ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

BRUNO FORNET

Viscous approach for Linear Hyperbolic Systems with Discontinuous Coefficients

Tome XVIII, nº 2 (2009), p. 397-443.

<http://afst.cedram.org/item?id=AFST_2009_6_18_2_397_0>

Annales de la Faculté des Sciences de Toulouse Vol. XVIII, n° 2, 2009
pp. 397–443

Viscous approach for Linear Hyperbolic Systems with Discontinuous Coefficients

Bruno Fornet[(1)]

Abstract. — We introduce small viscosity solutions of hyperbolic systems with discontinuous coefficients accross the fixed noncharacteristic hypersurface $\{x_d = 0\}$. Under a geometric stability assumption, our first result is obtained, in the multi-D framework, for piecewise smooth coefficients. For our second result, the considered operator is $\partial_t + a(x)\partial_x$, with $sign(xa(x)) > 0$ (expansive case not included in our first result), thus resulting in an infinity of weak solutions. Proving that this problem is uniformly Evans-stable, we show that our viscous approach successfully singles out a solution. Both results are new and incorporates a stability result as well as an asymptotic analysis of the convergence at any order, which results in an accurate boundary layer analysis.

Résumé. — On s'intéresse à des problèmes hyperboliques linéaires dont les coefficients sont discontinus au travers de l'hypersurface non-caractéristique $\{x_d = 0\}$. On prouve alors, sous une hypothèse de stabilité, la convergence, à la limite à viscosité évanescente, vers la solution d'un problème hyperbolique limite bien posé. Notre premier résultat concerne des systèmes multi-D, C^∞ par morceaux. Notre second résultat montre que, pour l'opérateur $\partial_t + a(x)\partial_x$, avec $sign(xa(x)) > 0$ (cas exclu de notre premier résultat), notre critère de stabilité est satisfait, et qu'une unique solution à petite viscosité se dégage de notre approche. Nos deux résultats sont nouveaux et incluent une analyse asymptotique à tout ordre ainsi qu'un théorème de stabilité.

(*) Reçu le 23/08/06, accepté le 23/01/08
(1) LATP, Université de Provence, 39 rue Joliot-Curie, 13453 Marseille cedex 13, France
LMRS, Université de Rouen, Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray, France
fornet@cmi.univ-mrs.fr

1. Introduction

Let us consider a linear hyperbolic system of the form:

$$\begin{cases} \partial_t u + \displaystyle\sum_{j=1}^{d} A_j(t,y,x)\partial_j u = f, & (t,y,x) \in \Omega \\ u|_{t=0} = h \quad , \end{cases} \tag{1.1}$$

where $\Omega = \{(t,y,x) \in (0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}\}$, with $T > 0$ fixed once for all. The unknown $u(t,y,x)$ belongs to $\mathbb{R}^N$ and the matrices A_j are valued in the set of $N \times N$ matrices with real coefficients $\mathcal{M}_N(\mathbb{R})$. Due to the discontinuity of the coefficients, the solution u is, in general, awaited to be discontinuous through $\{x = 0\}$. In such case, $\partial_x u$ has a Dirac measure supported on the hypersurface $\{x = 0\}$. Hence, if the coefficient of the normal derivative A_d is also discontinuous through $\{x = 0\}$, the nonconservative product $A_d \partial_x u$ cease to be well-defined in the sense of distributions; weak solutions for the considered problem thus cannot be defined in a classical way.

The definition of such nonconservative product is of course crucial for defining a notion of weak solutions for such problems. It is an interesting question by itself, solved for a quasi-linear analogous problem by Lefloch and Tzavaras ([15]). Adopting a viscous approach will allow us to avoid the difficult question of giving a sense to the nonconservative product in the linear framework.

The problematic investigated in this paper relates to many scalar works on analogous conservative problems. We can for instance refer to the works of Bouchut, James and Mancini in [3], [4]; by Poupaud and Rascle in [19] or by Diperna and Lions in [7]. Among other works on closely related topics, we can also refer to the works of Bachmann and al. ([1],[2]), Fornet ([8],[9]), Gallouët([10]), LeFloch and al. ([6],[14],[15], [13]). The common idea is that another notion of solution has to be introduced to deal with linear hyperbolic Cauchy problems with discontinuous coefficients. Note that almost all the papers cited before use a different approach to deal with the problem. Like in [8] and [9], we will opt for a small viscosity approach.

Let us now describe the first result obtained in this paper. We consider the following viscous hyperbolic-parabolic problem:

$$\begin{cases} \mathcal{H}^\varepsilon u^\varepsilon = f, & (t,y,x) \in \Omega, \\ u^\varepsilon|_{t<0} = 0, \end{cases} \tag{1.2}$$

where $\mathcal{H}^\varepsilon := \partial_t + \sum_{j=1}^{d-1} A_j \partial_j + A_d \partial_x - \varepsilon \sum_{1 \leqslant j,k \leqslant d} \partial_j (B_{j,k} \partial_k .)$, and the coefficients A_j, with $1 \leqslant j \leqslant d$, are piecewise smooth and constant outside a compact set. We assume that the discontinuity of the coefficients occurs only through the hypersurface $\{x = 0\}$. The unknown $u^\varepsilon(t, y, x) \in \mathbb{R}^N$, the source term f belongs to $H^\infty((0, T) \times \mathbb{R}^d)$, and satisfies $f|_{t<0} = 0$; this assumption allows to bypass the analysis of the compatibility conditions. In this problem, ε, commonly called viscosity, stands for a small positive parameter. We stress that, if we suppress the terms in $-\varepsilon\partial_x^2$ from our differential operator, the obtained hyperbolic problem has no obvious sense.

We make the classical hyperbolicity and hyperbolicity-parabolicity assumptions, plus we assume the boundary is noncharacteristic. Additionally, we make a transversality assumption and an assumption concerning the sign of the eigenvalues of A_d on each side of $\{x = 0\}$. Last, we suppose a spectral stability condition, which is a Uniform Evans Condition for a related problem, is satisfied.

Under these assumptions, we prove that, when $\varepsilon \to 0^+$, u^ε converges towards u in $L^2((0, T) \times \mathbb{R}^d)$, where $u := u^+ \mathbf{1}_{x \geqslant 0} + u^- \mathbf{1}_{x<0}$ is solution of a transmission problem of the form:

$$\begin{cases} \partial_t u^+ + \displaystyle\sum_{j=1}^{d} A_j^+ \partial_j u^+ = f^+, & (t, y, x) \in \Omega^+ \\ \partial_t u^- + \displaystyle\sum_{j=1}^{d} A_j^- \partial_j u^- = f^-, & (t, y, x) \in \Omega^- \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, & \\ u^+|_{t<0} = 0, \quad u^-|_{t<0} = 0 & . \end{cases}$$

where Σ is a linear subspace depending of the choice of the viscosity tensor $\sum_{1 \leqslant j,k \leqslant d} \partial_j (B_{j,k} \partial_k \cdot)$; $\Omega^\pm$ denotes $\Omega \bigcap \{\pm x > 0\}$ and the $\pm$ superscripts are used to indicate the restrictions of the concerned functions to $\Omega^\pm$.

A natural and very related question is the grasping of the nature of the interface. If $N = 1$, that is to say for scalar equations, three sort of discontinuities can arise depending on the sign of A_d (here scalar) on each side of $\{x_d = 0\}$. If A_d keeps the same sign in a neighborhood of $\{x_d = 0\}$, the discontinuity of the coefficient, also traducing the nature of the interface, is traversing. Always in a neighborhood of $\{x_d = 0\}$, if A_d has the same sign as x_d then the interface is expansive and if the sign is opposite the interface is compressive. In the case of systems, there is not one but several modes, each either traversing, expansive or compressive. For instance, considering Assumption 2.2, even if $A_d|_{x=0^-}$ and $A_d|_{x=0^+}$ can both be diagonalized in the same basis, this assumption does not prescribe whether or not expansive modes are present in our discontinuity (it depends on which eigenvalues of

$A_d|_{x=0^+}$ are associated to which eigenvalues of $A_d|_{x=0^-}$). If we also assume that, after change of basis, the eigenvalues of both $A_d|_{x=0^-}$ and $A_d|_{x=0^+}$ are both sorted by increasing (or decreasing) order then Assumption 2.2 states that there is exactly q compressive modes and $N-q$ traversing ones. Note well that, generally speaking, $A_d|_{x=0^+}$ and $A_d|_{x=0^-}$ cannot be diagonalized in the same basis and thus the acknowledgment of the different modes, traducing the nature of the interface, becomes difficult to grasp.

An important remark is that, for fixed positive ε, (1.2) can be put on the form of a parabolic problem on the half-space $\{x > 0\}$ with boundary conditions on $\{x = 0\}$ satisfying a **Uniform Evans Condition**. Moreover, the solution of this parabolic problem on a half-space tends, when ε goes to zero, towards the solution of a mixed hyperbolic problem, defined on $\{x > 0\}$, satisfying a **Uniform Lopatinski Condition**. An analogous theorem, in the nonlinear framework and for a shockwave solution, was proved by Rousset ([20]).

For our first result, with conciseness in mind, the proof of stability is exposed only for 1-D systems with piecewise constant coefficients and the artificial viscosity tensor $B = Id$. The goal is to check that the method introduced in [16] does apply to our boundary conditions. During this proof, accent is placed on the role played by the Uniform Evans Condition in the proof of our stability estimates via Kreiss-type Symmetrizers.

Let us now expose our second result, which concerns the sense to give to the solution of:

$$\begin{cases} \partial_t u + a(x)\partial_x u = f, & (t,x) \in (0,T) \times \mathbb{R}, \\ u|_{t=0} = h, \end{cases} \tag{1.3}$$

in the case where $a(x) = a^+\mathbf{1}_{x>0} + a^-\mathbf{1}_{x<0}$, where a^+ is a positive constant and a^- is a negative constant. The source term f belongs to $C_0^\infty((0,T)\times\mathbb{R})$ and the Cauchy data h belongs to $C_0^\infty(\mathbb{R})$. We assume that the coefficient is piecewise constant in order to simplify the proof of our stability estimates, which uses Kreiss-type symmetrizers. Referring to the sign of the coefficient on each side of $\{x = 0\}$, we call such discontinuity of the coefficient expansive. Note that such expansive case was excluded from our previous study on systems by our assumptions. An important point is that, compared to the cases studied for our first result, the expansive case has a quite different qualitative behavior. Indeed, for scalar equations, small amplitude characteristic boundary layers only form in the expansive case.

Our second result states the convergence in the vanishing viscosity limit and in $L^2((0,T)\times\mathbb{R})$ of u^ε, which is solution of

$$\begin{cases} \partial_t u^\varepsilon + a(x)\partial_x u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon = f, & (t,x) \in (0,T) \times \mathbb{R}, \\ u^\varepsilon|_{t=0} = h \quad . \end{cases}$$

towards $\underline{u} \in L^2((0,T) \times \mathbb{R})$, where $\underline{u} := \underline{u}^+ \mathbf{1}_{x \geqslant 0} + \underline{u}^- \mathbf{1}_{x<0}$ is the unique solution of the well-posed, even though **not classical**, transmission problem:

$$\begin{cases} \partial_t \underline{u}^+ + a^+ \partial_x \underline{u}^+ = f^+, & (t,x) \in (0,T) \times \mathbb{R}_+^*, \\ \partial_t \underline{u}^- + a^- \partial_x \underline{u}^- = f^-, & (t,x) \in (0,T) \times \mathbb{R}_-^*, \\ \underline{u}^+|_{x=0} - \underline{u}^-|_{x=0} = 0, \\ \partial_x \underline{u}^+|_{x=0} - \partial_x \underline{u}^-|_{x=0} = 0, \\ \underline{u}^+|_{t=0} = h^+, \quad \underline{u}^-|_{t=0} = h^- \quad . \end{cases}$$

Naturally, $\underline{u}$ is then what could be called the small viscosity solution of (1.3). The result seems to be completely new, since the main difficulty was to "select" a solution among all possible weak solutions. Remark that, this time, by performing explicit computations of the Evans function, we prove that the Uniform Evans Condition holds for our problem thus yielding the desired stability estimates.

2. Some results for multi-D nonconservative systems with "no expansive modes"

2.1. Description of the problem

We first expose our full set of assumptions for the problem involved in our first result.

We note $y := (x_1, \ldots, x_{d-1})$ and $x := x_d$ and consider the viscous equation:

$$\begin{cases} \mathcal{H}^\varepsilon u^\varepsilon = f, & (t,y,x) \in \Omega, \\ u^\varepsilon|_{t<0} = 0, \end{cases} \tag{2.1}$$

where $\mathcal{H}^\varepsilon := \partial_t + \sum_{j=1}^{d-1} A_j \partial_j + A_d \partial_x - \varepsilon \sum_{1 \leqslant j,k \leqslant d} \partial_j (B_{j,k} \partial_k \cdot)$, the unknown $u^\varepsilon(t,y,x) \in \mathbb{R}^N$, the source term f belongs to $H^\infty((0,T) \times \mathbb{R}^d)$, and satisfies $f|_{t<0} = 0$. All the matrices A_j, $1 \leqslant j \leqslant d$ are assumed smooth in (t,y,x) on $\pm x > 0$, discontinuous through $\{x = 0\}$ and constant outside a compact set. The matrices $B_{j,k}$ also depends smoothly of (t,y,x) and are constant outside a compact set. We will denote by $A_d^\pm$ the restriction of A_d to $\{\pm x > 0\}$. We assume that the boundary is noncharacteristic:

Assumption 2.1 (Noncharacteristic boundary). — *$A_d|_{x=0^+}$ and $A_d|_{x=0^-}$ are two nonsingular $N \times N$ matrices with real coefficients.*

Moreover, we make the following structure assumption on the discontinuity of A_d through $\{x = 0\}$:

Assumption 2.2 (Sign Assumption). —

- *The eigenvalues of $A_d^-(t, y, 0)$, sorted by increasing order are denoted by $(\lambda_i^-(t, y))_{1 \leqslant i \leqslant N}$, and are such that $\lambda_p^- < 0$ and $\lambda_{p+1}^- > 0$.*
- *The eigenvalues of $A_d^+(t, y, 0)$, sorted by increasing order are denoted by $(\lambda_i^+(t, y))_{1 \leqslant i \leqslant N}$, and satisfy $\lambda_{p+q}^+ < 0$ and $\lambda_{p+q+1}^+ > 0$, with $q \geqslant 0$.*

We make the following hyperbolicity assumption on the operator

$$\mathcal{H} := \partial_t + \sum_{j=1}^{d} A_j \partial_j :$$

Assumption 2.3 (Hyperbolicity with constant multiplicity). — *For all $(t, y, x) \in (0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ and $(\eta, \xi) \neq 0_{\mathbb{R}^d}$,*

$$\sum_{j=1}^{d-1} \eta_j A_j(t, y, x) + \xi A_d(t, y, x)$$

remains diagonalizable. Moreover, its eigenvalues keep constant multiplicities.

Let us now introduce the symbol of the parabolic part, B, defined by:

$$B(t, y, x, \eta, \xi) := \sum_{j,k<d} \eta_j \eta_k B_{j,k}(t, y, x)$$

$$+ \sum_{j<d} \xi \eta_j (B_{j,d}(t, y, x) + B_{d,j}(t, y, x)) + \xi^2 B_{d,d}(t, y, x).$$

We make then the following hyperbolicity-parabolicity assumption:

Assumption 2.4 (Hyperbolicity-Parabolicity). — *There is $c > 0$ such that for all $(t, y, x) \in (0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ and $(\eta, \xi) \in \mathbb{R}^d$, the eigenvalues of*

$$i \left(\sum_{j=1}^{d-1} \eta_j A_j(t, y, x) + \xi A_d(t, y, x) \right) + B(t, y, x, \eta, \xi)$$

satisfy $\Re e\ \mu \geqslant c(|\eta|^2 + \xi^2)$.

In what follows, $\eta := (\eta_1, \dots, \eta_{d-1})$ will denote the Fourier variable dual to y and ξ the Fourier variable dual to x. Let us now introduce some notations in view of writing the Uniform Evans Condition. $\mathbb{A}^{\pm}$ denotes the matrices of $\mathcal{M}_{2N}(\mathbb{C})$ defined by:

$$\mathbb{A}^{\pm}(t,y,x;\zeta) = \begin{pmatrix} 0 & Id \\ \mathcal{M}^{\pm}(t,y,x;\zeta) & \mathcal{A}^{\pm}(t,y,x;\eta) \end{pmatrix},$$

where $\zeta := (\tau, \gamma, \eta)$,

$$\mathcal{M}^{\pm}(t,y,x;\zeta) = B_{d,d}^{-1} A_d^{\pm}(t,y,x) A^{\pm}(t,y,x;\zeta) + B_{d,d}^{-1}(t,y,x) \sum_{j,k=1}^{d-1} \eta_j \eta_k B_{j,k}(t,y,x),$$

with $A^{\pm}$ standing for the symbol of the hyperbolic part defined by:

$$A^{\pm}(t,y,x;\zeta) := (A_d^{\pm})^{-1}(t,y) \left((i\tau + \gamma) Id + \sum_{j=1}^{d-1} i\eta_j A_j(t,y,x) \right).$$

and

$$\mathcal{A}^{\pm}(t,y,x;\eta) = B_{d,d}^{-1} A_d^{\pm}(t,y,x) - B_{d,d}^{-1}(t,y,x) \sum_{j=1}^{d-1} i\eta_j \left(B_{j,d}(t,y,x) + B_{d,j}(t,y,x) \right).$$

We introduce the weight $\Lambda(\zeta)$ used to deal with high frequencies:

$$\Lambda(\zeta) = \left(1 + \tau^2 + \gamma^2 + |\eta|^4\right)^{\frac{1}{4}}.$$

Let J_Λ be the mapping from $\mathbb{C}^N \times \mathbb{C}^N$ to $\mathbb{C}^N \times \mathbb{C}^N$ given by

$$(u,v) \mapsto (u, \Lambda^{-1} v).$$

The scaled negative and positive spaces of the matrices $\mathbb{A}^{\pm}(t,y,x;\eta)$ are defined by:

$$\tilde{\mathbb{E}}_{\pm}(\mathbb{A}^{\pm}) := J_\Lambda \mathbb{E}_{\pm}(\mathbb{A}^{\pm}).$$

If $\mathbb{E}$ and $\mathbb{F}$ are two linear subspaces of $\mathbb{C}^{2N}$ such that $\dim \mathbb{E} + \dim \mathbb{F} = 2N$, then $det(\mathbb{E}, \mathbb{F})$ stands for the determinant obtained by taking two direct orthonormal bases of $\mathbb{E}$ and $\mathbb{F}$. Our stability assumption writes then:

Assumption 2.5 (Uniform Evans Condition). — *We assume that $(\tilde{\mathcal{H}}^\varepsilon, \Gamma)$ satisfies the Uniform Evans Condition that is to say that, for all $(t,y) \in (0,T) \times \mathbb{R}^{d-1}$ and $\zeta = (\tau, \eta, \gamma) \in \mathbb{R}^d \times \mathbb{R}^+ - \{0_{\mathbb{R}^{d+1}}\}$, there holds:*

$$\tilde{D}(t,y,\zeta) = \left| det\left(\tilde{\mathbb{E}}_-(\mathbb{A}^+(t,y,0;\zeta)), \tilde{\mathbb{E}}_+(\mathbb{A}^-(t,y,0;\zeta)) \right) \right| \geqslant C > 0.$$

$\tilde{D}$ is called the scaled Evans function. The zeros of $\tilde{D}$ track down the instabilities of our problem.

Assumption 2.6 (Transversality). — *$\mathbb{E}_-(G_d^+|_{x=0})$ and $\mathbb{E}_+(G_d^-|_{x=0})$ intersects transversally in $\mathbb{R}^N$, which means that:*

$$\mathbb{E}_-(G_d^+|_{x=0}) + \mathbb{E}_+(G_d^-|_{x=0}) = \mathbb{R}^N.$$

Let G_d denote the matrix $G_d(t,y,x) := B_{d,d}^{-1}A_d(t,y,x)$. We have then the following Lemma:

Lemma 2.7. — *$B_{d,d}$ is nonsingular and its eigenvalues satisfy $\Re e\mu \geqslant c > 0$. Moreover, $G_d|_{x=0^+}$ and $G_d|_{x=0^-}$ have no eigenvalue on the imaginary axis, furthermore*

$$\dim \mathbb{E}_\pm(G_d|_{x=0^+}) = \dim \mathbb{E}_\pm(A_d^+|_{x=0})$$

and

$$\dim \mathbb{E}_\pm(G_d|_{x=0^-}) = \dim \mathbb{E}_\pm(A_d^-|_{x=0}).$$

Proof. — This lemma is a consequence of the hyperbolicity-parabolicity assumption. Fixing $\eta = 0$ and $\xi = \xi_0 \neq 0$ in the hyperbolicity-parabolicity assumption gives that the eigenvalues of: $\xi_0^2 B_{d,d} + i\xi_0 A_d$ satisfy $\Re e\mu \geqslant c\xi_0^2$, for some $c > 0$. Hence the eigenvalues of $B_{d,d} + \frac{i}{\xi_0}A_d$ are such that $\Re e\mu \geqslant c$. Making ξ_0 tends towards infinity, we check that $B_{d,d}$ is nonsingular and that its eigenvalues does not come near the imaginary axis. For all $\xi_0 \neq 0$ and $t \in [0,1]$, the eigenvalues of $tB_{d,d} + (1-t)Id + \frac{i}{\xi_0}A_d$ are such that $\Re e\mu > 0$. Thus $\left(tB_{d,d} + (1-t)Id + \frac{i}{\xi_0}A_d\right)^{-1} A_d$ has no eigenvalue on the imaginary axis. Indeed, if it was the case, it would mean that, for some $\xi_0' \neq 0$, $tB_{d,d}+(1-t)Id+\frac{i}{\xi_0'}A_d$ has also an eigenvalue on the imaginary axis. Since the eigenvalues of $\left(tB_{d,d} + (1-t)Id + \frac{i}{\xi_0}A_d\right)^{-1} A_d$ do not cross the imaginary axis, making ξ_0 tends to infinity and considering in succession $t = 0$ and $t = 1$, we have then proved that G_d has the same number of eigenvalues with positive [resp negative] real part than A_d. In particular, we get that $\dim \mathbb{E}_\pm(G_d|_{x=0^+}) = \dim \mathbb{E}_\pm(A_d^+|_{x=0})$ and $\dim \mathbb{E}_\pm(G_d|_{x=0^-}) = \dim \mathbb{E}_\pm(A_d^-|_{x=0})$. □

2.2. Construction of an approximate solution

We will begin by reformulating the problem (2.1). This viscous problem can be recast as a "doubled" problem on a half space.

Let the “+” [resp “-”] superscript denote the restriction of the concerned function to $\{x > 0\}$ [resp $\{x < 0\}$]. We begin by introducing

$$\tilde{u}^\varepsilon(t,y,x) = \begin{pmatrix} u^{\varepsilon+}(t,y,x) \\ u^{\varepsilon-}(t,y,-x) \end{pmatrix},$$

the new source term writes $\tilde{f}(t,y,x) = \begin{pmatrix} f^{+}(t,y,x) \\ f^{-}(t,y,-x) \end{pmatrix}$, and the new Cauchy data is $\tilde{h} = \begin{pmatrix} h^{+}(t,y,x) \\ h^{-}(t,y,-x) \end{pmatrix}$, the normal coefficient becomes:

$$\tilde{A}_d(t,y,x) = \begin{pmatrix} A_d^{+}(t,y,x) & 0 \\ 0 & -A_d^{-}(t,y,-x) \end{pmatrix}$$

We define then the tangential symbol $\tilde{A}$ as follows:

$$\tilde{A}(t,y,x;\zeta) = \begin{pmatrix} A^{+}(t,y,x;\zeta) & 0 \\ 0 & A^{-}(t,y,-x;\zeta) \end{pmatrix}.$$

For $1 \leqslant j \leqslant d-1$, we denote:

$$\tilde{A}_j(t,y,x) = \begin{pmatrix} A_j^{+}(t,y,x) & 0 \\ 0 & A_j^{-}(t,y,-x) \end{pmatrix}.$$

Moreover, if both $j \neq d, k \neq d$ or if $j = k = d$, we note:

$$\tilde{B}_{j,k}(t,y,x) = \begin{pmatrix} B_{j,k}^{+}(t,y,x) & 0 \\ 0 & B_{j,k}^{-}(t,y,-x) \end{pmatrix};$$

and, if $(j = d, k \neq d)$ or $(j \neq d, k = d)$, we write:

$$\tilde{B}_{j,k}(t,y,x) = \begin{pmatrix} B_{j,k}^{+}(t,y,x) & 0 \\ 0 & -B_{j,k}^{-}(t,y,-x) \end{pmatrix}.$$

Finally, the new boundary condition is:

$$\tilde{\Gamma} = \begin{pmatrix} Id & -Id \\ \partial_x & \partial_x \end{pmatrix},$$

we obtain then the following equivalent reformulation of the hyperbolic-parabolic viscous problem (2.1):

$$\begin{cases} \tilde{\mathcal{H}}^\varepsilon \tilde{u}^\varepsilon = \tilde{f}, \quad \{x > 0\}, \\ \tilde{\Gamma}\tilde{u}^\varepsilon|_{x=0} = 0, \\ \tilde{u}^\varepsilon|_{t<0} = 0. \end{cases} \tag{2.2}$$

where

$$\tilde{\mathcal{H}}^\varepsilon := \partial_t + \sum_{j=1}^{d-1} \tilde{A}_j \partial_j + \tilde{A}_d \partial_x - \varepsilon \sum_{1 \leqslant j,k \leqslant N} \partial_j (\tilde{B}_{j,k} \partial_k \cdot);$$

we will also note

$$\tilde{\mathcal{H}} := \partial_t + \sum_{j=1}^{d-1} \tilde{A}_j \partial_j + \tilde{A}_d \partial_x.$$

We construct an approximate solution of equation (2.2) along the following ansatz:

$$\tilde{u}^\varepsilon_{app}(t,y,x) := \sum_{n=1}^{M} U_n \left(t, y, x, \frac{x}{\varepsilon}\right) \varepsilon^n, \tag{2.3}$$

$$U_n(t,y,x,z) := \underline{U}_n(t,y,x) + U^*_n(t,y,x,z),$$

with $\underline{U}_n \in H^\infty((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*_+)$ and $U^*_n \in e^{-\delta z} H^\infty((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+)$, for some $\delta > 0$. Note that, due to our previous change of unknowns, we have $\underline{U}_n(t,y,x) \in \mathbb{R}^{2N}$ and $U^*_n(t,y,x,z) \in \mathbb{R}^{2N}$. Moreover, we will note:

$$\underline{U}_n(t,y,x) = \begin{pmatrix} \underline{U}^+_n(t,y,x) \\ \underline{U}^-_n(t,y,x) \end{pmatrix}, \qquad U^*_n(t,y,x,z) = \begin{pmatrix} U^{*+}_n(t,y,x,z) \\ U^{*-}_n(t,y,x,z) \end{pmatrix}.$$

Plugging our asymptotic expansion (2.3) into the doubled problem (2.2), we get the following profiles equations: to begin with, U^*_0 satisfies the following ODE in z :

$$\begin{cases} \tilde{A}_d(t,y,x)\partial_z U^*_0 - \tilde{B}_{d,d}(t,y,x)\partial_z^2 U^*_0 = 0, \\ U^{*+}_0|_{(z,x)=0} - U^{*-}_0|_{(z,x)=0} = -\left(\underline{U}^+_0|_{x=0} - \underline{U}^-_0|_{x=0}\right), \\ \partial_z U^{*+}_0|_{(z,x)=0} + \partial_z U^{*-}_0|_{(z,x)=0} = 0. \end{cases}$$

Denote $\tilde{G}_d = \tilde{B}^{-1}_{d,d}\tilde{A}_d$, the profile U^*_0 writes then:

$$U^*_0(t,y,x,z) = e^{\tilde{G}_d(t,y,x)z} U^*_0(t,y,x,0).$$

Going back to the transmission conditions satisfied by U^*_0, we obtain that $U^*_0|_{(z,x)=0}$ satisfies the relations:

$$\begin{cases} U^{*+}_0|_{(z,x)=0} - U^{*-}_0|_{(z,x)=0} = -\sigma_0(t,y), \\ G^+_d(t,y,0)U^{*+}_0|_{(z,x)=0} - G^-_d(t,y,0)U^{*-}_0|_{(z,x)=0} = 0, \\ U^{*+}_0|_{(z,x)=0} \in \mathbb{E}_-\left(G^+_d(t,y,0)\right), \\ U^{*-}_0|_{(z,x)=0} \in \mathbb{E}_+\left(G^-_d(t,y,0)\right), \end{cases}$$

where $\sigma_0 := \underline{U}_0^+|_{x=0} - \underline{U}_0^-|_{x=0}$, and $G_d^\pm := B_{d,d}^{-1} A_d^\pm$. This algebraic problem is well-posed for a fixed σ_0 iff it satisfies, for all $(t,y) \in (0,T) \times \mathbb{R}^{d-1}$:

$$\sigma_0(t,y) \in \Sigma(t,y),$$

with the linear subspace Σ defined by:

$$\Sigma := \left((G_d^+|_{x=0})^{-1} - (G_d^-|_{x=0})^{-1}\right)\left(\mathbb{E}_-(G_d^+|_{x=0}) \bigcap \mathbb{E}_+(G_d^-|_{x=0})\right).$$

The equation giving U_0^* has a unique solution iff:

$$\left[v \in \mathbb{E}_-\left(G_d^+(t,y,0)\right) \bigcap \mathbb{E}_+\left(G_d^-(t,y,0)\right), \left(G_d^+(t,y,0) - G_d^-(t,y,0)\right) v = 0\right]$$
$$\Rightarrow [v = 0],$$

which is equivalent to:

$$\dim \Sigma = \dim\ \mathbb{E}_-(G_d^+|_{x=0}) \bigcap \mathbb{E}_+(G_d^-|_{x=0}).$$

This property results from our assumptions, as we will prove now. As we shall see below, due to the Uniform Evans Condition holding, one gets:

$$\dim \Sigma = N - \dim \mathbb{E}_-(A_d^-|_{x=0}) - \dim \mathbb{E}_+(A_d^+|_{x=0}).$$

Since $A_d^-|_{x=0}$ and $A_d^+|_{x=0}$ are nonsingular, $\dim \mathbb{E}_-(A_d^-|_{x=0}) = N - \dim \mathbb{E}_+ (A_d^-|_{x=0})$ and $\dim \mathbb{E}_+(A_d^+|_{x=0}) = N - \dim \mathbb{E}_-(A_d^+|_{x=0})$. Plus, by Lemma 2.7, we have $\dim \mathbb{E}_-(G_d^+|_{x=0}) = \dim \mathbb{E}_-(A_d^+|_{x=0})$ and $\dim \mathbb{E}_+(G_d^-|_{x=0}) = \dim \mathbb{E}_+(A_d^-|_{x=0})$. We obtain thus:

$$N + \dim \Sigma = \dim \mathbb{E}_+(G_d^-|_{x=0}) + \dim \mathbb{E}_-(G_d^+|_{x=0}).$$

Thanks to our transversality assumption stated in Assumption 2.6, there holds:

$$\dim \mathbb{E}_+(G_d^-|_{x=0}) + \dim \mathbb{E}_-(G_d^+|_{x=0}) = N + \dim\ \mathbb{E}_-(G_d^+|_{x=0}) \bigcap \mathbb{E}_+(G_d^-|_{x=0}).$$

This ends the proof of:

$$\dim \Sigma = \dim\ \mathbb{E}_-(G_d^+|_{x=0}) \bigcap \mathbb{E}_+(G_d^-|_{x=0}).$$

We must however know $\sigma_0(t,y) \in \Sigma(t,y)$ in order to obtain U_0^*. σ_0 is deduced from the computation of the profile $\underline{U}_0$, which is solution of the following mixed hyperbolic problem:

$$\begin{cases} \tilde{\mathcal{H}} \underline{U}_0 = \tilde{f}, & \{x > 0\}, \\ \underline{U}_0^+|_{x=0} - \underline{U}_0^-|_{x=0} \in \Sigma, & \\ \underline{U}_0|_{t<0} = 0 \quad . & \end{cases} \tag{2.4}$$

We will now sketch a proof of the well-posedness of this equation. Some elements of it will be proved afterwards, in another subsection. The function $\underline{U}_0$ is also solution of the mixed hyperbolic problem:

$$\begin{cases} \tilde{\mathcal{H}}\underline{U}_0 = \tilde{f}, & \{x>0\}, \\ \Gamma^H \underline{U}_0|_{x=0} = 0, \\ \underline{U}_0|_{t<0} = 0, \end{cases}$$

where Γ^H denotes a linear operator such that:

$$\ker \Gamma^H = \mathcal{C}(t,y) := \left\{ \left(\begin{array}{c} U_0^{*+}|_{(z,x)=0} \\ U_0^{*-}|_{(z,x)=0} \end{array} \right) \; : \; U_0^{*+}|_{(z,x)=0} - U_0^{*-}|_{(z,x)=0} \in \Sigma \right\};$$

note that $\mathcal{C}$ is the stable manifold for the dynamical system U_0^* is solution of. The Uniform Lopatinski Condition writes that there is $C>0$, such that, for all $(t,y) \in (0,T) \times \mathbb{R}^{d-1}$ and ζ with $\gamma > 0$, there holds:

$$det(\mathbb{E}_+(A|_{x=0^-}), \mathbb{E}_-(A|_{x=0^+}))) \geqslant C > 0,$$

where we recall that:

$$A(t,y,x;\zeta) := -(A_d)^{-1}(t,y,x) \left((i\tau+\gamma)A_0(t,y,x) + i\sum_{j=1}^{d-1} \eta_j A_j(t,y,x) \right).$$

In particular, taking $\gamma = 1$ and $(\tau,\eta) = 0$, it induces that:

$$\mathbb{E}_-(A_d^-|_{x=0}) \bigcap \mathbb{E}_+(A_d^+|_{x=0}) = \{0\}.$$

We will prove in section 2.5 that this Uniform Lopatinski Condition holds. It is a result very similar to the one of Rousset in [20], established in the nonlinear framework, which states that the Uniform Lopatinski Condition holds for the limiting hyperbolic problem as the consequence of the Uniform Evans condition holding for the parabolic, viscously perturbed, problem. We underline that, in our case, our transversality assumption is necessary in order to prove this result. Remark that the Uniform Lopatinski Condition holds iff there is $C>0$ such that, for all $(t,y) \in (0,T) \times \mathbb{R}^{d-1}$ and ζ with $\gamma > 0$, there holds:

$$det \left(\mathbb{E}_+(A|_{x=0^-}) \bigoplus \mathbb{E}_-(A|_{x=0^+})(t,y,\zeta), \Sigma(t,y) \right) \geqslant C > 0.$$

It implies that $\dim \Sigma = N - \dim \mathbb{E}_-(A|_{x=0^+}) - \dim \mathbb{E}_+(A|_{x=0^-})$. Due to our hyperbolicity assumption, $\dim \mathbb{E}_-(A|_{x=0^+}) = \dim \mathbb{E}_+(A_d^+|_{x=0})$ and

$\dim \mathbb{E}_+(A|_{x=0^-}) = \dim \mathbb{E}_-(A_d^-|_{x=0})$. Hence $\dim \Sigma = N - \dim \mathbb{E}_-(A_d^-|_{x=0}) - \dim \mathbb{E}_+(A_d^+|_{x=0})$. Remark that, in the case of a 1-D problem with a piecewise constant coefficient, equal to $A^\pm$ on $\{\pm x > 0\}$, taking $B = Id$ as the viscosity tensor, the Uniform Lopatinski Condition writes:

$$\mathbb{E}_-(A^-) \bigoplus \mathbb{E}_+(A^+) \bigoplus \Sigma := \mathbb{R}^N.$$

For the sake of completeness, we will now show that the construction of the profiles can go on at any order. Let us assume the profiles up to order $n-1$, with $n \leqslant M$, have been computed. We will now proceed with the construction of the profiles $\underline{U}_n$ and U_n^*. To begin with, U_n^* satisfies the ODE in z :

$$\begin{cases} \tilde{A}_d(t,y,x)\partial_z U_n^* - \tilde{B}_{d,d}(t,y,x)\partial_z^2 U_n^* = \varphi_n^*, \\ U_n^{*+}|_{(z,x)=0} - U_n^{*-}|_{(z,x)=0} = -\sigma_n := -\left(\underline{U}_n^+|_{x=0} - \underline{U}_n^-|_{x=0}\right), \\ \partial_z U_n^{*+}|_{(z,x)=0} + \partial_z U_n^{*-}|_{(z,x)=0} = -\left(\partial_x \underline{U}_{n-1}^+|_{x=0} + \partial_x \underline{U}_{n-1}^-|_{x=0}\right), \end{cases}$$

with

$$\varphi_n^* = -\partial_t U_{n-1}^* - \sum_{j=1}^{d-1} \tilde{A}_j \partial_j U_{n-1}^* + \sum_{j=1}^{d} \partial_j (B_{j,d}\partial_z U_{n-1}^*)$$

$$+ \sum_{k=1}^{d} \partial_z (B_{d,k}\partial_k U_{n-1}^*) + \sum_{j,k<d} \partial_j (B_{j,k}\partial_k U_{n-2}^*).$$

As a consequence, there is $v_n^* \in e^{-\delta z} H^\infty((0,T)\times \mathbb{R}^{d-1} \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$ such that:

$$U_n^*(t,y,x,z) = e^{\tilde{G}_d(t,y,x)z}\left(U_n^*|_{z=0} - v_n^*|_{z=0}\right) + v_n^*(t,y,x,z).$$

Some more computations show that the ODE giving U_n^{*-} is well-posed for fixed σ_n, provided that σ_n belongs to Σ_n, where Σ_n is an affine space directed by Σ. More precisely, Σ_n writes:

$$\Sigma_n = q_n + \Sigma,$$

with $q_n \in H^\infty((0,T)\times \mathbb{R}^{d-1})$. $\underline{U}_n$ is then solution of the mixed hyperbolic problem satisfying a Uniform Lopatinski Condition:

$$\begin{cases} \tilde{\mathcal{H}}\underline{U}_n = \displaystyle\sum_{1\leqslant j,k\leqslant d} \partial_j (B_{j,k}\partial_k \underline{U}_{n-1}), \qquad \{x>0\}, \\ \underline{U}_n^+|_{x=0} - \underline{U}_n^-|_{x=0} \in \Sigma_n, \\ \underline{U}_n|_{t<0} = 0. \end{cases}$$

Indeed, there is $r_n \in H^\infty((0,T)\times\mathbb{R}^{d-1})$, such that the problem writes as well:

$$\begin{cases} \tilde{\mathcal{H}}\underline{U}_n = \sum_{1\leqslant j,k\leqslant d} \partial_j(B_{j,k}\partial_k \underline{U}_{n-1}), & \{x>0\}, \\ \Gamma^H \underline{U}_n|_{x=0} = \Gamma^H r_n, \\ \underline{U}_n|_{t<0} = 0. \end{cases}$$

$\sigma_n \in \Sigma_n$ is deduced from this equation and thus U_n^* can now be computed.

2.3. Stability Analysis and Main Result

The error equation writes, for $w^\varepsilon = u^\varepsilon_{app} - u^\varepsilon$:

$$\begin{cases} \mathcal{H}^\varepsilon w^\varepsilon = \varepsilon^M R^\varepsilon, \\ w^\varepsilon|_{t<0} = 0. \end{cases} \tag{2.5}$$

Our goal here is to prove that the error w^ε converges towards zero as the viscosity vanishes. To be more precise we will prove some uniform energy estimates in L^2 norm. The proof of these stability estimates is almost the same as the ones performed in [17]. In [16], Métivier gives a simplified version of the proof for constant coefficients. Assuming the coefficients are constant, the energy estimates can then be proved by performing a tangential Laplace-Fourier transform of the problem. In this special case, the symmetrizers are Fourier Multipliers hence avoiding the need of any pseudodifferential calculus. Moreover, we emphasize that the analysis of the stability of the problem for frozen coefficients is a crucial step in the proof of more general energy estimates.

For our part, some elements of proof have to be given since our assumptions differ of the ones in [16] or in [17]. In order to shorten a not so original proof, we will rather focus on showing that the scheme of proof exposed in [17] works for our present problem. We will proceed to do so on a very simplified example. In the process, we will reinvestigate the link existing between the Uniform Evans Condition holding and the construction of Kreiss-type symmetrizers. Our proof will be performed in the 1-D framework, for piecewise constant coefficients and for a viscosity tensor $B = Id$. Rather than giving a proof more simple but also more specific to our example, we aim at giving an easily generalized proof, which, even if exposed differently, relates clearly to [16], [17] and [11]. Note that a similar proof of stability can be proved in the multi-D framework thanks to the Theorem 2.12, which states the existence of a low frequency symmetrizer ([16]), be it for 1-D or multi-D systems. Remark that, in our special case, no glancing modes (i.e eigenvalues which becomes, after a rescaling focused on a neighborhood of $\zeta = 0$, purely imaginary and not semi-simple) appear, which makes the proof of

Theorem 2.12 become a lot easier to perform. These stability results can also be proved for multi-D systems with piecewise smooth coefficients, constant outside a compact set, through the use of pseudodifferential calculus. Let us now state the results obtained under our initial assumptions. Choosing M big enough, we get:

THEOREM 2.8 (Stability). — *There is $C > 0$ such that, for all $0 < \varepsilon < 1$, there holds*

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^2((0,T)\times\mathbb{R}^d)} \leqslant C\varepsilon.$$

Let u be $u := u^+\mathbf{1}_{x\geqslant 0} + u^-\mathbf{1}_{x<0}$, where (u^+, u^-) is the unique solution of the well-posed transmission problem:

$$\begin{cases} \mathcal{H}^+u^+ = f^+, & \{x > 0\}, \\ \mathcal{H}^-u^- = f^-, & \{x < 0\}, \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, \\ u^+|_{t<0} = 0, \quad u^-|_{t<0} = 0. \end{cases} \tag{2.6}$$

We obtain then the following convergence result, which is our main result:

THEOREM 2.9 (Convergence). — *There is $C > 0$ such that, for all $0 < \varepsilon < 1$, there holds:*

$$\|u^\varepsilon - u\|_{L^2((0,T)\times\mathbb{R}^d)} \leqslant C\varepsilon.$$

2.4. Simplified proof of stability estimates

We will prove stability estimates for the following viscous system in one space dimension:

$$\begin{cases} \partial_t u^\varepsilon + A(x)\partial_x u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon = f, & (t,x) \in (0,T)\times\Omega, \\ u^\varepsilon|_{t<0} = 0. \end{cases}$$

where the coefficient A is assumed piecewise constant, equal to A^+ on $\{x > 0\}$ and equal to A^- on $\{x < 0\}$. We still make the same assumptions as before on this system. We have constructed

$$u^\varepsilon_{app} := u^{\varepsilon+}_{app}(t,x)\mathbf{1}_{x>0} + u^{\varepsilon-}_{app}(t,-x)\mathbf{1}_{x<0}$$

such that, if we denote $w^\varepsilon = u^\varepsilon_{app} - u^\varepsilon$, there holds:

$$\begin{cases} \partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon\partial_x^2 w^\varepsilon = \varepsilon^M R^\varepsilon, & (t,x) \in \Omega, \\ w^\varepsilon|_{t<0} = 0. \end{cases}$$

where $\Omega = (0,T)\times\mathbb{R}$, R^ε belongs to $H^\infty((0,T)\times\mathbb{R}^*)$ and vanishes in the past. Since our method of estimation comes from pseudodifferential calculus, we have to perform a tangential Fourier-Laplace transform of the problem. To this aim, it is necessary to extend the definition of our error, in order for it to be defined for all time $t \in \mathbb{R}$. We denote by $\underline{\tilde{R}}^\varepsilon$, R^ε extended by 0 outside $(-\infty, T) \times \mathbb{R}$. Let us now proceed with the extension of our error to $t \geqslant T$. We call by $\underline{\tilde{w}}^\varepsilon$ the unique solution of:

$$\begin{cases} \mathcal{H}\underline{\tilde{w}}^\varepsilon - \varepsilon\partial_x^2\underline{\tilde{w}}^\varepsilon = \varepsilon^M \underline{\tilde{R}}^\varepsilon, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \underline{\tilde{w}}^\varepsilon|_{t<0} = 0. \end{cases} \tag{2.7}$$

Note well that the restriction of $\underline{\tilde{w}}^\varepsilon$ to Ω is w^ε. For the sake of simplicity, we will still denote $\underline{\tilde{w}}^\varepsilon$ [resp $\underline{\tilde{R}}^\varepsilon$] by w^ε [resp R^ε] in what follows.
We now come back to our error equation (2.7). To begin with, let us rewrite the problem (2.7) in a convenient form. w^ε is solution of:

$$\partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon\partial_x^2 w^\varepsilon = \varepsilon^M R^\varepsilon, \qquad (t,x) \in \mathbb{R} \times \mathbb{R},$$

Let γ stand for a positive parameter. We denote then by $\hat{w}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} w^{\varepsilon\pm})$ and $\hat{R}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} R^{\varepsilon\pm})$, where $\mathcal{F}$ stands for the tangential Fourier transform (with respect to t) and the $\pm$ superscripts indicates restrictions to $\{\pm x > 0\}$, we have then:

$$\begin{cases} (i\tau+\gamma)\hat{w}^{\varepsilon+} + A^+\partial_x\hat{w}^{\varepsilon+} - \varepsilon\partial_x^2\hat{w}^{\varepsilon+} = \varepsilon^M \hat{R}^{\varepsilon+}, & \{x>0\}, \\ (i\tau+\gamma)\hat{w}^{\varepsilon-} + A^-\partial_x\hat{w}^{\varepsilon-} - \varepsilon\partial_x^2\hat{w}^{\varepsilon-} = \varepsilon^M \hat{R}^{\varepsilon-}, & \{x<0\}, \\ \hat{w}^{\varepsilon+}|_{x=0} - \hat{w}^{\varepsilon-}|_{x=0} = 0, \\ \partial_x\hat{w}^{\varepsilon+}|_{x=0} - \partial_x\hat{w}^{\varepsilon-}|_{x=0} = 0. \end{cases} \tag{2.8}$$

Remark that, by taking γ big enough, the restrictions of the solution w^ε of (2.7) to $\{\pm x > 0\}$ are given by:

$$w^{\varepsilon\pm} = e^{\gamma t}\mathcal{F}^{-1}(\hat{w}^{\varepsilon\pm}),$$

where $(\hat{w}^{\varepsilon+}, \hat{w}^{\varepsilon-})$ are the solutions of the transmission problem (2.8).

Taking $W^{\varepsilon\pm}(i\tau+\gamma, x) = \begin{pmatrix} \hat{w}^{\varepsilon\pm} \\ \varepsilon\partial_x\hat{w}^{\varepsilon\pm} \end{pmatrix}$, we have then:

$$\begin{cases} \partial_x W^{\varepsilon+} = \begin{pmatrix} \partial_x\hat{w}^{\varepsilon+} \\ \varepsilon\partial_x^2\hat{w}^{\varepsilon+} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon}Id \\ (i\tau+\gamma) & \frac{1}{\varepsilon}A^+ \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon+} \\ \varepsilon\partial_x\hat{w}^{\varepsilon+} \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon^M\hat{R}^{\varepsilon+} \end{pmatrix}, \\ \partial_x W^{\varepsilon-} = \begin{pmatrix} \partial_x\hat{w}^{\varepsilon-} \\ \varepsilon\partial_x^2\hat{w}^{\varepsilon-} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon}Id \\ (i\tau+\gamma) & \frac{1}{\varepsilon}A^- \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon-} \\ \varepsilon\partial_x\hat{w}^{\varepsilon-} \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon^M\hat{R}^{\varepsilon-} \end{pmatrix}, \\ W^{\varepsilon+}|_{x=0} - W^{\varepsilon-}|_{x=0} = 0. \end{cases}$$

We note $\zeta = (\tau, \gamma)$ and $\tilde{\zeta} = (\varepsilon\tau, \varepsilon\gamma)$. Multiplying the previous equation by ε gives:

$$\begin{cases} \partial_z W^{\varepsilon+} - \mathbb{A}^+(\tilde{\zeta}) W^{\varepsilon+} = G^+, & \{z > 0\}, \\ \partial_z W^{\varepsilon-} - \mathbb{A}^-(\tilde{\zeta}) W^{\varepsilon-} = \tilde{G}^-, & \{z < 0\}, \\ W^{\varepsilon+}|_{z=0} = W^{\varepsilon-}|_{z=0}, \end{cases} \tag{2.9}$$

where $\mathbb{A}^\pm(\tilde{\zeta}) = \begin{pmatrix} 0 & Id \\ (i\tilde{\tau} + \tilde{\gamma}) Id & A^\pm \end{pmatrix}$ and $G^\pm = \begin{pmatrix} 0 \\ \varepsilon^{M+1} \hat{R}^{\varepsilon\pm} \end{pmatrix}$, and z stands for the fast variable $\frac{x}{\varepsilon}$. Note that the first energy estimates to be proved will concern this equation.

2.4.1. *Proof of the error estimate by symmetrizers*

We will now show how, thanks to the Uniform Evans condition holding, stability estimates can be proved by symmetrizers for the three different regimes of frequency: low, medium and high. In the construction of symmetrizers, for the sake of simplicity, we will drop the tildes in our notations and only introduce them back when needed.

An error estimate for medium frequencies

For $1 \leqslant |\zeta| \leqslant 2$, we will prove here Proposition 2.10. Denote $\tilde{\mathbb{A}}^- = -\mathbb{A}^-$, $\underline{W}^{\varepsilon-} := W^{\varepsilon-}(t, -z)$ and $\underline{G}^- = \tilde{G}^-(t, -z)$, $\underline{W}^{\varepsilon-}$ satisfies then the following ODE in z:

$$\begin{cases} \partial_z \underline{W}^{\varepsilon-} - \tilde{\mathbb{A}}^- \underline{W}^{\varepsilon-} = \underline{G}^-, & \{z > 0\}, \\ \lim\limits_{z\to\infty} \underline{W}^{\varepsilon-} = 0. \end{cases}$$

It implies that $\underline{W}^{\varepsilon-}|_{z=0}$ belongs to the stable manifold:

$$\mathcal{W}^{s-} = q_n^-|_{z=0} + \mathbb{E}_-\left(\tilde{\mathbb{A}}^-|_{z=0}\right),$$

where q_n^- is a bounded solution of the above ODE. Even if q_n^- can be chosen in several ways, the space $\mathcal{W}^{s-}$ is uniquely defined. In addition, $W^{\varepsilon+}$ is solution of:

$$\begin{cases} \partial_z W^{\varepsilon+} - \mathbb{A}^+ W^{\varepsilon+} = G^+, & \{z > 0\}, \\ \lim\limits_{z\to\infty} W^{\varepsilon+} = 0. \end{cases}$$

Therefore $W^{\varepsilon+}|_{z=0}$ belongs to the stable manifold:

$$\mathcal{W}^{s+} = q_n^+|_{z=0} + \mathbb{E}_-\left(\mathbb{A}^+|_{z=0}\right).$$

We have

$$\mathbb{C}^{2N} = \mathbb{E}_-(\mathbb{A}^+) \bigoplus \mathbb{E}_+(\mathbb{A}^+).$$

The projectors associated to this decomposition will respectively be Π_1^- and Π_1^+. Under our structure assumptions, as in [16], there is two hermitian symmetric, uniformly bounded, matrices S_1^+ and S_1^- such that:

- There is $C > 0$ such that, for all $q \in \mathbb{E}_+(\mathbb{A}^+)$,

  $$\langle \Re e S_1^+ \mathbb{A}^+ q, q \rangle \geqslant C|q|^2,$$

  and, for all $q \in \mathbb{E}_-(\mathbb{A}^+)$,

  $$-\langle \Re e S_1^- \mathbb{A}^+ q, q \rangle \geqslant C|q|^2.$$

- There is $c_1^+ > 0$ and $c_1^- > 0$ such that:

  $$\Pi_1^{+*}\Pi_1^+ \leqslant S_1^+ \leqslant c_1^+ \Pi_1^{+*}\Pi_1^+, \quad \Pi_1^{-*}\Pi_1^- \leqslant S_1^- \leqslant c_1^- \Pi_1^{-*}\Pi_1^-.$$

Note well that neither the Uniform Evans condition, nor our boundary conditions intervene in the proof of this result. In what follows, κ will always denote a positive parameter. We define then $\mathcal{S}_\kappa^+$ by

$$\mathcal{S}_\kappa^+ := \kappa S_1^+ - S_1^-.$$

We will prove further that, provided that we choose κ large enough, $\mathcal{S}_\kappa^+$ is a suitable Kreiss-type symmetrizer for our system if the Uniform Evans Condition holds. For now, we have constructed a hermitian symmetric, uniformly bounded matrix $\mathcal{S}_\kappa^+$ and there is $c_{1,\kappa} > 0$ such that:

$$2\Re e \mathcal{S}_\kappa^+ \mathbb{A}^+ \geqslant c_{1,\kappa} Id.$$

As we will see, our stability condition will play a role in the control of the traces $W^{\varepsilon+}|_{z=0}$ and $\underline{W}^{\varepsilon-}|_{z=0}$, which is the crucial step in the proof of our energy estimates. Those traces are linked together by the relations: $W^{\varepsilon+}|_{z=0} = \underline{W}^{\varepsilon-}|_{z=0}$, with $W^{\varepsilon+}|_{z=0} \in \mathcal{W}^{s+}$ and $\underline{W}^{\varepsilon-}|_{z=0} \in \mathcal{W}^{s-}$. Remark that there is uniqueness for the traces $W^{\varepsilon+}|_{z=0} = \underline{W}^{\varepsilon-}|_{z=0}$, satisfying the above relations, iff:

$$\mathbb{E}_-\left(\tilde{\mathbb{A}}^+|_{z=0}\right) \bigcap \mathbb{E}_-\left(\tilde{\mathbb{A}}^-|_{z=0}\right) = \{0\},$$

which is equivalent, for the range of frequencies we are presently considering, to our Uniform Evans Condition.

We perform an analogous construction of a potential symmetrizer for $\underline{W}^{\varepsilon-}$. The projectors associated to the decomposition:

$$\mathbb{C}^{2N} = \mathbb{E}_-(\tilde{\mathbb{A}}^-) \bigoplus \mathbb{E}_+(\tilde{\mathbb{A}}^-)$$

will respectively be Π_2^- and Π_2^+.

Under our structure assumptions, as in [16], there is two hermitian symmetric, uniformly bounded, matrices S_2^+ and S_2^- such that:

- There is $C > 0$ such that, for all $q \in \mathbb{E}_+(\tilde{\mathbb{A}}^-)$,

$$\langle \Re e S_2^+ \tilde{\mathbb{A}}^- q, q \rangle \geqslant C|q|^2,$$

  and, for all $q \in \mathbb{E}_-(\tilde{\mathbb{A}}^-)$,

$$-\langle \Re e S_2^- \tilde{\mathbb{A}}^- q, q \rangle \geqslant C|q|^2.$$

- There is $c_2^+ > 0$ and $c_2^- > 0$ such that:

$$\Pi_2^{+*}\Pi_2^+ \leqslant S_2^+ \leqslant c_2^+ \Pi_2^{+*}\Pi_2^+, \quad \Pi_2^{-*}\Pi_2^- \leqslant S_2^- \leqslant c_2^- \Pi_2^{-*}\Pi_2^-.$$

Like before, neither our stability condition, nor our boundary conditions intervene here. We define then $\mathcal{S}_\kappa^-$ by

$$\mathcal{S}_\kappa^- := \kappa S_2^+ - S_2^-.$$

The so constructed matrix $\mathcal{S}_\kappa^-$ is hermitian symmetric, uniformly bounded and satisfies, for some $c_{2,\kappa} > 0$:

$$2\Re e \mathcal{S}_\kappa^- \mathbb{A}^- \geqslant c_{2,\kappa} Id.$$

We recall that $W^{\varepsilon+}|_{z=0} = \underline{W}^{\varepsilon-}|_{z=0} = \underline{q}$. For the sake of clarity, we will drop the κ subscripts. Let us now prove our energy estimates.

$$-\left\langle \mathcal{S}^+ W^{\varepsilon+}|_{z=0}, W^{\varepsilon+}|_{z=0} \right\rangle = \int_0^\infty \left\langle \mathcal{S}^+ \frac{d}{dz} W^{\varepsilon+}, W^{\varepsilon+} \right\rangle + \left\langle \mathcal{S}^+ W^{\varepsilon+}, \frac{d}{dz} W^{\varepsilon+} \right\rangle dz$$

$$= \int_0^\infty \left\langle 2\Re e \mathcal{S}^+ \mathbb{A}^+ W^{\varepsilon+}, W^{\varepsilon+} \right\rangle \, dz + \int_0^\infty \left\langle 2\Re e \mathcal{S}^+ \tilde{G}^+, W^{\varepsilon+} \right\rangle \, dz$$

thus

$$c_1 \int_0^\infty \left\langle W^{\varepsilon+}, W^{\varepsilon+} \right\rangle \, dz \leqslant - \left\langle \mathcal{S}^+ W^{\varepsilon+}|_{z=0}, W^{\varepsilon+}|_{z=0} \right\rangle + \left| \int_0^\infty \left\langle 2\Re e \mathcal{S}^+ \tilde{G}^+, W^{\varepsilon+} \right\rangle \, dz \right|$$

Denoting by $\|u\| := \|u\|_{L^2(\mathbb{R}_+^*)} = \left(\int_0^\infty \langle u, u \rangle \, dz \right)^{\frac{1}{2}}$, we obtain then that there are $c_1' > 0$ and $C_1' > 0$ such that:

$$c_1' \|W^{\varepsilon+}\|^2 \leqslant - \left\langle \mathcal{S}^+ W^{\varepsilon+}|_{z=0}, W^{\varepsilon+}|_{z=0} \right\rangle + C_1' \|\Re e \mathcal{S}^+ \tilde{G}^+\|^2.$$

Performing the same steps once again, we get that:

$$c_2'\|\underline{W}^{\varepsilon -}\|^2 \leqslant -\left\langle \mathcal{S}^- \underline{W}^{\varepsilon -}|_{z=0}, \underline{W}^{\varepsilon -}|_{z=0}\right\rangle + C_2'\|\Re e\mathcal{S}^- \underline{G}^-\|^2.$$

Taking $c = min(c_1', c_2')$, and $C = min(C_1', C_2')$, we get then:

$$c\|W^\varepsilon\|^2_{L^2(\mathbb{R})} + \left\langle (\mathcal{S}^+ + \mathcal{S}^-)\underline{q}, \underline{q}\right\rangle \leqslant C\left(\|\Re e\mathcal{S}^+\tilde{G}^+\|^2 + \|\Re e\mathcal{S}^-\tilde{G}^-\|^2\right).$$

Proposition 2.10. — *For κ large enough, there is $\delta > 0$ such that, for all $\underline{q} \in \mathbb{C}^{2N}$, there holds:*

$$\left\langle \left(\mathcal{S}^+_\kappa + \mathcal{S}^-_\kappa\right)\underline{q}, \underline{q}\right\rangle \geqslant \delta\langle \underline{q}, \underline{q}\rangle. \tag{2.10}$$

Moreover, there is c, δ and C positive such that, for all $0 < \varepsilon < 1$, we have:

$$c\|W^\varepsilon\|^2_{L^2(\mathbb{R})} + \delta|W^\varepsilon|_{z=0}|^2 \leqslant C\|G\|^2_{L^2(\mathbb{R})}. \tag{2.11}$$

Proof. — As a preliminary, we have the next lemma:

Lemma 2.11. — *Suppose the uniform Evans condition satisfied, then, for all $|\zeta| \neq 0$ and for all $\underline{q} \in \mathbb{C}^{2N}$, we have either $\underline{q} = 0$ or $\Pi_1^+(\zeta)\underline{q} \neq 0$ or $\Pi_2^+(\zeta)\underline{q} \neq 0$.*

Proof. — Indeed, fixing $\zeta \neq 0$, if there exists $\underline{q} \neq 0$ such that $\Pi_1^+\underline{q} = 0$ or $\Pi_2^+\underline{q} = 0$, we get:

$$\Pi_1^-(\underline{q}) = \Pi_2^-(\underline{q}) = \underline{q}.$$

As a result $\underline{q}$ is nonzero and belongs to $\mathbb{E}_-(\mathbb{A}^+) \bigcap \mathbb{E}_+(\mathbb{A}^+)$, which contradicts our stability assumption. □

For $\underline{q} = 0$, the inequality is trivially satisfied. For $\underline{q} \in \mathbb{C}^{2N}$ such that $\Pi_1^+\underline{q} \neq 0$, taking κ large enough gives the result. Notice that, for $\underline{q} \in \mathbb{C}^{2N}$ with $\Pi_2^+\underline{q} \neq 0$, taking κ large enough also leads to the result. Now Lemma 2.11 states that either $q = 0$, either $\Pi_1^+\underline{q} \neq 0$ or $\Pi_2^+\underline{q} \neq 0$, which achieves the proof of the first part of Proposition 2.10, using the inequality (2.10), it follows that:

$$c\|W^\varepsilon\|^2_{L^2(\mathbb{R})} + \delta|W^\varepsilon|_{z=0}|^2 \leqslant C\left(\|\Re e\mathcal{S}^+\tilde{G}^+\|^2_{L^2(\mathbb{R}^+)} + \|\Re e\mathcal{S}^-\underline{G}^-\|^2_{L^2(\mathbb{R}^-)}\right),$$

thus leading to the estimate (2.11). □

An error estimate for high frequencies

Denote by

$$w_1^{\varepsilon+} := \begin{pmatrix} \Lambda \hat{w}^{\varepsilon+} \\ \partial_z \hat{w}^{\varepsilon+} \end{pmatrix},$$

and

$$w_1^{\varepsilon-} := \begin{pmatrix} \Lambda \hat{w}^{\varepsilon-} \\ \partial_z \hat{w}^{\varepsilon-} \end{pmatrix},$$

then, for Λ big enough, our problem is transformed in the study, for $\zeta \in \{|\zeta| = 1\} \bigcup \{\gamma \geqslant 0\}$ of the same equations than for medium frequencies, this time with unknown $(w_1^{\varepsilon+}, w_1^{\varepsilon-})$ instead of $(W^{\varepsilon+}, W^{\varepsilon-})$. We note $w_1^{\varepsilon} = w_1^{\varepsilon+}\mathbf{1}_{x>0} + w_1^{\varepsilon-}\mathbf{1}_{x<0}$. We obtain, the same way as for medium frequencies, that there are $c_h > 0$ and $\delta_h > 0$ such that for for all $|\zeta| > 2$ and for all $0 < \varepsilon < 1$, there holds:

$$c_h \left\| w_1^{\varepsilon} \right\|_{L^2(\mathbb{R})}^2 + \delta_h \left| w_1^{\varepsilon}|_{x=0} \right|^2 \leqslant C \left(\| Re \mathcal{S}^+ \tilde{G}^+ \|^2 + \| Re \mathcal{S}^- \underline{G}^- \|^2 \right). \quad (2.12)$$

An error estimate for low frequencies

For low frequencies, the study becomes much more delicate since some eigenvalues of $\mathbb{A}^{\pm}$ do not stay away from the imaginary axis, asymptotically when ζ tends to zero. As a result, the spectral projectors on the negative or positive eigenspaces of $\mathbb{A}^+$ and $\mathbb{A}^-$, which are needed in the construction of the symmetrizers are no longer well-defined. Hence, an appropriate rescaling has to be introduced for ζ in a neighborhood of zero, the important linear subspaces to consider are then the positive and negative spaces of the rescaled versions of $\mathbb{A}^+$ and $\mathbb{A}^-$. After rescaling, the spectral projectors on these spaces become perfectly well-defined, for $\check{\tau}^2 + \check{\gamma}^2 = 1$ and $\check{\gamma} > 0$, where $\check{\tau} = \frac{\tau}{|\zeta|}$ and $\check{\gamma} = \frac{\tau}{|\gamma|}$ are the frequencies rescaled for a low frequency analysis. A logical idea would be to prove a continuous extension of these linear subspaces to $\{\check{\gamma} = 0\}$, in order to help with the construction of low frequency symmetrizers. However, what happens is the converse, since the fact that those linear subspaces extends continuously to $\{\check{\gamma} = 0\}$ is a consequence of the construction of a Kreiss-type symmetrizer for low frequencies as defined by Theorem 2.12. This is shown in [18].

Let us now give a brief overview of the low frequency analysis of the problem. By a suitable change of basis, the matrix $\mathbb{A}^{\pm}$ becomes block diagonal. Constructing a symmetrizer for $\mathbb{A}^{\pm}$ reduces to the construction of a symmetrizer for each diagonal blocks. We group together the eigenvalues which do not come near the imaginary axis, forming what we will call the parabolic block. For this block, our treatment does not differ from the one previously described for medium frequencies. The other eigenvalues can be grouped together in the hyperbolic block. As explained in the beginning of this section, the construction of the symmetrizers for this hyperbolic block needs a specific approach. For 1-D systems, which is our present case, the construction of a low frequency symmetrizer is rather easy since all the eigenvalues in the hyperbolic block are strictly hyperbolic, which means that, even if they do cross the imaginary axis, they remain semi-simple. In general, for multi-D systems, glancing modes, that is to say purely imaginary, non semi-simple eigenvalues also do appear. Those need an elaborate analysis. For those part of the analysis, we can rely on Theorem 2.12 proved for instance in [16]. Indeed, compared to the problems studied in [16], we make the same structure assumptions (hyperbolicity, parabolicity and hyperbolicity-parabolicity), even though, our boundary conditions, and therefore the expression of our Uniform Evans Condition differs. As a consequence, the results of [16], proved by using only the structure assumptions, also holds here. It is in particular the case of Theorem 2.12.

$W^{\varepsilon+}$ and $\underline{W}^{\varepsilon-}$ satisfying almost the same equations, we will mostly describe the proof of the energy estimates involving $W^{\varepsilon+}$. Let us introduce some notations and some important properties involved in the low frequency study of the hyperbolic part. Using polar coordinates, we define:

$$\rho := |\tau + i\gamma|.$$

There is a nonsingular $N \times N$ matrix ν^+ and two $N \times N$ matrices H^+ and P^+, such that:

$$(\nu^+)^{-1}\mathbb{A}^+\nu^+ = \mathbb{A}_2^+ := \begin{pmatrix} H^+(\zeta) & 0 \\ 0 & P^+(\zeta) \end{pmatrix},$$

with the eigenvalues of P^+ staying away from he imaginary axis and the eigenvalues of H^+ vanishing for $|\zeta| = 0$. Indeed, $\mathbb{A}^\pm$ has got exactly N hyperbolic eigenvalues and N parabolic eigenvalues as proved for instance in [9]. In order to symmetrize properly H^+, we introduce the polar rescaling:

$$\zeta = \rho\check{\zeta} = \rho(\check{\tau}, \check{\gamma}),$$

we have thus $|\check{\zeta}| = 1$. The rescaled version of H^+, $\check{H}^+$ is then given by:

$$H^+(\zeta) = \rho\check{H}^+(\check{\zeta}, \rho).$$

Hence, $W_2^{\varepsilon+} = (\nu^+)^{-1}W^{\varepsilon+}$ satisfies the equation:

$$\begin{cases} \partial_z W_2^{\varepsilon+} - \mathbb{A}_2^+ W_2^{\varepsilon+} = (\nu^+)^{-1}\tilde{G}^+, \quad \{z > 0\}, \\ W_2^{\varepsilon+}|_{z=0} = (\nu^+)^{-1}\underline{q} := \underline{q}_2 \quad . \end{cases}$$

The symmetrizer for this problem will then be constructed by block, as follows:

$$\mathcal{S}_l^+ = \begin{pmatrix} \rho \check{\mathcal{S}}_H^+(\check{\zeta},\rho) & 0 \\ 0 & \mathcal{S}_P^+(\zeta) \end{pmatrix}.$$

The symmetrizer of P^+, $\mathcal{S}_P^+$ will not be detailed here since it is the exact analogous of the symmetrizer for medium frequencies.

For the hyperbolic part, we have:

$$\check{H}^+(\check{\zeta},0) = -(i\check{\tau}+\check{\gamma})(A^+)^{-1}.$$

For $\rho \geqslant C > 0$, H^+ has exactly N_1^+ eigenvalues with positive real part and N_1^- eigenvalues with negative real part while P^+ has exactly N_1^- eigenvalues with positive real part and N_1^+ eigenvalues with negative real part. For $\rho \geqslant C > 0$, we can construct $\mathcal{S}_H^+(\zeta) := \rho \check{\mathcal{S}}_H^+(\check{\zeta},\rho)$ the same way (we have the same qualitative behavior as for the medium frequencies previously treated). Under our assumptions, the following result, asserting that we can construct $\check{\mathcal{S}}_H^+(\check{\zeta},\rho)$, for $(\check{\zeta},\rho)$ in a neighborhood of $(\check{\zeta}_0,0)$ has been proved in [16]:

THEOREM 2.12. — *For all $\{|\check{\zeta}| = 1\} \bigcup \{\check{\gamma} \geqslant 0\}$, there are two linear subspaces $\underline{\mathbb{F}}_1^+$ and $\underline{\mathbb{F}}_1^-$ of constant dimension satisfying:*

$$\mathbb{C}^N = \underline{\mathbb{F}}_1^+ \bigoplus \underline{\mathbb{F}}_1^-, \tag{2.13}$$

with $dim(\underline{\mathbb{F}}_1^+) = N_1^+$, $dim(\underline{\mathbb{F}}_1^-) = N_1^-$, and such that for all $\kappa_1 \geqslant 1$ there exists a neighborhood $\check{\omega}$ of $(\check{\zeta},0)$ in $\mathbb{R}^2 \times \mathbb{R}$, a C^∞ mapping $\check{\mathcal{S}}_H^+$ from $\check{\omega}$ to the space of $N \times N$ matrices, and a constant $c > 0$ such that for all $(\check{\zeta},\rho) \in \check{\omega}$,

$$\check{\mathcal{S}}_H^+(\check{\zeta},\rho) = \left(\check{\mathcal{S}}_H^+(\check{\zeta},\rho)\right)^*$$

for all $h \in \mathbb{C}^N$, denoting by $\underline{\Pi}_1^+$ and $\underline{\Pi}_1^-$ the projectors associated to the decomposition (2.13) of $\mathbb{C}^N$:

$$\left\langle \check{\mathcal{S}}_H^+(\check{\zeta},\rho)h,h\right\rangle \geqslant \kappa_1 |\underline{\Pi}_1^+ h|^2 - |\underline{\Pi}_1^- h|^2$$

and, for all $(\check{\zeta},\rho) \in \check{\omega}$, with $\rho \geqslant 0$ and $\check{\gamma} \geqslant 0$:

$$2\Re e \left\langle \check{\mathcal{S}}_H^+(\check{\zeta},\rho)\check{H}^+(\check{\zeta},\rho)h,h\right\rangle \geqslant c(\check{\gamma}+\rho)|h|^2$$

Note that we have the analogous Theorem for $\underline{W}^{\varepsilon-}$:

THEOREM 2.13. — *For all $\{|\check{\zeta}| = 1\} \bigcup \{\check{\gamma} \geqslant 0\}$, there are two linear subspaces $\underline{\mathbb{F}}_2^+$ and $\underline{\mathbb{F}}_2^-$ of constant dimension satisfying:*

$$\mathbb{C}^N = \underline{\mathbb{F}}_2^+ \bigoplus \underline{\mathbb{F}}_2^-, \tag{2.14}$$

with $dim(\underline{\mathbb{F}}_2^+) = N_2^+$, $dim(\underline{\mathbb{F}}_2^-) = N_2^-$, and such that for all $\kappa_2 \geqslant 1$ there exists a neighborhood $\check{\omega}$ of $(\check{\zeta}, 0)$ in $\mathbb{R}^2 \times \mathbb{R}$, a C^∞ mapping $\check{\mathcal{S}}_H^-$ from $\check{\omega}$ to the space of $N \times N$ matrices, and a constant $c > 0$ such that for all $(\check{\zeta}, \rho) \in \check{\omega}$,

$$\check{\mathcal{S}}_H^-(\check{\zeta}, \rho) = \left(\check{\mathcal{S}}_H^-(\check{\zeta}, \rho)\right)^*$$

for all $h \in \mathbb{C}^N$, denoting by $\underline{\Pi}_2^+$ and $\underline{\Pi}_2^-$ the projectors associated to the decomposition (2.14) of $\mathbb{C}^N$:

$$\left\langle \check{\mathcal{S}}_H^-(\check{\zeta}, \rho)h, h \right\rangle \geqslant \kappa_2 |\underline{\Pi}_2^+ h|^2 - |\underline{\Pi}_2^- h|^2$$

and, for all $(\check{\zeta}, \rho) \in \check{\omega}$, with $\rho \geqslant 0$ and $\check{\gamma} \geqslant 0$:

$$2\Re e \left\langle \check{\mathcal{S}}_H^-(\check{\zeta}, \rho)\check{H}^-(\check{\zeta}, \rho)h, h \right\rangle \geqslant c(\check{\gamma} + \rho)|h|^2$$

We just expose here as a remark an important property linked to our current analysis.

Remark 2.14. — Let $\mathcal{H}^+(\zeta, \rho)$ be given by:

$$\mathcal{H}^+(\zeta, \rho) = \check{H}^+(\check{\zeta}, \rho).$$

There exists $e^+(\tau, \gamma, \xi, \rho)$ polynomial in ξ with smooth coefficients in (τ, γ, ρ) such that:

$$det\left((i\tau + \gamma)Id + i\xi A^+ + \rho Id\right) = e^+(\tau, \gamma, \xi, \rho) det\left(i\xi Id - \mathcal{H}^+(\tau, \gamma, \rho)\right)$$

and $e^+(\tau, \gamma, \xi, 0) \neq 0$. This shows the important link, for $\rho = 0$, existing between the spectral study of $\mathcal{H}^+$ and the spectral study of the symbol of the hyperbolic part of our equation.

For $\rho \geqslant 0$, we have, for all $h \in \mathbb{C}^N$:

$$2\Re e \langle \mathcal{S}_P^+ P^+ h, h\rangle \geqslant c\rho(\check{\gamma} + \rho)|h|^2.$$

As a result, For $\rho \geqslant 0$, we can construct $\mathcal{S}_l$ satisfying:

$$2\Re e \langle \mathcal{S}_l^+ \mathbb{A}^+ h, h\rangle \geqslant c(\gamma + \rho^2)|h|^2.$$

Mimicking what has been done for medium frequencies, after choosing for all $0 < \lambda < 2c'(\gamma + \rho^2)$, we get that, for all $\gamma > 0$, the following estimate holds:

$$\left(c'(\gamma + \rho^2) - \frac{\lambda}{2}\right) \|W_2^{\varepsilon +}\|^2 + \left\langle \mathcal{S}_l^+ W_2^{\varepsilon +}|_{x=0}, W_2^{\varepsilon +}|_{x=0}\right\rangle \leqslant \frac{2}{\lambda} \|Re\mathcal{S}^+ \tilde{G}^+\|^2.$$

Therefore, there are $c_1 > 0$ and $C_1 > 0$ such that:

$$c_1(\gamma+\rho^2)\|W_2^{\varepsilon+}\|^2 + \left\langle \mathcal{S}_l^+ W_2^{\varepsilon+}|_{x=0}, W_2^{\varepsilon+}|_{x=0}\right\rangle \leqslant \frac{C_1}{\gamma+\rho^2}\|Re\mathcal{S}^+\tilde{G}^+\|^2.$$

Adopting symmetric notations for $\underline{W_2^{\varepsilon-}}$ and adding the two estimates gives that there are $c > 0$ and $C > 0$, such that, for all $\gamma > 0$, there holds:

$$c(\gamma+\rho^2)\|W_2^{\varepsilon}\|^2_{L^2(\mathbb{R})} + \left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right)\underline{q}_2, \underline{q}_2\right\rangle$$

$$\leqslant \frac{C}{\gamma+\rho^2}\left(\|Re\mathcal{S}^+\tilde{G}^+\|^2 + \|Re\mathcal{S}^-\tilde{G}^-\|^2\right).$$

Proposition 2.15. — *For all $\underline{q} \in \mathbb{C}^{2N}$, there is $\delta > 0$, $\delta' > 0$ and a set of two symmetrizers $\mathcal{S}_l^+$ and $\mathcal{S}_l^-$ such that:*

$$\left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right)\underline{q}, \underline{q}\right\rangle \geqslant \min(\rho\delta', \delta)\left\langle \underline{q}, \underline{q}\right\rangle.$$

Proof. — Denote by q_H the N first coordinates of $\underline{q}$ and by q_P the N last ones. We have then:

$$\left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right)\underline{q}, \underline{q}\right\rangle = \rho\left\langle \left(\check{\mathcal{S}}_H^+ + \check{\mathcal{S}}_H^-\right)q_H, q_H\right\rangle + \left\langle \left(\mathcal{S}_P^+ + \mathcal{S}_P^-\right)q_P, q_P\right\rangle$$

The uniform Evans condition being satisfied, we get immediately the analogous of Proposition 2.10 for the parabolic part: there are two symmetrizers $\mathcal{S}_P^+$, $\mathcal{S}_P^-$ and a positive constant δ such that for all $q_P \in \mathbb{C}^N$, there holds:

$$\left\langle \left(\mathcal{S}_P^+ + \mathcal{S}_P^-\right)q_P, q_P\right\rangle \geqslant \delta\left\langle q_P, q_P\right\rangle.$$

For $\rho \geqslant C > 0$, we obtain the same way that there is a positive constant δ' such that, for all $q_H \in \mathbb{C}^N$,

$$\rho\left\langle \left(\check{\mathcal{S}}_H^+ + \check{\mathcal{S}}_H^-\right)q_H, q_H\right\rangle \geqslant \delta'\left\langle q_H, q_H\right\rangle.$$

Hence, for $\rho \geqslant C > 0$, there holds:

$$\left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right)\underline{q}, \underline{q}\right\rangle \geqslant \min(\delta', \delta)\left\langle \underline{q}, \underline{q}\right\rangle.$$

This inequality is true provided that the Evans Condition holds, **even if it is not uniformly**. For $\rho \geqslant C > 0$, due to our stability assumption holding, we had the following decomposition of $\mathbb{C}^N$:

$$\mathbb{C}^N = \mathbb{E}_-(H^+)\bigoplus\mathbb{E}_-(H^-).$$

Remark that, for all $\rho > 0$, $\mathbb{E}_-(\check{H}^+) = \mathbb{E}_-(H^+)$ and $\mathbb{E}_+(\check{H}^-) = \mathbb{E}_+(H^-)$. Moreover we had:

$$\mathbb{C}^N = \mathbb{E}_-(H^+) \bigoplus \mathbb{E}_+(H^+) = \mathbb{E}_-(H^-) \bigoplus \mathbb{E}_+(H^-).$$

For the frequencies in a neighborhood of zero, let us prove our result. By Theorem 2.12 and 2.13, we have: $\mathbb{C}^N = \underline{\mathbb{F}}_1^- \bigoplus \underline{\mathbb{F}}_1^+$, and $\mathbb{C}^N = \underline{\mathbb{F}}_2^- \bigoplus \underline{\mathbb{F}}_2^+$. For fixed $\rho > 0$, and $(\check{\tau}, \check{\gamma})$ such that $\check{\tau}^2 + \check{\gamma}^2 = 1$ with $\check{\gamma} \geqslant 0$, thanks to the Evans Condition holding, we have:

$$\mathbb{C}^N = \mathbb{E}_-(\check{H}^+) \bigoplus \mathbb{E}_-(\check{H}^-).$$

As a corollary of Theorem 2.12 and Theorem 2.13, as proven in [18] and [16], the vector bundles $\mathbb{E}_-(\check{H}^+)(\check{\zeta}, \rho)$ and $\mathbb{E}_-(\check{H}^-)(\check{\zeta}, \rho)$, defined for $\check{\zeta}$ such that $|\check{\zeta}| = 1$, with $\check{\gamma} \geqslant 0$ and $\rho > 0$, extends continuously to $\rho = 0$. As a matter of fact, these continuous extensions are the previously introduced linear subspaces $\underline{\mathbb{F}}_1^-$ and $\underline{\mathbb{F}}_2^-$. Since the Evans Condition holds **uniformly**, and the extensions of $\mathbb{E}_-(\check{H}^+)$ to $\underline{\mathbb{F}}_1^-$ and of $\mathbb{E}_-(\check{H}^-)$ to $\underline{\mathbb{F}}_2^-$ are continuous, we have then:

$$\underline{\mathbb{F}}_1^- \bigcap \underline{\mathbb{F}}_2^- = \{0\},$$

and therefore $\underline{\mathbb{F}}_1^- \bigoplus \underline{\mathbb{F}}_2^- = \mathbb{C}^N$

As a result, for all $q_H \in \mathbb{C}^N$, either $q_H = 0$, or $\underline{\Pi}_1^+ q_H \neq 0$ or $\underline{\Pi}_2^+ q_H \neq 0$. Moreover, by construction of $\check{\mathcal{S}}_H^\pm$:

$$\left\langle \check{\mathcal{S}}_H^+(\check{\zeta}, \rho) q_H, q_H \right\rangle \geqslant \kappa_1 |\underline{\Pi}_1^+ q_H|^2 - |\underline{\Pi}_1^- q_H|^2,$$

$$\left\langle \check{\mathcal{S}}_H^-(\check{\zeta}, \rho) q_H, q_H \right\rangle \geqslant \kappa_2 |\underline{\Pi}_2^+ q_H|^2 - |\underline{\Pi}_2^- q_H|^2.$$

For $q_H = 0$, the awaited inequality trivially holds. If it is not the case, since either $\underline{\Pi}_1^+ q_H \neq 0$ or $\underline{\Pi}_2^+ q_H \neq 0$, we obtain the desired result by choosing the two positive parameters κ_1 and κ_2 large enough. □

We get then the following estimate:

PROPOSITION 2.16. — *There are $\delta > 0$, $c > 0$ and $C > 0$ such that, for all nonzero frequencies, there holds:*

$$c(\gamma + \rho^2) \|W_2^\varepsilon\|^2_{L^2(\mathbb{R})} + \delta\rho |W_2^\varepsilon|_{x=0}|^2 \leqslant \tfrac{C}{\gamma + \rho^2} \|G\|^2_{L^2(\mathbb{R})}. \tag{2.15}$$

Note that this estimate needs that either $\gamma > 0$ or $\rho > 0$ to properly control our error. This shows the need to introduce the weight $e^{-\gamma t}$ with $\gamma > 0$.

The main error estimate

In the previous chapters, we have obtained three energy estimates, each concerning a different regime of frequencies. We recall that the frequencies were respectively divided in $\tilde{\zeta} < 1$ for the low frequencies, $1 \leqslant \tilde{\zeta} \leqslant 2$ for the medium frequencies and $\tilde{\zeta} > 2$ for the high frequencies. In a first step, we will rewrite our estimates (all the positive constants will be taken equal to one) for the different regimes of frequencies, this time for the original variables x and ζ instead of z and $\tilde{\zeta}$. To begin with, let us redefine here the notations $\|.\|$ and $|.|$ as follows:

$$\|f(\tau,x)\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle f(\tau,x), f(\tau,x)\rangle \; dx \; d\tau$$

and

$$|f(\tau)|^2 = \int_{-\infty}^{\infty} \langle f(\tau), f(\tau)\rangle d\tau.$$

We will integrate the previous estimations between $-\infty$ and ∞ with respect to τ. There is $C_m > 0$ such that, for all $1 \leqslant |\varepsilon\zeta| \leqslant 2$, the energy estimate writes:

$$\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + |\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon^2|\partial_x \hat{w}^\varepsilon|_{x=0}|^2 \leqslant C_m \varepsilon^{2M}$$

There is $C_h > 0$ such that, for all $|\varepsilon\zeta| > 2$, the following estimate holds:

$$\begin{aligned}&(1+\varepsilon\tau^2+\varepsilon\gamma^2)\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} \\ &+(1+\varepsilon\tau^2+\varepsilon\gamma^2)|\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon^2|\partial_x \hat{w}^\varepsilon|_{x=0}|^2 \leqslant C_h \varepsilon^{2M}.\end{aligned}$$

There is $C_l > 0$ such that, for all $|\varepsilon\zeta| < 1$, there holds:

$$\begin{aligned}(\varepsilon\gamma+\varepsilon^2\rho^2)\left(\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})}\right) + \varepsilon\rho\left(|\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon^2|\partial_x \hat{w}^\varepsilon|_{x=0}|^2\right) \\ \leqslant \frac{C_l}{\varepsilon\gamma+\varepsilon^2\rho^2}\varepsilon^{2M},\end{aligned}$$

and thus:

$$\begin{aligned}(\gamma+\varepsilon\rho^2)\left(\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})}\right) + \rho\left(|\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon|\partial_x \hat{w}^\varepsilon|_{x=0}|^2\right) \\ \leqslant \frac{C_l}{\gamma}\varepsilon^{2M-2}.\end{aligned} \tag{2.16}$$

Note that the estimates we proved for low frequencies were for the unknown $\tilde{W}^\varepsilon_2$. We explain here briefly how to come back to estimates on $\tilde{W}^\varepsilon$. $\tilde{W}^{\varepsilon\pm}_2$ are deduced from $\tilde{W}^\varepsilon$ by a change of basis described by $\nu^\pm$. There holds:

$$\nu^\pm|_{\zeta=0} = \begin{pmatrix} Id & (A^\pm)^{-1}B \\ 0 & Id \end{pmatrix}$$

ν^+ and ν^- are continuous in ζ. Thus, recalling that $\tilde{W}^{\varepsilon+} = \nu^+ \tilde{W}_2^{\varepsilon+}$ [resp $\tilde{W}^{\varepsilon-} = \nu^+ \tilde{W}_2^{\varepsilon-}$], both $\tilde{W}^{\varepsilon+}$ and $\tilde{W}_2^{\varepsilon+}$ satisfy estimates with coefficients of the same scale in ε and ζ. Thus, adjusting the symmetrizers to match the constants allows to obtain the low frequency estimate (2.16).

We have to keep in mind ε is destined to tend towards zero while looking at our estimates.

Since $\hat{w}^\varepsilon$ is continuous through $\{x = 0\}$, $\hat{w}^\varepsilon|_{x=0}$ is well-defined. Let us write the simplified estimates, not involving the traces on the boundary: there is C positive such that, for all $0 < \varepsilon < 1$, there holds:

$$\|\hat{w}^\varepsilon\|_{L^2(\mathbb{R})} \leqslant \frac{C}{\gamma} \varepsilon^{M-1},$$

where γ is a fixed positive parameter.

Recalling that $\hat{w}^\varepsilon(\tau, x) := \int_{-\infty}^{\infty} [e^{-\gamma t} w^\varepsilon(t, x)] e^{-2\pi i \tau t}\, dt$, and using Plancherel's Theorem, we get the following result: there is C positive independent of ε and γ, such that for all function w smooth with compact support satisfying our error equation, there holds:

$$\|e^{-\gamma t} w^\varepsilon\|_{L^2((0,T)\times\mathbb{R})} \leqslant \frac{C}{\gamma} \varepsilon^{M-1}.$$

Therefore, since γ is a positive parameter, by constructing our approximate solution at an order $M \geqslant 2$, we obtain the following stability result:

Theorem 2.17. — *There is $C > 0$ such that, for all $0 < \varepsilon < 1$:*

$$\|w^\varepsilon\|_{L^2((0,T)\times\mathbb{R})} \leqslant C\varepsilon.$$

2.5. Proof of the Uniform Lopatinski condition holding for the mixed hyperbolic problem (2.4)

We will now prove, by a detailed analysis of the Evans condition for low frequency, that the Uniform Lopatinski condition holds for (2.4) thus proving the well-posedness of the transmission problem (2.6).

$$\mathbb{A}(t, y, x; \zeta) := \begin{pmatrix} 0 & Id \\ \mathcal{M}(t, y, x; \zeta) & \mathcal{A}(t, y, x; \eta) \end{pmatrix}.$$

To begin with, let us fix the values of $(t, y, x) := (t_0, y_0, x_0)$ and study the behavior of $\mathbb{A}_0(\zeta) := \mathbb{A}(t_0, y_0, x_0; \zeta)$ for $|\zeta|$ in a neighborhood of zero.

Lemma 2.18. — *There is a nonsingular matrix $\nu(\zeta)$, smooth on a neighborhood ω_0, of 0 such that:*

$$\nu(\zeta)^{-1}\mathbb{A}_0(\zeta)\nu(\zeta) = \begin{pmatrix} H(\zeta) & 0 \\ 0 & P(\zeta) \end{pmatrix} := \mathcal{G}_0(\zeta).$$

At $\zeta = 0$, we have $P(0) = B^{-1}A_d(t_0, y_0, x_0; 0)$ and $H(0) = 0$.

$$\nu(0) = \begin{pmatrix} Id & (A_d)^{-1}B(t_0, y_0, x_0; 0) \\ 0 & Id \end{pmatrix} := \mathcal{G}(\zeta).$$

$H(\zeta)$ is often referred to as the hyperbolic block since it satisfies, for $\zeta \in \omega_0$:

$$H(\zeta) = A(t_0, y_0, x_0; \zeta) + \mathcal{O}(|\zeta|^2).$$

A proof of this Lemma can be found in [16]. Remark that:

$$\mathbb{E}_-(\mathbb{A}_0(\zeta)) = \nu(\zeta)\mathbb{E}_-(H(\zeta)) \times \mathbb{E}_-(P(\zeta)).$$

The Uniform Evans condition writes:

$$det\left(\mathbb{E}_-(\mathbb{A}^+|_{x=0}), \mathbb{E}_+(\mathbb{A}^-|_{x=0})\right) \geqslant C > 0.$$

When the two linear subspaces $\mathbb{E}_-(\mathbb{A}^+|_{x=0})$ and $\mathbb{E}_+(\mathbb{A}^-|_{x=0})$ extend continuously to $\zeta \neq 0$ with $\gamma > 0$, and if we denote by $\tilde{\mathbb{E}}_-(\mathbb{A}^+|_{x=0})$ and $\tilde{\mathbb{E}}_+(\mathbb{A}^-|_{x=0})$ the extended spaces, the Uniform Evans Condition consists in asking, for all $\zeta \neq 0$ that:

$$\tilde{\mathbb{E}}_-(\mathbb{A}^+|_{x=0}) \bigcap \tilde{\mathbb{E}}_+(\mathbb{A}^-|_{x=0}) = \{0\}.$$

Such extensions do exist in our case. Indeed in [18], Métivier and Zumbrun proves that, under our assumptions, the following result holds, as a direct consequence of the construction of a Kreiss-type Symmetrizer. Let us denote $\rho = |\zeta|$, we have then that $\zeta = \rho\check{\zeta}$. We have then the following result:

Theorem 2.19. — *The linear bundle $\check{\mathbb{E}}(t, y, \check{\zeta}, \rho) := \mathbb{E}_-(t, y, \rho\check{\zeta})$ has a continuous extension to $\rho = 0$, $\check{\gamma} \geqslant 0$.*

The Uniform Evans being satisfied for low frequencies, it implies that:

$$\left|det\left(\nu^+\left(\tilde{\mathbb{E}}_-(A^+|_{x=0}) \times \mathbb{E}_-(P^+|_{x=0})\right), \nu^-\left(\tilde{\mathbb{E}}_+(A^-|_{x=0}) \times \mathbb{E}_+(P^-|_{x=0})\right)\right)\right|$$
$$\geqslant C > 0.$$

where, for $|\zeta|$ in a neighborhood of zero:

$$\nu^{\pm}(t,y;\zeta)=\begin{pmatrix} Id & (A_d^{\pm})^{-1}B_{d,d}(t,y,0;\zeta) \\ 0 & Id \end{pmatrix}+\mathcal{O}(|\zeta|).$$

Let $\tilde{D}_0$ denote the following determinant:

$$\Big| det\left(\nu^{+}|_{\zeta=0}\left(\tilde{\mathbb{E}}_{-}(A^{+}|_{x=0})\times\mathbb{E}_{-}(P^{+}|_{x=0,\zeta=0})\right),\right.$$

$$\left.\nu^{-}|_{\zeta=0}\left(\tilde{\mathbb{E}}_{+}(A^{-}|_{x=0})\times\mathbb{E}_{+}(P^{-}|_{x=0,\zeta=0})\right)\right)\Big|$$

There is $\rho_0>0$ such that for all ζ such that $|\zeta|=\rho_0$, there holds:

$$\tilde{D}_0\geqslant C>0.$$

where $P^{\pm}|_{\zeta=0}=B_{d,d}^{-1}A_d^{\pm}$.

$\nu^{+}|_{\zeta=0}\left(\tilde{\mathbb{E}}_{-}(A^{+}|_{x=0})\times\mathbb{E}_{-}(P^{+}|_{x=0,\zeta=0})\right)$ is the linear subset composed of the $\begin{pmatrix} u^{+} \\ v^{+} \end{pmatrix}$ such that there are $u'^{+}\in\tilde{\mathbb{E}}_{-}(A^{+}|_{x=0})$ and $v'^{+}\in\mathbb{E}_{-}(B_{d,d}^{-1}A_d^{+})$ satisfying :

$$\begin{pmatrix} u^{+} \\ v^{+} \end{pmatrix}=\begin{pmatrix} u'^{+}+\left(A_d^{+}\right)^{-1}B_{d,d}|_{x=0}v'^{+} \\ v'^{+} \end{pmatrix}$$

The same way, $\nu^{-}|_{\zeta=0}\left(\tilde{\mathbb{E}}_{+}(A^{-}|_{x=0})\times\mathbb{E}_{+}(P^{-}|_{x=0,\zeta=0})\right)$ is the linear subset composed of the $\begin{pmatrix} u^{-} \\ v^{-} \end{pmatrix}$ such that there are $u'^{-}\in\tilde{\mathbb{E}}_{+}(A^{-}|_{x=0})$ and $v'^{-}\in\mathbb{E}_{+}(B_{d,d}^{-1}A_d^{-})$ satisfying :

$$\begin{pmatrix} u^{-} \\ v^{-} \end{pmatrix}=\begin{pmatrix} u'^{-}+\left(A_d^{-}\right)^{-1}B_{d,d}|_{x=0}v'^{-} \\ v'^{-} \end{pmatrix}$$

The low frequency Evans Condition rewrites then:

$$\nu^{-}|_{\zeta=0}\left(\tilde{\mathbb{E}}_{+}(A^{-}|_{x=0})\times\mathbb{E}_{+}(P^{-}|_{x=0,\zeta=0})\right)$$

$$\bigcap\nu^{+}|_{\zeta=0}\left(\tilde{\mathbb{E}}_{-}(A^{+}|_{x=0})\times\mathbb{E}_{-}(P^{+}|_{x=0,\zeta=0})\right)=\{0\},$$

which is equivalent to the following property: if there is $\lambda\in\mathbb{C}-\{0\}$ such that

$$\begin{cases} u'^{+}+(G_d^{+})^{-1}v'^{+}=\lambda\left(u'^{-}+(G_d^{-})^{-1}v'^{-}\right) \\ v'^{+}=\lambda v'^{-} \end{cases}$$

with $u'^{+} \in \tilde{\mathbb{E}}_-(A^+|_{x=0})$, $v'^{+} \in \mathbb{E}_-(B_{d,d}^{-1}A_d^+)$, $u'^{-} \in \tilde{\mathbb{E}}_+(A^-|_{x=0})$ and $v'^{-} \in \mathbb{E}_+(B_{d,d}^{-1}A_d^-)$, then this implies that $u'^{+} = u'^{-} = v'^{+} = v'^{-} = 0$. Easy algebraic considerations prove this is true iff:

$$((G_d^+)^{-1} - (G_d^-)^{-1}) \left(\mathbb{E}_-(G_d^+) \bigcap \mathbb{E}_+(G_d^-)\right) \bigoplus \tilde{\mathbb{E}}_-(A^+|_{x=0})$$

$$\bigoplus \tilde{\mathbb{E}}_+(A^-|_{x=0}) = \mathbb{C}^N.$$

We recall Σ denotes the space:

$$\Sigma = ((B_{d,d}^{-1}A_d^+)^{-1} - (B_{d,d}^{-1}A_d^-)^{-1}) \left(\mathbb{E}_-(B_{d,d}^{-1}A_d^+|_{x=0}) \bigcap \mathbb{E}_+(B_{d,d}^{-1}A_d^-|_{x=0})\right).$$

Thus for all $\zeta \neq 0$, such that $|\zeta| < \rho_0$, we have:

$$\tilde{\mathbb{E}}_-(A^+|_{x=0}) \bigoplus \tilde{\mathbb{E}}_+(A^-|_{x=0}) \bigoplus \Sigma = \mathbb{C}^N.$$

Since both of the tangential symbols $A^+|_{x=0}$ and $A^-|_{x=0}$ are homogeneous of order zero in ζ, this is equivalent to say that for all $\zeta \neq 0$, there holds:

$$\tilde{\mathbb{E}}_-(A^+|_{x=0}) \bigoplus \tilde{\mathbb{E}}_+(A^-|_{x=0}) \bigoplus \Sigma = \mathbb{C}^N,$$

which is an equivalent expression of the Uniform Lopatinski Condition for the mixed hyperbolic problem (2.4). Due to the hyperbolicity assumption, we get moreover that $\dim \mathbb{E}_-(A^+|_{x=0}) = \dim \mathbb{E}_+(A_d^+|_{x=0})$ and $\dim \mathbb{E}_+(A^-|_{x=0}) = \dim \mathbb{E}_-(A_d^-|_{x=0})$.

Remark 2.20. — In the 1-D framework, the Uniform Lopatinski Condition writes:

$$\mathbb{E}_+(A_d^+) \bigoplus \mathbb{E}_-(A_d^-) \bigoplus \Sigma = \mathbb{R}^N.$$

The role of our transversality Assumption, alongside the other structure assumptions, is to guarantee Σ has the suitable dimension. This Assumption is thus crucial since, if Σ has not the right dimension, the limiting mixed hyperbolic problem has no chance of satisfying a Lopatinski Condition even though its parabolic perturbation satisfies a Uniform Evans Condition.

3. An open scalar question: the scalar expansive case

For scalar hyperbolic problems of conservation laws with discontinuous coefficients, we saw in [8] that the expansive case was quite special to treat. This section is devoted to the open analogous nonconservative problem. To begin with, let us detail the current problematic: we have in mind to give a sense to the Cauchy problem for the hyperbolic operator $\mathcal{H} = \partial_t u + a(x)\partial_x u$

where a is piecewise constant, equal to a^+ on $\{x > 0\}$ and equal to a^- on $\{x < 0\}$, with $a^+ > 0$ and $a^- < 0$:

$$\begin{cases} \partial_t u + a(x)\partial_x u = f, \qquad x \in \mathbb{R}, \\ u|_{t=0} = h \quad , \end{cases} \tag{3.1}$$

where $f \in C_0^\infty((0,T)\times\mathbb{R})$ and $h \in C_0^\infty(\mathbb{R})$. By opting for a viscous approach, we will see that a solution of the above problem can be obtained in the vanishing viscosity limit. Moreover, our viscous approach successfully select a unique solution. Our main result is stated in Theorem 3.2.

Let us now describe our approach. We consider the viscous hyperbolic-parabolic problem:

$$\begin{cases} \partial_t u^\varepsilon + a(x)\partial_x u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon = f, \qquad x \in \mathbb{R}, \\ u^\varepsilon|_{t=0} = h \quad . \end{cases} \tag{3.2}$$

The stability of problem (3.2) has to be established via Kreiss-type Symmetrizers, thus explaining that we assume the coefficient to be piecewise constant in order to avoid the use of pseudodifferential calculus. We prove then a convergence result in $L^2((0,T)\times\mathbb{R})$, stating that the solution u^ε of (3.2) tends towards $\underline{u}$, deduced from an asymptotic analysis of the problem. More precisely, $\underline{u}$ is given by $\underline{u} := u_R\mathbf{1}_{x\geqslant 0} + u_L\mathbf{1}_{x<0}$, where (u_R, u_L) is the unique solution of the following problem:

$$\begin{cases} \partial_t u_R + a_R\partial_x u_R = f_R, \qquad \{x > 0\}, \\ \partial_t u_L + a_L\partial_x u_L = f_L, \qquad \{x < 0\}, \\ u_R|_{x=0} - u_L|_{x=0} = 0, \\ \partial_x u_R|_{x=0} - \partial_x u_L|_{x=0} = 0, \quad \forall t \in (0,T), \\ u_R|_{t=0} = h_R, u_L|_{t=0} = h_L \quad , \end{cases} \tag{3.3}$$

with f_R [resp h_R] denoting the restriction of f [resp h] to $\{x > 0\}$, and f_L [resp h_L] denoting the restriction of f [resp h] to $\{x < 0\}$. Note well that $\underline{u}$, deduced from this unusual, although well-posed, problem belongs to $C^0((0,T)\times\mathbb{R})\bigcap L^2((0,T)\times\mathbb{R})$. Indeed, as we shall prove below, the restriction of $\underline{u}$ to the side $\{x < 0\}$ is given by:

$$\begin{cases} \partial_t u_L + a_L\partial_x u_L = f_L, \qquad \{x < 0\}, \\ u_L|_{x=0} = h_L(0) + \displaystyle\int_0^t f|_{x=0}(s)\, ds, \quad \forall t \in (0,T). \\ u_L|_{t=0} = h_L \quad . \end{cases}$$

and the restriction of $\underline{u}$ to the side $\{x > 0\}$ satisfies:

$$\begin{cases} \partial_t u_R + a_R\partial_x u_R = f_R, \qquad \{x > 0\}, \\ u_R|_{x=0} = h_R(0) + \displaystyle\int_0^t f|_{x=0}(s)\, ds, \quad \forall t \in (0,T), \\ u_R|_{t=0} = h_R \quad . \end{cases}$$

Remark that, in general, the corner compatibilities are not satisfied here, and that $u \notin C([0,T] : H^s(\mathbb{R}))\ \forall s > \frac{3}{2}$ for example, even though the datas f and h are smooth.

Remark 3.1. — $\underline{u}$ is also given by:

$$\underline{u}(t,x) = h(x) + \int_0^t \underline{v}(s,x)\ ds,$$

where $\underline{v} := v_L \mathbf{1}_{x<0} + v_R \mathbf{1}_{x\geqslant 0}$ is the solution of the well-posed classical transmission problem:

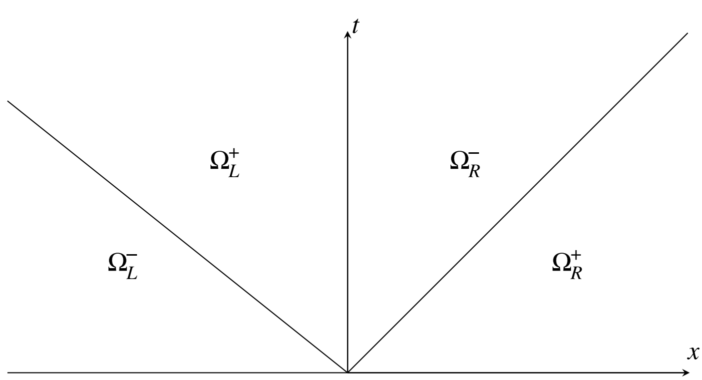
$$\begin{cases} \partial_t v_R + a_R \partial_x v_R = \partial_t f_R, & \{x>0\}, \\ \partial_t v_L + a_L \partial_x v_L = \partial_t f_L, & \{x<0\}, \\ \dfrac{1}{a_R} v_R|_{x=0} - \dfrac{1}{a_L} v_L|_{x=0} = \dfrac{1}{a_R}\partial_t f_R|_{x=0} - \dfrac{1}{a_L}\partial_t f_L|_{x=0}, \\ v_R|_{x=0} - v_L|_{x=0} = 0, \\ v_R|_{t=0} = f_R - a_R\partial_x h_R, \quad v_L|_{t=0} = f_L - a_L \partial_x h_L & . \end{cases}$$

This problem is labeled as classical since it is equivalent to a mixed hyperbolic problem satisfying a Uniform Lopatinski Condition.

As an illustration, let us compute $\underline{u}$ in the case where $f = 0$. We will first introduce some notations. We denote for instance:

$$\Omega_L^+ = \{(t,x) \in (0,T) \times \mathbb{R}^{*-} : x - a_L t > 0\},$$

where the "L" stands for "on left hand side of $\{x=0\}$" and the $+$ is related to the sign of $x - a_L t$. We define in the same manner: Ω_L^-, Ω_R^+ and Ω_R^-.



We get that, for all $(t,x) \in \Omega_L^+ \bigcup \Omega_R^- \bigcup \{x=0\}$,

$$\underline{u}(t,x) = h(0),$$

for all $(t,x) \in \Omega_R^+$,

$$\underline{u}(t,x) = h_R(x - a_R t),$$

and for all $(t,x) \in \Omega_L^-$,

$$\underline{u}(t,x) = h_L(x - a_L t).$$

This example shows clearly the discontinuity of $\partial_x \underline{u}$ occurring across the lines $\Gamma_R = \{(t,x) \in (0,T) \times \mathbb{R}_+^*, \quad x - a_R t = 0\}$ and $\Gamma_L = \{(t,x) \in (0,T) \times \mathbb{R}_-^*, \quad x - a_L t = 0\}$. The following Theorem is our main result:

THEOREM 3.2. — *There is $C > 0$ such that, for all $0 < \varepsilon < 1$, there holds:*

$$\|u^\varepsilon - \underline{u}\|_{L^2((0,T)\times\mathbb{R})} \leqslant C\varepsilon,$$

where u^ε is the solution of (3.2).

Remark 3.3. — The rate of convergence obtained here is better than the one we had on the analogous conservative problem treated in [8]. This is directly explained by a boundary layer analysis of the two problems, which shows that, in [8], strong amplitude boundary layers forms, whereas in our current case, only weak amplitude boundary layers form (we will prove that the boundary layer profile scaled in ε^0, denoted as $\mathbf{U}_0^c$, is equal to zero). Let us be more precise: in the conservative case investigated in [8], the rate of convergence was in $\mathcal{O}(\varepsilon^{1/4})$ while in our current framework, it is in $\mathcal{O}(\varepsilon)$. We will show here that the viscous solution u^ε behaves, when $\varepsilon \to 0^+$ as follows:

$$u^\varepsilon = \left(u_R + \sqrt{\varepsilon} e^{-\frac{|x-a_R t|}{\sqrt{\varepsilon}}} \theta_R\right) \mathbf{1}_{x>0} + \left(u_L + \sqrt{\varepsilon} e^{\frac{-|x-a_L t|}{\sqrt{\varepsilon}}} \theta_L\right) \mathbf{1}_{x<0} + \mathcal{O}(\varepsilon),$$

with θ_R and θ_L belonging to L^2 of their respective domains. By derivation with respect to x of the above asymptotic behavior, we get that the solution $v^\varepsilon := \partial_x u^\varepsilon$ of the dual conservative viscous problem behaves, when $\varepsilon \to 0^+$, as stated right below:

$$u^\varepsilon = \left(u_R + e^{-\frac{|x-a_R t|}{\sqrt{\varepsilon}}} \underline{\theta}_R\right) \mathbf{1}_{x>0} + \left(u_L + e^{-\frac{|x-a_L t|}{\sqrt{\varepsilon}}} \underline{\theta}_L\right) \mathbf{1}_{x<0} + \mathcal{O}(\sqrt{\varepsilon}),$$

with $\underline{\theta}_R$ and $\underline{\theta}_L$ belonging to L^2 of their respective domains. We recover then the rate of convergence given in [8] since $e^{-\frac{|x-a_R t|}{\sqrt{\varepsilon}}} \underline{\theta}_L$ and $e^{-\frac{|x-a_R t|}{\sqrt{\varepsilon}}} \underline{\theta}_R$ converges towards zero in L^2 norm of their respective domains with a rate in $\varepsilon^{\frac{1}{4}}$.

The proof of Theorem 3.2 is divided into two parts. First, we will construct an approximate solution of (3.2) at any order. Then, we will show that a Uniform Evans Condition holds for an equivalent problem, hence yielding the desired stability estimates.

3.1. Construction of an approximate solution

We shall begin by constructing an approximate solution of problem (3.2). As a first step, we will reformulate problem (3.2) in an equivalent manner. The restrictions of u^ε to $\{x > 0\}$ and $\{x < 0\}$, denoted respectively by u_L^ε and u_R^ε satisfy the following transmission problem:

$$\begin{cases} \partial_t u_R^\varepsilon + a_R \partial_x u_R^\varepsilon - \varepsilon \partial_x^2 u_R^\varepsilon = f_R, & \{x > 0\}, t \in (0,T), \\ \partial_t u_L^\varepsilon + a_L \partial_x u_L^\varepsilon - \varepsilon \partial_x^2 u_L^\varepsilon = f_L, & \{x < 0\}, t \in (0,T), \\ u_R^\varepsilon|_{x=0} - u_L^\varepsilon|_{x=0} = 0, \\ \partial_x u_R^\varepsilon|_{x=0} - \partial_x u_L^\varepsilon|_{x=0} = 0, \\ u_R^\varepsilon|_{t=0} = h_R, \\ u_L^\varepsilon|_{t=0} = h_L \quad . \end{cases} \tag{3.4}$$

Let us introduce $L_R^\varepsilon = \partial_t + a_R \partial_x - \varepsilon \partial_x^2$ and $L_L^\varepsilon = \partial_t + a_L \partial_x - \varepsilon \partial_x^2$. We perform the construction of the approximate solution separately on the four domains Ω_L^-, Ω_L^+, Ω_R^+ and Ω_R^-. . We will denote by $u_{app,L,+}^\varepsilon$ the restriction of u_{app}^ε to Ω_L^+ and so on. Let us present the different profiles and their ansatz:

$$u_{app,L,+}^\varepsilon(t,x) = \sum_{n=0}^{M} \left(\underline{\mathbf{U}}_{L,n,+}(t,x) + \mathbf{U}_{L,n,+}^c \left(t, \frac{x - a_L t}{\sqrt{\varepsilon}} \right) \right) \varepsilon^{\frac{n}{2}},$$

where the profiles $\underline{\mathbf{U}}_{L,n,+}$ belongs to $H^\infty(\Omega_L^+)$ and the characteristic boundary layer profiles $\mathbf{U}_{L,n,+}^c(t,\theta_L)$ belongs to $e^{-\delta|\theta_L|} H^\infty((0,T) \times \mathbb{R}^{*+})$, for some $\delta > 0$. We will take a similar ansatz for $u_{app,L,-}^\varepsilon$, $u_{app,R,-}^\varepsilon$ and $u_{app,R,+}^\varepsilon$ over their respective domains. Let us explain the different steps of the construction of the approximate solution. We begin by constructing the underlined profiles $\underline{\mathbf{U}}_n$ in cascade; the boundary layer profiles $\mathbf{U}_n^c$ are then computed as a last step. We construct our profiles such that, for all fixed $\varepsilon > 0$, u_{app}^ε belongs to $C^1((0,T) \times \mathbb{R})$. In what follows, we will note:

$$\underline{\mathbf{U}}_{R,j}(t,x) := \underline{\mathbf{U}}_{R,j,+}(t,x) \mathbf{1}_{(t,x) \in \Omega_R^+} + \underline{\mathbf{U}}_{R,j,-}(t,x) \mathbf{1}_{(t,x) \in \Omega_R^-}.$$

Next, we will write:

$$\mathbf{U}_{R,j}^c \left(t, x, \frac{x - a_R t}{\sqrt{\varepsilon}} \right) := \mathbf{U}_{R,j,+}^c \left(t, \frac{x - a_R t}{\sqrt{\varepsilon}} \right) \mathbf{1}_{(t,x) \in \Omega_R^+}$$
$$+ \mathbf{U}_{R,j,-}^c \left(t, \frac{x - a_R t}{\sqrt{\varepsilon}} \right) \mathbf{1}_{(t,x) \in \Omega_R^-}.$$

Note well that the dependence of $\mathbf{U}_{R,j}^c$ in x is a bit subtle. Actually, $\mathbf{U}_{R,j}^c$ is piecewise constant with respect to x on each side of Γ_R, which explains that $\mathbf{U}_{R,j,+}^c$ and $\mathbf{U}_{R,j,-}^c$ have no direct dependency in x. Due to their particular meaning, we prefer denoting the profiles $\underline{\mathbf{U}}_{R,0}$ and $\underline{\mathbf{U}}_{L,0}$ by u_R and u_L. Let us note $\mathcal{H}_R$ the differential operator

$$\mathcal{H}_R := \partial_t + a_R\partial_x$$

and $\mathcal{P}_R$ the differential operator

$$\mathcal{P}_R := \partial_t + a_R\partial_x - \partial^2_{\theta_R}.$$

We have

$$L^\varepsilon_R\ u^\varepsilon_{R,app}\left(t, x, \frac{x-a_Rt}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{R,j}\left(t, x, \frac{x-a_Rt}{\sqrt{\varepsilon}}\right)\varepsilon^{\frac{j}{2}}$$

where

$$L_{R,0} = \mathcal{H}_R u_R + \mathcal{P}_R U^c_{R,0},$$

$$L_{R,1} = \mathcal{H}_R\underline{\mathbf{U}}_{R,1} + \mathcal{P}_R U^c_{R,1} - 2\partial_x\partial_{\theta_R} U^c_{R,0},$$

and, for $2 \leqslant j \leqslant M-1$, we get:

$$L_{R,j} = \mathcal{H}_R\underline{\mathbf{U}}_{R,j} + \mathcal{P}_R U^c_{R,j} - \partial_x\left(2\partial_{\theta_R} U^c_{R,j-1} + \partial_x\underline{\mathbf{U}}_{R,j-2} + \partial_x U^c_{R,j-2}\right),$$

$$L_{R,M} = \mathcal{P}_R U^c_{R,M} - \partial_x\left(2\partial_{\theta_R} U^c_{R,M-1} + \partial_x\underline{\mathbf{U}}_{R,M-2} + \partial_x U^c_{R,M-2}\right),$$

$$L_{R,M+1} = -\partial_x\left(2\partial_{\theta_R} U^c_{R,M} + \partial_x\underline{\mathbf{U}}_{R,M-1} + \partial_x U^c_{R,M-1}\right).$$

Symmetrically, there holds:

$$L^\varepsilon_L\ u^\varepsilon_{L,app}\left(t, x, \frac{x-a_Lt}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{L,j}\left(t, x, \frac{x-a_Lt}{\sqrt{\varepsilon}}\right)\varepsilon^{\frac{j}{2}}$$

where, for instance, $L_{L,2}$ is given by:

$$L_{L,2} = \mathcal{H}_L\underline{\mathbf{U}}_{L,2} + \mathcal{P}_L U^c_{L,2} - \partial_x\left(2\partial_{\theta_L} + \partial_x u_L + \partial_x U^c_{L,0}\right),$$

with $\mathcal{H}_L$ and and $\mathcal{P}_L$ defined by:

$$\mathcal{H}_L := \partial_t + a_L\partial_x$$

$$\mathcal{P}_L := \partial_t + a_L\partial_x - \partial^2_{\theta_L}.$$

Plugging $u^\varepsilon_{L,app}$ and $u^\varepsilon_{R,app}$ in the problem (3.4) and identifying the terms with the same scale in ε, making then $|\theta_L|$ and $|\theta_R|$ tend to infinity, we obtain the profiles equations satisfied by the underlined profiles. Let us begin by writing the equations satisfied by $\underline{\mathbf{U}}_{L,j}$ and $\underline{\mathbf{U}}_{R,j}$ for all $0 \leqslant j \leqslant M-1$.

Thanks to the transmission conditions we had on the viscous problem (3.4), we get:

$$\begin{cases} u_{L,+}|_{x=0} - u_{R,-}|_{x=0} = 0, \\ \partial_x u_{L,+}|_{x=0} - \partial_x u_{R,-}|_{x=0} = 0, \end{cases}$$

and thus $(u_{R,-}, u_{L,+})$ satisfies the following transmission problem:

$$\begin{cases} \partial_t u_{R,-} + a_R \partial_x u_{R,-} = f_{R,-}, & (t,x) \in \Omega_R^-, \\ \partial_t u_{L,+} + a_L \partial_x u_{L,+} = f_{L,+}, & (t,x) \in \Omega_L^+, \\ u_{L,+}|_{x=0} - u_{R,-}|_{x=0} = 0, \\ \partial_x u_{L,+}|_{x=0} - \partial_x u_{R,-}|_{x=0} = 0 & . \end{cases} \tag{3.5}$$

As a result, the profile $u_{R,-}$ is the unique solution of:

$$\begin{cases} \partial_t u_{R,-} + a_R \partial_x u_{R,-} = f_{R,-}, & (t,x) \in \Omega_R^-, \\ u_{R,-}|_{x=0} = h(0) + \displaystyle\int_0^t f|_{x=0}(s)\, ds & , \end{cases}$$

and the profile $u_{L,+}$ is given by:

$$\begin{cases} \partial_t u_{L,+} + a_L \partial_x u_{L,+} = f_{L,+}, & (t,x) \in \Omega_L^+, \\ u_{L,+}|_{x=0} = h(0) + \displaystyle\int_0^t f|_{x=0}(s)\, ds & . \end{cases}$$

Proof. — The first boundary condition of (3.5) gives: $\partial_t u_{L,+}|_{x=0} = \partial_t u_{R,-}|_{x=0}$. Using then the equation, we obtain:

$$\partial_x u_{R,-}|_{x=0} = \frac{1}{a_R} \left(f_{R,-}|_{x=0} - \partial_t u_{R,-}|_{x=0} \right),$$

and

$$\partial_x u_{L,+}|_{x=0} = \frac{1}{a_L} \left(f_{L,+}|_{x=0} - \partial_t u_{L,+}|_{x=0} \right).$$

Using the second boundary condition, we have thus

$$a_L \left(f|_{x=0} - \partial_t u_{R,-}|_{x=0} \right) = a_R \left(f|_{x=0} - \partial_t u_{L,+}|_{x=0} \right),$$

therefore

$$\partial_t u_{L,+}|_{x=0} = \partial_t u_{R,-}|_{x=0} = f|_{x=0}.$$

Hence, there holds:

$$u_{L,+}|_{x=0} = u_{R,-}|_{x=0} = h(0) + \int_0^t f|_{x=0}(s)\, ds.$$

□

Moreover, as we could have forecasted, the profiles $u_{R,+}$ and $u_{L,-}$ satisfy the following well-posed hyperbolic problems:

$$\begin{cases} \partial_t u_{R,+} + a_R \partial_x u_{R,+} = f_{R,+}, & (t,x) \in \Omega_R^+, \\ u_R|_{t=0} = h_R & , \end{cases}$$

$$\begin{cases} \partial_t u_{L,-} + a_L \partial_x u_{L,-} = f_{L,-}, & (t,x) \in \Omega_L^-, \\ u_L|_{t=0} = h_L & . \end{cases}$$

Since these equations are well-posed, the function $\underline{u}$ is now perfectly defined. Let us go on with the construction of the next profiles. $\underline{\mathbf{U}}_{R,1}$ and $\underline{\mathbf{U}}_{L,1}$ are given by:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,1,-} + a_R \partial_x \underline{\mathbf{U}}_{R,1,-} = 0, & (t,x) \in \Omega_R^-, \\ \partial_t \underline{\mathbf{U}}_{L,1,+} + a_L \partial_x \underline{\mathbf{U}}_{L,1,+} = 0, & (t,x) \in \Omega_L^+, \\ \underline{\mathbf{U}}_{L,1,+}|_{x=0} = \underline{\mathbf{U}}_{R,1,-}|_{x=0} = 0 & . \end{cases}$$

Thus $\underline{\mathbf{U}}_{L,1,+} = 0$ and $\underline{\mathbf{U}}_{R,1,-} = 0$.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,1,+} + a_R \partial_x \underline{\mathbf{U}}_{R,1,+} = 0, & (t,x) \in \Omega_R^+, \\ \underline{\mathbf{U}}_{R,1,+}|_{t=0} = 0 & , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,1,-} + a_L \partial_x \underline{\mathbf{U}}_{L,1,-} = 0, & (t,x) \in \Omega_L^-, \\ \underline{\mathbf{U}}_{L,1,-}|_{t=0} = 0 & . \end{cases}$$

Hence $\underline{\mathbf{U}}_{R,1,+} = 0$ and $\underline{\mathbf{U}}_{L,1,-} = 0$. Actually, we see by induction that for all $n \in \mathbb{N}$, we have $\underline{U}^{\pm}_{R,2n+1,\pm} = 0$ and $\underline{U}_{L,2n+1,\pm} = 0$. On the other hand for $n \in \mathbb{N}^*$, the profiles $\underline{U}_{L,2n,\pm}$ and $\underline{U}_{R,2n,\pm}$ are given by the following well-posed hyperbolic problems. The first equation we get is:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,2n,-} + a_R \partial_x \underline{\mathbf{U}}_{R,2n,-} = \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,-}, & (t,x) \in \Omega_R^-, \\ \partial_t \underline{\mathbf{U}}_{L,2n,+} + a_L \partial_x \underline{\mathbf{U}}_{L,2n,+} = \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,+}, & (t,x) \in \Omega_L^+, \\ \underline{\mathbf{U}}_{R,2n,-}|_{x=0} - \underline{\mathbf{U}}_{L,2n,+}|_{x=0} = 0, \\ \partial_x \underline{\mathbf{U}}_{R,2n,-}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,2n,+}|_{x=0} = 0, \\ \underline{\mathbf{U}}_{R,2n,-}|_{t=0} = 0, \quad \underline{\mathbf{U}}_{L,2n,+}|_{t=0} = 0 & . \end{cases}$$

The same way as before, we obtain that $\underline{\mathbf{U}}_{R,2n,-}$ and $\underline{\mathbf{U}}_{L,2n,+}$ are the solutions of the following well-posed hyperbolic problems:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,2n,-} + a_R \partial_x \underline{\mathbf{U}}_{R,2n,-} = \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,-}, & (t,x) \in \Omega_R^-, \\ \underline{\mathbf{U}}_{R,2n,-}|_{x=0} = \displaystyle\int_0^t \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,-}|_{x=0}(s)\, ds & , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,2n,+} + a_L \partial_x \underline{\mathbf{U}}_{L,2n,+} = \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,+}, & (t,x) \in \Omega_L^+, \\ \underline{\mathbf{U}}_{L,2n,+}|_{x=0} = \displaystyle\int_0^t \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,+}|_{x=0}(s)\, ds & . \end{cases}$$

Moreover, there holds:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,2n,+} + a_R \partial_x \underline{\mathbf{U}}_{R,2n,+} = \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,+}, & (t,x) \in \Omega_R^+, \\ \underline{\mathbf{U}}_{R,2n,+}|_{t=0} = 0 \quad , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,2n,-} + a_L \partial_x \underline{\mathbf{U}}_{L,2n,-} = \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,-}, & (t,x) \in \Omega_L^-, \\ \underline{\mathbf{U}}_{L,2n,-}|_{t=0} = 0 \quad . \end{cases}$$

In conclusion, all the profiles $\underline{U}_n$ are constructed by induction.

We turn now to the construction of the boundary layer profiles $U^c_{L,j,\pm}(t,\theta_L)$ and $U^c_{R,j,\pm}(t,\theta_R)$. We will use the relations imposed on the profiles by the transmission conditions: $[u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[\partial_x u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[u^\varepsilon_{app}]_{\Gamma_L} = 0$, and $[\partial_x u^\varepsilon_{app}]_{\Gamma_L} = 0$; $[u^\varepsilon_{app}]_{\Gamma_R}$ stands for the jump of u^ε_{app} through Γ_R defined, for all $t \in (0,T)$ by:

$$[u^\varepsilon_{app}]_{\Gamma_R}(t) := \lim_{x \to a_R t, x > a_R t} u^\varepsilon_{app}\left(t, x, \frac{x - a_R t}{\sqrt{\varepsilon}}\right) - \lim_{x \to a_R t, x < a_R t} u^\varepsilon_{app}\left(t, x, \frac{x - a_R t}{\sqrt{\varepsilon}}\right).$$

$[u^\varepsilon_{app}]_{\Gamma_L}(t)$ is defined the same way. Because u^ε_{app} belongs to $C^1((0,T) \times \mathbb{R}^*)$, for all $0 \leqslant j \leqslant M$, we have:

$$[U^c_{L,j}]_L = -[\underline{\mathbf{U}}_{L,j}]_{\Gamma_L},$$

$$[U^c_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}.$$

Let $[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}$ be given, for all $t \in (0,T)$, by:

$$[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}(t) = \lim_{x \to a_R t, x > a_R t} \underline{\mathbf{U}}_{R,j,+}(t,x) - \lim_{x \to a_R t, x < a_R t} \underline{\mathbf{U}}_{R,j,-}(t,x)$$

and $[U^c_{R,j}]_R$ be defined, for all $t \in (0,T)$, by:

$$[U^c_{R,j}]_R(t) = \lim_{\theta_R \to 0^+} U^c_{R,j,+}(t,\theta_R) - \lim_{\theta_R \to 0^-} U^c_{R,j,-}(t,\theta_R).$$

To avoid writing the exact symmetric equations on $\{x > 0\}$ and $\{x < 0\}$, let us only proceed with the construction of the boundary layer profiles $U^c_{R,j,\pm}$. Referring to the computations above, for all $1 \leqslant j \leqslant M+1$, the following quantity must not have any Dirac measure in it:

$$\partial_x \left(\partial_{\theta_R} U^c_{R,j-1} + \frac{1}{2}(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2})) \right),$$

Our first boundary condition: $[U^c_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}$, ensures that, even if $\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2})$ is, in general, discontinuous on Γ_R, it has no Dirac Measure. $\partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2}))$ is the derivative of such a function and thus has a Dirac Measure. Let us describe this singularity: if we fix $t = t_0$, the Dirac measure forming is

$$\left([\partial_x \underline{\mathbf{U}}_{R,j-2}]|_{x=a_R t_0} + [\partial_x U^c_{R,j-2}]_R(t_0)\right)\ \delta_{x=a_R t_0},$$

where $[\omega]|_{x=a_R t_0} = \lim_{x\to a_R t_0, x>a_R t_0} \omega - \lim_{x\to a_R t_0, x<a_R t_0} \omega$. On the other hand, if $\partial_{\theta_R} U^c_{R,j-1}$ is discontinuous through Γ_R, $\partial_x \left(\partial_{\theta_R} U^c_{R,j-1}\right)$ has a Dirac measure given, for $t = t_0$ by:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R\ \delta_{x=a_R t_0}.$$

In order to ensure the sum of the two Dirac measure vanishes, the second boundary condition we get is that, $\forall t \in (0,T)$:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R(t) = -\frac{1}{2}\left([\partial_x \underline{\mathbf{U}}_{R,j-2}]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-2}(t)]_R\right).$$

The profiles $U^c_{R,0,+}$ and $U^c_{R,0,-}$ are solution of the following heat equation:

$$\begin{cases} \partial_t U^c_{R,0,+} - \partial^2_{\theta_R} U^c_{R,0,+} = 0 \quad t \in (0,T),\ \{\theta_R > 0\}, \\ \partial_t U^c_{R,0,-} - \partial^2_{\theta_R} U^c_{R,0,-} = 0 \quad t \in (0,T),\ \{\theta_R < 0\}, \\ [U^c_{R,0}]_R(t) = -[u_R]_{\Gamma_R}, \quad \forall t \in (0,T), \\ [\partial_{\theta_R} U^c_{R,j}]_R(t) = 0, \quad \forall t \in (0,T), \\ U^c_{R,j,+}|_{t=0} = 0, \\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Note well that, since $[u_R]_{\Gamma_R} = 0$, the profiles $U^c_{R,0}$ and $U^c_{L,0}$ are both equal to zero; this shows that the characteristic boundary layers forming are of weak amplitude. For all $1 \leqslant j \leqslant M$, the profiles $U^c_{R,j,+}$ and $U^c_{R,j,-}$ are given by:

$$\begin{cases} \partial_t U^c_{R,j,+} - \partial^2_{\theta_R} U^c_{R,j,+} = 0 \quad t \in (0,T),\ \{\theta_R > 0\}, \\ \partial_t U^c_{R,j,-} - \partial^2_{\theta_R} U^c_{R,j,-} = 0 \quad t \in (0,T),\ \{\theta_R < 0\}, \\ [U^c_{R,j}]_R(t) = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}, \quad \forall t \in (0,T), \\ [\partial_{\theta_R} U^c_{R,j}]_R(t) = -\dfrac{1}{2}\left([\partial_x \underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R\right), \quad \forall t \in (0,T), \\ U^c_{R,j,+}|_{t=0} = 0, \\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Let us now prove the well-posedness of these problems. We take $\psi_{R,j}$ in $H^\infty\left((0,T)\times\mathbb{R}^*\right)$ such that

$$[\psi_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R},$$

and

$$[\partial_{\theta_R} \psi_{R,j}]_R(t) = -\frac{1}{2}\left([\partial_x \underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R\right).$$

We can then compute $U^c_{R,j} := U^c_{R,j,+}\mathbf{1}_{\theta_R>0} + U^c_{R,j,-}\mathbf{1}_{\theta_R<0}$ by:

$$U^c_{R,j} := \psi_{R,j} + V^c_{R,j}.$$

$V^c_{R,j}$ is then the solution of the following heat equation with homogeneous transmission conditions straightforwardly satisfying a uniform Evans condition:

$$\begin{cases} \partial_t V^{c,\pm}_{R,j} - \partial^2_{\theta_R} V^{c,\pm}_{R,j} = \varphi^*_{R,j}, \quad t\in(0,T), \pm\theta_R > 0,\\ V^{c,+}_{R,j}|_{\theta_R=0} - V^{c,-}_{R,j}|_{\theta_R=0} = 0,\\ \partial_{\theta_R} V^{c,+}_{R,j}|_{\theta_R=0} - \partial_{\theta_R} V^{c,-}_{R,j}|_{\theta_R=0} = 0,\\ V^{c,\pm}_{R,j}|_{t=0} = 0. \end{cases}$$

and $\varphi^*_{R,j}$ is given by:

$$\varphi^*_{R,j} := -\left(\partial_t \psi_{R,j} - \partial^2_{\theta_R}\psi_{R,j}\right).$$

The profiles can thus be constructed by induction using the scheme just introduced.

3.2. Stability estimates

We will now prove stability estimates. We define the error $w^\varepsilon := u^\varepsilon_{app} - u^\varepsilon$. Let us denote by $w^{\varepsilon\pm}$ the restriction of w^ε to $\pm x > 0$. $(w^{\varepsilon+}, w^{\varepsilon-})$ is then solution of the transmission problem:

$$\begin{cases} \partial_t w^{\varepsilon+} + a_R\partial_x w^{\varepsilon+} - \varepsilon\partial^2_x w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, & x>0, t\in(0,T),\\ \partial_t w^{\varepsilon-} + a_L\partial_x w^{\varepsilon-} - \varepsilon\partial^2_x w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, & x<0, t\in(0,T),\\ w^{\varepsilon+}|_{x=0} - w^{\varepsilon-}|_{x=0} = 0,\\ \partial_x w^{\varepsilon+}|_{x=0} - \partial_x w^{\varepsilon-}|_{x=0} = 0,\\ w^{\varepsilon+}|_{t=0} = 0, \quad w^{\varepsilon-}|_{t=0} = 0. \end{cases}$$

By construction of our approximate solution, R^ε belongs to $L^2((0,T)\times\mathbb{R})$.

Like we have done previously for systems, we have to extend the definition of w^ε to $(t,x)\in\mathbb{R}^2$. Here for the sake of simplicity, we will make a slight abuse of notations and write:

$$\begin{cases} \partial_t w^{\varepsilon+} + a_R\partial_x w^{\varepsilon+} - \varepsilon\partial^2_x w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, & x>0, t\in\mathbb{R},\\ \partial_t w^{\varepsilon-} + a_L\partial_x w^{\varepsilon-} - \varepsilon\partial^2_x w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, & x<0, t\in\mathbb{R},\\ w^{\varepsilon+}|_{x=0} - w^{\varepsilon-}|_{x=0} = 0,\\ \partial_x w^{\varepsilon+}|_{x=0} - \partial_x w^{\varepsilon-}|_{x=0} = 0,\\ w^{\varepsilon+}|_{t<0} = 0, \quad w^{\varepsilon-}|_{t<0} = 0, \end{cases}$$

with R^ε belonging to $L^2(\mathbb{R}^2)$ and vanishing in the past. We prove in [9], in a more general framework, that we can do so.

We will now reformulate this problem into an equivalent problem, posed on one side of the boundary. Defining $\tilde{w}^\varepsilon := \begin{pmatrix} w^{\varepsilon+}(t,x) \\ w^{\varepsilon-}(t,-x) \end{pmatrix}$, the error equation rewrites as the doubled problem on one side of the boundary:

$$\begin{cases} \tilde{\mathcal{H}}^\varepsilon \tilde{w}^\varepsilon = \varepsilon^M \tilde{R}^\varepsilon, \quad \{x>0\}, \\ \Gamma \tilde{w}^\varepsilon|_{x=0} = 0, \\ \tilde{w}^\varepsilon|_{t<0} = 0. \end{cases}$$

where $\mathcal{H}^\varepsilon = \partial_t + \tilde{A}\partial_x - \varepsilon \partial_x^2$,

$$\tilde{A} = \begin{bmatrix} a_R & 0 \\ 0 & -a_L \end{bmatrix}, \ and \ \Gamma = \begin{bmatrix} 1 & -1 \\ \partial_x & \partial_x \end{bmatrix}.$$

Let us admit for now the following Proposition that will be proved in the next section.

PROPOSITION 3.4. — $(\tilde{\mathcal{H}}^\varepsilon, \Gamma)$ *satisfies a Uniform Evans Condition.*

As established earlier in the paper, if our linear mixed parabolic problem satisfies a Uniform Evans Condition, the following stability estimate holds:

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^2((0,T)\times\mathbb{R})} = \mathcal{O}\left(\varepsilon^{\frac{M-1}{2}}\right),$$

taking M large enough achieves then the proof of Theorem 3.2.

4. Proof of Proposition 3.4

In this section we will prefer using the notations a^+ and a^- instead of a_R and a_L. We refer to [9] for computations of the Evans function for 2×2 systems. In our present case, we have:

$$\mathbb{A}^\pm(\tilde{\zeta}) = \begin{pmatrix} 0 & 1 \\ i\tilde{\tau} + \tilde{\gamma} & a^\pm \end{pmatrix}$$

4.1. Computation of the Evans function for medium frequencies

There holds:

$$\mathbb{E}_-(\mathbb{A}^+(\tilde{\zeta})) = Span\left\{ \begin{pmatrix} 1 \\ \mu_-^+(\tilde{\zeta}) \end{pmatrix} \right\}$$

where μ_-^+ denotes the eigenvalue of $\mathbb{A}^+$ with negative real part and is given by:

$$\mu_-^+(\tilde{\zeta}) = \frac{1}{2}a^+ - \frac{1}{4}\left(((a^+)^2 + 4\tilde{\gamma})^2 + 16\tilde{\tau}^2\right)^{\frac{1}{4}} \left(\left(1 + \frac{16\tilde{\tau}^2}{((a^+)^2+4\tilde{\gamma})^2}\right)^{-\frac{1}{2}} + 1 \right)$$

$$-i\ sign(\tilde{\tau})\frac{1}{4}\left(((a^+)^2+4\tilde{\gamma})^2+16\tilde{\tau}^2\right)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tilde{\tau}^2}{\left((a^+)^2+4\tilde{\gamma}\right)^2}\right)^{-\frac{1}{2}}\right)$$

Moreover, we have:

$$\mathbb{E}_+(\mathbb{A}^-(\tilde{\zeta}))=Span\left\{\begin{pmatrix}1\\ \mu_+^-(\tilde{\zeta})\end{pmatrix}\right\}$$

where μ_+^- denotes the eigenvalue of $\mathbb{A}^-$ with positive real part and is given by:

$$\mu_+^-(\tilde{\zeta})=\frac{1}{2}a^-+\frac{1}{4}\left(((a^-)^2+4\tilde{\gamma})^2+16\tilde{\tau}^2\right)^{\frac{1}{4}}\left(\left(1+\frac{16\tilde{\tau}^2}{\left((a^-)^2+4\tilde{\gamma}\right)^2}\right)^{-\frac{1}{2}}+1\right)$$

$$+i\ sign(\tilde{\tau})\frac{1}{4}\left(((a^-)^2+4\tilde{\gamma})^2+16\tilde{\tau}^2\right)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tilde{\tau}^2}{\left((a^-)^2+4\tilde{\gamma}\right)^2}\right)^{-\frac{1}{2}}\right)$$

If we consider $\tilde{\zeta}$ such that $0<c\leqslant|\tilde{\zeta}|\leqslant C<\infty$, an Evans function is the modulus of the following determinant:

$$\begin{vmatrix}1 & 1\\ \mu_-^+(\tilde{\zeta}) & \mu_+^-(\tilde{\zeta})\end{vmatrix}$$

that is to say: $|\mu_+^-(\tilde{\zeta})-\mu_-^+(\tilde{\zeta})|$, since μ_+^- keeps a positive real part and μ_-^+ keeps a negative real part, for all $\tilde{\zeta}$ such that $0<c\leqslant|\tilde{\zeta}|\leqslant C<\infty$, there holds:

$$\left|\mu_+^-(\tilde{\zeta})-\mu_-^+(\tilde{\zeta})\right|>0.$$

Hence the Evans Condition is checked for medium frequencies.

4.2. Computation of the asymptotic Evans function when $|\tilde{\zeta}|\to\infty$

Λ is defined by:

$$\Lambda(\tilde{\zeta})=\left(1+\tilde{\tau}^2+\tilde{\gamma}^2\right)^{\frac{1}{2}}$$

We recall that the scaled eigenspaces for high frequencies write then:

$$\mathbb{E}_-(\mathbb{A}^+(\tilde{\zeta}))=Span\left\{\begin{pmatrix}1\\ \Lambda^{-1}\mu_-^+(\tilde{\zeta})\end{pmatrix}\right\}$$

$$\mathbb{E}_+(\mathbb{A}^-(\tilde{\zeta}))=Span\left\{\begin{pmatrix}1\\ \Lambda^{-1}\mu_+^-(\tilde{\zeta})\end{pmatrix}\right\}$$

An asymptotic Evans function for high frequencies writes:

$$\lim_{|\zeta|\to\infty}\left|\frac{\mu_+^-(\tilde{\zeta})-\mu_-^+(\tilde{\zeta})}{\Lambda(\tilde{\zeta})}\right|.$$

Since there is $C>0$ such that, for all $\rho\geqslant C>0$, $\Re e\frac{\mu_+^-(\tilde{\zeta})}{\Lambda(\tilde{\zeta})}\geqslant C$ and

$\Re e\frac{\mu_+^-(\tilde{\zeta})}{\Lambda(\tilde{\zeta})}\leqslant -C$, making $|\zeta|\to\infty$, we have:

$$\left|\frac{\mu_+^-(\tilde{\zeta})-\mu_-^+(\tilde{\zeta})}{\Lambda(\tilde{\zeta})}\right|\geqslant C'>0.$$

Therefore, the Evans Condition is checked for high frequencies.

4.3. Computation of the asymptotic Evans function when $|\tilde{\zeta}|\to 0^+$

Remark that

$$\mu_+^-|_{\tilde{\zeta}=0}=0,$$
$$\mu_-^+|_{\tilde{\zeta}=0}=0.$$

As a result, the linear subspaces $\mathbb{E}_-(\mathbb{A}^+(\tilde{\zeta}))$ and $\mathbb{E}_+(\mathbb{A}^-(\tilde{\zeta}))$ cease to be well-defined. $\mathbb{A}^\pm(\tilde{\zeta})$ appears in an ODE of the form:

$$\partial_z\begin{pmatrix} w^\pm \\ \partial_z w^\pm\end{pmatrix}=\mathbb{A}^\pm(\tilde{\zeta})\begin{pmatrix} w^\pm \\ \partial_z w^\pm\end{pmatrix}+F^\pm,$$

We have then:

$$\partial_z\begin{pmatrix} w^\pm \\ \rho^{-1}\partial_z w^\pm\end{pmatrix}:=\begin{pmatrix} 0 & \rho Id \\ \rho^{-1}(i\tilde{\tau}+\tilde{\gamma})Id & a^\pm\end{pmatrix}\begin{pmatrix} w^\pm \\ \rho^{-1}\partial_z w^\pm\end{pmatrix}$$
$$:=\rho\check{\mathbb{A}}(\check{\zeta},\rho)\begin{pmatrix} w^\pm \\ \rho^{-1}\partial_z w^\pm\end{pmatrix},$$

where

$$\check{\mathbb{A}}^\pm(\check{\zeta},\rho):=\begin{pmatrix} 0 & 1 \\ \rho^{-1}(i\check{\tau}+\check{\gamma}) & \rho^{-1}a^\pm\end{pmatrix}$$

with $\check{\tau}:=\frac{\tilde{\tau}}{\rho}$ and $\check{\gamma}:=\frac{\tilde{\gamma}}{\rho}$.
As reviewed earlier, a continuous extension of some positive and negative spaces of $\mathbb{A}^\pm$ has to be performed if we want to study the Evans function for low frequencies. These extended spaces are noted $\mathbb{E}_-^{lim}(\mathbb{A}^+)$ and $\mathbb{E}_+^{lim}(\mathbb{A}^-)$, and are computed as follows:

$$\mathbb{E}_-^{lim}(\mathbb{A}^+)=\mathbb{E}_-(\check{\mathbb{A}}^+)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0},$$

and

$$\mathbb{E}_{+}^{lim}(\mathbb{A}^{-}) = \mathbb{E}_{+}(\check{\mathbb{A}}^{-})|_{\check{\tau}=1,\check{\gamma}=0,\rho=0}.$$

The asymptotic Evans condition for low frequency writes then:

$$\mathbb{E}_{-}^{lim}(\mathbb{A}^{+}) \bigcap \mathbb{E}_{+}^{lim}(\mathbb{A}^{-}) = \{0\}.$$

Let us look at the negative eigenvalue of $\check{\mathbb{A}}^{+}(\check{\zeta},\rho)$ that we will note $\check{\lambda}^{+}(\check{\zeta},\rho)$ and compute its associated eigenvector:

$$\check{\mathbb{A}}^{+} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \check{\lambda}^{+} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

We get:

$$v_2 = \check{\lambda} v_1,$$

and multiplying by $\rho > 0$ the second coordinate of our vector gives:

$$(i\check{\tau} + \check{\gamma})v_1 + a^{+} v_2 = \rho \check{\lambda} v_2$$

Making $\rho \to 0^{+}$, we obtain that:

$$\check{\lambda^{+}}(\check{\zeta},\rho) = -\frac{i\check{\tau} + \check{\gamma}}{a^{+}}$$

As a result

$$\lim_{\rho \to 0^{+}} \mathbb{E}_{-}\left(\check{\mathbb{A}}^{+}(\check{\zeta},\rho)\right) = Span\left\{\begin{pmatrix} 1 \\ -\frac{i\check{\tau}+\check{\gamma}}{a^{+}} \end{pmatrix}\right\}$$

The same way, we have:

$$\lim_{\rho \to 0^{+}} \mathbb{E}_{+}\left(\check{\mathcal{G}}^{-}(\check{\zeta},\rho)\right) = Span\left\{\begin{pmatrix} 1 \\ -\frac{i\check{\tau}+\check{\gamma}}{a^{-}} \end{pmatrix}\right\}$$

Taking $\check{\gamma} = 0$ and $\check{\tau} = 1$, since, by assumption, $a^{-} < 0$ and $a^{+} > 0$ (otherwise the stability analysis for low frequencies would differ of the one we have just done), the Asymptotic Evans condition for low frequencies holds. This ends the proof of Proposition 3.4.

Bibliography

[1] Bachmann (F.). — Analysis of a scalar conservation law with a flux function with discontinuous coefficients, Adv. Diff. Eq. 11-12, p. 1317-1338 (2004).

[2] Bachmann (F.), Vovelle (J.). — Existence and uniqueness of entropy solution of scalar conservation laws with a flux function involving discontinuous coefficients, C.P.D.E, 31, p. 371-395 (2006).

[3] Bouchut (F.), James (F.). — One-dimensional transport equations with discontinuous coefficients, Nonlin. Anal., 32, p. 891-933 (1998).

[4] Bouchut (F.), James (F.), Mancini (S.). — Uniqueness and weak stability for multi-dimensional transport equations with one-sided Lipschitz coefficient, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), IV, p. 1-25 (2005).

[5] Chazarain (J.), Piriou (A.). — Introduction to the theory of linear partial differential equations. translated from the french, Studies in Mathematics and its Applications, 14 , North Holland Publishing Co., Amsterdam-New York,1982.

[6] Crasta (G.), LeFloch (P. G.). — Existence result for a class of nonconservative and nonstrictly hyperbolic systems, Commun. Pure Appl. Anal., 1(4), p. 513-530 (2002).

[7] DiPerna (R.J.), Lions (P.-L.). — Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98, p. 511-547 (1989).

[8] Fornet (B.). — Two Results concerning the Small Viscosity Solution of Linear Scalar Conservation Laws with Discontinuous Coefficients, HAL (2007).

[9] Fornet (B.). — The Cauchy problem for 1-D linear nonconservative hyperbolic systems with possibly expansive discontinuity of the coefficient: a viscous approach, Vol 245 pp. 2440-2476 (2008).

[10] Gallouët (T.). — Hyperbolic equations and systems with discontinuous coefficients or source terms, 10 pages, Proceedings of Equadiff-11, Bratislava, Slovaquia (July 25-29, 2005).

[11] Guès (O.), Métivier (G.), Williams (M.), Zumbrun (K.). — *Existence and stability of multidimensional shock fronts in the vanishing viscosity limit*, Arch. Rat. Mech. Anal. 175 p. 151-244 (2004).

[12] Guès (O.), Williams (M.). — Curved shocks as viscous limits: a boundary problem approach, Indiana Univ. Math. J., 51 (2002), 421-450.

[13] Hayes (B. T.), LeFloch (P. G.). — Measure solutions to a strictly hyperbolic system of conservation laws Nonlinearity, 9(6), p. 1547-1563 (1996).

[14] LeFloch (P. G.). — An existence and uniqueness result for two nonstrictly hyperbolic systems. In Nonlinear evolution 271 equations that change type, vol. 27 of IMA Vol. Math. Appl. (1990), 126-138. Springer, New York.

[15] LeFloch (P. G.), Tzavaras (A.E.). — Representation of weak limits and definition of nonconservartive products, SIAM J. Math. Anal. 30, p. 1309-1342 (1999).

[16] Métivier (G.). — Small Viscosity and Boundary Layer Methods : Theory, Stability Analysis, and Applications, Birkhauser (2003).

[17] Métivier (G.), Zumbrun (K.). — Large Viscous Boundary Layers for Noncharacteristic Nonlinear Hyperbolic Problems, Mem. Amer. Math. Soc. 175, no. 826, vi+107 pp (2005).

[18] Métivier (G.), Zumbrun (K.). — Symmetrizers and Continuity of Stable Subspaces for Parabolic-Hyperbolic Boundary Value Problems, Disc. Cont. Dyn. Syst., 11, p. 205-220 (2004).

[19] Poupaud (F.), Rascle (M.). — Measure solutions to the linear multidimensional transport equation with discontinuous coefficients, Comm. Diff. Equ. 22, p. 337-358 (1997).

[20] Rousset (F.). — Viscous approximation of strong shocks of systems of conservation laws, SIAM J. Math. Anal. 35 (2003), 492-519.