cap Z \cap V(x_0,x) \subset E_1 \cap \widehat{T} \cap V(x_0,x) = \emptyset$ by (6.43) and (6.28).

First suppose that $z \in P$. Then $z \in F = P \cap Z \cap V(x,b)$, and (6.47) shows that $w = f_3(z) \in \psi^{-1}\big(\big[[x,y] \times \partial J \times \{0\} \big] \cup \big[\{y\} \times J \times \{0\} \big] \big)$. This last set is the closure of $\partial F \setminus V(x,x)$ (where ∂F denotes the boundary of F in P), and it is contained in $Z_1 \cup Z_2$.

Now suppose that $z \notin P$; then (6.46) says that $z \in \partial Z$, so $w = f_3(z) = z$ by (6.48), and w lies in the set in $Z_1 \cup Z_2$ by (6.46) again. This proves (6.50).

Let us also record the fact that

$$f_3 \circ f_2 \circ f_1(z) = z \text{ for } z \notin Z, \text{ and } f_3 \circ f_2 \circ f_1(Z) \subset Z. \qquad (6.52)$$

When $d = n-1$, this is enough to conclude because $H^{n-1}(E_3 \cap Z) \leqslant H^{n-1}(Z_1 \cup Z_2) \leqslant C\varepsilon_0 r_y^d + C_{x,y} r_x^{d-1}$ (maybe see the proof of (6.80) if you are worried about the second term), and for this case the reader may set $f_4(z) = f_5(z) = z$ for all z, $Z^* = Z$, and go directly to the line below (6.89) for the final accounting that leads to a contradiction.

For the general case, we still need to compose $f_3 \circ f_2 \circ f_1$ with two last mappings f_4 and f_5, whose effect will be to project the potentially large sets $E_3 \cap Z_1$ and $E_3 \cap Z_2$ onto d-dimensional skeletons of cubes. In both cases, we shall use a standard Federer-Fleming argument.

Some amount of notation concerning dyadic cubes will be needed. Here dyadic cubes are closed cubes obtained from $[0, 2^k]^n$ for some $k \in Z$ by a translation in $2^k Z^n$. For such a dyadic cube Q, we shall denote by $\mathcal{S}_d(Q)$ the d-dimensional skeleton of Q, i.e., the union of the d-dimensional faces of Q.

Also let $j \in N$ be given; in this paper j will be chosen once and for all, sufficiently large for some geometric conditions explained below to hold, but for all practical purposes it is a geometric constant that depends only on n. We denote by $\Delta_j(Q)$ the set of dyadic cubes of sidelength 2^{k-j} that are contained in Q, and then by $\mathcal{S}_{d,j}(Q)$ the union of all the d-dimensional faces of dyadic cubes $R \in \Delta_j(Q)$. In other words, $\mathcal{S}_{d,j}(Q) = \bigcup_{R \in \Delta_j(Q)} \mathcal{S}_d(R)$.

We shall use the following lemma, which is part of Proposition 3.1 on page 13 of [DS]. We also refer to [DS] for additional background on dyadic cubes and skeletons.

LEMMA 6.53. — *Let Q be a dyadic cube in $\mathbb{R}^n$, and let $F \subset Q$ be closed, with $H^d(F) < +\infty$. Then there is a Lipschitz mapping $\varphi_Q : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\varphi_Q(z) = z \quad for \ z \in \overline{\mathbb{R}^n \setminus Q}. \tag{6.54}$$

$$\varphi_Q(F \cap Q) \subset \mathcal{S}_{d,j}(Q) \cup \partial Q, \tag{6.55}$$

$$\varphi_Q(R) \subset R \quad for \ R \in \Delta_j(Q), \tag{6.56}$$

and

$$H^d(\varphi_Q(F \cap R)) \leqslant C H^d(F \cap R) \quad for \ R \in \Delta_j(Q). \tag{6.57}$$

The constant C depends only on n and d, not on Q, j, or F.

We want to start with the function f_4 that will take care of $E_3 \cap Z_1$. Let $k \in Z$ be such that

$$r_x/32 \leqslant 2^k \sqrt{n} \leqslant r_x/16. \tag{6.58}$$

Let $\mathcal{D}_k$ denote the set of dyadic (closed) cubes Q of $\mathbb{R}^n$ with sidelength 2^k such that $\operatorname{dist}(Q, Z_1) \leqslant 2^{k+1} \sqrt{n}$. For each $Q \in \mathcal{D}_k$, we apply Lemma 6.53 to Q and the set $F = E_3 \cap Q$, and get a mapping φ_Q.

We compose all the mappings φ_Q, $Q \in \mathcal{D}_k$, (in some randomly decided order) and get a mapping ϕ_0. Notice that for a given $z \in \mathbb{R}^n$, at most one φ_Q may move z, the one (if it exists) for which z lies in the interior of Q. Even then, $\varphi_Q(z) \in Q$, and by (6.54), the other $\varphi_{Q'}$, $Q' \neq Q$, do not move z or $\varphi_Q(z)$. Because of this, ϕ_0 does not change if we compose the φ_Q, $Q \in \mathcal{D}_k$, in any other order.

Set $E_{3,0} = \phi_0(E_3)$. It should be a little better than E_3 near Z_1 because we were able to send lots of points on the $\mathcal{S}_{d,j}(Q)$, but there might still be some part of $\phi_0(E_3)$ left on the boundaries ∂Q. To fix this, we shall compose with variants of ϕ_0 for which the boundaries ∂Q lie in different places. Denote by $v_0, v_1, \cdots v_m$ the vectors of $\mathbb{R}^n$ whose coordinates are either 0 at 2^{k-1}. Thus $m = 2^n - 1$. To make the notation coherent, do this so that $v_0 = 0$.

We now construct ϕ_1. For each $Q \in \mathcal{D}_k$, we apply Lemma 6.53 to $Q + v_1$ and the set $F = E_{3,0} \cap [Q + v_1]$. The fact that Q is not a dyadic cube itself, but just a translation of such a cube by v_1, does not matter, but we shall use the fact that the cubes $R \in \Delta_j(Q + v_1)$ are still dyadic cubes of sidelength 2^{k-j}.

Call $\varphi_{Q,1}$ the map given by Lemma 6.53. Then compose the new $\varphi_{Q,1}$ in any order, and get a new mapping ϕ_1. Finally set $E_{3,1} = \phi_1(E_{3,0})$.

We iterate this construction in the obvious way: once we have ϕ_{l-1} and $E_{3,l-1}$ for some $l \leqslant m$, we apply Lemma 6.53 to the $Q + v_l$, $Q \in \mathcal{D}_k$, with $F = E_{3,l-1} \cap [Q + v_l]$. This gives a mapping $\varphi_{Q,l}$. We compose all these mappings to get ϕ_l, and set $E_{3,l} = \phi_l(E_{3,l-1})$.

We end with ϕ_m, and set

$$f_4 = \phi_m \circ \cdots \phi_0 \text{ and } E_4 = f_4(E_3) = E_{3,m}. \tag{6.59}$$

Our next goal is to control E_4 away from Z_2, as follows.

LEMMA 6.60. — *Let R be a dyadic cube of sidelength 2^{k-j} such that*

$$\operatorname{dist}(R, Z_1 \cup Z_2) \leqslant r_x/2 \text{ and } \operatorname{dist}(R, Z_2) \geqslant 2^k. \tag{6.61}$$

Then

$$H^d(R \cap E_4) \leqslant C2^{kd}. \tag{6.62}$$

Let us first see how $H^d(E_{3,l} \cap R)$ evolves with time. Set $E_{3,-1} = E_3$ and $\varphi_{Q,0} = \varphi_Q$ to make the notation uniform. Recall from (6.57) with $F = E_{3,l-1} \cap [Q + v_l]$ that

$$H^d(\varphi_{Q,l}(R \cap E_{3,l-1})) \leqslant CH^d(R \cap E_{3,l-1}) \tag{6.63}$$

for $0 \leqslant l \leqslant m$ and $Q \in \mathcal{D}_k$. This is only useful when Q is the cube such that $Q + v_l$ contains R, because for the other cubes Q', (6.54) says that $\varphi_{Q',l}(z) = z$ on R. Thus

$$H^d(\phi_l(R \cap E_{3,l-1})) \leqslant CH^d(R \cap E_{3,l-1}). \tag{6.64}$$

Observe also that $\phi_l(R') \subset R'$ for every dyadic cube R' of sidelength 2^{k-j}, because (6.54) and (6.56) say that this is true for every $\varphi_{Q,l}$. Thus, if $z \in R'$ and $\phi_l(z)$ lies in R, R' meets R. That is,

$$R \cap E_{3,l} = R \cap \phi_l(E_{3,l-1}) \subset \bigcup_{R' \in \mathcal{N}(R)} \phi_l(R' \cap E_{3,l-1}), \tag{6.65}$$

where we denote by $\mathcal{N}(R)$ the set of dyadic cubes R' of sidelength 2^{k-j} that touch R. Consequently,

$$H^d(R \cap E_{3,l}) \leqslant \sum_{R' \in \mathcal{N}(R)} H^d(\phi_l(R' \cap E_{3,l-1})) \leqslant C \sum_{R' \in \mathcal{N}(R)} H^d(R' \cap E_{3,l-1}), \tag{6.66}$$

by (6.64). Iterations of this yield

$$H^d(R \cap E_4) = H^d(R \cap E_{3,m}) \leqslant C \sum_{R' \in \mathcal{N}_{m+1}(R)} H^d(R' \cap E_3), \qquad (6.67)$$

where the set $\mathcal{N}_l(R)$ of neighbors of order l of R is just defined by induction by $\mathcal{N}_1(R) = \mathcal{N}(R)$ and $\mathcal{N}_{l+1}(R) = \bigcup_{R' \in \mathcal{N}_l(R)} \mathcal{N}(R')$. An obvious induction yields

$$\mathrm{dist}(R', R) \leqslant (l-1)2^{k-j}\sqrt{n} \ \ \text{for } R' \in \mathcal{N}_l(R), \qquad (6.68)$$

so, if z_0 is any point of R,

$$\bigcup_{R' \in \mathcal{N}_{m+1}(R)} R' \subset B\big(z_0, (m+2)2^{k-j}\sqrt{n}\big) = B\big(z_0, (2^n+1)\,2^{k-j}\sqrt{n}\big) \subset B\big(z_0, 2^{k-2}\big)$$

$$(6.69)$$

because $m = 2^n - 1$, and if j is chosen large enough.

When R lies far enough from $Z_1 \cap Z_2$, all this will lead to (6.62) rather directly. Let us first check that for a dyadic cube S of sidelength 2^{k-j},

$$H^d(E_3 \cap S) \leqslant C2^{kd} \text{ when } \mathrm{dist}(S, Z_1 \cup Z_2) \leqslant r_x \text{ and } S \text{ does not meet } Z_1 \cup Z_2.$$
$$(6.70)$$

Indeed S does not meet $E_3 \cap Z$ by (6.50), and then $S \cap E_3 \subset S \cap E$ by (6.52). So (6.70) follows from the local Ahlfors-regularity of E (Lemma 2.15 above, but only the easier upper bound and in the easier special case of locally minimal sets); this is why we assumed that $\mathrm{dist}(S, Z_1 \cup Z_2) \leqslant r_x$, to make sure that $S \subset B$ for some small ball B such that $2B \subset A \subset B(0,b)$. See in particular (6.14), the definition of A above (6.5), and the statement of Theorem 6.2.

Now we can already check (6.62) when

$$2^k \leqslant \mathrm{dist}(R, Z_1 \cup Z_2) \leqslant r_x/2. \qquad (6.71)$$

Indeed, by (6.69) and (6.58), all the cubes $S \in \mathcal{N}_{m+1}(R)$ satisfy the condition in (6.70), and (6.62) follows from (6.67). We are thus left with the case of cubes R such that $\mathrm{dist}(R, Z_1 \cup Z_2) < 2^k$ and hence, by (6.61)

$$\mathrm{dist}(R, Z_1) \leqslant 2^k. \qquad (6.72)$$

Pick $z_0 \in R$, denote its coordinates by $z_{0,i}$, $1 \leqslant i \leqslant n$, and then choose $\varepsilon_i \in \{0, 2^{k-1}\}$ so that $z_{0,i} - \varepsilon_i$ lies at distance at least 2^{k-2} from $2^k\mathbb{Z}$. Recall that there is a v_l such that $v_l = (\varepsilon_1, \cdots, \varepsilon_n)$, and then every coordinate of $z_0 - v_l$ lies at distance at least 2^{k-2} from $2^k\mathbb{Z}$. Now

$$\mathrm{dist}(z_0, v_l + \partial Q) \geqslant 2^{k-2} \ \text{ for } Q \in \mathcal{D}_k, \qquad (6.73)$$

because the points in the faces that compose ∂Q have at least one coordinate in $2^k \mathbb{Z}$.

Now let S denote any cube of $\mathcal{N}_{m+1}(R)$, and let Q be such that $Q + v_l$ contains S. Observe that

$$
\begin{aligned}
\text{dist}(Q, Z_1) &\leqslant |v_l| + \text{dist}(S, Z_1) \leqslant |v_l| + 2^{k-2} + \text{dist}(z_0, Z_1) \\
&\leqslant |v_l| + 2^{k-2} + \text{diam}(R) + \text{dist}(R, Z_1) \\
&\leqslant 2^{k-1}\sqrt{n} + 2^{k-2} + 2^{k-j}\sqrt{n} + \text{dist}(R, Z_1) \leqslant 2^{k+1}\sqrt{n}
\end{aligned}
\tag{6.74}
$$

by (6.69) and (6.72) in particular. So $Q \in \mathcal{D}_k$ (see the definition below (6.58)), and (6.55) says that

$$
\varphi_{Q,l}(E_{3,l-1} \cap S) \subset \varphi_{Q,l}(E_{3,l-1} \cap [v_l + Q]) \subset v_l + [\mathcal{S}_{d,j}(Q) \cup \partial Q]. \tag{6.75}
$$

Now $S \in \mathcal{N}_{m+1}(R)$, so $S \subset B(z_0, 2^{k-2})$ by (6.69), and (6.73) says that S does not meet $v_l + \partial Q$. Recall from (6.56) that $\varphi_{Q,l}(S) \subset S$; then $\varphi_{Q,l}(E_{3,l-1} \cap S) \subset v_l + \mathcal{S}_{d,j}(Q)$, and $H^d(\varphi_{Q,l}(E_{3,l-1} \cap S)) \leqslant C2^{kd}$. Now

$$
H^d(\phi_l(E_{3,l-1} \cap S)) \leqslant C2^{kd} \tag{6.76}
$$

too, because by (6.54) all the other $\varphi_{Q',l}$ that compose ϕ_l leave the points of S alone. We shall remember that (6.76) holds for all $S \in \mathcal{N}_{m+1}(R)$.

We may now conclude as before, with repeated uses of (6.64). That is,

$$
\begin{aligned}
H^d(R \cap E_4) = H^d(R \cap E_{3,m}) &= H^d(R \cap \phi_m(E_{3,m-1})) \\
&\leqslant \sum_{S \in \mathcal{N}_1(R)} H^d(\phi_m(S \cap E_{3,m-1})) \leqslant C \sum_{S \in \mathcal{N}_1(R)} H^d(S \cap E_{3,m-1}) \\
&= C \sum_{S \in \mathcal{N}_1(R)} H^d(S \cap \phi_{m-1}(E_{3,m-2})) \\
&\leqslant C \sum_{S \in \mathcal{N}_2(R)} H^d(\phi_{m-1}(S \cap E_{3,m-2})) \\
&\leqslant C^2 \sum_{S \in \mathcal{N}_2(R)} H^d(S \cap E_{3,m-2}) \\
\cdots \quad &\leqslant C \sum_{S \in \mathcal{N}_{m-l+1}(R)} H^d(\phi_l(S \cap E_{3,l-1})) \leqslant C2^{kd}
\end{aligned}
\tag{6.77}
$$

by (6.76). This completes our proof of Lemma 6.60. $\qquad \square$

Let us record a few facts about f_4 before we move to f_5. Recall that if R is a dyadic cube of sidelength 2^{k-j}, then all the mappings that compose f_4 send R to R, so $f_4(z) \in R$ for $z \in R$. Consequently,

$$
|f_4(z) - z| \leqslant 2^{k-j}\sqrt{n} \text{ for } z \in \mathbb{R}^n. \tag{6.78}
$$

Also set $Z_1^* = \left\{ z \in \mathbb{R}^n \, ; \, \mathrm{dist}(z, Z_1) \leqslant 2^{k+2} \sqrt{n} \right\}$. Notice that Z_1^* contains all the cubes $Q + v_l$, $Q \in \mathcal{D}_k$. [See below (6.58) for the definition of $\mathcal{D}_k$.] Then

$$f_4(z) = z \text{ for } z \notin Z_1^* \quad \text{and} \quad f_4(Z_1^*) \subset Z_1^* \tag{6.79}$$

because the ϕ_l only act nontrivially on the cubes $Q + v_l$, and preserve them. Let us also check that

$$H^d\big(\left\{ z \in E_4 \cap Z_1^* \, ; \, \mathrm{dist}(z, Z_2) \geqslant 2^{k+1} \right\}\big) \leqslant C_{x,y} \, r_x^{d-1}, \tag{6.80}$$

where $C_{x,y}$ depends on x, y, but not on r_x and k.

Call Y the set on the left-hand side of (6.80). Cover Y with dyadic cubes R of sidelength 2^{k-j}. Notice that (6.61) holds when T meets Y, so $H^d(E_4 \cap R) \leqslant C 2^{kd}$ by Lemma 6.60. Recall from (6.51) that $Z_1 = V(x,y) \cap \partial \widehat{T} \setminus B_y$ is contained in a thin tube of width Cr_x around the segment $[x, y]$, where C may depend on x and y (typically, through the ratio $|y|/|x|$). Then Z_1, and also Y, can be covered by less than $C|x-y|/r_x$ cubes R as above (recall from (6.58) that r_x and 2^k are comparable); each cube contributes at most $C 2^{kd} \leqslant Cr_x^d$ by (6.62), so (6.80) follows. The reader should not be shocked by the crude estimate above; we shall compensate by taking r_x really small.

Lemma 6.60 only controls $E_4 = f_4(E_3)$ far from Z_2, and indeed E_4 could still contain a big piece of $Z_2 = \partial B_y \cap H$. So we shall compose with a very last mapping f_5. The strategy is the same as for Z_1, but this time we shall work at the scale 2^t, where $t \in \mathbb{Z}$ is such that

$$2^{t-1} \leqslant \varepsilon_0 r_y \leqslant 2^t, \tag{6.81}$$

instead of 2^k. Notice that we shall choose r_x very small, depending on r_y and ε_0, so we can assume that k is much smaller than t.

Denote by $\mathcal{D}_t$ the set of dyadic cubes Q of sidelength 2^t such that

$$\mathrm{dist}(Q, Z_2) \leqslant 2^{t+1} \sqrt{n} \tag{6.82}$$

We shall repeat exactly the same construction as before. Denote by $w_0, \cdots w_m$ the 2^n vectors with coordinates in $\{0, 2^{t-1}\}$. We inductively build successive mappings ϕ_l, $0 \leqslant l \leqslant m$, and each one is obtained by composing Lipschitz functions $\varphi_{l,Q}$, $Q \in \mathcal{D}_t$. We start with $E_{4,-1} = E_4$, and later set $E_{4,l} = \phi_l(E_{4,l-1})$. Each $\varphi_{l,Q}$ is obtained by applying Lemma 6.53 to $Q + w_l$ and the set $F = E_{4,l-1} \cap [Q + w_l]$. All this defines the mappings $\varphi_{l,Q}$ and ϕ_l by induction. At the end, we set $f_5 = \phi_m \circ \cdots \phi_0$ and $E_5 = f_5(E_4) = \phi_m(E_{4,m-1})$. Here is the analogue of Lemma 6.60.

LEMMA 6.83. — *Let R be a dyadic cube of sidelength 2^{t-j} such that* $\operatorname{dist}(R, Z_2) \leqslant r_y/2$. *Then*

$$H^d(R \cap E_5) \leqslant C2^{td}. \tag{6.84}$$

We shall start with the easier cases where we can prove that

$$H^d(E_4 \cap S) \leqslant C2^{td} \quad \text{for } S \in \mathcal{N}_{m+1}(R), \tag{6.85}$$

where $\mathcal{N}_{m+1}(R)$ still denotes the set of dyadic cubes of sidelength 2^{t-j} that are neighbors of $(n+1)^{st}$ order of R.

Indeed, once we have (6.85), we can obtain (6.84) by the same argument as before: the analogue of (6.67) holds and yields (6.84).

Let us first take care of the contribution of E in (6.85). Choose $z_0 \in R$ and let $S \in \mathcal{N}_{m+1}(R)$ be given; the proof of (6.69) says that $S \subset B(z_0, 2^{t-2})$. Since we want to apply the local Ahlfors-regularity of E, we should check that $B(z_0, 2^{t-1}) \subset B(0, b)$ (because E is minimal in $B(0, b)$). And indeed $B(z_0, 2^{t-1}) \subset 2B_y$ because $\operatorname{dist}(R, \overline{B}_y) \leqslant \operatorname{dist}(R, Z_2) \leqslant r_y/2$ and both 2^t and $\operatorname{diam}(R)$ are much smaller than r_y, and $2B_y \subset A \subset B(0, b)$ by definition of r_y (below (6.13)). The (easier) upper bound in Lemma 2.15 says that

$$H^d(E \cap S) \leqslant H^d(E \cap B(z_0, 2^{t-2})) \leqslant C2^{td}; \tag{6.86}$$

so it will be enough to control $S \cap E_4 \setminus E$.

We start with the case when $\operatorname{dist}(R, Z_1 \cup Z_2) > 2^{t+1}$. Recall that $S \subset B(z_0, 2^{t-2})$, so $\operatorname{dist}(S, Z_1 \cup Z_2) \geqslant 2^t$. In particular S does not meet the set Z_1^* of (6.79), and (6.79) says that $E_4 \cap S = E_3 \cap S$. In addition, $E_3 \cap S$ does not meet $Z_1 \cup Z_2$, so (6.50) says that it does not meet Z. By (6.52), $E_3 \cap S = E_3 \cap S \setminus Z \subset E$. Altogether, $E_4 \cap S \subset E$, and (6.85) follows from (6.86).

We may now assume that $\operatorname{dist}(R, Z_1 \cup Z_2) \leqslant 2^{t+1}$, and our second case will be when $\operatorname{dist}(R, Z_2) \geqslant 2^{t+1}$. Let $z \in E_4 \cap S$ be given. Then $\operatorname{dist}(z, Z_2) \geqslant \operatorname{dist}(z_0, Z_2) - 2^{t-2} \geqslant \operatorname{dist}(R, Z_2) - \operatorname{diam}(R) - 2^{t-2} \geqslant 2^t$ because $S \subset B(z_0, 2^{t-2})$. If $z \in Z_1^*$, it lies in the set of (6.80); therefore,

$$E_4 \cap S \cap Z_1^* \leqslant C_{x,y} \, r_x^{d-1}, \tag{6.87}$$

and we can assume that $z \notin Z_1^*$. Then $z \in E_3$, by (6.79). If $z \in Z$, (6.50) says that $z \in Z_1 \cup Z_2$. This is impossible, because $Z_1 \subset Z_1^*$ and $\operatorname{dist}(z, Z_2) \geqslant 2^t$. So $z \notin Z$. Let $w \in E$ be such that $z = f_3 \circ f_2 \circ f_1(w)$. By (6.52), $w \in Z$, and then $z = w$. So $z \in E$, and (6.86) already takes care of it. This proves (6.85) and (6.84) in our second case.

We are left with the case when $\text{dist}(R, Z_2) \leqslant 2^{t+1}$. In this case we cannot prove (6.85), but instead we proceed as in the last case of Lemma 6.60 (when (6.72) holds). The point is that R is so close to Z_2 that it lies in the middle of the grid $\mathcal{D}_t$, and there is an $l \leqslant m$ such that none of the $S \in \mathcal{N}_{m+1}(R)$ meets a face $w_l + \partial Q$, $Q \in \mathcal{D}_t$. Then the corresponding images $\phi_l(E_{4,l-1} \cap S)$ are contained in d-dimensional grids $\mathcal{S}_{d,j}(Q)$, whose H^d-measures are controlled. The proof is the same as before, and completes our proof of Lemma 6.83. $\qquad\square$

Set $Z_2^* = \{z \in \mathbb{R}^n \,;\, \text{dist}(z, Z_2) \leqslant 2^{t+2}\sqrt{n}\}$. Notice that Z_2^* contains all the cubes $Q + w_l$, $Q \in \mathcal{D}_t$; the argument here and below is just the same as for Z_1^*, near (6.79). Then

$$f_5(z) = z \text{ for } z \notin Z_2^* \text{ and } f_5(Z_2^*) \subset Z_2^* \tag{6.88}$$

because the ϕ_l only act nontrivially on the cubes $Q + w_l$, and preserve them.

Recall from (6.51) that $Z_2 = \partial B_y \cap H$, where H is a small neighborhood of the d-plane P, as in (6.39); then we can cover Z_2 by less than $C\varepsilon_0^{-d+1}$ balls of radius $\varepsilon_0 r_y$. By the same argument, and because (6.81) says that 2^t and $\varepsilon_0 r_y$ are comparable, we need less than $C\varepsilon_0^{-d+1}$ dyadic cubes of sidelength 2^{t-j} to cover Z_2^*. All these cubes R, if they touch Z_2^*, are such that $\text{dist}(R, Z_2) \leqslant r_y/2$, so Lemma 6.83 says that $H^d(R \cap E_5) \leqslant C2^{td} \leqslant C\varepsilon_0^d r_y^d$. Altogether,

$$H^d(E_5 \cap Z_2^*) \leqslant C\varepsilon_0 r_y^d. \tag{6.89}$$

We are ready for the final comparison. Set $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, and then $\varphi_t(z) = tf(z) + (1-t)z$ for $0 \leqslant t \leqslant 1$ and $z \in \mathbb{R}^3$. The maps φ_t satisfy the properties (2.2)-(2.4).

Set $Z^* = Z \cup Z_1^* \cup Z_2^*$. Notice that

$$f(z) = z \text{ for } z \notin Z^*, \text{ and } f(Z^*) \subset Z^* \tag{6.90}$$

by (6.52), (6.79), and (6.88). So, if we define the W_t and $\widehat{W}$ as in (2.5), we get that $\widehat{W}$ is contained in the convex hull of Z^*, and (2.6) holds (with $U = B(0, b)$) because $Z^* \subset B(0, b)$.

We can apply (4.9) or (4.4) (depending on which definition of minimality we use), with $h(\delta) = 0$. Let us first see how to derive the final contradiction if we use (4.9). We get that $H^d(E \setminus E_5) \leqslant H^d(E_5 \setminus E)$ (because here $F = \varphi_1(E) = f(E) = E_5$), or equivalently

$$H^d(E \cap Z^*) \leqslant H^d(E_5 \cap Z^*), \tag{6.91}$$

because E and E_5 coincide out of Z^*. Let us check that

$$H^d(E_5 \cap Z^*) \leqslant C\varepsilon_0 r_y^d + C_{x,y}\, r_x^{d-1}\,. \tag{6.92}$$

Let $z \in E_5 \cap Z^*$ be given. The case when $z \in Z_2^*$ is taken care of by (6.89). If $z \in Z_1^* \setminus Z_2^*$, (6.88) says that $z \in E_4$; then $\mathrm{dist}(z, Z_2) \geqslant 2^{t+2}\sqrt{n}$ because $z \notin Z_2^*$, so z lies in the set of (6.80); (6.80) takes care of this case. So we may assume that $z \in Z^* \setminus Z_1^* \cup Z_2^*$. By (6.88) and (6.79), $z \in E_3$. Also, $z \in Z$, so (6.50) says that $z \in Z_1 \cup Z_2$. This case is impossible, because $Z_i \subset Z_i^*$; (6.92) follows.

On the other hand,

$$H^2(E \cap Z^*) \geqslant H^2(E \cap B_y) \geqslant C^{-1} r_y^d, \tag{6.93}$$

by Lemma 2.15, because $y \in E$, E is reduced, and $2B_y \subset B(0, b)$.

Incidentally, (6.93) would be easier to prove here because we have (6.13). Or we could have used the fact that the upper density of E at almost every point is larger than some positive constant, restricted to such an y in Proposition 6.11, and then chosen r_y specifically so that (6.93) holds.

We get the desired contradiction with (6.91) or (6.92) by choosing ε_0 small, and then r_x very small (depending on x, y, and r_y).

The reader may be worried about the case when $d = 1$; this case is sufficiently simple to be treated directly, but let us say a few words about it for the sake of completeness. In this case, $P' = \{0\}$ and lemma 6.15 says that there is a $\rho \in (r_x/2, 2r_x/3)$ such that the cone $\widehat{S}_\rho$ over the $(n-2)$-dimensional sphere $S_\rho = x + [Q \cap \partial B(0, \rho)]$ in the vertical hyperplane $x + Q$, does not meet $A \cap E$. At the same time, (6.13) says that the $(n-1)$-dimensional cone Σ centered at y and defined by $\mathrm{dist}(z, P) = 2\varepsilon_0|z - y|$ does not meet $E \cap B_y \setminus \{y\}$. We choose r_x so small that $\widehat{S}_\rho$ meets Σ well inside B_y. Denote by Ω the region of $V(x, y)$ bounded by $\widehat{S}_\rho \cup \Sigma$; thus Ω is the intersection of $V(x, y)$ with a thin double-cone centered at 0 and y. Notice that $\Omega \cap V(x, x)$ does not meet E, by (6.14). It is easy to contract Ω onto the line segment $[x, y]$ by radial vertical motion, and since E does not meet $\partial\Omega$ except at y, we can make sure that the contraction f is Lipschitz and does not move the points of $E \setminus \Omega$. In addition, $f(E \cap \Omega)$ does not meet B_x (again by (6.14)), so we can contract the whole piece $f(E \cap \Omega) \subset [x, y]$ to the point y along $[x, y]$. The final contradiction is now obtained as in the general case.

Return to the case when $d > 1$. If we use (4.4) (instead of (4.9)), we get that

$$H^d(E \cap W) \leqslant H^d(f(E \cap W)), \tag{6.94}$$

with $W = \{w \in \mathbb{R}^3 \, ; \, f(w) \neq w\}$. Recall that $f(z) = z$ out of Z^* (by (6.90)), so $W \subset Z^*$ and

$$H^d(f(E \cap W)) \leqslant H^d(f(E \cap Z^*)) \leqslant H^d(E_5 \cap Z^*) \qquad (6.95)$$

(by (6.90) again). Notice that

$$|f_5(z) - z| \leqslant 2^{t-j}\sqrt{n} \ \text{ for } z \in \mathbb{R}^n, \qquad (6.96)$$

by the proof of (6.78). Then $E_5 = f_5 \circ f_4(E_3)$ does not meet $B(y, r_y/2)$, by (6.78), (6.96), and because E_3 does not meet B_y (by (6.50) and the definition (6.43)). Hence $E \cap B(y, r_y/2) \subset E \cap W \setminus E_5$, just because $f(z) \neq z$ if $z \in E \setminus E_5$, and

$$\begin{aligned} H^d(E \cap B(y, r_y/2)) &\leqslant H^d(E \cap W) \leqslant H^d(f(E \cap W)) \\ &\leqslant H^d(E_5 \cap Z^*) \leqslant C\varepsilon_0 r_y^2 + C_{x,y} \, r_x^{d-1}, \end{aligned} \qquad (6.97)$$

by (6.94), (6.95) and (6.92). On the other hand, $H^d(E \cap B(y, r_y/2)) \geqslant C^{-1} r_y^d$ by Lemma 2.15, so we also get a contradiction when we use (4.4) in the accounting.

This completes our proof of Proposition 6.11. As was explained before, (6.12) follows and so E coincides with a cone inside A. $\qquad \square$

We return to the proof of Theorem 6.2. So far we proved that there is a cone $\mathcal{C}$ centered at the origin, that satisfies (6.4). We still need to check that $\mathcal{C}$ is a reduced minimal set. Notice that it is reduced, by (6.4) and because E is reduced. [See Definition 2.12 for the definition.] Suppose it is not minimal. Then we can find a competitor $F = \varphi_1(\mathcal{C})$ where $\{\varphi_t\}$ satisfies (2.2)-(2.6) with $U = \mathbb{R}^3$, and such that

$$H^2(F \setminus \mathcal{C}) < H^2(\mathcal{C} \setminus F). \qquad (6.98)$$

Observe that we use Definition 4.8 for minimality, but Proposition 4.10 tells us that we could also have used Definition 4.3 (or even Definition 4.1, since here $h(r) = 0$).

Recall from (2.6) that the set $\widehat{W}$ of (2.5) is bounded. Since $\mathcal{C}$ is a cone, we may always dilate everything by a small constant (this does not change (6.98)), so we may assume that $\widehat{W} \subset B(0, a)$.

We want to use F and the φ_t to construct a deformation of E and contradict its minimality. First observe that $\mathcal{C} \cap B(0, b)$ is a deformation of E in $B(0, b)$. Indeed, pick any $b' \in (a, b)$, and set $g(x) = 0$ for $x \in \overline{B}(0, a)$, $g(x) = \frac{b'(|x|-a)}{b'-a} \frac{x}{|x|}$ for $x \in B(0, b') \setminus \overline{B}(0, a)$, and $g(x) = x$ for

$x \in B(0, b) \setminus B(0, b')$. Then set $f_t(x) = t g(x) + (1 - t) x$ for $0 \leqslant t \leqslant 1$. It is easy to see that the g_t satisfy (2.2)-(2.6) with $U = B(0, b)$, and that $g_1(E) = \mathcal{C} \cap B(0, b)$ (by (6.4)). We now put our two deformations together. That is, we set $\psi_t(x) = g_{2t}(x)$ for $0 \leqslant t \leqslant 1/2$ and $\psi_t(x) = \varphi_{2t-1}(g_1(x))$ for $1/2 \leqslant t \leqslant 1$, and get a new family that satisfies (2.2)-(2.6) with $U = B(0, b)$. By minimality of E,

$$H^2(\psi_1(E) \setminus E) \geqslant H^2(E \setminus \psi_1(E)). \tag{6.99}$$

But $\psi_1(E) = F \cap B(0, b)$, so (6.99) just says that $H^2(E) \leqslant H^2(F \cap B(0, b))$.

Recall from (6.3) that $H^2(E \cap B(0, r)) = \theta_0 r^2$ for $a < r < b$. Also, $H^2(\mathcal{C} \cap A) = H^2(E \cap A)$ by (6.4), so $H^2(\mathcal{C} \cap B(0, r)) = \theta_0 r^2$ for $r > 0$ and $H^2(\mathcal{C} \cap B(0, b)) = H^2(E)$. Altogether, $H^2(\mathcal{C} \cap B(0, b)) = H^2(E) \leqslant H^2(F \cap B(0, b))$, which contradicts (6.98). This completes the proof of Theorem 6.2. $\square$

7. Almost constant density

In this section we use Theorem 6.2 to show that if E is a reduced almost-minimal set of dimension d in $B(0, b) \subset \mathbb{R}^n$, with sufficiently small gauge function, and if $\theta(r) = r^{-d} H^d(E \cap B(0, r))$ varies sufficiently little between a and b, then E is very close to a reduced minimal cone in $A = B(0, b) \setminus \overline{B}(0, a)$.

We also have a (simpler) result when $a = 0$ (Proposition 7.24), and we shall see in Proposition 7.31 that every blow-up limit of a reduced almost-minimal set of dimension d in $U \subset \mathbb{R}^n$ is a reduced minimal cone.

PROPOSITION 7.1. — *Let n, d (the dimensions), a, and b be given, with $0 < a < b < +\infty$. For each $\tau > 0$, we can find $\varepsilon > 0$ so that the following holds. Let E be a reduced almost-minimal set of dimension d in $B(0, b) \subset \mathbb{R}^n$, with gauge function h, and assume that $h(2b) \leqslant \varepsilon$. Also assume that $\theta(b) \leqslant \theta(a) + \varepsilon$. Then there is a reduced minimal cone $\mathcal{C}$ centered at the origin such that*

$$\operatorname{dist}(x, \mathcal{C}) \leqslant \tau \quad \text{for } x \in E \cap B(0, b - \tau) \setminus B(0, a + \tau), \tag{7.2}$$

$$\operatorname{dist}(x, E) \leqslant \tau \quad \text{for } x \in \mathcal{C} \cap B(0, b - \tau) \setminus B(0, a + \tau), \tag{7.3}$$

$$|H^d(E \cap B(x, r)) - H^d(\mathcal{C} \cap B(x, r))| \leqslant \tau$$
$$\text{for } x \in \mathbb{R}^n \text{ and } r > 0 \text{ such that} \quad B(x, r) \subset B(0, b - \tau) \setminus B(0, a + \tau), \tag{7.4}$$

and

$$|\theta(r) - r^{-d} H^d(\mathcal{C} \cap B(0, r))| \leqslant \tau \quad \text{for } a + \tau \leqslant r \leqslant b - \tau. \tag{7.5}$$

See Definition 2.12 for the notion of reduction, and observe that if E is not reduced, Remark 2.14 allows us to apply Proposition 7.1 to E^*. Our statement allows $\mathcal{C} = \emptyset$, but only if $\theta(r)$ is very small (by (7.5)). Notice that (7.5) forces $\theta(r)$ to be very close to the constant density for $\mathcal{C}$. This is a constraint on $\theta(r)$, but also can be a way to determine $\mathcal{C}$ in terms of $\theta(r)$.

We shall prove the proposition by compactness. Fix n, d, a, and b, and consider a sequence $\{E_n\}$ of reduced almost-minimal sets on $B(0, b)$, with gauge functions h_n such that $\varepsilon_n = h_n(2b)$ tends to 0. It is a standard fact about the Hausdorff distance that we can extract a subsequence $\{n_k\}$ so that the sets E_{n_k} converge to a limit $F \subset B(0, b)$, as in the beginning of Section 3 (see for instance [D2], Proposition 34.6 on page 214).

Set $\widetilde{h}_l = \sup_{n \geqslant l} h_n$ for $l \geqslant 0$. Notice that $\widetilde{h}_l$ is nondecreasing (just like the h_n), and $\lim_{r \to 0} \widetilde{h}_l(r) = 0$, because each h_n tends to 0 and $h_n(r) \leqslant \varepsilon_n$ for $r \leqslant 2b$. Then Lemma 4.7 (applied to the E_{n_k}) says that for each l, F is almost-minimal in $B(0, b)$, with gauge function $\widetilde{h}_l$. But since we can take l as large as we want and $\widetilde{h}_l(2b)$ tends to 0, we get that

$$F \text{ is a minimal set in } B(0, b). \tag{7.6}$$

We also get that

$$H^d(F \cap V) \leqslant \liminf_{k \to +\infty} H^d(E_{n_k} \cap V) \tag{7.7}$$

for every open set $V \subset B(0, b)$, by (3.4) in Lemma 3.3 (applied with $U = B(0, b)$, $M = 1$, $\delta = 2b$, and $h = \widetilde{h}_l(2b)$ for some large l). We also need an inequality in the other direction, so we apply Lemma 3.12 to E_{n_k}, with the same choice of U, M, δ, and h, and then let l tend to $+\infty$; this yields

$$\limsup_{k \to +\infty} H^d(E_{n_k} \cap H) \leqslant H^d(F \cap H)$$
$$\text{for every compact set } H \subset B(0, b). \tag{7.8}$$

Notice that $H^d(F \cap \partial B(x, r)) = 0$ for almost-every $r \in (a, b)$; (7.7) and (7.8) yield

$$\limsup_{k \to +\infty} H^d(E_{n_k} \cap B(x, r)) \leqslant H^d(F \cap B(x, r)) = H^d(F \cap \overline{B}(x, r))$$
$$\leqslant \liminf_{k \to +\infty} H^d(E_{n_k} \cap B(x, r)) \tag{7.9}$$

for such r, so that

$$H^d(F \cap B(0, r)) = \lim_{k \to +\infty} H^d(E_{n_k} \cap B(0, r))$$
$$\text{for almost-every } r \in (a, b). \tag{7.10}$$

Set $\theta_k(r) = r^{-d} H^d(E_{n_k} \cap B(0,r))$ for $r \leqslant b$, and suppose in addition that $\theta_k(b) \leqslant \theta_k(a) + \varepsilon_k$. Pick a' and b', with $a < a' < b < b'$, such (7.10) holds for a' and b'. Then

$$
\begin{aligned}
H^d(F \cap B(0,b')) &= \lim_{k \to +\infty} H^d(E_{n_k} \cap B(0,b')) \leqslant \liminf_{k \to +\infty} H^d(E_{n_k} \cap B(0,b)) \\
&= b^d \liminf_{k \to +\infty} \theta_k(b) \leqslant b^d \liminf_{k \to +\infty} [\theta_k(a) + \varepsilon_k] \\
&= b^d \liminf_{k \to +\infty} \theta_k(a) = b^d a^{-d} \liminf_{k \to +\infty} H^d(E_{n_k} \cap B(0,a)) \\
&\leqslant b^d a^{-d} \lim_{k \to +\infty} H^d(E_{n_k} \cap B(0,a')) = b^d a^{-d} H^d(F \cap B(0,a')).
\end{aligned}
$$
(7.11)

We let b' tend to b and get that

$$
\theta(b) =: b^{-d} H^d(F \cap B(0,b)) \leqslant a^{-d} H^d(F \cap B(0,a')) = a^{-d}(a')^d \theta(a'). \quad (7.12)
$$

Recall from Proposition 5.16 and (7.6) that θ is nondecreasing on (a,b), so $\theta(a^+) = \lim_{a' \to a^+} \theta(a')$ exists, and (7.12) says that $\theta(b) \leqslant \theta(a^+)$. But $\theta(b) \geqslant \theta(a^+)$ by Proposition 5.16, so $\theta(b) = \theta(a^+)$ and θ is constant on (a,b). By Theorem 6.2, there is a reduced minimal cone $\mathcal{C}$ centered at the origin such that

$$
F \cap A = \mathcal{C} \cap A. \quad (7.13)
$$

In particular, (7.2) and (7.3) hold for E_{n_k} and k large enough (and with any $\tau > 0$ given in advance).

It will be good to know that

$$
H^d(\mathcal{C} \cap \partial B) = 0 \text{ for every ball } B, \quad (7.14)
$$

but since this is easy to believe, we shall only check it in Lemma 7.34, at the end of the section. Now let a ball B be given, with $B \subset B(0, b - \tau/2) \setminus B(0, a + \tau/2)$. By (7.14), (7.13), (7.7), (7.8), and the proof of (7.10),

$$
\lim_{k \to +\infty} H^d(E_{n_k} \cap B) = H^d(F \cap B) = H^d(\mathcal{C} \cap B), \quad (7.15)
$$

so the analogue of (7.4) for E_{n_k} and B holds for k large.

We are now ready to prove Proposition 7.1. Let a, b, and τ be given, and suppose that we cannot find ε so that the proposition holds. Then we can find $E_n \subset B(0,b)$ that satisfies the hypotheses with $\varepsilon_n = 2^{-n}$, but not the conclusion. We extract a convergent subsequence, as above, and get a set F such that (7.6)-(7.13) hold. Let us show that (7.2)-(7.5) hold for k large; this will prove the desired contradiction.

We already know that (7.2) and (7.3) hold for k large. For (7.4) we only know that it holds for k large when x and r are fixed, so we need a

little bit of uniformity. By (7.14), $H^d(\mathcal{C} \cap B(x,r))$ is a continuous function of x and r, so we can find a finite collection of balls B_i, $i \in I$, such that $B_i \subset B(0, b - \tau/2) \setminus B(0, a + \tau/2)$ for $i \in I$, and such that whenever $B(x,r)$ is a ball such that $B(x,r) \subset B(0, b - \tau) \setminus B(0, a + \tau)$ (as in (7.4)), we can find $i, j \in I$ such that

$$B_i \subset B(x,r) \subset B_j \ \text{ and } \ H^d(\mathcal{C} \cap B_j \setminus B_i) \leqslant \tau/5. \tag{7.16}$$

We also know from (7.15) that for k large enough,

$$|H^d(\mathcal{C} \cap B_i) - H^d(E_{n_k} \cap B_i)| \leqslant \tau/5 \ \text{ for } i \in I. \tag{7.17}$$

Now let $B(x,r)$ be as in (7.4), and let $i, j \in I$ be such that (7.16) holds. Then

$$H^d(\mathcal{C} \cap B_i) \leqslant H^d(\mathcal{C} \cap B(x,r)) \leqslant H^d(\mathcal{C} \cap B_j) \leqslant H^d(\mathcal{C} \cap B_i) + \tau/5 \tag{7.18}$$

and, for k large (depending on I, but not on $B(x,r)$),

$$H^d(\mathcal{C} \cap B_i) - \tau/5 \leqslant H^d(E_{n_k} \cap B_i) \leqslant H^d(E_{n_k} \cap B(x,r)) \leqslant H^d(E_{n_k} \cap B_j)$$
$$\leqslant H^d(\mathcal{C} \cap B_j) + \tau/5 \leqslant H^d(\mathcal{C} \cap B_i) + 2\tau/5, \tag{7.19}$$

by (7.17) and (7.16). Thus $|H^d(\mathcal{C} \cap B(x,r)) - H^d(E_{n_k} \cap B(x,r))| < \tau$, by (7.18) and (7.19), so (7.4) holds.

We are left with (7.5) to prove. Call θ_0 the constant density of $\mathcal{C}$; that is, $\theta_0 = r^{-d} H^d(\mathcal{C} \cap B(0,r))$ for $r > 0$. Notice that $H^d(\mathcal{C} \cap \partial B(0,r)) = 0$ for $a < r < b$, so

$$\lim_{k \to +\infty} H^d(E_{n_k} \cap B(0,r)) = H^d(F \cap B(0,r)) = H^d(\mathcal{C} \cap B(0,r)) = \theta_0 r^d, \tag{7.20}$$

by (7.7), (7.8), and (7.13). As before, we need some uniformity, so we pick a finite (but large) collection $\{r_i\}_{i \in I}$ in (a,b). We know that for k large enough,

$$|r_i^{-d} H^d(E_{n_k} \cap B(0,r_i)) - \theta_0| \leqslant \tau/2 \ \text{ for } i \in I. \tag{7.21}$$

Let $r \in [a + \tau, b - \tau]$ be given. If our collection $\{r_i\}$ is dense enough we can find $i, j \in I$ such that $r_i \leqslant r \leqslant r_j$ and r_j/r_i is as close to 1 as we want. Then for k large

$$\theta_k(r) = r^{-d} H^d(E_{n_k} \cap B(0,r)) \leqslant r^{-d} H^d(E_{n_k} \cap B(0,r_j))$$
$$\leqslant r^{-d} r_j^d (\theta_0 + \tau/2) \leqslant \theta_0 + \tau, \tag{7.22}$$

by (7.21). Similarly,

$$\theta_k(r) \geqslant r^{-d} H^d(E_{n_k} \cap B(0,r_i)) \geqslant r^{-d} r_i^d (\theta_0 - \tau/2) \leqslant \theta_0 - \tau, \tag{7.23}$$

and (7.5) holds for k large. This completes the proof of Proposition 7.1 by contradiction. $\square$

Proposition 7.1 easily extends to the case when $a = 0$. Let us give a slightly more invariant statement for this to simplify later uses.

PROPOSITION 7.24. — *For each $\tau > 0$, we can find $\varepsilon > 0$ so that the following holds. Let E be a reduced almost-minimal set in the open set $U \subset \mathbb{R}^n$, with gauge function h. Let $x \in E$ and $r > 0$ be such that $B(x, r) \subset U$, $h(2r) \leqslant \varepsilon$, and*

$$\theta(x, r) \leqslant \inf_{0 < t < r/100} \theta(x, t) + \varepsilon, \tag{7.25}$$

where we set $\theta(x, t) = t^{-d} H^d(E \cap B(x, t))$. Then there is a reduced minimal cone $\mathcal{C}$ centered at x, such that

$$\mathrm{dist}(y, \mathcal{C}) \leqslant \tau r \quad \text{for } y \in E \cap B(x, (1 - \tau)r), \tag{7.26}$$

$$\mathrm{dist}(y, E) \leqslant \tau r \quad \text{for } y \in \mathcal{C} \cap B(x, (1 - \tau)r), \tag{7.27}$$

and

$$|H^d(E \cap B(y, t)) - H^d(\mathcal{C} \cap B(y, t))| \leqslant \tau r^d$$
$$\text{for } y \in \mathbb{R}^n \text{ and } t > 0 \quad \text{such that } B(y, t) \subset B(x, (1 - \tau)r). \tag{7.28}$$

By scale invariance, is enough to prove this when $x = 0$, $r = 1$, and E is almost-minimal in (an open set U that contains) $B(0, 1)$. We can proceed as in the proof of Proposition 7.1 (with $a = 0$ and $b = 1$), up to (7.10) included. Then pick $a_1 \in (0, 1/100)$ and $b' \in (a_1, 1)$ such that (7.10) holds for a_1 and b'. By (7.25), $\theta_k(1) \leqslant \theta_k(a_1) + \varepsilon_k$, so

$$\begin{aligned}
H^d(F \cap B(0, b')) &= \lim_{k \to +\infty} H^d(E_{n_k} \cap B(0, b')) \leqslant \liminf_{k \to +\infty} H^d(E_{n_k} \cap B(0, 1)) \\
&= \liminf_{k \to +\infty} \theta_k(1) \leqslant \liminf_{k \to +\infty} [\theta_k(a_1) + \varepsilon_k] = \liminf_{k \to +\infty} \theta_k(a_1) \\
&= a_1^{-d} \liminf_{k \to +\infty} H^d(E_{n_k} \cap B(0, a_1)) \\
&= a_1^{-d} H^d(F \cap B(0, a_1)),
\end{aligned} \tag{7.29}$$

by (7.10). We let b' tend to 1 and get that

$$H^d(F \cap B(0, 1)) \leqslant a_1^{-d} H^d(F \cap B(0, a_1)). \tag{7.30}$$

Now F is minimal in $B(0, 1)$, by (7.6), so Proposition 5.16 says that $\theta(r) = r^{-d} H^d(F \cap B(0, r))$ is nondecreasing in $(0, 1)$. Then $\theta(1) = \lim_{r \to 1} \theta(r) \geqslant \theta(a_1)$. But (7.30) says that $\theta(1) \leqslant \theta(a_1)$, so $\theta(r)$ is constant on $[a_1, 1]$, hence

also on $(0,1]$ because we can take a_1 as close to the origin as we want. Theorem 6.2 says that F is a reduced minimal cone, and we can continue the argument as in Proposition 7.1. □

Let us also give a baby version of Proposition 7.24.

PROPOSITION 7.31. — *Let E be a reduced almost-minimal set in the open set U, and let $x \in E$ be such that*

$$\theta(x) = \lim_{r \to 0} r^{-d} H^d(E \cap B(x,r)) \quad \text{exists.} \qquad (7.32)$$

Then every blow-up limit of E at x is a reduced minimal cone F centered at the origin, and $H^d(F \cap B(0,1)) = \theta(x)$.

See Definition 2.12 for "reduced", and Definition 4.8 or 4.3 for almost-minimality. We do not care about the precise values of the gauge function $h(r)$ here, we just need it to tend to 0 when r tends to 0, but of course (7.32) is easier to obtain when h is small, by Proposition 5.24. A blow-up limit of E at x is any closed set in $\mathbb{R}^n$ that can be obtained as the limit (as in (3.1)) of a sequence $\{r_k^{-1}(E - x)\}$, with $\lim_{k \to +\infty} r_k = 0$.

To prove the proposition, assume that the sequence $\{r_k^{-1}(E - x)\}$ converges to some limit F. Notice that for every small $a > 0$, $F_k = r_k^{-1}(E - x)$ is, for k large enough, a reduced almost-minimal set with gauge function $\widetilde{h}(t) = h(at)$. Then Lemma 4.7 says that F is a reduced almost-minimal set, with gauge function $\widetilde{h}$. But since we can take a as small as we want, and $\lim_{a \to 0} \widetilde{h}(t) = 0$ for each t, F is a reduced minimal set in $\mathbb{R}^n$.

We also get that

$$H^d(F \cap B(0,r)) = \lim_{k \to +\infty} H^d(F_k \cap B(0,r)) \quad \text{for almost every } r > 0, \ (7.33)$$

by the proof of (7.10). But $H^d(F_k \cap B(0,r)) = r_k^{-d} H^d(E \cap B(x, r_k r))$, which tends to $r^d \theta(x)$ by (7.32). Thus $H^d(F \cap B(0,r)) = r^d \theta(x)$ for almost every $r > 0$. Theorem 6.2 says that F is a cone, and Proposition 7.31 follows. □

Propositions 7.1 and 7.24 will help us find many places where E looks a lot like a minimal cone. We shall then use a variant of Reifenberg's topological disk theorem to get a local parametric description of E.

We end this section with the proof of (7.14).

LEMMA 7.34. — *Let $\mathcal{C}$ be a closed cone in $\mathbb{R}^n$ (centered at the origin), and assume that $H^d(\mathcal{C} \cap B(0,1)) < +\infty$. Then $H^d(\mathcal{C} \cap \partial B) = 0$ for every ball B.*

Let $0 < a < b < +\infty$ be given, and set $A = \{x \in \mathbb{R}^n \,;\, a \leqslant |x| \leqslant b\}$; it is enough to show that $H^d(\mathcal{C} \cap A \cap \partial B) = 0$. Denote by $S = \partial B(0,1)$ the unit sphere, and set $\varphi(r,\theta) = r\theta$ for $r \in [a,b]$ and $\theta \in S$. Notice that φ is a bilipschitz mapping from $[a,b] \times S$ to A, and $\varphi^{-1}(\mathcal{C} \cap A) = [a,b] \times T$, where $T = S \cap \mathcal{C}$. Then

$$H^d([a,b] \times T) = H^d(\varphi^{-1}(\mathcal{C} \cap A)) \leqslant CH^d(\mathcal{C} \cap A) < +\infty. \qquad (7.35)$$

Let us first check that

$$H^{d-1}(T) \leqslant CH^d([a,b] \times T) < +\infty, \qquad (7.36)$$

where C depends also on a and b, but we don't care.

Let $\varepsilon > 0$ be given. By (7.35), there is a covering of $[a,b] \times T$ by balls $B_i = B(x_i, r_i)$, $i \in I$, with $r_i \leqslant \varepsilon$ for $i \in I$ and

$$\sum_{i \in I} r_i^d \leqslant CH^d([a,b] \times T) + \varepsilon. \qquad (7.37)$$

By compactness, we may assume that I is finite. For $t \in [a,b]$, set $P_t = \{t\} \times S$. Set $B_{i,t} = B_i \cap P_t$, and denote by I_t the set of $i \in I$ such that $B_{i,t} \neq \emptyset$. Notice that the balls $B_{i,t} = B_i \cap P_t$, $i \in I_t$ form a covering of T by balls of radius at most ε, so

$$H^{d-1}_\varepsilon(T) \leqslant c_{d-1} \sum_{i \in I_t} \operatorname{diam}(B_{i,t})^{d-1} \leqslant C \sum_{i \in I_t} H^{d-1}(B_{i,t}) = C \sum_{i \in I} H^{d-1}(B_{i,t}) \qquad (7.38)$$

where $H^{d-1}_\varepsilon(T)$ is the usual infimum over coverings that is used to define $H^{d-1}(T)$. We average this over $[a,b]$ and get that

$$\begin{aligned} H^{d-1}_\varepsilon(T) &\leqslant C \int_{[a,b]} \sum_{i \in I} H^{d-1}(B_{i,t}) dt = C \sum_{i \in I} \int_{[a,b]} H^{d-1}(B_{i,t}) dt \\ &\leqslant C \sum_{i \in I} r_i^d \leqslant CH^d([a,b] \times T) + C\varepsilon \end{aligned} \qquad (7.39)$$

by (7.37). Then we let ε tend to 0 and we get (7.36).

By (7.36), we can apply Theorem 2.10.45 in [Fe], which says that there are positive constants C_1 and C_2 such that

$$C_1 H^d(U \times V) \leqslant H^1(U) H^{d-1}(V) \leqslant C_2 H^d(U \times V) \qquad (7.40)$$

for $U \subset [a,b]$ and $V \subset T$ measurable. In other words, the restriction of H^d to $[a,b] \times T$ is comparable to the product of the Lebesgue measure on $[a,b]$ with the restriction of H^{d-1} to F.

Now let B be any ball; notice that ∂B never meets a radial line more than twice, which means that for $v \in T$, $\varphi^{-1}(\mathcal{C} \cap \partial B \cap A))$ does not meet $[a, b] \times \{v\}$ more than twice. Then

$$H^d(\mathcal{C} \cap \partial B \cap A) \leqslant C H^d(\varphi^{-1}(\mathcal{C} \cap \partial B \cap A))$$

$$\leqslant C \int_{v \in T} H^1\big([a, b] \times \{v\} \bigcap \varphi^{-1}(\mathcal{C} \cap \partial B \cap A)\big) dH^{d-1}(v)$$

$$= 0$$

$$(7.41)$$

because φ is bilipschitz and by Fubini. Lemma 7.34 follows. $\square$

C. MINIMAL CONES OF DIMENSION 2

In this part we study the reduced minimal cones E of dimension 2 in $\mathbb{R}^n$. The goal is not to obtain a finite list as when $n = 3$, but a fairly simple description of $K = E \cap \partial B(0, 1)$ as a finite collection of geodesic arcs of circles in $\partial B(0, 1)$, with lengths bounded from below, and that meet only by sets of three with 120° angles. See Proposition 14.11.

We start with a description of the tangent cones to E, where we show that if x lies in $K = E \cap \partial B(0, 1)$, every blow-up limit of K at x is a one-dimensional reduced minimal set. See Theorem 8.23 for a more precise statement where K is approximated by minimal sets in small balls centered at x. This description is based on the fact that if $E \times \mathbb{R}^m$ is a reduced $(m + d)$-dimensional minimal set in $\mathbb{R}^n \times \mathbb{R}^m$, then E is a d-dimensional minimal set in $\mathbb{R}^n$. See Proposition 8.3.

With this and the simple version of Reifenberg's topological disk theorem which we describe in Section 12, we could easily obtain information on the bi-Höder structure of E, but since it is preferable to use C^1 information, we will show that K satisfies a weak form of almost-minimization (see Definition 9.1 and Proposition 9.4), and then follow some of the general theory to prove that these weak almost-minimal sets of dimension 1 have some form of C^1 regularity. This is done in Sections 11-13, and for the local description of K by curves we use a simple version of Reifenberg's Topological Disk Theorem (Proposition 12.6).

In the mean time we show in Section 10 that the reduced minimal sets of dimension 1 are cones. See Theorem 10.1.

We state and prove the description of $K = E \cap \partial B(0, 1)$ when E is a reduced minimal cone of dimension 2 in Section 14. The fact that K

is composed of C^1 curves comes from the description of the weak almost-minimal sets given in Sections 11-13, and then a construction of harmonic competitors for a cone, which is done in Section 13, allows us to check that all the little arcs that compose K are arcs of great circles of $\partial B(0,1)$.

8. Minimality of a product and tangent sets

We start our study of minimal cones with what would be the most natural attempt to get a nice description of the d-dimensional minimal sets by induction on d. It will be easy to control the minimal sets of dimension 1. Then assume that we have a reasonable description of the minimal sets of dimension d; we can try to use this to get information on minimal cones of dimension $d+1$, because we shall see in this section that blow-up limits of these cones are products of a minimal set of dimension d by a line. In turn a good control on the minimal cones may lead to valuable information on minimal sets of the same dimension, by the results of Section 7. We shall only be able to make this plan work for 2-dimensional minimal sets, but we shall write down the first stage in full generality.

In this section, we first show that if the product $E \times \mathbb{R}^m$ is a reduced minimal set of dimension $d + m$, then E is a reduced dimensional minimal set of dimension d. We shall then use this to show that if the cone over K is a reduced minimal set of dimension $d + 1$, then the blow-up limits of K are minimal sets of dimension d, and even that K is close to minimal sets of dimension d in small balls.

Let us recall what we mean by minimal sets. This is the same as almost-minimal sets as in Definition 4.3 or 4.8, with $h(\delta) = 0$ for all δ, but since the definition simplifies slightly, let us state it again for future reference.

DEFINITION 8.1. — *The closed set $E \subset \mathbb{R}^n$ is a minimal set of dimension d if, whenever $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function such that $\varphi(x) = x$ out of some compact set of $\mathbb{R}^n$, then*

$$H^d(E \setminus \varphi(E)) \leqslant H^d(\varphi(E) \setminus E). \tag{8.2}$$

We say that E is reduced when $H^d(E \cap B(x,r)) > 0$ for $x \in E$ and $r > 0$.

See Definition 2.12 for the original definition of reduced sets. Concerning our new definition of minimality, it is easily seen to be equivalent to Definition 4.3 with the gauge function $h = 0$. The point is that if φ is as in Definition 8.1, we can set $\varphi_t(x) = t\varphi(x) + (1-t)x$ for $0 \leqslant t \leqslant 1$; the family $\{\varphi_t\}$ satisfies (2.2)-(2.6) for some $\delta > 0$, and then (8.2) is the same as (4.9) with $h(\delta) = 0$. Let us also recall that Definitions 4.3 and 4.8 are equivalent, by Proposition 4.10.

The first result of this section is a simple observation on the structure of some minimal product sets.

PROPOSITION 8.3. — *Let m, n, and d be integers, and let $E \times R^m$ be a reduced $(d+m)$-dimensional minimal set in $\mathbb{R}^{n+m}$. Then E is a reduced d-dimensional minimal set in $\mathbb{R}^n$.*

Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a lipschitz function such that $\varphi(x) = x$ out of some compact set; we want to show that (8.2) holds. Our plan is to use φ to construct a Lipschitz function $f : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ and apply the minimality of $E \times R^m$.

Let $R > 0$ be large, to be chosen near the end, and choose a smooth radial cut-off function ψ on $\mathbb{R}^m$, with

$$\begin{aligned}
\psi(y) &= 1 & \text{for} \quad &|y| \leqslant R, \\
0 \leqslant \psi(y) &\leqslant 1 & \text{for} \quad &R < |y| < R+1, \\
\psi(y) &= 0 & \text{for} \quad &|y| \geqslant R+1,
\end{aligned} \tag{8.4}$$

and $|\nabla \psi(y)| \leqslant 2$ everywhere. Then define $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ by

$$g(x,y) = \psi(y)\varphi(x) + (1 - \psi(y))\, x \quad \text{for } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^m \tag{8.5}$$

and set $f(x,y) = (g(x,y), y)$. Notice that $g(x,y) = x + \psi(y)(\varphi(x) - x)$, and since $\varphi(x) - x$ is bounded,

$$g \text{ and } f \text{ are Lipschitz, with bounds that do not depend on } R. \tag{8.6}$$

Set $W^\sharp = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \,;\, f(x,y) \neq (x,y)\}$. If $(x,y) \in W^\sharp$, then $\psi(y) \neq 0$ and $\varphi(x) \neq x$. That is, $W^\sharp \subset W \times B(0, R+1)$, where $W = \{x \in \mathbb{R}^n \,;\, \varphi(x) \neq x\}$. In addition, if B is a ball that contains $W \cup \varphi(W)$, then

$$W^\sharp \cup f(W^\sharp) \subset \Sigma, \text{ where we set } \Sigma = B \times B(0, R+1). \tag{8.7}$$

By assumption, $E^\sharp = E \times R^m$ is minimal, and (8.2) says that

$$H^{d+m}(E^\sharp \setminus f(E^\sharp)) \leqslant H^{d+m}(f(E^\sharp) \setminus E^\sharp). \tag{8.8}$$

By (8.7), both sets are contained in Σ; we add $H^{d+m}(E^\sharp \cap f(E^\sharp) \cap \Sigma)$ to both sides and get that

$$H^{d+m}(E^\sharp \cap \Sigma) \leqslant H^{d+m}(f(E^\sharp \cap \Sigma)). \tag{8.9}$$

We shall need to be a little careful when we compute Hausdorff measures of product spaces, because the Hausdorff measure of a product is not always exactly the product of the Hausdorff measures. Fortunately, $E^\sharp$ is rectifiable

(with dimension $d+m$), by Section 2 (or directly [Al2]), so it will be possible to use the following variant of the coarea formula, which we borrow from [Fe], Theorem 3.2.22. We change the notation a little, and reduce slightly the generality.

LEMMA 8.10. — *Let d, m, n be nonnegative integers, let $Z \subset \mathbb{R}^n$ be a rectifiable set of dimension $m + d$, with $H^{m+d}(E) < +\infty$, and let $\pi : Z \to \mathbb{R}^m$ be Lipschitz. Then*

$$\pi^{-1}(y) \text{ is a } d\text{-dimensional rectifiable set for } H^m\text{-almost every } y \in \mathbb{R}^m, \tag{8.11}$$

and

$$\int_{Z \cap A} apJ_m\pi(z) \, dH^{d+m}(z) = \int_{y \in \mathbb{R}^m} \left\{ \int_{z \in \pi^{-1}(y)} \mathbf{1}_{A \cap Z}(z) \, dH^d(z) \right\} dH^m(y) \tag{8.12}$$

for every Borel set $A \subset \mathbb{R}^n$, where the approximate Jacobian $apJ_m\pi$ of dimension m is defined (almost-everywhere) by $apJ_m\pi(z) = || \wedge_m D\pi(z)||$; see 1.3.1 in [Fe].

We apply Lemma 8.10 to $Z = E^\sharp \cap \Sigma$, and where π is the restriction to Z of the second projection $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^m$. Here π is even linear, and since π is 1-Lipschitz and the tangent plane to $E^\sharp$ always contains the m-plane $\{0\} \times R^m$ where $D\pi$ is the identity, the approximate Jacobian $apJ_m\pi$ is equal to 1.

Also, $\pi^{-1}(y) = E \cap B$ for $y \in B(0, R+1) \subset \mathbb{R}^m$, and $\emptyset$ otherwise, so (8.11) says that

$$E \cap B \text{ is rectifiable}, \tag{8.13}$$

and (8.12) with $A = \Sigma$ says that

$$H^{d+m}(E^\sharp \cap \Sigma) = H^d(E \cap B) \, H^m(\mathbb{R}^m \cap B(0, R+1)). \tag{8.14}$$

Similarly set $\Sigma_1 = B \times B(0, R)$, and observe that $f(x, y) = (\varphi(x), y)$ for $(x, y) \in \Sigma_1$, by (8.5) and because $\psi(y) = 1$ for $y \in B(0, R)$. Then

$$f(E^\sharp \cap \Sigma_1) = f\big[(E \cap B) \times B(0, R)\big] = \varphi(E \cap B) \times B(0, R) = [\varphi(E) \cap B] \times B(0, R) \tag{8.15}$$

since $\varphi(E \cap B) = \varphi(E) \cap B$ because by definition of B, $\varphi(B) \subset B$ and $\varphi(x) = x$ out of B.

Notice that $f(E^\sharp)$ is rectifiable, because $E^\sharp$ is rectifiable and f is Lipschitz. Apply Lemma 8.10 to the product set $f(E^\sharp \cap \Sigma_1) = [\varphi(E) \cap B] \times$

$B(0, R)$ (which is rectifiable because it is contained in $f(E^\sharp)$), and with the same projection π as before. The proof of (8.14) yields

$$
\begin{aligned}
H^{d+m}(f(E^\sharp \cap \Sigma_1)) &= H^{d+m}\big([\varphi(E) \cap B] \times B(0, R)\big) \\
&= H^d(\varphi(E) \cap B) \, H^m(B(0, R)).
\end{aligned}
\tag{8.16}
$$

Set $\Xi = E^\sharp \cap (\Sigma \setminus \Sigma_1) = [E \cap B] \times [B(0, R+1) \setminus B(0, R)]$. Now

$$
\begin{aligned}
H^{d+m}(f(\Xi)) &\leqslant C H^{d+m}(\Xi) \\
&= C H^d(E \cap B) \, H^m(B(0, R+1) \setminus B(0, R)) \leqslant C R^{m-1}
\end{aligned}
\tag{8.17}
$$

because f is Lipschitz and by Lemma 8.10 (applied to the rectifiable product set Ξ; the proof is again the same as for (8.14)). Here C depends on the Lipschitz constant for φ through $|\nabla f|$, and on $E \cap B$. This is unusual, but the main point is that C does not depend on R, by (8.6).

Return to the right-hand side of (8.9). Let z lie in $f(E^\sharp \cap \Sigma)$, and let $(x, y) \in E^\sharp \cap \Sigma$ be such that $z = f(x, y)$. If $y \in B(0, R+1) \setminus B(0, R)$, then $(x, y) \in \Xi$ and $z \in f(\Xi)$. Otherwise, $\psi(y) = 1$, $g(x, y) = \varphi(x)$, and $z = (\varphi(x), y) \in [\varphi(E) \cap B] \times B(0, R)$. Thus $f(E^\sharp \cap \Sigma) \subset f(\Xi) \cup [\varphi(E) \cap B] \times B(0, R)$, and (8.9) yields

$$
\begin{aligned}
H^{d+m}(E^\sharp \cap \Sigma) &\leqslant H^{d+m}(f(E^\sharp \cap \Sigma)) \\
&\leqslant H^{d+m}(f(\Xi)) + H^{d+m}([\varphi(E) \cap B] \times B(0, R)) \\
&\leqslant C R^{m-1} + H^{d+m}(f(E^\sharp \cap \Sigma_1)) \\
&\leqslant C R^{m-1} + H^d(\varphi(E) \cap B) \, H^m(B(0, R))
\end{aligned}
\tag{8.18}
$$

by (8.17), (8.15), and (8.16). Now apply (8.14), divide by $H^m(B(0, R+1))$, and use (8.18) to get that

$$
H^d(E \cap B) = H^m(B(0, R+1))^{-1} H^{d+m}(E^\sharp \cap \Sigma) \leqslant H^d(\varphi(E) \cap B) + C R^{-1}.
\tag{8.19}
$$

Then let R tend to $+\infty$, remove from both sides $H^d(E \cap \varphi(E) \cap B)$, and get (8.2).

We still need to show that E is reduced. So let $x \in E$ and $r > 0$ be given. The same computations as above show that

$$
H^{d+m}(E^\sharp \cap [B(x, r) \times B(0, 1)]) = H^d(E \cap B(x, r)) \, H^m(B(0, 1) \cap \mathbb{R}^m).
\tag{8.20}
$$

The left-hand side is positive because $E^\sharp$ is reduced, so $H^d(E \cap B(x, r)) > 0$, as needed. This completes our proof of Proposition 8.3. $\quad\square$

Remark 8.21. — The proof of Proposition 8.3 relies on the fact that we work with minimal sets, so that we were able to use the Lipschitz norm of

φ in the argument above, compensate by taking R enormous, and get away with it. There will be an argument of the same type in the next section, but we will then be working with almost-minimal sets, and have to pay by including the Lipschitz properties of φ in the definition of weak almost-minimal sets.

Remark 8.22. — Probably the proof of Proposition 8.3 allows us to show that if a product $E^\sharp = E \times F$ of a d_1-dimensional closed set $E \subset \mathbb{R}^{n_1}$ with a d_2-dimensional closed set $F \subset \mathbb{R}^{n_2}$ is a $(d_1 + d_2)$-dimensional minimal set in $\mathbb{R}^{n_1+n_2}$, and if some mild assumptions on the behavior of $H^{d_i}(E_i \cap B(0,R))$ for R large are satisfied, then each E_i is a minimal set of dimension d_i in $\mathbb{R}^{n_i}$.

I don't know whether the converse is true, i.e., whether products of minimal sets are necessarily minimal. There is an argument in [D2] for the product of a minimal set with $\mathbb{R}^{n_2}$ (see page 530 and Exercise 76.16 in [D2]), but in the different context of Mumford-Shah minimal sets in codimension 1; the argument does not apply here, even for $E^\sharp = E \times \mathbb{R}^{n_2}$, because the sections $\varphi_1(E) \cap [\mathbb{R}^{n_1} \times \{y\}]$, $y \in \mathbb{R}^{n_2}$, are not always competitors for $E \times \{y\}$ in $\mathbb{R}^{n_1} \times \{y\}$.

Note that by contrast, products of area-minimizing normal currents are not always area-minimizing; see Remarks 1.2 in [Mo1].

THEOREM 8.23. — *Let the integers $0 < d < n$ be given. For each $\varepsilon > 0$, we can find $r_0 \in (0,1)$ such if E is a reduced minimal cone of dimension $d + 1$ in $\mathbb{R}^n$, x is a point of $K = E \cap \partial B(0,1)$, $0 < r \leqslant r_0$, and P denotes the tangent hyperplane to $\partial B(0,1)$ at x, then there is a reduced minimal set L of dimension d in P such that*

$$
\begin{aligned}
&\operatorname{dist}(y, L) \leqslant \varepsilon r \text{ for } y \in K \cap B(x,r) \text{ and} \\
&\operatorname{dist}(z, K) \leqslant \varepsilon r \text{ for } z \in L \cap B(x,r),
\end{aligned}
\tag{8.24}
$$

and

$$
\begin{aligned}
H^d(L \cap B(y, t - \varepsilon r)) - \varepsilon r^d &\leqslant H^d(K \cap B(y,t)) \\
&\leqslant H^d(L \cap B(y, t + \varepsilon r)) + \varepsilon r^d
\end{aligned}
\tag{8.25}
$$

for every ball $B(y,t)$ contained in $B(x,r)$.

We start with a few comments before the proof. The definition of minimal sets was recalled (and slightly simplified) in Definition 8.1. A minimal cone is simply a minimal set which happens to be a cone centered at the origin. The two possible definitions of reduced minimal sets in P (where we see L as a subset of P, or directly as a subset of $\mathbb{R}^n$) are equivalent. Indeed it is obvious that if L is minimal in $\mathbb{R}^n$, then it is minimal in P because

every deformation in P is a deformation in $\mathbb{R}^n$. Conversely, suppose that L is not minimal in $\mathbb{R}^n$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz, with $\varphi(x) = x$ out of a compact set, and such that $H^d(E \setminus \varphi(E)) > H^d(\varphi(E) \setminus E)$. Set $\widetilde{\varphi} = \pi \circ \varphi$, where π denotes the orthogonal projection on P. Let B be a large ball centered on P such that $\varphi(x) = x$ out of B and $\varphi(B) \subset B$. Thus E, $\varphi(E)$, and $\widetilde{\varphi}(E)$ coincide out of B. In addition,

$$\begin{aligned}
H^d(\widetilde{\varphi}(E) \cap B) &= H^d(\pi(\varphi(E)) \cap B) = H^d(\pi(\varphi(E) \cap B)) \\
&\leqslant H^d(\varphi(E) \cap B) < H^d(E \cap B),
\end{aligned} \tag{8.26}$$

so E is not minimal in P either.

Observe also that Theorem 8.23 contains the slightly easier observation that any blow-up limit of $K = E \cap \partial B(0,1)$ at a point $x \in K$ is a reduced minimal set of dimension d in the tangent plane to $\partial B(0,1)$ at x. For a direct proof of this observation, simply follow the proof of the theorem up to (8.30).

Our estimate (8.25) looks a little ugly, compared for instance to its analogue (7.28); the point is that we don't know that $H^d(L \cap \partial B(y,t)) = 0$ (as happened with Lemma 7.34). Incidentally, if $t \leqslant \varepsilon r$, simply understand $B(y, t - \varepsilon r)$ as the empty set (and the first half of (8.25) is trivial). We do not really need (8.25) for the rest of this text (we use it for (14.2), but we could also use (8.24) and a compactness argument using Lemma 3.12 there, and the argument would be a tiny bit simpler), but we decided to include (8.25) for the sake of completeness and to avoid potential unpleasant surprises later. We shall start with the simpler proof of (8.24) anyway.

We now prove the theorem, by contradiction and compactness. Suppose that we can find ε such that no r_0 works. Then for each integer $k \geqslant 0$, the statement fails with $r_0 = 2^{-k}$ and there is a reduced minimal cone E_k, a point $x_k \in K_k = E_k \cap \partial B(0,1)$, and a radius $r_k \in (0, 2^{-k})$ such that no minimal set L_k satisfies (8.24) and (8.25). We want to derive a contradiction.

Set $A = (1, 0, \cdots, 0)$. By rotation invariance, we can assume that $x_k = A$ for every k, and then P is always the same vertical plane through A. Set $F_k = r_k^{-1}(E_k - A)$, so that now $\{F_k\}$ is a sequence of reduced minimal sets through the origin. We can extract a subsequence, so that after extraction $\{F_k\}$ converges to some closed set $F \subset \mathbb{R}^n$. See (3.1) for the definition, and recall that the existence of converging subsequences is a standard consequence of the fact that the Hausdorff distance on the set of compact subsets of a compact metric space makes this set compact too. Then Lemma 4.7 (with the gauge function $h = 0$) says that

$$F \text{ is a reduced minimal set of dimension } d + 1. \tag{8.27}$$

Next denote by P_0 the hyperplane parallel to P through the origin and by D the first axis (the vector line perpendicular to P_0). Let us check that

$$F = D \times F^\sharp, \quad \text{where } F^\sharp = F \cap P_0. \qquad (8.28)$$

Let $y \in F$ be given, and let us verify that $y + D \subset F$. Since the F_k tend to F, we can find $y_k \in F_k$ so that $\{y_k\}$ tends to y. Set $z_k = r_k y_k + A$; notice that $z_k \in E_k$ by definition of F_k, and that z_k tends to A because r_k tends to 0. Fix $\lambda \in \mathbb{R}$, and set $v_k = (1 + \lambda r_k) z_k$. Then $v_k \in E_k$ for k large, because E_k is a cone, and so $w_k = r_k^{-1}(v_k - A)$ lies in F_k. But

$$
\begin{aligned}
w_k &= r_k^{-1}((1 + \lambda r_k) z_k - A) = r_k^{-1}((1 + \lambda r_k)(r_k y_k + A) - A) \\
&= r_k^{-1}(r_k y_k + \lambda r_k^2 y_k + \lambda r_k A) = y_k + \lambda A + \lambda r_k y_k,
\end{aligned}
\qquad (8.29)
$$

so w_k tends to $y + \lambda A$ (because y_k tends to y and r_k tends to 0), and $y + \lambda A \in F$ because F is the limit of the F_k. Thus $y + D \subset F$ when $y \in F$; this proves (8.28).

We deduce from (8.28), (8.27), and Proposition 8.3 (with $m = 1$) that

$$F^\sharp \text{ is a reduced minimal set of dimension } d \text{ in } P_0. \qquad (8.30)$$

[This is where a proof of our description of blow-up limits of K would end.]

Set $L_k = r_k F^\sharp + A$; this is a reduced minimal set of dimension d in P. Let us check that for k large, L_k satisfies (8.24) with respect to $K_k = E_k \cap \partial B(0, 1)$.

First let $y \in L_k \cap B(A, r_k)$ be given, and set $z = r_k^{-1}(y - A)$. Thus $z \in F^\sharp \cap B(0, 1) \subset F \cap B(0, 1)$. Let $v \in F_k$ minimize the distance to z; thus $|v - z| \leqslant \varepsilon_k$, where ε_k measures the distance from F_k to F in $B(0, 2)$, say, and the sequence $\{\varepsilon_k\}$ converges to 0. Set $w = r_k v + A$. Then $w \in E_k$, and $|w - y| = r_k |v - z| \leqslant r_k \varepsilon_k$. Also, $y \in P \cap B(A, r_k)$, so $|y - 1| \leqslant Cr_k^2$ (because P is tangent to $\partial B(0, 1)$ at A), and $|w - 1| \leqslant |w - y| + |y - 1| \leqslant r_k \varepsilon_k + Cr_k^2$. Now $\frac{w}{|w|} \in K_k$, and $|\frac{w}{|w|} - y| \leqslant |w - 1| + |w - y| \leqslant 2 r_k \varepsilon_k + Cr_k^2$, which is less than εr_k for k large. This proves the second half of (8.24).

Now consider $y \in K_k \cap B(A, r_k)$, set $z = r_k^{-1}(y - A)$, and observe that $z \in F_k \cap B(0, 1)$ because $y \in E_k$. We can find $v \in F$ such that $|v - z| \leqslant \varepsilon_k$, with ε_k as above. Observe that $\text{dist}(y, P) \leqslant Cr_k^2$ because P is tangent to $\partial B(0, 1)$ at A, so $\text{dist}(z, P_0) \leqslant Cr_k$ and $\text{dist}(v, P_0) \leqslant Cr_k + \varepsilon_k$. Recall from (8.28) that $F = D \times F^\sharp$, so the point v' at the intersection of $v + D$ and P_0 lies in $F^\sharp$, and of course $|v' - v| = \text{dist}(v, P_0) \leqslant Cr_k + \varepsilon_k$. Now $w = r_k v' + A$ lies in L_k, and $|w - y| = r_k |v' - z| \leqslant r_k(Cr_k + 2\varepsilon_k) < \varepsilon r_k$ for k large. Thus (8.24) holds for k large. Our proof of Theorem 8.23 would end here with a contradiction if we did not include (8.25) in the statement.

By assumption, there is a ball $B_k = B(y_k, t_k) \subset B(A, r_k)$ such that (8.25) fails, i.e.,

$$H^d(L_k \cap B(y_k, t_k - \varepsilon r_k)) > H^d(K_k \cap B_k) + \varepsilon r_k^d \qquad (8.31)$$

or

$$H^d(L_k \cap B(y_k, t_k + \varepsilon r_k)) < H^d(K_k \cap B_k) - \varepsilon r_k^d. \qquad (8.32)$$

We want to produce a contradiction. Set

$$z_k = r_k^{-1}(y_k - A), \quad \rho_k = r_k^{-1} t_k, \quad \text{and } B_k' = B(z_k, \rho_k) = r_k^{-1}(B_k - A). \qquad (8.33)$$

Notice that $B_k' \subset B(0, 1)$ because $B_k \subset B(A, r_k)$. Since we may replace $\{E_k\}$ with a subsequence, we may assume that $\{B_k'\}$ converges to some ball $B(z, \rho) \subset \overline{B}(0, 1)$. That is,

$$\{z_k\} \text{ converges to } z \text{ and } \{\rho_k\} \text{ converges to } \rho \geqslant 0. \qquad (8.34)$$

We want to use the results of Section 4 to estimate the measure of some sets, but we need a small amount of space to move around. So let $\tau > 0$ be very small (to be chosen later). We shall start with upper bounds for $H^d(E_k \cap B_k)$. Identify D with the real line, set $I = [0, 1] \subset D$, and apply Lemma 3.12 to the sequence $\{F_k\}$ of $(d+1)$-minimal sets, in the compact set

$$D_1 = I \times \overline{B}(z, \rho + \tau). \qquad (8.35)$$

We can take $M = 1$ and $h = 0$ in (3.13), and we get that

$$\limsup_{k \to +\infty} H^{d+1}(F_k \cap D_1) \leqslant H^{d+1}(F \cap D_1). \qquad (8.36)$$

By (8.27), F is rectifiable, and we can apply Lemma 8.10 (or Theorem 3.2.22 in [Fe]) to the product $F \cap D_1 = [I \times F^\sharp \cap \overline{B}(z, \rho + \tau)]$, in the same conditions as above. This yields

$$H^{d+1}(F \cap D_1) = H^1(I) \, H^d(F^\sharp \cap \overline{B}(z, \rho + \tau)) = H^d(F^\sharp \cap \overline{B}(z, \rho + \tau)). \qquad (8.37)$$

Let us check that for k large,

$$H^d(F^\sharp \cap \overline{B}(z, \rho + \tau)) \leqslant r_k^{-d} H^d(L_k \cap B(y_k, t_k + \varepsilon r_k)). \qquad (8.38)$$

Indeed $\overline{B}(z, \rho + \tau) \subset B(z_k, \rho_k + \varepsilon)$ for k large, by (8.34) and if we choose $\tau < \varepsilon$. Set $\psi(x) = A + r_k x$ for a moment; we know that $L_k = \psi(F^\sharp)$, and $\psi(B(z_k, \rho_k + \varepsilon)) = B(y_k, t_k + \varepsilon r_k)$ by (8.33). So

$$\psi(F^\sharp \cap \overline{B}(z, \rho + \tau)) \subset \psi(F^\sharp \cap B(z_k, \rho_k + \varepsilon)) = L_k \cap B(y_k, t_k + \varepsilon r_k), \qquad (8.39)$$

and (8.38) follows. We deduce from (8.36), (8.37), and (8.38) that for k large,

$$H^{d+1}(F_k \cap D_1) \leqslant \tau + r_k^{-d} H^d(L_k \cap B(y_k, t_k + \varepsilon r_k)). \tag{8.40}$$

We also need to compare $H^{d+1}(F_k \cap D_1)$ with $H^d(K_k \cap B_k)$. Set

$$E_k' = \{\lambda v \,; 1 + \tau r_k \leqslant \lambda \leqslant 1 + r_k - \tau r_k \text{ and } v \in K_k \cap B_k\}. \tag{8.41}$$

and $E_k'' = r_k^{-1}(E_k' - A)$. We claim that for k large,

$$E_k'' \subset F_k \cap D_1. \tag{8.42}$$

Indeed let λ and v be as in the definition of E_k'; we want to show that $w = r_k^{-1}(\lambda v - A)$ lies in $F_k \cap D_1$. First, $\lambda v \in E_k$ because $v \in K_k \subset E_k$ and E_k is a cone, so $w \in F_k = r_k^{-1}(E_k - A)$. We still need to check that $w \in D_1$.

Write $v = A + (y_k - A) + (v - y_k)$, and also denote by π the orthogonal projection onto P_0. Observe that

$$|\pi(y_k - A) - (y_k - A)| \leqslant Cr_k|y_k - A| \leqslant Cr_k^2 \tag{8.43}$$

because $y_k \in B(A, r_k)$ and P is tangent to $\partial B(0, 1)$ at A, and similarly

$$|\pi(v - y_k) - (v - y_k)| \leqslant Cr_k|v - y_k| \leqslant Cr_k t_k \tag{8.44}$$

because $v \in B_k = B(y_k, t_k)$ and both points lie in $K_k \subset \partial B(0, 1)$. Then

$$\begin{aligned} w &= r_k^{-1}(\lambda v - A) = r_k^{-1}[(\lambda - 1)v + (v - A)] \\ &= r_k^{-1}[(\lambda - 1)v + (y_k - A) + (v - y_k)] \\ &= z_k + r_k^{-1}[(\lambda - 1)v + (v - y_k)], \end{aligned} \tag{8.45}$$

because $z_k = r_k^{-1}(y_k - A)$ by (8.33). Then

$$\begin{aligned} \big|w - z_k &- r_k^{-1}[(\lambda - 1)A - \pi(v - y_k)]\big| \\ &\leqslant r_k^{-1}[(\lambda - 1)|v - A| + |\pi(v - y_k) - (v - y_k)|] \\ &\leqslant |v - A| + Ct_k \leqslant r_k + Ct_k \end{aligned} \tag{8.46}$$

because $1 \leqslant \lambda \leqslant 1 + r_k$, by (8.44), and because $v \in B_k \subset B(A, r_k)$ by definition of $B_k = B(y_k, t_k)$. Set $w' = z + r_k^{-1}(\lambda - 1)A + r_k^{-1}\pi(v - y_k)$. Then

$$|w - w'| \leqslant |z - z_k| + r_k + Ct_k. \tag{8.47}$$

Observe that $z \in P_0$, for instance because it is the limit of the $z_k = r_k^{-1}(y_k - A)$ and by (8.43). Thus the projection of w' on D is $r_k^{-1}(\lambda - 1)A \in [\tau, 1 - \tau]$, and by (8.47) the first coordinate of w lies in I for k large.

Similarly, $\pi(w') = z + r_k^{-1}\pi(v - y_k)$, and $r_k^{-1}|v - y_k| < r_k^{-1}t_k = \rho_k$ because $v \in B_k$ and by (8.33), so $\pi(w') \in B(z, \rho_k)$. Since ρ_k tends to ρ, we deduce from (8.47) that $\pi(w) \in B(z, \rho + \tau)$ for k large. Altogether, $w \in D_1 = I \times \overline{B}(z, \rho + \tau)$ (compare with the definition (8.35)), and we completed our proof of (8.42).

Next we evaluate $H^{d+1}(E_k'')$. Recall that $E_k'' = r_k^{-1}(E_k' - A)$, so

$$H^{d+1}(E_k'') = r_k^{-d-1}H^{d+1}(E_k'). \tag{8.48}$$

Notice that $E_k' \subset E_k$, which is rectifiable, so we may apply Lemma 8.10 with $m = 1$, $Z = E_k'$ and to the Lipschitz function $\pi_r(x) = |x|$. Here (8.12) can be written as

$$\int_{E_k'} apJ_1\pi_r(x)\,dH^{d+1}(x) = \int_\lambda \left\{ \int_{|x|=\lambda} \mathbf{1}_{E_k'}(x)\,dH^d(x) \right\} dH^1(\lambda). \tag{8.49}$$

The approximate Jacobian is $apJ_1\pi_r(x) = 1$, because π_r is 1-Lipschitz, and the tangent plane to E_k at x always contains the radial direction where the differential $D\pi_r$ has a norm exactly equal to 1. Thus (8.49) and (8.41) say that

$$\begin{aligned} H^{d+1}(E_k') &= \int_{1+\tau r_k \leqslant \lambda \leqslant 1+r_k-\tau r_k} H^d(\lambda(K_k \cap B_k))\,dH^1(\lambda) \\ &= \int_{1+\tau r_k \leqslant \lambda \leqslant 1+r_k-\tau r_k} \lambda^d H^d(K_k \cap B_k)\,dH^1(\lambda) \\ &\geqslant (r_k - 2\tau r_k)H^d(K_k \cap B_k). \end{aligned} \tag{8.50}$$

Altogether,

$$\begin{aligned} (r_k - 2\tau r_k)H^d(K_k \cap B_k) &\leqslant H^{d+1}(E_k') = r_k^{d+1}H^{d+1}(E_k'') \\ &\leqslant r_k^{d+1}H^{d+1}(F_k \cap D_1) \\ &\leqslant r_k^{d+1}\tau + r_k H^d(L_k \cap B(y_k, t_k + \varepsilon r_k)) \end{aligned} \tag{8.51}$$

for k large, by (8.50), (8.48), (8.42), and (8.40). Thus

$$\begin{aligned} H^d(K_k \cap B_k) &\leqslant (1 - 2\tau)^{-1}[r_k^d\tau + H^d(L_k \cap B(y_k, t_k + \varepsilon r_k))] \\ &\leqslant H^d(L_k \cap B(y_k, t_k + \varepsilon r_k)) + \varepsilon r_k^d, \end{aligned} \tag{8.52}$$

because $H^d(L_k \cap B(y_k, t_k + \varepsilon r_k)) \leqslant Cr^d$ (recall from (8.30) that $L_k = A + r_k F^\sharp$ is a d-dimensional minimal set, and use Lemma 2.15), and if τ is small enough. We just proved that (8.32) fails for k large.

Now we want to show that (8.31) fails too. The argument will be quite similar to the one above. This time we set $I = (0,1) \subset D$ and

$D_2 = I \times B(z, \rho - \tau)$. [If $\rho < \tau$, then $\rho_k < \varepsilon$ and hence $t_k < \varepsilon r_k$ for k large, and (8.31) fails trivially.] Apply Lemma 3.3 to the open set D_2 and the sequence $\{F_k\}$ of $(d+1)$-minimal sets. We get that

$$H^{d+1}(F \cap D_2) \leqslant \liminf_{k \to +\infty} H^{d+1}(F_k \cap D_2). \qquad (8.53)$$

As for (8.37), F is rectifiable and we can apply Lemma 8.10 to the product $F \cap D_2 = I \times [F^\sharp \cap B(z, \rho - \tau)]$. We get that

$$H^{d+1}(F \cap D_2) = H^d(F^\sharp \cap B(z, \rho - \tau)). \qquad (8.54)$$

Next we check that for k large,

$$H^d(F^\sharp \cap B(z, \rho - \tau)) \geqslant r_k^{-d} H^d(L_k \cap B(y_k, t_k - \varepsilon r_k)). \qquad (8.55)$$

Observe that for k large, $B(z, \rho - \tau)$ contains $B(z_k, \rho_k - \varepsilon)$, by (8.34). Again set $\psi(x) = A + r_k x$ for $x \in \mathbb{R}^n$; thus $L_k = \psi(F^\sharp)$ and $\psi(B(z_k, \rho_k - \varepsilon)) = B(y_k, t_k - \varepsilon r_k)$ by (8.33). So

$$\psi(F^\sharp \cap \overline{B}(z, \rho - \tau)) \supset \psi(F^\sharp \cap B(z_k, \rho_k - \varepsilon)) = L_k \cap B(y_k, t_k - \varepsilon r_k), \ (8.56)$$

and (8.55) follows. Then

$$H^{d+1}(F_k \cap D_2) \geqslant H^{d+1}(F \cap D_2) - \tau \geqslant r_k^{-d} H^d(L_k \cap B(y_k, t_k - \varepsilon r_k)) - \tau \ (8.57)$$

for k large, by (8.53), (8.54), and (8.55).

We again compare $H^{d+1}(F_k \cap D_2)$ with $H^d(K_k \cap B_k)$. We still set $E_k'' = r_k^{-1}(E_k' - A)$, but this time with the larger

$$E_k' = \big\{ \lambda v \, ; \, 1 - \tau r_k \leqslant \lambda \leqslant 1 + r_k + \tau r_k \text{ and } v \in K_k \cap B_k \big\}. \qquad (8.58)$$

We claim that then

$$F_k \cap D_2 \subset E_k'' \quad \text{for } k \text{ large}. \qquad (8.59)$$

Let $x \in F_k \cap D_2$ be given; we need show that $w = A + r_k x$ lies in E_k'. We already know that $w \in E_k$, so $w/|w| \in K_k$, and it is enough to show that

$$w/|w| \in B_k \quad \text{and} \quad 1 - \tau r_k \leqslant |w| \leqslant 1 + r_k + \tau r_k \qquad (8.60)$$

Write $x = \alpha A + z + v$, where αA is the projection of x on D. Thus $0 < \alpha < 1$, and also $v \in P_0 \cap B(0, \rho - \tau)$ (because $x \in D_2$ and $z \in P_0$). Now $w = A + r_k x = (1 + \alpha r_k)A + r_k(z + v)$ so

$$|w| = 1 + \alpha r_k + O(r_k^2) \qquad (8.61)$$

by Pythagoras (recall that $z + v \in P_0$). Then

$$w/|w| = (1 - \alpha r_k)\, w + O(r_k^2) = A + r_k(z + v) + O(r_k^2) \qquad (8.62)$$

and

$$\bigl|w/|w|-y_k\bigr| \leqslant r_k|z-z_k|+r_k|v|+O(r_k^2) \leqslant r_k|z-z_k|+r_k(\rho-\tau)+O(r_k^2) \quad (8.63)$$

because $y_k = A + r_k z_k$ and $v \in B(0, \rho - \tau)$. In these estimates, the error terms $O(r_k^2)$ are less than Cr_k^2, with C independent of x. If k is large enough, (8.61) implies that $1 - \tau r_k \leqslant |w| \leqslant 1 + r_k + \tau r_k$ (because $0 \leqslant \alpha \leqslant 1$), and (8.63) says that $\bigl|w/|w| - y_k\bigr| < t_k = r_k\rho_k$ (because z_k tends to z and ρ_k tends to ρ, by (8.34)). This proves (8.60) and (8.59).

We still have that $H^{d+1}(E_k'') = r_k^{-d-1}H^{d+1}(E_k')$ as in (8.48), and Lemma 8.10 now leads to

$$
\begin{aligned}
H^{d+1}(E_k') &= \int_{1-\tau r_k \leqslant \lambda \leqslant 1+r_k+\tau r_k} H^d(\lambda(K_k \cap B_k))\, dH^1(\lambda)\\
&= \int_{1-\tau r_k \leqslant \lambda \leqslant 1+r_k+\tau r_k} \lambda^d H^d(K_k \cap B_k)\, dH^1(\lambda) \qquad (8.64)\\
&\leqslant (r_k + 2\tau r_k)(1 + r_k + \tau r_k)^d H^d(K_k \cap B_k).
\end{aligned}
$$

as in (8.49) and (8.50). Thus

$$
\begin{aligned}
H^d(K_k \cap B_k) &\geqslant (1 - Cr_k - C\tau)\, r_k^{-1} H^{d+1}(E_k')\\
&= (1 - Cr_k - C\tau)\, r_k^d\, H^{d+1}(E_k'')\\
&\geqslant (1 - Cr_k - C\tau)\, r_k^d\, H^{d+1}(F_k \cap D_2) \qquad (8.65)\\
&\geqslant (1 - Cr_k - C\tau)H^d(L_k \cap B(y_k, t_k - \varepsilon r_k)) - \tau r_k^d\\
&\geqslant H^d(L_k \cap B(y_k, t_k - \varepsilon r_k)) - \varepsilon r_k^d
\end{aligned}
$$

by (8.59) and (8.57), because $H^d(L_k \cap B(y_k, t_k - \varepsilon r_k)) \leqslant Cr_k^d$ by Ahlfors regularity of the minimal set L_k (see the proof of (8.52)), if τ is small enough, and for k large. That is, (8.31) fails for k large.

We finally managed to show that (8.31) and (8.32) fail for k large; this contradiction with the definition of our sequence $\{K_k\}$ completes the proof of Theorem 8.23. $\square$

Theorem 8.23 gives some information on the local behavior of $K = E \cap \partial B(0,1)$ when E is a reduced minimal cone of dimension $d + 1$. For instance, if we have a good control on the minimal sets of dimension d, we can try to use a generalization of Reifenberg's topological disk theorem to get local bi-Hölder parameterizations of K. This may not be enough to control K precisely, but it could give some useful topological control.

We shall be interested in the case when $d = 1$; then we'll see in Section 10 that minimal sets of dimension 1 are lines and sets Y, and a simple case of the generalized Reifenberg theorem shows that K is locally bi-Hölder equivalent to a line or a Y. See Section 15 for the existence of generalized Reifenberg parameterizations, Section 12 for the simpler special case when $d = 1$, and Section 16 for similar arguments in dimension 2.

But we also want to know more about K when $d = 1$. In particular, we want to check that it is composed of a finite collection of arcs of great circles, and for this it will be simpler to have a Lipschitz control, rather than merely bi-Hölder.

So we shall start with another approach, where we show that K satisfies a weak almost-minimality property, and first prove reasonable properties for weak almost-minimal sets. This will give slightly more precise results than Theorem 8.23, at least when $d = 1$, and maybe the method can be used in slightly different contexts. But to be fair, one could also start from Theorem 8.23, use Proposition 12.6 to show that K is locally bi-Hölder equivalent to a line or a Y, and then find more direct ways to show that K is composed of arcs of great circles in the regions where it looks a lot like a line. Some arguments of this type will be used in the next paper [D4], to deduce the C^1-regularity of almost-minimal sets of dimension 2 in $\mathbb{R}^3$ from a local biHölder description, and they would be simpler for one-dimensional sets.

9. Weak almost-minimality of $\mathbf{K} = \mathbf{E} \cap \partial \mathbf{B(0,1)}$

Let E be a reduced minimal cone of dimension $d + 1$ in $\mathbb{R}^n$, and set $K = E \cap \partial B(0,1)$. We want to show that E is weakly almost-minimal, as in the next definition. Later on we shall use this property to study E when $d = 1$.

The following definition is a little awkward, because it is a compromise between what we can easily prove for K and what will be needed in later arguments. We state the definition for subsets of $\partial B(0,1)$, but we could also work directly in $\mathbb{R}^n$, or in some affine subspace, with only minor modifications.

DEFINITION 9.1. — *Let K be a closed subset of $\partial B = \partial B(0,1) \subset \mathbb{R}^n$. We say that K is a weak almost-minimal set of dimension d, with gauge function h if, for each choice of $x \in \partial B$, $r \in (0,1)$, and a Lipschitz function $f : \partial B \to \partial B$ such that*

$$W_f = \big\{ y \in \partial B \,;\, f(y) \neq y \big\} \subset B(x,r) \ \ and \ f(W_f) \subset \overline{B}(x,r), \qquad (9.2)$$

we have that

$$H^d(K \cap \overline{B}(x,r)) \leqslant H^d(f(K) \cap \overline{B}(x,r)) + (1 + M^d)\, h(r) r^d, \qquad (9.3)$$

whenever there is a partition of $K \cap B$ into a finite number of pieces where f is M-Lipschitz.

As before, our convention is that a gauge function is a nonnegative non-decreasing function h, with $\lim_{r \to 0} h(r) = 0$. There are two differences with the definitions of Section 4. The small one is that we work on ∂B instead of an open set in $\mathbb{R}^n$. This is just because our examples will live on the sphere, but the same definition would work in $\mathbb{R}^n$ or in some other smooth manifolds.

The main difference is that here we pay an additional price that depends on the Lipschitz properties of f. The story about the partition of $K \cap B$ is a mostly a way to avoid discussing local Lipschitz constants or gradients. Note that M may be significantly smaller than the Lipschitz norm of f, or even of its restriction to $K \cap B$; we shall use this advantage in some proofs.

We would have a slightly weaker notion of weak almost-minimal set if we replaced M^d in (9.3) with some larger functions of M; I don't think that this would make much difference in the properties of weak almost-minimal set that will be proved below. Also M^{d+1} would be a little easier to get in the proof of Proposition 9.4. On the other hand M^d looks more natural, so we shall keep it in the definition.

PROPOSITION 9.4. — *Let E be a reduced minimal cone of dimension $d+1$ in $\mathbb{R}^n$, and set $K = E \cap \partial B(0,1)$. Then K is a reduced weak almost-minimal set of dimension d, with gauge function $h(r) = Cr$.*

Let us first check that K is reduced. Recall from Definition 2.12 that this means that

$$H^d(K \cap B(x,r)) > 0 \qquad (9.5)$$

for $x \in K$ and $r \in (0,1)$. Let us check this brutally. We apply Lemma 8.10 (i.e., Theorem 3.2.22 in [Fe]) with $Z = E \cap B(x, r/2)$, $m = 1$, $A = Z$, and for π the radial projection defined by $\pi(x) = |x|$. The rectifiability of E was discussed in Section 2, in a more general context. So (8.12) yields

$$\int_Z ap J_1 \pi(x) dH^{d+1}(x) = \int_\rho \left\{ \int_{Z \cap \pi^{-1}(\rho)} dH^d \right\} d\rho. \qquad (9.6)$$

The approximate Jacobian $ap J_1 \pi(x)$ is easily computed to be 1, because π is 1-Lipschitz and the tangent plane to E always contains the radial direction

where the Lipschitz constant for π is exactly 1. Thus (9.6) says that

$$H^{d+1}(E \cap B(x, r/2)) = H^{d+1}(Z) = \int_\rho H^d(Z \cap \pi^{-1}(\rho)) \, d\rho$$

$$= \int_\rho H^d(E \cap B(x, r/2) \cap \partial B(0, \rho)) \, d\rho. \tag{9.7}$$

In addition, the only contribution in the right-hand side comes from $1 - r/2 < \rho < 1 + r/2$, for which $E \cap B(x, r/2) \cap \partial B(0, \rho)$ is contained in $\rho \cdot (K \cap B(x, r))$ because E is a cone and $K = E \cap \partial B(0, 1)$. Thus

$$H^{d+1}(E \cap B(x, r/2)) \leqslant \int_{1-r/2 < \rho < 1+r/2} H^d(K \cap B(x, r)) \, \rho^d d\rho. \tag{9.8}$$

The left-hand side of (9.8) is positive because E is reduced, hence $H^d(K \cap B(x, r)) > 0$, (9.5) holds, and K is reduced.

We now prove that K is weakly almost-minimal. Let $x \in \partial B$, $r \in (0, 1)$, and $f : \partial B \to \partial B$ be as in Definition 9.1, and let us use f to define a competitor for E.

Let $\psi : [0, +\infty) \to [0, 1]$ be such that

$$\psi(t) = 0 \quad \text{for } 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$\psi(t) = 1 \quad \text{for } \frac{1}{2} + \frac{r}{10} \leqslant t \leqslant 1 - \frac{r}{10}, \tag{9.9}$$
$$\psi(t) = 0 \quad \text{for } t \geqslant 1,$$

ψ is continuous, and ψ is linear in the two missing interval.

Write every point of $\mathbb{R}^n$ as $z = \rho y$, with $\rho = |z|$ and $y = y/|y|$ (the choice of y when $z = 0$ does not matter), and set

$$\varphi(z) = \psi(\rho)\rho f(y) + (1 - \psi(\rho))\rho y \quad \text{for } z \in \mathbb{R}^n. \tag{9.10}$$

Since E is minimal and φ is Lipschitz, Definition 8.1 yields

$$H^{d+1}(E \setminus \varphi(E)) \leqslant H^{d+1}(\varphi(E) \setminus E). \tag{9.11}$$

Let us compute what this means in terms of K and $f(K)$. Observe that

$$\varphi(z) = z \text{ for } z \in \mathbb{R}^n \setminus B(0, 1) \quad \text{and} \quad \varphi(B(0, 1)) \subset B(0, 1), \tag{9.12}$$

so $\varphi(E)$ coincides with E out of $B(0, 1)$ and there will be no contribution to (9.11) coming from $\mathbb{R}^n \setminus B(0, 1)$. The most interesting contributions will come from the annulus

$$A = \Big\{ z \in \mathbb{R}^n \, ; \, \frac{1}{2} + \frac{r}{10} \leqslant |z| \leqslant 1 - \frac{r}{10} \Big\}. \tag{9.13}$$

Let us again apply Lemma 8.10 to the mapping $\pi : z \to |z|$, from $Z = E \cap A$ to $I = [\frac{1}{2} + \frac{r}{10}, 1 - \frac{r}{10}]$. We get (9.6) as before, again with $apJ_1\pi(x) = 1$, so

$$H^{d+1}(E \cap A) = H^{d+1}(Z) = \int_I H^d(Z \cap \pi^{-1}(t)) \, dt = \int_I H^d(E \cap \partial B(0,t)) \, dt$$
$$= \int_I H^d(t \cdot K) \, dt = \int_I t^d \, H^d(K) \, dt = \alpha H^d(K), \tag{9.14}$$

because $\pi^{-1}(t) = \partial B(0, \rho) \subset A$ for $t \in I$, and E is a cone. The precise value of $\alpha = \int_I t^d dt$ won't matter much.

Observe that $\psi(|z|) = 1$ when $z \in A$, so $\varphi(z) = \rho f(y)$. Thus

$$\varphi(E \cap A) \cap \partial B(0, \rho) = \rho f(K) \text{ for } \rho \in I, \tag{9.15}$$

and it is empty otherwise. The same computation as for $E \cap A$, but with the other rectifiable set $\varphi(E \cap A)$, yields

$$H^{d+1}(\varphi(E \cap A)) = \int_I H^d(\varphi(E \cap A) \cap \partial B(0,t)) \, dt = \alpha H^d(f(K)). \tag{9.16}$$

We shall also need to estimate the contributions of the two annuli

$$A_1 = \Big\{ z \in \mathbb{R}^n \, ; \, \frac{1}{2} \leqslant |z| \leqslant \frac{1}{2} + \frac{r}{10} \Big\} \text{ and } A_2 = \Big\{ z \in \mathbb{R}^n \, ; \, 1 - \frac{r}{10} \leqslant |z| \leqslant 1 \Big\}. \tag{9.17}$$

Set $Z_i = \big\{ z \in E \cap A_i \, ; \, \varphi(z) \neq z \big\}$. We want to estimate the Lipschitz properties of φ on Z_i, and these will depend on the Lipschitz properties of f on $K \cap B(x, r)$. Let us assume, as in Definition 9.1, that we have a partition $K \cap B(x, r) = \cup_j K_j$ of $K \cap B(x, r)$ into a finitely number of sets K_j where f is M-Lipschitz. Note that we may assume that the K_j are Borel sets, because f is automatically M-Lipschitz on the $\overline{K}_j$, which enables us to first replace K_j with $\overline{K}_j$, and then build a Borel partition out of the covering. If $z \in Z_i$, $y = z/|z|$ lies in K, because $z \in E$ and E is a cone, and $f(y) \neq y$ because $\varphi(z) \neq z$ (see (9.10)). Then $y \in K \cap B(x, r)$, by (9.2). Thus the sets

$$Z_{i,j} = \big\{ z \in Z_i \, ; \, y = z/|z| \in K_j \big\} \tag{9.18}$$

form a partition of Z_i. Let us check that

$$\varphi \text{ is } (25 + 2M)\text{-Lipschitz on each } Z_{i,j}. \tag{9.19}$$

It will be enough to control $g(z) = \varphi(z) - z = \rho \psi(\rho)(f(y) - y)$. We write that

$$g(z_1) - g(z_2) = \rho_1 \psi(\rho_1)(f(y_1) - y_1) - \rho_2 \psi(\rho_2)(f(y_2) - y_2) = \Delta_1 - \Delta_2 \,, \tag{9.20}$$

with $\Delta_1 = [\rho_1\psi(\rho_1) - \rho_2\psi(\rho_2)](f(y_1) - y_1)$ and $\Delta_2 = \rho_2\psi(\rho_2)[(f(y_2) - y_2) - (f(y_1) - y_1)]$.

Since $|\nabla(\rho\psi(\rho))| \leqslant \psi(\rho) + \rho|\psi'(\rho)| \leqslant 1 + \frac{10}{r}$, we get that

$$|\Delta_1| \leqslant \frac{11}{r}|\rho_2 - \rho_1||f(y_1) - y_1| \leqslant 22|\rho_2 - \rho_1|, \qquad (9.21)$$

because y_1 and $f(y_1)$ lie in $B(x, r)$, by (9.2). Also, z_1 and z_2 both lie in $Z_{i,j}$, so y_1 and y_2 lie in K_j. Then

$$|\Delta_2| \leqslant |(f(y_2) - y_2) - (f(y_1) - y_1)| \leqslant |y_2 - y_1| + |f(y_2) - f(y_1)| \leqslant (1+M)|y_2 - y_1|, \qquad (9.22)$$

by definition of M and the K_j. Altogether

$$|g(z_1) - g(z_2)| \leqslant |\Delta_1| + |\Delta_2| \leqslant 22|\rho_2 - \rho_1| + (1+M)|y_2 - y_1| \leqslant (24 + 2M)|z_2 - z_1| \qquad (9.23)$$

and (9.19) follows because $\varphi(z) = z + g(z)$.

An immediate consequence of (9.19) is that

$$H^{d+1}(\varphi(Z_{i,j})) \leqslant C(1 + M)^{d+1}H^{d+1}(Z_{i,j}). \qquad (9.24)$$

This would be enough to prove (9.3) with M^d replaced by M^{d+1}, but since we decided to go for the more restrictive definition, we shall need to improve (9.24) a little bit. Observe that $Z_{i,j}$ is rectifiable, because it is contained in E and by the rectifiability results of Section 2, so we can use the area formula (see for instance Corollary 3.2.20 in [Fe]). That is,

$$H^{d+1}(\varphi(Z_{i,j})) \leqslant \int_{Z_{i,j}} J_\varphi(z)\, dH^{d+1}(z). \qquad (9.25)$$

Our improvement on (9.24) will come from a better control of the $(d+1)$-dimensional Jacobian J_φ due to a better control on the derivative of φ in the radial direction.

It may comfort the reader to notice that since E is Ahlfors-regular and rectifiable, it even has a true tangent $(d+1)$-plane at z for H^{d+1}-almost every z. This plane $z + P(z)$ contains the radial direction because E is a cone, and if $z + Q(z)$ denotes the d-plane in $P(z)$ perpendicular to the radial direction, it is easy to see that K has a tangent plane at $y = z/|z|$, which is $y + Q(z)$. Note that we already knew from the first part of Lemma 8.10 (applied as before to the radial projection π) that K is rectifiable.

The derivative of $g(z) = \rho\psi(\rho)(f(y) - y)$ in the radial direction is bounded by

$$[\psi(\rho) + \rho\psi'(\rho)]|f(y) - y| \leqslant Cr^{-1}|f(y) - y| \leqslant C, \qquad (9.26)$$

by (9.2). The derivative in all the tangential directions is bounded by $25 + 2M$, because we restrict our attention to $Z_{i,j}$ and by (9.19). So $J_\varphi(z) \leqslant C(1 + M)^d$. Indeed, after some manipulations, we can compute $J_\varphi(z)$ as a determinant of $D\varphi$ in some set of bases, and if we start the basis of $P(z)$ with a radial vector, the first row of the corresponding matrix has bounded terms, while the terms in the other rows are bounded by $C(1 + M)$. Thus (9.25) yields

$$H^{d+1}(\varphi(Z_{i,j})) \leqslant C(1 + M)^d H^{d+1}(Z_{i,j}). \qquad (9.27)$$

Incidentally, we have been trying to avoid talking about approximate differentials of f and φ on K and E, but the natural way to deal with (9.25) would use these. Our discussion leads to $J_\varphi(z) \leqslant C|D_E\varphi(z)|^d$, where $|D_E\varphi(z)|$ is the norm of the approximate differential of the restriction of φ to E (which exits H^{d+1}-almost everywhere on E). Then the proof of (9.19) yields $|D_E\varphi(z)| \leqslant C + |D_K f(y)|$, where $|D_K f(y)|$ is the norm of the approximate differential of the restriction of f to K. Thus a more precise version of (9.27) is that

$$H^{d+1}(\varphi(Z_i)) \leqslant C \int_{Z_i} (1 + |D_K\varphi(y)|^d) \, dH^{d+1}(z). \qquad (9.28)$$

Notice that we don't need to cut $K \cap B(x,r)$ or the Z_i into pieces in this context, because $|D_K\varphi(y)|$ is local.

We return to (9.27) and estimate $H^{d+1}(Z_{i,j})$. Recall from (9.18) that $Z_{i,j}$ is contained in the cone over $K_j \subset K \cap B(x,r)$. It is also contained in the thin annulus A_i of (9.17), so $\cup_j Z_{i,j}$ is contained in a ball B_i of radius $2r$. Then

$$\sum_j H^{d+1}(Z_{i,j}) = H^{d+1}\big(\cup_j Z_{i,j}\big) \leqslant H^{d+1}(E \cap B_i) \leqslant Cr^{d+1} \qquad (9.29)$$

because the $Z_{i,j}$ are disjoint and contained in E, and by the easiest part of Lemma 2.15 (the Ahlfors-regularity of the minimal set E).

We are almost ready to return to (9.11). Set

$$S = E \cap B(0, 1/2) \cup \big[\cup_{i \in \{1,2\}} E \cap A_i \setminus Z_i \big], \qquad (9.30)$$

and check that $\varphi(z) = z$ on S. When $z \in B(0, 1/2)$, this comes directly from (9.10) and the fact that $\psi(\rho) = 0$ by (9.9). On $E \cap A_i \setminus Z_i$, this is the definition of Z_i (just below (9.17)). Next

$$\begin{aligned} \varphi(E) \cap B(0, 1) &= \varphi(E \cap B(0, 1)) \subset \varphi(S) \cup \varphi(E \cap A) \cup \big[\cup_i \varphi(Z_i) \big] \\ &= S \cup \varphi(E \cap A) \cup \big[\cup_i \varphi(Z_i) \big] \end{aligned} \qquad (9.31)$$

by (9.12), because $E \cap B(0,1) = S \cup (E \cap A) \cup [\cup_i Z_i]$, and because $\varphi(z) = z$ on S. So

$$H^{d+1}(\varphi(E) \cap B(0,1)) \leqslant H^{d+1}(S) + H^{d+1}(\varphi(E \cap A)) + C(1+M)^d r^{d+1} \quad (9.32)$$

by (9.27) and (9.29). Similarly,

$$H^{d+1}(E \cap B(0,1)) \geqslant H^{d+1}(S) + H^{d+1}(E \cap A) \qquad (9.33)$$

because these sets are disjoint. On the other hand, $H^{d+1}(E \setminus \varphi(E)) \leqslant H^{d+1}(\varphi(E) \setminus E)$ by (9.11), and since $\varphi(E)$ coincides with E out of $B(0,1)$ by (9.12), we get that

$$H^{d+1}(E \cap B(0,1)) \leqslant H^{d+1}(\varphi(E) \cap B(0,1)) \qquad (9.34)$$

by adding $H^{d+1}(E \cap \varphi(E) \cap B(0,1))$ to both sides. Altogether,

$$\begin{aligned} H^{d+1}(E \cap A) &\leqslant H^{d+1}(E \cap B(0,1)) - H^{d+1}(S) \\ &\leqslant H^{d+1}(\varphi(E) \cap B(0,1)) - H^{d+1}(S) \qquad (9.35) \\ &\leqslant H^{d+1}(\varphi(E \cap A)) + C(1+M)^d r^{d+1} \end{aligned}$$

by (9.33), (9.34), and (9.32). By (9.14) and (9.16), this implies that

$$\alpha H^d(K) \leqslant \alpha H^d(f(K)) + C(1+M)^d r^{d+1}, \qquad (9.36)$$

with $\alpha = \int_I t^d dt \geqslant \int_{6/10}^{9/10} t^d dt$ (recall that $I = [\frac{1}{2} + \frac{r}{10}, 1 - \frac{r}{10}]$, see above (9.14)). This is just the same thing as (9.3), with $h(r) = Cr$. Proposition 9.4 follows. $\square$

Remark 9.37. — We can improve slightly our estimate of $H^{d+1}(\varphi(Z_i))$, by using (9.28) instead of (9.27). The same computations as above then lead to

$$H^d(K) \leqslant H^d(f(K)) + C \int_{Z_1 \cup Z_2} (1 + |D_K \varphi(y)|^d) \, dH^{d+1}(z), \qquad (9.38)$$

where we still use the convention that $y = z/|z|$. Recall from just above (9.18) that if $z \in Z_1 \cup Z_2$, then $f(y) \neq y$. In particular, $y \in K \cap B(x,r)$, by (9.2). Also, $z \in A_1 \cup A_2$. We can apply Lemma 8.10 to $Z = Z_1 \cup Z_2$ (which is rectifiable because it is contained in E) and the radial projection π from Z to $J = [\frac{1}{2}, \frac{1}{2} + \frac{r}{10}] \cup [1 - \frac{r}{10}, 1]$. This time we use sets $A \subset Z$ to show that both sides of (8.12) define equal measures, and we get that

$$\int_Z (1 + |D_K \varphi(y)|^d) \, dH^{d+1}(z)$$
$$= \int_{\rho \in J} \left\{ \int_{z \in Z \cap \pi^{-1}(\rho)} (1 + |D_K \varphi(y)|^d) \, dH^d(z) \right\} d\rho, \qquad (9.39)$$

where we omitted the approximate Jacobian from the left-hand side because it is equal to 1, as for (9.6) and (9.14). Recall that $\pi^{-1}(\rho) = \partial B(0, \rho)$ and that Z is contained in the cone over $W_f = \{y \in \partial B(0,1) \, ; \, f(y) \neq y\} \subset K \cap B(x, r)$. Thus

$$\int_{z \in Z \cap \pi^{-1}(\rho)} (1 + |D_K \varphi(y)|^d) \, dH^d(z) \leqslant \int_{\rho W_f} (1 + |D_K \varphi(y)|^d) \, dH^d(z)$$
$$= \rho^d \int_{W_f} (1 + |D_K \varphi(y)|^d) \, dH^d(y).$$
(9.40)

We integrate this over J and get that

$$H^d(K) \leqslant H^d(f(K)) + Cr \int_{W_f} (1 + |D_K \varphi(y)|^d) \, dH^d(y), \qquad (9.41)$$

by (9.38) and (9.39). This is slightly more precise and elegant than (9.36), but will not be used here.

10. One-dimensional minimal sets in $\mathbb{R}^n$

Here we want to describe the reduced minimal sets of dimension 1 in $\mathbb{R}^n$. The main result (Theorem 10.1) is not new, see [Mo3]. We still give a direct proof, mostly for self-containedness.

There are two simple examples minimal sets of dimension 1 in $\mathbb{R}^n$, the lines and the Y-sets. In this section, a Y-set will be a union of three half lines contained in an affine plane, with the same endpoint, and that make angles of 120° at that point.

It is fairly easy to see that lines and Y-sets are indeed minimal. See [D2], Lemma 59.1 on page 392, but here is a rapid sketch of the argument for the typical case when Y is a Y-set centered at the origin. Let $Z = \varphi(Y)$ be a deformation of Y, choose R so large that $Y = Z$ out of $B(0, R)$, and denote by a_1, a_2, a_3 the three points of $Y \cap \partial B(0, R)$. The images under φ of the three segments of $Y \cap \overline{B}(0, R)$ are three arcs that connect $\varphi(0)$ to the a_j. Then (if $H^1(Z \cap B(0, R)) < +\infty$) we can find a point $z \in Z \cap \overline{B}(0, R)$ and three simple arcs in $\overline{B}(0, R) \cap Z$ that go from z to the a_j, and whose interiors are disjoint. Replacing these by line segments makes Z shorter, and a variational argument shows that the optimal case is when $z = 0$ and $Z = Y$.

To the mild surprise of the author (but again this was known), there is no other one-dimensional reduced minimal set, even in large ambient dimensions.

THEOREM 10.1. — *Every nonempty reduced minimal set of dimension 1 in $\mathbb{R}^n$ is a line or a Y-set.*

The rest of this section is devoted to the proof. Even though a direct approach would be possible, it will be easier for us to apply some general arguments, and in particular use blow-down limits. We start with the simpler case of minimal cones.

LEMMA 10.2. — *Every nonempty reduced minimal cone of dimension 1 is a line or a Y-set.*

Let E be a reduced minimal cone of dimension 1 (centered at the origin), and set $K = E \cap \partial B(0,1)$. Notice that K is finite, because otherwise $H^1(E \cap B(0,1))$ is infinite. Write $K = \{a_1, \cdots, a_m\}$, and denote by L_j the half line through a_j. Let us first show that

$$\text{Angle}\,(L_i, L_j) \geqslant \frac{2\pi}{3} \quad \text{for } i \neq j. \tag{10.3}$$

Choose $i \neq j$ so that $\theta = \text{Angle}\,(L_i, L_j)$ is minimal, and suppose that $\theta < \frac{2\pi}{3}$. We want to get a contradiction by replacing $[L_i \cup L_j] \cap \overline{B}(0,1)$ with a shorter fork F_t defined as follows. Set $z_t = t(a_i + a_j)$ for $0 < t < 10^{-1}$, say, and $F_t = [0, z_t] \cup [z_t, a_i] \cup [z_t, a_j]$. A simple differentiation of $\ell(t) = H^1(F_t)$ at $t = 0$ shows that $H^1(F_t) < H^1([L_i \cup L_j] \cap \overline{B}(0,1))$ for t small (because $\theta < \frac{2\pi}{3}$). We choose t so that this holds, and set

$$\widetilde{E} = \big[E \setminus \big([L_i \cup L_j] \cap \overline{B}(0,1)\big)\big] \cup F_t. \tag{10.4}$$

Notice that $H^1(\widetilde{E} \cap B(0,1)) < H^1(E \cap B(0,1))$ and the two sets coincide out of $B(0,1)$, so $H^1(\widetilde{E} \setminus E) < H^1(E \setminus \widetilde{E})$.

We want to construct a Lipschitz mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(x) = x$ out of $B(0,1)$ and $\varphi(E) \subset \widetilde{E}$. Since E is minimal, Definition 8.1 will allow us to apply (8.2) to φ, and get that $H^1(E \setminus \varphi(E)) \leqslant H^1(\varphi(E) \setminus E)$, hence $H^1(E \setminus \widetilde{E}) \leqslant H^1(\widetilde{E} \setminus E)$. This is just the opposite of what we have, so we shall get the desired contradiction as soon as we have φ.

Let L denote the half line through $a_i + a_j$, and set $\mathcal{C}_\alpha = \big\{y \in \mathbb{R}^n\,;\, \text{dist}(y, L) \leqslant \alpha|y|\big\}$, with $\alpha = \text{dist}(a_i, L)$. That is, $\mathcal{C}_\alpha$ is the smallest revolution cone around L that contains L_i and L_j. Notice that since $|a_i - a_j|$ is minimal, $\text{dist}(a_l, L_i) \geqslant \text{dist}(a_j, L_i)$ for $l \neq i$. Also, a_i and a_j are furthest possible in $\mathcal{C}_\alpha \cap \partial B(0,1)$, so the only point of $\mathcal{C}_\alpha \cap \partial B(0,1)$ such that $\text{dist}(y, L_i) \geqslant \text{dist}(a_j, L_i)$ is $y = a_j$. Put in other words, $a_l \notin \mathcal{C}_\alpha$ unless $l = j$. Let us choose $\alpha' > \alpha$ so close to α that no a_l, $l \neq i, j$, lies in $\mathcal{C}_{\alpha'}$ either.

Set $\varphi(y) = y$ for $y \in \mathbb{R}^n \setminus [\mathcal{C}_{\alpha'} \cap B(0,1)]$. Thus we still need to define φ on $\mathcal{C}_{\alpha'} \cap B(0,1)$, and we start with $(L_i \cup L_j) \cap B(0,1)$. Define φ on $L_i \cap \overline{B}(0,1)$ so that it is Lipschitz and maps $L_i \cap \overline{B}(0,1)$ into $F_t \cap \overline{B}(0,1)$, with $\varphi(0) = 0$ and $\varphi(x) = x$ near a_i. Proceed similarly with L_j. Because there are small security cones around L_i and L_j that are contained in $\mathcal{C}_{\alpha'}$, our map φ is Lipschitz on $[\mathbb{R}^n \setminus [\mathcal{C}_{\alpha'} \cap B(0,1)]] \cup L_i \cup L_j$. We simply use the Whitney extension Theorem to extend it to the rest of $\mathbb{R}^n$, so that it stays Lipschitz.

Notice that φ does not move the half lines L_l, $l \neq i, j$; then $\varphi(E) \subset \widetilde{E}$, where $\widetilde{E}$ is as in (10.4). This completes our construction of φ, and we have noted earlier that the ensuing contradiction proves (10.3).

Next we claim that if j, k, l are distinct, the three lines L_j, L_k, L_l lie in a plane, and $L_j \cup L_k \cup L_l$ is a Y-set. Denote by P the plane that contains L_j and L_k, and identify P with $\mathbb{C}$. By symmetry, we may assume that $a_j = e^{i\theta}$ and $a_k = e^{-i\theta}$, with $0 < \theta \leqslant \pi/2$. By (10.3), $\theta \geqslant \pi/3$. Denote by ξ the orthogonal projection of a_l on P; then (10.3) says that $\langle \xi, a_j \rangle = \langle a_l, a_j \rangle \leqslant \cos(2\pi/3) = -1/2$, and similarly $\langle \xi, a_j \rangle \leqslant -1/2$. Since ξ lies in the closed unit ball, this forces the half line through ξ to make angles at least $2\pi/3$ with L_i and L_j. The only possibility is that $\theta = \pi/3$, $\xi = -1$ and $a_l = -1$. Our claim follows.

We are now ready to prove the lemma. If $E \cap \partial B(0,1)$ is empty, then E is empty. Next $E \cap \partial B(0,1)$ cannot be reduced to one point, because a half line is not minimal (we can contract it along its end to make it shorter). If it has exactly two points, it is easy to check that E is a line (otherwise, deform $E \cap B(0,1)$ into a segment with the same endpoints, and you'll save some length). If it has more than two points, choose two of them (say, a_i and a_j); the argument above shows that all the other points a_k lie just opposite to $a_i + a_j$, so there is only one of them, and E is a Y-set. This proves Lemma 10.2. □

Now we want to deal with the general case of Theorem 10.1. We shall use results of the previous parts to save time, but more direct proofs would be possible (and also the theorems that we use are easier for one-dimensional sets).

So let E be a nonempty reduced minimal set of dimension 1. Let us assume for simplicity that the origin lies in E, and set $\theta(r) = r^{-1} H^1(E \cap B(0,r))$. By Lemma 2.15, θ is bounded. By Proposition 5.16, θ is nondecreasing, so $\theta_\infty = \lim_{r \to +\infty} \theta(r)$ exists and is finite.

Let $\tau > 0$ be small, and let $\varepsilon > 0$ be as in Proposition 7.1, with $a = 1$ and $b = 2$. Then let $R \geqslant 1$ be such that $\theta(r) \geqslant \theta_\infty - \varepsilon$ for $r \geqslant R$. For $r \geqslant R$,

apply Proposition 7.1 to $r^{-1}E$; the density assumption is satisfied because $\theta_\infty - \varepsilon \leqslant \theta(r) \leqslant \theta(2r) \leqslant \theta_\infty$. The conclusion is that there is a minimal cone $\mathcal{C}$ centered at the origin, that is close to $E' = r^{-1}E$ in various respects, but in particular such that

$$|\rho^{-1}H^1(E' \cap B(0,\rho)) - \rho^{-1}H^1(\mathcal{C} \cap B(0,\rho))| \leqslant \tau \text{ for } 1+\tau \leqslant \rho \leqslant 2-\tau, \quad (10.5)$$

as in (7.5). Take $\rho = 3/2$; we get that

$$\theta(3r/2) = \rho^{-1}H^1(E' \cap B(0,\rho)) \leqslant \rho^{-1}H^1(\mathcal{C} \cap B(0,\rho)) + \tau \leqslant 3 + \tau \quad (10.6)$$

by Lemma 10.2. Denote by $N(t)$ the number of points in $E \cap \partial B(0,t)$. By [D2], Lemma 26.1 on page 160, or [Fe], Theorem 3.2.22,

$$\int_0^{3r/2} N(x)dx \leqslant H^1(E \cap B(0, 3r/2)) = (3r/2)\theta(3r/2) \leqslant 3(3+\tau)r/2 \quad (10.7)$$

so we can find $\rho \in (r/10, 3r/2)$ such that $N(\rho) \leqslant \frac{10}{14r} \frac{3(3+\tau)r}{2} < 4$ (if τ is small enough). That is,

$$E \cap \partial B(0,\rho) \text{ has at most three points.} \quad (10.8)$$

LEMMA 10.9. — *If E is a reduced minimal set that contains the origin, and $\rho > 0$ is such that (10.8) holds, then E coincides with a line or a Y-set in $B(0,\rho)$.*

Theorem 10.1 will follow as soon as we prove this lemma. Indeed, we found arbitrarily large radii ρ such that (10.8) holds, hence such that E coincides with a line or a Y-set in $B(0,\rho)$. It is easy to see that E is then a line or a Y-set.

Our proof of Lemma 10.9 will take some time, in particular because we want to be able to use the same construction with weak almost-minimal sets.

Before we really start, let us construct retractions h_F onto some simple sets F. Our first case is when $F = Y \cap B$, where Y is a Y-set and B is a closed ball that contains the center of Y. Let P denote the plane that contains Y. We first construct a retraction $h_1 : P \to Y$. Denote by L_1, L_2, L_3 the three half lines that compose Y, and by V_i the connected component of $P \setminus Y$ opposite to L_i. For $y \in V_i$, $h_1(y)$ is the projection of y onto $L_j \cup L_k$ along the direction of L_i, where L_j and L_k are the two other half lines. In other words, $h_1(y)$ is the intersection of $L_j \cup L_k$ with the line through y parallel to L_i. It is easy to see that h_1 is 2-Lipschitz on V_i. For instance, observe that we can split V_i into two equal sectors where h_1 is the oblique projection onto

the line L_j or L_k, along a direction that makes an angle $\pi/3$ with that line. Then h_1 is also 2-Lipschitz on P, because the various definitions coincide with the identity on Y.

We compose h_1 with the retraction h_2 from Y to F, which maps $y \in F$ to itself, and maps $y \in Y \setminus F$ to the point of $Y \cap \partial B$ that lies on the same branch of Y as y. Obviously h_2 is 1-Lipschitz. Finally set

$$h_F = h_2 \circ h_1 \circ \pi_P \qquad (10.10)$$

on $\mathbb{R}^n$, where π_P denotes the orthogonal projection onto P. Then

$$h_F \text{ is a 2-Lipschitz retraction of } \mathbb{R}^n \text{ onto } F. \qquad (10.11)$$

We also need to construct h_F when $F = B \cap (L_1 \cup L_2)$, where L_1 and L_2 are half lines with the same endpoint and that make an angle at least $2\pi/3$ at that point, and B is a closed ball that contains that point. Let us first define a retraction h_1 on the plane P that contains Z (if $L_1 \cup L_2$ is a line L, choose any plane that contains L; anyway $h_1 \circ \pi_P$ will end up being the orthogonal projection onto L). Call D the line in P that goes through the common endpoint of L_1 and L_2 and lies at equal distance from L_1 and L_2 and, for $y \in P$, let $h_1(y)$ be the intersection of $L_1 \cup L_2$ with the line through y parallel to D. Now h_1 is still 2-Lipschitz on P, because D makes an angle of $\pi/3$ or more with the lines that contain L_1 and L_2. We then define h_F as before, by (10.10), and (10.11) holds for the same reasons.

Let E be as in Lemma 10.9. We want to treat a few cases at the same time, so some notation will be useful. Set $B = \overline{B}(0, \rho)$, and denote by a_j, $1 \leqslant j \leqslant N$, the points of $E \cap \partial B$. Let us still keep the number N of points in $E \cap \partial B$ open, except that (10.8) says that $N \leqslant 3$.

Then denote by E_j the connected component of $E \cap B$ that contains a_j. Call M the number of components that we get, and renumber the a_j so that the E_j, $1 \leqslant j \leqslant M$, are disjoint, and each a_k, $k > M$, lies in some E_j such that $j \leqslant M$. Notice that the E_j are compact, so

$$\text{dist}(E_i, E_j) > 0 \quad \text{for } 1 \leqslant i, j \leqslant M, i \neq j. \qquad (10.12)$$

When $M > 1$, we shall also need to dispatch the rest of $E \cap B$ among the E_j, as follows.

LEMMA 10.13. — *We can find a partition of $E \cap B$ into sets E'_j, $1 \leqslant j \leqslant M$, such that $E_j \subset E'_j$ for $j \leqslant M$ and*

$$\text{dist}(E'_i, E'_j) > 0 \quad \text{for } i, j \leqslant M, i \neq j. \qquad (10.14)$$

We leave the proof of Lemma 10.13 for the end of the section, because it is a pure fact about connected components, and it does not interfere with the rest of the proof.

Let us return to the proof of Lemma 10.9. For $1 \leqslant j \leqslant M$, denote by F_j the shortest connected set that contains all the points of $E_j \cap \partial B$. We claim that if $E_j \cap \partial B$ has three points, i.e., if $E \cap \partial B$ has three points which lie in the same component of $E \cap B$, then F_1 is either the intersection of B with a Y-set whose center lies in B, or else we can rename the three points a_j so that $F = [a_1, a_2] \cup [a_2, a_3]$, and these segments make an angle at least $2\pi/3$ at a_2.

To prove the claim, we consider any connected set G with finite length that contains the three a_j. By Proposition 10.37 in [D2] (for instance), we can find a simple arc γ in G from a_1 to a_2, and a simple arc γ_1 in G from a_3 to a_1. We denote by z the first point of $\gamma_1 \cap \gamma$ when we start from a_3 and run along γ_1. Possibly z is one of the a_i, but this is all right. Call g_1 the arc of γ from a_1 to z, g_2 the arc of γ from z to a_2, and g_3 the arc of γ_1 from a_3 to z. These three arcs are disjoint, except for z, and they are contained in G. Then $H^1(G) \geqslant H^1(\cup_j g_j) = \sum_j H^1(g_j) \geqslant H^1(\cup_i [a_i, z])$, because the g_i are disjoint. So it is enough to minimize length among sets of the form $\cup_i [a_i, z]$. By differentiating in z, it is easy to see that $H^1(\cup_j g_j)$ is minimal only when $\cup_j g_j$ is contained in a Y-set centered at z, or else z is one of the a_j. In this last case, if for instance $z = a_2$, it is easy to show directly that the segments $[a_1, z]$ and $[z, a_3]$ make an angle at least $2\pi/3$ at z (otherwise, a Y-set would be shorter). This proves our claim. Notice that in both cases, we constructed a Lipschitz retraction h_{F_j} onto F_j.

When E_j contains exactly two points a_i, F_j is the line segment between these points, and we also constructed h_{F_j} above (this is the case when L_2 lies opposite to L_1). Finally, when $E_j \cap \partial B$ is reduced to a_j, $F_j = \{a_j\}$ and we simply take h_{F_j} constant equal to a_j.

We are now ready to define a competitor $\varphi(E)$. First set

$$\varphi(y) = y \quad \text{for } y \in \mathbb{R}^n \setminus B(0, \rho). \tag{10.15}$$

We would have liked to set $\varphi(y) = h_{F_j}$ on E'_j, but then φ would not necessarily be Lipschitz, because $E \cap B(0, \rho)$ may approach the a_j tangentially, and the definitions of φ from inside and outside B would not match well. So we shall need a small cut-off function ψ. For this reason we introduce a small positive number ε, set $d(y) = \text{dist}(y, E \cap \partial B)$, and then

$$\begin{aligned}
\psi(y) \quad &= 0 && \text{for } 0 \leqslant d(y) \leqslant \varepsilon, \\
\psi(y) \quad &= \varepsilon^{-1}d(y) - 1 && \text{for } \varepsilon \leqslant d(y) \leqslant 2\varepsilon, \\
\psi(y) \quad &= 1 && \text{for } d(y) \geqslant 2\varepsilon.
\end{aligned} \tag{10.16}$$

Next set

$$\varphi_j(y) = y + \psi(y)[h_{F_j}(y) - y] \text{ for } y \in \mathbb{R}^n, \tag{10.17}$$

and

$$\varphi(y) = \varphi_j(y) \text{ for } y \in E'_j. \tag{10.18}$$

Notice that $\varphi_j(y) = y$ on F_j, which contains all the points of $E'_j \cap \partial B$, because F_j contains all the points of $E_j \cap \partial B$, and $E'_j \setminus E_j$ does no meet ∂B (every point of $E \cap \partial B$ lies in some E_j). Thus the two definitions of $\varphi(y)$ coincide on $E \cap \partial B$. Let us check that for each j,

$$\varphi_j \text{ is 10-Lipschitz.} \tag{10.19}$$

Fix j, and start the verification in the small balls $B_k = \overline{B}(a_k, 2\varepsilon)$ centered at the points of $E \cap \partial B$. Notice that $|h_{F_j}(y) - y| \leqslant 6\varepsilon$ for $y \in B_k$, because h_{F_j} is 2-Lipschitz and $h_{F_j}(a_k) = a_k$ (see (10.11)). Then

$$\left|\psi(y)[h_{F_j}(y) - y] - \psi(y')[h_{F_j}(y') - y']\right|$$

$$\leqslant \left|[\psi(y) - \psi(y')][h_{F_j}(y) - y]\right| + \psi(y')\left|[h_{F_j}(y) - y] - [h_{F_j}(y) - y]\right| \tag{10.20}$$

$$\leqslant \varepsilon^{-1}|y - y'|\,|h_{F_j}(y) - y| + 3\psi(y')|y - y'| \leqslant 9|y - y'|$$

for $y, y' \in B_k$, again because h_{F_j} is 2-Lipschitz. Thus $\psi(y)[h_{F_j}(y) - y]$ is 9-Lipschitz on B_k, and φ_j is 10-Lipschitz there. Out of the B_k, $\psi(y) = 1$ by (10.16), so $\varphi_j(y) = h_{F_j}(y)$, which is 2-Lipschitz. This proves (10.19).

Next we check that φ is Lipschitz on $[\mathbb{R}^n \setminus B(0, \rho)] \cup [E \cap B(0, \rho)]$, i.e., that

$$|\varphi(x) - \varphi(y)| \leqslant C|x - y| \tag{10.21}$$

for some (very large) C. If x and y lie out of $B(0, \rho)$, this is clear because of (10.15); so let us assume that $y \in B(0, \rho)$. Notice that $y \in E'_j$ for some $j \leqslant M$, by Lemma 10.13, so $\varphi(y) = \varphi_j(y)$ by (10.18), hence $\varphi(y)$ lies in the segment $[y, h_{F_j}(y)]$ by (10.17). In particular, we just checked that

$$\varphi(y) \in B = \overline{B}(0, \rho) \text{ for } y \in E \cap B(0, \rho) \tag{10.22}$$

because $F_j \subset B$.

If x lies out of $B(0, 2\rho)$, (10.21) holds because $|\varphi(x) - \varphi(y)| = |x - \varphi(y)| \leqslant |x - y| + |y - \varphi(y)| \leqslant |x - y| + 2\rho \leqslant 3|x - y|$. Next assume that $x \in B(0, 2\rho) \setminus B(0, \rho)$. If y lies in one of the $B(a_k, \varepsilon)$, then

$\psi(y) = 0$, $\varphi(y) = y$, and (10.21) is trivial. Otherwise, y lies in the compact set $K_\varepsilon = E \cap B \setminus \cup_k B(a_k, \varepsilon)$. Notice that K_ε does not meet $\mathbb{R}^n \setminus B(0, \rho)$, because $E \cap \partial B$ reduces to the a_k, so $|y - x| \geqslant d_\varepsilon = \mathrm{dist}(K_\varepsilon, \mathbb{R}^n \setminus B(0, \rho)) > 0$, and $|\varphi(x) - \varphi(y)| \leqslant 4\rho \leqslant 4\rho d_\varepsilon^{-1} |y - x|$ because $\varphi(x)$, $\varphi(y) \in B(0, 2\rho)$, so (10.21) holds.

We are left with the case when $x, y \in E \cap B(0, \rho)$. Then x also lies in some E_i'. If $i = j$, we just use (10.18) and (10.19). Otherwise, $|\varphi(x) - \varphi(y)| \leqslant 2\rho$, and $|y - x| \geqslant \mathrm{dist}(E_j', E_i') > 0$ by Lemma 10.13, so (10.21) holds in this last case as well.

Let us now use the Whitney extension theorem and extend φ to the rest of $\mathbb{R}^n$, so that it stays Lipschitz. For Lemma 10.9 and Theorem 10.1, we don't really need that

$$\varphi(B(0, \rho)) \subset B \tag{10.23}$$

because there will be no gauge function to pay for, but let us require this anyway, for later uses of the argument. This is easy to arrange; simply compose the restriction of φ to $B(0, \rho)$ with the radial contraction on B defined by $f(z) = z$ for $z \in B$, and $f(z) = z/|z|$ otherwise. This does not change the values of $\varphi(y)$ when $y \in E \cap B(0, \rho)$, by (10.22), and φ stays Lipschitz on $\mathbb{R}^n$ because $\varphi(y) = y = f(y)$ on ∂B.

Our argument so far would work with minor modifications for weak almost-minimal sets. We shall return to this soon, but for the moment let us do the accounting in the situation of Lemma 10.9 and Theorem 10.1, where E is a reduced minimal set. Since φ is Lipschitz and $\varphi(x) = x$ out of B, we can apply (8.2), which says that $H^1(E \setminus \varphi(E)) \leqslant H^1(\varphi(E) \setminus E)$. Since E and $\varphi(E)$ coincide out of B (by (10.15) and (10.22)), we get that

$$H^1(E \cap B) \leqslant H^1(\varphi(E) \cap B) = H^1(\varphi(E \cap B)) \tag{10.24}$$

(by (10.15) and (10.22) again). Let $y \in E \cap B$ be given. Thus y lies in some E_j'. If y lies out of the $B(a_k, 2\varepsilon)$, $\psi(y) = 1$, and $\varphi(y) = \varphi_j(y) = h_{F_j}(y) \in F_j$ (by (10.11)). Hence

$$
\begin{aligned}
H^1(\varphi(E \cap B)) &\leqslant \sum_{j \leqslant M} H^1(F_j) + \sum_j \sum_k H^1(\varphi_j(E_j' \cap B \cap B(a_k, 2\varepsilon))) \\
&\leqslant \sum_{j \leqslant M} H^1(F_j) + 10 \sum_j \sum_k H^1(E_j' \cap B \cap B(a_k, 2\varepsilon)) \\
&= \sum_{j \leqslant M} H^1(F_j) + 10 \sum_k H^1(E \cap B \cap B(a_k, 2\varepsilon))
\end{aligned}
\tag{10.25}
$$

by (10.18) and (10.19). Hence

$$H^1(E \cap B) \leqslant \sum_{j \leqslant M} H^1(F_j) + 10 \sum_k H^1(E \cap B \cap B(a_k, 2\varepsilon)) \qquad (10.26)$$

by (10.24). We may now let ε go to 0, and we get that

$$H^1(E \cap B) \leqslant \sum_{j \leqslant M} H^1(F_j) \qquad (10.27)$$

because $E \cap \partial B$ is finite. Now recall that F_j is the connected set that contains the points of $E_j \cap \partial B$ for which $H^1(F_j)$ is minimal (the description above gives the uniqueness). In particular, $H^1(F_j) \leqslant H^1(E_j)$ because E_j is connected. The E_j are distinct and contained in $E \cap B$ by construction, so (10.27) says that $H^1(E_j) = H^1(F_j)$, whence $E_j = F_j$. In addition, $H^1(E'_j \setminus E_j) = 0$.

Suppose for a moment that $E'_j \setminus E_j$ is not empty. Then we can find $y \in E'_j \setminus E_j$ and $r > 0$ so small that $B(y, r)$ does not meet the closed sets $E_j = F_j$, nor $\mathbb{R}^n \setminus B(0, \rho)$ (because $y \in E \cap \partial B$ is impossible). Then $E \cap B(y, r)$ is contained in the union of the $E'_j \setminus E_j$, so $H^1(E \cap B(y, r)) = 0$, which contradicts our assumption that E is reduced. So $E'_j \setminus E_j$ is empty, and $E \cap B$ is the union of the F_j.

If $F_1 \cap \partial B$ contains three points, then there is no other F_j (by (10.8)), and $E \cap B = F_1 \cap B$. In addition, the origin lies in F_1 because we assumed that it lies in E. If F_j does not come from a Y-set, it is of the form $[a_1, a_2] \cup [a_2, a_3]$. Assume for instance that $0 \in [a_1, a_2]$. Then $[a_1, a_2]$ is a diameter of B, and then its angle with $[a_2, a_3]$ it less than $\pi/2$. We saw that this is not possible (by minimality of F_j). So E coincides with a Y-set in B, and we are happy.

If $F_1 \cap \partial B$ contains two points a_1 and a_2, F_2 either does not exist (if $N{=}2$) or is reduced to one point, and $E \cap B(0, \rho) = F_1 \cap B(0, \rho) = (a_1, a_2)$. This case is fine too. The same thing happens if some other $F_j \cap \partial B$ has exactly two points.

The other cases where no E_j contains more than one point of ∂B are impossible, because then the F_j are reduced to one point, and the origin cannot lie in E.

This completes our proof of Lemma 10.9 and Theorem 10.1, modulo Lemma 10.13 which we prove now. $\qquad \square$

Proof of Lemma 10.13. — For each small $\eta > 0$, denote by $G_{j,\eta}$ the set of points $y \in E \cap B$ that can be connected to E_j by an η-string in $E \cap B$,

i.e., a finite sequence $\{y_k\}$ in $E \cap B$, with $y_0 = y$, $|y_k - y_{k-1}| \leqslant \eta$, and the last y_k lies in E_j. Let $i \leqslant M$ be given, with $i \neq j$, and let us check that for η small enough, $G_{j,\eta}$ does not meet E_i.

Suppose not. We can find a sequence $\{\eta_l\}$ that tends to 0, and for each l an η_l-string $\{y_k\}$ in $E \cap B$ that goes from some $z_l \in E_i$ to some $w_l \in E_j$. Set $\Gamma_l = \cup_{k \geqslant 1}[y_{k-1}, y_k]$; thus Γ_l is compact, connected, and contained in B. Replace $\{\eta_l\}$ with a subsequence for which $\{z_l\}$ converges to some z, $\{w_l\}$ converges to some w, and $\{\Gamma_l\}$ converges to some compact set $\Gamma \subset B$.

Notice that $y \in E_i$ (because E_i is closed), $z \in E_j$, and also $\Gamma \subset E$, because each point of Γ_l lies within η_l of E, and E is closed. In addition, we claim that Γ is connected.

Again, suppose this is not the case. Split Γ as $\Gamma = G_1 \cup G_2$, where the G_i are disjoint nonempty closed subsets. Set $d = \text{dist}(G_1, G_2) > 0$, and take l so large that every point of Γ_l lies within $d/3$ of Γ, and every point of Γ lies within $d/3$ of Γ_l. For $m = 1, 2$, set $G_{l,m} = \{y \in \Gamma_l \, ; \, \text{dist}(y, G_m) \leqslant d/3\}$. First, $G_{l,m}$ is closed. It is not empty, because G_m contains some y, then there is a $y' \in \Gamma_l$ such that $|y' - y| \leqslant d/3$, and obviously $y' \in G_{l,m}$. Also, $\text{dist}(G_{l,1}, G_{l,2}) \geqslant d/3$ by definitions, and finally $\Gamma_l = G_{l,1} \cup G_{l,2}$ because every point of Γ_l lies within $d/3$ of $\Gamma = G_1 \cup G_2$. All this contradicts the connectedness of Γ_l, and our claim that Γ is connected follows.

We managed to find a connected set $\Gamma \subset E \cap B$ that contains $z \in E_i$ and $w \in E_j$, but this contradicts our assumption that E_i and E_j are different connected components of $E \cap B$. Thus $G_{j,\eta}$ does not meet E_i for η small. Then $\text{dist}(G_{j,\eta}, G_{i,\eta}) \geqslant \eta$, because otherwise we could put two η-strings together to go from E_i to E_j.

Now choose η so small that all the $G_{j,\eta}$ lie at distance $\geqslant \eta$ from each other, and set $E'_j = G_{j,\eta}$ for $j > 1$ and $E'_1 = E \cap B \setminus \cup_{j>1} G_{j,\eta}$. These sets are disjoint because the $G_{j,\eta}$ don't meet. It is obvious that $E'_j \supset E_j$ for $j > 1$, and for $j = 1$ this is true because $G_{1,\eta}$ does not meet the other $G_{j,\eta}$. Their union is $E \cap B$ by definition of E'_1. We still need to check (10.14). Let us even check that $\text{dist}(E'_i, E'_j) \geqslant \eta$ for $i \neq j$.

When i and j are larger than 1, this comes directly from our choice of η. If $i = 1$, we simply observe that if $y \in E'_1$, then $dist(y, E'_j) = dist(y, G_{j,\eta}) \geqslant \eta$ because otherwise y would lie in $G_{j,\eta}$. This completes our proof of Lemma 10.13. Lemma 10.9 and Theorem 10.1 follows as well. $\qquad\square$

11. Local regularity for 1-dimensional weak almost-minimal sets

We still want to describe the two-dimensional minimal cones in $\mathbb{R}^n$, and to this effect we shall need information on the reduced weak almost-minimal sets of dimension 1. In this section we let $E \subset \partial B(0,1) \subset \mathbb{R}^n$ be a closed set, and we assume that

$$E \text{ is a reduced weak almost-minimal set of dimension 1,} \atop \text{with gauge function } h \tag{11.1}$$

[see Definition 9.1 for weak almost-minimal sets, and Definition 2.12 (or (9.5)), for the notion of reduction], and we try to derive some amount of regularity from mild assumptions on $E \cap \partial B(x, \rho)$. The proof will be very close to what we would do for almost-minimal sets (a case that was treated in [Mo3]). The main result of this section is the following.

PROPOSITION 11.2. — *Let $E \subset \partial B(0,1) \subset \mathbb{R}^n$ satisfy (11.1), and suppose that $E \cap \partial B(x, \rho)$ has at most three points. Also suppose that $r + h(r)$ is small enough (depending only on n). Then for $y \in E \cap B(x, \rho/10)$ and $0 < r \leqslant \rho/10$, we can find a one-dimensional minimal cone Z that contains y (i.e., a line or a Y-set through y) such that*

$$\text{dist}\,(z, Z) \leqslant C[r + h(10r)]^{1/2} r \;\; \text{for } z \in E \cap B(y, r)$$
$$\text{and } \text{dist}(z, E) \leqslant C[r + h(10r)]^{1/2} r \;\; \text{for } z \in Z \cap B(y, r). \tag{11.3}$$

Observe that we do not require Z to be centered at y when it is a Y, and the type of Z may depend on y and r (but in fact not wildly). If E, x, and ρ are as in the proposition and $h(\rho)$ is small enough, it follows from Proposition 12.6 (the one-dimensional case of the generalization of Reifenberg' theorem that will be stated in Section 15) that E is biHölder-equivalent to a line or a Y-set in a ball that contains $B(y, \rho/20)$.

In the case when $E = K \cup \partial B(0,1)$ for a reduced minimal set K of dimension 2, we could also get the biHölder equivalence from Theorem 8.23 and Theorem 10.1, but in this case Proposition 9.4 says that (11.1) and then (11.3) hold with $h(r) \leqslant Cr$; we shall see in Proposition 12.27 that (11.3) then gives a C^1 control of E near y. This better control will make it easier to prove that E is composed of arcs of great circles which only meet with $120°$ angles; see Section 14.

Let us return to the general case where (11.1) holds, forget about Proposition 11.2 for the moment, and see what the proof of Lemma 10.9 yields in this context. We let $x \in \partial B(0,1)$ and $\rho \in (0, 1/2)$ be given, and start with

the only additional assumption that

$$E \cap \partial B(x,\rho) \text{ is finite.} \tag{11.4}$$

As before, we denote by a_j, $j \leqslant N$, the points of $E \cap \partial B(x,\rho)$ (if $E \cap \partial B(x,\rho) \neq \emptyset$), and by E_j the connected component of a_j in $E \cap \overline{B}(x,\rho)$. We number the points a_j so that the first E_j, $j \leqslant M$, are distinct, while each E_k, $k > M$, coincides with some previous E_j. And we denote by F_j the shortest connected set in $\mathbb{R}^n$ that contains all the points of $E_j \cap \partial B(x,\rho)$.

LEMMA 11.5. — *There is a constant $C \geqslant 1$, that depends only on n, such that*

$$H^1(E \cap B(x,\rho)) \leqslant \sum_{j \leqslant M} H^1 F_j) + C\left[h(\rho) + \rho\right]\rho. \tag{11.6}$$

Let us repeat the construction of the function φ as in Lemma 10.9, except that we replace the origin with x, and we only define φ on $\partial B(0,1)$. We cannot use φ directly, because of the constraint that competitors lie in $\partial B(0,1)$, so we shall need to project back onto $\partial B(0,1)$.

Denote by ξ the point of $\mathbb{R}^n$ such that the cone centered at ξ over $\partial B(x,\rho) \cap \partial B(0,1)$ is tangent to B along $\partial B(x,\rho) \cap \partial B(0,1)$. That is, ξ lies on the line through 0 and x and $z - \xi$ is orthogonal to $z - x$ for $z \in \partial B(x,\rho) \cap \partial B(0,1)$. Notice that z lies on the half line $[0, -x)$ because we may assume that $\rho \leqslant 1$. Set $\eta(z) = z$ for $z \in \partial B(0,1) \backslash B(x,\rho)$ and, for $z \in B(z,\rho)$, denote by $\eta(z)$ the radial projection of z onto $\partial B(0,1) \cap B(x,\rho)$ centered at ξ. That is, $\eta(z)$ is the only point of $\partial B(0,1) \cap B(x,\rho)$ that lies on the line through 0 and z. Observe that η is 2-Lipschitz on $[\partial B(0,1) \setminus B(x,\rho)] \cup B(x,\rho)$; we shall not need to define it on a larger set.

We set $f(y) = \eta(\varphi(y))$ for $y \in \partial B(0,1)$. Observe that we still have that

$$f(y) = \varphi(y) = y \ \text{ for } y \in \partial B(0,1) \setminus B(x,\rho) \tag{11.7}$$

by (10.15), and $\varphi(y) \in B = \overline{B}(x,\rho)$ for $y \in B(x,\rho)$, by (10.23), so that in particular $f(y)$ is well defined, and

$$f(B(x,\rho)) \subset \overline{B}(x,\rho) \cap \partial B(0,1). \tag{11.8}$$

The composition with η does not radically change the Lipschitz properties of φ, so f is Lipschitz (by (10.21)), and

$$\text{the restriction of } f \text{ to each } E_j' \text{ is 20-Lipschitz,} \tag{11.9}$$

by (10.19) and (10.18).

Let us apply the weak almost-minimality of E, as defined in Definition 9.1. The set $W_f = \{y \in \partial B(0,1) \,;\, f(y) \neq y\}$ is contained in $B(x, \rho)$ by (11.7), and $f(W_f) \subset \overline{B}(x, \rho)$ by (11.8). In addition, we have a partition of $E \cap \overline{B}(x, \rho)$ into sets E'_j, as in Lemma 10.13, and f is 20-Lipschitz on each E'_j by (11.9). Thus we can take $M = 20$ in (9.3), which yields

$$H^1\big(E \cap \overline{B}(x,\rho)\big) \leqslant H^1\big(f(E) \cap \overline{B}(x,\rho)\big) + C\,h(\rho)\rho. \qquad (11.10)$$

We want to estimate $H^1\big(f(E) \cap \overline{B}(x,\rho)\big)$. We get no contribution from the complement of $\overline{B}(x,\rho)$, by (11.7). So it is enough to control the sets $f(E'_j)$ (because $E \cap \overline{B}(x,\rho) = \cup_j E'_j$).

Recall that $E \cap \partial B(x,\rho)$ is composed of a finite number of points a_k. Observe that

$$\sum_j \sum_k H^1(f(E'_j \cap B(a_k, 2\varepsilon))) \leqslant 20 \sum_j \sum_k H^1(E'_j \cap B(a_k, 2\varepsilon))$$
$$\leqslant 20 \sum_k H^1(E \cap B(a_k, 2\varepsilon)) \qquad (11.11)$$

by (11.9). When $y \in E'_j$ lies out of the $B(a_k, 2\varepsilon)$, we know that $\varphi(y) = h_{F_j}(y) \in F_j$, where F_j still denotes the shortest connected set that contains the points of $E_j \cap \partial B(x,\rho)$. Then $f(y) \in \eta(F_j)$. Altogether,

$$H^1\big(f(E) \cap \overline{B}(x,\rho)\big) \leqslant \sum_j H^1(\eta(F_j)) + \sum_j \sum_k H^1\big(f(E'_j \cap B(a_k, 2\varepsilon))$$
$$\leqslant \sum_j H^1(\eta(F_j)) + 20 \sum_k H^1(E \cap B(a_k, 2\varepsilon)) \qquad (11.12)$$

by (11.11). We replace in (10.10), let ε tend to 0, and get that

$$H^1(E \cap \overline{B}(x,\rho)) \leqslant \sum_j H^1(\eta(F_j)) + C\,h(\rho)\rho$$
$$\leqslant \sum_j H^1 F_j) + C\,[h(\rho) + \rho]\,\rho \qquad (11.13)$$

because η barely deforms F_j (recall that the center ξ of the radial projection is far from x, and that F_j lies in the convex hull of the points a_k of $E \cap \partial B(x,\rho)$, hence close to the surface $\partial B(0,1)$). This proves Lemma 11.5. $\square$

LEMMA 11.14. — *There is a constant $C_0 \geqslant 1$, that depends only on n, such that if (11.1) holds, then*

$$H^1(E \cap B(x_0, r)) \geqslant r/2 \quad \text{for } x_0 \in E$$
$$\text{and } r > 0 \text{ such that } r + h(r) \leqslant C_0^{-1}. \qquad (11.15)$$

Indeed suppose that $x_0 \in E$ and $r > 0$ are such that $H^1(E \cap B(x_0, r)) < r/2$. Pick any $x \in \partial B(0,1) \cap B(x_0, r/4)$, and set $\delta(y) = |y - x|$ for $y \in \mathbb{R}^n$; then $H^1(\delta(E \cap B(x_0, r))) < r/2$ because δ is 1-Lipschitz, so we can find $\rho \in (r/4, 3r/4)$ such that $\rho \notin \delta(E \cap B(x_0, r))$. Since $B(x, \rho) \subset B(x_0, r)$, this means that $\partial B(x, \rho)$ does not meet E.

Thus we can apply Lemma 11.5, with no set F_j, and get that

$$
\begin{aligned}
H^1(E \cap B(x, r/4)) \leqslant H^1(E \cap B(x, \rho)) &\leqslant C\left[h(\rho) + \rho\right]\rho \\
&\leqslant C\left[h(r) + r\right]r < r/8
\end{aligned}
\tag{11.16}
$$

because h is nondecreasing (it is a gauge function) and if $h(r) + r$ is small enough.

We can iterate this argument, this time keeping the same point x, and we get that

$$
H^1(E \cap B(x, 4^{-k}r)) < 4^{-k-1}r \quad \text{for } k \geqslant 1
\tag{11.17}
$$

and even, by (11.16) again,

$$
H^1(E \cap B(x, 4^{-k-1}r)) \leqslant C[h(4^{-k}r) + 4^{-k}r]\, 4^{-k}r.
\tag{11.18}
$$

Then, if t is small and k is chosen such that $4^{-k-2} \leqslant t \leqslant 4^{-k-1}$, we get that

$$
\begin{aligned}
t^{-1}H^1(E \cap B(x, t)) &\leqslant t^{-1}H^1(E \cap B(x, 4^{-k-1})) \\
&\leqslant Ct^{-1}[h(4^{-k}r) + 4^{-k}r]\, 4^{-k}r \\
&\leqslant 4C[h(16t) + 16t].
\end{aligned}
\tag{11.19}
$$

We proved that

$$
\lim_{t \to 0} t^{-1}H^1(E \cap B(x, t)) = 0
\tag{11.20}
$$

for every $x \in \partial B(0,1) \cap B(x_0, r/4)$. That is, $B(x_0, r/4)$ contains no point of positive density for E. But it is a simple and standard result that as soon as $H^1(E) < +\infty$, H^1-almost every point of E has positive upper density; see for instance [Ma], Theorem 6.2 on page 89 or Lemma 19.1 in [D2]. Thus $H^1(E \cap B(x_0, r/4)) = 0$, and $x \notin E$ because E is reduced. Lemma 11.14 follows. $\square$

We start the local study of E that will lead to Proposition 11.2 with the simpler case when (11.1) holds and $E \cap \partial B(x, \rho)$ has two points.

LEMMA 11.21. — *There is a constant $C_1 \geqslant 1$ such that the following holds. Let $E \subset \partial B(0,1)$ satisfy (11.1), and let $x \in \partial B(0,1)$ and $\rho \in (0, 1/2)$*

be such that $E \cap \partial B(x, \rho) = \{a_1, a_2\}$, $E \cap B(x, \rho/2) \neq \emptyset$, and $r + h(r) \leqslant C_1^{-1}$. Then

$$H^1(E \cap B(x, \rho)) \leqslant |a_2 - a_1| + C[\rho + h(\rho)] \rho, \qquad (11.22)$$

where C is as in (11.6),

$$\operatorname{dist}(y, [a_1, a_2]) \leqslant C_1 [\rho + h(\rho)]^{1/2} \rho \ \text{ for } y \in E \cap B(x, 9\rho/10), \qquad (11.23)$$

and

$$\operatorname{dist}(z, E \cap B(x, \rho)) \leqslant C_1 [\rho + h(\rho)]^{1/2} \rho \ \text{ for } z \in [a_1, a_2]. \qquad (11.24)$$

Let E, x, and ρ be as in the lemma. First observe that if we take $C_1 \geqslant C_0$, we can apply Lemma 11.14 to the pair $(x_0, \rho/2)$, where x_0 is any point of $E \cap B(x, \rho/2)$, and get that

$$H^1(E \cap B(x, \rho)) \geqslant H^1(E \cap B(x_0, \rho/2)) \geqslant \rho/4. \qquad (11.25)$$

Denote by E_1 the connected component of $E \cap \overline{B}(x, \rho)$ that contains a_1. First suppose that E_1 does not contain a_2. Then Lemma 11.5 applies, with $F_1 = \{a_1\}$ and $F_2 = \{a_2\}$, and (11.6) says that $H^1(E \cap B(x, \rho)) \leqslant C[\rho + h(\rho)] \rho$. This is impossible (if $\rho + h(\rho)$ is small enough), by (11.25). So E_1 contains a_2, we can Lemma 11.5 with $F_1 = [a_1, a_2]$, and we get (11.22).

Next let $y \in E_1$ be given, set $d = \operatorname{dist}(y, [a_1, a_2])$, and denote by Z the shortest connected set that contains y, a_1, and a_2. Thus

$$H^1(Z) \leqslant H^1(E_1) \leqslant |a_2 - a_1| + C[\rho + h(\rho)] \rho, \qquad (11.26)$$

by (11.22). If $Z = [a_1, a_2] \cup [a_i, y]$ for some i, we get that $d \leqslant |y - a_i| \leqslant C[\rho + h(\rho)] \rho$, by (11.26). If $Z = [a_1, y] \cup [y, a_2]$, (11.26) and Pythagoras yield $d \leqslant C[\rho + h(\rho)]^{1/2} \rho$. The last case is when $Z = [y, z] \cup [a_1, z] \cup [a_2, z]$ is a piece of Y-set, but then $\operatorname{dist}(z, [a_1, a_2]) \leqslant C[\rho + h(\rho)]^{1/2} \rho$ because $H^1([a_1, z] \cup [a_2, z]) \leqslant H^1(Z)$ and by (11.26), and $|y - z| = H^1(Z) - H^1([a_1, z] \cup [a_2, z]) \leqslant H^1(Z) - |a_2 - a_1| \leqslant C[\rho + h(\rho)] \rho$. In this last case too, $d \leqslant C[\rho + h(\rho)]^{1/2} \rho$. So we proved that

$$\operatorname{dist}(y, [a_1, a_2]) \leqslant C_1 [\rho + h(\rho)]^{1/2} \rho \ \text{ for } y \in E_1. \qquad (11.27)$$

This proves (11.24), because if $z \in [a_1, a_2]$ is given, the hyperplane through z which is orthogonal to $[a_1, a_2]$ meets E_1 (because E_1 connects a_1 to a_2), and the intersection lies within $C_1 [\rho + h(\rho)]^{1/2} \rho$ of z by (11.27).

We still need to prove (11.23). Let $y \in E \cap B(x, 9\rho/10)$ be given, and set $\delta = \text{Min}(\text{dist}(y, E_1), \rho/10)$. Thus $B(y, \delta) \subset B(x, \rho) \setminus E_1$, and

$$
\begin{aligned}
H^1(E \cap B(y, \delta)) &\leqslant H^1(E \cap B(x, \rho)) - H^1(E_1) \\
&\leqslant H^1(E \cap B(x, \rho)) - |a_2 - a_1| \\
&\leqslant C[\rho + h(\rho)] \rho,
\end{aligned} \tag{11.28}
$$

because E_1 is connected and contains a_1 and a_2 and by (11.22). On the other hand, Lemma 11.14 says that $H^1(E \cap B(y, \delta)) \geqslant \delta/2$ (if $C_1 \geqslant C_0$), so $\delta \leqslant C[\rho + h(\rho)] \rho$. In particular, $\delta < \rho/10$, so $\text{dist}(y, E_1) = \delta \leqslant C[\rho + h(\rho)] \rho$ and then $dist(y, [a_1, a_2]) \leqslant C[\rho + h(\rho)]^{1/2} \rho$ by (11.27). This proves (11.23) (with a slightly larger C_1); Lemma 11.21 follows. $\square$

Lemma 11.21 can be iterated to give more information. The following lemma gives a proof of Proposition 11.2 in the special case when $E \cap B(x, \rho)$ has at most two points.

LEMMA 11.29. — *Let $E \subset \partial B(0, 1) \subset \mathbb{R}^n$ satisfy (11.1), and suppose that $E \cap \partial B(x, \rho)$ has at most two points. Also suppose that $r + h(r)$ is small enough (depending only on n). Then for $y \in E \cap B(x, \rho/10)$ and $0 < r \leqslant \rho/10$, we can find a line L that contains y such that*

$$
\begin{aligned}
\text{dist}(z, L) &\leqslant C[r + h(10r)]^{1/2} r \quad \text{for } z \in E \cap B(y, r) \\
\text{and } \text{dist}(z, E) &\leqslant C[r + h(10r)]^{1/2} r \quad \text{for } z \in L \cap B(y, r).
\end{aligned} \tag{11.30}
$$

Let E, x, ρ, and y be as in the lemma. Then we are in the situation of Lemma 11.21 (because $E \cap B(x, \rho/2)$ contains y). By (11.22),

$$
H^1(E \cap B(y, 9\rho/10)) \leqslant 2\rho + C[\rho + h(\rho)] \rho < 21\rho/10, \tag{11.31}
$$

so we can find $r \in (\rho/10, 9\rho/10)$ such that $E \cap \partial B(y, r)$ has at most two points (otherwise $H^1(E \cap B(y, 9\rho/10)) \setminus B(y, \rho/10)) \geqslant 24\rho/10$ by the same computation as for (10.7)). In fact, $E \cap \partial B(y, r)$ has exactly two points, because otherwise Lemma 11.5 says that $H^1(E \cap B(y, r)) \leqslant C[r + h(r)] r$, which contradicts Lemma 11.14. So we can apply Lemma 11.21 again to the pair (y, r).

Let us iterate this with the same y. We get a sequence of radii r_k, with $r_0 = r$ and $r_k \in (r_{k-1}/10, 9r_{k-1}/10)$ for $k \geqslant 1$, such that Lemma 11.21 applies to the pair (y, r_k). For each k, we get a line segment I_k with extremities on $\partial B(y, r_k)$, such that

$$
\begin{aligned}
\text{dist}(y, I_k) &\leqslant C_1[r_k + h(r_k)]^{1/2} r_k \quad \text{for } y \in E \cap B(y, 9r_k/10) \\
\text{and } \text{dist}(z, E \cap B(y, r_k)) &\leqslant C_1[r_k + h(r_k)]^{1/2} r_k \quad \text{for } z \in I_k.
\end{aligned} \tag{11.32}
$$

The sequence $\{r_k\}$ is dense enough; we get that for $0 < t < 8\rho/10$, we can find a line L such that

$$\text{dist}(z, L) \leqslant C[t + h(10t)]^{1/2}t \quad \text{for } z \in E \cap B(y, t)$$
$$\text{and } \text{dist}(z, E) \leqslant C[t + h(10t)]^{1/2}t \quad \text{for } z \in L \cap B(y, t). \tag{11.33}$$

Notice that $\text{dist}(y, L) \leqslant C[t + h(10t)]^{1/2}t$ because $y \in E$, so the line L' through y and parallel to L also satisfies (11.33) (with a larger constant). This proves Lemma 11.29. $\quad\square$

In the case of Lemma 11.29, the classical topological disk theorem of Reifenberg, in the simple case of sets of dimension 1, says that E is a Hölder curve near x. If in addition $h(t) \leqslant Ct^\alpha$ for some $\alpha > 0$, say, then (11.30) implies that E has a tangent everywhere on $B(x, \rho/10)$, and the direction of this tangent is Hölder continuous, so that E is in fact a $C^{1,\alpha/2}$ curve near x.

Let us now turn to the case when (11.1) holds and $E \cap \partial B(x, \rho)$ has three points, and discuss the analogue of Lemmas 11.21 and 11.29. Assume that (1.1) holds, that

$$E \cap B(x, \rho/10) \neq \emptyset, \tag{11.34}$$

and that

$$E \cap \partial B(x, \rho) = \{a_1, a_2, a_3\}. \tag{11.35}$$

Recall that E_j is the component of a_j in $E \cap \overline{B}(x, \rho)$. If the three E_j are different, Lemma 11.5 applies with $F_j = \{a_j\}$, and yields $H^1(E \cap B(x, \rho)) \leqslant C[h(\rho) + \rho]\rho$. This is impossible, by (11.34) and Lemma 11.14. So we may assume that E_1 contains a_1 and a_2.

If E_1 does not contain a_3, we can apply the proof of Lemma 11.21, with $F_1 = [a_1, a_2]$ and $F_2 = \{a_3\}$, and get the conclusions (11.22)-(11.24), and then (11.31)-(11.33). In particular, the conclusions of Lemma 11.29 and Proposition 11.2 are valid.

So let us assume that E_1 contains the three a_j. Let F_1 be the shortest connected set that contains the three a_j. We know from Lemma 11.5 that

$$H^1(E \cap B(x, \rho)) \leqslant H^1(F_1) + C[h(\rho) + \rho]\rho, \tag{11.36}$$

and we claim that this implies that

$$\text{dist}(y, F_1) \leqslant C[h(\rho) + \rho]^{1/2}\rho \quad \text{for } y \in E_1. \tag{11.37}$$

Let G_1 be a connected set that contains y and the three a_j, and whose length is minimal. Thus (11.36) says that

$$H^1(G_1) \leqslant H^1(E_1) \leqslant H^1 F_1) + C[h(\rho) + \rho]\rho. \tag{11.38}$$

Let us describe G_1 as we did for F_j in Lemma 10.13. As before, we can find a simple arc $\gamma_1 \subset G_1$ from a_1 to a_2, then a simple arc $\gamma_2 \subset G_1$ from a_3 to some point $z_1 \in \gamma_1$ (where z_1 is the first point of $\gamma_1 \cap \gamma_2$ when we run along γ_2 starting from a_3), and then a simple arc γ_3 that starts from y and hits $\gamma_1 \cup \gamma_2$ at some point z_2 (and not before). We do not exclude the cases when z_1 is one of the a_i or $z_2 = y$. Set $\gamma = \cup_{1 \leqslant i \leqslant 3} \gamma_i$. Since G_1 has minimal length and γ connects y and the three a_j, $G_1 = \gamma$. In addition, the arcs of γ between two consecutive nodes or endpoints are line segments. That is, $G_1 = \gamma$ is composed of five or less line segments. The question is how far from F_1 can it get, knowing (11.38).

Set $G_2 = G_1 \setminus [y, z_2[$. Observe that G_2 still connects the three a_i, and $H^1(G_2) = H^1(G_1) - |y - z_2|$. By minimality of F_1, $H^1(F_1) \leqslant H^1(G_2)$, so $|y - z_2| = H^1(G_1) - H^1(G_2) \leqslant H^1(G_1) - H^1(F_1) \leqslant [h(\rho) + \rho] \rho$, by (11.38). Thus (11.37) will follow as soon as we check that

$$\operatorname{dist}(z_2, F_1) \leqslant C \left[h(\rho) + \rho \right]^{1/2} \rho. \tag{11.39}$$

Recall that z_2 lies in $\gamma_1 \cup \gamma_2$. First suppose that $z_2 \in \gamma_2$. Thus G_2 is composed of the four arc segments $[a_1, z_1]$, $[a_2, z_1]$, $[z_1, z_2]$, and $[z_2, a_3]$, where we do not exclude the possibility that some of these segments are reduced to one point. If we replace $[z_1, z_2] \cup [z_2, a_3]$ with $[z_1, a_3]$ in G_2, we get a shorter connected set G_3 that contains the three a_j, and so $H^1(G_2) - H^1(G_3) \leqslant H^1(G_2) - H^1(F_1) \leqslant [h(\rho) + \rho] \rho$, by (11.38). Thus $\operatorname{dist}(z_2, [z_1, a_3]) \leqslant C \left[h(\rho) + \rho \right]^{1/2} \rho$, and it will be enough to show that

$$\operatorname{dist}(z, F_1) \leqslant C \left[h(\rho) + \rho \right]^{1/2} \rho. \quad \text{for } z \in G_3. \tag{11.40}$$

In the other case where $z_2 \in \gamma_1$, G_2 is either composed of the four segments $[a_1, z_2]$, $[z_2, z_1]$, $[z_1, a_2]$, and $[z_1, a_3]$ (if z_2 lies between a_1 and z_1 on γ_1), or $[a_1, z_1]$, $a_3, z_1]$, $[z_1, z_2]$, and $[z_2, a_2]$ (if z_2 lies between z_1 and a_2). In both cases we set $G_3 = \cup_j [a_j, z_1]$ as before, and it is still enough to prove (11.40). Notice that

$$H^1(G_3) \leqslant H^1(G_1) \leqslant H^1 F_1) + C \left[h(\rho) + \rho \right] \rho \tag{11.41}$$

by (11.38). Also, $F_1 = \cup_j [a_j, z]$, for some z that may also be equal to one of the a_j, and it is enough to show that

$$|z - z_1| \leqslant C \left[h(\rho) + \rho \right]^{1/2} \rho. \tag{11.42}$$

Suppose that $z_1 \neq z$, set $v = (z_1 - z)/|z_1 - z|$, and set $f(t) = \sum_j |z + tv - a_j|$ for $t \geqslant 0$. Let us also assume that $z + tv \neq a_j$ for $t \neq 0$, so that we can differentiate f.

Let us compute some derivatives. Set $r_j(t) = |z + tv - a_j|$ and $u_j(t) = (z + tv - a_j)/r_j(t)$. Then $r'_j(t) = \langle v, u_j(t) \rangle$ and $u'_j(t) = v/r_j(t) - (z + tv - a_j)\, r_j(t)^{-2} \langle v, u_j(t) \rangle$, so

$$r''_j(t) = \langle v, u'_j(t) \rangle = r_j(t)^{-1} - \langle v, z + tv - a_j \rangle \, r_j(t)^{-2} \langle v, u_j(t) \rangle$$
$$= r_j(t)^{-1} \big(1 - \langle v, u_j(t) \rangle^2\big) \tag{11.43}$$

(where we used again the definition of $u_j(t)$). Thus $f'(t) = \sum_j \langle v, u_j(t) \rangle$ and $f''(t) = \sum_j r_j(t)^{-1}\big(1 - \langle v, u_j(t) \rangle^2\big) \geqslant 0$.

Notice that f has a minimum at $t = 0$. Then the half derivative $f'(0_+)$ of f at 0 is nonnegative. As long as

$$\sum_j \big(1 - \langle v, u_j(t) \rangle^2\big) > 10^{-2} \tag{11.44}$$

for $0 \leqslant t < t_0$, we get that $f''(t) \geqslant (200\rho)^{-1}$ (because $r_j \leqslant 2\rho$, since all our points lie in $\overline{B}(x, \rho)$). Then

$$f'(t) \geqslant (200\rho)^{-1} t \ \text{ for } 0 \leqslant t \leqslant t_0. \tag{11.45}$$

If (11.44) fails for some t_0, the three $\langle v, u_j(t_0) \rangle^2$ are close to 1, which means that the three $u_j(t_0)$ are close to v or $-v$. Recall that $f'(t) = \sum_j \langle v, u_j(t) \rangle$, and $f'(t) \geqslant 0$ (because $f'(0_+) \geqslant 0$ and $f'' \geqslant 0$), so at least two of the $u_j(t_0)$ are close to v. This is even more so for $t > t_0$ (we move $z + tv$ away from the corresponding two a_j), so $f'(t) \geqslant 1/2$ for $t \geqslant t_0$, which is even better than (11.45) if $t \leqslant 2\rho$.

We may now integrate (11.45) or the better estimate between 0 and $t_1 = |z_1 - z|$ and get that $f(t_1) - f(0) \geqslant (400\rho)^{-1} t_1^2$. But

$$f(t_1) - f(0) = \sum_j |z_1 - a_j| - \sum_j |z - a_j|$$
$$= H^1(G_3) - H^1(F_1) \leqslant C\,[h(\rho) + \rho]\,\rho \tag{11.46}$$

by (11.41), so $|z_1 - z|^2 = t_1^2 \leqslant C\,[h(\rho) + \rho]\,\rho^2$, and (11.42) follows. This completes our proof of (11.37).

Observe that $H^1(E \cap B(x, \rho)) \leqslant H^1(F_1) + C\,[h(\rho) + \rho]\,\rho \leqslant H^1(E_1) + C\,[h(\rho) + \rho]\,\rho$, by (11.36) and because E_1 also connects the three a_j, so $H^1(E \cap B(x, \rho) \setminus E_1) \leqslant C\,[h(\rho) + \rho]\,\rho$. Then

$$\operatorname{dist}(y, F_1) \leqslant C\,[h(\rho) + \rho]^{1/2}\rho \ \text{ for } y \in E \cap B(x, 9\rho/10) \tag{11.47}$$

by (11.37) and Lemma 11.14; the proof is the same as for the proof of (11.23) at the end of Lemma 11.21. Next

$$\operatorname{dist}(y, E \cap B(x, \rho)) \leqslant \operatorname{dist}(y, E_1) \leqslant C\,[h(\rho) + \rho]^{1/2}\rho \ \text{ for } y \in F_1, \tag{11.48}$$

because E_1 connects the three a_i and is contained in a small strip around F_1, by (11.37). [Cut the strip transversally by a segment of length $C\,[h(\rho) + \rho]^{1/2}\rho$, and observe that if the segment did not meet E_1, it would separate some a_j from the other ones.]

Finally notice that since E meets $B(x, \rho/10)$ by (11.34), F_1 gets very close to $B(x, \rho/10)$. If $h(r)$ is small enough, this actually prevents F_1 from being a simple union of two intervals $[a_i, a_j]$, because one of them would be close to a diagonal, and then they would make an angle smaller than $2\pi/3$. So F_1 is the intersection of $\overline{B}(x, \rho)$ with a Y-set.

We just finished a verification of the analogue of Lemma 11.21, and we are ready to complete the verification of Proposition 11.2 in the present remaining case. Let $y \in E \cap B(x, \rho/10)$ be given; as for (11.31),

$$
\begin{aligned}
H^1(E \cap B(y, 9\rho/10)) &\leqslant H^1(E \cap B(x, \rho)) \\
&\leqslant H^1(F_1) + C\,[h(\rho) + \rho]\,\rho, \qquad (11.49) \\
&\leqslant 3\rho + C[\rho + h(\rho)]\,\rho < 31\rho/10,
\end{aligned}
$$

by (11.36) and if $h(r)$ is small enough. Then we can find $r \in (\rho/10, 9\rho/10)$ such that $E \cap \partial B(y, r)$ has at most three points (otherwise $H^1(E \cap B(y, 9\rho/10) \setminus B(y, \rho/10)) \geqslant 32\rho/10$, by the same computation as for (10.7)). If $E \cap \partial B(y, r)$ has two points, we can apply Lemma 11.21 again to the pair (y, r), and conclude as in Lemma 11.29. Otherwise, the pair (y, r) satisfies (11.34) and (11.35), and we can iterate our argument (with the same y).

In all cases we can find a sequence of radii r_k, with $r_0 = r$ and $r_k \in (r_{k-1}/10, 9r_{k-1}/10)$ for $k \geqslant 1$, so that for each k, we can apply Lemma 11.21 or the discussion above to the pair (y, r_k). By (11.23) and (11.24) or (11.47) and (11.48), there is minimal cone Z_k (a line or a Y-set) such that

$$
\operatorname{dist}(z, Z_k) \leqslant C_1[r_k + h(r_k)]^{1/2}r_k \ \text{ for } z \in E \cap B(y, 9r_k/10)
$$
$$
\text{and } \operatorname{dist}(z, E \cap B(y, r_k)) \leqslant C_1[r_k + h(r_k)]^{1/2}r_k \qquad (11.50)
$$
$$
\text{for } z \in Z_k \cap B(y, r_k).
$$

If Z_k does not go through y, we can replace it with a translation of Z_k, without affecting (11.50). We also get a similar result (with a constant C_1 ten times larger) for every $r \in (0, \rho/10)$, by the same argument as for (11.33). [Apply (11.50) to r_k, where $r \leqslant r_k \leqslant 10r$.] This completes our proof of Proposition 11.2. $\qquad\square$

Remark 11.51. — It is very likely that we can use Lemma 11.5 to prove a rapid decay estimate on the density of E when $E \cap \partial B(x, \rho)$ has more

than four points. Indeed, if $E \cap \partial B(x, \rho) = \{a_1, \cdots, a_N\}$, with $N \geqslant 4$, the length of the shortest connected set that contains the a_j is at most $\tau N \rho$, where the constant $\tau < 1$ depends on the dimension, and can be taken very small when N is large.

Once we get such a decay, we can probably find an effective radius $r_0 > 0$ (that depends on the dimension) such that for $y \in E$ and $0 < r \leqslant r_0$, we can find $\rho > r$ so that $E \cap \partial B(x, \rho)$ has at most three points. Then there is also an effective radius $r_1 > 0$ such that for $\in E$, E is biHölder-equivalent to a line or a Y in a ball that contains $B(y, r_1)$.

12. A baby extension of Reifenberg's theorem

If we want to deduce a good description of $E \cap B(x, \rho/10)$ from the conclusion of Proposition 11.2, the simplest is to use a slight improvement of Reifenberg's topological disk theorem, in the special case of one-dimensional sets, but where we only assume that E is close enough to lines or Y-sets. The statement below is a special case of the generalization of Reifenberg's theorem from [DDT] that will be described in Section 15, but for the convenience of the reader, we shall try to give a short account of how it could be proved.

In this section, $E \subset \mathbb{R}^n$ is a compact set. Denote by $\mathcal{Y}$ the set of nonempty one-dimensional minimal cones in $\mathbb{R}^n$. That is, $Y \in \mathcal{Y}$ if Y is a line or a Y-set (three half lines in a same plane, with the same endpoint, and that make 120° angles at that point). We measure the closeness of E to sets of $\mathcal{Y}$ with

$$\beta_Y(x, r) = \inf_{Y \in \mathcal{Y}} d_{x,r}(E, Y), \tag{12.1}$$

where

$$d_{x,r}(E, Y) = \frac{1}{r} \sup \left\{ \mathrm{dist}(y, Y) \, ; \, y \in E \cap B(x, r) \right\}$$
$$+ \frac{1}{r} \sup \left\{ \mathrm{dist}(y, E) \, ; \, y \in Y \cap B(x, r) \right\}. \tag{12.2}$$

We shall only use $\beta_Y(x, r)$ and $d_{x,r}(E, Y)$ when E and Y meet $B(x, r)$, so we don't need to worry about the definition of the supremums in (12.2) when some sets are empty. Notice that then $\beta_Y(x, r) \leqslant 2$ (take any set Y through x). We could easily restrict to Y-sets (and forget about lines), because for each line L, there is a Y that coincides with L in $B(x, 4r)$, but it is more natural to keep the lines. If $\mathcal{Y}$ was reduced to the set of lines, the $\beta_Y(x, r)$ would be the bilateral P. Jones numbers that are often used to study

the uniform rectifiability of sets. We shall assume that the origin lies in E (a normalization), and that

$$\beta_Y(x, r) \leqslant \varepsilon \ \text{ for } x \in E \cap B(0, 2) \text{ and } 0 < r \leqslant 2, \tag{12.3}$$

where $\varepsilon > 0$ is a small constant that we can choose later. Let us immediately choose, for $x \in E \cap B(0, 2)$ and $0 < r \leqslant 2$, a set $Y(x, r) \in \mathcal{Y}$ such that

$$d_{x,r}(E, Y(x, r)) \leqslant \beta_Y(x, r) \leqslant \varepsilon. \tag{12.4}$$

Our first statement will be a description of E near the origin as a biHölder image of a line or a Y-set. Later in this section we will prove the C^1 equivalence under a suitable decay assumption on the $\beta_Y(x, r)$. Our initial description will be slightly different, depending on whether or not

$$Y(0, 2) \text{ is a } Y\text{-set centered in } B(0, 12/10). \tag{12.5}$$

PROPOSITION 12.6. — *For each small constant $\tau > 0$, there is a constant $\varepsilon > 0$, that depends only on n and τ, such that if $E \subset \mathbb{R}^n$ is a compact set such that $0 \in E$ and (12.3) holds, we have the following description. If (12.5) does not hold, there is a line L through the origin and a biHölder mapping $f : L \cap B(0, 11/10) \to \mathbb{R}^n$, such that*

$$|f(x) - x| \leqslant \tau \ \text{ for } x \in L \cap B(0, 11/10), \tag{12.7}$$

$$(1 - \tau)|x - y|^{1+\tau} \leqslant |f(x) - f(y)| \leqslant (1 + \tau)|x - y|^{1-\tau}$$
$$\text{for } x, y \in L \cap B(0, 11/10), \tag{12.8}$$

$$E \cap B(0, 1 - \tau) \subset f(L \cap B(0, 1)) \subset E \cap B(0, 1 + \tau). \tag{12.9}$$

If (12.5) holds, there is a Y-set Y centered at some $z \in E \cap B(0, 13/10)$, and a biHölder mapping $f : Y \cap B(0, 14/10) \to \mathbb{R}^n$, such that $f(z) = z$,

$$|f(x) - x| \leqslant \tau \ \text{ for } x \in Y \cap B(0, 14/10), \tag{12.10}$$

$$(1 - \tau)|x - y|^{1+\tau} \leqslant |f(x) - f(y)| \leqslant (1 + \tau)|x - y|^{1-\tau}$$
$$\text{for } x, y \in Y \cap B(0, 13/10), \tag{12.11}$$

$$E \cap B(0, 13/10 - \tau) \subset f(L \cap B(0, 13/10)) \subset E \cap B(0, 13/10 + \tau). \tag{12.12}$$

Of course the precise values of the explicit constants are not so important. The main point is a local description of E in $B(0, 1 - \tau)$ as a simple curve or a union of three curves that we control well. We can also extend our

mapping f to $\mathbb{R}^n$, so that it is still a biHölder homeomorphism of $\mathbb{R}^n$ that satisfies (12.7) and (12.8) (or (12.10) and (12.11)) for $x, y \in B(0, 2)$, but we shall not need this. See Section 15 for a more general statement and [DDT] for a proof of that result. Also see Proposition 12.27 for a C^1 description.

For the proof of Proposition 12.6, we suppose that the reader knows a proof of the standard Reifenberg theorem, and we explain how to deduce the Proposition 12.6 from that proof. We start with the observation that the set $Y(x, r) \in \mathcal{Y}$ that we chose in (12.4) does not depend too wildly on x and r. Let us check that

$$d_{y, 9t/10}(Y(x, r), Y(y, t)) \leqslant 100\beta_Y(x, r) + 2\beta_Y(y, t) \leqslant 102\varepsilon \qquad (12.13)$$

when

$$x, y \in E \cap B(0, 2),\ 0 \leqslant r \leqslant 1,\ B(y, t) \subset B(x, r),\ \text{and}\ t \geqslant r/90. \qquad (12.14)$$

The point will be that $Y(x, r)$ and $Y(y, t)$ are both very close to E in $B(y, t)$. First let $z \in Y(y, t) \cap B(y, 9t/10)$ be given. By (12.4), we can find $z_1 \in E$ such that $|z_1 - z| \leqslant \beta_Y(y, t)t$. Then $z_1 \in B(y, t) \subset B(x, r)$, and we can find $z_2 \in Y(x, r)$ such that $|z_2 - z_1| \leqslant \beta_Y(x, r)r$. Altogether, $|z_2 - z| \leqslant \beta_Y(y, t)t + \beta_Y(x, r)r \leqslant \beta_Y(y, t)t + 90\beta_Y(x, r)t$ (by (12.14)). Similarly, if $z \in Y(x, r) \cap B(y, 9t/10)$, we can find $z_1 \in E$ such that $|z_1 - z| \leqslant \beta_Y(x, r)r \leqslant 90\beta_Y(x, r)t$, so $z_1 \in B(y, t)$ and we can find $z_2 \in Y(y, t)$ so that $|z_2 - z_1| \leqslant \beta_Y(y, t)t$ and $|z_2 - z| \leqslant \beta_Y(y, t)t + 90\beta_Y(x, r)t$; (12.13) follows.

Denote by $c(x, r)$ the center of $Y(x, r)$ if $Y(x, r)$ is a Y-set, and set $c(x, r) = \infty$ otherwise. A direct consequence of (12.13) is that $c(x, r)$ does not vary too rapidly, and in particular that

$$|c(x, r) - c(y, t)| \leqslant C\big[\beta_Y(x, r) + \beta_Y(y, t)\big]r \leqslant C\varepsilon r \qquad (12.15)$$

when (12.14) holds and $c(x, r)$ or $c(y, t)$ lies in $B(y, t/2)$.

Let us first prove the proposition when (12.5) fails, i.e., when $c(0, 2) \notin B(0, 12/10)$. Then (12.15) says that $c(y, 1/10) \notin B(y, 1/20)$ for $y \in E \cap B(0, 11/10)$. Repeated applications of (12.15) show that $c(y, t) \notin B(y, t/2)$ for $y \in E \cap B(0, 11/10)$ and $0 < t \leqslant 1/10$. In this case we have a good approximation of E by a line in $B(y, t/4)$ for each $y \in E \cap B(0, 11/10)$ and $0 < t \leqslant 1/10$, and we can conclude with the standard Reifenberg theorem.

So we may now assume that (12.5) holds. Let us first look for the point $z \in E$ where the three curves that compose E near $B(0, 1)$ meet. By definition of $Y(0, 2)$, we can find $z_0 \in E$ such that $|z_0 - c(0, 2)| \leqslant 2\beta_Y(0, 2) \leqslant 2\varepsilon$.

Then

$$|c(z_0, 1) - c(0, 2)| \leqslant C\big(\beta_Y(0, 2) + \beta_Y(z_0, 1)\big) \leqslant C\varepsilon, \qquad (12.16)$$

by (12.15). Notice that $|c(z_0, 1) - z_0| \leqslant C\varepsilon$, so $c(z_0, 1)$ lies in the middle of $B(z_0, 1)$ and we can find $z_1 \in E$ such that $|z_1 - c(z_0, 1)| \leqslant \beta_Y(z_0, 1) \leqslant C\varepsilon$. Then we iterate. By induction, we can find a sequence $\{z_k\}$ in E, so that

$$|z_{k+1} - c(z_k, 2^{-k})| \leqslant 2^{-k}\beta_Y(z_k, 2^{-k}) \leqslant C\varepsilon 2^{-k}, \qquad (12.17)$$

hence

$$|c(z_{k+1}, 2^{-k-1}) - c(z_k, 2^{-k})| \leqslant C\varepsilon 2^{-k} \qquad (12.18)$$

by (12.15), then $|c(z_{k+1}, 2^{-k-1}) - z_{k+1}| \leqslant 2C\varepsilon 2^{-k} < 2^{-k-2}$, we can find z_{k+2}, and so on. Notice also that

$$\begin{aligned} |z_{k+2} - z_{k+1}| \quad &\leqslant |z_{k+2} - c(z_{k+1}, 2^{-k-1})| + |z_{k+1} - c(z_k, 2^{-k})| \\ &\quad + |c(z_{k+1}, 2^{-k-1}) - c(z_k, 2^{-k})| \leqslant C\varepsilon 2^{-k} \end{aligned} \qquad (12.19)$$

by (12.17) and (12.18), so the z_k converge to some $z \in E \cap B(0, 13/10)$. We shall prove that near $B(0, 13/10)$, E is composed of three Reifenberg-flat curves that meet at z.

Let $k \geqslant 0$ be given. Observe that $|z - c(z_k, 2^{-k})| \leqslant |z - z_{k+1}| + |z_{k+1} - c(z_k, 2^{-k})| \leqslant C\varepsilon 2^{-k}$, by (12.19) and (12.17). Denote by Y_k the translation of $Y(z_k, 2^{-k})$ by $z - c(z_k, 2^{-k})$. Thus Y_k is a Y-set centered at z, and it also gives a very good approximation of E in $B(z, 2^{-k-1})$. Indeed,

$$\begin{aligned} d_{z, 2^{-k-1}}(Y_k, E) &\leqslant 2d_{z_k, 2^{-k}}(Y(z_k, 2^{-k}), E) + 2^{k+1}|z - c(z_k, 2^{-k})| \\ &\leqslant C\varepsilon, \end{aligned} \qquad (12.20)$$

by the same proof as for (12.13). We also have a similar control out of $B(z, 1/2)$. Indeed

$$d_{0,2}(Y_0, E) \leqslant C\varepsilon \qquad (12.21)$$

because $d_{0,2}(Y(0, 2), E) \leqslant C\varepsilon$ and Y_0 is so close to $Y(0, 2)$. This last comes from (12.13) which says that $d_{z_0, 9/10}(Y(0, 2), Y(z_0, 1)) \leqslant C\varepsilon$, and because $|z - z_0| \leqslant C\varepsilon$.

Next we use (12.20) and (12.21) to get a good description of E in the annuli

$$A_0 = B(0, 5/3) \setminus B(z, 2^{-6})$$
$$\text{and} \quad A_k = B(z, 2^{-k-3}) \setminus B(z, 2^{-k-6}), \ k \geqslant 1. \qquad (12.22)$$

We claim that

$$c(x, t) \notin B(x, t/2) \quad \text{for } x \in A_k \text{ and } t \leqslant 2^{-k-7}. \qquad (12.23)$$

We start with the case when $2^{-k-8} \leqslant t \leqslant 2^{-k-7}$. Let us first check that

$$\operatorname{dist}(w, Y_k) \leqslant C\varepsilon 2^{-k} \quad \text{for } w \in Y(x,t) \cap B(x,t). \qquad (12.24)$$

By (12.4), we can find $w_1 \in E$ so that $|w_1 - w| \leqslant \varepsilon t \leqslant \varepsilon 2^{-k-7}$; hence $w_1 \in B(z, 2^{-k-2})$ (because $x \in B(z, 2^{-k-3})$), and (12.20) says that $\operatorname{dist}(w_1, Y_k) \leqslant C\varepsilon 2^{-k}$; (12.24) follows.

Recall that Y_k is a Y-set centered at z and that $|z - x| \geqslant 2^{-k-6}$; then (12.24) prevents $Y(x,t)$ from being a Y-set with a center in $B(x, t/2)$. That is, (12.23) holds for $2^{-k-8} \leqslant t \leqslant 2^{-k-7}$.

Finally, if (12.23) holds for some $t \leqslant 2^{-k-7}$, but $c(x, t/2) \in B(x, t/4)$, (12.15) says that $|c(x, t/2) - c(x, t)| \leqslant C\varepsilon t$, and $c(x, t) \in B(x, t/2)$, a contradiction. This proves (12.23).

Because of (12.23), we can use the standard Reifenberg theorem to control E in each annulus A_k. We get three biHölder parameterizations $f_{i,k}$, defined (for instance) on the three intervals $J_{i,k}$ that compose $Y_k \cap A_k$, with properties similar to (12.7) and (12.8) above, and such that $E \cap A_k \subset \cup_i f_{i,k}(J_{i,k}) \subset E$. The argument even works on A_0, even though A_0 is not perfectly symmetric.

Recall that the A_k have large overlaps, so we can easily put together our descriptions of E in the various A_k. We get three biHölder curves γ_i, that leave from y and end out of $B(0, 3/2)$, such that $E \cap B(0, 3/2) \setminus \{y\} \subset \cup_i \gamma_i \subset E$. These curves are disjoint (they are even far from each other on each A_k), and they can easily be parameterized (with gluing of the $f_{i,k}$), so that they have a single parameterization defined on some Y-set centered at z, and that satisfies the constraints (12.10)-(12.12). [Incidentally, we shall not really need (12.11) here.] Proposition 12.6 follows. $\qquad \square$

Next we want to study the regularity of the branches of E when the numbers $\beta_Y(x, r)$ tend to 0 uniformly and sufficiently fast. Let E satisfy the assumptions of Proposition 12.6, say, with $\tau = 10^{-2}$, and assume in addition that there are numbers $\varepsilon_k > 0$, $k \in \mathbb{N}$, such that

$$\sum_{k \geqslant 0} \varepsilon_k < +\infty \qquad (12.25)$$

and

$$\beta_Y(x, 2^{-k}) \leqslant \varepsilon_k \quad \text{for } y \in E \cap B(0,2) \text{ and } k \geqslant 0. \qquad (12.26)$$

Let L or Y and f be as in Proposition 12.6. Set $\gamma_1 = f(L \cap B(0, 11/10))$ when (12.5) does not hold. When (12.5) holds, denote by ℓ_1, ℓ_2, and ℓ_3, the three components of $Y \cap B(14/10) \setminus \{z\}$, and set $\gamma_j = f(\ell_j)$ for $1 \leqslant j \leqslant 3$.

PROPOSITION 12.27. — *Let E satisfy the assumptions of Proposition 12.6, and also (12.25) and (12.26). Then each γ_j is of class C^1, and the tangent direction $\tau_j(x)$ of γ_j at $x \in \gamma_j$ is such that*

$$\operatorname{dist}(\tau_j(x), \tau_j(y)) \leqslant C \sum_{k \in \mathbb{N}\, ;\, 2^{-k} \leqslant 16|x-y|} \varepsilon_j \quad \text{for } x, y \in \gamma_j \qquad (12.28)$$

[see (12.31) for the definition of $\operatorname{dist}(\tau_j(x), \tau_j(y))$]. In the case when (12.5) holds, the three γ_j have tangent half lines at z that are coplanar and make $120°$ angles.

In particular, the curves γ_j are of class $C^{1,\alpha}$ if $\varepsilon_k \leqslant C2^{-\alpha k}$ for some $\alpha \in (0, 1)$.

Let us only prove the proposition when (12.5) holds; the other case is merely simpler. Let $x \in \gamma_j$ be given, and let $k \geqslant 0$ be such that $x \in A_k$. Also let $t \leqslant 2^{-k-7}$ be given. Recall from (12.23) that $c(x, t) \notin B(x, t/2)$. Also observe that $Y(x, t)$ comes very near x, just by (12.4). Then $Y(x, t)$ coincides with a line in $B(x, t/2)$, and we denote this line by $L(x, t)$. Thus

$$d_{x,t/4}(E, L(x, t)) \leqslant 4d_{x,t}(E, Y(x, t)) \leqslant 4\beta_Y(x, t) \qquad (12.29)$$

by definition of $Y(x, t)$. It will be enough to restrict to powers of 2. We just proved that

$$d_{x,2^{-l-2}}(E, L(x, 2^{-l})) \leqslant 4\beta_Y(x, 2^{-l}) \leqslant 4\varepsilon_l \quad \text{for } l \geqslant k + 7, \qquad (12.30)$$

by (12.26).

Denote by $\tau_l(x)$ the direction of $L(x, 2^{-l})$ for $l \geqslant k + 7$. We see $\tau_l(x)$ as a vector in the unit sphere $\mathbb{S}^{n-1}$, determined modulo multiplication by ± 1, and we use the natural distance

$$\operatorname{dist}(\tau, \tau') = \operatorname{Min}(|\tau - \tau'|, |\tau + \tau'|) \qquad (12.31)$$

on the sphere modulo ± 1.

By (12.30) and its analogue for $l + 1$, every point of $L(x, 2^{-l-1}) \cap B(x, 2^{-l-1})$ lies within $\varepsilon_{l+1}2^{-l+1}$ from E, and then within $\varepsilon_{l+1}2^{-l+1} + \varepsilon_l 2^{-l+2}$ from $L(x, 2^{-l})$; hence

$$\operatorname{dist}(\tau_{l+1}(x), \tau_l(x)) \leqslant C\varepsilon_l + C\varepsilon_{l+1}. \qquad (12.32)$$

The triangle inequality yields $\operatorname{dist}(\tau_l(x), \tau_j(x)) \leqslant C \sum_{m \geqslant l} \varepsilon_m < +\infty$ when $j \geqslant l$, where the finiteness comes from (12.25). Thus $\tau(x) = \lim_{l \to +\infty} \tau_l(x)$ exists, and

$$\operatorname{dist}(\tau(x), \tau_l(x)) \leqslant C \sum_{m \geqslant l} \varepsilon_m \quad \text{for } l \geqslant k + 7. \qquad (12.33)$$

Denote by $L(x)$ the line through x with direction $\tau(x)$, and let us check that

$$d_{x,2^{-l-3}}(E, L(x)) \leqslant C \sum_{m \geqslant l} \varepsilon_m \quad \text{for } l \geqslant k+7. \tag{12.34}$$

First let $w \in L(x) \cap B(x, 2^{-l-3})$ be given. Observe that $\operatorname{dist}(x, L(x, 2^{-l})) \leqslant 2^{-l}\varepsilon_l$ by (12.30), so $\operatorname{dist}(w, L(x, 2^{-l})) \leqslant 2^{-l}\varepsilon_l + 2^{-l-3}\operatorname{dist}(\tau(x), \tau_l(x)) \leqslant C2^{-l} \sum_{m \geqslant l} \varepsilon_m$ by (12.33). So we can find $w_1 \in L(x, 2^{-l})$ such that $|w_1 - w| \leqslant C2^{-l} \sum_{m \geqslant l} \varepsilon_m$. Notice that $w_1 \in B(x, 2^{-l-2})$ (or else some point near x will do even better), so by (12.30) we can find $y \in E$ such that $|y - w_1| \leqslant 2^{-l}\varepsilon_l$. Altogether, $|y - w| \leqslant C2^{-l} \sum_{m \geqslant l} \varepsilon_m$, as needed.

Conversely, let $y \in E \cap B(x, 2^{-l-3})$ be given. By (12.30), we can find $w \in L(x, 2^{-l})$ such that $|w - y| \leqslant 2^{-l}\varepsilon_l$. Then we can find $w_1 \in L(x)$ such that $|w_1 - w| \leqslant C2^{-l} \sum_{m \geqslant l} \varepsilon_m$. This completes our verification of (12.34).

The right-hand side of (12.34) tends to 0 when l tends to $+\infty$, hence $L(x)$ is tangent to E at x. That is, our curves γ_j have tangents everywhere.

We still need to estimate the oscillation of $\tau(x)$. Let y be another point of the same γ_j as x, and assume for the moment that y lies in the same annulus A_k as x, and that $|y - x| \leqslant 2^{-k-11}$. Choose l so that $2^{-l-4} \leqslant |y - x| < 2^{-l-3}$; then $l \geqslant k+7$, and we can apply (12.30). We also have the analogue of (12.30) for y, i.e., $d_{y,2^{-l-2}}(E, L(y, 2^{-l})) \leqslant 4\varepsilon_l$. So E is very close to $L(x, 2^{-l})$ and $L(y, 2^{-l})$ in $B(x, 2^{-l-3}) \subset B(x, 2^{-l-2}) \cap B(y, 2^{-l-2})$, and hence

$$\operatorname{dist}(\tau_l(x), \tau_l(y)) \leqslant C\varepsilon_l, \tag{12.35}$$

for instance by the proof of (12.32). Then

$$\begin{aligned} \operatorname{dist}(\tau(x), \tau(y)) &\leqslant \operatorname{dist}(\tau(x), \tau_l(x)) + \operatorname{dist}(\tau(y), \tau_l(y)) + C\varepsilon_l \\ &\leqslant C \sum_{m \geqslant l} \varepsilon_m \leqslant C \sum_{2^{-m} \leqslant 2^{-l}} \varepsilon_m \\ &\leqslant C \sum_{2^{-m} \leqslant 16|z-x|} \varepsilon_m, \end{aligned} \tag{12.36}$$

by (12.35) and (12.33), which is enough for (12.28).

Next assume that y lies in the same A_k as x, but that $|y - x| > 2^{-k-11}$. By (12.20) or (12.21) and because y lies in the same γ_j as x, there is a chain of points x_ℓ, $0 \leqslant \ell \leqslant N$, in $E \cap A_k$, with $x_0 = x$, $|x_\ell - x_{\ell-1}| \leqslant 2^{-k-11}$ for $\ell \geqslant 1$, $x_N = y$, and $N \leqslant 2^{15}$. Then $\operatorname{dist}(\tau_{k+7}(x_\ell), \tau_{k+7}(x_{\ell-1})) \leqslant C\varepsilon_{k+7}$, by the proof of (12.35), and hence

$$\operatorname{dist}(\tau_{k+7}(x), \tau_{k+7}(y)) \leqslant C\varepsilon_{k+7}. \tag{12.37}$$

Now

$$\operatorname{dist}(\tau(x), \tau(y)) \leqslant \operatorname{dist}(\tau_{k+7}(x), \tau_{k+7}(y)) + \operatorname{dist}(\tau_{k+7}(x), \tau(x))$$
$$+ \operatorname{dist}(\tau_{k+7}(y), \tau(y)) \tag{12.38}$$
$$\leqslant C\varepsilon_{k+7} + C \sum_{m \geqslant k+7} \varepsilon_m \leqslant C \sum_{2^{-m} \leqslant 16|x-y|} \varepsilon_m$$

by (12.37), (12.33), and because $|y - x| > 2^{-k-11}$. Again this is enough for (12.28).

We are left with the case when y lies in an annulus $A_l \neq A_k$. Without loss of generality, we may assume that $l \geqslant k$. We may also assume that k was chosen as large as possible. Thus, by (12.22), $x \in A_k \setminus A_{k+1} = B(z, 2^{-k-3}) \setminus B(z, 2^{-k-4})$ if $k \geqslant 1$, and $x \in B(0, 5/2) \setminus B(z, 1/16)$ if $k = 0$.

We can use the fairly large overlap of the annuli A_m to find a chain of points x_m, $k \leqslant m \leqslant l$, that all lie in the same γ_j as x and y, and so that each x_m lies in $A_m \cap A_{m+1}$. Observe that for $k \leqslant m \leqslant l - 1$,

$$\operatorname{dist}(\tau_{m+7}(x_m), \tau_{m+8}(x_{m+1})) \leqslant \operatorname{dist}(\tau_{m+7}(x_m), \tau_{m+7}(x_{m+1}))$$
$$+ \operatorname{dist}(\tau_{m+7}(x_{m+1}), \tau_{m+8}(x_{m+1})) \tag{12.39}$$
$$\leqslant C\varepsilon_{m+7} + C\varepsilon_{m+8}$$

by (12.32) and (12.37) or the proof of (12.35). We also have similar estimates on

$\operatorname{dist}(\tau_{k+7}(x), \tau_{k+7}(x_k))$ and $\operatorname{dist}(\tau_{l+7}(y), \tau_{l+7}(x_l))$, hence

$$\operatorname{dist}(\tau_{k+7}(x), \tau_{l+7}(y)) \leqslant C \sum_{m \geqslant k+7} \varepsilon_m, \tag{12.40}$$

and then

$$\operatorname{dist}(\tau(x), \tau(y)) \leqslant C \sum_{m \geqslant k+7} \varepsilon_m, \tag{12.41}$$

by (12.33). Recall that $|z - x| \geqslant 2^{-k-4}$ because $x \in A_k \setminus A_{k+1}$. At the same time, $y \in A_l$ for some $l > k$, so $|z - y| \leqslant 2^{-k-4}$. Since in addition $z \notin A_k$, we get that $|z - y| \leqslant 2^{-k-6}$ (compare with (12.22)). Altogether, $16|x - y| \geqslant 16|x - z| - 16|z - y| \geqslant 2^{-k-7}$, and (12.41) implies (12.28).

We still need to show that the three tangent half lines to the γ_j at z are coplanar and make a 120° angle. Recall from (12.20) that $d_{z, 2^{-k-1}}(Y_k, E) \leqslant C\varepsilon$, but the same proof shows that

$$\lim_{k \to +\infty} d_{z, 2^{-k-1}}(Y_k, E) = 0 \tag{12.42}$$

(that is, (12.25) allows us to replace $\varepsilon > 0$ with any small number in the proof of (12.20)).

Fix a γ_j and an annulus A_k, and denote by $L_{j,k}$ the branch of Y_k that gets close to $\gamma_j \cap A_k$. Let $\tau_{j,k}$ denote the direction of $L_{j,k}$. Pick any $x \in A_k \cap \gamma_j$, and apply (12.34) with $l = k + 7$. We get that

$$\begin{aligned}
\operatorname{dist}(\tau_{j,k}, \tau(x)) &\leqslant C d_{z,2^{-k-1}}(Y_k, E) + C d_{x,2^{-k-10}}(E, L(x)) \\
&\leqslant C d_{z,2^{-k-1}}(Y_k, E) + C \sum_{m \geqslant k+7} \varepsilon_m .
\end{aligned} \tag{12.43}$$

Now let k tend to $+\infty$. The right-hand side of (12.43) tends to 0, by (12.25) and (12.42). Also, $\tau(x)$ has a limit, by (12.28), so Y_k has a limit Y. The three branches of Y are tangent half lines for the corresponding γ_j at z, by (12.42), and they make 120° angles by definition. In addition, Y lies in a plane because each Y_k lies in a plane. This completes our proof of Proposition 12.27. $\square$

13. Harmonic improvement over a small Lipschitz cone

In this section we consider the cone over a small Lipschitz graph defined on an arc of circle, and show that we almost always reduce its surface when we replace it with the graph of a harmonic function with the same boundary values. This will be useful in the next section, to show that the reduced two-dimensional minimal cones in $\mathbb{R}^n$ are composed of arcs of great circles.

We start with a Lipschitz function $f : [0, T] \to \mathbb{R}^m$, and we assume that

$$0 < T < \pi , \quad f(0) = f(T) = 0, \quad \text{and} \quad ||\nabla f||_\infty \leqslant \tau \tag{13.1}$$

for some small $\tau > 0$. We associate to f a Lipschitz function F_0 on the sector

$$D_T = \{(\rho \cos t, \rho \sin t) \, ; \, 0 \leqslant \rho \leqslant 1 \text{ and } 0 \leqslant t \leqslant T\} \subset \mathbb{R}^2 \tag{13.2}$$

by

$$F(\rho \cos t, \rho \sin t) = \rho f(t) \quad \text{for } 0 \leqslant \rho \leqslant 1 \text{ and } 0 \leqslant t \leqslant T. \tag{13.3}$$

Thus the graph of F is a part of the cone over the graph of the lipschitz function $(\cos t, \sin t) \to f(t)$ that lies above D_T. We want to compare the surface measure of this graph with what we get with a harmonic function G with the same values on ∂D_T.

Since $f(0) = 0$, we can extend f to $[-T, T]$ so that it is odd, and then $f(-T) = f(T) = 0$ by (13.1). So we may write f as a sum of sines, i.e.,

$$f(t) = \sum_{k \geqslant 1} \beta_k \sin(\pi k t / T), \tag{13.4}$$

for some coefficients $\beta_k \in \mathbb{R}^m$ such that $\sum_k |k\beta_k|^2 < +\infty$, because the periodized extension is Lipschitz. Let us check that

$$\frac{\pi^2}{2T} \sum_{k \geqslant 1} k^2 |\beta_k|^2 = ||f'||_2^2 = \int_{[0,T]} |f'(t)|^2 dt. \tag{13.5}$$

Indeed $f'(t) = \sum_{k \geqslant 1} \frac{\pi k}{T} \beta_k \cos(\pi k t/T)$ and the cosine functions are orthogonal on $[0,T]$ because $\int_{[0,\pi]} \cos kt \cos lt\, dt = \frac{1}{2} \int_{[-\pi,\pi]} \cos kt \cos lt\, dt = 0$ for $k \neq l$, so

$$||f'||_2^2 = \frac{T}{2} \sum_{k \geqslant 1} \left(\frac{\pi k}{T}\right)^2 |\beta_k|^2 = \frac{\pi^2}{2T} \sum_{k \geqslant 1} k^2 |\beta_k|^2. \tag{13.6}$$

Our new extension G is defined on D_T by

$$G(\rho \cos t, \rho \sin t) = \sum_{k \geqslant 1} \beta_k \rho^{\pi k/T} \sin(\pi k t/T). \tag{13.7}$$

Notice that it has the same boundary values as F, i.e., that

$$G(z) = F(z) \quad \text{for } z \in \partial D_T. \tag{13.8}$$

Indeed $G(\cos t, \sin t) = f(t)$ by (13.4), and $G(\rho \cos t, \rho \sin t) = 0$ when $t = 0$ or $t = T$.

We do not really need to know that G is harmonic on D_T, but this is the case, essentially because (13.7) defines a harmonic function when $T = \pi$ and D_T is a half disk, while in general G_T is obtained by composition with the conformal mapping $z \to z^{\pi/T}$ which sends D_T to a half-disk. Of course it was reasonable to pick a harmonic function G, because we want its energy to be as small as possible.

LEMMA 13.9. — *The functions F and G lie in $W^{1,2}(D_T)$, with*

$$\int_{D_T} |\nabla G|^2 = \frac{\pi}{2} \sum_{k \geqslant 1} k |\beta_k|^2 \leqslant ||f'||_2^2 \tag{13.10}$$

and

$$\frac{1}{2} ||f'||_2^2 \leqslant \int_{D_T} |\nabla F|^2 = \sum_{k \geqslant 1} \frac{k^2 \pi^2 + T^2}{4T} |\beta_k|^2 \leqslant ||f'||_2^2. \tag{13.11}$$

Let us first compute the energy of G. We start with the radial derivative

$$\frac{\partial G}{\partial \rho}(\rho \cos t, \rho \sin t) = \sum_{k \geqslant 1} \beta_k \frac{\pi k}{T\rho} \rho^{k\pi/T} \sin(\pi kt/T). \qquad (13.12)$$

For ρ fixed, the sine functions are orthogonal on $[0, T]$ (again because $\int_{[0,\pi]} \sin kt \sin lt\, dt = \frac{1}{2} \int_{[-\pi,\pi]} \sin kt \sin lt\, dt = 0$ for $k \neq l$ by evenness), so

$$\int_0^T \left| \frac{\partial G}{\partial \rho}(\rho \cos t, \rho \sin t) \right|^2 dt = \frac{T}{2} \sum_{k \geqslant 1} |\beta_k|^2 \left(\frac{\pi k}{T\rho} \right)^2 \rho^{2\pi k/T}$$
$$= \frac{\pi^2}{2T} \sum_{k \geqslant 1} k^2 |\beta_k|^2 \, \rho^{2\pi k/T} \rho^{-2} \qquad (13.13)$$

and then

$$\int_{D_T} \left| \frac{\partial G}{\partial \rho} \right|^2 = \frac{\pi^2}{2T} \sum_{k \geqslant 1} k^2 |\beta_k|^2 \int_0^1 \rho^{2\pi k/T} \rho^{-1} d\rho$$
$$= \frac{\pi^2}{2T} \sum_{k \geqslant 1} k^2 |\beta_k|^2 \frac{T}{2\pi k} = \frac{\pi}{4} \sum_{k \geqslant 1} k|\beta_k|^2 . \qquad (13.14)$$

Similarly, the derivative in the tangential direction is

$$\frac{1}{\rho} \frac{\partial G}{\partial t}(\rho \cos t, \rho \sin t) = \frac{1}{\rho} \sum_{k \geqslant 1} \beta_k \rho^{k\pi/T} \frac{\pi k}{T} \cos(\pi kt/T), \qquad (13.15)$$

and the same computation (only with sines replaced by cosines) leads to

$$\int_{D_T} \left| \frac{1}{\rho} \frac{\partial G}{\partial t} \right|^2 = \frac{\pi}{4} \sum_{k \geqslant 1} k|\beta_k|^2. \qquad (13.16)$$

We add (13.14) and (13.16) and get the first half of (13.10). The fact that the result does not depend on T is not too surprising, because energy integrals are invariant under the conformal mapping $z \to z^{\pi/T}$. For the second half of (13.10), we simply use (13.5) and the fact that $k\pi/2 \leqslant k^2\pi^2/2T$ for $k \geqslant 1$, by (13.1).

Let us turn to F. Recall from (13.3) et (13.4) that

$$F(\rho \cos t, \rho \sin t) = \rho f(t) = \rho \sum_{k \geqslant 1} \beta_k \sin(\pi kt/T). \qquad (13.17)$$

The radial derivative is $\sum_{k \geqslant 1} \beta_k \sin(\pi k t/T)$ and the integral of its square is $\dfrac{T}{4} \sum_{k \geqslant 1} |\beta_k|^2$. The tangential derivative is $\dfrac{1}{\rho} \dfrac{\partial F}{\partial t} = \sum_{k \geqslant 1} \beta_k \dfrac{\pi k}{T} \cos(\pi k t/T)$, and its contribution to the energy is $\dfrac{T}{4} \sum_{k \geqslant 1} |\beta_k|^2 \left(\dfrac{\pi k}{T}\right)^2$. Altogether

$$\int_{D_T} |\nabla F|^2 = \sum_{k \geqslant 1} \frac{k^2 \pi^2 + T^2}{4T} |\beta_k|^2. \tag{13.18}$$

This is the main part of (13.11); for the rest, just use (13.5) and notice that $0 \leqslant T^2 \leqslant \pi^2 k^2$. Lemma 13.9 follows. $\qquad \square$

Observe that when $T = \pi$ and $k = 1$, the energy terms in (13.10) and (13.11) are both equal to $\frac{\pi}{2} |\beta_1|^2$, which is not surprising because for $f(t) = \sin t$, the harmonic extension is the same as the homogeneous extension. Here $T < \pi$, and we shall see now that all the values of k give an advantage to G. This is not surprising, since the sine function in no longer possible, by the constraints in (13.1) and because a plane that passes through the origin, $(1, 0, 0)$ and $(\cos T, \sin T, 0)$, has to be the horizontal plane through the origin.

Set $\lambda = T/\pi < 1$ (by (13.1)), and let us check that

$$\int_{D_T} |\nabla G|^2 \leqslant \frac{2\lambda}{1 + \lambda^2} \int_{D_T} |\nabla F|^2 \quad \text{and} \quad \frac{2\lambda}{1 + \lambda^2} < 1. \tag{13.19}$$

For the first inequality, we just need to compare the coefficients in (13.10) and (13.11), and show that for $k \geqslant 1$,

$$\frac{k\pi}{2} \leqslant \frac{2\lambda}{1 + \lambda^2} \frac{k^2 \pi^2 + T^2}{4T} = \frac{2\lambda}{1 + \lambda^2} \frac{k^2 \pi^2 + \lambda^2 \pi^2}{4\pi\lambda}. \tag{13.20}$$

So we just need to check that $k \leqslant \dfrac{k^2 + \lambda^2}{1 + \lambda^2}$, but $k(1 + \lambda^2) \leqslant k^2 + \lambda^2$ because $k^2 + \lambda^2 - k(1 + \lambda^2) = k(k - 1) - \lambda^2(k - 1) \geqslant 0$, so the first part of (13.19) holds. For the second part, we just need to observe that $\frac{2\lambda}{1+\lambda^2}$ is an increasing function of $\lambda \in [0, 1]$.

Let Σ_F and Σ_G denote the respective graphs of F and G (both defined on D_T). We want to use the computations above to compare $H^2(\Sigma_F)$ and $H^2(\Sigma_G)$. This will be easy because we can assume that $||f'||_\infty \leqslant \tau$, which is as small as we want.

LEMMA 13.21. — *We have that*

$$H^2(\Sigma_G) - H^2(D_T) \leqslant \frac{1}{2} \int_{D_T} |\nabla G|^2 + C\tau \, ||f'||_2^2$$

$$\leqslant \frac{\lambda + C\tau}{1 + \lambda^2} \int_{D_T} |\nabla F|^2 \tag{13.22}$$

and

$$H^2(\Sigma_F) - H^2(D_T) \geqslant \frac{1}{2} \left(1 - \frac{(1+T)^2 \tau^2}{2} \right) \int_{D_T} |\nabla F|^2. \tag{13.23}$$

Before we prove the lemma, observe that since we can take τ as small as we want (and $T \leqslant \pi$ anyway), $1 - \frac{(1+T)^2 \tau^2}{2}$ is as close to 1 as we want. Since $\frac{\lambda}{1+\lambda^2} < \frac{1}{2}$ by (13.19), for each choice of $T < \pi$ we can find $\eta \in (0,1)$ such that (for τ small enough, depending on T through λ), (13.22) and (13.23) imply that

$$H^2(\Sigma_G) - H^2(D_T) \leqslant \eta \big[H^2(\Sigma_F) - H^2(D_T) \big]. \tag{13.24}$$

Also,

$$H^2(\Sigma_F) - H^2(D_T) \geqslant \frac{1}{3} \int_{D_T} |\nabla F|^2 \geqslant \frac{1}{6} ||f'||_\infty^2 \tag{13.25}$$

if τ is small enough, by (13.11).

Now we prove the lemma. Notice that G is smooth in the interior of D_T: the series in (13.12) and (13.15) that give ∇G converge rather brutally because of the term in $\rho^{k\pi/T}$. By the area formula,

$$H^2(\Sigma_G) = \int_{D_T} J(z) dz, \tag{13.26}$$

where J is the jacobian of the parameterization $H : z \to (z, G(z))$, which we compute now. Fix $z \in D_T$, denote by e_1 and e_2 the first two vectors of the canonical basis of $\mathbb{R}^n$ (so e_1 and e_2 form a basis of the plane that contains D_T), and set $v_1 = \frac{\partial G}{\partial x}(z)$ and $v_2 = \frac{\partial G}{\partial y}(z)$. Thus, if we set $w_1 = DH(z)(e_1)$ and $w_2 = DH(z)(e_2)$, we get that $w_i = e_i + v_i$ for $i = 1, 2$. Now $J(z) = |w_1||w_2| \sin \alpha$, where $\alpha \in [0, \pi/2]$ is the angle of w_1 with w_2, so

$$\begin{aligned}
J(z)^2 &= |w_1|^2 |w_2|^2 \sin^2 \alpha = |w_1|^2 |w_2|^2 [1 - \cos^2 \alpha] \\
&= |w_1|^2 |w_2|^2 - \langle w_1, w_2 \rangle^2 = (1 + |v_1|^2)(1 + |v_2|^2) - \langle v_1, v_2 \rangle^2 \\
&= 1 + |v_1|^2 + |v_2|^2 + |v_1|^2 |v_2|^2 - \langle v_1, v_2 \rangle^2 \\
&\leqslant 1 + |v_1|^2 + |v_2|^2 + |v_1|^2 |v_2|^2 \leqslant 1 + |\nabla G(z)|^2 + |\nabla G(z)|^4
\end{aligned} \tag{13.27}$$

We expect $|\nabla G(z)|^4$ to be much smaller than $|\nabla G(z)|^2$ on average, but our estimates will be slightly worse near $\partial B(0,1)$, so we'll cut the integral in two.

Set $D_1 = D_T \cap B(0, 1 - \tau)$. For $z \in D_1$, (13.12) and (13.15) yield

$$
\begin{aligned}
|\nabla G(z)| &\leqslant C \sum_{k \geqslant 1} k |\beta_k| \, \rho^{k\pi/T} \rho^{-1} \leqslant C \sum_{k \geqslant 1} k |\beta_k| \, \rho^{k-1} \\
&\leqslant C \Big\{ \sum_{k \geqslant 1} k^2 |\beta_k|^2 \Big\}^{1/2} \Big\{ \sum_{k \geqslant 1} \rho^{2k-2} \Big\}^{1/2} \\
&\leqslant C \|f'\|_2 \left[1 - \rho^2 \right]^{-1/2} \leqslant C \sqrt{\tau}
\end{aligned}
\tag{13.28}
$$

because $T < \pi$, by (13.11), and because $\|f'\|_\infty \leqslant \tau$ and $\rho \leqslant 1 - \tau$.

Set $u(z) = J(z)^2 - 1$. Observe that $u(z) \leqslant |\nabla G(z)|^2 + |\nabla G(z)|^4 \leqslant (1 + C\tau)|\nabla G(z)|^2$ for $z \in D_1$, by (13.27) and (13.28). Since $(1+u)^{1/2} \leqslant 1 + \frac{u}{2}$ for $u \geqslant 0$. we get that

$$
\begin{aligned}
\int_{D_1} J(z) \, dz = \int_{D_1} (1 + u(z))^{1/2} dz &\leqslant \int_{D_1} (1 + \frac{u(z)}{2}) \, dz \\
&= H^2(D_1) + \frac{1}{2} \int_{D_1} u(z) \, dz \\
&\leqslant H^2(D_1) + \frac{(1 + C\tau)}{2} \int_{D_1} |\nabla G(z)|^2 dz \\
&\leqslant H^2(D_1) + \frac{1}{2} \int_{D_1} |\nabla G(z)|^2 dz + C\tau \, \|f'\|_2^2
\end{aligned}
\tag{13.29}
$$

by (13.10). On $D_T \setminus D_1$, we say that

$$
\begin{aligned}
J(z) &\leqslant (1 + |\nabla G(z)|^2 + |\nabla G(z)|^4)^{1/2} \\
&\leqslant (1 + |\nabla G(z)|^2)^{1/2} + |\nabla G(z)|^2 \\
&\leqslant 1 + \frac{1}{2} |\nabla G(z)|^2 + |\nabla G(z)|^2 = 1 + \frac{3}{2} |\nabla G(z)|^2,
\end{aligned}
\tag{13.30}
$$

hence

$$\int_{D_T \setminus D_1} J(z)\,dz \leqslant H^2(D_T \setminus D_1) + \frac{3}{2} \int_{D_T \setminus D_1} |\nabla G(z)|^2$$

$$\leqslant H^2(D_T \setminus D_1) + C \sum_{k \geqslant 1} k^2 |\beta_k|^2 \int_{1-\tau}^{1} \rho^{2\pi k/T} \rho^{-2} d\rho \qquad (13.31)$$

$$\leqslant H^2(D_T \setminus D_1) + C\tau \sum_{k \geqslant 1} k^2 |\beta_k|^2$$

$$\leqslant H^2(D_T \setminus D_1) + C\tau \, ||f'||_2^2$$

by (13.13), its analogue for the tangential derivative, the same computation as in (13.14), and (13.11). We add (13.31) to (13.29), use (13.26) and get the first inequality in (13.22). The second one comes from (13.19) and (13.11).

For (13.23), it will be useful to know that F is Lipschitz with a small norm. And indeed $F(\rho \cos t, \rho \sin t) = \rho f(t)$ by (13.3), so

$$|\nabla F(\rho \cos t, \rho \sin t)| \leqslant |f(t)| + |f'(t)| \leqslant (1+T)\, ||f'||_\infty \leqslant (1+T)\, \tau, \ (13.32)$$

where we also used the fact that $f(0) = 0$ (by (13.1)). Now se use the analogue of (13.26), with a Jacobian $J(z)$ such that $J(z)^2 \geqslant 1 + |v_1|^2 + |v_2|^2 = 1 + |\nabla F(z)|^2$ by the second line of (13.27). Set $u = |\nabla F|^2$; then

$$(1+u)^{1/2} \geqslant 1 + \frac{u}{2} - \frac{u^2}{4} \geqslant 1 + \frac{u}{2}\left(1 - \frac{(1+T)^2\tau^2}{2}\right) \qquad (13.33)$$

because $0 \leqslant u \leqslant (1+T)^2\tau^2$. Then

$$H^2(\Sigma_G) - H^2(D_T) = \int_{D_T} J(z) - 1$$

$$\geqslant \int_{D_T} \left\{ \{1 + |\nabla F|^2\}^{1/2} - 1 \right\} \qquad (13.34)$$

$$\geqslant \frac{1}{2}\left(1 - \frac{(1+T)^2\tau^2}{2}\right) \int_{D_T} |\nabla F|^2$$

by (13.26). This proves (13.23); Lemma 13.21 follows. $\qquad \square$

14. Structure of the two-dimensional minimal cones

We are finally ready to give a description of the reduced two-dimensional minimal cones in $\mathbb{R}^n$. Recall that by minimal cone, we simply mean a minimal set (as in Definition 8.1), which is a cone. We say that the closed set

E is reduced when $E^* = E$; see Definition 2.12. We shall denote by $\mathcal{Z}$ the set of non empty reduced minimal cones of dimension 2 in $\mathbb{R}^n$, not necessarily centered at the origin, and by $\mathcal{Z}_0$ the subset of elements of $\mathcal{Z}$ that are centered at the origin.

PROPOSITION 14.1. — *Let $E \in \mathcal{Z}$ be given, and set $K = E \cap \partial B(0,1)$. Then K is a finite union of great circles or arcs of great circles C_j, $j \in J$. The C_j can only meet at their extremities, and each extremity is a common extremity of exactly three C_j, which meet with $120°$ angles. In addition, the length of each C_j is a least l_0, where $l_0 > 0$ depends only on n.*

See (14.10) and Lemma 14.12 for additional constraints on K. Thus K is an array of arcs of great circles, with some constraints, but starting from the ambient dimension $n = 4$, we do not know exactly which arrays give rise to minimal cones. When $n = 3$, the answer was given by E. Lamarle [La], A. Heppes [He], and J. Tayor [Tay], and indeed some arrays in $\mathbb{S}^2$ do not give a minimal cone. See [Tay] and Ken Brakke's home page for descriptions. When $n \geqslant 4$, it is expected that the disjoint union of two disjoint great circles, contained in almost orthogonal two-planes, sometimes gives a minimal cone, but the precise condition for this to happen is not known. There may even be other, combinatorially more complicated, continuous families of non mutually isometric minimal cones.

With the help of Proposition 14.1, we shall be able to give a local biHölder description of the two-dimensional almost-minimal sets E, but if we want a C^1 description near a point x, as in the original Jean Taylor theorem, it is probable that more information on the blow-up limits of E at x will be needed.

Let us prove the proposition. Take $\varepsilon > 0$ small; $\varepsilon = 10^{-2}$ will be more than enough. Apply Theorem 8.23 with $d = 1$. We get $r_0 > 0$, that depends only on n and ε, such that if $E \in \mathcal{Z}$, then for $x \in K = E \cap \partial B(0,1)$ and $0 < r < r_0$, we can find a reduced minimal set L of dimension 1 such that (8.24) and (8.25) hold (that is, K is close to L in $B(x,r)$, both in terms of Hausdorff distance and measure).

Continue with a given choice of x and r. Observe that L is not empty, by (8.24) and because $x \in E$, so Theorem 10.1 says that L is a line or a Y-set (not necessarily centered at x). By (8.25) with $y = x$ and $t = r$, we get that

$$H^1(K \cap B(x,r)) \leqslant H^1(L \cap B(x, r + \varepsilon r)) + \varepsilon r \leqslant 34r/10. \qquad (14.2)$$

Denote by $N(t)$ the number of points in $K \cap \partial B(x,t)$. By [D2], Lemma 26.1 on page 160, or [Fe], Theorem 3.2.22,

$$\int_0^r N(t)dt \leqslant H^1(K \cap B(x,r)) \leqslant 34r/10, \qquad (14.3)$$

so by Chebyshev we can find $\rho \in (r/10, r)$ such that $N(\rho) < 4$.

By Proposition 9.4, K is a reduced weak almost-minimal set, with gauge function $h(s) \leqslant Cs$. If r_0 was chosen small enough, Proposition 11.2 applies and yield the good approximation of K by one-dimensional minimal cones in every $B(y,t)$, $y \in K \cap B(x, \rho/10)$ and $0 < t < \rho/10$, as in (11.3). This is enough to apply Proposition 12.6, with the unit ball replaced with $B(x, \rho/20) \supset B(x, r/200)$ (and again, if r_0 is small enough). We get that there is an open set V, with $B(x, r/300) \subset V \subset B(x, r/100)$, where K is either composed of a single biHölder curve, or of three biHölder curves that end at a some point. In addition, (11.3) says that (12.25) and (12.26) hold, with $\varepsilon_k \leqslant C[2^{-k} + h(2^{-k})]^{1/2} \leqslant C2^{-k/2}$. So Proposition 11.27 says that each of these curve is of class $C^{1+\alpha}$. If we prove that every point of these curves C_j has a small neighborhood where C_j is a great circle, Proposition 14.1 will follow, with $l_0 = r_0/1000$, say.

So let x lie in the interior of a curve C_j, and choose $r > 0$ so small that $K \cap B(x,r) = C_j \cap B(x,r)$ is a single $C^{1+\alpha}$ curve with two endpoints on $C_j \cap \partial B(x,r)$, and the oscillation of the direction of the tangent line to C_j along this curve is very small (how small will be decided soon). Without loss of generality, we can assume that the two points of $K \cap \partial B(x,r)$ are $(1,0,0)$ and $(\cos T, \sin T, 0)$ for some small positive $T < \pi$ (and where the last coordinate 0 lies in $\mathbb{R}^{n-2}$). If the oscillation of the tangent direction of K in $B(x,r)$ is small enough, $K \cap B(x,r)$ is a small lipschitz graph over the arc of circle $J = \{(\cos t, \sin t, 0) \,;\, 0 \leqslant t \leqslant T\}$.

Define the Lipschitz function f on $[0,T]$ by the fact that for $0 \leqslant t \leqslant T$, $(\cos t, \sin t, f(t))$ is the point of $E \cap B(x,r)$ above $(\cos t, \sin t, 0) \in J$, or equivalently by the fact that $[1 + |f(t)|^2]^{-1/2}(\cos t, \sin t, f(t)) \in K \cap B(x,r)$. We can assume that $||f'||_\infty$ is as small as we want, by taking r very small (recall that C_j is $C^{1+\alpha}$).

Define F on the cone over J by $F(\rho \cos t, \rho \sin t) = \rho f(\cos t, \sin t)$ for $\rho \geqslant 0$ and $0 \leqslant t \leqslant T$. Thus F is the homogeneous extension of the function defined in (13.3). Denote by $C(x,r)$ the cone over $\partial B(0,1) \cap B(x,r)$ and by Γ the graph of F. Thus

$$E \cap C(x,r) = \Gamma, \qquad (14.4)$$

because E and Γ are cones, and both coincide on the unit sphere with $K \cap B(x,r)$. We want to use section 13 to construct a competitor for E.

Let G be as in (13.7). Also denote by π the orthogonal projection on the horizontal plane, and by $D_T = \{(\rho\cos t, \rho\sin t, 0)\,;\, 0 \leqslant t \leqslant T \text{ and } 0 \leqslant \rho \leqslant 1\}$ the same sector as in (13.2). We define a function φ on Γ by

$$\varphi(z) = (\pi(z), G(\pi(z))) \text{ for } z \in \Gamma \cap \pi^{-1}(D_T)$$
$$\text{and } \varphi(z) = z \text{ for } z \in \Gamma \setminus \pi^{-1}(D_T). \tag{14.5}$$

Notice that $\varphi(z) = z$ on $\Gamma \cap \pi^{-1}(\partial D_T)$, by (13.8), so φ is continuous across $\Gamma \cap \pi^{-1}(\partial D_T)$. We should also check that φ is Lipschitz on Γ, or equivalently that G is Lipschitz, but we don't need to have precise bounds. One could easily get rid of this difficulty by replacing G with a lipschitz function G', which is so close to G in the Sobolev $W^{1,2}$-norm that the strict inequalities of Section 13 stay true for G'. But we can also observe that G is Lipschitz in the present situation. Indeed, f is $C^{1+\alpha}$ (by Proposition 11.27), and G is obtained from f by a composition with a dilation (to get a map defined on $[0, \pi]$), an extension by antisymmetry (to get a function on $[-\pi, \pi]$ whose 2π-periodic extension is still $C^{1+\alpha}$), then a Poisson extension to the disk (which is Lipschitz because f is $C^{1+\alpha}$), which we restrict to the half disk and compose with the conformal mapping $z \to z^{\pi/T}$ to get a function defined on D_T, which is still Lipschitz because $T < \pi$. Set

$$\varphi(z) = z \quad \text{for } z \in \mathbb{R}^n \setminus C(x, r). \tag{14.6}$$

Recall that $\Gamma \cap \partial C(x, r)$ is the union of the two half lines through $(1, 0, 0)$ and $(\cos T, \sin T, 0)$, so (13.8) says that $\varphi(z) = z$ there, and φ is continuous across $\partial C(x, r)$. In fact, due to the fact that Γ is transverse to $\partial C(x, r)$, our mapping φ, which is now defined on $\Gamma \cup [\mathbb{R}^n \setminus C(x, r)]$, is Lipschitz there. In addition, $\varphi(z) = z$ for $|z| \geqslant 2$. So we can extend φ to $\mathbb{R}^n$ in a Lipschitz way, and so that $\varphi(z) = z$ for $|z| \geqslant 2$.

Now E is a minimal set, and Definition 8.1 says that

$$H^2(E \setminus \varphi(E)) \leqslant H^2(\varphi(E) \setminus E). \tag{14.7}$$

Set $E_1 = E \cap C(x, y) \cap \pi^{-1}(D_T) = \Gamma \cap \pi^{-1}(D_T)$ (by (14.4)). Observe that $\varphi(z) = z$ for $z \in E \setminus E_1$, by (14.5) and (14.6). Then $E \setminus \varphi(E) = E_1 \setminus \varphi(E)$ and $\varphi(E) \setminus E = \varphi(E_1) \setminus E \subset \varphi(E_1) \setminus E_1$, so (14.7) says that

$$H^2(E_1 \setminus \varphi(E)) = H^2(E \setminus \varphi(E)) \leqslant H^2(\varphi(E) \setminus E) \leqslant H^2(\varphi(E_1) \setminus E_1), \tag{14.8}$$

and hence

$$\begin{aligned}
H^2(E_1) &= H^2(E_1 \setminus \varphi(E)) + H^2(E_1 \cap \varphi(E)) \\
&\leqslant H^2(\varphi(E_1) \setminus E_1) + H^2(E_1 \cap \varphi(E)) \\
&= H^2(\varphi(E_1) \setminus E_1) + H^2(E_1 \cap \varphi(E_1)) = H^2(\varphi(E_1)),
\end{aligned} \tag{14.9}$$

because $\varphi(E) \setminus \varphi(E_1) \subset \varphi(E \setminus E_1)$ does not meet E_1.

On the other hand, $E_1 = \Gamma \cap \pi^{-1}(D_T)$ is the graph of F over D_T, and by (14.5) $\varphi(E_1)$ is the graph of G over D_T. With the notation of Section 13, $E_1 = \Sigma_F$ and $\varphi(E_1) = \Sigma_G$ (see above Lemma 13.21). If we chose r above small enough, (13.1) holds with a τ so small that by Lemma 13.21, (13.24) holds for some $\eta \in (0, 1)$. That is, $H^2(\Sigma_G) - H^2(D_T) \leqslant \eta[H^2(\Sigma_F) - H^2(D_T)]$. But (14.9) says that $H^2(\Sigma_F) = H^2(E_1) \leqslant H^2(\varphi(E_1)) = H^2(\Sigma_G)$, so we get that $H^2(\Sigma_F) = H^2(D_T)$. Then $\int_{D_T} |\nabla F|^2 = 0$, by (13.23) (recall that τ is small), and $||f'||_2 = 0$, by (13.11). That is, $f = 0$ (by (13.1)).

We just proved that the intersection of the arc C_j with $B(x, r)$ is contained in the horizontal plane. That is, $C_j \cap B(x, r)$ is an arc of great circle, as needed. Proposition 14.1 follows, as was noted before. $\qquad \square$

We complete this section with a little more information on the minimal cones in $\mathcal{Z}_0$. Let $E \subset \mathcal{Z}_0$ be given, and set $K = E \cap \partial B(0, 1)$. Recall from Proposition 14.1 that K is composed of some great circles C_j, $j \in J_0$, and some arcs of great circles C_j, $j \in J_1$.

Let H denote the set of extremities of the arcs C_j, $j \in J_1$. For each $x \in H$, there are exactly three C_j that leave from x; thus there are $3N/2$ arcs, where N is the number of points in H. In particular, N is even and the number of arcs is a multiple of 3.

Next we claim that there is a constant $\eta_0 > 0$, that depends only on n, such that

$$\text{if } i, j \in J_0 \cup J_1, \, i \neq j, \, x \in C_i,$$
$$\text{and } \operatorname{dist}(x, C_j) \leqslant \eta_0, \text{ then } i, j \in J_0, \tag{14.10}$$

and C_i and C_j have a common endpoint in $\overline{B}(x, \operatorname{dist}(x, C_j))$.

We shall take $\eta_0 = r_0/300$, where r_0 is as in the proof of Proposition 14.1. Let i, j, and $x \in C_i$ be such that $\operatorname{dist}(x, C_j) \leqslant \eta_0$. As before (see below (14.3)), there is a neighborhood V of $B(x, r_0/300)$ where E is composed of one simple curve, or of three simple arcs of curves that meet at one point $a \in B(x, r_0/100)$. In addition, these arcs are arcs of great circles, and each one is contained in a C_k, $k \in J_0 \cup J_1$. One of these arcs is contained in C_i (because $x \in C_i$), and our assumption that $\operatorname{dist}(x, C_j) \leqslant \eta_0$ for some other j implies that there are three arcs, and another one is contained in C_j. These three arcs meet at a with $120°$ angles, so $|a - x| \leqslant \operatorname{dist}(x, C_j)$; (14.10) follows.

The simplest set $E \in \mathcal{Z}_0$ is a plane, and then K is a great circle.

In dimension $n \geqslant 2p$, $p \geqslant 2$, we may select p planes P_j that are orthogonal (or nearly orthogonal) to each other, and set $E = \cup_j P_j$ (or equivalently, K is the union of the great circles $P_j \cap \partial B(0,1)$). It seems that the precise conditions under which $E \in \mathcal{Z}_0$ are not known. More generally, we may always start from a set $E \in \mathcal{Z}_0$ that lies in $\mathbb{R}^{n-2}$, and add to it the plane $\{0\} \times \mathbb{R}^2$; this gives a set E' that fits the description above; I don't know whether E' is necessarily a minimal cone.

The next simplest example is a set of type $\mathbb{Y}$, the product of a Y-set in a plane by a line perpendicular to that plane. It is thus contained in a space of dimension 3, and is composed of three half planes with a common boundary L, that make $120°$ angles along L (see Figure 1). Notice that it is easy to see that

$$\text{if } E \in \mathcal{Z}_0, K \text{ is connected, and } H \text{ has 2 points,}$$
$$\text{then } E \text{ is a set of type } \mathbb{Y};$$

$$(14.11)$$

indeed E is composed of exactly three arcs of great circles that connect the two points of H, these arcs are determined by their directions at any of the two points of H, and we know that these directions make $120°$ angles.

The simplest set for which H has 4 points is the cone over the union of vertices of a regular tetrahedron in $\mathbb{R}^3$ centered at the origin (see Figure 2).

LEMMA 14.12. — *There is a constant $d_T > 3\pi/2$, that depends only on n, such that*

$$H^2(E \cap B(0,1)) \geqslant d_T$$

$$(14.13)$$

when $E \in \mathcal{Z}_0$ is neither a plane nor a set of type $\mathbb{Y}$.

Let us first check that

$$H^2(E \cap B(0,1)) > 3\pi/2 \tag{14.14}$$

for such sets E. If E is composed of more than one full great circles, $H^2(E \cap B(0,1)) \geqslant 2\pi$. Otherwise, H is not empty. Let $y \in H$ be given, and set $\theta(y,r) = r^{-2}H^2(E \cap B(y,r))$ for $r > 0$. Proposition 5.16 says that $\theta(y,r) = r^{-2}H^2(E \cap B(y,r))$ is nondecreasing, so the limits $\theta(y) = \lim_{r \to 0} \theta(y,r)$ and $\theta_\infty(y) = \lim_{r \to +\infty} \theta(y,r)$ exist, and $\theta_\infty(y) \geqslant \theta(y)$. In addition, E does not coincide with a cone centered at y in big annuli, so Theorem 6.2 says that $\theta_\infty(y) > \theta(y)$.

By Proposition 14.1, the sets $k^{-1}(E - y)$ converge to a minimal cone of type $\mathbb{Y}$, and Proposition 7.31 (or directly Lemmas 3.3 and 3.12) says that $\theta(y) = 3\pi/2$ (the density of a set of type $\mathbb{Y}$). Hence $\theta_\infty(y) > 3\pi/2$. Finally,

$$
\begin{aligned}
H^2(E \cap B(0,1)) &= R^{-2} H^2(E \cap B(0,R)) \\
&\geqslant R^{-2} H^2(E \cap B(y, R - |y|)) \qquad (14.15) \\
&= R^{-2}(R - |y|)^2\, \theta(y, R - |y|)
\end{aligned}
$$

for R large, and the last term tends to $\theta_\infty(y) > 3\pi/2$, so (14.14) holds.

Now suppose that (14.13) fails. Then there is a sequence of sets E_k in $\mathcal{Z}_0$, that are neither planes nor sets of type $\mathbb{Y}$, and for which $a_k = H^2(E_k \cap B(0,1))$ tends to $3\pi/2$. We can replace $\{E_k\}$ by a subsequence that converges to some limit E. Obviously, E is a cone, and Lemma 4.7 (with $h = 0$) says that $E \in \mathcal{Z}_0$. In addition, $H^2(E \cap B(0,1)) = 3\pi/2$ by Lemmas 3.3 and 3.12, so E is a set of type $\mathbb{Y}$ by (14.14).

Denote by H_k the set of extremities of the arcs of great circles in E_k. By Proposition 14.1 or (14.10), the points of H_k lie at distance at least $\eta_1 = \mathrm{Min}(l_0, \eta_0)$ from each other. If H_k has more than 2 points and k is large enough, then some $y_k \in H_k$ lies at distance at least $\eta_1/2$ from the two singular points of $K = E \cap \partial B(0,1)$, and K is composed of at most an arc of great circle in $B(y_k, \eta_1/3)$. At the same time, Proposition 14.1 says that $E_k \cap B(y_k, \eta_1/3)$ is very close, in Hausdorff distance, to a Y-set. This is impossible, because $\{E_k\}$ converges to E.

So H_k has at most two points for k large. If it has exactly two points, E_k contains a set of type $\mathbb{Y}$ by the proof of (14.11), and since it is not a set of type $\mathbb{Y}$, Proposition 14.1 says that it needs to contain at least another arc of circle of length at least l_0. Thus $H^2(E_k \cap B(0,1)) \geqslant 3\pi/2 + l_0/2$, which does not tend to $3\pi/2$. If H_k is empty, E_k is composed of at least two great circles, and $H^2(E_k \cap B(0,1)) \geqslant 2\pi$. In every case we reach a contradiction, which proves Lemma 4.12. $\qquad\square$

It is believable that we can take $d_T = 3\mathrm{Argcos}(-1/3) \approx 1.82 \cdot \pi$ in Lemma 4.12, which corresponds to the set T based on a tetrahedron in $\mathbb{R}^3$ (see Section 20), but I did not try to check.

D. LOCAL REGULARITY OF 2-DIMENSIONAL ALMOST-MINIMAL SETS

We want to generalize J. Taylor's theorem and give a local biHölder description of E when E is a reduced almost-minimal set of dimension 2 in $\mathbb{R}^n$, with sufficiently small gauge function.

The proof will rely on a generalization of Reifenberg's Topological Disk Theorem stated in the next section, together with the approximation results of Part B and a small argument with a topological degree. Most of the local regularity results are stated and proved in Section 16, except for a topological argument which is left for Section 17. In Section 18 we prove a regularity result for almost-minimal sets of dimension 2 in $\mathbb{R}^3$, which is valid in balls where E is close to a set of type $\mathbb{T}$, but under an additional separation assumption. This is enough to establish Theorem 1.9 (i.e., the fact that every reduced MS-minimal set of dimension 2 in $\mathbb{R}^3$ is a minimal cone). Section 19 contains the description of a set $E \subset \mathbb{R}^3$ that looks like a set of type $\mathbb{T}$ at large scales, but does not have the same topology. We rapidly compute the density of the set of type $\mathbb{T}$ of Figure 1.2 in Section 20.

15. An extension of Reifenberg's Topological Disk Theorem

Our statement of the desired extension of Reifenberg's theorem will use some amount of notation. We shall consider sets of dimension $d \geqslant 2$. The most important case for us is when $d = 2$; we could also take $d = 2$, forget about general sets of type $\mathbb{T}$, and get a statement as in Section 12, but let us not do this. We shall allow $d > 2$, but we shall keep a collection of cones that are just obtained as products of a 2-dimensional cone with an orthogonal $(d-2)$-dimensional plane.

Let us describe the collection of cones that will be used to approximate our set E. We start with the collection of d-dimensional (affine) planes, which we call $\mathcal{P}$.

Next we define the collection $\mathcal{Y}$ of sets of type $\mathbb{Y}$. We start with propellers in a plane, which we called Y-sets in the previous sections, and which are just unions of three half lines with a same endpoint and that make $120°$ angles at that point. In [DDT], we even allow sets Y where the three half lines are only required to make angles at least $\pi/2$, but we won't need that generality here. We obtain a first set of type $\mathbb{Y}$ as the product $\mathbb{Y}_0 = Y \times V$, where Y is a Y-set centered at the origin, and V is a $(d-1)$-dimensional vector space that is orthogonal to the plane that contains Y. We shall call V the spine of $\mathbb{Y}_0$. Finally, $\mathcal{Y}$ is the collection of sets $\mathbb{Y}$ of the form $\mathbb{Y} = j(\mathbb{Y}_0)$, where j is an isometry of $\mathbb{R}^n$. Then the spine of $\mathbb{Y}$ is the image by j of the spine of $\mathbb{Y}_0$.

The description of the collection $\mathcal{T}$ of sets of type $\mathbb{T}$ (the sets of type TG3 in [DDT]) will take a little longer. Our set $\mathcal{T}$ will be the collection of sets $T = j(T_0 \times V)$, where T_0 lies in a set $\mathcal{T}_0$ of 2-dimensional cones in $\mathbb{R}^{n-d+2}$, V is the plane of dimension $d-2$ orthogonal to $\mathbb{R}^{n-d+2}$ in $\mathbb{R}^n$, and j is an isometry of $\mathbb{R}^n$.

Each $T \in \mathcal{T}_0$ will be the cone over a set $K \subset \partial B(0,1)$, with the following properties. First, $K = \cup_{j \in J} C_j$ is a finite union of great circles, or closed arcs of great circles. Denote by H the collection of extremities of arcs C_j, $j \subset J$; each point x of H lies in exactly three C_j, x is an endpoint for each such C_j, and the three C_j make $120°$ angles at x. The C_j can only meet at their endpoints (and hence the full arcs of circles are disjoint from the rest of K). In addition, there is a constant $\eta_0 > 0$ such that

$$H^1(C_j) \geqslant \eta_0 \ \text{ for } j \in J, \tag{15.1}$$

and

if $x \in C_i$, and $\text{dist}(x, C_j) \leqslant \eta_0$ for some other j, then

C_i and C_j have a common extremity in $B(x, \text{dist}(x, C_j))$. $\qquad(15.2)$

In addition, we exclude the case when T is a plane or a set of type $\mathbb{Y}$.

When $Y \in \mathcal{T}_0$, the spine of E is the cone over H or, if K is uniquely composed of great circles, the origin. In general, when $T = j(T_0 \times V)$, the spine of T is $j(S \times V)$ where S is the spine of T_0.

The definition of sets of type $TG3$ given in [DDT] was a little different. We also allowed angles larger than $\pi/2$ instead of exactly $2\pi/3$, but we don't need this here. We also decided to take the full collection of sets K as above, but our statement also allows a subcollection (both in the assumptions and the conclusion). See the last sentence of the statement of Theorem 2.2 in [DDT].

Before we start, let us record the fact that the class $\mathcal{T}_0$ of reduced minimal cones of dimension 2 in $\mathbb{R}^{n-d+2}$ that are neither planes nor sets of type $\mathbb{Y}$ satisfies the requirements above, with a constant η_0 that depends only on m. This is a consequence of Proposition 4.1, except for (15.2) which was stated separately in (14.10).

Let $\mathcal{P}$, $\mathcal{Y}$, and some choice of $\mathcal{T}$ be as above. Set $\mathcal{Z} = \mathcal{P} \cup \mathcal{Y} \cup \mathcal{T}$, For the statement below, we are given a closed set $E \subset \mathbb{R}^n$, and we assume for simplicity that the origin lies in E. As in Section 12, we measure how close E is to sets of $\mathcal{Z}$ with

$$\beta_Z(x, r) = \inf_{Z \in \mathcal{Z}} d_{x,r}(E, Z), \tag{15.3}$$

where $d_{x,r}(E, Z)$ is defined as in (12.2). (Compare with (12.1).) We assume that

$$\beta_Z(x, r) \leqslant \varepsilon \ \text{ for } x \in E \cap B(0, 3) \text{ and } 0 < r \leqslant 3, \tag{15.4}$$

where $\varepsilon \geqslant 0$ is small, depending on n, d, and the constant η_0. This means that for $x \in E \cap B(0,3)$ and $0 < r \leqslant 3$, we can choose $Z(x,r) \in \mathcal{Z}$ such that $d_{x,r}(E, Z(x,r)) \leqslant \varepsilon$. We want a bilipschitz parameterization of E by a set of $\mathcal{Z}$ near the unit ball.

THEOREM 15.5 [DDT]. — *Let $\mathcal{Z}$ be as above, and let $\tau > 0$ be given. Then there is a constant $\varepsilon > 0$ that depends only on n, d, η_0, and τ, such that if $E \subset \mathbb{R}^n$ is a closed set that contains the origin and if (15.4) holds, we can find $Z \in \mathcal{Z}$ and a a biHölder mapping $f : B(0,2) \to \mathbb{R}^n$, such that*

$$|f(x) - x| \leqslant \tau \ \ for \ x \in B(0,2), \tag{15.6}$$

$$(1 - \tau)|x - y|^{1+\tau} \leqslant |f(x) - f(y)| \leqslant (1 + \tau)|x - y|^{1-\tau}$$
$$for \ x, y \in B(0,2), \tag{15.7}$$

$$B(0, 2 - \tau) \subset f(B(0,2)), \tag{15.8}$$

and

$$E \cap B(0, 2 - \tau) \subset f(Z \cap B(0,2)) \subset E. \tag{15.9}$$

Compared to [DDT], we just changed a little some of the constants (such as 2 and 3), but this does not matter.

It will be useful for us that in addition to giving a parameterization of E by Z near 0, f extends to a local homeomorphism of $\mathbb{R}^n$. In particular, we know that the topology of $\mathbb{R}^n \setminus E$ near 0 is the same as for $\mathbb{R}^n \setminus Z$.

We can say a little more on the choice of Z. First observe that Z has to be fairly close to $Z(0,3)$ in Hausdorff distance, because (15.6) and (15.9) say that it is fairly close to E.

If $Z(0,3)$ is a plane, or coincides with a plane near $B(0, 29/10)$, say, we can show that for $x \in E \cap B(0, 27/10)$ and $r \leqslant 1/10$, say, $Z(x,r)$ is a plane, or at least its spine does not get close to x. Then we are in the standard Reifenberg situation, and we can take $Z = Z(0,3) \in \mathcal{P}$.

Similarly, if $Z(0,3)$ is a set of type $\mathbb{Y}$ with a spine through the origin, we can take $Z = Z(0,3) \in \mathcal{Y}$. Slightly more generally (but we won't need that case in the present paper), if $Z(0,3) \in \mathcal{Y}$ or coincides with a set of type $\mathbb{Y}$ in $B(0, 29/10)$, we can show that for $x \in E \cap B(0, 27/10)$ and $r \leqslant 1/10$, $Z(x,r)$ is never a set of type $\mathbb{T}$ centered near x. Then we can take $Z \in \mathcal{Y}$, except perhaps in the situation above where it is more clever to take a plane.

If $Z(0,3)$ is a set of type $\mathbb{T}$ centered at the origin, then again we can take $Z = Z(0,3)$. More generally, if $Z(0,3) \in \mathcal{T}$, with a center in $B(0, 25/10)$, we

can first show (by checking things at smaller and smaller scales) that there is a single point $x_0 \in B(26/10)$ such that every $Z(x_0, r)$ is a set of type T centered in $B(x_0, r/10)$. Then we use $Z = Z(0,3)$ (or a translation of it if we want f to map the center of Z to x_0).

If our class $\mathcal{T}$ (modulo identification of isometric sets) is not discrete, it could be that when r varies, the set $Z(x_0, r)$ changes. This is all right, it just means that some other sets $Z' \in \mathcal{T}$ could be used instead of Z (to parameterize E), and they are all biHölder equivalent to each other. This is not surprising, since $Z(x_0, r)$ is only allowed to vary very slowly with r (because $Z(x_0, r)$ stays so close to E). The situation would be different if we wanted C^1 parameterizations. Then we would probably need bounds on how fast the sets $Z(x_0, r)$ vary and (in the case of minimal sets), it can be expected that these bounds will be harder to prove than when $d = 2$ and $n = 3$.

The remaining case when the center of $Z(0,3)$ lies out of $B(0, 25/10)$, but $Z(0,3)$ does not coincide with a set of type $\mathbb{Y}$ in $B(0, 29/10)$, could be treated similarly, or just eliminated by replacing 3 with a larger constant. We shall not need it here anyway.

See [DDT] for a proof of Theorem 15.5. The proof is a little ugly because there are more cases to study, but we essentially use Reifenberg's scheme, plus an organization of the construction in layers (first find the center if there is one, then find the image of the spine, then define f on Z, and finally extend to the rest of $\mathbb{R}^n$).

There is no immediate obstruction to having an additional layer in the complexity of our sets, but the situation is already complicated enough, and we were lucky that, just because there is at most one center x_0, we did not have to understand what happens when the isometry class of $Z(x_0, r)$ changes with r.

We shall get our local descriptions of the almost-minimal sets by brutal applications of Theorem 15.5 in the case when $Z(0,3)$ is a cone centered at 0. The following definition will make it more convenient.

DEFINITION 15.10. — *Let E be a closed set in $\mathbb{R}^n$. We way that $B(0,1)$ is a biHölder ball for E, with constant $\tau \in (0,1)$, if we can find a cone $Z \in \mathcal{Z}$ centered at 0, and $f : B(0,2) \to \mathbb{R}^n$, with the properties (15.6)-(15.9). We say that $B(0,1)$ is of type $\mathbb{P}$, $\mathbb{Y}$, or $\mathbb{T}$, depending on whether Z lies in $\mathcal{P}$, $\mathcal{Y}$, or $\mathcal{T}$. Finally, we say that $B(x,r)$ is a biHölder ball for E (with the same parameters) when $B(0,1)$ is a biHölder ball for $r^{-1}(E - x)$.*

With this definition, we have the following consequence of Theorem 15.5 and the discussion above. We keep the same assumptions on $\mathcal{Z}$.

COROLLARY 15.11. — *For each small $\tau > 0$, we can find $\varepsilon > 0$, that depends only on n, d, η_0, and τ, such that if $E \subset \mathbb{R}^n$ is a closed set, $x \in E$, and $r > 0$ are such that*

$$\text{for } y \in E \cap B(x, 3r) \text{ and } 0 < t \leqslant 3r, \text{ we can find}$$
$$Z(y, t) \in \mathcal{Z} \text{ such that } d_{y,t}(E, Z(y, t)) \leqslant \varepsilon, \tag{15.12}$$

and in addition $Z(x, 3r)$ is a cone centered at x, then $B(x, r)$ is a biHölder ball for E, with constant τ, of the same type as $Z(x, 3r)$, and we can even take Z isometric to $Z(x, 3r)$ in Definition 15.10.

16. A local regularity theorem for two-dimensional almost-minimal sets in $\mathbb{R}^n$

We are now ready to state a generalization to higher dimensions of the biHölder part of Jean Taylor's local regularity theorem [Tay].

THEOREM 16.1. — *Let U be an open set in $\mathbb{R}^n$ and E a closed subset of U. Suppose that E is a reduced A-almost-minimal set in U, with a gauge function h such that*

$$\int_0^{r_1} h(r) \frac{dr}{r} < +\infty \quad \text{for some } r_1 > 0. \tag{16.2}$$

Then for each $x_0 \in E$ and every choice of $\tau \in (0, 1)$, there is an $r_0 > 0$ such that $B(x_0, r_0)$ is a biHölder ball for E, with constant τ.

See Definition 4.3 for A-almost-minimal sets. Notice that we could equivalently have assumed that E is a reduced A'-almost-minimal set with the same gauge function h, as in Definition 4.8, because Proposition 4.10 says that the two definitions are equivalent. Recall that we say that E is reduced when $E^* = E$, where E^* is the closed support of the restriction of H^2 to E; see Definition 2.12.

The notion of biHölder ball is defined in Definition 15.10, and here we take for $\mathcal{Z}$ the collection of two-dimensional reduced minimal cones in $\mathbb{R}^n$, as described in Section 14. Thus, in short, Theorem 16.1 says that every point of E has a small neighborhood where E is biHölder equivalent to a minimal cone of dimension 2.

Jean Taylor [Tay] also gives the C^1 equivalence when $n = 3$, under conditions on h that are stronger than (16.2) (but it is not clear that she

tried to optimize this). We shall see in [D4] how to deduce this from Theorem 16.1 (or some weaker separation result). It is not clear to the author that this more precise result will generalize almost-minimal sets of dimension 2 in $\mathbb{R}^n$, $n \geqslant 4$; the answer may depend on which type of blow-up limits for E at x exist.

Curiously, the condition (16.2) is only used to prove control the density, as in (16.6) at the beginning of the argument; I don't know whether it is sharp.

It should be observed that neither here nor in [Tay] (I think) do we get any precise lower bound for r_0. The proof will say that for each $x \in E$, the density $\theta(x,r)$ eventually gets very close to its limit at $r = 0$, and then $B(x,r)$ is a biHölder ball for E, but we do not know in advance when this happens. Some of our intermediate results, such as Lemma 16.19 and 16.48, Proposition 16.24, Lemmas 16.25, 16.51, and 16.56, and later on Proposition 18.1, are more quantitative, though.

We shall prove Theorem 16.1 now, except for Proposition 16.24, which uses a little bit of degree theory and will be treated in Section 17. We shall systematically assume in this section that

$$E \text{ is a reduced } A\text{-almost-minimal set in } U,$$
$$\text{with a gauge function } h \text{ such that (16.2) holds.} \tag{16.3}$$

Set

$$\theta(x,r) = r^{-2} H^2(E \cap B(x,r)) \tag{16.4}$$

for $x \in E$ and $r > 0$. Set

$$A(r) = \int_0^r h(2t)\frac{dt}{t}. \tag{16.5}$$

By (16.2), we can apply Proposition 5.24 and get that

$$\theta(x,r)e^{\lambda A(r)} \text{ is a nondecreasing function of } r, \tag{16.6}$$

as long as $B(x,r) \subset U$ and $h(r)$ stays small enough, and where λ is a positive constant.

Notice that $A(r)$ tends to 0 when r tends to 0 (by (16.2)), so (16.6) allows us to define

$$\theta(x) = \lim_{r \to 0} \theta(x,r) \tag{16.7}$$

for $x \in E$. As we shall see soon, there are restrictions on the values that $\theta(x)$ can take.

Proposition 7.31 says that every blow-up limit of E at x is a reduced minimal cone, whose density is $\theta(x)$.

If $\theta(x) = \pi$, all the blow up limits are planes, and we shall call x a P-point. If $\theta(x) = 3\pi/2$, all the blow up limits are minimal cones of type $\mathbb{Y}$, and we shall call x a Y-point. Otherwise, we shall say that x is a T-point. In this case do not know whether the various blow-up limits of E at x are all isometric to each other, but they all have the same density $\theta(x)$. Lemma 14.12 then says that their density is at least d_T (where $d_T > 3\pi/2$ is a constant that may depend on n), so

$$\theta(x) \geqslant d_T \quad \text{when } x \text{ is a } T\text{-point.} \tag{16.8}$$

When $n = 3$, there is only one type of minimal cone of type $\mathbb{T}$ (i.e., which is not a plane or a set of type $\mathbb{Y}$), which is the one introduced in Remark 1.8 and Theorem 1.9. Then $\theta(y) = 3\mathrm{Argcos}(-1/3) \approx 1.82 \cdot \pi$ when x is a T-point. See the computation in Section 20. In higher dimensions, there may be lots of minimal cones with different densities $\geqslant d_T$. But this will not matter much here, because the T-points are isolated.

We start with a simple consequence of (16.6) that will be used repeatedly. We claim that

$$\theta(x) \leqslant e^{\lambda A(r)} \theta(x,r) \tag{16.9}$$

for $x \in E$ and $r > 0$ such that $B(x,r) \subset U$ and $h(r)$ is small enough. Indeed notice that for $0 < s \leqslant r$,

$$\theta(x,s) \leqslant e^{\lambda A(s)} \theta(x,s) \leqslant e^{\lambda A(r)} \theta(x,r); \tag{16.10}$$

we let s tend to 0 and get the result.

The following consequence of Proposition 7.24 will also be used quite often.

LEMMA 16.11. — *For each $\tau > 0$, we can find $\varepsilon(\tau) > 0$ such that if $x \in E$ and $r > 0$ are such that*

$$B(x,r) \subset U, \quad h(2r) \leqslant \varepsilon(\tau), \quad \int_0^{2r} h(r)\frac{dr}{r} < \varepsilon(\tau), \tag{16.12}$$

and

$$\theta(x,r) \leqslant \theta(x) + \varepsilon(\tau), \tag{16.13}$$

then for every $\rho \in (0, 9r/10]$ there is a reduced minimal cone $Z(x,\rho)$ centered at x, such that

$$d_{x,\rho}(E, Z(x,\rho)) \leqslant \tau \tag{16.14}$$

and

$$\left|H^2(E \cap B(y,t)) - H^2(Z(x,\rho) \cap B(y,t))\right| \leqslant \tau\rho^2$$
$$\textit{for } y \in \mathbb{R}^n \textit{ and } t > 0 \textit{ such that } B(y,t) \subset B(x,\rho).$$

(16.15)

If $\theta(x) = \pi$, $Z(x,\rho)$ *is a plane; if* $\theta(x) = 3\pi/2$, $Z(x,\rho)$ *is a set of type* $\mathbb{Y}$, *and if* $\theta(x) > 3\pi/2$, $Z(x,\rho)$ *is a set of type* $\mathbb{T}$.

Let $\rho \in (0, 9r/10]$ be given, and let us try to apply Proposition 7.24, with the constant $\tau' = \tau/2$, to the ball $B(x, 10\rho/9)$. The main assumption is (7.25), which holds because

$$\begin{aligned}
\theta(x, 10\rho/9) &\leqslant e^{\lambda A(10\rho/9)}\,\theta(x, 10\rho/9) \\
&\leqslant \theta(x,r)e^{\lambda A(r)} \leqslant \theta(x,r)e^{\lambda\varepsilon(\tau)} \\
&\leqslant [\theta(x) + \varepsilon(\tau)]\,e^{\lambda\varepsilon(\tau)} \leqslant \theta(x) + C\varepsilon(\tau) \\
&\leqslant e^{\lambda A(t)}\,\theta(x,t) + C\varepsilon(\tau) \leqslant e^{\lambda\varepsilon(\tau)}\,\theta(x,t) + C\varepsilon(\tau) \\
&\leqslant \theta(x,t) + C\varepsilon(\tau) \leqslant \theta(x,t) + \tau'
\end{aligned}$$

(16.16)

for $t < \rho$, by (16.6), (16.5), (16.12), (16.13), (16.9), and (16.12), and if $\varepsilon(\tau)$ is small enough.

The other assumptions are satisfied because of (16.12), so Proposition 7.24 applies and gives a reduced minimal cone $\mathcal{C}$ with the properties (7.26)-(7.28). We take $Z(x,\rho) = \mathcal{C}$, (16.14) follows from (7.26) and (7.27), and (16.15) follows from (7.28), and because we may assume that $(1-\tau') > 9/10$.

We still need to show that the type of $Z(x,\rho)$ is determined by $\theta(x)$. Denote by $D = H^2(Z(x,\rho) \cap B(x,1))$ the density of $Z(x,\rho)$. Then

$$\left|D - \theta(x,\rho)\right| = \rho^{-2}\left|H^2(\mathcal{C} \cap B(x,\rho)) - H^2(E \cap B(x,\rho))\right| \leqslant \tau \quad (16.17)$$

by (16.15), applied with $B(y,t) = B(\rho,r)$.

But $\theta(x,\rho) \leqslant \theta(x) + C\varepsilon(\tau)$ by the first two lines of (16.16) (with $10\rho/9$ replaced with ρ) and $\theta(x) \leqslant e^{\lambda A(\rho)}\theta(x,\rho) \leqslant \theta(x,\rho) + C\varepsilon(\tau)$ by (16.9) and (16.2), we get that $|\theta(x,\rho) - \theta(x)| \leqslant C\varepsilon(\tau)$, and then

$$|D - \theta(x)| \leqslant |D - \theta(x,\rho)| + |\theta(x,r) - \theta(x)| \leqslant \tau + C\varepsilon(\tau). \quad (16.18)$$

So $|D-\theta(x)|$ is as small as we want. But $Z(x,\rho)$ is a plane as soon as $D < \pi$; by Lemma 14.12, it is a set of type $\mathbb{Y}$ if $\pi < D < d_T$, and it is of type $\mathbb{T}$ if $D > 3\pi/2$; Lemma 16.11 follows. $\qquad\square$

The following lemma will allow us to deal with P-points in Theorem 16.1

LEMMA 16.19. — *For each choice of $\tau \in (0,1)$ there is a constant $\varepsilon_1 > 0$ such that if $x_0 \in E$ and $r_0 > 0$ are such that $B(x_0, 9r_0) \subset U$, $h(18r_0) \leqslant \varepsilon_1$,*

$$\int_0^{18r_0} h(t) \frac{dt}{t} \leqslant \varepsilon_1, \quad and \quad \theta(x_0, 9r_0) \leqslant \pi + \varepsilon_1, \tag{16.20}$$

then $B(x, r)$ is a biHölder ball of type P for E, with constant τ, for every choice of $x \in E \cap B(x_0, r_0)$ and $0 < r \leqslant r_0$.

Most of the time we shall use this with $x = x_0$, but we shall need the additional uniformity provided by the present statement in Section 17.

Let x_0, r_0, x, and r be as in the statement. We want to apply Corollary 15.11 to the pair (x, r). Notice however that here all the sets $Z(y, t)$ in (15.12) will be planes, so Corollary 15.11 comes directly from Reifenberg's original theorem.

So we need to show that (15.12) holds for the pair (x, r). That is, for each $y \in E \cap B(x, 3r)$ and $0 < t \leqslant 3r$, we need to find a plane $Z(y, t)$ such that $d_{y,t}(E, Z(y, t)) \leqslant \varepsilon$, where ε comes from Corollary 15.11 and depends on τ. For the rest of the argument, we shall only need to know that $y \in E \cap B(x_0, 4r_0)$ and $0 < t \leqslant 3r_0$ (which is true because $x \in E \cap B(x_0, r_0)$ and $0 < r \leqslant r_0$).

Let τ_1 be small (much smaller than τ, and to be chosen soon), and apply Lemma 16.11 to $B(x_0, 9r_0)$. The assumptions are satisfied if $\varepsilon_1 \leqslant \varepsilon(\tau_1)$, and we get that for $0 < \rho \leqslant 8r_0$, there is a plane $Z(x_0, \rho)$ through x_0, with the properties (16.14) and (16.15). We are only interested in $P = Z(x_0, 8r_0)$, and (16.15) which says that

$$\left| H^2(E \cap B(y, t)) - H^2(P \cap B(y, t)) \right| \leqslant 64\tau_1 r_0^2 \tag{16.21}$$

for $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y, t) \subset B(x_0, 8r_0)$.

Take $y \in E \cap B(x_0, 4r_0)$ as above, and $t = 4r_0$. We get that

$$\begin{aligned}
\theta(y, 4r_0) &= (4r_0)^{-2} H^2(E \cap B(y, 4r_0)) \\
&\leqslant (4r_0)^{-2} \left[H^2(P \cap B(y, 4r_0)) + 64\tau_1 r_0^2 \right] \tag{16.22} \\
&\leqslant \pi + 4\tau_1
\end{aligned}$$

by (16.21) and because P is a plane.

Notice that $\theta(y) \geqslant \pi$ because $y \in E$ and E is reduced (see the discussion below (16.7)). On the other hand,

$$\begin{aligned}
\theta(y) &\leqslant e^{\lambda A(4r_0)} \theta(y, 4r_0) \leqslant e^{\lambda \varepsilon_1} \theta(y, 4r_0) \\
&\leqslant e^{\lambda \varepsilon_1} (\pi + 4\tau_1) \leqslant \pi + 4\tau_1 + C\varepsilon_1 < 3\pi/2
\end{aligned} \tag{16.23}$$

by (16.9) and (16.20), (16.22), and if τ_1 and ε_1 are small enough. Hence $\theta(y) = \pi$.

Apply Lemma 16.11 to the ball $B(y, 4r_0)$ and with $\tau = \varepsilon$, where ε comes from Corollary 15.11 as above. The hypotheses are satisfied, by (16.19) and (16.22) (and if ε_1 and τ_1 are small enough). We obtain that for $0 < t < 36r_0/10$, there is a plane $Z(y, t)$ through y such that (16.15) holds. This means that $d_{y,t}(E, Z(y, t)) \leqslant \varepsilon$, and this is exactly what was needed to apply Corollary 15.11. This completes our proof of Lemma 16.19. $\qquad\square$

Lemma 16.19 gives the local regularity of E near every P-point (and for this we may even take $x = x_0$). Next we want to take care of the Y-points. The proof above does not work directly in this case, because we used once or twice the fact that the current density was close to π, and π is the smallest possible density at a point of E. Thus it will be good to know that when E is close to a set of type $\mathbb{Y}$, we have Y-points around to which we shall be able to apply Lemma 16.11 because their density is not too small. This is the point of the next proposition.

PROPOSITION 16.24. — *There is constant $\varepsilon_2 > 0$, that depends only on n, such that if (16.3) holds, $B(x, r) \subset U$, $h(2r) \leqslant \varepsilon_2$, $\int_0^{2r} h(t)dt/t \leqslant \varepsilon_2$, and there is a reduced minimal cone Y of type $\mathbb{Y}$ centered at x and such that $d_{x,r}(E, Y) \leqslant \varepsilon_2$, then $E \cap B(x, r/100)$ contains (at least) a Y-point.*

The proof of Proposition 16.24 relies on Lemma 16.11 (which we use to get the local regularity of E if there is no Y-point), and a topological argument that counts the points of intersection of E with some $(n-2)$-dimensional spheres. We shall do the proof in Section 17, and in the mean time we proceed with the regularity of E near a Y-point.

LEMMA 16.25. — *For each choice of $\tau \in (0, 1)$, there is a constant $\varepsilon_3 > 0$ such that if $x_0 \in E$ is a Y-point, $r_0 > 0$,*

$$B(x_0, 50r_0) \subset U, \quad h(100r_0) \leqslant \varepsilon_3, \quad \int_0^{100r_0} h(t)\frac{dt}{t} \leqslant \varepsilon_3, \qquad (16.26)$$

and

$$\theta(x_0, 50r_0) \leqslant 3\pi/2 + \varepsilon_3, \qquad (16.27)$$

then $B(x_0, r_0)$ is a biHölder ball of type $\mathbb{Y}$ for E, with constant τ.

It is enough to prove the lemma when $x_0 = 0$ and $r_0 = 1$, because it is invariant under rotations and dilations. Let τ_2 be small, to be chosen later, and apply Lemma 16.11 to the ball $B(0, 50)$; the assumptions (16.12) and (16.13) are satisfied if ε_3 is small enough. We get that for $0 < \rho \leqslant 45$, there

is a reduced minimal cone $Y(\rho) = Z(0, \rho)$ of type $\mathbb{Y}$, centered at 0, such that (16.14) and (16.15) hold. That is,

$$d_{0,\rho}(E, Y(\rho)) \leqslant \tau_2 \tag{16.28}$$

and

$$\left| H^2(E \cap B(y, t)) - H^2(Y(\rho) \cap B(y, t)) \right| \leqslant \tau_2 \rho^2$$
$$\text{for } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y, t) \subset B(0, \rho). \tag{16.29}$$

Let us check that

$$\theta(y, 10) \leqslant 3\pi/2 + C\tau_2 \quad \text{for } y \in E \cap B(0, 35). \tag{16.30}$$

Set $Y = Y(45)$, and observe that $H^2(Y \cap B(y, t)) \leqslant 3\pi r^2/2$ for all y and t because the density is largest for balls centered on the spine of Y, for instance by Proposition 5.16. Then apply (16.29) with $\rho = 45$ and $t = 10$ (so that $B(y, t) \subset B(0, 45)$). We get that

$$\theta(y, 10) = 10^{-2} H^2(E \cap B(y, t))$$
$$\leqslant 10^{-2} H^2(Y \cap B(y, t)) + 10^{-2} \tau_2 \rho^2 \tag{16.31}$$
$$\leqslant 3\pi/2 + C\tau_2,$$

as needed for (16.30). In addition, (16.9) says that

$$\theta(y) \leqslant e^{\lambda A(10)} \theta(y, 10) \leqslant e^{\lambda \varepsilon_3} \theta(y, 10) \leqslant \theta(y, 10) + C\varepsilon_3 \tag{16.32}$$

by (16.5) and (16.26), so (16.31) says that $\theta(y) \leqslant 3\pi/2 + C\tau_2 + C\varepsilon_3 < d_T$ if τ_2 and ε_3 are small enough. By (16.8), y cannot be a T-point. So we just checked that

$$\text{there is no } T\text{-point in } E \cap B(0, 35). \tag{16.33}$$

We shall need some control on

$$E_Y = \left\{ x \in E \,;\, x \text{ is a } Y\text{-point of } E \right\} = \left\{ x \in E \,;\, \theta(x) = \frac{3\pi}{2} \right\}. \tag{16.34}$$

Let τ_3 be small, to be chosen soon, and let us apply Lemma 16.11 with $\tau = \tau_3$, and to the ball $B(y, 10)$, where we now assume that $y \in E_Y \cap B(0, 35)$. If ε_3 and τ_2 are small enough, the assumption (16.12) follows from (16.26), and (16.13) follows from (16.31) because $\theta(y) = 3\pi/2$ when $y \in E_Y$. We get that for

$$y \in E_Y \cap B(0, 35) \quad \text{and} \quad 0 < \rho \leqslant 9, \tag{16.35}$$

there is a reduced minimal cone $Y(y, \rho)$ of type $\mathbb{Y}$, centered at y, such that

$$d_{y,\rho}(E, Y(y, \rho)) \leqslant \tau_3 \qquad (16.36)$$

and

$$\left| H^2(E \cap B(x, t)) - H^2(Y(y, \rho) \cap B(x, t)) \right| \leqslant \tau_3 \rho^2$$
$$\text{for } x \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(x, t) \subset B(y, \rho). \qquad (16.37)$$

Notice that (16.36) gives the sort of control on $E \cap B(y, \rho)$ that we need in order to apply Corollary 15.11, but only in balls $B(y, \rho)$ that are centered on E_Y. We also need to control the balls that are centered on P-points.

Let $x \in E \cap B(0, 3) \setminus E_Y$ be given, and set

$$d(x) = \text{dist}(x, E_Y). \qquad (16.38)$$

Observe that $d(x) \leqslant 3$, because the origin lies in E_Y by assumption. Also, $d(x) > 0$, because the proof of Lemma 16.19 gives a neighborhood of x which contains no Y-point (see below (16.23)). We do not even need to know this, because if $d(x) = 0$ we can use the cones $Y(y, \rho)$ to control E in the balls centered at x, so we do not need the construction below. See the small argument above (16.42).

Pick $y \in E_Y$ such that $|y - x| \leqslant 11d(x)/10$, and set $\rho = 2d(x)$. Then $y \in E_Y \cap B(0, 10)$ and $\rho \leqslant 6$, so (16.35) holds and we can set $Y = Y(y, \rho)$, where $Y(y, \rho)$ is as in (16.36) and (16.37). Denote by L the spine of Y; that is, L is the common boundary of the three half planes that compose Y. It is a line through y. Let us check that

$$\text{dist}(x, L) \geqslant 3d(x)/4. \qquad (16.39)$$

Suppose that (16.39) fails. Choose $z \in L$ so that $|z - x| \leqslant 3d(x)/4$, and apply Proposition 16.24 to the ball $B(z, d(x)/10)$. We need to check the assumption that $d_{z,d(x)/10}(E, Y) \leqslant \varepsilon_2$. Indeed, $B(z, d(x)/10) \subset B(y, \rho)$ because $|z - y| \leqslant |z - x| + |x - y| \leqslant 3d(x)/4 + 11d(x)/10$ and $\rho = 2d(x)$. Then $d_{z,d(x)/10}(E, Y) \leqslant 10d(x)^{-1}\rho d_{y,\rho}(E, Y) \leqslant 20\tau_3 < \varepsilon_2$ if τ_3 is small enough, as needed.

So Proposition 16.24 applies, and there is a point of $\xi \in E_Y$ inside $B(z, 10^{-3}d(x))$. But then $|\xi - x| \leqslant 10^{-3}d(x) + |z - x| \leqslant [10^{-3} + 3/4]\,d(x) < d(x)$. This is incompatible with the definition of $d(x)$, and so (16.39) holds.

Notice that $B(x, d(x)/2)$ is contained in $B(y, \rho)$ (because $|y - x| \leqslant 11d(x)/10$ and $\rho = 2d(x)$), so (16.37) says that

$$
\begin{aligned}
\theta(x, d(x)/2) &= (d(x)/2)^{-2} H^2(E \cap B(x, d(x)/2)) \\
&\leqslant (d(x)/2)^{-2} H^2(Y \cap B(x, d(x)/2)) + \tau_3 (d(x)/2)^{-2} \rho^2 \quad (16.40) \\
&\leqslant \pi + 4\tau_3 d(x)^{-2} \rho^2 \leqslant \pi + 16\tau_3,
\end{aligned}
$$

because (16.39) says that Y coincides with a plane in $B(x, d(x)/2)$. Recall that $\theta(x) = \pi$ because we assumed that $x \in E \setminus E_Y$ (and by (16.33)). Hence, if τ_3 is small enough, (16.40) allows us to apply Lemma 16.11 with $\tau = \varepsilon$, where ε comes from Corollary 15.11. We get that for $0 < r < d(x)/3$, there is a plane $Z(x, r)$ through x such that $d_{x,r}(E, Z(x, r)) \leqslant \varepsilon$, as in (16.14).

We are now ready to deduce Lemma 16.25 from Corollary 15.11. We want to get the constant τ in the conclusion of Lemma 16.25, so we need to prove (15.12), with a constant ε that depends on τ. For each choice of $x \in E \cap B(0, 3)$ and $0 \leqslant r \leqslant 3$, we need to find a minimal cone $Z(x, r)$ such that

$$
d_{x,r}(E, Z(x, r)) \leqslant \varepsilon. \tag{16.41}
$$

When $x \in E_Y$, the pair (x, r) satisfies (16.35), so we can use $Z(x, r) = Y(x, r)$, and (16.41) follows from (16.36) if $\tau_3 < \varepsilon$. Note that $Z(0, 3)$ is a set of type $\mathbb{Y}$ centered at the origin, as required by the last assumption of Corollary 15.11.

When $x \in E \cap B(0, 3) \setminus E_Y$ and $r < d(x)/3$, we can use the set $Z(x, r)$ that was found a few lines ago, below (16.40).

We are left with the case when $x \in E \cap B(0, 3) \setminus E_Y$ and $d(x)/3 \leqslant r \leqslant 3$. Our argument also applies when $d(x) = 0$, if we pretend that we don't know that $x \in E_Y$ when $d(x) = 0$. Let $y \in E_Y$ be such that $|y - x| \leqslant d(x) + r$, and set $\rho = 2r + d(x)$. Observe that $|y| \leqslant |x| + d(x) + r \leqslant 9$ because $x \in B(0, 3)$, $d(x) \leqslant |x|$, and $r \leqslant 3$, and also $\rho \leqslant 9$, so (16.35) holds, and we can set $Z(x, r) = Y(y, \rho)$. Now $B(x, r) \subset B(y, \rho)$ because $\rho \geqslant r + |y - x|$, so

$$
d_{x,r}(E, Z(x, r)) \leqslant r^{-1} \rho \, d_{y,\rho}(E, Z(x, r)) \leqslant r^{-1} \rho \tau_3 \leqslant 5\tau_3 \leqslant \varepsilon \tag{16.42}
$$

by (16.36) and if τ_3 is small enough. This completes our proof of Lemma 16.25. $\square$

The next lemmas are a little bit of a digression, and are not needed for the proof of Theorem 16.1. We shall use a compactness argument to replace the density assumptions in Lemmas 16.19 and 16.25 with assumptions of good approximation of E by a minimal cones.

LEMMA 16.43. — *For each small $\delta > 0$, there is a constant $\varepsilon > 0$ (that depends only on n and d) such that if E and F are reduced almost minimal sets of dimension d in the open set $U \subset \mathbb{R}^n$, with gauge function h, and if*

$$B(x, 10r/9) \subset U, \quad h(20r/9) \leqslant \varepsilon, \quad \text{and} \quad d_{x,10r/9}(E, F) \leqslant \varepsilon, \qquad (16.44)$$

then

$$H^d(E \cap B(x, r)) \leqslant H^d(F \cap B(x, (1 + \delta)r)) + \delta r^d. \qquad (16.45)$$

Suppose that the lemma fails, and choose for each $k \geqslant 0$ sets E_k and F_k that provide a counterexample with $\varepsilon = 2^{-k}$. By homogeneity, we can assume that $x = 0$ and $r = 1$ for all k. Thus E_k and F_k are reduced almost minimal sets in $B(0, 10/9)$. It is a standard fact about the Hausdorff metric that we can find a subsequence (which we shall still denote by $\{E_k\}$ to save notation) such that $\{E_k\}$ converges to a limit E locally in $B(0, 10/9)$. See for instance Proposition 34.6 on page 214 of [D2]. Since $d_{0,10/9}(E_k, F_k) \leqslant 2^{-k}$ by (16.44), $\{F_k\}$ also converges to E locally in $B(0, 10/9)$ (and we don't even need to extract a second subsequence).

Our assumption is that E_k and F_k are almost minimal in a domain that contains $B(0, 10/9)$, and with a gauge function h_k such that $h_k(20/9) \leqslant 2^{-k}$. The sequence $\{F_k\}$ satisfies the assumptions of Lemma 3.3 (and even of Theorem 3.4 in [D1]; we do not need generalized quasiminimal sets here), so

$$H^d(E \cap B(0, 1 + \delta)) \leqslant \liminf_{k \to +\infty} H^d(F_k \cap B(0, 1 + \delta)) \qquad (16.46)$$

by (3.3), applied with $V = B(0, 1)$. Similarly, $\{F_k\}$ satisfies the assumptions of Lemma 3.12, with $M = 1$ and the number h as small as we want. This is because (the end of) Remark 4.5 says that $F_k \in GAQ(M, \delta, U, h)$ with $M = 1$, $\delta = 20/9$, $U = B(0, 10/9)$, and $h = h_k(20/9)$. Then (3.13) with $H = \overline{B}(0, 1)$ says that

$$\limsup_{k \to +\infty} H^d(E_k \cap \overline{B}(0, 1)) \leqslant H^d(E \cap \overline{B}(0, 1)). \qquad (16.47)$$

Altogether, $\limsup_{k \to +\infty} H^d(E_k \cap B(0, 1)) \leqslant \liminf_{k \to +\infty} H^d(F_k \cap B(0, 1 + \delta))$, and (16.45) holds for k large. This contradiction completes our proof of Lemma 16.43. $\square$

Here is a variant of Lemma 16.19.

LEMMA 16.48. — *For each choice of $\tau \in (0, 1)$, there is a constant $\varepsilon > 0$ such that if $x_0 \in E$, $r_0 > 0$,*

$$B(x_0, 10r_0) \subset U, \quad h(20r_0) \leqslant \varepsilon, \quad \int_0^{18r_0} h(t) \frac{dt}{t} \leqslant \varepsilon, \qquad (16.49)$$

and there is a plane P such that

$$d_{x_0,10r_0}(E,P) \leqslant \varepsilon\,, \tag{16.50}$$

then for every choice of $x \in B(x_0,r_0)$ and $0 < r \leqslant r_0$, $B(x,r)$ is a biHölder ball of type $\mathbb{P}$ for E, with constant τ.

Indeed, let ε_1 be as in Lemma 16.19. If ε is chosen small enough we can apply Lemma 16.43 to $B(x_0,9r_0)$, $\delta = \varepsilon_1/10$, and $F = P$, and get that $\theta(x_0,9r_0) \leqslant \pi + \varepsilon_1$. Then the conclusion follows from Lemma 16.19 (if in addition $\varepsilon \leqslant \varepsilon_1$). $\square$

Now we give a variant of Lemma 16.25.

LEMMA 16.51. — *For each choice of $\tau \in (0,1)$, there is a constant $\varepsilon_4 > 0$ such that if $x_0 \in E$, $r_0 > 0$,*

$$B(x_0,60r_0) \subset U, \;\; h(100r_0) \leqslant \varepsilon_4, \;\; \int_0^{100r_0} h(t)\frac{dt}{t} \leqslant \varepsilon_4, \tag{16.52}$$

and there is a reduced minimal cone Y of type $\mathbb{Y}$, centered at x_0, and such that

$$d_{x_0,60r_0}(E,Y) \leqslant \varepsilon_4\,, \tag{16.53}$$

then $B(x_0,r_0)$ is a biHölder ball of type $\mathbb{Y}$ for E, with constant τ.

Let x_0 and r_0 be as in the statement. Let ε_5 be small, to be chosen soon, and apply Proposition 16.24 to the ball $B(x_0,\varepsilon_5 r_0)$. Notice that

$$d_{x_0,\varepsilon_5 r_0}(E,Y) \leqslant 60\varepsilon_5^{-1}d_{x_0,60r_0}(E,Y) \leqslant 60\varepsilon_5^{-1}\varepsilon_4 < \varepsilon_2 \tag{16.54}$$

if ε_4 is small enough, and the other assumptions of Proposition 16.24 are satisfied more trivially, so there is a point $x_1 \in E \cap B(x_0,\varepsilon_5 r_0)$ which is of type $\mathbb{Y}$. If ε_4 is small enough, (16.53) and (16.52) allow us to apply Lemma 16.43 to $B(x_1,500r_0/9)$ and get that

$$\theta(x_1,50r_0) \leqslant 3\pi/2 + \varepsilon_3\,, \tag{16.55}$$

where ε_3 is as in Lemma 16.25.

The pair $(x_1,50r_0)$ satisfies the hypotheses of Lemma 16.25, so $B(x_1,r_0)$ is a bihölder ball of type $\mathbb{Y}$. Since this is not exactly the conclusion of Lemma 16.43, let us say a little more. The proof of Lemma 16.25 gives that for $x \in E \cap B(x_1,3r_0)$ and $0 < r \leqslant 3r_0$, there is a reduced minimal cone $Z(x,r)$ such that $d_{x,r}(E,Z(x,r)) \leqslant \varepsilon$ (see (16.41) and below), and then we use Corollary 15.11 to conclude. But if ε_5 is small enough (and $\varepsilon_5 = 10^{-2}$ would be more

than enough), we also get $Z(x, r)$ for $x \in E \cap B(x_0, 3r_0) \subset B(x_1, 3r_0 + \varepsilon_5 r_0)$. In addition, (16.44) allows us to use Y itself (if ε_4 is small enough) as the set $Z(x_0, r_0)$, so the last condition in Corollary 15.11 is also fulfilled. Finally Corollary 15.11 says that $B(x_0, r_0)$ also is a bihölder ball of type $\mathbb{Y}$, and Lemma 16.51 follows. $\quad\square$

Return to the slow proof of Theorem 16.1. Lemmas 16.19 and 16.25 show that E has some biHölder regularity near P-points and Y-points. We deal with remaining case of T-points now.

LEMMA 16.56. — *For each choice of $\tau \in (0, 1)$, there is a constant $\varepsilon_6 > 0$ such that if $x_0 \in E$ is a T-point, $r_0 > 0$,*

$$B(x_0, 10r_0) \subset U, \ h(20r_0) \leqslant \varepsilon_6, \ \int_0^{20r_0} h(t)\frac{dt}{t} \leqslant \varepsilon_6, \tag{16.57}$$

and

$$\theta(x_0, 10r_0) \leqslant \theta(x_0) + \varepsilon_6 \tag{16.58}$$

then $B(x_0, r_0)$ is a biHölder ball of type $\mathbb{T}$ for E, with constant τ.

The proof will go as for Lemma 16.25. By rotation and dilation invariance, we may assume that $x_0 = 0$ and $r_0 = 1$. Let τ_4 be small, to be chosen later, and apply Lemma 16.11 with $\tau = \tau_4$ and to $B(0, 10)$. If $\varepsilon_6 \leqslant \varepsilon(\tau_4)$, we get that for $0 < \rho \leqslant 9$, there is a reduced minimal cone $T(\rho)$ of type $\mathbb{T}$, centered at 0, and such that

$$d_{0,\rho}(E, T(\rho)) \leqslant \tau_4 \tag{16.59}$$

and

$$\begin{aligned} \left| H^2(E \cap B(y, t)) - H^2(T(\rho) \cap B(y, t)) \right| &\leqslant \tau_4 \rho^2 \\ \text{for } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y, t) &\subset B(0, \rho). \end{aligned} \tag{16.60}$$

as in (16.14) and (16.15).

We want to control the density of $T(\rho)$ in small balls. Set $\eta = 10^{-1} \operatorname{Min}(\eta_0, l_0)$, where η_0 is as in (14.10) and l_0 comes from Proposition 14.1. It will be useful to know that for $\rho > 0$ and $x \neq 0$,

$$T(\rho) \text{ coincides with a cone of type } \mathbb{Y} \text{ in } B(x, \eta|x|). \tag{16.61}$$

Indeed suppose that $B(x, \eta|x|)$ meets two of the faces that compose $T(\rho)$. Call F_1 and F_2 these faces, and pick y_j in $F_j \cap B(x, \eta|x|)$. Set $z_j = y_j/|y_j|$; thus z_j lies in the arc of great circle $C_j = F_j \cap \partial B(0, 1)$. Observe that $|z_1 - z_2| < 3\eta < \eta_0$, so (14.10) says that the C_j have a common endpoint $a \in$

$B(z_1, 3\eta)$. If $B(x, \eta|x|)$ meets some other face F_3 of $T(\rho)$, the corresponding arc C_3 also has a common endpoint with F_1 in $B(z_1, 3\eta)$. This endpoint is a, because the length of C_1 is at least l_0. In other words, all the faces that meet $B(x, \eta|x|)$ are contained in the set of type $\mathbb{Y}$ which coincides with $T(\rho)$ in a small neighborhood of a; (16.61) follows.

Next let $x \in E \cap B(0, 8) \setminus \{0\}$ be given, and set $\rho(x) = 9|x|/8$. Thus $T(\rho(x))$ is defined, and

$$
\begin{aligned}
\theta(x, \eta|x|) &= (\eta|x|)^{-2} H^2(E \cap B(x, \eta|x|)) \\
&\leqslant (\eta|x|)^{-2} H^2(T(\rho(x)) \cap B(x, \eta|x|)) + (\eta|x|)^{-2} \tau_4 \rho(x)^2 \quad (16.62) \\
&\leqslant 3\pi/2 + 2\eta^{-2} \tau_4
\end{aligned}
$$

by (16.60), because $B(x, \eta|x|) \subset B(0, \rho(x))$, and by (16.61). Then

$$
\begin{aligned}
\theta(x) &\leqslant e^{\lambda A(\eta|x|)} \theta(x, \eta|x|) \leqslant e^{\lambda \varepsilon_6} [3\pi/2 + 2\eta^{-2} \tau_4] \\
&\leqslant 3\pi/2 + C\varepsilon_6 + 2\eta^{-2} \tau_4
\end{aligned}
\quad (16.63)
$$

by (16.9), (16.57), and (16.62). In particular, $\theta(x) < d_T$ if ε_6 and τ_4 are small enough, and

$$
0 \text{ is the only } T\text{-point of } B(0, 8). \quad (16.64)
$$

Next assume that our $x \in E \cap B(0, 8)$ is a Y-point. Let τ_5 be small, and apply Lemma 16.11 with the constant τ_5, and to the ball $B(x, \eta|x|)$. The assumption (16.12) follows from (16.57), and (16.13) follows from (16.62) because $\theta(x) = 3\pi/2$ (if ε_6 and τ_4 are small enough, depending on τ_5). We get that for $0 < r \leqslant 9\eta|x|/10$, there is a reduced minimal cone $Y(x, r)$ of type $\mathbb{Y}$, centered at x, and such that

$$
d_{x,r}(E, Y(x, r)) \leqslant \tau_5 \quad (16.65)
$$

and

$$
\begin{aligned}
&\left| H^2(E \cap B(y, t)) - H^2(Y(x, r) \cap B(y, t)) \right| \leqslant \tau_5 r^2 \\
&\quad \text{for } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y, t) \subset B(x, r).
\end{aligned}
\quad (16.66)
$$

We also need to approximate E near the points of type P. So let $z \in E \cap B(0, 3)$ be a P-point, and set $d(z) = \operatorname{dist}(z, \{0\} \cup E_Y)$. Let us show that for $0 < r < 9\eta d(z)/10$,

there is a plane $P(z, r)$ through z such that $d_{z,r}(E, P(z, r)) \leqslant \varepsilon$. (16.67)

Let us first assume that

$$d(z) \geqslant \eta|z|/2. \tag{16.68}$$

Set $\rho = 2|z| \leqslant 6$, and $T = T(\rho)$. We shall use T to compute the density of E near z. Denote by L the spine of T (the cone over the the union of the extremities of the arcs of circles that compose $T \cap \partial B(0,1)$). Let us show that

$$\mathrm{dist}(z, L) \geqslant d(z)/2. \tag{16.69}$$

Suppose instead that we can find $\xi \in L$, with $|\xi - z| = \mathrm{dist}(z, L) < d(z)/2$. Set $t = \eta d(z)/4$; we want to apply Proposition 16.24 to $B(\xi, t)$, so we need a set of type $\mathbb{Y}$. By (16.61), there is a cone Y of type $\mathbb{Y}$ that coincides with T in $B(\xi, \eta|\xi|)$. Observe that Y is centered at ξ, because ξ lies in the spine of T. Next $|\xi| \geqslant |z| - |\xi - z| \geqslant |z| - d(z)/2 \geqslant d(z)/2$ (because $d(z) \leqslant |z|$), so $t = \eta d(z)/4 \leqslant \eta|\xi|/2$ and Y coincides with T in $B(\xi, 2t)$. Thus $d_{\xi,t}(E, Y) = d_{\xi,t}(E, T)$.

In addition, $|\xi| \leqslant |z| + |\xi - z| \leqslant |z| + d(z)/2 \leqslant 3|z|/2$ and $\rho = 2|z|$, so $B(\xi, t) \subset B(0, \rho)$. Hence

$$\begin{aligned} d_{\xi,t}(E, Y) = d_{\xi,t}(E, T) &\leqslant t^{-1}\rho\, d_{0,\rho}(E, T) \\ &\leqslant t^{-1}\rho\, \tau_4 = 8\eta^{-1}d(z)^{-1}|z|\,\tau_4 \leqslant 16\eta^{-2}\tau_4 \end{aligned} \tag{16.70}$$

by (16.59) and (16.68).

If τ_4 is small enough, (16.70) allows us to apply Proposition 16.24 to $B(\xi, t)$, and get a Y-point y in $B(\xi, t/100)$. Recall that $|\xi - z| < d(z)/2$; then $|y - z| \leqslant d(z)/2 + t < d(z)$, which contradicts the definition of $d(z)$. So (16.69) holds.

Next we check that

$$T \text{ coincides with a plane in } B(z, \eta d(z)). \tag{16.71}$$

Otherwise, $B(z, \eta d(z))$ meets two faces of T. Let $z_1, z_2 \in B(z, \eta d(z))$ lie in different faces F_1 and F_2, and set $w_i = z_i/|z_i|$. First observe that

$$|z_i - z| \leqslant \eta d(z) \leqslant \eta|z|, \tag{16.72}$$

so $|z_i| \geqslant |z|/2$, and then

$$|w_1 - w_2| \leqslant 2|z_1 - z_2|/|z| \leqslant 4\eta d(z)/|z| \tag{16.73}$$

because the radial projection onto $\partial B(0,1)$ is $(2/|z|)$-Lipschitz on $\mathbb{R}^n \setminus B(0, |z|/2)$.

Next, w_1 and w_2 lie in different arcs of circles $C_i = F_i \cap \partial B(0,1)$ in the description of $T \cap \partial B(0,1)$, and

$$\operatorname{dist}(w_1, C_2) \leqslant |w_1 - w_2| \leqslant 4\eta d(z)/|z| \leqslant 4\eta < \eta_0 \qquad (16.74)$$

because $\eta \leqslant \eta_0/10$ (see below (16.60)). So (14.10) says that C_1 and C_2 have a common endpoint a such that $|a - w_1| \leqslant |w_1 - w_2| \leqslant 4\eta d(z)/|z|$. Of course $|z_1|a$ lies in the spine L, so

$$\operatorname{dist}(z_1, L) \leqslant |z_1 - |z_1|a| \leqslant |z_1||w_1 - a| \leqslant 4\eta|z_1|d(z)/|z| \leqslant 8\eta d(z). \quad (16.75)$$

This is impossible, because $z_1 \in B(z, \eta d(z))$ and $\operatorname{dist}(z, L) \geqslant d(z)/2$ (by (16.69)). So (16.71) holds, and $H^2(T \cap B(z, \eta d(z))) \leqslant (\eta d(z))^2 \pi$. Recall that $B(z, \eta d(z)) \subset B(0, \rho)$ because $\rho = 2|z|$ and $d(z) \leqslant |z|$, so (16.60) says that

$$\begin{aligned}
\theta(z, \eta d(z)) &\leqslant (\eta d(z))^{-2} H^2(T \cap B(z, \eta d(z))) + (\eta d(z))^{-2} \tau_4 \rho^2 \\
&\leqslant \pi + (\eta d(z))^{-2} \tau_4 \rho^2 \leqslant \pi + 4\eta^{-4}|z|^{-2} \tau_4 \rho^2 \qquad (16.76) \\
&\leqslant \pi + 16\eta^{-4} \tau_4
\end{aligned}$$

by (16.68).

Recall that z is a P-point; if τ_4 is small enough, (16.76) allows us to apply Lemma 16.11 to $B(z, \eta d(z))$, where this time we take $\tau = \varepsilon$, with ε coming from Corollary 15.11. We get that for $0 < r \leqslant 9\eta d(z)/10$, there is a plane $P(z, r)$ through z such that $d_{z,r}(E, P(z, r)) \leqslant \varepsilon$. In other words, (16.67) holds.

We are left with the case when $z \in E \cap B(0,3)$ is a P-point as before, but (16.68) fails, i.e., when $d(z) \leqslant \eta|z|/2$.

Let $x \in E_Y$ be such that $|x - z| \leqslant 11d(z)/10 \leqslant \eta|z|/20$. Thus $x \in B(0,4)$. Also set $r = 3d(z)/2$. Then $r \leqslant 3\eta|z|/4 < 9\eta|x|/10$, and we have a set $Y = Y(x, r)$ of type $\mathbb{Y}$ that satisfies (16.65) and (16.66). Let L denote the spine of Y; then $\operatorname{dist}(z, L) \geqslant d(z)/4$, by the proof of (16.39), and so Y coincides with a plane in $B(z, d(z)/4)$. Now

$$\begin{aligned}
\theta(z, d(z)/4) &\leqslant (d(z)/4)^{-2} H^2(Y \cap B(z, d(z)/4)) + (d(z)/4)^{-2} \tau_5 r^2 \\
&\leqslant \pi + (d(z)/4)^{-2} \tau_5 (3d(z)/2)^2 = \pi + 36\tau_5
\end{aligned} \qquad (16.77)$$

because $B(z, d(z)/4) \subset B(x, r) = B(x, 3d(z)/2)$, and by (16.66). By Lemma 16.11, we can find planes $P(z, t)$, $0 < t \leqslant d(z)/5$, such that $d_{z,t}(E, P(z, t)) \leqslant \varepsilon$. That is, we have (16.67) for a slightly larger range of radii.

We are now ready to use our various cones to check the assumptions of Corollary 15.11. We need to define a minimal cone $Z(x, r)$ for $x \in E \cap B(0, 3)$ and $0 < r \leqslant 3$, so that

$$d_{x,r}(E, Z(x, r)) \leqslant \varepsilon \tag{16.78}$$

as in (15.12).

When $x = 0$, we take $Z(x, t) = T(\rho)$, and (16.78) follows from (16.59). Notice, by the way, that $Z(0, 3)$ is a set of type $\mathbb{T}$ centered at 0, as required by Corollary 15.11 if we want a biHölder ball of type $\mathbb{T}$.

Now suppose that $x \in E_Y$. If $0 < r < 9\eta|x|/10$, we can use the set $Y(x, r)$ constructed near (16.65), and (16.78) follows from (16.65). Otherwise, we take $Z(x, r) = T(|x| + r)$, which is defined because $|x| + r \leqslant 6$. Notice that $B(x, r) \subset B(0, |x| + r)$, so

$$\begin{aligned} d_{x,r}(E, Z(x, r)) &\leqslant r^{-1}(|x| + r) \, d_{0,|x|+r}(E, T(|x| + r)) \\ &\leqslant r^{-1}(|x| + r) \, \tau_4 \leqslant 2\eta^{-1}\tau_4 \end{aligned} \tag{16.79}$$

by (16.59) and because $|x| \leqslant 10\eta^{-1}r/9$. Again (16.78) holds if τ_4 is small enough.

We are left with the case when y is a P-point. If $0 < r < 9\eta d(x)/10$, we can use the plane $P(x, r)$ of (16.67), and (16.78) follows directly from (16.67). So we may assume that $r \geqslant 9\eta d(x)/10$.

If $r + 2d(x) \leqslant \eta|x|/3$, we choose $y \in E_Y$ such that $|x - y| < 2d(x)$, and try to take $Z(x, r) = Y(y, r+2d(x))$. [If $d(x) = 0$, apply the argument below with $d(x)$ replaced with a very small number (compared to r and $|x|$).] Let us check that $Z(x, r)$ is defined. First, $|x - y| \leqslant 2d(x) \leqslant \eta|x|/3$, so $y \in B(0, 4)$ (because $x \in B(0, 3)$), and also $|y| \geqslant 2|x|/3$. Then $r + 2d(x) \leqslant \eta|x|/3 \leqslant \eta|y|/2$, which is enough to define $Y(y, r + 2d(x))$ (see above (16.65)). Now

$$\begin{aligned} d_{x,r}(E, Z(x, r)) &\leqslant r^{-1}(r + 2d(x)) \, d_{y,r+2d(x)}(E, Z(x, r)) \\ &= r^{-1}(r + 2d(x)) \, d_{y,r+2d(x)}(E, Y(y, r + 2d(x))) \quad (16.80) \\ &\leqslant r^{-1}(r + 2d(x)) \, \tau_5 \leqslant 3\eta^{-1}\tau_5 \end{aligned}$$

because $B(x, r) \subset B(y, r + 2d(x))$, by (16.65), and because $r \geqslant 9\eta d(x)/10$. Thus (16.78) holds if τ_5 is small enough.

Finally assume that $r + 2d(x) > \eta|x|/3$. Take $Z(x, r) = T(r + |x|)$, which is defined because $r \leqslant 3$ and $x \in B(0, 3)$. Then

$$\begin{aligned} d_{x,r}(E, Z(x, r)) &\leqslant r^{-1}(r + |x|) \, d_{0,r+|x|}(E, T(r + |x|)) \\ &\leqslant 2r^{-1}(r + |x|) \, \tau_4 \end{aligned} \tag{16.81}$$

by (16.59). Since $|x| \leqslant 3\eta^{-1}(r + 2d(x)) \leqslant 3\eta^{-1}(r + 3\eta^{-1}r)$ because $r \geqslant 9\eta d(x)/10$, (16.78) follows from (16.81).

We completed the verification of (16.78) in all cases, Corollary 15.11 applies, and Lemma 16.56 follows. □

Let us now see how to deduce Theorem 16.1 from our various lemmas. Let U and E be as in the statement, and let $x_0 \in E$ be given. If x_0 is a P-point, we can choose r_0 so small that the assumptions of Lemma 16.19 are satisfied, and then $B(x_0, r_0)$ is a biHölder ball of type $\mathbb{P}$. If x_0 is a Y-point, we use Lemma 16.25 instead. Otherwise, x_0 is a T-point, and we use Lemma 16.56. This completes our proof of Theorem 16.1, except for the fact that the two last lemmas depend on Proposition 16.24, which will be proved in the next section. □

Remark 16.82. — We do not know whether the obvious analogue of Lemma 16.43 for T-points holds. That is, suppose that the locally minimal set E is very close to a set of type $\mathbb{T}$ in a ball; is there necessarily a point of type $\mathbb{T}$ near the center? However, the answer is yes if $n = 3$ and we also assume that E separates locally $\mathbb{R}^3$ into at least three big connected components. See Proposition 18.1.

17. A topological argument for the existence of Y-points

This section is devoted to the proof of Proposition 16.24. The author was very proud of "finding" the topological argument below, but it seems that this too could be found in Taylor's paper [Tay]. The generalization to $\mathbb{R}^n$, $n > 3$ is a little more technical, but the idea is similar.

So we are given a reduced almost-minimal set E of dimension 2 in the domain $U \subset \mathbb{R}^n$, with a gauge function h such that (16.2) holds, and a pair (x, r) such that

$$B(x, r) \subset U, \ h(2r) \leqslant \varepsilon_2, \text{ and } \int_0^{2r} h(t)dt/t \leqslant \varepsilon_2. \tag{17.1}$$

We also assume that there is a reduced minimal cone Y of type $\mathbb{Y}$, centered at x and such that

$$d_{x,r}(E, Y) \leqslant \varepsilon_2, \tag{17.2}$$

where ε_2 is s small constant that depends on n, and we want to prove that $E \cap B(x, r/100)$ contains (at least) a Y-point. By rotation and dilation invariance, we may assume that $x = 0$ and $r = 200$. Thus

$$\int_0^{400} h(t)dt/t \leqslant \varepsilon_2 \text{ and } d_{0,200}(E, Y) \leqslant \varepsilon_2, \tag{17.3}$$

by (17.1) and (17.2).

We shall prove the proposition by contradiction. So we assume that $B(0, 2)$ contains no Y-point, and try to prove a contradiction.

Let us check that for any given $\varepsilon > 0$, we have that

$$H^2(E \cap B(y, t)) \leqslant H^2(Y \cap B(y, t)) + \varepsilon$$
$$\text{for } y \in B(0, 2) \text{ and } 10^{-1} \leqslant t \leqslant 1$$
(17.4)

if ε_2 is small enough.

Let ε' denote the small constant that we get when we apply Lemma 16.43 with $\delta = \varepsilon/10$, and let us check that (17.4) holds as soon as $\varepsilon_2 \leqslant \varepsilon'/1800$. Let $y \in B(0, 2)$ and $10^{-1} \leqslant t \leqslant 1$ be given; observe that

$$d_{y,10t/9}(E, Y) \leqslant \frac{9 \cdot 200}{10t} d_{0,200}(E, Y) \leqslant 9\varepsilon \cdot 200 < \varepsilon' \qquad (17.5)$$

because $B(10t/9) \subset B(0, 200)$ and by (17.3), and more trivially $h(20t/9) \leqslant h(20/9) \leqslant \int_0^{400} h(t)dt/t \leqslant \varepsilon_2 \leqslant \varepsilon'$ by (17.3), so Lemma 16.43 applies (with $F = Y$) and we get that

$$H^2(E \cap B(y, t)) \leqslant H^2(Y \cap B(y, (1+\delta)t)) + \delta t^2 \leqslant H^2(Y \cap B(y, t)) + 10\delta t^2$$
(17.6)

(the measure of $Y \cap B(y, (1+\delta)t)) \setminus B(y, t)$ is easy to control, in particular because y lies very close to Y by (17.3)). Now (17.4) follows from (17.6) (recall that $\delta = \varepsilon/10$).

Let us use (17.4) to check that

$$\text{every point of } E \cap B(0, 2) \text{ is a } P\text{-point}, \qquad (17.7)$$

or equivalently that $B(0, 2)$ contains no T-point. Let $x \in B(0, 2)$ be given; then

$$\theta(x) \leqslant e^{\lambda A(1)} \theta(x, 1) = e^{\lambda \int_0^2 h(t)dt/t} \theta(x, 1) \leqslant e^{\lambda \varepsilon_2} \theta(x, 1)$$
$$\leqslant e^{\lambda \varepsilon_2} [H^2(Y \cap B(x, 1)) + \varepsilon] \leqslant e^{\lambda \varepsilon_2} (3\pi/2 + \varepsilon) < d_T$$
(17.8)

by (16.9), (16.5), (17.3), (17.4), because the density of a set of type $\mathbb{Y}$ never exceeds $3\pi/2$, and if ε, and then ε_2 are small enough. Recall that d_T is the constant from Lemma 14.12 and that $d_T > 3\pi/2$. Now (16.8) says that x is not a T-point; (17.7) follows.

Next choose a small constant $\tau > 0$; we claim that we can find $\eta \in (0, 1)$ such that

$$B(x, r) \text{ is a biHölder ball of type } P \text{ for } E, \text{ with constant } \tau,$$
$$\text{for every } x \in B(0, 3/2) \text{ and } 0 < r \leqslant 10\eta.$$
(17.9)

Indeed, set $K = E \cap \overline{B}(0, 3/2)$ and, for $x_0 \in K$, choose $r_0 = r(x_0)$ so small that the assumptions of Lemma 16.19 are satisfied. This is possible, because $\theta(x_0) = \pi$ by (17.7). Then cover K by a finite number of balls $B(x_0, r(x_0))$. Finally take η smaller than the tenth of the smallest radius $r(x_0)$. If $x \in K$, we can find x_0 as above such that $x \in B(x_0, r(x_0))$, and then Lemma 16.19 says that $B(x, r)$ is a biHölder ball of type P for E, with constant τ, for $0 < r \leqslant r_0$. This proves the claim (17.9).

Let us first give a vague idea of the plan that we would like to follow to get the desired contradiction. First, we would find a sphere S of codimension 2 in $B(0, 3/2)$ that meets E exactly three times. We would deform this sphere through a family S_t of spheres that eventually collapse to a point. Then we would use (17.9) to show that along this deformation, the number of points of $E \cap S_t$ can only jump by multiples of two; this is impossible when we start with 3 intersections and end with none.

Things will be a little more complicated because we need to put our objects in general position before we compute the number of intersections; we shall also need to relate our numbers to degrees of mappings on spheres to get some stability.

To compute the number of points in $E \cap S_t$ (maybe after we modify S_t slightly), one way would be to define a one-parameter family of functions f_t on S, so that $S_t = f_t(S)$, give equations of E locally, and compute how many points $x \in S$ satisfy the equations. We shall find it more convenient to proceed in the dual way, i.e., give equations of the sets S_t, and count the number of solutions of these equations in E. It seems a little more convenient because it is apparently easier to localize in E.

I wish to thank Christopher Colin for his help in my initial struggle with degrees, and in particular the suggestion that I use equations.

Let us fix the setting and define a sphere S. Write $\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^d$, with $d = n - 3$. By rotation invariance, we may assume that $Y = Y_0 \times \{0\}$, where $Y_0 \subset \mathbb{R}^3$ is a vertical propeller (or Y-set). That is, we select three points a_1, a_2, and a_3 in the unit circle of $\mathbb{R}^2$, so that if L_j denotes the half line that starts from 0 and goes trough a_j, the three L_j make $120°$ angles at the origin, and then we set $Y_0 = [L_1 \cup L_2 \cup L_3] \times \mathbb{R} \subset \mathbb{R}^3$.

Denote by P the horizontal hyperplane $P = \mathbb{R}^2 \times \{0\} \times \mathbb{R}^d$, and set $S = \partial B(0, 1) \cap P$. Observe that $S \cap Y = \{a_1, a_2, a_3\}$; S is the sphere of codimension 2 that we would like to use in the argument suggested above, but we need to modify it slightly near the a_j if we want it to meet E exactly three times.

Fix $j \leqslant 3$ and use (17.3) to choose $z_j \in E$ such that $|z_j - a_j| \leqslant 300\varepsilon_2$. Choose ε in (17.4) smaller than $\varepsilon_1/4$, where ε_1 is as in Lemma 16.19 (and where, as before, the small constant τ just needs to be reasonably small, depending on the geometry). Then $\theta(z_j, 1/2) \leqslant 4H^2(Y \cap z_j, 1/2) + 4\varepsilon$ $\leqslant \pi + 4\varepsilon$ if ε_2 is small enough, by (17.4) and because $B(z_j, 1/2)$ only meets one face of Y. Then we can apply Lemma 16.19 to the pair $(z_j, 1/18)$, and we get that

$$B(z_j, r) \text{ is a biHölder ball of type } P \text{ for } E \text{ for } 0 < r \leqslant 1/18. \qquad (17.10)$$

We start with a rough model, which will be used as a pattern, where the S_t is $(d+1)$-dimensional sphere with radius $(1-t)$, and centered at $b_t = t(a_1 + a_2)/100$, say. We choose a final center $b_1 = (a_1 + a_2)/100$ purposefully away from Y, to be sure that S_t does not meet E for t close to 1. A formula for the pair of equations of S_t is $f_t(x) = 0$, where

$$f_t(x) = (x_2, |x - b_t|^2 - (1-t)^2). \qquad (17.11)$$

Let us record the fact that

$$|f_1(x)| \geqslant |x - b_1|^2 \geqslant \mathrm{dist}(b_1, E)^2 \geqslant 10^{-5} \quad \text{for } x \in E, \qquad (17.12)$$

by (17.3).

We shall not use the one-parameter family $\{f_t\}$ to go from f_0 to f_1, and instead we shall use a finite number of intermediate functions, so that each time the modifications occur on a small biHölder ball for E. So we shall use a partition of unity. Denote by $\{x_j\}$, $1 \leqslant j \leqslant l$, a maximal collection of points in $E \cap B(0, 3/2)$, with the property that $|x_i - x_j| \geqslant \eta$ when $i \neq j$. Let $\widetilde{\varphi}_j$ be a smooth bump function supported in $B(x_j, 2\eta)$ and such that $\widetilde{\varphi}_j(x) = 1$ for $x \in \overline{B}(x_j, \eta)$ and $0 \leqslant \widetilde{\varphi}_j(x) \leqslant 1$ everywhere. Notice that $\sum_j \widetilde{\varphi}_j(x) \geqslant 1$ for $x \in E \cap B(0, 3/2)$, because x lies in some $\overline{B}(x_j, \eta)$ by maximality of the family $\{x_j\}$. Also let $\widetilde{\varphi}_0$ be a smooth function on $\mathbb{R}^n$ such that $\widetilde{\varphi}_0(x) = 0$ when $|x| \leqslant 3/2 - \eta$, $\widetilde{\varphi}_0(x) = 1$ when $|x| \geqslant 3/2$, and $0 \leqslant \widetilde{\varphi}_0(x) \leqslant 1$ otherwise. Now $\sum_{j=0}^{l} \widetilde{\varphi}_j(x) \geqslant 1$ on E, and we set

$$\varphi_j(x) = \widetilde{\varphi}_j(x) \left\{ \sum_{j=0}^{l} \widetilde{\varphi}_j(x) \right\}^{-1} \quad \text{for } x \in E \text{ and } 0 \leqslant j \leqslant l. \qquad (17.13)$$

Thus the φ_j keep the same support properties as the the $\widetilde{\varphi}_j$, and in particular

$$\varphi_j \text{ is supported in } B(x_j, 2\eta) \text{ for } j \geqslant 1, \qquad (17.14)$$

and in addition

$$\sum_{j=0}^{l} \varphi_j(x) = 1 \text{ for } x \in E, \quad \text{and}$$

$$\sum_{j=1}^{l} \varphi_j(x) = 1 \text{ for } x \in E \cap B(0, 3/2 - \eta). \tag{17.15}$$

(because $\varphi_0(x) = 0$ there). Our first approximation of a discrete path from f_0 to f_1 is given by the maps

$$g_k = f_0 + \sum_{0 < j \leqslant k} \varphi_j \, (f_1 - f_0), \tag{17.16}$$

with $0 \leqslant k \leqslant l$. Thus $g_0 = f_0$ and

$$g_l(x) = f_1(x) \text{ for } x \in E \cap B(0, 3/2 - \eta), \tag{17.17}$$

by (17.15). Notice that for $k \geqslant 1$,

$$g_k(x) - g_{k-1}(x) = \varphi_k(x)(f_1(x) - f_0(x)) \text{ is supported in } B(x_k, 2\eta), \tag{17.18}$$

by (17.14). Recall that we would like to compute the number of solutions in $E \cap B(0, 3/2)$ of the systems $g_k(x) = 0$. This will be easier to do with the small modifications of f_0 and the g_k that we shall construct next, so that they only have a finite number of simple roots. We start with a modification of f_0

LEMMA 17.19. — *There is a continuous function h_0 on E, such that*

$$|h_0(x) - f_0(x)| \leqslant 10^{-6} \quad \text{for } x \in E, \tag{17.20}$$

h_0 has exactly three zeroes ξ_1, ξ_2, and ξ_3 in E, one in each $B(a_i, 1/60)$, and in addition

$$\text{each } \xi_j \text{ is a non-degenerate simple zero of } h_0. \tag{17.21}$$

Let us first define (17.21). We shall say that $\xi \in E$ is a non-degenerate simple zero of some continuous function h on E if $h(\xi) = 0$ and there is a ball $B(\xi, \rho)$ and a bi-Hölder map γ from $E \cap B(\xi, \rho)$ to an open set V of a plane, such that $h \circ \gamma^{-1}$ is of class C^1 on V and the differential of $h \circ \gamma^{-1}$ at $\gamma(\xi)$ is of rank 2.

Notice that when this holds, there is a small neighborhood W of $\gamma(\xi)$ such that $h \circ \gamma^{-1}$ is a homeomorphism from W to its image in $\mathbb{R}^2$, so if we replace ρ with a sufficiently small radius $t > 0$, we get that

the restriction of h to $E \cap B(\xi, t)$ is a homeomorphism

from $E \cap B(\xi, t)$ to the open subset $h(E \cap B(\xi, t))$ of $\mathbb{R}^2$. $\tag{17.22}$

We now prove the lemma. We shall only modify f_0 near the three points a_j of $S \cap Y$. Recall from (17.10) that $B(z_j, 1/18)$ is a biHölder ball of type $\mathbb{P}$. Thus there is a plane P_j centered at z_j and a mapping $\psi_j : B(z_j, 1/9) \to \mathbb{R}^n$, with properties like (15.6)-(15.9) (see Definition 15.10). In particular, $|\psi_j(x) - x| \leqslant \tau/18$ for $x \in B(z_j, 1/9)$ and

$$E \cap B(z_j, 10^{-1}) \subset \psi_j(P_j \cap B(z_j, 1/9)) \subset E. \qquad (17.23)$$

We keep $h_0 = f_0$ out of the three $B(z_j, 1/30)$. In $B(z_j, 1/60)$, we replace f_0 with $h_0 = f_0 \circ \psi_j^{-1}$. In the intermediate regions, we interpolate, i.e., we set

$$h_0(x) = \alpha(x) f_0(x) + (1 - \alpha(x)) f_0 \circ \psi_j^{-1}(x), \qquad (17.24)$$

with $\alpha(x) = 60|x - z_j| - 1$. Notice that $|h_0(x) - f_0(x)| \leqslant |f_0(x) - f_0 \circ \psi_j^{-1}(x)| \leqslant C\tau$ for $x \in B(z_j, 1/30)$, because $|\psi_j(x) - x| \leqslant \tau/18$ and the derivative of f_0 is bounded there; thus (17.20) holds.

Recall from (17.11) that $f_0(x) = (x_2, |x|^2 - 1)$; then $|f_0(x)| \geqslant 1/500$ for $x \in Y \setminus \cup_j B(x_j, 10^{-2})$ (essentially, because $S \cap Y = \{a_1, a_2, a_3\}$ and $|z_j - a_j| \leqslant 300\varepsilon_2$ (see above (17.10)). This implies that

$$|h_0(x)| \geqslant 10^{-3} \text{ for } x \in E \cap B(0, 2) \setminus \cup_j (B(z_j, 1/60)), \qquad (17.25)$$

because f_0 varies slowly, and by (17.3) and (17.20). Thus h_0 only has zeroes in the $B(z_j, 1/60)$.

We still need to check that h_0 has exactly one zero in each $B(z_j, 1/60)$, and that (17.21) holds. Set $\gamma_j(x) = \psi_j^{-1}(x)$ for $x \in B(z_j, 1/60)$. By (17.23) and because ψ_j is biHölder, γ_j is a homeomorphism from $E \cap B(z_j, 1/60)$ to its image, which is an open subset of P_j.

Recall that $h_0 = f_0 \circ \psi_j^{-1} = f_0 \circ \gamma_j$ in $B(z_j, 1/60)$. Thus, for $\xi \in E \cap B(z_j, 1/60)$, ξ is a zero of h_0 if and only if $\gamma_j(\xi)$ is a zero of $f_0(x) = (x_2, |x|^2 - 1)$ in P_j. So we study f_0 in P_j.

By (17.3), we can find $z'_j \in E$ such that $|z'_j - (1 + 10^{-2}) a_j| \leqslant 300\varepsilon_3$, and $z''_j \in E$ such that $|z''_j - a_j - 10^{-2} e_2| \leqslant 300\varepsilon_3$, where $e_2 = (0, 1, 0, \cdots, 0)$. Set $w = \gamma_j(z_j)$, $w' = \gamma_j(z'_j)$, and $w'' = \gamma_j(z''_j)$. These three points lie in P_j, so P_j is the plane that contains w and the two vectors $v_1 = 100(w' - w)$ and $v_2 = 100(w'' - w)$.

Notice that $|w - z_j| \leqslant \tau/18$, by (15.6), and similarly for w' and w''. Then $|f_0(w)| \leqslant 10|w - a_j| \leqslant C\varepsilon_3 + C\tau$, because $f_0(a_j) = 0$.

Since $|v_1 - a_j| \leqslant C\varepsilon_3 + C\tau$ and $|v_2 - e_2| \leqslant C\varepsilon_3 + C\tau$, when we can compute the differential of f_0 in P_j, we can replace v_1 with a_j and v_2 with

e_2, and make errors smaller than $C\varepsilon_3 + C\tau$. But $D_{a_j} f_0(x) = (0, 2\langle x, a_j \rangle)$ and $D_{e_2} f_0(x) = (1, 2x_2)$. These stay within 10^{-1} of $(0, 2)$ and $(1, 0)$ when $x \in P_j \cap B(a_j, 1/50)$, so, if ε_3 and τ are small enough, the differential of f_0 in $P_j \cap B(a_j, 1/50)$ is invertible and almost constant, and f_0 has a unique zero ζ_j in $P_j \cap B(a_j, 1/50)$, which even lies in $B(a_j, 10^{-3})$ (recall that $|f_0(w)| \leqslant C\varepsilon_3 + C\tau$).

Thus $h_0 = f_0 \circ \psi_j^{-1}$ has a unique zero in $B(z_j, 1/60)$, namely, $\xi_j = \psi_j(\zeta_j)$. In addition, (17.21) holds, precisely because $f_0 = h_0 \circ \psi_j = h_0 \circ \gamma_j^{-1}$ is C^1 on P_j, with a differential of full rank at ζ_j. This completes our proof of Lemma 17.19. $\qquad \square$

LEMMA 17.26. — *We can find continuous functions θ_k, $1 \leqslant k \leqslant l$, such that*

$$\theta_k \text{ is supported in } B(x_k, 3\eta), \tag{17.27}$$

$$||\theta_k||_\infty \leqslant 2^{-k} 10^{-6}, \tag{17.28}$$

and, if we set

$$h_k = h_{k-1} + \varphi_k (f_1 - f_0) + \theta_k \tag{17.29}$$

for $1 \leqslant k \leqslant l$, then

$$\begin{aligned} &\text{each } h_k \text{ has only a finite number of zeroes in } E, \\ &\text{which are all non-degenerate simple zeroes of } h_k. \end{aligned} \tag{17.30}$$

Thus we keep the same formula as for g_k in (17.18), but we add small perturbations θ_k to obtain (17.30).

We shall construct the h_k by induction. Let $k \geqslant 1$ be given, and suppose that we already constructed h_{k-1} so that (17.30) holds. Notice that this is the case when $k = 1$, by Lemma 17.19.

Notice that $h_{k-1} + \varphi_k (f_1 - f_0)$ coincides with h_{k-1} out of $B(x_k, 2\eta)$, by (17.14). We select a very thin annulus

$$A = \overline{B}(x_k, \rho_2) \setminus B(x_k, \rho_1), \text{ with } 2\eta < \rho_1 < \rho_2 < 3\eta, \tag{17.31}$$

that does not meet the finite set of zeroes of h_{k-1}. Recall from (17.9) that $B(x_k, 10\eta)$ is a biHölder ball of type $\mathbb{P}$, so there is a biHölder mapping $\psi_k : B(x_k, 20\eta) \to \mathbb{R}^n$ and a plane P_k through x_k such that $|\psi_k(x) - x| \leqslant 10\eta\tau$ for $x \in B(x_k, 20\eta)$, as in (15.6), and

$$E \cap B(x_k, 19\eta) \subset \psi_k(P_k \cap B(x_k, 20\eta)) \subset E, \tag{17.32}$$

as in (15.9).

We shall take θ_k supported in $B(x_k, \rho_2)$, so (17.27) will hold, and also the desired control on the zeroes of h_k out of $B(x_k, \rho_2)$ will come from the induction assumption on h_{k-1}, since $h_k = h_{k-1}$ out of $B(x_k, \rho_2)$.

We shall also take $||\theta_k||_\infty$ smaller than $2^{-k}10^{-6}$, so that (17.28) holds, and smaller than $\inf_{x \in A} |h_{k-1}(x)| > 0$, so that (17.29) (with the fact that $0 \leqslant \varphi_k(x) \leqslant 1$ everywhere) implies that h_k has no zero in A.

We still need to control h_k in $B(x_k, \rho_1)$. Set $\gamma(x) = \psi_k^{-1}(x)$ for $x \in E \cap B(x_k, \rho_1)$. By (17.32) and because ψ_k is biHölder-continuous on $B(x_k, 20\eta)$, γ is a biHölder homeomorphism from $E \cap B(x_j, \rho_1)$ to some open subset V of the plane P_k, and its inverse is the restriction of ψ_k to V.

By density of the C^1 functions (in the space of bounded continuous functions on V, with the "sup" norm), we can choose θ_k with the constraints above, and such that

$$h_k \circ \psi_k \text{ is of class } C^1 \text{ on } V. \tag{17.33}$$

We can even keep the option of adding a very small constant $w \in \mathbb{R}^2$ to θ_k in $E \cap B(x_j, \rho_1)$ (and interpolating continuously on A). Let us first check that for almost every choice of w,

$$h_k \text{ has a finite number of zeros in } E \cap B(x_j, \rho_1) \tag{17.34}$$

for the modified h_k. Set

$$Z_y = \big\{ z \in V \,;\, h_k \circ \psi_k(z) = y \big\} \tag{17.35}$$

for $y \in \mathbb{R}^2$. By (17.33), we can apply the coarea theorem to $h_k \circ \psi_k$ on V. We get that

$$\int_V J(z) \, dH^2(z) = \int_{y \in \mathbb{R}^2} H^0(Z_y) \, dH^2(y), \tag{17.36}$$

where J is a bounded Jacobian that we don't need to compute (apply (8.12) with $m = 2$ and $d = 0$). Then Z_y is finite for almost every $y \in \mathbb{R}^2$. If we choose w such that Z_w is finite and add $-w$ to θ_k in $E \cap B(x_j, \rho_1)$, then the new Z_0 will be finite, i.e., (17.34) will hold.

We also care about the rank of the differential. Sard's theorem says that the set of critical values of $h_k \circ \psi_k$ has vanishing Lebesgue measure in $\mathbb{R}^2$; if we choose $w \in \mathbb{R}^2$ such that w is not a critical value, and add $-w$ to θ_k in $E \cap B(x_j, \rho_1)$, then the differential of the new $h_k \circ \psi_k$ at every zero of $h_k \circ \psi_k$ has rank 2.

We select w very small, with the properties above, and add $-w$ t o θ_k in $E \cap B(x_j, \rho_1)$; then h_k has a finite number of zeros in $E \cap B(x_j, \rho_1)$, and

they are all non-degenerate simple zeroes of h_k (see the definition below Lemma 17.19 and recall that $h_k \circ \gamma^{-1} = h_k \circ \psi_k$). So (17.30) holds, and Lemma 17.26 follows. $\quad\square$

Denote by $N(k)$ the number of zeroes of h_k in $E \cap B(0, 2)$. By Lemma 17.9, $N(0) = 3$. Let us check that for the last index l, $N(l) = 0$. First observe that

$$h_l - h_0 = \sum_{1 \leqslant k \leqslant l} (h_k - h_{k-1}) = \sum_{1 \leqslant k \leqslant l} \varphi_k \, (f_1 - f_0) + \sum_{1 \leqslant k \leqslant l} \theta_k \qquad (17.37)$$

by (17.29). If $x \in E \cap B(0, 4/3)$, (17.15) says that $\sum_{1 \leqslant k \leqslant l} \varphi_k(x) = 1$, so

$$h_l(x) = h_0(x) + f_1(x) - f_0(x) + \sum_{1 \leqslant k \leqslant l} \theta_k(x) \qquad (17.38)$$

and

$$|h_l(x)| \geqslant |f_1(x)| - |h_0(x) - f_0(x)| - \sum_{1 \leqslant k \leqslant l} |\theta_k(x)|$$
$$\geqslant 10^{-5} - 10^{-6} - \sum_{1 \leqslant k \leqslant l} 2^{-k} 10^{-6} > 0 \qquad (17.39)$$

by (17.12), (17.20), and (17.28).

If $x \in E \cap B(0, 2) \backslash B(0, 4/3)$, (17.15) says that $\sum_{1 \leqslant k \leqslant l} \varphi_k(x) = 1 - \varphi_0(x)$, so

$$h_l(x) = h_0(x) + [1 - \varphi_0(x)][f_1(x) - f_0(x)] + \sum_{1 \leqslant k \leqslant l} \theta_k(x), \qquad (17.40)$$

and

$$|h_l(x) - f_0(x) - [1 - \varphi_0(x)][f_1(x) - f_0(x)]|$$
$$\leqslant |h_0(x) - f_0(x)| + \sum_{1 \leqslant k \leqslant l} |\theta_k(x)| \leqslant 2 \cdot 10^{-6} \qquad (17.41)$$

by (17.20) and (17.28). But the second coordinate of $f_0(x) + [1 - \varphi_0(x)][f_1(x) - f_0(x)]$ is

$$|x|^2 - 1 + [1 - \varphi_0(x)][|x - b_1|^2 - |x|^2 + 1]$$
$$= \varphi_0(x)[|x|^2 - 1] + [1 - \varphi_0(x)]|x - b_1|^2 \geqslant 1/3 \qquad (17.42)$$

by (17.11) and because $|x| \geqslant 4/3$. So $h_l(x) \neq 0$ in this case as well, h_l has no zero in $E \cap B(0, 2)$, and $N(l) = 0$.

We shall reach the desired contradiction as soon as we prove that

$$N(k) - N(k - 1) \text{ is even for } 1 \leqslant k \leqslant l. \qquad (17.43)$$

This is the moment where we shall use some degree theory. Since it is easier to study the degree for mappings between spheres, we first replace h_k and h_{k-1} with mappings between 2-spheres.

Recall that h_{k-1} does not vanish on A (by definition of A), and that we chose $||\theta_k||_\infty$ so small that this stays true for h_k (see (17.29) and recall that $\varphi_k = 0$ on A, by (17.31) and (17.14)). Set

$$m_t(x) = h_{k-1}(x) + t[h_k(x) + h_{k-1}(x)] = h_{k-1}(x) + t\theta_k(x)$$
$$\text{for } x \in E \cap \overline{B}(x_k, \rho_2) \text{ and } 0 \leqslant t \leqslant 1. \tag{17.44}$$

Thus $m_0 = h_{k-1}$ and $m_1 = h_k$ on $E \cap \overline{B}(x_k, \rho_2)$. Notice that $m_t(x) = h_{k-1}(x) + t\theta_k(x)$ for $x \in E \cap A$ and $0 \leqslant t \leqslant 1$, by (17.29), so $m_t(x) \neq 0$ if we took $||\theta_k||_\infty$ small enough.

Choose $\beta_k > 0$ such that $|m_t(x)| \geqslant \beta_k$ for $x \in E \cap A$ and $0 \leqslant t \leqslant 1$. Set $S_\infty = \mathbb{R}^2 \cup \{\infty\}$, which we see as a 2-dimensional sphere, and define $\pi : \mathbb{R}^2 \to S_\infty$ by

$$\pi(x) = \infty \ \text{ if } |x| \geqslant \beta_k \ \text{ and } \ \pi(x) = \frac{x}{\beta_k - |x|} \ \text{ otherwise.} \tag{17.45}$$

Then set

$$p_t(x) = \pi(m_t(x)) \ \text{ for } x \in E \cap \overline{B}(x_k, \rho_2) \text{ and } 0 \leqslant t \leqslant 1. \tag{17.46}$$

Notice that $p_t(x)$ is a continuous function of x and t, with values in S_∞. By definition of β_k,

$$p_t(x) = \infty \ \text{ for } x \in E \cap A \text{ and } 0 \leqslant t \leqslant 1. \tag{17.47}$$

It will also be more convenient to replace the domain $E \cap \overline{B}(x_k, \rho_2)$ with a subset of the plane. Let us use the same mapping ψ_k as above (near (17.32)), and its inverse γ, which is a biHölder homeomorphism from $E \cap B(x_k, \rho_2)$ to some open subset V' of P_k (for the same reason as before). For $0 \leqslant t \leqslant 1$, set

$$q_t(x) = p_t(\psi_k(x)) \ \text{ for } x \in V' \ \text{ and } \ q_t(x) = \infty \ \text{ for } x \in P_k \setminus V'. \tag{17.48}$$

Let us check that q_t is continuous on $P_k \times [0, 1]$. It is continuous on $V' \times [0, 1]$, because p_t is continuous on $E \cap B(x_k, \rho_2) \times [0, 1]$, and also on $[P_k \setminus \overline{V}'] \times [0, 1]$, because it is constant there. We are left with $\partial V' \times [0, 1]$. But for $x \in \partial V'$, $\psi_k(x)$ lies in $E \cap \partial B(x_k, \rho_2)$ (again by (17.32) and because ψ_k is biHölder), so some neighborhood of $\psi_k(x)$ in $\overline{B}(x_k, \rho_2)$ is contained in A, hence $p_t(w) = \infty$ there (by (17.47)), and $q_t(y) = \infty$ near x (and for all t).

Extend q_t to the sphere $S' = P_k \cup \infty$ by setting $q_t(\infty) = \infty$; obviously q_t is still continuous on $S' \times [0,1]$.

Now we have two mappings q_0 and q_1 from the 2-sphere S' to the 2-sphere S_∞, so we can compute their degrees. See for instance in [Do], to which we shall systematically refer because it contains the needed information rather explicitly. First observe that since q_0 and q_1 are homotopic, they have the same degree (see Proposition 4.2 on page 63 of [Do]).

Fix $t \in \{0, 1\}$, and try to compute the degree of q_t. First observe that the degree (which roughly speaking counts the total number of inverse images of any point of the target space S_∞, counted with orientation and multiplicity) can be computed locally near any point Q of the target space; see Corollary 5.6 on page 67 of [Do]. Here we shall use $Q = 0$. Let us check that for $x \in S'$,

if $q_t(x) = 0$, then $\psi_k(x)$ is a zero of h_{k-1} if $t = 0$, and of h_k if $t = 1$.
$$(17.49)$$
Indeed, if $q_t(x) = 0$, then $x \in V'$ (otherwise $q_t(x) = \infty$), hence $q_t(x) = p_t(\psi_k(x))$ (by (17.48), so $p_t(\psi_k(x)) = 0$, and then $m_t(\psi_k(x)) = 0$ (by (17.46) and (17.45)). Now (17.49) follows from (17.44).

By (17.30), both h_{k-1} and h_k have a finite number of zeroes, so the same thing holds for q_t. Then the degree of q_t (computed at 0) can be computed locally: Proposition 5.8 of [Do] says that if we cover S' by finitely many open sets Y_l, so that each Y_l contains exactly one zero y_l of q_t, then the degree of q_t is the sum of the degrees of the restrictions of q_t to Y_l. In addition, Proposition 5.5 of [Do] allows us to replace each Y_l with any smaller open neighborhood W_l of y_l. Let us check that if W_l is small enough,

$$q_t \text{ is a homeomorphism from } W_l \text{ to } q_t(W_l). \qquad (17.50)$$

Indeed, set $\xi_l = \psi_k(y_l)$; by (17.49), ξ_l is a zero of $h = h_{k-1}$ or h_k, and (17.30) says that it is a non-degenerate simple zero. This means that there is a small ball $B(\xi_l, \rho)$ and a bi-Hölder map γ_l from $E \cap B(\xi_l, \rho)$ to an open set V of a plane, such that $h \circ \gamma_l^{-1}$ is of class C^1 on V and the differential of $h \circ \gamma_l^{-1}$ at $\gamma_l(\xi_l)$ is of rank 2. But it was observed near (17.22) that if we take ρ small enough, the restriction of h to $E \cap B(\xi, \rho)$ is a homeomorphism from $E \cap B(\xi, \rho)$ to the open subset $h(E \cap B(\xi, \rho))$ of $\mathbb{R}^2$. Recall that $h(\xi) = 0$, so if ρ is small enough, $\pi \circ h$ is a homeomorphism from $E \cap B(\xi, \rho))$ to $\pi \circ h(E \cap B(\xi, \rho))$.

Take $W_l = \gamma(E \cap B(x, \rho)) = \psi_k^{-1}(E \cap B(x, \rho))$ above; then $\pi \circ h \circ \psi_k$ is a homeomorphism from W_l to $\pi \circ h(E \cap B(x, \rho)) \subset S_\infty$.

At the same time, y_l is a zero of q_t, so by the proof of (17.49), $q_t(x) = p_t \circ \psi_k(x) = \pi \circ m_t \circ \psi_k(x) = \pi \circ h \circ \psi_k(x)$ near y_l (by (17.48), (17.46), and (17.44)); (17.50) follows.

By (17.50) and Example 5.4 in [Do], the degree at 0 of the restriction of q_t to W_l is ± 1. We add up all these degrees and get that the degree of q_t (on the sphere) is equal, modulo 2, to the number of zeroes of q_t. Since q_0 and q_1 are homotopic, they have the same degree, and q_0 and q_1 have the same number of zeroes modulo 2.

We are now ready to prove (17.43). Since h_{k-1} and h_k coincide out of $B(x_k, \rho_2)$ and have no zero in $E \cap A$, we just need to consider the zeroes in $E \cap B(x_k, \rho_1)$. If $\xi \in E \cap B(x_k, \rho_1)$ is such that $h_{k-1}(x) = 0$, then $p_0(x) = 0$ by (17.44) and (17.46), and $q_0(\psi_k^{-1}(x)) = p_0(x) = 0$ by 0 by (17.48). So every zero of h_{k-1} gives a zero for q_0. The converse is true, by (17.49). Similarly, the number of zeroes for h_k in $E \cap B(x_k, \rho_1)$ is the same as the number of zeroes for q_1. This proves (17.43) and, as explained near (17.43), Proposition 16.24 follows. $\square$

18. MS-minimal sets in $\mathbb{R}^3$

In this section we return to sets of dimension 2 in $\mathbb{R}^3$, and in particular prove Theorem 1.9, which says that every nonempty reduced MS-minimal set in $\mathbb{R}^3$ is a plane, a cone of type $\mathbb{Y}$, or a cone of type $\mathbb{T}$.

We start with a variant of Proposition 16.24 for points of type T, which is valid for (standard) almost-minimal sets, but needs an extra separation assumption. This proposition yields an analogue of Theorem 1.9 for Al-minimal sets, again with an extra separation assumption when E looks like a minimal cone of type $\mathbb{T}$ near infinity (see Proposition 18.29).

We can then apply Proposition 18.29 to MS-minimal sets, because the separation assumption is easy to get in this context, and obtain Theorem 1.9.

The proof of Proposition 16.24 relies on another topoogical argument, which again could already be found in [Tay], when she proves by contradiction that the set $\mathbb{T}$ is a minimal cone. Also, there is a statement similar to Theorem 1.9 in [Mo2], but for integral currents. In this context the separation property comes from the boundary constraints. The proof starts the same way (use a large ball where E looks like a $\mathbb{T}$, assume that there is no point of type $\mathbb{T}$ in the ball, and get a topological contradiction), but either the situation is much simpler with currents, or there is something that I did not get.

The results of this section and the previous ones also apply to MS-almost-minimal sets, but to avoid new discussions about definitions, we simply mention this in Remark 18.44 at the end of the section.

We start with a variant of Proposition 16.24 that finds a T-point when the reduced almost-minimal set E is close enough to a set of type $\mathbb{T}$ in a ball.

PROPOSITION 18.1. — *There is constant $\varepsilon_7 > 0$ with the following property. Let E be a reduced A-almost-minimal set of dimension 2 in an open set $U \subset \mathbb{R}^3$, with the gauge function h. Suppose that*

$$B(x,r) \subset U, \ \ h(2r) \leqslant \varepsilon_7, \ \ \int_0^{2r} h(t)dt/t \leqslant \varepsilon_7, \qquad (18.2)$$

and that there is a minimal cone T of type $\mathbb{T}$, centered at x, such that

$$d_{x,r}(E,T) \leqslant \varepsilon_7. \qquad (18.3)$$

Denote by a_1, a_2, a_3, and a_4 the points of $\partial B(x,r/2)$ whose distance to T is largest, and by $V(a_j)$ the connected component of a_j in $B(x,r) \setminus E$. Suppose in addition that at least three of the $V(a_j)$ are distinct. Then $E \cap B(x, 3r/4)$ contains at least a T-point.

See Definition 4.3 for A-almost-minimal sets, and recall that Definition 4.8 gives an equivalent notion, by Proposition 4.10. See Definition 2.12 for the "reduced sets". Also recall that in $\mathbb{R}^3$, there is only one sort of set of type $\mathbb{T}$: the cones based on a tetrahedron that are described near Figure 2. A T-point is therefore a point $x \in E$ such that every blow-up limit of E at x is of this type (see the definition and discussion near (16.8)).

The author doesn't know whether the separation assumption can be removed. At least the most obvious attempt fails: see Section 19.

By scale invariance, it is enough to prove the proposition when $x = 0$ and $r = 2$. Our plan is to proceed by contradiction, so we shall assume that there is no point of type T in $E \cap B(0, 3/2)$ and try to reach a contradiction. This assumption will make it easier for us to study $E_Y \cap B(0,1)$, where E_Y still denotes the set of Y-points of E.

Let $\alpha > 0$ be small, to be chosen later. Let us first prove that if ε_7 is small enough,

$$\operatorname{dist}(y, L) \leqslant 2\alpha \ \text{ for } y \in E_Y \cap B(0, 3/2), \qquad (18.4)$$

where L denotes the spine of T (a union of four half lines). First observe that if ε_7 is small enough,

$$H^2(E \cap B(y,\alpha)) \leqslant H^2(T \cap B(y,\alpha)) + 10^{-1}\alpha^2 \ \text{ for } y \in E \cap B(0, 3/2), \ (18.5)$$

because we can apply Lemma 16.43 to $B(y, \alpha)$ with $\delta > 0$ as small as we want. The details are the same as for (17.4). If $y \in E \cap B(0, 3/2)$ and $\mathrm{dist}(y, L) \geqslant 2\alpha$, then $H^2(T \cap B(y, \alpha)) \leqslant \pi\alpha^2$, and (18.5) says that $\theta(y, \alpha) \leqslant 11\pi/10$. Then

$$\theta(y) \leqslant e^{\lambda A(\alpha)} \theta(y, \alpha) = e^{\lambda \int_0^\alpha h(2t) \frac{dt}{t}} \theta(y, \alpha) \leqslant e^{\lambda \varepsilon_7} \theta(y, \alpha) < 3\pi/2 \quad (18.6)$$

by (16.9), (16.5), and (18.2), so y is a P-point. So (18.4) holds.

Next we want to show that every point of $E_Y \cap B(0, 4/3)$ has a neighborhood where E_Y is a nice curve. So let $y \in E_Y \cap B(0, 4/3)$ be given. Lemma 16.25 says that for r small, $B(y, r)$ is a biHölder ball of type $\mathbb{Y}$ for E (with $\tau = 10^{-4}$, say) Let $r = r_y$ be such a radius, denote by $f_y : B(y, 2r) \to \mathbb{R}^3$ the corresponding biHölder mapping, and let Y_y be the minimal cone of type Y centered at y that goes with it. Here (15.6)-(15.9) say that

$$|f_y(x) - x| \leqslant 2\tau r \quad \text{for } x \in B(y, 2r), \quad (18.7)$$

$$(1 - \tau)|(x - z)/r|^{1+\tau} \leqslant |(f(x) - f(z))/r| \leqslant (1 + \tau)|(x - z)/r|^{1-\tau}$$
$$\text{for } x, z \in B(y, 2r), \quad (18.8)$$

$$B(y, 19r/10) \subset f(B(y, 2r)), \quad (18.9)$$

and

$$E \cap B(y, 19r/10) \subset f(Y_y \cap B(y, 2r)) \subset E. \quad (18.10)$$

LEMMA 18.11. — *Denote by L_y the spine of Y_y. Then*

$$E_Y \cap B(y, r/2) \subset f_y(L_y \cap B(y, r)) \subset E_Y \cap B(y, 3r/2). \quad (18.12)$$

We start with the first inclusion. Let $z \in E_Y \cap B(y, r/2)$ be given, and set $w = f_y^{-1}(z)$. Clearly $w \in B(y, r)$ (by (18.7)); let us assume that $w \notin L_y$ and find a contradiction. The idea will be that this implies that $\mathbb{R}^3 \setminus E$ has only two local connected components near w, while the fact that $z \in E_Y$ implies that there are three. But we want to check this more carefully. Let us first check that

$$\text{for each } \rho_2 > 0, \text{ we can find } \rho > 0 \text{ such that}$$

$$\text{for } z_1, z_2, z_3 \in B(z, \rho) \setminus E, \text{ we can find a path} \quad (18.13)$$

$$\gamma \subset \overline{B}(z, \rho_2) \setminus E, \text{ that connects two of the } z_j.$$

So let z_1, z_2, and z_3 lie in $B(z, \rho) \setminus E$. Set $w_j = f_y^{-1}(z_j)$ (these are well defined if $\rho \leqslant r$, by (18.9)). By the first half of (18.8), $(1 - \tau)(|w_j - w|/r)^{1+\tau} \leqslant |z_j - z|/r \leqslant \rho/r$, or equivalently

$$|w_j - w| \leqslant \rho_1, \quad \text{with } \rho_1 = r\left[(1-\tau)^{-1}\rho/r\right]^{1/(1+\tau)}. \qquad (18.14)$$

If ρ is small enough, $4\rho_1 \leqslant \text{dist}(w, L_y)$, and Y_y coincides with a plane in $B(w, 2\rho_1)$. Notice that the w_j lie out of Y_y, because they lie in $B(y, 2r)$ and otherwise $z_j = f_y(w_j)$ would lie in E by the second part of (18.10). So one of the segments $[w_j, w_k]$ does not meet Y_y (pick j and k, $j \neq k$, so that w_j and w_k lie on the same side of the plane that coincides with Y_y in $B(w, 2\rho_1)$).

Set $\gamma = f_y([w_j, w_k])$; this is (the support of) a path from z_j to z_k. Recall that $z = f_y(w)$. By (18.14) and the second half of (18.8), $\gamma \subset \overline{B}(z, \rho_2)$, where

$$\rho_2 = (1+\tau)\, r\, (\rho_1/r)^{1-\tau} \leqslant (1+\tau)\, r\left[(1-\tau)^{-1}\rho/r\right]^{(1-\tau)/(1+\tau)}. \qquad (18.15)$$

It is a little unpleasant that ρ_2 is not equivalent to ρ (because f_y is merely biHölder-continuous, and not quasisymmetric), but at least it is as small as we want, which will be enough for (18.13).

Suppose that γ meets E at some point x. Since $\gamma \subset B(y, 19r/10)$, the first part of (18.10) says that $x = f_y(\xi)$ for some $\xi \in Y_y \cap B(y, 2r)$. But f_y is injective, so $\xi \in [w_j, w_k]$ and this is impossible because $[w_j, w_k]$ does not meet Y_y. So γ does not meet E. This proves (18.13).

Next we use the fact that $z \in E_Y$, so Lemma 16.25 says that for t small, $B(z, t)$ is a biHölder ball of type Y for E. Choose ρ_2 so small that this is the case for $t \leqslant 2\rho_2$. Denote by $f_{z,t} : B(z, 2t) \to \mathbb{R}^3$ the corresponding biHölder mapping and by $Z_{z,t}$ the associated cone of type $\mathbb{Y}$ centered at z.

Let ρ be as in (18.13); we can safely assume that $\rho < \rho_2$, so $f_{z,\rho}$ and $Z_{z,\rho}$ are defined. Pick three points $\zeta_j \in B(z, \rho/2)$, lying in the three different connected components of $\mathbb{R}^3 \setminus Z_{z,\rho}$, and set $z_j = f_{z,\rho}(\zeta_j)$. By the analogue of (18.7) the z_j lie in $B(z, \rho) \setminus E$, and by (8.13) there is a path $\gamma \subset \overline{B}(z, \rho_2) \setminus E$ that connects two of the z_j.

Since γ may get way out of $B(z, \rho)$, we cannot find a contradiction immediately in $B(z, \rho)$, and we need to zoom out again. But we want to do this carefully, because we want to keep the information that E should separate the three z_j because $Z_{z,\rho}$ separates the ζ_j.

Let us construct three new paths leaving from the z_j. Denote by $\zeta_{0,j}$ the point of $\partial B(z, \rho)$ that lies in the same connected component of $\mathbb{R}^3 \setminus Z_{z,\rho}$ as ζ_j, and as far as possible from $Z_{z,\rho}$. Set $l_{0,j} = [\zeta_j, \zeta_{0,j}]$. Thus $l_{0,j}$ does not meet $Z_{z,\rho}$ (the components of $\mathbb{R}^3 \setminus Z_{z,\rho}$ are convex), so $g_{0,j} = f_{z,\rho}(l_{0,j})$ does

not meet E (because it is contained in $B(z, 3\rho/2)$ by the analogue of (18.7), and by the first part of the analogue of (18.10)).

Now we iterate, and construct, for $k \geqslant 1$ such that $2^k \rho \leqslant 2\rho_2$, line segments $l_{k,j}$ and arcs $g_{k,j}$, so that in particular

$$g_{k,j} \subset B(z, 3 \cdot 2^{k-1}\rho) \setminus E. \qquad (18.16)$$

Let $k \geqslant 1$ be given, with $2^k \rho \leqslant 2\rho_2$. Thus we can use $f = f_{z,2^k\rho}$ and $Z = Z_{z,2^k\rho}$. Denote by $z_{k-1,j}$ the final endpoint of $g_{k-1,j}$. If (18.16) holds for $k - 1$ (which, by the way, is the case when $k = 1$), then $z_{k-1,j} \in B(z, 3 \cdot 2^{k-2}\rho) \setminus E$, and $\zeta'_{k-1,j} = f^{-1}(z_{k-1,j})$ is defined and lies in $B(z, 2^k\rho) \setminus Z$, by the analogues for f of (18.7) and (18.10). Choose $l_{k,j} = [\zeta'_{k-1,j}, \zeta_{k,j}]$, where $\zeta_{k,j}$ is the point of $\partial B(z, 2^k\rho)$ that lies in the same component of $\mathbb{R}^3 \setminus Z$ as $\zeta'_{k-1,j}$ and as far as possible from Z. As before (for $l_{0,j}$), $l_{k,j}$ does not meet Z and is contained in $\overline{B}(z, 2^k\rho)$. Finally set $g_{k,j} = f(l_{k,j})$; then (18.16) holds, with the same proof as for $g_{0,j}$. Thus we can construct the $g_{k,j}$ by induction.

Next we check that for each k, the three $\zeta_{k,j}$ lie in different components of $\mathbb{R}^3 \setminus Z_{z,2^k\rho}$. This is true for $k = 0$, because the three ζ_j lie in different components of $\mathbb{R}^3 \setminus Z_{z,\rho}$, and $\zeta_{0,j}$ lies in the same component as ζ_j. Now suppose that $k \geqslant 1$ and that this is true for $k - 1$. Observe that $|\zeta'_{k-1,j} - \zeta_{k-1,j}| \leqslant |\zeta'_{k-1,j} - z_{k-1,j}| + |z_{k-1,j} - \zeta_{k-1,j}| \leqslant 2^{k+1}\tau\rho$, by (18.7) for f and $f_{z,2^{k-1}\rho}$. In addition, Z is quite close to $Z_{z,2^{k-1}\rho}$, because they are both close to E in $B(z, 2^{k-1}\rho)$ (use the analogue of (18.10)). The three $\zeta_{k-1,j}$ lie far from $Z_{z,2^{k-1}\rho}$ (by construction), and in different components of $\mathbb{R}^3 \setminus Z_{z,2^{k-1}\rho}$ (by induction assumption). Then the $\zeta'_{k-1,j}$ also lie far from Z, and in different components of $\mathbb{R}^3 \setminus Z$. This is also true for the $\zeta_{k,j}$, because $\zeta_{k,j}$ lies in the same component as $\zeta'_{k-1,j}$. This is what we wanted.

Let m denote the largest exponent k such that $2^k \rho \leqslant 2\rho_2$; thus $2^m \rho > \rho_2$. We just proved that the three $\zeta_{m,j}$ lie in different components of $\mathbb{R}^3 \setminus Z_{z,2^m\rho}$. Denote by g_j the path from z_j to $z_{j,m}$ obtained by concatenation of all the $g_{j,k}$, $0 \leqslant k \leqslant m$. Then $g_j \subset B(z, 3 \cdot 2^{m-1}\rho) \setminus E$, by (18.16). We also have a path $\gamma \subset \overline{B}(z, \rho_2) \setminus E$ that connects two of the z_j, and when we add to it the two corresponding g_j, we get a path $\widehat{\gamma} \subset B(z, 3 \cdot 2^{m-1}\rho) \setminus E$ that connects two $z_{j,m}$. Set $\Gamma = f_{z,2^m\rho}^{-1}(\widehat{\gamma})$. This path does not meet $Z_{z,2^m\rho}$, by (18.10) and the usual argument. Yet it connects two different $\zeta_{m,j}$, a contradiction. This completes our proof of the first inclusion in (18.12).

Now we consider $w \in L_y \cap B(y, r)$ and check that $z = f_y(w) \in E_Y$ (it is clear that $f_y(w) \in B(y, 3r/2)$, by (18.7)). By (18.10), E is locally biHölder-equivalent to a set of type $\mathbb{Y}$ near z, and in particular $\mathbb{R}^3 \setminus E$ has three

local components near z. More precisely, choose three points ζ_1, ζ_2 and ζ_3 in $B(y,r) \setminus Y_y$, that lie in different connected components of $\mathbb{R}^3 \setminus Y_y$. For each small ρ and each $j \leqslant 3$, choose a point $\zeta_{j,\rho} \in B(w,\rho) \setminus Y_y$, that lies in the same component of $\mathbb{R}^3 \setminus Y_y$ as ζ_j. The segment $l_{j,\rho} = [\zeta_j, \zeta_{j,\rho}]$ does not meet Y_y and is contained in $B(y,r)$, so (18.10) says that $\gamma_{j,\rho} = f_y(l_{j,\rho})$ does not meet E. In addition, (18.8) says that the extremities $z_{j,\rho} = f_y(\zeta_{j,\rho})$ lie in $\overline{B}(z,\rho_1)$, where $\rho_1 = r(1+\tau)|\rho/r|^{1-\tau}$. Notice that ρ_1 tends to 0 when ρ tends to 0.

If z is a P-point, Lemma 16.19 says that for ρ_1 small enough, $B(z,\rho_1)$ is a biHölder ball of type P for E. Denote by $f : B(z,2\rho_1) \to \mathbb{R}^3$ the corresponding mapping, and P the corresponding plane through z. The three $f^{-1}(z_{j,\rho})$ lie in $B(z,11\rho_1/10) \setminus P$, by the analogues of (18.7) and (18.10) and because $z_{j,\rho} \notin E$, and at least two of them lie on the same side of P. Choose j and $k \neq j$ such that $\Gamma = [f^{-1}(z_{j,\rho}), f^{-1}(z_{k,\rho})]$ does not meet P. Then $f(\Gamma) \subset B(z,3\rho_1/2) \setminus E$ (by (18.7) and (18.10)). Then $\gamma = \gamma_{j,\rho} \cup f(\Gamma) \cup \gamma_{k,\rho} \subset B(y,r) \setminus E$, so its inverse image $f_y^{-1}(\gamma)$ does not meet Y_y. But γ contains ζ_j and ζ_k, which lie in different components of $\mathbb{R}^3 \setminus Y_y$. This contradiction proves that z is not a P-point.

We assumed that there is no T-point in $B(0,3/2)$, and $y \in E \cap B(0,4/3)$, so it is a Y-point. [We could also exclude the case when y is a T-point, by an argument similar to the proof of the first inclusion in (18.12).] This proves the second half of (18.12), and Lemma 18.11 follows. $\qquad\square$

We shall also need to know that near $\partial B(0,1)$, E_Y is composed of three nice arcs of curves. Let T be the minimal cone of (18.3), and denote by L the spine of T. Thus L is composed of four half-line L_j, and we choose the names so that the point a_j in the statement of Proposition 18.1 lies exactly opposite to L_j. Call z_j the point of $L_j \cap \partial B(0,1)$; thus $z_j = -a_j$. Observe that T coincides with a set of type $\mathbb{Y}$ in $B(z_j, 10^{-1})$. Choose $y_j \in E$ such that

$$|y_j - z_j| \leqslant 2d_{0,2}(E,T) \leqslant 2\varepsilon_7 \tag{18.17}$$

(by (18.3)). If $\varepsilon_7 < 10^{-4}\varepsilon_4$, where ε_4 comes from Lemma 16.51 (applied with $\tau = 10^{-4}$, say), we can apply Lemma 16.51 to $B(y_j, 10^{-3})$, where we use a translation by $y_j - z_j$ of the set of type $\mathbb{Y}$ that coincides with T near z_j, to make sure that the spine goes through y_j. We get that $B(y_j, 10^{-3})$ is a biHölder ball of type $\mathbb{Y}$ for E, with constant τ.

Denote by $f_j : B(y_j, 2 \cdot 10^{-3}) \to \mathbb{R}^3$ the corresponding biHölder mapping, by Y_j the associated minimal cone of type $\mathbb{Y}$ and centered at y_j, and by ℓ_j the spine of Y_j. The proof of Lemma 18.11 yields

$$E_Y \cap B(y_j, 10^{-3}/2) \subset f_j(\ell_j \cap B(y_j, 10^{-3})) \subset E_Y \cap B(y_j, 3 \cdot 10^{-3}/2). \tag{18.18}$$

Set $g_j = \ell_j \cap B(y_j, 10^{-3})$ and $\gamma_j = f_j(g_j)$; these are nice little arcs, and (18.18) says that $\gamma_j \subset E_Y$. Notice that γ_j crosses the annulus $A = B(0, 1 - 10^{-4}) \setminus B(0, 1 + 10^{-4})$, because $|f_j(x) - x| \leqslant \tau 10^{-3}$ for $x \in g_j$ and the extremities of g_j lie on both sides of A (and reasonably far from it). Let us also check that

$$E_Y \cap A \subset \bigcup_{j=1}^{4} \gamma_j. \tag{18.19}$$

Indeed, if $y \in E_Y \cap A$, (18.4) (with $\alpha = 10^{-4}/2$) says that $\operatorname{dist}(y, L) \leqslant 10^{-4}$, so y lies in one of the $B(y_j, 10^{-3}/2)$ (because it lies in A). Hence (18.19) follows from (18.18).

The situation in $B(0, 1)$ will also be easy to control. First observe that

$$E_Y \cap \overline{B}(0, 1) \text{ is compact.} \tag{18.20}$$

Indeed, let $\{y_k\}$ be a sequence in $E_Y \cap \overline{B}(0, 1)$, and assume that it converges to y. Then $y \in E$, because E is closed in U. If y is a P-point, Lemma 16.19 says that every point $x \in E$ near y is a P-point again. [Incidentally, we do not need the whole proof; we could deduce the same result from Lemma 16.11 (and in particular (16.15) with the plane $Z(x, \rho)$), plus (16.9) to control $\theta(x)$.] This is impossible, because $y_k \in E_Y$, so y is not a P-point. It cannot be a T-point either, because we assumed that there is no T-point in $B(0, 3/2)$. So $y \in E_Y$, as needed for (18.20).

Select a small biHölder ball $B(y, r_y)$ for each $y \in E_Y \cap \overline{B}(0, 1)$, as we did for (18.7)-(18.10), and then cover $E_Y \cap \overline{B}(0, 1)$ by a finite collection of $B(y, r_y/10)$, $y \in J$. Thus J is finite and contained in $E_Y \cap \overline{B}(0, 1)$. Call η the minimum of 10^{-5} and the smallest of the $r_y/10$, $y \in J$.

Now pick one of the four γ_j. Recall that γ_j crosses A. Let us continue the branch of γ_j that goes inside $B(0, 1)$, by little steps of diameter comparable to η, by a curve in E_Y. For each extension that we get, and if the endpoint ξ of the extension lies in $\overline{B}(0, 1)$, we can find $y \in J$ such that $\xi \in B(y, r_y/10)$. Then by Lemma 18.11 there is a neighborhood of ξ, that contains $B(\xi, \eta)$, where E_Y coincides with the nice simple curve $f_y(L_y \cap B(y, r_y/2))$. We continue our extension of γ_j by an arc of $f_y(L_y \cap B(y, r_y/2))$ of diameter η, in the direction that was not covered (if it exists). That is, since the curve comes from the outside boundary of A, it has to cover at least one side of $f_y(L_y \cap B(y, r_y/2))$. We continue the path in the other direction.

We continue the construction like this, as long as we do not leave $B(0, 1)$. Observe that when we hit a point x, our curve really crosses a neighborhood of x in E_Y. As a result, our curve is simple: otherwise, consider the first

time where we hit a point x that was already on the curve; we cannot come from any of the two branches of E_Y that leave from x, because they were covered some time ago and x is the first point of return, but then there is no access left, so we cannot have returned to x. In fact, this argument shows that as soon as our curve meets x, our next extension leaves $B(x, \eta)$, and then the curve never returns to $B(x, \eta)$ again.

As a result of this (and because $B(0, 1)$ does not contain too many disjoint balls of radius $\eta/2$), our extension process has to stop after a finite number of steps. This means that our extension of γ_j eventually hits $\partial B(0, 1)$ again. By (18.19), this happens in one of the γ_k. Since our curve is simple, $k \neq j$. So we proved that for each j,

$$\text{there is a path in } E_Y \cap \overline{B}(0, 1)$$
$$\text{that connects } \gamma_j \text{ to some other } \gamma_k. \tag{18.21}$$

Next we want to study the connected components of the complement of E near E_Y. For $y \in E_Y \cap B(0, 4/3)$, denote by $\mathcal{A}(y)$ the collection of connected components V of $B(0, 2) \setminus E$ such that $y \in \overline{V}$. Recall that each $y \in E_Y \cap B(0, 4/3)$ has a small neighborhood where E_Y is biHölder-equivalent to a set of type $\mathbb{Y}$ as in (18.7)-(18.10). Consequently, $\mathcal{A}(y)$ has at most 3 elements, and is a locally constant function of $y \in E_Y \cap B(0, 4/3)$.

We can control $\mathcal{A}(y)$ near the four y_j. For $1 \leqslant j \leqslant 4$, pick $w_j \in \gamma_j \cap \partial B(0, 1)$.

Lemma 18.22. — *For each j,*

$$\mathcal{A}(w_j) = \big\{ V(a_i) \, ; \, i \neq j \big\}, \tag{18.23}$$

where the $V(a_i)$, $1 \leqslant i \leqslant 4$, are as in the statement of Proposition 18.1.

We do not say here that the three components mentioned in (18.23) are different; we shall take care of this later.

Recall that for $1 \leqslant i \leqslant 4$, $a_i = -z_i$, where z_i is the point of the spine of T that was used to define y_i; see a little above (18.17). And $V(a_i)$ is the connected component of a_i in $B(0, 2) \setminus E$.

Fix j and $i \neq j$. Denote by b_i the point of $[a_i, z_j] \cap \partial B(z_j, 10^{-5})$. Notice that

$$\text{dist}([a_i, b_i], E) \geqslant \text{dist}([a_i, b_i], T) - 2\varepsilon_7 \geqslant 10^{-6} \tag{18.24}$$

by simple geometry and (18.3), so $b_i \in [a_i, b_i] \subset V(a_i)$.

To relate $V(b_i) = V(a_i)$ to $\mathcal{A}(w_j)$, we shall use the same biHölder ball $B(y_j, 10^{-3})$ as for (18.18), and the corresponding mapping f_j : $B(y_j, 2 \cdot 10^{-3}) \to \mathbb{R}^3$. Recall that

$$|f_j(x) - x| \leqslant \tau 10^{-3} \ \text{ for } x \in B(y_j, 2 \cdot 10^{-3}), \tag{18.25}$$

$$B(y_j, (2 - \tau)10^{-3}) \subset f_j(B(y_j, 2 \cdot 10^{-3})), \tag{18.26}$$

and

$$E \cap B(y_j, (2 - \tau)10^{-3}) \subset f_j(Y_j \cap B(y_j, 2 \cdot 10^{-3}))) \subset E. \tag{18.27}$$

by (15.6), (15.8) , and (15.9).

Recall that $b_i \in \partial B(z_j, 10^{-5}) \setminus E$ (by (18.24)), and that $|y_j - z_j| \leqslant 2\varepsilon_7$ (by (18.17)). Then $c_i = f_j^{-1}(b_i)$ is well defined (by (18.26)), lies in $B(y_j, 10^{-4})$ (by (18.25)), and out of Y_j (by (18.27)). Denote by W_i the connected component of $B(y_j, 10^{-4}) \setminus Y_j$ that contains c_i, and let us check that

$$\text{the three } W_i, \ i \neq j, \text{ are different.} \tag{18.28}$$

Indeed Y_j is quite close to T near y_j: every point $\xi \in Y_j \cap B(y_j, 2 \cdot 10^{-3})$ lies within $\tau 10^{-3}$ of $f_j(\xi) \in E$, by (18.27), hence $\operatorname{dist}(\xi, T) \leqslant \tau 10^{-3} + 2\varepsilon_7$, by (18.3). Now the b_i lie reasonably far from T and Y_y, and on different sides of Y_y (by (18.24)), and this stays true for the c_i (by (18.25)); this proves (18.28).

Recall that $w_j \in \partial B(0, 1) \cap \gamma_j$, and $\gamma_j = f_j(g_j)$, where $g_j = \ell_j \cap B(y_j, 10^{-3})$ (see below (18.18)). Let $\zeta_j \in g_j$ be such that $f_j(\zeta_j) = w_j$. Then $|\zeta_j - 1| \leqslant \tau 10^{-3}$, by (18.25). The distance from ζ_j to the spine of T (and here this means the half line through z_j) is less than $C(\tau + \varepsilon_7)$, because $\zeta_j \in g_j$ and Y_j is quite close to T near y_j. Then $|\zeta_j - z_j| \leqslant C(\tau + \varepsilon_7)$ (because $|z_j| = |-a_j| = 1$), so ζ_j and w_j both lie in $B(y_j, 10^{-4})$ (recall that $|y_j - z_j| \leqslant 2\varepsilon_7$).

For each $i \neq j$, $c_i \in B(y_j, 10^{-4}) \setminus Y_j$ and $\zeta_j \in B(y_j, 10^{-4}) \cap \ell_j$; then $I = [c_i, \zeta_j) \subset B(y_j, 10^{-4}) \setminus Y_j$ and $f_j(I) \subset B(0, 2) \setminus E$, by (18.27). Hence $f_j(I) \subset V(b_i) = V(a_i)$ (because $f_j(c_i) = b_i$). Since $f_j(I)$ gets as close to $w_j = f(\zeta_j)$ as we want, $V(a_i)$ is one of the elements of $\mathcal{A}(w_j)$.

Conversely, let V be one of the elements of $\mathcal{A}(w_j)$, and pick $x \in V \cap B(w_j, 10^{-4})$. Then $f_j^{-1}(x)$ is defined by (18.26), lies in $B(y_j, 10^{-3})$ by (18.25) and because $w_j \in B(y_j, 10^{-4})$, and lies out of Y_j by (18.27). Thus $f_j^{-1}(x)$ lies in some W_i, by (18.28). Then $J = [f_j^{-1}(x), c_i]$ lies in W_i by convexity of W_i, and $f(J)$ does not meet E by (18.27). Its extremities x

and $b_i = f_i(c_i)$ lie in the same component of $B(0,2) \setminus E$. That is, b_i lies in V, hence $V = V(b_i) = V(a_i)$.

This completes our proof of Lemma 18.22.   □

We are ready to conclude. By assumption, at least three of the four $V(a_i)$ are different. Let us assume that these are the $V(a_i)$, $1 \leqslant i \leqslant 3$. By (18.21), we can connect $w_4 \in \gamma_4$ to some other w_k by a path in $E_Y \cap \overline{B}(0,1)$; without loss of generality, we can assume that $k = 3$. Since $\mathcal{A}(x)$ is locally constant on E_Y, $\mathcal{A}(w_4) = \mathcal{A}(w_3)$ or, by Lemma 18.22, $\{V(a_1), V(a_2), V(a_3)\} = \{V(a_1), V(a_2), V(a_4)\}$. Since the $V(a_j)$, $1 \leqslant j \leqslant 3$, are distinct, $V(a_4) = V(a_3)$.

Let us also apply (18.21) with $j = 2$; we get that $\mathcal{A}(w_2) = \mathcal{A}(w_l)$ for some $l \neq 2$. Since $V(a_2) \in \mathcal{A}(w_l)$, it lies in $\mathcal{A}(w_2)$ as well, so it is equal to $V(a_1)$ or $V(a_3)$ (we already know that $V(a_4) = V(a_3)$). This contradicts our assumption that three $V(a_j)$ are different, and completes the proof of Proposition 18.1.   □

Next we prove a variant of Theorem 1.9 for Al-minimal sets, but with an additional assumption that will allow us to apply Proposition 18.1. Theorem 1.9 will then follow rather easily.

PROPOSITION 18.29. — *Let E be a reduced Al-minimal set of dimension 2 in $\mathbb{R}^3$ (as in Definition 1.6). If there is a minimal cone T of type $\mathbb{T}$, centered at the origin, and a sequence $\{t_k\}$ such that*

$$\lim_{k \to +\infty} t_k = +\infty \quad and \quad \lim_{k \to +\infty} d_{0,t_k}(E,T) = 0, \qquad (18.30)$$

assume in addition that there is a subsequence $\{t_{k_l}\}$ for which the following separation assumption holds. Denote by a_i, $1 \leqslant i \leqslant 4$, the four points of $\partial B(0, t_{k_l}/2)$ that lie the furthest from T. Thus the $-a_i$ are the intersections of $\partial B(0, t_{k_l}/2)$ with the spine of T. Denote by $V(a_i)$ the connected component of a_i in $B(0, t_{k_l}) \setminus E_{k_l}$. Then (assume that)

$$at\ least\ three\ of\ the\ four\ components\ V(a_i)\ are\ distinct. \qquad (18.31)$$

Then E is a reduced minimal cone, i.e., E is empty, a plane, or a cone of type $\mathbb{Y}$ or $\mathbb{T}$.

Since we deal with two dimensional sets in $\mathbb{R}^3$, the reduced minimal cones of type $\mathbb{T}$ are the cones based on the edges of a regular tetrahedron, as defined near Figure 2.

Recall that an Al-minimal set in $\mathbb{R}^3$ is also the same as an almost-minimal set in $\mathbb{R}^3$, with gauge function $h = 0$, as in Definition 4.8 (or even

Definitions 4.1 and 4.3, see Proposition 4.10), so we may apply to E all the results from the previous sections.

By Proposition 5.16, $\theta(x, r) = r^{-2} H^2(E \cap B(x, r))$ is a nondecreasing function of r for each $x \in \mathbb{R}^3$. It is also bounded, by the easy part of (2.16), so it has finite limits

$$\theta(x) = \lim_{r \to 0} \theta(x, r) \text{ and } \theta_\infty = \lim_{r \to +\infty} \theta(x, r). \tag{18.32}$$

It is easy to see that θ_∞ does not depend on x; indeed $B(0, r - |x|) \subset B(x, r) \subset B(0, r + |x|)$ for r large, so $(r - |x|)^2 \theta(0, r - |x|) \leqslant r^2 \theta(x, r) \leqslant (r + |x|)^2 \theta(0, r + |x|)$, which yields $\lim_{r \to +\infty} \theta(x, r) = \lim_{r \to +\infty} \theta(0, r)$ after dividing by r^2 and taking limits.

We know from the discussion below (16.7) that $\theta(x)$ can only take the values $0, \pi, 3\pi/2$, and $d_+ = \text{Argcos}(-1/3)$, corresponding to the various types of blow-up limits that E may have at x. When $x \in E$, $\theta(x) = 0$ is excluded by (2.16).

But we also have that $\theta_\infty \in \{0, \pi, 3\pi/2, d_+\}$. Indeed consider a blow-in sequence $\{E_k\}$, where $E_k = t_k^{-1} E$ for some sequence $\{t_k\}$ that tends to $+\infty$. We can extract a subsequence (which we shall still denote by $\{E_k\}$), which converges to a limit F (in Hausdorff distance on compact subsets of $\mathbb{R}^3$, as in Section 3). Recall that each E_k is a minimal set, by homogeneity. By Lemma 4.7, F is a reduced minimal set in $\mathbb{R}^3$. By (3.4),

$$\begin{aligned} H^2(F \cap B(0, r)) &\leqslant \liminf_{k \to +\infty} H^2(E_k \cap B(0, r)) \\ &= \liminf_{k \to +\infty} t_k^{-2} H^2(E \cap B(0, t_k r)) \\ &= \liminf_{k \to +\infty} r^2 \theta(0, t_k r) = r^2 \theta_\infty. \end{aligned} \tag{18.33}$$

Similarly, Lemma 3.12 says that

$$\begin{aligned} H^2(F \cap \overline{B}(0, r)) &\geqslant \limsup_{k \to +\infty} H^2(E_k \cap \overline{B}(0, r)) \\ &= \limsup_{k \to +\infty} t_k^{-2} H^2(E \cap \overline{B}(0, t_k r)) = r^2 \theta_\infty, \end{aligned} \tag{18.34}$$

so $H^2(F \cap B(0, r)) = r^2 \theta_\infty$ for $r > 0$. By Theorem 6.2, F is a reduced minimal cone. By [Tay], F is the empty set, a plane, or a cone of type $\mathbb{Y}$ or $\mathbb{T}$. Then $\theta_\infty \in \{0, \pi, 3\pi/2, d_+\}$, as announced.

Let us distinguish between cases, depending on the value of θ_∞. If $\theta_\infty = 0$, $\theta(0, r) = 0$ for all r, $E = \emptyset$, and we don't need to worry. If $\theta_\infty = \pi$,

$\theta(0, r) > 0$ for some r, and E is not empty. Pick $x \in E$. We know that $\theta(x) > 0$, by (2.16), so $\theta(x) \geqslant \pi = \theta_\infty$. Then $\theta(x, \cdot)$ is constant and Proposition 6.2 says that E is a cone (in fact, a plane).

Now suppose that $\theta_\infty = 3\pi/2$. As before, we can find a sequence $\{t_k\}$ that tends to $+\infty$, so that the sets $E_k = t_k^{-1}E$ converge to a limit F, and then F is a reduced minimal cone. In fact, (18.33) and (18.34) tell us that the density of F is $3\pi/2$, so F is a cone of type $\mathbb{Y}$ centered at the origin.

Each $t_k^{-1}E$ is a reduced minimal set, and the sequence converges to F. For k large, it satisfies the assumptions of Proposition 16.24 (in the unit ball), so there is a Y-point in $t_k^{-1}E$, hence also in E. For this point x, $\theta(x) = 3\pi/2 = \theta_\infty$. Again, $\theta(x, \cdot)$ is constant and Proposition 6.2 says that E is a minimal cone (here a set of type $\mathbb{Y}$).

We are left with the case when $\theta_\infty = d_+$. The same argument as before gives a blow-in sequence $\{E_k\} = \{t_k^{-1}E\}$ that converges to a reduced minimal cone T of type $\mathbb{T}$ centered at the origin. This time, we do not have a good enough analogue of Proposition 16.24 that would allow us to conclude, and this is why we added an extra hypothesis.

The sequence $\{t_k\}$ satisfies the assumption (18.30), so we have (18.31). This is exactly what we need to apply Proposition 18.1 to the ball $B(0, t_{k_l})$ (for l large enough, so that (18.3) holds). We get the existence of a point of type T in $E \cap B(0, t_{k_l})$. Then $\theta(x) = d_+ = \theta_\infty$ at this point, $\theta(x, \cdot)$ is constant, and E is a minimal cone (of type $\mathbb{T}$), as before. This completes the proof of Proposition 18.29. $\qquad \square$

Proof of Theorem 1.9. — Let us prove Theorem 1.9. Let E be a MS-minimal set in $\mathbb{R}^3$ (see Definition 1.4). By Remark 1.8, E is Al-minimal. By Remark 2.14, E^* (the closed support of the restriction to E of H^2, as in Definition 2.12) is a reduced Al-minimal set. We just need to show that E^* is a reduced minimal cone (see (2.13)).

We want to apply Proposition 18.29 to E^*, so we need to check the separation assumption. So we assume that $\{t_k\}$ is a sequence such that (18.30) holds for E^*, and we want to prove (18.31) (for the same sequence, but k large; we won't need to take a real subsequence).

Let us show the dependence in k in our notation. Call $a_{j,k}$, $1 \leqslant j \leqslant 4$, the four points of $\partial B(0, t_k/2)$ that lie furthest from T, and denote by $V_{j,k}$ the connected component of $a_{j,k}$ in $B(0, t_k) \setminus E^*$. We want to show that for k large, the $V_{j,k}$ are all different.

Suppose instead that two of the $V_{j,k}$ are the same, for instance $V_{1,k}$ and $V_{2,k}$. We want to show that this is impossible, and for this we intend to construct a better competitor F, obtained by removing a big piece of the wall near $a_k = \frac{1}{2}(a_{1,k} + a_{2,k})$. The comparison will be easier once we know a little more about E near a_k.

Set $E_k = t_k^{-1}E^*$, and $a = t_k^{-1}a_k$ (this is a fixed point). By (18.30), the E_k converge to T in $B(0,1)$, so Lemma 3.3 says that

$$\limsup_{k \to +\infty} H^2(E_k \cap B(a, 10^{-1})) \leqslant H^2(T \cap B(a, 10^{-1})) \leqslant 10^{-2}\pi, \quad (18.35)$$

where the last inequality holds because T coincides with a plane in $B(a, 10^{-1})$.

Pick $x_k \in E_k$ as close to a as possible, and apply Lemma 16.19 to E_k in $B(x_k, r_k)$, where we choose r_k such that $9r_k = 10^{-1} - |x_k - a|$ and hence $B(x_k, 9r_k) \subset B(a, 10^{-1})$. The assumption (16.20) holds for k large, because

$$\limsup_{k \to +\infty} H^2(E_k \cap B(x_k, 9r_k)) \leqslant \limsup_{k \to +\infty} H^2(E_k \cap B(a, 10^{-1}))$$
$$\leqslant 10^{-2}\pi \quad (18.36)$$

by (18.35), and $9r_k$ tends to 10^{-1}. So Lemma 16.19 applies, and says that $B(x_k, 10^{-2})$ is a biHölder ball of type $\mathbb{P}$ for E_k. Let $f_k : B(x_k, 2/100) \to \mathbb{R}^3$ be the corresponding mapping, and P_k the associated plane. Let us use f_k to show that

$$B(x_k, 10^{-2}) \setminus E_k \subset 2^{-k}V_{1,k} ; \quad (18.37)$$

this is believable because we assumed that the components $V_{1,k}$ and $V_{2,k}$ coincide, but let us check the details. Let ζ_1 and ζ_2 be the two points of $\partial B(x_k, 10^{-2}) \setminus P_k$ that lie at distance 10^{-2} from P_k (so they are on opposite sides of P_k). If $x \in B(x_k, 10^{-2}) \setminus E_k$, $f_k^{-1}(x)$ is defined (by (15.8)), lies in $B(x_k, \frac{11}{1000})$ (by (15.6)) and away from P_k (by (15.9)), so one of the two line segments $I = [f_k^{-1}(x), \zeta_i]$ does not meet P_k. Then $f_k(I)$ is an arc in $B(x_k, \frac{12}{1000}) \setminus E$ (by (15.6) and (15.9)) that goes from x to $z_i = f(\zeta_i)$. Thus (18.37) will follow if we show that both z_i lie in $2^{-k}V_{1,k}$. Now $\text{dist}(\zeta_i, P_k) = 10^{-2}$, so

$$\text{dist}(z_i, E_k \cap B(x_k, (2 - \tau)10^{-2}))$$
$$\geqslant \text{dist}(z_i, f(P_k \cap B(x_k, 2 \cdot 10^{-2})))$$
$$\geqslant \text{dist}(\zeta_i, P_k \cap B(x_k, 2 \cdot 10^{-2})) - 2\tau 10^{-2} \quad (18.38)$$
$$\geqslant (2 - 2\tau)10^{-2}$$

by (15.9), (15.6), and the definition of ζ_i. Since

$$\text{dist}(z_i, E_k \setminus B(x_k, (2 - \tau)10^{-2})) \geqslant (2 - \tau)10^{-2} - |z_i - x_k| \geqslant (1 - 2\tau)10^{-2}$$
$$(18.39)$$

by (15.6) and because $\zeta_i \in \partial B(x_k, 10^{-2})$, we get that

$$\operatorname{dist}(z_i, T) \geqslant \operatorname{dist}(z_i, E_k) - d_{0,1}(E_k, T) \geqslant 10^{-2}/2 \tag{18.40}$$

for k large, because the E_k converge to T in $B(0,1)$ (by (18.30)). Then z_i lies in the same component as a_1 or a_2 in $B(0,1) \setminus E_k$, or equivalently $2^k z_i \in V_{1,k} \cup V_{2,k}$. But we assumed that $V_{1,k} = V_{2,k}$, so $z_i \in 2^{-k}V_{1,k}$, as needed for (18.37).

Set $F_k = E \setminus B(r_k x_k, 10^{-3}r_k)$. This set may be a little larger than $r_k(E_k \setminus B(x_k, 10^{-3}))$ if E is not reduced, and we need to pay some attention to the difference because did not verify that E^* is a MS-minimal set. Let us check that

$$F_k \text{ is a } MS\text{-competitor for } E, \tag{18.41}$$

as in Definition 1.1. Let us take $R = r_k$ in that definition; (1.2) is clear, and we just need to prove (1.3). Let z_1, z_2 lie in $\mathbb{R}^3 \setminus [E \cup B(0, r_k)]$. We need to show that if E separates z_1 from z_2, F_k separates them too. Suppose it does not, and pick a path γ from z_1 to z_2, that does not meet F_k. Since γ meets E and the only difference is $E \cap B(r_k x_k, 10^{-3}r_k)$, γ has to go through ∂B, where we set $B = B(r_k x_k, 2 \cdot 10^{-3}r_k)$.

Call z_1' the first point of $\gamma \cap \partial B$ when we go from z_1 to z_2 along γ, and z_2' the last point of $\gamma \cap \partial B$. The part of γ between z_1 and z_1' does not meet E, because it does not meet F_k (and F_k coincides with E out of B). Similarly, the part of γ between z_2 and z_2' does not meet E either. So we just need to show that E does not separate z_1' from z_2' and we shall reach the desired contradiction with our assumption that E separates z_1 from z_2.

Notice that the z_i' lie on $\partial B \setminus E$, hence also on $\partial B \setminus E^*$. Then the $r_k^{-1}z_i'$ lie in $\partial B(x_k, 2 \cdot 10^{-3}) \setminus E_k$ (recall that $E_k = r_k^{-1}E^*$). By (18.37), the z_i' lie in $V_{1,k}$, which means that there is a path $\gamma_1 \subset B(0, r_k) \setminus E^*$ that goes from z_1' to z_2'.

We can even make γ_1 piecewise linear. Now γ_1 may meet $\Delta = E \setminus E^*$, but observe that $H^2(\Delta) = 0$ by (2.13). Let us modify γ_1 slightly, so that it does not meet E. First we move the endpoints of the segments that compose γ_1 very slightly (to make sure that the new path will not meet E^*), so that the endpoints lie out of Δ too. This is very easy to arrange, because the complement of Δ is dense. Note that we can leave z_1' and z_2' alone, since they already lie out of E.

For each of the segments that compose our new path, the orthogonal projection of Δ on a plane perpendicular to that segment has vanishing Lebesgue measure because $H^2(\Delta) = 0$, so we may translate the segment a

tiny bit so that it no longer meets Δ. We do this for each segment, and then connect the new endpoints together by very short line segments to make a new path γ_2 from z'_1 to z'_2. Now γ_2 does not meet E. So z'_1 and z'_2 lie in the same component of $\mathbb{R}^3 \setminus E$, we reached the desired contradiction, and (18.41) holds.

Since E is MS-minimal, Definition 1.4 says that $H^2(E \setminus F_k) \leqslant H^2(F_k \setminus E) = 0$. But this is not true:

$$\begin{aligned} H^2(E \setminus F_k) &= H^2(E \cap B(r_k x_k, 10^{-3} r_k)) \\ &= r_k^2 H^2(E_k \cap B(x_k, 10^{-3})) > 0 \end{aligned} \tag{18.42}$$

by (2.13) and (2.16), for instance.

This last contradiction completes our proof of (18.31); Proposition 18.29 applies, and Theorem 1.9 follows. $\square$

Remark 18.43. — So we got a complete description of MS-minimal sets in $\mathbb{R}^3$, but unfortunately not of Al-minimal sets. It is tempting to try to deduce the separation assumption in Proposition 18.29 from the closeness of E to a cone of type $\mathbb{T}$ and its local description of E in Theorem 16.1. But we shall see in Section 19 that it is not too hard to construct sets $E \subset \mathbb{R}^3$ that are very close to a cone of type $\mathbb{T}$ in the unit ball, satisfy the conclusions of Theorem 16.1, but do not have any T-point and only separate $\mathbb{R}^3$ locally into two connected components only. So a proof of the analogue of Theorem 1.9 for Al-minimal sets would need to use Al-minimality one more time.

Remark 18.44. — It is possible to define a notion of MS-almost-minimal sets in $U_1 \mathbb{R}^3$, as we did for A'-almost-minimal sets in Definition 4.8. That is, if E is a closed subset of $U \subset \mathbb{R}^3$ such that $H^2(E \cap \overline{B}) < +\infty$ for every compact ball $\overline{B} \subset U$, we say that E is a MS-almost-minimal set in U, with gauge function h, if

$$H^2(E \setminus F) \leqslant H^2(F \setminus E) + h(r) r^2 \tag{18.45}$$

whenever $\overline{B}(x, r) \subset U$ and F is another closed set in U such that $F \setminus \overline{B}(x, r) = E \setminus \overline{B}(x, r)$, and

$$\begin{aligned} &F \text{ separates } y \text{ from } z \text{ whenever } y, z \in U \setminus (\overline{B}(x, r) \cup E) \\ &\qquad \text{are separated by } E. \end{aligned} \tag{18.46}$$

When $h = 0$ and $U = \mathbb{R}^3$, we get the minimal sets of Definition 1.4.

We claim (but we shall not check) that the following things are true. First, E^* is a MS-almost-minimal set in $U \subset \mathbb{R}^3$, with the same gauge

function as E. See "Coral pairs" on pages 58-60 of [D2], but the main point is our proof of (18.41).

Next, all the results from the previous sections apply to MS-almost-minimal sets. We cannot say directly that this is because all MS-almost-minimal sets are A'-almost-minimal, even though we can localize the proof of Remark 1.8. Indeed the Al-competitors $F = \varphi_1(E)$, where the family $\{\varphi_t\}$ satisfies (2.2)-(2.6), are not necessarily MS-competitors as above, because even though $\widehat{W}$ in (2.6) is compact and contained in U, it is not sure that it is contained in a compact ball $\overline{B}(x,r) \subset U$. But we claim that this is not a problem, that all sets $\widehat{W}$ used in and for this paper are contained in compact balls $\overline{B}(x,r) \subset U$ with a diameter that is not too large.

Finally, we can even remove the separation assumption in Proposition 18.29 when we deal with MS-almost-minimal sets, by the final argument in our proof of Theorem 1.9.

19. A set $E \subset \mathbb{R}^3$ that looks like a cone of type $\mathbb{T}$ but has no T-point

In this section we try to describe a closed set $E \subset \mathbb{R}^3$ that coincides outside of the unit ball with a set of type $\mathbb{T}$ centered at the origin, but is locally C^1-equivalent to a plane or a set of type $\mathbb{Y}$ near each of its points. This set is probably never minimal, so it is only a counterexample to what would be a wrong way to prove an analogue of Theorem 1.9 for Al-minimal sets. It is quite possible that this set is the same as the one depicted in Figure 1.1.1 in [LM] and Figure 3 of [Mo3], following a suggestion of R. Hardt.

Even though we shall try to give a reasonable description, it is probable that the best way for the reader to convince herself that nothing goes wrong with the topology is to construct her own paper model with scissors and glue.

So we start from the minimal set T centered at the origin, and call L its spine. Thus L is composed of four half lines L_j, $1 \leqslant j \leqslant 4$. Call x_j the point of $L_j \cap \partial B(0,1)$. We first construct two faces of E and the set E_Y of Y-points of E.

Call P_1 the plane that contains $L_1 \cup L_2$, and then pick a nice smooth curve $\Gamma_1 \subset P_1$ that has $L_1 \setminus B(0,1/2)$ and $L_2 \setminus B(0,1/2)$ as its ends. The simplest is to add to $L_1 \setminus B(0,1/2)$ and $L_2 \setminus B(0,1/2)$ the arc of circle that goes from $L_1 \cap \partial B(0,1/2)$ to $L_2 \cap \partial B(0,1/2)$ and is tangent to each L_i at the point of $L_i \cap \partial B(0,1/2)$. But the reader could take a smoother, equivalent

curve. Call F_1 the convex component of $P_1 \setminus \Gamma_1$; this will be one of the faces of E; it is just a little smaller than the corresponding face of T.

Similarly, call P_4 the plane that contains $L_3 \cup L_4$, and pick a nice curve $\Gamma_4 \subset P_4$ that contains $L_3 \setminus B(0, 1/2)$ and $L_4 \setminus B(0, 1/2)$. Also call F_4 the convex component of $P_4 \setminus \Gamma_4$.

At this point, we have two faces of E (F_1 and F_4), and we also decide that $E_Y = \Gamma_1 \cup \Gamma_4$. Observe that here Γ_1 and Γ_4 are neither knotted nor braided; this will make our discussion and the construction of paper models simplest, but the case when the Γ_i are knotted and linked could be more interesting , because then it could be harder to prove that E is not minimal.

Before we continue with the construction, we should observe that when we do all the construction below, we shall be able to preserve the two symmetries, with respect to the planes P_1 and P_4 respectively.

There is a little more of E that we could construct without thinking. We know that a neighborhood of each Γ_i will be composed of three thin stripes that touch Γ_i. We already know one of these stripes, which comes from F_i , and we could brutally construct the two other ones: we would just need to draw two little curves vaguely parallel to Γ_i, and make sure that the two new stripes meet F_i neatly along Γ_i, with 120° angles. This is what is suggested by Figure 10. We shall not really need this, but maybe knowing this will help the reader follow the description of F below.

The remaining set $F = E \setminus [\overline{F}_1 \cup \overline{F}_4]$ will be a little harder to describe. The contact with F_1 and F_4 is along the Γ_i, and through the four stripes alluded to above. Now F is actually a single connected face (we shall see that it is even oriented). Let us use the symmetry of E and F with respect to P_1 to simplify our description.
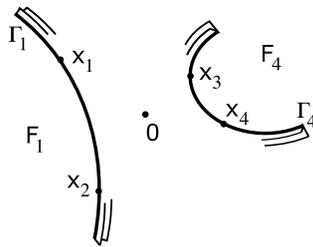

Figure 10
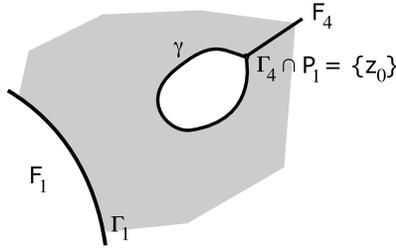
Figure 11. — Intersection with P_1

First, the intersection $F \cap P_1$ is just a vaguely round closed curve γ, with a singularity at the point of $\Gamma_4 \cap P_1$ (let us denote it by z_0), which is actually excluded (because it lies in F_4) and where γ has two half tangents that make $120°$ angles. See Figure 11, but do not pay attention to the shaded area yet. The face F crosses γ perpendicularly to P_1; it also approaches Γ_1 from both sides of P_1, with $60°$ angles.

Call F_+ the part of F that lies on the same side of P_1 as x_3. Out of $B(0,1)$, F_+ coincides with the union of two faces of T (namely the face $T_{1,3}$ bounded by L_1 and L_3, and the face $T_{2,3}$). Let us look at $F_+ \cap B(0,1)$. Topologically, it is a disk, and its boundary is composed as follows. We may start with the curve $\Gamma_1 \cap B(0,1)$, say from x_1 to x_2. Then we follow the arc $\partial B(0,1) \cap T_{2,3}$, up to x_3. Then we follow Γ_4 from x_3 to z_0, then we turn around γ, and return along Γ_4 from z_0 to x_3. After this we run along the arc $\partial B(0,1) \cap T_{1,3}$ up to x_1, and we are back where we started. Notice that if we include the part of Γ_4 from x_3 to z_0, $\overline{F}_+ \cap B(0,1)$ is a topological tube (because of the gluing along Γ_4). Similarly, the whole $\overline{F}_+$ is a half-infinite tube, with a gluing along half of Γ_4 and only one "circular" boundary along γ. The reader may try to get a mental picture of $\overline{F}_+$ by looking at the shaded part of Figure 19.2 that lives above Γ_1 and outside of γ, and imagining that it is lifted in the third direction. In particular, imagine that $F_4 \cap P_1$ is lifted until it becomes the upper half of Γ_4. Also recall that $\overline{F}_+$ leaves from Γ_1 and both sides of Γ_4 with $60°$ angles, and vertically from γ.

Let F_- denote the part of F that lies on the other side of P_1. Since $\overline{F}_-$ is obtained from $\overline{F}_+$ by symmetry with respect to P_1, it is also a tube, bounded by γ and glued along the other half of Γ_4. Finally $\overline{F}$ is an infinite tube, obtained from the two pieces by gluing along γ. The fact that F_+ and F_- live on different sides of P_1 makes it easy to believe that nothing weird (like self-intersections) happens when we do the gluing. It is easy to see that $\overline{F}$ can be oriented too.

Hopefully the properties above give a precise idea of the set F, and of $E = \overline{F}_1 \cup \overline{F}_4 \cup F$. The fact that E is locally equivalent to a plane or a Y is easy to check, and E coincides with T out of $B(0,1)$.

Let us say a few words about the connected components of $\mathbb{R}^3 \setminus E$. It should be expected from the proof of Theorem 1.9 that E does not separate $\mathbb{R}^3$ into three or four components, and indeed $\mathbb{R}^3 \setminus E$ has only two components. More precisely, call $y_j = -x_j$ the point of $\partial B(0,1)$ which is opposite to L_j, and denote by U_j the connected component of $\mathbb{R}^3 \setminus E$ that contains y_j. It is easy to see that $U_3 = U_4$, because there is a path from y_3 to y_4 that does not meet E and crosses P_1 inside γ. Similarly, $U_1 = U_2$; rather than trying to use the fact that P_1 and P_4 can be interchanged in the construction, the reader should just look at Figure 11 and imagine a curve that runs in P_1 along Γ_1, at a short distance from but just opposite to F_1. This curve does not meet E, and it goes from U_3 to U_4. Of course the two ways to see that $U_1 = U_2$ are similar, since the analogue of γ here is a loop in $P_4 \setminus F_1$ that ends at the point of $P_4 \cap \Gamma_1$ and passes through the closest point of γ on the other side.

It is also true that $U_1 \neq U_3$; this is not easy to see without a paper model, and not so important, so we skip the proof. Notice that once we know that $U_1 = U_2$ and $U_3 = U_4$, there is no contradiction with the fact that the list $\mathcal{A}(y)$ of components that touch y is locally constant along the curves Γ_i; in both cases, $\mathcal{A}(y) = \{U_1, U_3\}$ (with one of the components counted twice).

This completes our description of E. Unless we are extremely lucky, our example E will not be minimal. We can try to make it minimal inside $B(0,1)$ under the constraint that E coincides with T out of $B(0,1)$; the author does not know whether this is really possible. If we can make $H^2(E \cap B(0,1))$ smaller than $H^2(T \cap B(0,1))$, then we can probably minimize $H^2(E \cap B(0,1))$, but then E is not globally minimal in $\mathbb{R}^3$, because a dilation would do better. If $H^2(E \cap B(0,1)) > H^2(T \cap B(0,1))$ for E as above, the only minimum is T (obtained as limit of contractions of sets E). But there is still the unlikely possibility that the minimal value of $H^2(E \cap B(0,1))$ is precisely $H^2(T \cap B(0,1))$, and then we could have an exotic minimal set E. The case when E is obtained with curves Γ_i that are linked in a complicated way could even be more fun.

20. Simple computations of density

For the convenience of the reader, we do the simple computations of density for the minimal cones of types $\mathbb{Y}$ and $\mathbb{T}$.

Let us first remind the reader that $H^2(Y \cap B(0,r)) = 3\pi r^2/2$ when Y is a set of type $\mathbb{Y}$ centered at the origin, simply because $Y \cap B(0,r)$ is composed of three half disks of radius r.

Next let T be a set of type $\mathbb{T}$ centered at the origin. We want to check that

$$H^2(T \cap B(0,r)) = d_+ r^2, \text{ with } d_+ = 3\text{Argcos}(-1/3). \qquad (20.1)$$

Now T is composed of six equal angular sectors, so if $\alpha \in (\pi/2, \pi)$ denotes the angle of these sectors, $H^2(T \cap B(0,r)) = 3\alpha r^2$. We just need to show that $\cos \alpha = -1/3$.

Recall that T is the cone over the edges of a regular tetrahedron centered at the origin; we may obtain such a tetrahedron by starting from the cube $[-1,1]^3$, and selecting four vertices (out of the eight) that are not mutually contiguous (see Figure 12). The distance between two vertices is $2\sqrt{2}$, and the distance form a vertex to the origin is $\sqrt{3}$. Thus (see Figure 13) $\sin(\alpha/2) = \frac{\sqrt{2}}{\sqrt{3}}$ and so $\cos \alpha = 1 - 2\sin^2(\alpha/2) = 1 - 4/3 = -1/3$, as needed. This proves (20.1).
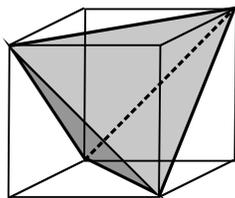

Figure 12. — A tetrahedron in its box
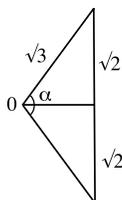
Figure 13. — A face

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