ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XVIII, n° 3 (2009), p. 561-576.

<http://afst.cedram.org/item?id=AFST_2009_6_18_3_561_0>

Annales de la Faculté des Sciences de Toulouse Vol. XVIII, n° 3, 2009
pp. 561–576

Some addition to the generalized Riemann-Hilbert problem[1]

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Abstract. — We consider the generalized Riemann-Hilbert problem for linear differential equations with irregular singularities. If one weakens the conditions by allowing one of the Poincaré ranks to be non-minimal, the problem is known to have a solution. In this article we give a bound for the possibly non-minimal Poincaré rank. We also give a bound for the number of apparent singularities of a scalar equation with prescribed generalized monodromy data.

Résumé. — Nous considérons le problème de Riemann-Hilbert généralisé pour des équations différentielles linéaires avec singularités irrégulières. Si on affaiblit les conditions en autorisant que l'un des rangs de Poincaré ne soit pas minimal, il est connu que le problème a une solution. Dans cet article nous donnons une borne pour le rang de Poincaré ainsi obtenu. Nous donnons aussi une borne pour le nombre de singularités apparentes de l'équation scalaire avec une donnée de monodromie généralisée prescrite.

1. Introduction

Consider a system

$$\frac{dy}{dz} = B(z)y, \qquad y(z) \in \mathbb{C}^p, \tag{1.1}$$

of p linear differential equations whose matrix $B(z)$ is meromorphic on the Riemann sphere $\overline{\mathbb{C}}$ and holomorphic outside the set of singular points $a_1, \dots, a_n$.

(*) Reçu le 15 octobre 2007, accepté le 30 avril 2008

(1) This research was carried out with the support of the Russian Foundation for Basic Research (grant no. 08-01-00342-a) and the RF President Programmes of Support of Young Russian Scientists (grant no. MK-3178.2009.1) and Leading Scientific Schools (grant no. N.Sh.-3038.2008.1).

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By the *monodromy representation* or the *monodromy* of this system we mean the representation

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \dots, a_n\}) \to GL(p, \mathbb{C}) \tag{1.2}$$

of the fundamental group of the punctured sphere in the space of invertible complex matrices of order p. (A loop γ is mapped to a matrix G_γ such that $Y(z) = \widetilde{Y}(z)G_\gamma$, where $Y(z)$ is a fundamental matrix of the system and $\widetilde{Y}(z)$ its analytic continuation along γ.)

Since the fundamental group of the punctured sphere is generated by the homotopy classes of all simple loops γ_i (each γ_i encircles the only singular point a_i, and by convention we assume the loop $\gamma_1 \dots \gamma_n$ is contractible), the representation χ is defined by local *monodromy matrices* G_i corresponding to these loops.

A singular point a_i of the system (1.1) is said to be *Fuchsian* if the matrix differential 1-form $B(z)dz$ has a simple pole at this point. By Sauvage's theorem (see [9], Th. 11.1) a Fuchsian singularity is always *regular* (i. e., each solution has at most power growth near it), although a regular singularity is not necessarily Fuchsian. The system (1.1) is said to be *Fuchsian* if all its singular points are Fuchsian.

The classical Riemann-Hilbert problem asks for conditions under which it is possible to construct a Fuchsian system (1.1) with prescribed singular points $a_1, \dots, a_n$ and prescribed monodromy (1.2). (In the general case the problem has a negative solution, a counterexample was found by A. Bolibrukh, see [1], Sect. 2.) One knows various sufficient conditions for the affirmative solution of this problem. One such condition is the irreducibility of the representation (1.2) (see [1], Th. 4.2.1). And by Plemelj's theorem the problem always has a solution if one allows the point a_1 to be regular rather than Fuchsian (see [1], Th. 3.2.1).

A generalization of the Riemann-Hilbert problem was formulated by A. Bolibrukh, S. Malek and C. Mitschi in [6], under the denomination of the *generalized Riemann-Hilbert problem* (the GRH-problem). Before presenting this problem we first recall the notions of local holomorphic and meromorphic transformations, the Poincaré rank and the minimal Poincaré rank of the system (1.1).

If the coefficient matrix $B(z)$ of the system (1.1) has the Laurent expansion of the form

$$B(z) = \frac{B_{-r-1}}{(z-a)^{r+1}} + \dots + \frac{B_{-1}}{z-a} + B_0 + \dots \qquad (B_{-r-1} \neq 0)$$

in a neighbourhood of a singularity $a = a_i$ then one refers to the integer r as the *Poincaré rank* of the system at this point.

A local linear transformation (in a neighbourhood O_i of a point a_i)

$$y' = \Gamma(z)y$$

is said to be *holomorphic* (more precisely, *holomorphically invertible*) if the matrix $\Gamma(z)$ is holomorphic in O_i and $\det \Gamma(a_i) \neq 0$. This transformation is said to be *meromorphic* (more precisely, *meromorphically invertible*) if the matrix $\Gamma(z)$ is meromorphic at a_i, holomorphic in $O_i \setminus \{a_i\}$ and $\det \Gamma(z) \not\equiv 0$.

Such transformations take (1.1) to the system

$$\frac{dy'}{dz} = B'(z)y', \qquad B'(z) = \frac{d\Gamma}{dz}\Gamma^{-1} + \Gamma B(z)\Gamma^{-1}. \tag{1.3}$$

Under such transformations the systems (1.1) and (1.3) are said to be *holomorphically* (resp. *meromorphically*) *equivalent*.

A holomorphic transformation does not change the Poincaré rank of the original system, while a meromorphic one may increase or decrease the Poincaré rank. The *minimal Poincaré rank* of the system (1.1) at the point a_i is the smallest Poincaré rank of local systems (1.3) in the meromorphic equivalence class of (1.1) at the point a_i.

Now the GRH-problem can be formulated as follows.

Let for each $i = 1, \dots, n$ a local system

$$\frac{dy}{dz} = B_i(z)y \tag{1.4}$$

be given in the neighbourhood O_i of an (irregular) singular point a_i of the Poincaré rank r_i which is minimal. Assume that a monodromy matrix of (1.4) (with respect to a suitable fundamental solution) coincides with G_i (recall that $G_i = \chi(\gamma_i)$ for the given representation (1.2)).

Does there exist a global system (1.1) with singularities $a_1, \dots, a_n$ of the Poincaré ranks $r_1, \dots, r_n$, with the prescribed monodromy (1.2) and such that it is meromorphically equivalent to the system (1.4) in each O_i?

We will refer to the monodromy representation (1.2) and the family of local systems (1.4) as *generalized monodromy data*.

These data are called *reducible* if the representation (1.2) is reducible and the local systems (1.4) are simultaneously reducible, i. e., they can be reduced via meromorphic transformations to systems with coefficient matrices of the same block upper-triangular form. Otherwise we say that the generalized monodromy data are *irreducible*.

Generalizing A. Bolibrukh's method of solution of the classical Riemann-Hilbert problem to the case of irregular singularities, the authors of [6] obtained some sufficient conditions for an affirmative solution of the GRH-problem. One such condition is the irreducibility of the generalized monodromy data in the case if one at least of the singularities is *unramified* (the definition of ramified and unramified singular points see in §2).

An analogue of Plemelj's theorem is that the problem has always a solution if one allows the Poincaré rank of a global system at the point a_1 not to be minimal. We here obtain an estimate for the Poincaré rank at this point.

THEOREM 1.1. — *Any generalized monodromy data can be realized by a global system (1.1) that has the minimal Poincaré ranks at all points but one (a_1 for instance), at which it has Poincaré rank not greater than $r_1 + (p-1)(n+R-2)$, where $R = \sum_{i=1}^{n} r_i \geqslant 1$.*

Remark. — We assume that $R \geqslant 1$ because $R = 0$ is a case of Fuchsian singularities $a_1, \ldots, a_n$, which was considered in [11]. It was shown there that the Poincaré rank of unique non-Fuchsian (regular) singularity from Plemelj's theorem can be reduced to a value that is not greater than $(p-1)(n-1)$.

Section 4 is devoted to the GRH-problem for scalar equations, for which we recall and reformulate results already proved in [11]. The problem is to construct a scalar linear differential equation

$$\frac{d^p y}{dz^p} + b_1(z)\frac{d^{p-1} y}{dz^{p-1}} + \ldots + b_p(z)y = 0$$

with prescribed singular points $a_1, \ldots, a_n$ and generalized monodromy data. In the construction there necessary arise *apparent singularities* (at which coefficients of an equation are singular, but solutions are meromorphic, so that a monodromy is trivial), the number of which was estimated in [11]. We present these results in the language of the GRH-problem for systems, although Section 4 is independent of the previous sections.

We are very thankful to the prof. D. Bertrand for the offering to publish this paper and to the referee for a lot of useful comments which allowed us to improve the presentation of the paper.

2. Irregular systems and holomorphic vector bundles

In this paragraph we recall the main results which we will need from the theory of irregular singularities and their relations with vector bundles. Our main reference is the article [6] by A. Bolibrukh, S. Malek and C. Mitschi.

In a neighbourhood of an irregular singularity $a = a_i$ of Poincaré rank r the system (1.1) has a formal fundamental matrix $\widehat{Y}(z)$ of the form (see [3], Th. 1)

$$\widehat{Y}(z) = \widehat{F}(z)(z-a)^E U e^{Q(z)}, \tag{2.1}$$

where

$\widehat{F}(z)$ is a formal (matrix) Laurent series in $z-a$ (in general divergent) with a finite principal part and $\det \widehat{F}(z)$ is distinct from the zero series;

$Q(z)$, E and U are block-diagonal matrices with diagonal blocks $Q^j(z)$, E^j and U^j of the same size, $j = 1, \dots, N$;

the blocks $Q^j(z)$ and E^j too are block-diagonal of the form

$$Q^j(z) = \operatorname{diag}\left(q_j(t)I_{m_j}, q_j(t\zeta_j)I_{m_j}, \dots, q_j(t\zeta_j^{s_j-1})I_{m_j}\right),$$

where $q_j(t)$ is a polynomial in $t = (z-a)^{-1/s_j}$ with no constant term, $\zeta_j = e^{-2\pi i/s_j}$ for some integer s_j and $\deg q_j \leqslant r s_j$, I_{m_j} denotes the identity matrix of size m_j;

$$E^j = \operatorname{diag}\left(\widehat{E}^{m_j}, \widehat{E}^{m_j} + \frac{1}{s_j}I_{m_j}, \dots, \widehat{E}^{m_j} + \frac{s_j-1}{s_j}I_{m_j}\right),$$

where $\widehat{E}^{m_j}$ is a constant matrix of size m_j in canonical Jordan form and its eigenvalues ρ satisfy the condition $0 \leqslant \operatorname{Re}\rho < 1/s_j$;

the matrix U^j decomposes into blocks $\left[U^j\right]^{kl}$ of the form

$$\left[U^j\right]^{kl} = \zeta_j^{-(k-1)(l-1)} I_{m_j}, \qquad 1 \leqslant k, l \leqslant s_j,$$

with respect to the block structure of the matrices $Q^j(z)$ and E^j.

Let the matrix $Q(z)$ be thought of as the (matrix) polynomial in $1/(z-a)$ of fractional degree $\deg Q$. Then this degree is called the *Katz rank* of a singularity $z=a$.

Since the matrix $Q(z)$ is a meromorphic invariant of the system (1.1), it follows from the properties of this matrix that the Katz rank is not greater than the minimal Poincaré rank of a singularity. Moreover, the minimal Poincaré rank is the least integer greater than or equal to the Katz rank of a singularity.

Definition 2.1. — *An irregular singularity of the system (1.1) is called* unramified *(or a* singularity without roots*) if for every block $Q^j(z)$ of the matrix $Q(z)$ from (2.1) the corresponding integer s_j is equal to one.*

In the opposite case a singularity is called ramified *(or a* singularity with roots*).*

Now we will describe briefly the method of solution for the GRH-problem given in [6].

From the representation (1.2) one constructs over the punctured Riemann sphere $\overline{\mathbb{C}}\setminus\{a_1,\dots,a_n\}$ a holomorphic vector bundle F of rank p with a holomorphic connection ∇ having the prescribed monodromy (1.2). This bundle is defined by a set $\{U_\alpha\}$ of sufficiently small discs covering $\overline{\mathbb{C}}\setminus\{a_1,\dots,a_n\}$ and a set $\{g_{\alpha\beta}\}$ of constant matrices defining a gluing cocycle. A connection ∇ is defined by a set $\{\omega_\alpha\}$ of matrix differential 1-forms $\omega_\alpha\equiv 0$. So in the intersections $U_\alpha\cap U_\beta\neq\varnothing$ the gluing conditions

$$\omega_\alpha=(dg_{\alpha\beta})g_{\alpha\beta}^{-1}+g_{\alpha\beta}\omega_\beta g_{\alpha\beta}^{-1}$$

hold.

Further one extends the pair (F,∇) to the whole Riemann sphere by means of the local matrix differential 1-forms $\omega_i=B_i(z)dz$ of the coefficients of the systems (1.4) defined each in the neighbourhood O_i of the point a_i, $i=1,\dots,n$. This is the so-called *canonical* extension (F^0,∇^0) of the pair (F,∇) in the sense of Deligne.

Then one constructs a family $\mathcal{F}$ of extensions of the pair (F,∇) replacing the forms ω_i in the construction of (F^0,∇^0) by the forms

$$\omega_i'=(d\Gamma_i)\Gamma_i^{-1}+\Gamma_i\omega_i\Gamma_i^{-1}, \tag{2.2}$$

where $y'=\Gamma_i(z)y$ are all possible meromorphic transformations of a system (1.4) which do not increase its Poincaré rank r_i, $i=1,\dots,n$ (see (1.3)).

The family $\mathcal{F}$ contains all holomorphic vector bundles over $\overline{\mathbb{C}}$ with connections having the prescribed monodromy data (1.2), (1.4). Since a connection on a holomorphically trivial vector bundle defines a global system of linear differential equations on the Riemann sphere, one gets that

The GRH-problem has a positive solution for the given generalized monodromy data (1.2), (1.4) if and only if at least one of the vector bundles of the family $\mathcal{F}$ is holomorphically trivial.

The Birkhoff-Grothendieck theorem states that each holomorphic vector bundle F' of rank p on the Riemann sphere is holomorphically equivalent to a sum of line bundles

$$F' \cong \mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_p),$$

where $\{k_1 \geqslant \ldots \geqslant k_p\}$ is a system of integers called the *splitting type* of the bundle F'.

From this theorem one concludes that for every $(F', \nabla') \in \mathcal{F}$ there exists a global system $dy = \omega y$ holomorphically equivalent to the system $dy = \omega_i' y$ in O_i for $i = 2, \ldots, n$ and

$$\omega = -\frac{K}{z - a_1} dz + (z - a_1)^{-K} \tilde{\omega}_1 (z - a_1)^K, \qquad K = \operatorname{diag}(k_1, \ldots, k_p), \tag{2.3}$$

in O_1, where $\operatorname{ord}_{a_1} \tilde{\omega}_1 = -(r_1 + 1)$. This global system has the prescribed generalized monodromy data, but its Poincaré rank r_1' at the singular point a_1 may be greater than r_1. The relation (2.3) implies that r_1' is not greater than $r_1 + k_1 - k_p$. Further to prove Theorem 1.1 we will estimate the integers $k_1 - k_p$ for some bundles F' from $\mathcal{F}$.

Let us recall that the *degree* $\deg F'$ of a bundle F' with a connection ∇' defined by the forms ω_i' from (2.2) is the sum

$$\deg F' = \sum_{i=1}^{n} \operatorname{res}_{a_i} \operatorname{tr} \omega_i'.$$

The degree of a bundle is an integer equal to the sum of the coefficients k_i of its splitting type.

Now we consider a subset $\mathcal{E} \subset \mathcal{F}$ of the family $\mathcal{F}$ constructed by means of meromorphic transformations with matrices $\Gamma_i(z)$ from (2.2) of some special form. For this construction one needs the following definition.

Definition 2.2. — *Consider a system (1.4) with an (irregular) singular point a_i and its formal fundamental matrix $\widehat{Y}_i(z)$ of the form (2.1), where all*

matrices are supplied with subscript i. An admissible matrix for this system is an integer-valued diagonal matrix $\Lambda_i = \operatorname{diag}(\Lambda_i^1, \dots, \Lambda_i^N)$ blocked in the same way as $Q_i(z)$ and such that

$(z-a_i)^{\Lambda_i^j} E_i^j (z-a_i)^{-\Lambda_i^j}$ is holomorphic at the point a_i if the block $Q_i^j(z)$ has no ramification;

Λ_i^j is a scalar matrix if the block $Q_i^j(z)$ has ramification.

Let us write the matrix $\widehat{Y}_i(z)$ as follows:

$$\widehat{Y}_i(z) = \widehat{F}_i(z)(z-a_i)^{-\Lambda_i}(z-a_i)^{\Lambda_i}(z-a_i)^{E_i}\, U_i e^{Q_i(z)}. \tag{2.4}$$

By an analogue of Sauvage's lemma (see [9], L. 11.2) for formal series, there exists a meromorphically invertible matrix $\Gamma_i'(z)$ in O_i, such that

$$\Gamma_i'(z)\widehat{F}_i(z)(z-a_i)^{-\Lambda_i} = (z-a_i)^D \widehat{F}_0(z), \tag{2.5}$$

where D is a diagonal integer-valued matrix and $\widehat{F}_0(z)$ is an invertible formal (matrix) Taylor series in $z-a_i$.

Meromorphic transformations (required for the construction of a subset $\mathcal{E} \in \mathcal{F}$) for an irregular singularity a_i are now defined by the matrices $\Gamma^{\Lambda_i}(z) = (z-a_i)^{-D}\,\Gamma_i'(z)$ depending on Λ_i (because $\Gamma_i'(z)$ depends on Λ_i). It follows from (2.4), (2.5) that the transformation $y' = \Gamma^{\Lambda_i}(z)y$ takes (1.4) to the system with formal fundamental matrix

$$\widehat{Y}_i'(z) = \widehat{F}_0(z)(z-a_i)^{\Lambda_i}(z-a_i)^{E_i}\, U_i e^{Q_i(z)}.$$

One needs to verify now that such transformations do not increase the Poincaré ranks r_i of the systems (1.4). This is provided by the following lemma, all statements of which are proved in [6], Sect. 2.

Lemma 2.3. — *Consider the set $\mathcal{E}$ of the extensions $(F^\Lambda, \nabla^\Lambda)$ of the pair (F, ∇) to the whole Riemann sphere obtained by means of all possible systems $\Lambda = \{\Lambda_1, \dots, \Lambda_n\}$ of admissible matrices for the singularities $a_1, \dots, a_n$. This set is a subset of the family $\mathcal{F}$, i. e. for each pair $(F^\Lambda, \nabla^\Lambda)$ the Poincaré rank of the connection ∇^Λ at the point a_i is equal to r_i.*

Moreover, for the degree of the bundle F^Λ the following relation holds:

$$\deg F^\Lambda = \sum_{i=1}^{n} \operatorname{tr}(\Lambda_i + E_i).$$

Let us call the eigenvalues $\beta_i^j = \lambda_i^j + \rho_i^j$ of the matrix $\Lambda_i + E_i$ *(formal) exponents of the connection ∇^Λ* at the (irregular) singular point a_i.

3. Proof of Theorem 1.1

Theorem 1.1 is a direct consequence of the following result (which is based on the proof of Lemma 2 from [6]).

PROPOSITION 3.1. — *Consider a pair* $(F^\Lambda, \nabla^\Lambda) \in \mathcal{E}$ *such that the exponents of* ∇^Λ *satisfy the condition* $0 \leqslant \operatorname{Re}\beta_i^j < M$, $M \in \mathbb{N}^*$, *for all* i, j. *Assume* $R = \sum_{i=1}^n r_i \geqslant 1$.

Then the following inequalities hold for the splitting type $(k_1^\Lambda, \dots, k_p^\Lambda)$ *of the bundle* F^Λ*:*

$$k_j^\Lambda - k_{j+1}^\Lambda \leqslant (n+R)M - 2, \qquad j = 1, \dots, p-1.$$

Proof. — We consider two separate cases.

Case 1. For the splitting type of the bundle F^Λ one has the inequalities

$$k_j^\Lambda - k_{j+1}^\Lambda \leqslant n + R - 2, \qquad j = 1, \dots, p-1.$$

Since $M \in \mathbb{N}^*$, the announced result in this case follows from these inequalities.

Case 2. For some l one has $k_l^\Lambda - k_{l+1}^\Lambda > n + R - 2$.

Consider the system (1.1) with the singularities $a_1, \dots, a_n$ and generalized monodromy data (1.2), (1.4) such that the Poincaré ranks of singularities $a_2, \dots, a_n$ are equal to $r_2, \dots, r_n$ respectively and the differential form $\omega = B(z)dz$ of the coefficients in the neighbourhood O_1 of the point a_1 has the form

$$\omega = -\frac{K}{z - a_1} dz + (z - a_1)^{-K} \tilde{\omega}^{\Lambda_1} (z - a_1)^K, \tag{3.1}$$

where $K = \operatorname{diag}(k_1^\Lambda, \dots, k_p^\Lambda)$ and $\operatorname{ord}_{a_1} \tilde{\omega}^{\Lambda_1} = -(r_1 + 1)$ (see (2.3)).

By (3.1) the entries ω_{mj} and $\tilde{\omega}_{mj}$ of the matrix differential 1-forms ω and $\tilde{\omega}^{\Lambda_1}$ are connected for $m \neq j$ by the equality

$$\omega_{mj} = (z - a_1)^{-k_m^\Lambda + k_j^\Lambda} \, \tilde{\omega}_{mj}.$$

By assumption $k_l^\Lambda - k_{l+1}^\Lambda > n + R - 2$ for some l. Therefore we have $k_j^\Lambda - k_m^\Lambda > n + R - 2$ for $j \leqslant l$, $m > l$, hence the orders $\operatorname{ord}_{a_1} \omega_{mj}$ at the point a_1 of the differential 1-forms ω_{mj} with indicated indices are greater than $n + R - r_1 - 3$,

whereas the sum of the orders $\operatorname{ord}_{a_i}\omega_{mj}$ at the singular points distinct from a_1 is at least $-n-R+r_1+1$.

We thus obtain for meromorphic forms ω_{mj} with indicated indices that the sum of their orders over all singularities and zeros is greater than -2, although this sum is known to be -2 for a non-trivial differential 1-form on $\overline{\mathbb{C}}$ (see [8], Prop. 17.12). These forms are therefore identically equal to zero, so that the matrix differential 1-forms ω, $\tilde{\omega}^{\Lambda_1}$ are block upper-triangular:

$$\omega=\begin{pmatrix}\omega^1 & * \\ 0 & \omega^2\end{pmatrix},\qquad \tilde{\omega}^{\Lambda_1}=\begin{pmatrix}\tilde{\omega}^1 & * \\ 0 & \tilde{\omega}^2\end{pmatrix},$$

where the matrix forms ω^1, $\tilde{\omega}^1$ have size $l\times l$.

This means that the bundle F^Λ has a subbundle $F^1\cong\mathcal{O}(k_1^\Lambda)\oplus\ldots\oplus\mathcal{O}(k_l^\Lambda)$ of rank l with a connection ∇^1 defined by the forms ω^1, $\tilde{\omega}^1$ satisfying the required gluing conditions (in view of (3.1)).

From results of [4] it follows that the local system $dy=\tilde{\omega}^{\Lambda_1}y$ (which is holomorphically equivalent to the system $dy=\omega^{\Lambda_1}y$ in O_1) and system $dy=\omega y$ (which is holomorphically equivalent to the systems $dy=\omega^{\Lambda_i}y$ in O_i, $i=2,\ldots,n$) have formal fundamental matrices $\widetilde{Y}_i$ in O_i of a block upper-triangular structure similar to $\tilde{\omega}^{\Lambda_1}$, ω respectively:

$$\widetilde{Y}_i=\begin{pmatrix}\widetilde{Y}_i^1 & * \\ 0 & \widetilde{Y}_i^2\end{pmatrix}.$$

Furthermore they have the form

$$\widetilde{Y}_i(z)=\widetilde{F}_0(z)(z-a_i)^{\widetilde{\Lambda}_i}(z-a_i)^{\widetilde{E}_i}\,\widetilde{U}_i e^{\widetilde{Q}_i(z)},$$

where $\widetilde{\Lambda}_i=S^{-1}\Lambda_i S$, $\widetilde{E}_i=S^{-1}E_iS$, $\widetilde{U}_i=S^{-1}U_iS$, $\widetilde{Q}_i(z)=S^{-1}Q_i(z)S$ for some constant invertible matrix S, the matrices $\widetilde{\Lambda}_i$ and $\widetilde{Q}_i(z)$ are diagonal and obtained by suitable permutations of the diagonal elements of Λ_i and $Q_i(z)$ respectively, the matrix $\widetilde{E}_i$ is upper-triangular and the matrix $\widetilde{U}_i=\operatorname{diag}(\widetilde{U}_i^1,\widetilde{U}_i^2)$ is block-diagonal with respect to the block-structure of the matrix $\widetilde{Y}_i$. Moreover, the invertible formal (matrix) Taylor series $\widetilde{F}_0(z)$ has the same block upper-triangular structure as the matrix $\widetilde{Y}_i$.

Thus, one gets that the set $\{{}^1\beta_i^1,\ldots,{}^1\beta_i^l\}$ of the (formal) exponents of the connection ∇^1 at the singularity a_i is a subset of the (formal) exponents of the connection ∇^Λ at this point.

Assume that $k_l^\Lambda - k_{l+1}^\Lambda \geqslant (n+R)M-1$. Then for the mean value of the exponents ${}^1\beta_i^j$ of the connection ∇^1 we have the lower bound

$$\frac{1}{ln}\sum_{i=1}^{n}\sum_{j=1}^{l}{}^1\beta_i^j = \frac{\deg F^1}{ln} = \frac{k_1^\Lambda+\ldots+k_l^\Lambda}{ln} \geqslant \frac{k_{l+1}^\Lambda}{n}+M+\frac{MR-1}{n} \geqslant \frac{k_{l+1}^\Lambda}{n}+M$$

(recall that M and R are elements of $\mathbb{N}^*$), while for the mean value of the other exponents ${}^2\beta_i^j$ of the connection ∇^Λ we have the upper bound

$$\frac{1}{(p-l)n}\sum_{i=1}^{n}\sum_{j=1}^{p-l}{}^2\beta_i^j = \frac{\deg F^\Lambda - \deg F^1}{(p-l)n} = \frac{k_{l+1}^\Lambda+\ldots+k_p^\Lambda}{(p-l)n} \leqslant \frac{k_{l+1}^\Lambda}{n}.$$

We obtain that the mean value of the exponents ${}^1\beta_i^j$ is larger by M at least than the mean value of the exponents ${}^2\beta_i^j$, while by assumption the real parts of all the exponents of the connection ∇^Λ are strictly less than M. We arrive to a contradiction, hence $k_l^\Lambda - k_{l+1}^\Lambda \leqslant (n+R)M-2$ for each l. $\square$

Proof of Theorem 1.1. — Consider the pair $(F^{\Lambda^0},\nabla^{\Lambda^0}) \in \mathcal{E} \subset \mathcal{F}$ corresponding to the system $\Lambda^0 = \{0,\ldots,0\}$ of zero matrices. In that case the exponents β_i^j of the connection ∇^{Λ^0} satisfy the condition

$$0 \leqslant \operatorname{Re}\beta_i^j = \operatorname{Re}\rho_i^j < 1,$$

therefore by Proposition 3.1 we have the inequalities

$$k_j^0 - k_{j+1}^0 \leqslant n+R-2, \qquad j=1,\ldots,p-1,$$

for the coefficients k_j^0 of the splitting type of the bundle F^{Λ^0}. Hence

$$k_1^0 - k_p^0 = \sum_{j=1}^{p-1}(k_j^0 - k_{j+1}^0) \leqslant (p-1)(n+R-2).$$

The coefficient matrix $B(z)$ of the global system (1.1) corresponding to the connection ∇^{Λ^0} has the form

$$B(z) = -\frac{K^0}{z-a_1} + (z-a_1)^{-K^0}\widetilde{B}(z)(z-a_1)^{K^0}$$

in the neighbourhood O_1 of the point a_1, where $K^0 = \operatorname{diag}(k_1^0,\ldots,k_p^0)$ and $\operatorname{ord}_{a_1}\widetilde{B}(z) = -(r_1+1)$ (see (3.1)). Then the Poincaré rank of this system at the point a_1 is not greater than the quantity

$$r_1 + k_1^0 - k_p^0 \leqslant r_1 + (p-1)(n+R-2)$$

(recall that this system has the prescribed singularities $a_1, \dots, a_n$, generalized monodromy data (1.2), (1.4) and the Poincaré ranks $r_2, \dots, r_n$ at the points $a_2, \dots, a_n$ respectively). $\square$

Let us say a few words about the problem of the meromorphic transformation of a local system

$$\frac{dy}{dz} = C(z)y, \qquad C(z) = \frac{C_{-r-1}}{z^{r+1}} + \dots + \frac{C_{-1}}{z} + C_0 + \dots, \tag{3.2}$$

of p linear differential equations to a *Birkhoff standard form* in a neighbourhood of an irregular singularity $z = 0$ of Poincaré rank r (not necessarily minimal), i.e. to a system with coefficient matrix $C'(z)$ of the form

$$C'(z) = \frac{C'_{-r'-1}}{z^{r'+1}} + \dots + \frac{C'_{-1}}{z}, \qquad r' \leqslant r \tag{3.3}$$

(note that such a system is defined on the whole Riemann sphere and ∞ is a Fuchsian singularity for it).

This problem is not yet resolved, though it is known that the problem has an affirmative answer in dimensions $p = 2$ and $p = 3$; one also knows various sufficient conditions for a positive solution in an arbitrary dimension p (for instance, the problem has a positive solution if the system (3.2) is irreducible (A. Bolibrukh) or if all the eigenvalues of the matrix C_{-r-1} are distinct (H. L. Turrittin); see Balser's survey [2] for details).

Denote by $r^0 > 0$ the minimal Poincaré rank of the system (3.2) and consider the GRH-problem for the following generalized monodromy data:

i) an irregular singularity $a_1 = 0$ of local system meromorphically equivalent to (3.2) and with Poincaré rank r^0;

ii) a Fuchsian singularity $a_2 = \infty$.

There exists a global system on the whole Riemann sphere that is Fuchsian at infinity and meromorphically equivalent to the system (3.2) in a neighbourhood of the point $a_1 = 0$. By Theorem 1.1 (where $n = 2$ and $R = r^0 > 0$) the coefficient matrix of this system has the form (3.3), where $r' \leqslant r^0 + (p-1)r^0 = pr^0$. Thus, one gets the following statement.

Corollary 3.2. — *If for the minimal Poincaré rank r^0 of the system (3.2) the inequality $r^0 \leqslant r/p$ holds then it can be meromorphically transformed into a Birkhoff standard form (with Poincaré rank not greater than r).*

4. The GRH-problem for scalar linear differential equations

Consider a linear differential equation

$$\frac{d^p u}{dz^p} + b_1(z)\frac{d^{p-1}u}{dz^{p-1}} + \ldots + b_p(z)u = 0 \tag{4.1}$$

of order p with coefficients $b_1(z), \ldots, b_p(z)$ meromorphic on the Riemann sphere $\overline{\mathbb{C}}$ and holomorphic outside the set of singular points $a_1, \ldots, a_n$.

One defines the monodromy representation

$$\chi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_n\}) \to GL(p, \mathbb{C}) \tag{4.2}$$

of this equation in the same way as for a system (1.1); one merely needs to consider in place of a fundamental matrix $Y(z)$ a row $(u_1, \ldots, u_p)$, where the functions $u_1(z), \ldots, u_p(z)$ form a basis in the solution space of the equation. This representation is defined by local monodromy matrices G_i corresponding to simple loops γ_i.

A singular point a_i of the equation (4.1) is said to be *Fuchsian* if the coefficient $b_j(z)$ has at this point a pole of order j or lower ($j = 1, \ldots, p$). By Fuchs's theorem (see [9], Th. 12.1) a singular point of the equation (4.1) is Fuchsian if and only if it is regular. The equation (4.1) is said to be *Fuchsian* if all its singular points are Fuchsian.

Using a standard change

$$y^1 = u, \quad y^2 = \frac{du}{dz}, \quad \ldots, \quad y^p = \frac{d^{p-1}u}{dz^{p-1}}$$

one can go over from the equation (4.1) to a *companion system* (1.1) with coefficient matrix $B(z)$ of the form

$$B(z) = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -b_p & \ldots & \ldots & -b_1 \end{pmatrix}. \tag{4.3}$$

Definition 4.1. — *Let us call two linear differential equations* meromorphically equivalent *in a neighbourhood of a singular point if their companion systems are.*

The Katz rank K_i of the equation (4.1) at a singularity a_i is equal to the Katz rank of the companion system at this point and it is known that

$$\mathrm{ord}_{a_i} b_j(z) \geqslant -j(K_i+1), \qquad j=1,\dots,p \tag{4.4}$$

(see [10], Sect. 3, especially Th. 3.2, Th. 3.3).

Let us now formulate the GRH-problem for scalar linear differential equations as follows.

Let for each $i=1,\dots,n$ a local equation

$$\frac{d^p u}{dz^p} + b_1^i(z)\frac{d^{p-1}u}{dz^{p-1}} + \dots + b_p^i(z)u = 0 \tag{4.5}$$

be given in the neighbourhood O_i of a singular point a_i with Katz rank K_i and such that its monodromy matrix (with respect to a suitable fundamental solution) coincides with G_i (recall that $G_i=\chi(\gamma_i)$ for the given representation (4.2)).

Does there exist a global equation (4.1) with singularities $a_1,\dots,a_n$, prescribed monodromy (4.2) and such that it is meromorphically equivalent to the equation (4.5) in each O_i?

We will again refer to the monodromy representation (4.2) and local equations (4.5) as the generalized monodromy data.

Note that in view of (4.4) coefficients $b_j(z)$ of a global equation solving the GRH-problem have bounded orders of poles.

If a_i is a Fuchsian singularity of the local equation (4.5) for each $i=1,\dots,n$ then a global equation (if it exists) is Fuchsian by Fuchs's theorem (or, if the reader prefers, by the equalities $K_i=0$). Thus, in this case one gets a classical problem of the construction of a Fuchsian equation with prescribed singularities and monodromy. Even in this case the problem has a negative solution in general because for $p>2$, $n>2$, and for $p=2$, $n>3$ the number of parameters determining a Fuchsian equation is less than the number of parameters determining the set of conjugacy classes of representations χ (see [1], pp. 158–159). Therefore, to construct such an equation with given monodromy, one needs so-called apparent singular points. In the case of *irreducible* representations an expression for the smallest possible number of apparent singular points has been obtained by A. Bolibrukh [5]. Estimates for this number in the case of an arbitrary monodromy were presented in [11], as well as similar estimates in the case of non-Fuchsian singularities.

From results of J. Plemelj it follows that the classical Riemann-Hilbert problem (for linear systems) has a positive solution if one at least of the monodromy matrices of the representation (4.2) is diagonalisable (see [1], p. 10 and p. 62). Thus, any monodromy representation can be realized by a Fuchsian system with one apparent singularity (one only needs to consider the representation χ^* obtained from (4.2) by the addition of a singular point a_{n+1} with identity monodromy matrix). In the same way one obtains that each generalized monodromy data can be realized by a global system with prescribed singularities of minimal Poincaré rank and one possible apparent Fuchsian singularity.

Let us consider the representation (4.2) and the local companion systems for the local equations (4.5). For each local system (with Katz rank K_i) consider a meromorphically equivalent one

$$\frac{dy}{dz} = B_i'(z)y \tag{4.6}$$

with minimal Poincaré rank r_i. Recall that r_i is the least integer greater than or equal to the Katz rank K_i, i. e. $r_i = -[-K_i]$, where [ ] stands for the integer part.

We now realize the generalized monodromy data (4.2), (4.6) by a global system with singularities $a_1, \dots, a_n$ of Poincaré ranks $r_1, \dots, r_n$ respectively and apparent Fuchsian singularity a_{n+1}. By Deligne's lemma from [7] (p. 163) this system can be meromorphically transformed (globally) to a system with coefficient matrix $B(z)$ of the form (4.3), where $b_1(z), \dots, b_p(z)$ are meromorphic functions on the Riemann sphere. Besides $a_1, \dots, a_{n+1}$, the transformed system has apparent singularities, the number m of which satisfies the inequality

$$m \leqslant \frac{(R+n+1)p(p-1)}{2},$$

where $R = \sum_{i=1}^{n} r_i$ (this estimate is presented in [11], Lemma 2).

One readily sees that the first component of a solution of the last system with coefficient matrix $B(z)$ of the form (4.3) is a solution of an equation (4.1). By construction this equation has the prescribed singularities $a_1, \dots, a_n$, monodromy (4.2) and it is meromorphically equivalent to the equation (4.5) in each O_i. Furthermore, the number of its apparent singularities is $m+1$ (note that a_{n+1} is also an apparent singularity of the equation with respect to the originally prescribed singular points $a_1, \dots, a_n$). Bearing in mind that $R = -\sum_{i=1}^{n}[-K_i]$, we obtain that

Each generalized monodromy data (4.2), (4.5) can be realized by an equation (4.1) such that the number of its apparent singularities is not greater than

$$\frac{(K+n+1)p(p-1)}{2}+1,$$

where $K=-\sum_{i=1}^{n}[-K_i]$ *and* $[\,]$ *stands for the integer part.*

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