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MARCELLO D'ABBICCO, GIOVANNI TAGLIALATELA

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Some results on the well-posedness for systems with time dependent coefficients

Marcello D'Abbicco[(1)] and Giovanni Taglialatela[(2)]

Abstract. — We consider hyperbolic systems with time dependent coefficients and size 2 or 3. We give some sufficient conditions in order the Cauchy Problem to be well-posed in $\mathcal{C}^\infty$ and in Gevrey spaces.

Résumé. — On considère des systèmes hyperboliques dont les coefficients ne dépendent que du temps. On donne des conditions suffisantes pour que le problème de Cauchy soit bien posé en $\mathcal{C}^\infty$ et dans les espaces de Gevrey.

1. Introduction

In this note we consider the well-posedness of the Cauchy Problem for weakly hyperbolic systems whose coefficients depend only on the time variable of dimension 2 and 3:

$$\begin{cases} LU \equiv \partial_t U - \displaystyle\sum_{j=1}^n A_j(t)\partial_{x_j} U - B(t)U = f(t,x) & (t,x) \in [0,T] \times \mathbb{R}^n\,, \\ U(0,x) = U_0(x)\,, & x \in \mathbb{R}^n\,. \end{cases} \tag{1.1}$$

We assume that

$$A(t,\xi) := |\xi|^{-1} \sum_{j=1}^n A_j(t)\xi_j$$

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(1) Dipartimento di Matematica, Università di Bari, via E. Orabona 4, 70125 Bari – Italy.
(2) Dipartimento di Scienze Economiche e Metodi Matematici, Facoltà di Economia, Università di Bari, via C. Rosalba 53, 70124 Bari – Italy

is real valued and (weakly) hyperbolic, that is its eigenvalues are real (not necessarily distinct) for any $\xi \in \mathbb{R}^n \setminus \{0\}$. The coefficients of B may be complex valued.

We start with 2×2 systems. Let:

$$A(t,\xi) = \begin{pmatrix} d_1(t,\xi) & a(t,\xi) \\ b(t,\xi) & d_2(t,\xi) \end{pmatrix}, \qquad B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix};$$

in this case the assumption of hyperbolicity means that:

$$h := \frac{(d_1 - d_2)^2}{4} + ab \geqslant 0 .$$

In [N], Nishitani gave a necessary and sufficient condition in order the Cauchy Problem for any 2×2 system in one space variable and analytic coefficients to be well-posed in $\mathcal{C}^\infty$. On the other hand, Mencherini and Spagnolo [MS2] proved that Nishitani's condition is sufficient for the well-posedness in all Gevrey classes, if the coefficients of $A = A(t)$ are $\mathcal{C}^\infty$ and $B \equiv 0$.

In this note we prove an analogous result if the coefficients are $\mathcal{C}^\chi$, and we do not assume that $B \equiv 0$.

To state our result, we recall some notations introduced in [N] and [MS2]. Let:

$$\delta = \delta(t,\xi) := \frac{d_1 - d_2}{2}, \qquad \tau(t,\xi) := \frac{d_1 + d_2}{2},$$

$$\widetilde{A} = \widetilde{A}(t,\xi) := A - \tau I = \begin{pmatrix} \delta & a \\ b & -\delta \end{pmatrix}$$

$$k = k(t,\xi) := \frac{1}{2}\,\|\widetilde{A}\|^2 = \frac{1}{2}(a^2 + b^2) + \delta^2 ,$$

$$a^\# = a^\#(t,\xi) := \frac{a+b}{2} + i\frac{d_1 - d_2}{2}, \qquad c^\# = c^\#(t,\xi) := i\frac{a-b}{2},$$

$$D^\# = D^\#(t,\xi) := c^\# a^{\#\prime} - a^\# c^{\#\prime} = \frac{1}{2}[(a' - b')\delta - (a-b)\delta'] - \frac{i}{2}[a'b - ab'] .$$

Simple calculations show that:

$$0 \leqslant h = |a^\#|^2 - |c^\#|^2 \leqslant |a^\#|^2 + |c^\#|^2 = k , \tag{1.2}$$

$$\frac{1}{2}\,k \leqslant \left|a^\#(t)\right|^2 = \frac{1}{2}\,k + \frac{1}{2}\,h \leqslant k . \tag{1.3}$$

Notations. — 1) Let $f(t)$ and $g(t)$ be defined in $[0,T]$; we will write $f \lesssim g$ to mean that there exists a positive constant C such that

$$f(t) \leqslant C\, g(t)\,, \quad \text{for } t \in [0,T]\,.$$

Analogously, if $f(t,\xi)$ and $g(t,\xi)$ are symbols defined in $[0,T] \times \mathbb{R}^n$, we will write $f \lesssim g$ to mean that there exists a positive constant C such that

$$f(t,\xi) \leqslant C\, g(t,\xi) \quad \text{for } (t,\xi) \in [0,T] \times \mathbb{R}^n\,.$$

In both cases, we will write $f \approx g$ to mean that $f \lesssim g$ and $g \lesssim f$.

2) We write for brevity $A \in \mathcal{C}^\chi$ (resp. $A \in \mathcal{C}^\infty$, resp. $A \in \mathcal{A}$) if A_j belong to $\mathcal{C}^\chi\big([0,T]\big)$ (resp. to $\mathcal{C}^\infty\big([0,T]\big)$, resp. are analytic on $[0,T]$) for $j = 1, \ldots, n$.

Then we have:

Theorem 1.1. — *Assume that $A \in \mathcal{C}^\chi$, $\chi \geqslant 2$, $B \in \mathcal{C}^1\big([0,T]\big)$, and suppose that there exists $\alpha \geqslant 0$ such that for any $\varepsilon \in \,]0,\varepsilon_0]$ we have:*

$$\int_0^T \frac{\big|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,B)\big| + \big|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,\overline{B})\big|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\, dt \lesssim \varepsilon^{-\frac{1}{\alpha}}\,. \tag{1.4}$$

Then the Cauchy Problem (1.1) is well-posed in γ^s for any $s < s_0$, where:

$$s_0 := 1 + \frac{\min(\chi,\alpha)}{2}\,.$$

Moreover, if the coefficients of A are analytic, and

$$\int_0^T \frac{\big|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,B)\big| + \big|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,\overline{B})\big|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\, dt \lesssim \log\frac{1}{\varepsilon}\,, \tag{1.5}$$

then the Cauchy Problem (1.1) is well-posed in $\mathcal{C}^\infty$.

Remark 1.2. — Since

$$\big|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,B)\big| + \big|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,\overline{B})\big| \lesssim \sqrt{k}\,, \tag{1.6}$$

we may assume $\alpha \geqslant 2$ in (1.4), and the Cauchy Problem (1.1) is always well-posed in γ^s, for any $s < 2$.

Remark 1.3. — If the coefficients do not depend on x, Nishitani's condition [N] reduces to the following:

$$t\big|D^{\#}-a\mathrm{tr}(\widetilde{A}\,B)\big|+t\big|D^{\#}-a\mathrm{tr}(\widetilde{A}\,\overline{B})\big|\lesssim\sqrt{h\,k}\,. \tag{1.7}$$

It's easy to see that (1.7) implies (1.5). Indeed, using (1.3) and (1.6), we have:

$$\begin{aligned}\int_0^T\frac{\big|D^{\#}-a^{\#}\mathrm{tr}(\widetilde{A}\,\overline{B})\big|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt &\lesssim \int_0^{\sqrt{\varepsilon}}\frac{\sqrt{k}}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt\\ &\quad+\int_{\sqrt{\varepsilon}}^T\frac{1}{t}\,\frac{\sqrt{h\,k}}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt\\ &\lesssim \int_0^{\sqrt{\varepsilon}}\frac{1}{\sqrt{\varepsilon}}\,dt+\int_{\sqrt{\varepsilon}}^T\frac{1}{t}\,dt\lesssim\log\frac{1}{\varepsilon}\,,\end{aligned}$$

for $\varepsilon\in\,]0,\varepsilon_0]$.

Hypothesis (1.4) is obviously satisfied if the following conditions hold true:

$$\int_0^T\frac{|D^{\#}|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt\lesssim\varepsilon^{-1/\alpha}\,, \tag{1.8}$$

$$\int_0^T\frac{\big|\mathrm{tr}(\widetilde{A}\,B)\big|}{\sqrt{h+\varepsilon}}\,dt\lesssim\varepsilon^{-1/\alpha}\,, \tag{1.9}$$

whereas, hypothesis (1.5) is satisfied if:

$$\int_0^T\frac{|D^{\#}|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt\lesssim\log\frac{1}{\varepsilon}\,, \tag{1.10}$$

$$\int_0^T\frac{\big|\mathrm{tr}(\widetilde{A}\,B)\big|}{\sqrt{h+\varepsilon}}\,dt\lesssim\log\frac{1}{\varepsilon}\,. \tag{1.11}$$

Conditions (1.8) and (1.10) correspond to the homogeneous case (i.e. $B\equiv 0$), as treated in [MS2]. Indeed our proof of Theorem 1.1 is the natural generalization of that in [MS2].

The assumptions on the regularity of the coefficients of A can be relaxed in the following way. Let

$$\begin{aligned}\mathcal{L}_\chi &:=\left\{f\in\mathcal{C}^1\Big([0,T]\times\mathbb{R}^n\setminus\{0\}\Big)\;\Big|\;\int_0^T\frac{\big|f'(t,\xi)\big|}{\big|f(t,\xi)\big|+\varepsilon}\,dt\lesssim\varepsilon^{-2/\chi}\right\},\\ \mathcal{L}_\infty &:=\left\{f\in\mathcal{C}^1\Big([0,T]\times\mathbb{R}^n\setminus\{0\}\Big)\;\Big|\;\int_0^T\frac{\big|f'(t,\xi)\big|}{\big|f(t,\xi)\big|+\varepsilon}\,dt\lesssim\log\frac{1}{\varepsilon}\right\}.\end{aligned}$$

If we assume

$$a\,,\,b\,,\,\delta\in\mathcal{L}_{\chi}\,,\quad h\in\mathcal{L}_{2\chi}\,,\tag{1.12}$$

instead of $A\in\mathcal{C}^{\chi}$, and

$$a\,,\,b\,,\,\delta\,,h\in\mathcal{L}_{\infty}\,,\tag{1.13}$$

instead of $A\in\mathcal{A}$, Theorem 1.1 still holds true.

Indeed, a simple calculation shows that if we assume (1.12) (resp. (1.13)), we have:

$$\int_0^T\frac{|a^{\#'}|+|c^{\#'}|}{\sqrt{k+\varepsilon}}+\frac{|h'|}{h+\varepsilon}\lesssim\varepsilon^{-\frac{1}{\chi}}\qquad(\text{resp.}\lesssim\log\frac{1}{\varepsilon})\,,\tag{1.14}$$

and, as we will show in §2, these estimates are enough to prove our results.

In [CJS] it is proved that if $f\in\mathcal{C}^{\chi}$ is a real positive function then $f\in\mathcal{L}_{2\chi}$; more precisely for any $\varepsilon\in\,]0,\varepsilon_0]$ we have:

$$\int_0^T\frac{|f'(t,\xi)|}{f(t,\xi)+\varepsilon}\,dt\lesssim\varepsilon^{-\frac{1}{\chi}}\big\|f(\cdot,\xi)\big\|_{\chi}^{\frac{1}{\chi}}\,,\tag{1.15}$$

whereas, in [S], it is shown that if f is a complex valued $\mathcal{C}^{\chi}$ function, then $f\in\mathcal{L}_{\chi}$, and for any $\varepsilon\in\,]0,\varepsilon_0]$:

$$\int_0^T\frac{|f'(t,\xi)|}{|f(t,\xi)|+\varepsilon}\,dt\lesssim\varepsilon^{-\frac{2}{\chi}}\big\|f(\cdot,\xi)\big\|_{\chi}^{\frac{2}{\chi}}\,.\tag{1.16}$$

A general sufficient condition in order f to belong to $\mathcal{L}_{\infty}$ is given in [CN1] [O, Lemma 1], where it is shown that if there exists $N\in\mathbb{N}$ such that $f'(\cdot,\xi)=0$ has at most N roots for any $\xi\in\mathbb{R}^n$, then $f\in\mathcal{L}_{\infty}$. Examples of functions in $\mathcal{L}_{\infty}$ not verifying the above conditions are given in [DAR].

If $A\in\mathcal{C}^{\infty}$ has *finite degeneracy*, Theorem 1.1 can be improved. By *finite degeneracy* we mean that the discriminant h has only finite order zeroes:

$$\sum_{j=0}^{\infty}\Big|\partial_t^j h(t,\xi)\Big|\neq 0\,,\quad\text{for all } t\in[0,T]\text{ and }\xi\neq 0\,.$$

By a compactness argument, if A has finite degeneracy we may find $\varkappa\in\mathbb{N}$ such that:

$$\sum_{j=0}^{\varkappa}\Big|\partial_t^j h(t,\xi)\Big|\neq 0\,,\quad\text{for all } t\in[0,T]\text{ and }\xi\neq 0\,.\tag{1.17}$$

Theorem 1.4. — *Assume that $A \in \mathcal{C}^\infty$ has finite degeneracy, and that*

$$\left|D^\# - a^\# \operatorname{tr}(\widetilde{A}\, B)\right| + \left|D^\# - a^\# \operatorname{tr}(\widetilde{A}\, \overline{B})\right| \lesssim h^\gamma \sqrt{k}\,, \tag{1.18}$$

holds true for some $\gamma > 0$. Then

1. *if $\gamma + \dfrac{1}{\varkappa} \geqslant \dfrac{1}{2}$, the Cauchy Problem (1.1) is well-posed in all Gevrey spaces and in $\mathcal{C}^\infty$;*
2. *if $\gamma + \dfrac{1}{\varkappa} < \dfrac{1}{2}$, the Cauchy Problem (1.1) is well-posed in all Gevrey spaces γ^s with*

$$1 \leqslant s < \frac{1-\gamma}{\dfrac{1}{2} - \left(\gamma + \dfrac{1}{\varkappa}\right)}\,.$$

If (1.10) is satisfied we can replace (1.18) with

$$\left|\operatorname{tr}(\widetilde{A}\, B)\right| \lesssim h^\gamma\,.$$

Since (1.18) is always satisfied with $\gamma = 0$, we have the following consequence of Theorem 1.4:

Corollary 1.5. — *Assume that $A \in \mathcal{C}^\infty$ has finite degeneracy. Then*

1. *if $\varkappa \leqslant 2$, the Cauchy Problem (1.1) is well-posed in all Gevrey spaces and in $\mathcal{C}^\infty$ for any B;*
2. *if $\varkappa > 2$, the Cauchy Problem (1.1) is well-posed in all Gevrey spaces γ^s with*

$$1 \leqslant s < \frac{2\varkappa}{\varkappa - 2}\,,$$

for any B.

We will give other applications of Theorem 1.1 and 1.4 in section 4.

Next, we consider the Cauchy Problem for 3×3 first order systems.

Theorem 1.6. — *Assume that $A \in \mathcal{C}^3$, $B \in \mathcal{C}^2$ and the eigenvalues $\{\lambda_1(t,\xi), \lambda_2(t,\xi), \lambda_3(t,\xi)\}$ of $A(t,\xi)$ satisfy the estimates:*

$$|\lambda_1 - \lambda_2| \approx t^\ell\,, \tag{1.19}$$

$$|\lambda_1 - \lambda_3| \approx t^\varkappa \quad \text{and} \quad |\lambda_2 - \lambda_3| \approx t^\varkappa\,, \tag{1.20}$$

for some $1 \leqslant \varkappa \leqslant \ell$.

Then the Cauchy Problem (1.1) is well-posed in γ^s *for any* $s < s_0$, *where:*

$$s_0 := \begin{cases} \dfrac{3\varkappa}{2\varkappa - 1} & \text{if } \varkappa \geqslant 2 \quad , \\ 2 + \dfrac{1}{\ell} & \text{if } \varkappa = 1 \quad . \end{cases} \tag{1.21}$$

Remark 1.7. — Theorem 1.6 extends to first order systems as result by Colombini [C], for third order equations.

Remark 1.8. — The hypothesis (1.19) and (1.20) can be weakened. Instead of (1.19) and (1.20), we can assume that

$$\begin{gathered} |\lambda_1 - \lambda_2| \gtrsim t^\ell \\ |\lambda_1 - \lambda_3| \gtrsim t^\varkappa \quad \text{and} \quad |\lambda_2 - \lambda_3| \gtrsim t^\varkappa . \end{gathered}$$

If we assume some additional conditions on the structure of A, then the result of Theorem 1.6 can be improved.

Theorem 1.9. — *Assume that* $A \in \mathcal{C}^3$, $B \in \mathcal{C}^2$ *and the eigenvalues* $\{\lambda_1(t,\xi), \lambda_2(t,\xi), \lambda_3(t,\xi)\}$ *of* $A(t,\xi)$ *satisfy conditions (1.19) and (1.20), for some* $1 \leqslant \varkappa \leqslant \ell$.

Let us define

$$Q := \Big(\frac{\mathrm{tr} A}{3}\, I - A\Big)^2 ,$$

and assume that there exists $\alpha > 0$ *such that:*

$$\|Q\|_\infty + \|Q'\|_\infty \lesssim t^\alpha . \tag{1.22}$$

Then the Cauchy Problem (1.1) is well-posed in γ^s *for any* $s < s_\alpha$, *where:*

$$s_\alpha := \begin{cases} \dfrac{3\varkappa - \alpha}{2\varkappa - \alpha - 1} & \text{if } \alpha \leqslant \varkappa - 2 , \\ \dfrac{2\ell + \varkappa - \alpha}{\ell + \varkappa - \alpha - 1} & \text{if } \varkappa - 2 < \alpha \leqslant \varkappa , \\ \dfrac{2\ell}{\ell - 1} & \text{if } \alpha \geqslant \varkappa ,\ \ell > 1 , \\ +\infty & \text{if } \alpha \geqslant \ell = \varkappa = 1 . \end{cases} \tag{1.23}$$

If moreover $\varkappa = \ell = 1$, *and* $\alpha \geqslant 1$, *then the Cauchy Problem is well-posed in* $\mathcal{C}^\infty$ *too.*

Remark 1.10. — Assuming a Levi-type condition on B:

$$\|BQ - Q'\|_\infty + \|QB + Q'\|_\infty \lesssim t^\alpha \,, \tag{1.24}$$

instead of condition (1.22), we get the same conclusion of Theorem 1.9.

If we assume more specific conditions on the structure of $A(t,\xi)$, then the result of Theorem 1.9 can be improved.

Example 1.11. — We assume that $A(t,\xi)$ is triangular:

$$A(t,\xi) = \begin{pmatrix} \lambda_1 & a_1 & a_2 \\ 0 & \lambda_2 & a_3 \\ 0 & 0 & \lambda_3 \end{pmatrix} ,$$

and that conditions (1.19)-(1.20) hold true for some $1 \leqslant \varkappa \leqslant \ell$.

If we assume that:

$$|a_1| + |a_2| + |a_3| \lesssim t^\alpha \,, \tag{1.25}$$

with $\alpha \leqslant \varkappa$, then condition (1.22) holds true.

More generally, if we replace condition (1.22) by:

$$|a_1| + |a_3| \lesssim t^\alpha \,, \tag{1.26}$$

then the conclusion of Theorem 1.9 holds true (we remark that conditions (1.22) and (1.26) are independent).

Remark 1.12. — The hypothesis $B \in \mathcal{C}^{m-1}$, $m = 2, 3$, in our Theorems is related to the fact that in both cases we transform the system into a system of order m with diagonal principal part. This procedure is carried out by composing the operator with an operator whose principal part is the cofactor matrix of A, which is of order $m-1$.

We expect that with a different technique one may require only $B \in \mathcal{C}^0$.

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2. Proof of Theorem 1.1

By Duhamel principle, we can assume $f \equiv 0$.

Let U be a solution of the system (1.1) and let $V(t,\xi) := \widehat{U}(t,\xi)$ be the Fourier transform with respect to the x variables of U; V satisfies the system

$$\begin{cases} V' = iA(t,\xi)\,|\xi|\,V + B(t)V\,, \\ V(0,\xi) = V_0(\xi)\,. \end{cases}$$

Let

$$\widetilde{V}(t,\xi) := \exp\big(i\,\theta(t,\xi)\,|\xi|\big)\,V(t,\xi)\,,$$

where

$$\theta(t,\xi) := \int_0^t \tau(s,\xi)\,ds\,, \qquad \tau(t,\xi) := \frac{d_1(t,\xi)+d_2(t,\xi)}{2}\,,$$

so that the Cauchy Problem (1.1) is transformed into

$$\begin{cases} \widetilde{V}' = i\,\widetilde{A}(t,\xi)\,|\xi|\;\widetilde{V} + B(t)\widetilde{V}\,, \\ \widetilde{V}(0,\xi) = V_0(\xi)\,. \end{cases}$$

Next, we make the substitution $\widetilde{V} = P\,V^{\#}$, with:

$$P := \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\,,$$

in order to obtain the equivalent system:

$$\begin{cases} {V^{\#}}' = iA^{\#}(t,\xi)\,|\xi|\,V^{\#} + B^{\#}(t)V^{\#}\,, \\ V^{\#}(0,\xi) = V_0^{\#}(\xi)\,, \end{cases}$$

where

$$A^{\#} := P^{-1}\widetilde{A}P = \begin{pmatrix} c^{\#} & a^{\#} \\ \overline{a}^{\#} & -c^{\#} \end{pmatrix}\,, \qquad B^{\#} := P^{-1}\widetilde{B}P\,,$$

with $a^{\#}$ and $c^{\#}$ defined in the introduction.

Note that, since $\mathrm{tr}(A^{\#}) = 0$, we have:

$${A^{\#}}^2 = hI_2\,,$$

$$A^{\#}B^{\#} - {B^{\#}}^{co}A^{\#} = A^{\#}B^{\#} + (A^{\#}B^{\#})^{co} = \mathrm{tr}(A^{\#}B^{\#})I_2 = \mathrm{tr}(\widetilde{A}B)I_2\,.$$

To simplify the notations, in the following we omit the # and we write A, B and V instead of $A^{\#}$, $B^{\#}$ and $V^{\#}$.

Let:

$$M := \partial_t + iA(t,\xi)\,|\xi| + C_\varepsilon(t,\xi) - B^{co}(t)\,,$$

where C_ε will be chosen in the following. It's clear that if we prove the well-posedness of ML and LM, then the well-posedness for L follows. We show only the well-posedness of ML, the proof for LM being completely analogue.

We have:

$$\begin{aligned} ML &= I_2\partial_t^2 - iA'\,|\xi| - iA\,|\xi|\,\partial_t - B' - B\partial_t + iA\,|\xi|\,\partial_t + h - iAB\,|\xi| \\ &\qquad + C_\varepsilon\partial_t - iC_\varepsilon A\,|\xi| - C_\varepsilon B - B^{co}\partial_t + iB^{co}A\,|\xi| + B^{co}B \\ &= I_2\partial_t^2 + h\,|\xi|^2\,I_2 - iK\,|\xi| + \Gamma\partial_t + \widetilde{B}\,, \end{aligned}$$

where:

$$\begin{aligned} K &:= A' + \mathrm{tr}(AB)I_2 + C_\varepsilon A\,, \\ \Gamma &:= C_\varepsilon - B - B^{co}\,, \\ \widetilde{B} &:= -C_\varepsilon B - B' + B^{co}B\,. \end{aligned}$$

Let us define the *approximated energy*:

$$E_\varepsilon(t,\xi) := |V'(t,\xi)|^2 + (h+\varepsilon)\,|\xi|^2\,|V(t,\xi)|^2\;, \qquad |\xi| \geqslant 1\,,$$

We have:

$$\begin{aligned} E_\varepsilon'(t,\xi) &= 2\mathrm{Re}(V'',V') + 2h'\,|\xi|^2\,\mathrm{Re}(V,V) + 2(h+\varepsilon)\,|\xi|^2\,\mathrm{Re}(V,V') \\ &= 2\,|\xi|\,\mathrm{Re}(iKV,V') - 2\mathrm{Re}(\Gamma V',V') - 2\mathrm{Re}(\widetilde{B}V,V') \\ &\qquad + 2h'\,|\xi|^2\,\mathrm{Re}(V,V) + 2\varepsilon\,|\xi|^2\,\mathrm{Re}(V,V')\,. \end{aligned}$$

Now, since $|V'| \leqslant \sqrt{E_\varepsilon}$ and $|\xi|\,|V| \leqslant \dfrac{1}{\sqrt{h+\varepsilon}}\sqrt{E_\varepsilon}$, we get:

$$E_\varepsilon'(t,\xi) \leqslant C\left[\frac{|K|}{\sqrt{h+\varepsilon}} + |\Gamma| + \frac{|\widetilde{B}|}{|\xi|\,\sqrt{h+\varepsilon}} + \frac{|h'|}{h+\varepsilon} + \frac{\varepsilon\,|\xi|}{\sqrt{h+\varepsilon}}\right] E_\varepsilon(t,\xi)\,,$$

and, using Grönwall's Lemma:

$$E_\varepsilon(t,\xi) \lesssim \exp\left[\int_0^T\!\left(\frac{|K|}{\sqrt{h+\varepsilon}} + |\Gamma| + \frac{|\widetilde{B}|}{|\xi|\,\sqrt{h+\varepsilon}} + \frac{|h'|}{h+\varepsilon} + \frac{|\xi|\,\varepsilon}{\sqrt{h+\varepsilon}}\right)dt\right] E_\varepsilon(0,\xi)\,.$$

In order to estimate the first three terms, we choose

$$C_\varepsilon := \operatorname{diag}(c_{11}, \overline{c}_{11}), \quad \text{where} \quad c_{11} := -\frac{2\, a'\overline{a}}{2\,|a|^2 + \varepsilon},$$

so that:

$$K = \begin{pmatrix} c' + \operatorname{tr}(AB) - \dfrac{2\, c\overline{a}a'}{2\,|a|^2+\varepsilon} & \dfrac{a'\,\varepsilon}{2\,|a|^2+\varepsilon} \\ \dfrac{\overline{a}'\,\varepsilon}{2\,|a|^2+\varepsilon} & \overline{c}' + \operatorname{tr}(AB) - \dfrac{2\,\overline{c}a\overline{a}'}{2\,|a|^2+\varepsilon} \end{pmatrix}.$$

We have:

$$\int_0^T \frac{|K_{11}|}{\sqrt{h+\varepsilon}}\,dt \leqslant 2\int_0^T \frac{|\overline{a}|\,|ca' - ac' - a\operatorname{tr}(AB)|}{(2\,|a|^2+\varepsilon)\sqrt{h+\varepsilon}}\,dt$$
$$+\int_0^T \frac{|c'|\,\varepsilon}{(2\,|a|^2+\varepsilon)\sqrt{h+\varepsilon}}\,dt + \int_0^T \frac{|\operatorname{tr}(AB)|\,\varepsilon}{(2\,|a|^2+\varepsilon)\sqrt{h+\varepsilon}}\,dt = I + II + III\,.$$

Now assume that Assumption (1.4) holds true. Using (1.3) and (1.14) we have:

$$\begin{aligned} I &\leqslant 2\int_0^T \frac{\left|D^{\#} - a^{\#}\operatorname{tr}(\widetilde{A}\,B)\right|}{\sqrt{(k+\varepsilon)\,(h+\varepsilon)}}\,dt \lesssim \varepsilon^{-\frac{1}{\alpha}}\,, \qquad (2.1)\\ II &\leqslant \int_0^T \frac{|c'|\,\varepsilon}{(k+\varepsilon)\sqrt{h+\varepsilon}}\,dt \leqslant \int_0^T \frac{|c'|}{\sqrt{k+\varepsilon}}\,dt \lesssim \varepsilon^{-\frac{1}{\chi}}\\ III &\lesssim \int_0^T \frac{\varepsilon\sqrt{k}}{(k+\varepsilon)\sqrt{h+\varepsilon}}\,dt \leqslant \frac{T}{2}\,. \end{aligned}$$

The term $\displaystyle\int_0^T \frac{|K_{22}|}{\sqrt{h+\varepsilon}}\,dt$ is estimated in the same way thanks to condition (1.4) whereas the terms

$$\int_0^T \frac{|K_{12}|}{\sqrt{h+\varepsilon}}\,dt = \int_0^T \frac{|K_{21}|}{\sqrt{h+\varepsilon}}\,dt$$

are estimated thanks to (1.14).

Using again (1.14), we have:

$$\int_0^T |\Gamma| + \frac{|h'|}{h+\varepsilon}\,dt \lesssim \varepsilon^{-\frac{1}{\chi}}\,,$$

whereas:

$$\int_0^T \frac{|\widetilde{B}|}{|\xi|\sqrt{h+\varepsilon}}\,dt \lesssim \frac{\varepsilon^{-\frac{1}{\chi}}}{\sqrt{\varepsilon}\,|\xi|}\,, \qquad \int_0^T \frac{|\xi|\,\varepsilon}{\sqrt{h+\varepsilon}}\,dt \lesssim \sqrt{\varepsilon}\,|\xi|\,. \tag{2.2}$$

Hence we get:

$$E_\varepsilon(t,\xi) \lesssim \exp\left[\varepsilon^{-\frac{1}{\chi}} + \varepsilon^{-\frac{1}{\alpha}} + \frac{\varepsilon^{-\frac{1}{\chi}}}{\sqrt{\varepsilon}\,|\xi|} + \sqrt{\varepsilon}\,|\xi|\right] E_\varepsilon(0,\xi)\,. \tag{2.3}$$

Choosing $\varepsilon = |\xi|^{-\frac{2\beta}{\beta+2}}$, with $\beta := \min(\chi,\alpha)$, we get:

$$E_\varepsilon(t,\xi) \lesssim \exp\bigl(|\xi|^{\frac{2}{\beta+2}}\bigr) E_\varepsilon(0,\xi)\,,$$

and we conclude the proof by standard argument based on Paley-Wiener Theorem (see e.g. [CJS]).

If (1.5) holds true and the coefficients of A satisfy (1.13), we have:

$$I\,,\ II\,,\ \int_0^T |\Gamma| + \frac{|h'|}{h+\varepsilon}\,dt \lesssim \log\frac{1}{\varepsilon}\,,$$

which gives

$$E_\varepsilon(t,\xi) \lesssim \exp\left[\log\frac{1}{\varepsilon} + \frac{1}{\sqrt{\varepsilon}\,|\xi|} + \sqrt{\varepsilon}\,|\xi|\right] E_\varepsilon(0,\xi)\,,$$

instead of (2.3). Choosing $\varepsilon = |\xi|^{-2}$, we get:

$$E_\varepsilon(t,\xi) \lesssim |\xi|^M\, E_\varepsilon(0,\xi)\,,$$

for some $M > 0$, and this gives the well-posedness in $\mathcal{C}^\infty$.

3. Proof of Theorem 1.4

The proof of Theorem 1.4 is similar to that of Theorem 1.1, but we use the following Lemma to improve some estimates.

Lemma 3.1 ([CIO]). — *If $h(t,\xi)$ verifies (1.17), then, for any $\eta \geqslant 0$, there exists M_η and ε_0 positive such that for any $\varepsilon \in (0,\varepsilon_0]$, we have:*

$$\int_0^T \frac{1}{\bigl(h(t,\xi)+\varepsilon\bigr)^\eta}\,dt \leqslant \begin{cases} M_\eta & \text{if } \eta < 1/\varkappa\,, \\ M_\eta \log\dfrac{1}{\varepsilon} & \text{if } \eta = 1/\varkappa\,, \\ M_\eta\,\varepsilon^{\frac{1}{\varkappa}-\eta} & \text{if } \eta > 1/\varkappa\,. \end{cases}$$

Assume at first $\gamma + \frac{1}{\varkappa} < \frac{1}{2}$.

If (1.18) holds true, then, using Lemma 3.1, we have:

$$I \leqslant \int_0^T \frac{\left|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,B)\right|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt \lesssim \int_0^T \frac{dt}{(h+\varepsilon)^{1/2-\gamma}} \lesssim \varepsilon^{\frac{1}{\varkappa}+\gamma-\frac{1}{2}}\,, \tag{3.1}$$

instead of (2.1), and

$$\int_0^T \frac{|\xi|\,\varepsilon}{\sqrt{h+\varepsilon}}\,dt \lesssim \varepsilon^{\frac{1}{\varkappa}+\frac{1}{2}}\,|\xi|\;,$$

rather than (2.2).

Hence we get:

$$E_\varepsilon(t,\xi) \quad \lesssim \exp\left[\varepsilon^{\frac{1}{\varkappa}+\gamma-\frac{1}{2}} + \frac{1}{\sqrt{\varepsilon}\,|\xi|} + \varepsilon^{\frac{1}{\varkappa}+\frac{1}{2}}\,|\xi|\right] E_\varepsilon(0,\xi)\,,$$

instead of (2.3). Choosing $\varepsilon = |\xi|^{-1/(1-\gamma)}$, we get:

$$E_\varepsilon(t,\xi) \quad \lesssim \exp\left[|\xi|^{\left(\frac{1}{2}-(\frac{1}{\varkappa}+\gamma)\right)/(1-\gamma)}\right] E_\varepsilon(0,\xi)\,.$$

If $\gamma + \frac{1}{\varkappa} \geqslant \frac{1}{2}$, we have:

$$I \leqslant \int_0^T \frac{\left|D^\# - a^\# \mathrm{tr}(\widetilde{A}\,B)\right|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt \lesssim \int_0^T \frac{dt}{(h+\varepsilon)^{1/2-\gamma}} \lesssim \log\frac{1}{\varepsilon}\,,$$

hence:

$$E_\varepsilon(t,\xi) \lesssim \exp\left[\log\frac{1}{\varepsilon} + \frac{1}{\sqrt{\varepsilon}\,|\xi|} + \varepsilon^{\frac{1}{\varkappa}+\frac{1}{2}}\,|\xi|\right] E_\varepsilon(0,\xi)\,,$$

instead of (2.3). Choosing $\varepsilon = |\xi|^{-2\varkappa/(\varkappa+2)}$, we get:

$$E_\varepsilon(t,\xi) \lesssim |\xi|^M\; E_\varepsilon(0,\xi)\,,$$

for some constant M.

4. Examples

A simple sufficient condition for (1.10) is given in [MS2, Prop. 2].

We give here some equivalent forms.

LEMMA 4.1. — *Assume that one of the following equivalent conditions is satisfied:*

$$C_1\,\delta^2 + ab \geqslant 0\,, \qquad \textit{for some } C_1 \in [0,1[, \tag{4.1}$$

$$|\delta| \leqslant C_2\sqrt{h}\,, \qquad \textit{for some } C_2 > 1, \tag{4.2}$$

$$|ab| \leqslant C_3\,h\,, \qquad \textit{for some } C_3 > 1. \tag{4.3}$$

If $a, b, \delta \in \mathcal{L}_\chi$ (resp. $\in \mathcal{L}_\infty$), then (1.8) with $\alpha = \chi$ (resp. (1.10)) holds true.

Proof. — To prove the equivalence between the conditions (4.1), (4.2) and (4.3), we note that if $ab \geqslant 0$ then all the conditions are satisfied, hence we can assume $ab < 0$.

To prove the equivalence between (4.1) and (4.2), we remark that condition (4.1) is equivalent to

$$(1 - C_1)\delta^2 \leqslant \delta^2 + ab = h\,,$$

which is equivalent to (4.2) with $C_2 := (1 - C_1)^{-1/2}$.

We prove the equivalence between (4.1) and (4.3). If (4.1) is satisfied, then

$$|ab| = -ab = \frac{1}{1 - C_1}(C_1\,ab - ab) \leqslant \frac{1}{1 - C_1}(C_1\,ab + C_1\,\delta) = \frac{C_1}{1 - C_1}\,h\,,$$

hence (4.3) is verified with $C_3 := \dfrac{C_1}{1 - C_1}$.

Conversely, if (4.3) is satisfied, then

$$C_1\,\delta^2 + ab = C_1\,h + (1 - C_1)ab \geqslant [C_1 - (1 - C_1)C_3]h = 0\,,$$

if we choose $C_1 := \dfrac{C_3}{C_3 - 1} < 1$.

Using (1.16), we have:

$$\int_0^T \frac{\big|(a-b)'\delta\big|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt \quad \leqslant \int_0^T \frac{|a'| + |b'|}{|a| + |b| + \sqrt{\varepsilon}}\,\frac{|\delta|}{\sqrt{h+\varepsilon}}\,dt \leqslant \varepsilon^{-\frac{1}{\chi}}$$

$$\int_0^T \frac{|a'b|}{\sqrt{(h+\varepsilon)(k+\varepsilon)}}\,dt \quad \leqslant \int_0^T \frac{|a'|}{|a| + \sqrt{\varepsilon}}\,\frac{|a\,b| + |b|\sqrt{\varepsilon}}{\sqrt{h+\varepsilon}}\,dt \leqslant \varepsilon^{-\frac{1}{\chi}}\,,$$

and the other terms are estimated in a similar way. □

Condition (4.1) is satisfied, in particular, if A is *pseudosymmetric* [DAS1], that is if $ab \geqslant 0$. Pseudosymmetric matrices have been introduced in [DAS1], where the corresponding homogeneous Cauchy problem is studied. Non homogeneous Cauchy problem has been studied in [MS2] (see Example 4.4 below).

Condition (4.1) is satisfied also if A is *uniformly symmetrizable*, that is if there exists an invertible matrix $P = P(t, \xi)$ such that

$$\|P\| + \|P^{-1}\| \leqslant M \,,$$

and $P^{-1}AP$ is Hermitian. Indeed A is uniformly symmetrizable, if, and only if (cf. [MS1, Prop. 1]),

$$h(t) \geqslant \eta\, k(t)\,, \tag{4.4}$$

for some constant $\eta \leqslant 1$ and condition (4.4) is equivalent to

$$(1-\eta)\delta^2 + ab \geqslant \frac{\eta}{2}\,(a^2+b^2)\,,$$

hence (4.1) holds true. Then from Theorem 1.1 we get the following result:

Corollary 4.2. — *Assume that A is uniformly symmetrizable, and $B \in \mathcal{C}^1$.*

1. *If the coefficients of A satisfy (1.13)* (*in particular if they are analytic*)*, then (1.1) is well-posed in $\mathcal{C}^\infty$ [N], [CN1], [DAS3].*
2. *If the coefficients of A satisfy (1.12)* (*in particular if they are $\mathcal{C}^\chi$, $\chi \geqslant 2$*)*, then (1.1) is well-posed in γ^s for $s < 1 + \dfrac{\chi}{2}$ [CN1].*

We should remark that in [N] and [CN1] the coefficients of B may depend also on x, whereas in [DAS3] the same result is proved for $N \times N$ systems.

Note that A is symmetric if, and only if, $h = k$.

Corollary 4.3. — *If $A \in \mathcal{C}^\infty\big([0,T]\big)$ verifies the condition*

$$C_0\delta^2 + ab \geqslant \eta\, t^{2p}(a^2+b^2)\,, \tag{4.5}$$

for some $C_0 \in [0,1[$, $\eta > 0$, $p \geqslant 0$, and $B(t) \in \mathcal{C}^1\big([0,T]\big)$, then

1. *if $p \leqslant 1$ the Cauchy Problem (1.1) is well-posed in all Gevrey classes; if the coefficients of A are analytic it is also $\mathcal{C}^\infty$ well-posed;*
2. *if $p > 1$ the Cauchy Problem (1.1) is γ^s well-posed, for any s such that $1 \leqslant s < \dfrac{p}{p-1}$.*

If $A \in \mathcal{C}^\infty\big([0,T]\big)$ *verifies the condition*

$$C_0\delta^2 + ab \geqslant \eta\,(a^2+b^2)^{1/(2\gamma)}\,, \tag{4.6}$$

h *vanishes at order at most* $\varkappa$ *and* $B \in \mathcal{C}^1$, *then:*

1. *if* $\gamma + \dfrac{1}{\varkappa} \geqslant \dfrac{1}{2}$ *the Cauchy Problem (1.1) is* $\mathcal{C}^\infty$ *well-posed;*
2. *if* $\gamma + \dfrac{1}{\varkappa} < \dfrac{1}{2}$ *the Cauchy Problem (1.1) is* γ^s *well-posed, for any* s *such that*

$$1 \leqslant s < \frac{1-\gamma}{\dfrac{1}{2} - \Big(\gamma + \dfrac{1}{\varkappa}\Big)}\,.$$

Proof. — First of all, we note that (4.5) and (4.6) imply (4.1) and hence (1.10). Moreover, since

$$\big|\mathrm{tr}(\widetilde{A}\,B)\big| \lesssim \sqrt{k}\,,$$

we have:

$$\int_0^T \frac{\big|\mathrm{tr}(\widetilde{A}\,B)\big|}{\sqrt{h+\varepsilon}}\,dt \lesssim \int_0^T \sqrt{\frac{k}{h+\varepsilon}}\,dt\,.$$

If assumption (4.5) holds true, then we can assume, with no loss of generality, that η is small with respect to T so that

$$2\,\eta\,t^{2p}k \leqslant C_0\delta^2 + ab + 2\,\eta\,t^{2p}\,\delta^2 \leqslant h\,, \tag{4.7}$$

hence we have:

$$\int_0^T \sqrt{\frac{k}{h+\varepsilon}}\,dt = \int_0^{\sqrt{\varepsilon}} \sqrt{\frac{k}{\varepsilon}}\,dt + \int_{\sqrt{\varepsilon}}^T \frac{1}{\sqrt{2\,\eta}\;t^p}\sqrt{\frac{h}{h+\varepsilon}}\,dt$$

$$\lesssim \begin{cases} \varepsilon^{(1-p)/2}\,, & \text{if } p>1\quad, \\ \log\Big(1+\dfrac{1}{\varepsilon}\Big)\,, & \text{if } p=1\quad, \end{cases}$$

which shows that (1.9) is satisfied with $\alpha = \dfrac{2}{p-1}$, if $p>1$, and (1.11) is satisfied if $p=1$. The result follows from Theorem 1.1.

If (4.6) holds true, using (4.2) we have:

$$2\,k = a^2+b^2+2\,\delta^2 \leqslant \eta^{-2\,\gamma}\,(C_0\delta^2+ab)^{2\,\gamma}+2\,\delta^2 \leqslant \eta^{-2\,\gamma}\,h^{2\,\gamma}+2\,C_2^2\,h \leqslant C_\eta\,h^{2\,\gamma}\,.$$

Hence, using Lemma 3.1:

$$\int_0^T \sqrt{\frac{k}{h+\varepsilon}}\,dt \lesssim \int_0^T \frac{1}{(h+\varepsilon)^{1/2-\gamma}}\,dt = \begin{cases} M(h)\,, & \text{if } \gamma+\frac{1}{\varkappa}>\frac{1}{2}\,, \\ M(h)\log\frac{1}{\varepsilon}\,, & \text{if } \gamma+\frac{1}{\varkappa}=\frac{1}{2}\,, \\ M(h)\,\varepsilon^{1/\varkappa+\gamma-1/2} & \text{if } \gamma+\frac{1}{\varkappa}<\frac{1}{2}\,, \end{cases}$$

and we get easily the result from Theorem 1.4. □

Example 4.4. — If $a=t^\alpha$ and $b=t^\beta$, we can choose

$$\gamma=\frac{\min(\alpha,\beta)}{\alpha+\beta}=\frac{\alpha+\beta-|\alpha-\beta|}{2(\alpha+\beta)}\,,$$

in (4.6), and we get:

1. *if $|\alpha-\beta|\leqslant 2$ the Cauchy Problem (1.1) is $\mathcal{C}^\infty$ well-posed;*
2. *if $|\alpha-\beta|>2$ the Cauchy Problem (1.1) is γ^s well-posed, for any s such that*

$$1\leqslant s<\frac{\alpha+\beta+|\alpha-\beta|}{|\alpha-\beta|-2}\,,$$

meanwhile in [MS2] they proved well-posedness for $1\leqslant s<\dfrac{|\alpha-\beta|}{|\alpha-\beta|-2}$.

We can derive from Theorem 1.1 the following result for second order scalar equations.

Corollary 4.5. — *Let*

$$Pu\equiv\partial_t^2u-2a_1\partial_t\partial_xu-a_2\partial_x^2u-b_0\partial_tu-b_1\partial_xu=0$$

be a second order equation with $\mathcal{C}^\chi$ coefficients. If

$$\int_0^T \frac{\left|a_1'+a_1b_0+b_1\right|}{\sqrt{h+\varepsilon}}+\frac{\left|h'\right|}{h+\varepsilon}\,dt \quad \lesssim \varepsilon^{-1/\alpha}\,, \tag{4.8}$$

where $h:=a_1^2+a_2$, then the Cauchy problem for P is well-posed in γ^s, for $1\leqslant s<1+\dfrac{\min(\chi,\alpha)}{2}$.

If moreover the coefficients are analytic and

$$\int_0^T \frac{\left|a_1'+a_1b_0+b_1\right|}{\sqrt{h+\varepsilon}}+\frac{\left|h'\right|}{h+\varepsilon}\,dt \quad \lesssim \log\frac{1}{\varepsilon}\,, \tag{4.9}$$

then the Cauchy problem for P is well-posed in $\mathcal{C}^\infty$.

Proof. — By standard method, we reduce the Cauchy Problem for P to a Cauchy Problem for a 2×2 system by setting $U = (\partial_x u, \partial_t u)$:

$$\partial_t U = A\partial_x U + BU\,,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ a_2 & 2a_1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_0 \end{pmatrix} \tag{4.10}$$

We have

$$a^{\#} = \frac{1}{2}(1+a_2) - ia_1\,, \qquad c^{\#} = \frac{i}{2}(1-a_2)\,,$$

$$h = a_1^2 + a_2\,, \qquad k = \frac{1}{2}(1+a_2^2) + a_1^2\,,$$

$$D^{\#} = a_1'\big(a^{\#} - h\big) + \frac{1}{2}(a_1+i)h'\,, \qquad \mathrm{tr}(AB) = b_0a_1 + b_1\,,$$

hence:

$$\begin{aligned} \frac{|D^{\#} - a^{\#}\mathrm{tr}(AB)|}{\sqrt{h+\varepsilon}\sqrt{k+\varepsilon}} &\leqslant \frac{\big|a_1'\big(a^{\#}-h\big) + \frac{1}{2}(a_1+i)h' - a^{\#}(b_0a_1+b_1)\big|}{\sqrt{h+\varepsilon}\sqrt{k+\varepsilon}} \\ &\leqslant \frac{|a^{\#}|\,\big|a_1' - a_1b_0 - b_1\big|}{\sqrt{h+\varepsilon}\sqrt{k+\varepsilon}} + \frac{\big|ha_1'\big|}{\sqrt{h+\varepsilon}\sqrt{k+\varepsilon}} + \frac{\big|(a_1+i)h'\big|}{\sqrt{h+\varepsilon}\sqrt{k+\varepsilon}} \\ &\lesssim \Big[\frac{\big|a_1' - a_1b_0 - b_1\big|}{\sqrt{h+\varepsilon}} + \frac{|a_1'|}{|a_1| + \sqrt{\varepsilon}} + \frac{|h'|}{h+\varepsilon}\Big]\,, \end{aligned}$$

hence Corollary 4.5 follows form Theorem 1.1. □

Note that (4.8) (resp. (4.9)) is equivalent to

$$\int_0^T \frac{\big|\lambda_1^{(\varepsilon)\prime} + b_0\lambda_1^{(\varepsilon)} + b_1\big| + \big|\lambda_2^{(\varepsilon)\prime} + b_0\lambda_2^{(\varepsilon)} + b_1\big|}{\big|\lambda_1^{(\varepsilon)} - \lambda_2^{(\varepsilon)}\big|}\,dt \leqslant \varepsilon^{-1/\alpha} \quad (\text{resp.} \leqslant \log\frac{1}{\varepsilon})\,,$$

where $\lambda_1^{(\varepsilon)}, \lambda_2^{(\varepsilon)}$ are the solutions of the perturbed characteristic equation

$$\lambda^2 - 2a_1\lambda - (a_2+\varepsilon) = 0\,.$$

Note also that if λ_1, λ_2 are the solutions of the characteristic equation

$$\lambda^2 - 2a_1\lambda - a_2 = 0\,,$$

and satisfy the condition (cf. [CO], [KS])

$$\frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1 - \lambda_2)^2} \leqslant C\,,$$

then the matrix A in (4.10) satisfies the condition (4.1).

Example 4.6 (Systems with constant coefficients). — If A and B have constant coefficients, then (1.8) and (1.10) are trivially satisfied, whereas (1.9) for $\alpha \geqslant 2$ and (1.11) are satisfied if, and only if,

$$\left|\mathrm{tr}(\widetilde{A}\,B)\right| \lesssim \sqrt{h}\,. \tag{4.11}$$

On the other side, we know that in order the Cauchy Problem to be well-posed in $\mathcal{C}^\infty$ [G] (resp. γ^s [L]) it is necessary and sufficient that

$$\det\bigl(-\lambda\,I+A(\xi)+B\bigr)=0 \quad \Rightarrow \quad \bigl|\mathrm{Im}\lambda(\xi)\bigr| \leqslant M\ \bigl(\text{resp. } \bigl(1+|\xi|\bigr)^{1/s'},\ s<s'\bigr)\,.$$

A simple calculation shows that:

$$\det(-\lambda\,I+A+B)=\lambda^2-\bigl(\mathrm{tr}(A)+\mathrm{tr}(B)\bigr)\lambda+\det(A)-\mathrm{tr}\bigl((A-2\tau\,I)\,B\bigr)+\det(B)\,.$$

Clearly the imaginary parts of the roots of this polynomial are bounded if, and only if, its discriminant is bounded from below. After some calculations this condition can be written as

$$\left[2\sqrt{h}+\frac{\mathrm{tr}(\widetilde{A}\,B)}{\sqrt{h}}\right]^2-\frac{\left|\mathrm{tr}(\widetilde{A}\,B)\right|^2}{h}+\mathrm{tr}(B)^2-4\det(B)\geqslant -C\,,$$

which is equivalent to (4.11).

Example 4.7 (Systems with constant multiplicity). — If $h(t,\xi)\equiv 0$, then condition (1.5) is always satisfied with $\alpha=2$ and the Cauchy Problem (1.1) is well-posed in γ^s, $s<2$. Moreover If the coefficients of A belong to $\mathcal{L}_\chi$ and

$$D(t,\xi)-a(t,\xi)\mathrm{tr}(\widetilde{A}\,B)\equiv D(t,\xi)-a(t,\xi)\mathrm{tr}(\widetilde{A}\,\overline{B})\equiv 0\,, \tag{4.12}$$

then the Cauchy Problem (1.1) is well-posed in γ^s, for each:

$$s<s_0=1+\frac{\chi}{2}\,.$$

If the coefficients of A belong to $\mathcal{L}_\infty$ and condition (4.12) holds, then the Cauchy Problem (1.1) is well-posed in $\mathcal{C}^\infty$.

As it is shown in [N], (4.12) is the usual Levi condition for systems with double characteristic [Ma] [D].

5. Proof of Theorem 1.6

As in the proof of Theorem 1.1, by Fourier transform with respect to the space variables, (1.1) yields:

$$\begin{cases} V_t - i\,|\xi|\,A(t,\xi)V - B(t)V = \hat{f}\,, \\ V(0,\xi) = V_0(\xi)\,, \end{cases} \tag{5.1}$$

where $V(t,\xi) = \widehat{U}(t,\xi)$, $V_0(\xi) = \widehat{U}_0(\xi)$, and the hat $\widehat{\ }$ denotes the Fourier transform with respect to the x variable.

We divide the proof in two Steps.

Step I: $A(t,\xi)$ is a Sylvester matrix. We assume that $A(t,\xi)$ is a Sylvester matrix, that is:

$$A(t,\xi) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ H_3 & H_2 & H_1 \end{pmatrix},$$

where:

$$H_1 = \sum_{1\leqslant i\leqslant 3} \lambda_i\,, \qquad H_2 = -\sum_{1\leqslant i<j\leqslant 3} \lambda_i\lambda_j\,, \qquad H_3 = \prod_{1\leqslant i\leqslant 3} \lambda_i\,.$$

We define for any $i,j = 1,2,3$, $i \neq j$, the row vectors:

$$\begin{array}{rcl} \omega & := & (1,0,0)\,, \\ \omega_i & := & (-\lambda_i, 1, 0)\,, \\ \omega_{ij} & := & \big(\lambda_i\lambda_j, -(\lambda_i+\lambda_j), 1\big)\,. \end{array} \tag{5.2}$$

Note that if $\{i,j,k\} = \{1,2,3\}$, then ω_{ij} is the left eigenvector of A related to λ_k, that is:

$$\omega_{ij}A = \lambda_k\omega_{ij}\,, \tag{5.3}$$

whereas:

$$\omega_i A = \omega_{ij} + \lambda_i\omega_j\,, \tag{5.4}$$

$$\omega A = \omega_i + \lambda_i\omega\,. \tag{5.5}$$

For $t \in [0,T]$ and $|\xi| \geqslant 1$, we define the energy:

$$E(t) := \sum_{i<j} |\omega_{ij}V|^2 + \varepsilon_1^2\big(|\omega_1 V|^2 + |\omega_2 V|^2\big) + \varepsilon_2^2\,|\omega_3 V|^2 + (\mu^2t^2 + \varepsilon_1^4)\,|\omega V|^2\,,$$

where $\varepsilon_1 = |\xi|^{-\delta_1}$, $\varepsilon_2 = |\xi|^{-\delta_2}$, $\mu = |\xi|^{-\nu}$, and δ_1, δ_2, ν are positive constants that we will choose in the following.

In order to obtain the energy estimate for E, we prepare some useful estimations.

Estimate of $|\omega_{ij} V|$*.* It's clear that:

$$|\omega_{ij} V| \leqslant \sqrt{E}\,. \tag{5.6}$$

Estimate of $|\omega_i V|$*.* From the definition of E, we have

$$\varepsilon_2 \, |\omega_3 V| \leqslant \sqrt{E}\,, \tag{5.7}$$

whereas, since

$$\omega_{ij} - \omega_{ik} = (\lambda_k - \lambda_j)\, \omega_i\,,$$

using Assumption (1.19) we have:

$$t^\ell \, |\omega_3 V| \lesssim |\lambda_2 - \lambda_1| \, |\omega_3 V| = |\omega_{13} V - \omega_{23} V| \lesssim \sqrt{E}\,. \tag{5.8}$$

Combining (5.7) and (5.8), we get:

$$|\omega_3 V| \lesssim \frac{\sqrt{E}}{t^\ell + \varepsilon_2}\,, \tag{5.9}$$

and, analogously:

$$|\omega_1 V|\,, |\omega_2 V| \lesssim \frac{\sqrt{E}}{t^\kappa + \varepsilon_1}\,. \tag{5.10}$$

Estimate of $|\omega V|$*.* We can estimate $|\omega V|$ in two different ways; first of all, from the definition of E, we get immediately:

$$|\omega V| \lesssim \frac{\sqrt{E}}{\varepsilon_1^2 + \mu t}\,,$$

whereas from the identity

$$\omega_i - \omega_j = (\lambda_j - \lambda_i)\, \omega\,,$$

using Assumption (1.20), we obtain:

$$t^\kappa \, |\omega| \lesssim |\lambda_3 - \lambda_2| \, |\omega| = |\omega_3 - \omega_2|\,,$$

and by (5.9) and (5.10), we get:

$$|\omega V| \lesssim \frac{\sqrt{E}}{t^\kappa (t^\ell + \varepsilon_2)} + \frac{\sqrt{E}}{t^\kappa (t^\kappa + \varepsilon_1)}\,.$$

Thus, we have the estimate:

$$|\omega V| \lesssim \varphi(t,\xi)\sqrt{E}\,,$$

$$\text{where}\quad \varphi(t,\xi) := \min\left\{\frac{1}{\varepsilon_1^2+\mu t}\,,\ \frac{1}{t^\kappa(t^\ell+\varepsilon_2)}+\frac{1}{t^\kappa(t^\kappa+\varepsilon_1)}\right\}. \tag{5.11}$$

Estimate of $|V|$. We remark that $\{\omega,\omega_1,\omega_{12}\}$ is a base of $\mathbb{R}^3$. Indeed, by a simple calculations, we see that if $\{e_1,e_2,e_3\}$ is the canonical row base of $\mathbb{R}^3$, then:

$$\begin{aligned} e_1 &= \omega\,,\\ e_2 &= \omega_1+\lambda_1\omega\,,\\ e_3 &= \omega_{12}+(\lambda_1+\lambda_2)\omega_1+\lambda_1^2\omega\,, \end{aligned}$$

hence:

$$\begin{aligned} |V|^2 &= |e_1V|^2+|e_2V|^2+|e_3V|^2\\ &\lesssim [1+|\lambda_1|^2+|\lambda_1|^4]\,|\omega V|^2+[1+(\lambda_1+\lambda_2)^2]\,|\omega_1V|^2+|\omega_{12}V|^2\,. \end{aligned}$$

Using (5.6), (5.10) and (5.11), we get:

$$|V| \lesssim |\omega V|+|\omega_1V|+|\omega_{12}V| \lesssim \Big(\varphi+\frac{1}{t^\kappa+\varepsilon_1}+1\Big)\sqrt{E}\,. \tag{5.12}$$

Estimate of $\partial_t|\omega_{ij}V|^2$. Let i,j,k be such that $\{i,j,k\}=\{1,2,3\}$, we have:

$$\partial_t\omega_{ij} = -\lambda_i'\omega_j-\lambda_j'\omega_i\,,$$

hence, using (5.3):

$$\begin{aligned} \partial_t(\omega_{ij}V) &= \partial_t\omega_{ij}V+\omega_{ij}\partial_tV\\ &= -\lambda_i'\omega_jV-\lambda_j'\omega_iV+i\xi\lambda_k\omega_{ij}V+\omega_{ij}BV+\omega_{ij}\hat f\,, \end{aligned}$$

thus:

$$\begin{aligned} \partial_t|\omega_{ij}V|^2 &= 2\mathrm{Re}\big(\partial_t(\omega_{ij}V),\omega_{ij}V\big)\\ &= 2\mathrm{Re}\big(-\lambda_i'\omega_jV-\lambda_j'\omega_iV+i\,|\xi|\,\lambda_k\omega_{ij}V+\omega_{ij}BV+\omega_{ij}\hat f,\omega_{ij}V\big)\\ &= 2\mathrm{Re}\big(-\lambda_i'\omega_jV-\lambda_j'\omega_iV+\omega_{ij}BV+\omega_{ij}\hat f,\omega_{ij}V\big)\,, \end{aligned}$$

since the characteristic roots are real.

Now we recall that by *Bronšteĭn's Lemma* ([B]; cf.[W], [M], [T2]) the roots λ_j are Lipschitz continuous functions of t, hence, using (5.6), (5.9) and (5.10):

$$\begin{aligned}\mathrm{Re}\big(-\lambda_i'\omega_j V - \lambda_j'\omega_i V, \omega_{ij} V\big) &\leqslant \Big(|\lambda_i'|\,|\omega_j V| + \big|\lambda_j'\big|\,|\omega_i V|\Big)\,|\omega_{ij} V| \\ &\lesssim \Big(\frac{1}{t^\kappa+\varepsilon_1}+\frac{1}{t^\ell+\varepsilon_2}\Big)E\,.\end{aligned}$$

Since $|\omega_{ij} BV| \lesssim |V|$, and $\big|(\omega_{ij}\hat f, \omega_{ij} V)\big| \lesssim \hat f^2 + E$, using (5.12), we get:

$$\partial_t\,|\omega_{ij} V|^2 \lesssim \Big(\varphi + \frac{1}{t^\kappa+\varepsilon_1}+\frac{1}{t^\ell+\varepsilon_2}+1\Big)E + \hat f^2\,. \tag{5.13}$$

Estimate of $\partial_t\,|\omega_i V|^2$. We have $\partial_t\omega_i = -\lambda_i'\omega$, hence, using (5.4):

$$\partial_t(\omega_i V) \quad = \partial_t\omega_i V + \omega_i\partial_t V = -\lambda_i'\omega V + i\,|\xi|\,(\omega_{ij}+\lambda_j\omega_i)V + \omega_i BV + \omega_i\hat f\,,$$

and, using the same arguments as above:

$$\begin{aligned}\varepsilon_1^2\partial_t\,|\omega_1 V|^2 &= 2\varepsilon_1^2\mathrm{Re}\big(\partial_t(\omega_1 V), \omega_1 V\big) \\ &= 2\varepsilon_1^2\mathrm{Re}\big(-\lambda_1'\omega V + i\,|\xi|\,(\omega_{i1}+\lambda_1\omega_i)V + \omega_1 BV + \omega_1\hat f, \omega_1 V\big) \\ &= 2\varepsilon_1^2\mathrm{Re}\big(-\lambda_1'\omega V + i\,|\xi|\,\omega_{i1} V + \omega_1 BV + \omega_1\hat f, \omega_1 V\big) \\ &\lesssim \Big(\varphi + |\xi| + \frac{1}{t^\kappa+\varepsilon_1}+1\Big)\,\frac{\varepsilon_1^2}{t^\kappa+\varepsilon_1}\,E + \hat f^2\,,\end{aligned}$$

which gives:

$$\varepsilon_1^2\partial_t\,|\omega_1 V|^2 \quad \lesssim \Big(\varphi + \frac{\varepsilon_1^2\,|\xi|+1}{t^\kappa+\varepsilon_1}+1\Big)E + \hat f^2\,, \tag{5.14}$$

and similarly:

$$\varepsilon_1^2\partial_t\,|\omega_2 V|^2 \quad \lesssim \Big(\varphi + \frac{\varepsilon_1^2\,|\xi|+1}{t^\kappa+\varepsilon_1}+1\Big)E + \hat f^2\,, \tag{5.15}$$

$$\varepsilon_2^2\partial_t\,|\omega_3 V|^2 \quad \lesssim \Big(\varphi + \frac{\varepsilon_2^2\,|\xi|+1}{t^\ell+\varepsilon_2}+1\Big)E + \hat f^2\,. \tag{5.16}$$

Estimate of $\partial_t\,|\omega V|^2$. Using (5.5), we have:

$$\begin{aligned}\partial_t(\omega V) &= \omega\partial_t V = i\,|\xi|\,\omega AV + \omega BV + \hat f \\ &= i\,|\xi|\,(\omega_i + \lambda_i\omega)V + \omega BV + \omega\hat f\,,\end{aligned} \tag{5.17}$$

hence:

$$\begin{aligned}\partial_t |\omega V|^2 &= 2\mathrm{Re}\big(i\,|\xi|\,(\omega_1+\lambda_1\omega)V+\omega BV+\omega\hat{f},\omega V\big)\\ &= 2\mathrm{Re}\big(i\,|\xi|\,\omega_1 V+\omega BV+\omega\hat{f},\omega V\big)\\ &\lesssim \big(|\xi|\,|\omega_1 V|+|\omega BV|+|\omega\hat{f}|\big)\,|\omega V|\\ &\lesssim \big(|\xi|\,|\omega_1 V|+|V|+|\omega\hat{f}|\big)\,|\omega V|\,, \end{aligned}\tag{5.18}$$

and using (5.10) and (5.12):

$$\begin{aligned}(\mu^2t^2+\varepsilon_1^4)\partial_t|\omega V|^2 &\lesssim (\varepsilon_1^2+\mu t)\Big(\frac{|\xi|}{t^\kappa+\varepsilon_1}+\varphi+\frac{1}{t^\kappa+\varepsilon_1}+1\Big)E+\hat{f}^2\\ &\lesssim \Big(\varphi+\frac{1+\varepsilon_1^2\,|\xi|}{t^\kappa+\varepsilon_1}+\frac{\mu t\,|\xi|}{t^\kappa+\varepsilon_1}+1\Big)E+\hat{f}^2\,.\end{aligned}\tag{5.19}$$

Now we can derive the energy estimate. Differentiating E we get:

$$\begin{aligned}E'(t) &= \sum_{i<j}\partial_t|\omega_{ij}V|^2+\varepsilon_1^2\big(\partial_t|\omega_1V|^2+\partial_t|\omega_2V|^2\big)+\varepsilon_2^2\partial_t|\omega_3V|^2\\ &\quad +(\mu^2t^2+\varepsilon_1^4)\partial_t|\omega V|^2+2\mu^2t\,|\omega V|^2\,,\end{aligned}$$

and, using the above estimates, we derive:

$$E'(t)\lesssim\Big(\varphi+\frac{\mu}{\varepsilon_1^2+\mu t}+\frac{1+\varepsilon_1^2\,|\xi|}{t^\kappa+\varepsilon_1}+\frac{1+\varepsilon_2^2\,|\xi|}{t^\ell+\varepsilon_2}+\frac{\mu\,|\xi|\,t}{t^\kappa+\varepsilon_1}+1\Big)E(t)+\hat{f}^2\tag{5.20}$$

thus, applying Grönwall's Lemma, we get:

$$\begin{aligned}E(t_2) &\lesssim \exp\Big[\int_{t_1}^{t_2}\Big(\varphi+\frac{\mu}{\varepsilon_1^2+\mu t}+\frac{1+\varepsilon_1^2\,|\xi|}{t^\kappa+\varepsilon_1}+\frac{1+\varepsilon_2^2\,|\xi|}{t^\ell+\varepsilon_2}+\frac{\mu\,|\xi|\,t}{t^\kappa+\varepsilon_1}+1\Big)\,dt\Big]E(t_1)\\ &\qquad +\int_{t_1}^{t_2}\hat{f}^2(s,\xi)\,ds\\ &= \exp\big[(I)+(II)+(III)+(IV)+(V)+T\big]E(t_1)+\int_{t_1}^{t_2}\hat{f}^2(s,\xi)\,ds\,.\end{aligned}$$

Now, it's easy to see that if $p>-1$ and $q>0$, then:

$$\int_0^T\frac{t^p\,dt}{\eta+t^q}\lesssim\eta^{-\frac{[q-p-1]^+}{q}}\,\log(1+\eta^{-1})\,,\tag{5.21}$$

for any $\eta\in\,]0,1[$. Here $[a]^+:=\max(a,0)$, is the positive part of a.

Indeed, if $q < p+1$ then

$$\int_0^T \frac{t^p\,dt}{\eta + t^q} \leqslant \int_0^T \frac{t^p\,dt}{t^q} \leqslant C(p,q)\,,$$

whereas, if $q \geqslant p+1$ then

$$\int_0^T \frac{t^p\,dt}{\eta + t^q} = \eta^{-1+\frac{p+1}{q}} \int_0^{T\eta^{-\frac{1}{q}}} \frac{\sigma^p\,d\sigma}{1+\sigma^q} \leqslant \begin{cases} C(p,q)\,\log(1+\eta^{-1})\,, & \text{if } q = p+1 \\ C(p,q)\,\eta^{-1+\frac{p+1}{q}} & \text{if } q > p+1 \end{cases}\,,$$

for any $\eta \in \,]0,1[$.

If $\kappa \geqslant 2$, we have the estimates:

$$(I) \leqslant \int_0^T \frac{1}{\varepsilon_1^2 + \mu t} \lesssim \frac{1}{\mu}\,\log(1+\varepsilon_1^{-2}\mu)\,, \tag{5.22}$$

$$(II) \leqslant \int_0^T \frac{\mu\,dt}{\varepsilon_1^2 + \mu t} \lesssim \log(1+\varepsilon_1^{-2}\mu)\,, \tag{5.23}$$

$$(III) \leqslant \int_0^T \frac{1+\varepsilon_1^2\,|\xi|}{t^\kappa + \varepsilon_1}\,dt \lesssim \big(1+\varepsilon_1^2\,|\xi|\big)\varepsilon_1^{-1+\frac{1}{\kappa}}\,, \tag{5.24}$$

$$(IV) \leqslant \int_0^T \frac{\varepsilon_2^2\,|\xi|+1}{t^\ell + \varepsilon_2}\,dt \lesssim \big(1+\varepsilon_2^2\,|\xi|\big)\varepsilon_2^{-1+\frac{1}{\ell}}\,, \tag{5.25}$$

$$(V) \leqslant \int_0^T \frac{\mu\,|\xi|\,t}{t^\kappa + \varepsilon_1}\,dt \lesssim \mu\,|\xi|\,\varepsilon_1^{-1+\frac{2}{\kappa}}\,\log(1+\varepsilon_1^{-1})\,, \tag{5.26}$$

which give:

$$\begin{aligned} E(t_2) \lesssim\ & \exp\bigg[\frac{1}{\mu}\,\log(1+\varepsilon_1^{-2}\mu) + \big(1+\varepsilon_1^2\,|\xi|\big)\varepsilon_1^{-1+\frac{1}{\kappa}} \\ & +\big(1+\varepsilon_2^2\,|\xi|\big)\varepsilon_2^{-1+\frac{1}{\ell}} + \mu\,|\xi|\,\varepsilon_1^{-1+\frac{2}{\kappa}}\,\log(1+\varepsilon_1^{-1}) + T\bigg]E(t_1) \\ & + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\,ds\,. \end{aligned}$$

Choosing:

$$\varepsilon_1 := |\xi|^{-\frac{1}{3}}\,, \qquad \varepsilon_2 := |\xi|^{-\frac{1}{2}}\,, \qquad \mu := |\xi|^{-\frac{2\kappa-1}{3\kappa}}\,,$$

we obtain:

$$\begin{aligned} E(t_2) &\lesssim \exp\Big[|\xi|^{\frac{2\kappa-1}{3\kappa}}\,\log(1+|\xi|^{1/2}) + |\xi|^{\frac{\ell-1}{2\ell}}\Big]E(t_1) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\,ds \\ &\lesssim \exp\Big[|\xi|^{\frac{2\kappa-1}{3\kappa}}\,\log(1+|\xi|)\Big]E(t_1) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\,ds\,, \end{aligned}$$

since $\dfrac{2\kappa-1}{3\kappa} \geqslant \dfrac{\ell-1}{2\ell}$, if $\kappa \leqslant \ell$.

Now, we remark that E is equivalent to $|V(t,\xi)|^2$ for $|\xi| \geqslant 1$, in the sense that there exists $M_1, M_2 > 0$ such that:

$$E(t) \lesssim |\xi|^{2M_1} |V(t,\xi)|^2 \,, \qquad \text{and} \qquad |V(t,\xi)|^2 \lesssim |\xi|^{2M_2} E(t) \,. \tag{5.27}$$

Indeed the first inequality in (5.27) follows easily from the choice of ε_1, ε_2 and μ, taking $M_1 := \max(2\,\delta_1, \delta_2, \nu)$, whereas the second inequality follows easily from (5.12). Thus we get:

$$\big|V(t_2,\xi)\big| \lesssim |\xi|^M \exp\Big[|\xi|^{\frac{2\kappa-1}{3\kappa}}\Big]\big|V(t_1,\xi)\big| + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\,ds\,,$$

if $1 < \kappa \leqslant \ell$.

If $\kappa = 1$, we split the phase space $[0,T] \times \mathbb{R}^n$ in two zones:

$$\begin{aligned} Z^{(1)} &= \Big\{(t,\xi) \in [0,T] \times \mathbb{R}^n \;\Big|\; 0 \leqslant t \leqslant t_\xi \,, |\xi| \geqslant 1\Big\}\,, \\ Z^{(2)} &= \Big\{(t,\xi) \in [0,T] \times \mathbb{R}^n \;\Big|\; t \geqslant t_\xi \,, |\xi| \geqslant 1\Big\}\,, \end{aligned} \tag{5.28}$$

where $t_\xi := |\xi|^{-\frac{1}{2\ell+1}}$.

In both zones we use the energy E as in the case $\kappa \geqslant 2$, but with a different choice of ε_1, ε_2 and μ. Clearly these different energies are equivalent each other.

For $(t,\xi) \in Z^{(1)}$ we have:

$$\begin{aligned} (III) &\leqslant \int_0^T \frac{1+\varepsilon_1^2\,|\xi|}{t+\varepsilon_1}\,dt \lesssim \big(1+\varepsilon_1^2\,|\xi|\big)\log(1+\varepsilon_1^{-1})\,, \\ (V) &\leqslant \int_0^{t_\xi} \frac{\mu\,|\xi|\,t}{t+\varepsilon_1}\,dt \lesssim \mu\,|\xi|\,t_\xi\,, \end{aligned}$$

instead of (5.24) and (5.26), whereas (5.25) will be replaced by

$$(IV) \leqslant \int_0^T \frac{\varepsilon_2^2\,|\xi|+1}{t^\ell+\varepsilon_2}\,dt \lesssim \big(1+\varepsilon_2^2\,|\xi|\big)\log(1+\varepsilon_2^{-1})\,,$$

if $\ell = 1$. Hence we get:

$$\begin{aligned} E(t_2,\xi) \;\lesssim\; & \exp\Big[\frac{1}{\mu}\log(1+\varepsilon_1^{-2}\mu) + \big(1+\varepsilon_1^2\,|\xi|\big)\log(1+\varepsilon_1^{-1}) \\ & + \big(1+\varepsilon_2^2\,|\xi|\big)\varepsilon_2^{-1+\frac{1}{\ell}} + \mu\,|\xi|\,t_\xi + T\Big]E(t_1,\xi) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\,ds\,. \end{aligned} \tag{5.29}$$

Choosing:

$$\mu := |\xi|^{-\frac{\ell}{2\ell+1}} \,, \qquad \varepsilon_1 := |\xi|^{-\frac{\ell+1}{2(2\ell+1)}} \,, \qquad \varepsilon_2 := |\xi|^{-\frac{\ell}{2\ell+1}} \,, \tag{5.30}$$

we obtain:

$$E(t_2,\xi) \lesssim \exp\Big[|\xi|^{\frac{\ell}{2\ell+1}}\, \log(1+|\xi|)\Big] E(t_1,\xi) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\, ds \,,$$

for $(t_1,\xi),(t_2,\xi) \in Z^{(1)}$.

For $(t,\xi) \in Z^{(2)}$, we choose $\mu \equiv 0$ so that $(V) \equiv 0$. Moreover, we have:

$$\begin{aligned}(I) \;&\leqslant\; \int_{t_\xi}^{T} \Big[\frac{1}{t(t^\ell+\varepsilon_2)} + \frac{1}{t(t+\varepsilon_1)}\Big]\, dt \leqslant \frac{1}{t_\xi}\int_{t_\xi}^{T} \Big[\frac{1}{t^\ell+\varepsilon_2} + \frac{1}{t+\varepsilon_1}\Big]\, dt \\ &\lesssim\; \frac{1}{t_\xi}\big[\varepsilon_2^{-1+\frac{1}{\ell}} + \log(1+\varepsilon_1^{-1})\big] \lesssim \varepsilon_2^{-1}\, \log\big(1+|\xi|\big)\,,\end{aligned}$$

hence we get:

$$\begin{aligned}E(t_2,\xi) \;&\lesssim\; \exp\Big[\varepsilon_2^{-1}\, \log\big(1+|\xi|\big) + \big(1+\varepsilon_1^2\,|\xi|\big)\log(1+\varepsilon_1^{-1}) \\ &\qquad + \big(1+\varepsilon_2^2\,|\xi|\big)\varepsilon_2^{-1+\frac{1}{\ell}} + T\Big] E(t_1,\xi) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\, ds\,,\end{aligned}$$

and choosing ε_1 and ε_2 as in (5.30) we obtain:

$$E(t_2,\xi) \;\lesssim \exp\Big[|\xi|^{\frac{\ell}{2\ell+1}}\, \log(1+|\xi|)\Big] E(t_1,\xi) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\, ds\,,$$

for $(t_1,\xi),(t_2,\xi) \in Z^{(2)}$.

Step II: The general case. As in [DAS2], we transform the 3×3 system (1.1) into a 9×9 system whose principal part is a block Sylvester matrix.

Using Duhamel principle, we can assume $f \equiv 0$.

Let:

$$\Lambda(t,\lambda,\xi) = L_1(t,\lambda,\xi)^{co} \equiv \big(\lambda - |\xi|\, A(t,\xi)\big)^{co}$$

be the cofactor matrix of the principal symbol $L_1(t,\lambda,\xi) := \big(\lambda - |\xi|\, A(t,\xi)\big)$ of L. It's clear that the well-posedness of both ΛL and $L\Lambda$ implies the well-posedness for L.

We prove the well-posedness for ΛL, the well-posedness of $L\Lambda$ is proved analogously.

ΛL is a third order system with diagonal principal part:

$$\sigma_3(\Lambda L) = I_3\, P(t,\lambda,\xi)\,,$$

where $P(t,\lambda,\xi)$ is the characteristic polynomial of $A(t,\lambda,\xi)$.

Let

$$W^{(j)} := \begin{pmatrix} -\,|\xi|^2\, V^{(j)} \\ i\,|\xi|\,\partial_t V^{(j)} \\ \partial_t^2 V^{(j)} \end{pmatrix}\,, \quad j=1,2,3\,, \quad \text{and} \quad \mathcal{W} := \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ W^{(3)} \end{pmatrix} \in \mathbb{C}^9\,,$$

then the Cauchy Problem for ΛL is equivalent to the Cauchy Problem:

$$\partial_t \mathcal{W} - i\,|\xi|\,\mathcal{A}(t,\xi)\mathcal{W} - \mathcal{B}(t,\xi)\mathcal{W} = 0\,, \tag{5.31}$$

where:

$$\mathcal{A}(t,\xi) = A(t,\xi) \oplus A(t,\xi) \oplus A(t,\xi)\,,$$

$A(t,\xi)$ is a Sylvester matrix as in Step I, and $\mathcal{B}$ a 9×9 matrix with the following block structure:

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_{[1,1]} & \mathcal{B}_{[1,2]} & \mathcal{B}_{[1,3]} \\ \mathcal{B}_{[2,1]} & \mathcal{B}_{[2,2]} & \mathcal{B}_{[2,3]} \\ \mathcal{B}_{[3,1]} & \mathcal{B}_{[3,2]} & \mathcal{B}_{[3,3]} \end{pmatrix}\;;$$

each 3×3 block $\mathcal{B}_{[j,k]}$ has non zero elements only on the last line.

We remark that $W^{(j)}$ satisfies the system

$$W_t^{(j)} - i\,|\xi|\,A(t,\xi)W^{(j)} - \mathcal{B}_{[j,j]}(t,\xi)W^{(j)} = \sum_{k\neq j} \mathcal{B}_{[j,k]}(t,\xi)W^{(k)}\,,$$

hence we may regard $\sum_{k\neq j} \mathcal{B}_{[j,k]}(t,\xi)W^{(k)}$ as a second member and proceed as in Step I. Let $E[W^{(j)}]$ be the energy of $W^{(j)}$; from (5.20), we get:

$$E[W^{(j)}]' \lesssim \mathcal{K}\, E[W^{(j)}] + \Big[\sum_{k\neq j} \mathcal{B}_{jk} W^{(k)}\Big]^2\,,$$

where $\mathcal{K} = \mathcal{K}(t,\xi,\varepsilon_1,\varepsilon_2,\mu)$ denotes the sum of the terms in the bracket in (5.20). Now, using (5.12):

$$\Big[\sum_{k\neq j} \mathcal{B}_{jk} W^{(k)}\Big]^2 \lesssim \sum_{k\neq j} [W^{(k)}]^2 \lesssim \Big(\varphi + \frac{1}{t^\kappa+\varepsilon_1} + 1\Big) \sum_{k\neq j} E[W^{(j)}] \lesssim \mathcal{K} \sum_{k\neq j} E[W^{(j)}]\,,$$

hence we have:

$$E[W^{(j)}]' \lesssim \mathcal{K} \sum_{k=1}^{3} E[W^{(k)}]\,,$$

thus, defining the energy for $\mathcal{W}$

$$\mathcal{E}[\mathcal{W}] := \sum_{j=1}^{3} E[W^{(j)}]\,,$$

we get:

$$\mathcal{E}'[\mathcal{W}] \lesssim \mathcal{K}\,\mathcal{E}[\mathcal{W}]\,,$$

and applying Grönwall Lemma we get the a priori estimate as in Step I.

6. Proof of Theorem 1.9

In order to prove Theorem 1.9, we use the following Lemma.

LEMMA 6.1. — *Let $A(t,\xi)$ be as in Theorem 1.6, and assume the following hypothesis on $B(t,\xi)$:*

$$B(t,\xi) = B^{(0)}(t,\xi) + B^{(1)}(t,\xi)$$

with

$$\left|B^{(0)}_{j,1} + \tau\, B^{(0)}_{j,2} + \tau^2\, B^{(0)}_{j,3}\right| \lesssim t^{\alpha}\,, \quad \tau := \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\,, \tag{6.1}$$

for some $\alpha \geqslant 0$ and any $j = 1,2,3$, and $|\xi|\,\left|B_1(t,\xi)\right|$ is bounded for $|\xi|$ large.

Let:

$$E(t) := \sum_{i<j} |\omega_{ij}V|^2 + \varepsilon_1^2\big(|\omega_1 V|^2 + |\omega_2 V|^2\big) + \varepsilon_2^2\,|\omega_3 V|^2 + \big(\varepsilon_1^4 + \mu^2 t^{2(\alpha+1)}\big)\,|\omega V|^2\ ;$$

if $\alpha < 1$ or $\ell > 1$, then E verifies:

$$E(t,\xi) \lesssim \exp|\xi|^{\frac{1}{s_\alpha}}\, E(0,\xi)\,, \tag{6.2}$$

where:

$$s_\alpha := \begin{cases} \dfrac{3\kappa - \alpha}{2\kappa - \alpha - 1} & \alpha \leqslant \kappa - 2 \\[2mm] \dfrac{2\ell + \kappa - \alpha}{\ell + \kappa - \alpha - 1} & \kappa - 2 < \alpha \leqslant \kappa \\[2mm] \dfrac{2\ell}{\ell - 1} & \alpha \geqslant \kappa\,,\ \ell > 1\,; \end{cases}$$

whereas if $\alpha \geqslant 1$ and $\ell = 1$, then E verifies:

$$E(t,\xi) \lesssim \big(1 + |\xi|\big)^{\exists N} E(0,\xi)\,.$$

Example 6.2 If $\kappa = 1$, then:

$$s_\alpha = \begin{cases} \dfrac{2\ell + 1 - \alpha}{\ell - \alpha} & \alpha \leqslant 1 \\ \dfrac{2\ell}{\ell - 1} & \alpha \geqslant 1\,,\ \ell > 1 \end{cases}$$

If $\kappa = \ell$, then:

$$s_\alpha = \begin{cases} \dfrac{3\kappa - \alpha}{2\kappa - \alpha - 1} & \alpha \leqslant \kappa - 2 \\ \dfrac{2\kappa}{\kappa - 1} & \alpha \geqslant \kappa\,,\ \kappa > 1\,. \end{cases}$$

Remark 6.3. — If $B \equiv 0$, we can choose α arbitrarily large, in particular $\alpha \geqslant \kappa$, and we get $s_\alpha = \dfrac{2\ell}{\ell - 1}$.

Proof. — First of all we remark that we can assume, with no loss of generality, that $\alpha \leqslant \kappa$.

The proof of Lemma 6.1, is similar to the proof of Step I of Theorem 1.6: the only difference concerns the estimation of $|\omega V|$. Using similar arguments, we find:

$$|\omega V| \lesssim \varphi_\alpha \sqrt{E}\,, \qquad \varphi_\alpha := \min\left\{ \frac{1}{\varepsilon_1^2 + \mu t^{\alpha+1}}\,, \frac{1}{t^\kappa (t^\ell + \varepsilon_2)} + \frac{1}{t^\kappa (t^\kappa + \varepsilon_1)} \right\},$$

instead of (5.11). Using (5.2), we have:

$$\begin{aligned} (B^{(0)}V)_j &= B^{(0)}_{j,1} V_1 + B^{(0)}_{j,2} V_2 + B^{(0)}_{j,3} V_3 \\ &= B^{(0)}_{j,1}(e_1 \cdot V) + B^{(0)}_{j,2}(e_2 \cdot V) + B^{(0)}_{j,3}(e_3 \cdot V) \\ &= \Big[B^{(0)}_{j,1} + \lambda_1\, B^{(0)}_{j,2} + \lambda_1^2\, B^{(0)}_{j,3} \Big] (\omega \cdot V) \\ &\quad + \big[B^{(0)}_{j,2} + (\lambda_1 + \lambda_2)\, B^{(0)}_{j,3} \big] (\omega_1 \cdot V) + B^{(0)}_{j,3} (\omega_{1,2} \cdot V)\,, \end{aligned}$$

and, since

$$\Big| \Big(B^{(0)}_{j,1} + \lambda_1\, B^{(0)}_{j,2} + \lambda_1^2\, B^{(0)}_{j,3} \Big) - \Big(B^{(0)}_{j,1} + \tau\, B^{(0)}_{j,2} + \tau^2\, B^{(0)}_{j,3} \Big) \Big|$$

$$= (\lambda_1 - \tau) \Big| B^{(0)}_{j,2} + (\lambda_1 + \tau)\, B^{(0)}_{j,3} \Big| \lesssim t^\alpha\,,$$

we have:

$$|B^{(0)}V| \quad \lesssim t^\alpha\, |\omega \cdot V| + |\omega_1 \cdot V| + |\omega_{1,2} \cdot V|\,,$$

hence:

$$|BV| \lesssim \Big(\big(t^\alpha + |\xi|^{-1}\big)\, |\omega V| + |\omega_1 V| + |\omega_{12} V| \Big).$$

Proceeding as in the proof of Theorem 1.6, we get:

$$\begin{aligned} &E(t_2) \\ &\lesssim \exp\bigg[\int_0^T \Big(t^\alpha \varphi_\alpha + \frac{1}{\varepsilon_1^2 |\xi|} + \frac{\mu t^\alpha}{\varepsilon_1^2 + \mu t^{\alpha+1}} + \frac{1+\varepsilon_1^2 |\xi|}{t^\kappa + \varepsilon_1} + \frac{1+\varepsilon_2^2 |\xi|}{t^\ell + \varepsilon_2} + \frac{\mu |\xi|\, t^{\alpha+1}}{t^\kappa + \varepsilon_1} \Big)\, dt \bigg] E(t_1) \\ &\qquad\qquad + \int_{t_1}^{t_2} \hat f^2(s,\xi)\, ds \\ &= \exp\big[(I)_a + (I)_b + (II) + (III) + (IV) + (V) \big] E(t_1) + \int_{t_1}^{t_2} \hat f^2(s,\xi)\, ds\,. \end{aligned}$$

We can estimate the terms (III) and (IV) as before, and

$$(II) \lesssim \log\Big(1 + \frac{1}{\varepsilon_1}\Big).$$

To estimate the other terms, we have to distinguish different cases.

Case I. If $\alpha \leqslant \kappa - 2$, then we have:

$$\begin{aligned} (I)_a &\leqslant \int_0^T \frac{t^\alpha}{\varepsilon_1^2 + \mu t^{\alpha+1}}\, dt \lesssim \frac{1}{\mu} \log\Big(1 + \frac{\mu}{\varepsilon_1^2}\Big), \\ (V) &= \int_0^T \frac{\mu |\xi|\, t^{\alpha+1}}{t^\kappa + \varepsilon_1}\, dt \lesssim \mu |\xi|\, \varepsilon_1^{-1+\frac{\alpha+2}{\kappa}}, \qquad \text{if } k > \alpha + 2\,, \\ (V) &= \int_0^T \frac{\mu |\xi|\, t^{\alpha+1}}{t^\kappa + \varepsilon_1}\, dt \lesssim \mu |\xi| \log(1 + \varepsilon_1), \quad \text{if } k = \alpha + 2\,, \end{aligned}$$

so that:

$$\begin{aligned} E(t_2) \lesssim\ & \exp\bigg[\frac{1}{\mu} \log\Big(1 + \frac{\mu}{\varepsilon_1^2}\Big) + \frac{1}{\varepsilon_1^2 |\xi|} + \big(1 + \varepsilon_1^2 |\xi|\big) \varepsilon_1^{-1+\frac{1}{\kappa}} \\ & + \big(1 + \varepsilon_2^2 |\xi|\big) \varepsilon_2^{-1+\frac{1}{\ell}} + \mu |\xi|\, \varepsilon_1^{-1+\frac{\alpha+2}{\kappa}} \bigg] E(t_1) + \int_{t_1}^{t_2} \hat f^2(s,\xi)\, ds\,. \end{aligned}$$

Choosing:

$$\mu := |\xi|^{-\frac{2\kappa-\alpha-1}{3\kappa-\alpha}}, \qquad \varepsilon_1 := |\xi|^{-\frac{\kappa}{3\kappa-\alpha}}, \qquad \varepsilon_2 := |\xi|^{-\frac{1}{2}},$$

we obtain:

$$\begin{aligned} E(t_2) &\lesssim \exp\Big[|\xi|^{\frac{2\kappa-\alpha-1}{3\kappa-\alpha}} + |\xi|^{\frac{\ell-1}{2\ell}} \Big] E(t_1) + \int_{t_1}^{t_2} \hat f^2(s,\xi)\, ds \\ &\lesssim \exp\big(|\xi|^{\frac{2\kappa-\alpha-1}{3\kappa-\alpha}}\big) E(t_1) + \int_{t_1}^{t_2} \hat f^2(s,\xi)\, ds\,, \end{aligned}$$

since

$$\frac{2\kappa-\alpha-1}{3\kappa-\alpha} \geqslant \frac{\ell-1}{2\ell}\,.$$

Case II. If $\kappa-2 \leqslant \alpha \leqslant \kappa$, we split the strip $[0,T]\times\mathbb{R}^n$ into two zones as in (5.28), choosing $\tau := |\xi|^{-\frac{1}{2\ell+\kappa-\alpha}}$.

For $(t,\xi)\in Z^{(1)}$ we have the estimates:

$$(V) \leqslant \int_0^\tau \frac{\mu\,|\xi|\,t^{\alpha+1}}{t^\kappa+\varepsilon_1}\,dt \leqslant \mu\,|\xi|\,\tau^{\alpha+2-\kappa}\int_0^\tau \frac{t^{\kappa-1}}{t^\kappa+\varepsilon_1}\,dt \lesssim \mu\,|\xi|\,\tau^{\alpha+2-\kappa}\,\log\Big(\frac{\tau^\kappa+\varepsilon_1}{\varepsilon_1}\Big)\,.$$

Choosing:

$$\mu = |\xi|^{-\frac{\ell+\kappa-\alpha-1}{2\ell+\kappa-\alpha}}\,, \qquad \varepsilon_1 = |\xi|^{-\frac{\kappa(\ell+1)}{(2\ell+\kappa-\alpha)(\kappa+1)}}\,, \qquad \varepsilon_2 = |\xi|^{-\frac{\ell}{2\ell+\kappa-\alpha}}\,,$$

so that:

$$E(t_2) \lesssim \exp\Big[|\xi|^{\frac{\ell+\kappa-\alpha-1}{2\ell+\kappa-\alpha}}\,\log\big(1+|\xi|\big)\Big]E(t_1) + \int_{t_1}^{t_2}\hat f^2(s,\xi)\,ds\,,$$

for $(t_1,\xi),(t_2,\xi)\in Z^{(1)}$.

For $(t,\xi)\in Z^{(2)}$, we choose $\mu\equiv 0$, so that $(II)\equiv(V)\equiv 0$, and we have:

$$(I)_a \leqslant \int_\tau^T \frac{dt}{t^{\kappa-\alpha}(t^\ell+\varepsilon_2)} + \int_\tau^T \frac{dt}{t^{-(\kappa-\alpha)}(t^\kappa+\varepsilon_1)} \lesssim \tau^{\kappa-\alpha}\Big(\varepsilon_2^{-1+\frac{1}{\ell}} + \varepsilon_1^{-1+\frac{1}{\kappa}}\Big)\,.$$

Choosing:

$$\varepsilon_2 := |\xi|^{-\frac{\ell}{2\ell+\kappa-\alpha}}\,, \qquad \varepsilon_1 := \varepsilon_2^{\frac{\kappa(\ell+1)}{\ell(\kappa+1)}} = |\xi|^{-\frac{\kappa(\ell+1)}{(2\ell+\kappa-\alpha)(\kappa+1)}}\,,$$

so that:

$$\tau^{-(\kappa-\alpha)}\varepsilon_2^{-1+\frac{1}{\ell}} = \varepsilon_2^{1+\frac{1}{\ell}}\,|\xi| = \varepsilon_1^{1+\frac{1}{\kappa}}\,|\xi| = |\xi|^{\frac{\ell+\kappa-\alpha-1}{2\ell+\kappa-\alpha}}\,,$$

we get

$$E(t_2) \lesssim \exp\big[|\xi|^{\frac{\ell+\kappa-\alpha-1}{2\ell+\kappa-\alpha}}\,\log\big(1+|\xi|\big)\big]E(t_1) + \int_{t_1}^{t_2}\hat f^2(s,\xi)\,ds\,,$$

since

$$\frac{2\ell+\kappa-\alpha}{\ell+\kappa-\alpha-1} \leqslant \frac{2\ell}{\ell-1}\,.$$

Case III. If $\alpha \geqslant \kappa$, we choose $\mu = 0$, and (V) vanishes. Moreover, choosing $\varepsilon_1 = \varepsilon_2 = |\xi|^{-\frac{1}{2}}$, we have:

$$(I)_a \quad \leqslant \int_0^T \frac{t^{\alpha-\kappa}}{\varepsilon_2 + t^\ell}\, dt \lesssim \int_0^T \frac{1}{\varepsilon_2 + t^\ell}\, dt \lesssim \varepsilon_2^{-1+\frac{1}{\ell}} = |\xi|^{\frac{\ell-1}{2\ell}} \;,$$

if $\ell > 1$, and

$$(I)_a \quad \leqslant \int_0^T \frac{t^{\alpha-\kappa}}{\varepsilon_2 + t^\ell}\, dt \leqslant \int_0^T \frac{1}{\varepsilon_2 + t^\ell}\, dt \lesssim \log\Big(1 + \frac{1}{\varepsilon_2}\Big)\,,$$

if $\ell = 1$. Thus we obtain:

$$E(t_2) \quad \lesssim \exp\big(|\xi|^{\frac{\ell-1}{2\ell}}\big) E(t_1) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\, ds\,,$$

if $\ell > 1$, and

$$E(t_2) \quad \lesssim \big(1 + |\xi|\big)^N E(t_1) + \int_{t_1}^{t_2} \hat{f}^2(s,\xi)\, ds\,,$$

for some $N > 0$, if $\ell = 1$.

This concludes the proof of Lemma 6.1. $\square$

LEMMA 6.4. — *Let*

$$h := I_3\, P(t, \partial_t, \partial_x) - \sum_{j+k \leqslant 2} \beta_{j,k}(t)\, \partial_t^j \partial_x^k$$

be a 3×3 third order system with diagonal principal part. Assume that the characteristic roots of h verify the conditions (1.19) and (1.20), and moreover:

$$\big\|\beta_{2,0}\tau^2 + \beta_{1,1}\tau + \beta_{0,2}\big\| \lesssim t^\alpha\,. \tag{6.3}$$

Then the Cauchy problem for h is well-posed in γ^s for any $s < s_\alpha$, where s_α is defined in (1.23).

Proof. — We transform h to a 9×9 system as in (5.31). $\mathcal{B}$ has the same block structure, and moreover:

$$\mathcal{B}_{[j,k]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (\beta_{0,2})_{j,k} + |\xi|^{-1}(\beta_{0,1})_{j,k} + |\xi|^{-2}(\beta_{0,0})_{j,k} & (\beta_{1,1})_{j,k} + |\xi|^{-1}(\beta_{1,0})_{j,k} & (\beta_{2,0})_{j,k} \end{pmatrix},$$

hence thanks to (6.3) each block $\mathcal{B}_{[j,k]}$ satisfies (6.1), and we get the proof repeating the same arguments of Step II of the proof of Theorem 1.6. $\square$

Now we prove Theorem 1.9. To this end we prove that LM and NL verify the hypothesis of Lemma 6.4, where

$$M := \Lambda(t, \partial_t, i\xi) - i\,|\xi|\,C(t,\xi)\,, \qquad N := \Lambda(t, \partial_t, i\xi) - i\,|\xi|\,D(t,\xi)\,,$$

and C and D are suitable 0-order matrices.

First of all, we have the following Lemma, whose proof is obtained by direct calculation.

Lemma 6.5. — *Let A be any matrix 3×3 matrix, we have:*

$$(\lambda\,I - A)^{co} \;=\; \lambda^2 I + \lambda\widetilde{A} + A^{co}\,, \tag{6.4}$$

$$(\lambda\,I - A)^{co} \;=\; (\lambda\,I - A)^2 - \mathrm{tr}(\lambda\,I - A)\,(\lambda\,I - A) + \Xi(\lambda)\,, \tag{6.5}$$

where

$$\widetilde{A} := A - \mathrm{tr}(A)\,, \qquad \Xi(\lambda) := \sum_{j<k} (\lambda - \lambda_j)(\lambda - \lambda_k)\,I\,.$$

It follows from (6.4) that the cofactor matrix $\Lambda(t,\lambda,\xi)$ of L_1 can be written as:

$$\Lambda(t,\lambda,\xi) := \big(\lambda\,I - |\xi|\,A(t,\xi)\big)^{co} = \lambda^2 I + |\xi|\,\widetilde{A}(t,\xi)\lambda + |\xi|^2\,A^{co}(t,\xi)\,,$$

with $\widetilde{A}(t,\xi) := \big(A(t,\xi) - \mathrm{tr}A(t,\xi)\,I\big)$. Hence, we have:

$$L(t,\partial_t,\xi)M(t,\partial_t,\xi) = I_3 P(t,\partial_t,\xi) - B\partial_t^2 - |\xi|\,G(t,\xi)\partial_t - |\xi|^2\,F(t,\xi)\,.$$

where:

$$\begin{aligned} F &= F^{(0)} + F^{(1)} \\ F^{(0)} &= (F^{(0)}_{(i,j)})_{i,j=1,2,3} := BA^{co} - A^{co\prime} - AC\,, \\ F^{(1)} &= (F^{(1)}_{(i,j)})_{i,j=1,2,3} := -i\,|\xi|^{-1}\,(C' - BC)\,, \\ G &= (G_{(i,j)})_{i,j=1,2,3} := B\widetilde{A} - \widetilde{A}' + C\,. \end{aligned} \tag{6.6}$$

Let

$$Y := \tau^2\,B + \tau\,G + F^{(0)} = B\,[\tau^2 + \tau\,\widetilde{A} + A^{co}] - (\tau\,\widetilde{A}' + A^{co\prime}) + (\tau\,I - A)\,C\,,$$

we seek for $C \in \mathcal{C}^1$ such that:

$$\big\|Y(t,\xi)\big\|_\infty \lesssim t^\alpha\,, \tag{6.7}$$

From (6.4) and (6.5) we get:

$$B[\tau^2 + \tau\,\widetilde{A} + A^{co}] = B[\tau\, I - A]^{co} = B[Q + \Xi(\tau)]\,,$$

since $\operatorname{tr}(\tau\, I - A) \equiv 0$. Differentiating (6.4) we get:

$$(\tau\,\widetilde{A}' + A^{co\prime}) = \big[(\tau\, I - A)^{co}\big]' + \tau'(\tau\, I - A) = Q' + \Xi(\tau)' + \tau'(\tau\, I - A)\,,$$

hence:

$$Y = B\, Q - Q' + B\,\Xi(\tau) - \Xi(\tau)' - \tau'(\tau\, I - A) + (\tau\, I - A)C\,.$$

Now we remark that $|\Xi(\tau)| + |\Xi(\tau)'| \lesssim t^\kappa$, hence it is sufficient to choose $C = \tau'\, I$ (which has the same regularity of A'), and we have (6.7) by hypothesis (1.24).

On the other hand, if we consider the operator $N(t, \partial_t, i\xi)L(t, \partial_t, i\xi)$, we have to prove (6.7) for

$$Y = (\tau\, I - A)^{co} B + D(\tau\, I - A) - (\tau\, I - A)A'\,.$$

hence choosing $D = 2\,\tau'\, I - A'$, we have:

$$Y = Q\, B + Q' + \Xi(\tau)\, B + \mathcal{O}(t^\alpha)\,.$$

Applying Lemma 6.4 we conclude the proof of Theorem 1.9.

7. Appendix

Let us consider the Example 1.11. We note that:

$$Q = \begin{pmatrix} (\lambda - \lambda_1)^2 & (\lambda_3 - \lambda)a_1 & (\lambda_2 - \lambda)a_2 + a_1 a_3 \\ 0 & (\lambda - \lambda_2)^2 & (\lambda_1 - \lambda)a_3 \\ 0 & 0 & (\lambda - \lambda_3)^2 \end{pmatrix} \approx t^\alpha\,,$$

and:

$$Q' = (\lambda_2 - \lambda)' a_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(t^\alpha)\,.$$

Moreover:

$$\lambda\, I - A = -a_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(t^\alpha)\,.$$

We follow the proof of theorem 1.9, but we choose:

$$C = \lambda' I - (\lambda_2 - \lambda)' \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D = 2\lambda' I - A' + (\lambda_2 - \lambda)' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that we can apply Lemma 6.4.

Bibliography

[B] Bronšteĭn (M. D.). — Smoothness of roots of polynomials depending on parameters, Siberian Math. J. 20 (1980) 347-352, translation from Sibirsk. Mat. Zh. 20 p. 493-501 (1979).

[C] Colombini (F.). — Quelques remarques sur le problème de Cauchy pour des équations faiblement hyperboliques, Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1992), École Polytech., Palaiseau, 1992, Exp. No. XIII.

[CIO] Colombini (F.), Ishida (H.), and Orrú (N.). — On the Cauchy problem for finitely degenerate hyperbolic equations of second order, Ark. Mat. 38, p. 223-230 (2000).

[CJS] Colombini (F.), Jannelli (E.) and Spagnolo (S.). — Well posedness in the Gevrey Classes of the Cauchy Problem for a Non Strictly Hyperbolic Equation with Coefficients Depending On Time, Ann. Scu. Norm. Sup. Pisa, 10, p. 291-312 (1983).

[CN1] Colombini (F.) and Nishitani (T.). — Two by two strongly hyperbolic systems and Gevrey classes, Ann. Univ. Ferrara Sez. VII (N.S.) 45, suppl. (1999), p. 79-108 (2000).

[CN2] ______. — Systèmes 2 fois 2 fortement hyperboliques dans $\mathcal{C}^\infty$ et dans les classes de Gevrey, C. R. Acad. Sci. Paris Sér. I Math. 330, p. 969-972 (2000).

[CO] Colombini (F.) and Orrú (N.). — Well-posedness in $\mathcal{C}^\infty$ for some weakly hyperbolic equations, J. Math. Kyoto Univ. 39, p. 399-420 (1999).

[CS] Colombini (F.) and Spagnolo (S.). — An example of a weakly hyperbolic Cauchy problem not well posed in $\mathcal{C}^\infty$, Acta Math. 148, p. 243-253 (1982).

[CT] Colombini (F.) and Taglialatela (G.). — Levi conditions for higher order operators with finite degeneracy, J. Math. Kyoto Univ. 46 (2006).

[DAK] D'Ancona (P.) and Kinoshita (T.). — On the wellposedness of the Cauchy problem for weakly hyperbolic equations of higher order, Math. Nach. 278, p. 1147-1162 (2005).

[DAKS1] D'Ancona (P.), Kinoshita (T.), and Spagnolo (S.). — Weakly hyperbolic systems with Hölder continuous coefficients, J. Differ. Equations 203, p. 64-81 (2004).

[DAKS2] ______. — On the Well-Posedness of the Cauchy Problem for 2 × 2 weakly hyperbolic systems, Osaka J. Math. 45, p. 921-939 (2008).

[DAR] D'Ancona (P.) and Racke (R.). — Weakly hyperbolic equations in domains with boundaries, Nonlinear Anal. 33, p. 455-472 (1998).

[DAS1] D'Ancona (P.) and Spagnolo (S.). — On pseudosymmetric hyperbolic systems, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), p. 397-418 (1998).

[DAS2] ______. — Quasi-symmetrization of hyperbolic systems and propagation of the analytic regularity, Boll. Unione Mat. Ital., Sez. B Artic Ric. Mat. (8) 1, p. 169-185 (1998).

[DAS3] ______. — A remark on uniformly symmetrizable systems, Adv. in Math. 158, p. 18-25 (2001).

[D] Demay (Y.). — Paramètrix pour des systèmes hyperboliques du premier ordre à multiplicité constante, J. Math. Pures Appl., IX. Sér. 56, p. 393-422 (1977).

[G] Gårding (L.). — Linear hyperbolic partial differential equations with constant coefficients, Acta Math. 85, p. 1-62 (1951).

[IY] Ishida (H.) and Yagdjian (K.). — On a sharp Levi condition in Gevrey classes for some infinitely degenerate hyperbolic equations and its necessity, Publ. Res. Inst. Math. Sci. 38, no. 2, p. 265-287 (2002).

[I] Ivrii (V.Ja.). — Cauchy problem conditions for hyperbolic operators with characteristics of variable multiplicity for Gevrey classes, Sib. Math. J. 17, p. 921-931 (1977).

[KS] Kinoshita (T.) and Spagnolo (S.). — Hyperbolic equations with non analytic coefficients, Math. Ann. 336, p. 551-569 (2006).

[L] Larsson (E.). — Generalized hyperbolicity, Arkiv för Matematik 7, p. 11-32 (1967).

[M] Mandai (T.). — Smoothness of roots of hyperbolic polynomials with respect to one-dimensional parameter, Bull. Fac. Gen. Ed. Gifu Univ. 21, p. 115-118 (1985).

[Ma] Matsumoto (W.). — On the condition for the hyperbolicity of systems with double characteristics roots. I and II, J. Math. Kyoto Univ. 21, p. 47-84, p. 251-271 (1981).

[Me] Mencherini (L.). — Il Problema di Cauchy per sistemi lineari di primo ordine debolmente iperbolici, Ph.D. Thesis, University of Pisa.

[MS1] Mencherini (L.) and Spagnolo (S.). — Gevrey well-posedness for pseudosymmetric systems with lower order terms, Hyperbolic differential operators and related problems (V. Ancona and J. Vaillant, eds.), Lecture Notes in Pure and Appl. Math., vol. 233, Marcel Dekker, New York, NY, (2003), p. 67-81.

[MS2] ______. — Well-posedness of 2×2 systems with $\mathcal{C}^\infty$-coefficients, Hyperbolic problems and related topics (F. Colombini and T. Nishitani, eds.), Grad. Ser. Anal, International Press, Somerville, MA, 2003, Proceedings of the conference, Cortona, Italy, September 10-14, (2002), p. 235-241.

[MS3] ______. — Uniformly symmetrizable 3×3 matrices, Linear Algebra Appl. 382, p. 25-38 (2004).

[N] Nishitani (T.). — Hyperbolicity of two by two systems with two independent variables, Comm. P.D.E. 23, p. 1061-1110 (1998).

[O] Orrù (N.). — On a weakly hyperbolic equation with a term of order zero, Annales de la Faculté des Sciences de Toulouse Sér. 6, 6, p. 525-534 (1997).

[ST] Shinkai (K.) and Taniguchi (K.). — Fundamental solution for a degenerate hyperbolic operator in Gevrey classes, Publ. Res. Inst. Math. Sci. 28, p. 169-205 (1992).

[S] Spagnolo (S.). — On the Absolute Continuity of the Roots of Some Algebraic Equation, Ann.Univ.Ferrara, Suppl. XLV, p. 327-337 (1999).

[T1] Tarama (S.). — On the second order hyperbolic equations degenerating in the infinite order. Example, Math. Japon. 42, p. 523-533 (1995).

[T2] ______. — Note on the Bronshtein theorem concerning hyperbolic polynomials, Sci. Math. Jpn. 63, p. 247-285 (2006).

[W] Wakabayashi (S.). — Remarks on hyperbolic polynomials, Tsukuba J. Math. 10, p. 17-28 (1986).