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Local Peak Sets in Weakly Pseudoconvex Boundaries in $\mathbb{C}^n$ (*)

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Abstract. — We give a sufficient condition for a C^ω (resp. C^∞)-totally real, complex-tangential, $(n-1)$-dimensional submanifold in a weakly pseudoconvex boundary of class C^ω (resp. C^∞) to be a local peak set for the class $\mathcal{O}$ (resp. A^∞). Moreover, we give a consequence of it for Catlin's multitype.

Résumé. — On donne une condition suffisante pour qu'une sous variété C^ω (resp. C^∞), totalement réelle, complexe-tangentielle, de dimension $(n-1)$ dans le bord d'un domaine faiblement pseudoconvexe de $\mathbb{C}^n$, soit un ensemble localement pic pour la classe $\mathcal{O}$ (resp. A^∞). De plus, on donne une conséquence de cette condition en terme de multitype de D. Catlin.

1. Introduction and basic definitions

This article is a part of the Ph.D thesis of the author. The $\mathcal{O}$ part was motivated by the paper of Boutet de Monvel and Iordan [B-I] and A^∞ part by the methods of Hakim and Sibony [H-S]. Let D be a domain in $\mathbb{C}^n$ with C^ω (resp. C^∞)-boundary. We denote for an open set $\mathcal{U}$ by $\mathcal{O}$ (resp. A^∞) the class of holomorphic functions on $\mathcal{U}$ (resp. the class of holomorphic functions in $\mathcal{U}$ which have a C^∞-extension to $\overline{\mathcal{U}}$).

We say that $\mathbf{M} \subset bD$ is a local peak set at a point $p \in \mathbf{M}$ for the class $\mathcal{O}$ (resp. A^∞), if there exist a neighborhood $\mathcal{U}$ of p in $\mathbb{C}^n$ and a function

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$f \in \mathcal{O}(\mathcal{U})$ (resp. $A^\infty(D \cap \mathcal{U})$) such that $|f| < 1$ on $(\overline{D} \cap \mathcal{U})\backslash\mathbf{M}$ and $f = 1$ on $\mathbf{M} \cap \mathcal{U}$. Or equivalently, if there exists a function $g \in \mathcal{O}(\mathcal{U})$ (resp. $A^\infty(D \cap \mathcal{U})$) such that $g = 0$ on $\mathbf{M} \cap \mathcal{U}$ and $\Re g < 0$ on $(\overline{D} \cap \mathcal{U})\backslash\mathbf{M}$.

We say that $\mathbf{M} \subset bD$ is a local interpolation set at a point $p \in \mathbf{M}$ for the class A^∞, if there exists a neighborhood $\mathcal{U}$ of p such that each function $f \in C^\infty(\mathbf{M} \cap \mathcal{U})$ is the restriction to $\mathbf{M} \cap \mathcal{U}$ of a function $F \in A^\infty(D \cap \mathcal{U})$. A submanifold $\mathbf{M}$ of bD is complex-tangential if for every $p \in \mathbf{M}$ we have $T_p(\mathbf{M}) \subseteq T_p^{\mathbb{C}}(bD)$, where $T_p^{\mathbb{C}}(bD)$ is the complex tangent space of $T_p(bD)$. If for every $p \in \mathbf{M}$, $T_p(\mathbf{M}) \cap iT_p(\mathbf{M}) = \{0\}$, we say that $\mathbf{M}$ is totally real. Let $\rho : \mathcal{U} \longrightarrow \mathbb{R}$ be a local C^∞ defining function of D, $D \cap \mathcal{U} = \{z \in \mathcal{U}/\rho(z) < 0\}$, $d\rho(p) \neq 0$, where $\mathcal{U}$ is a neighborhood of $p \in bD$. We say D is (Levi) pseudoconvex at p if

$$\mathcal{L}ev\rho(p)[t] = \sum_{1 \leqslant i,j \leqslant n} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j}(p) t_i \bar{t}_j \geqslant 0,$$

for every $t \in T_p^{\mathbb{C}}(bD)$.$\mathcal{L}ev\rho(p)[t]$ is called the Levi form or the complex hessian of ρ.

Let D be Levi pseudoconvex at p. The point p is said to be strongly pseudoconvex if the Levi form is positive definite whenever $t \neq 0$, $t \in T_p^{\mathbb{C}}(bD)$. Otherwise it is said to be weakly pseudoconvex. A domain is called pseudoconvex if its boundary points are pseudoconvex.

We need the following terminology due to L. Hörmander. A function $\phi \in C^\infty(\mathcal{U})$ is almost-holomorphic with respect to a set $E \subset \overline{\mathcal{U}}$ if $\overline{\partial}\phi$ vanishes to infinite order at points of E.

The paper is organized as follows: In §2, we introduce the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. In §3 and §4, we give the equivalent more handy sufficient condition $(\mathcal{H})$ for the existence of local peak set for the class $\mathcal{O}$ and for the class A^∞. In the final section, we give some consequences for the multitype on $\mathbf{M}$ of the sufficient hypotheses.

2. Preliminaries

Let D be a pseudoconvex domain with C^ω (resp. C^∞)-boundary. Let $\mathbf{M}$ be an $(n-1)$ dimensional submanifold of bD which is totally real and complex- tangential in a neighborhood of a point $p \in \mathbf{M}$. Let (V, γ) be a C^ω (resp. C^∞)-parametrization of $\mathbf{M}$ at p, where V is a neighborhood of the origin in $\mathbb{R}^{n-1}$ such that $\gamma(0) = p$. Let $\mathbf{X}$ be a C^ω (resp. C^∞)-vector field on $\mathbf{M}$ such that $\mathbf{X}(p) = 0$. Denote by $\zeta = (\zeta_1, \ldots, \zeta_{n-1})$ the coordinates of a point in V. Then $\mathbf{X}$ can be written as $\mathbf{X} = \sum_i d_i(\zeta)\dfrac{\partial}{\partial \zeta_i}$ where d_i are C^ω (resp. C^∞)-functions on V. We set D_0 the Jacobian matrix at the origin: $\left\{\dfrac{\partial d_i}{\partial \zeta_i}(0)\right\}_{i \leqslant i,j \leqslant n-1}$. Now, we introduce our first hypothesis:

($\mathcal{H}_1$) The matrix D_0 is diagonalizable and has $\widetilde{m}_1 \geqslant \ldots \geqslant \widetilde{m}_{n-1}$ eigenvalues with $\widetilde{m}_i \in \mathbb{N}^*$ for all i.

We say that $\mathbf{M}$ admits a peak-admissible C^ω (resp. C^∞)-vector field $\mathbf{X}$ of weights $(\widetilde{m}_1, \ldots, \widetilde{m}_{n-1})$ at $p \in \mathbf{M}$ for the class $\mathcal{O}$ (resp. A^∞). ($\mathcal{H}_1$) is independent of the choice of the parametrization and the $\widetilde{m}_i$ and their multiplicities are uniquely determined. Using hypothesis ($\mathcal{H}_1$), one can easily prove that there exists a C^ω (resp. C^∞)-change of coordinates on V such that $\mathbf{X} = \sum_i \widetilde{m}_i \zeta_i \dfrac{\partial}{\partial \zeta_i}$. This representation of $\mathbf{X}$ is invariant if we apply a "weight-homogeneous" polynomial transformation of coordinates as below:

Lemma 2.1. — *Let $\Lambda = (\Lambda_1, \ldots, \Lambda_{n-1})$ be a C^ω (resp. C^∞)-change of coordinates on V such that $\Lambda(0) = 0$ and $d\Lambda(\mathbf{X}) = \mathbf{X}$. Then Λ is a polynomial map. More precisely, if $\zeta = (\zeta_1, \ldots, \zeta_{n-1}) \in V$, $I = (i_1, \ldots, i_{n-1}) \in \mathbb{N}^{n-1}$ and we set $|I|_* = \sum_\nu i_\nu \widetilde{m}_\nu$ then for every $1 \leqslant j \leqslant n-1$, $\Lambda_j(\zeta) = \sum_{|I|_* = \widetilde{m}_j} a_I^j \zeta_1^{i_1} \ldots \zeta_{n-1}^{i_{n-1}}$ with $a_I^j \in \mathbb{R}$. Conversely, any Λ of this form preserves $\mathbf{X}$.*

Proof. — The integral curves of $\mathbf{X}$ are $\kappa_\zeta(\lambda) = (\lambda^{\widetilde{m}_1}\zeta_1, \ldots, \lambda^{\widetilde{m}_{n-1}}\zeta_{n-1})$, $\lambda \in \mathbb{R}$. Since $d\Lambda(\mathbf{X}) = \mathbf{X}$, Λ transforms an integral curve passing through ζ to an integral curve passing through $\eta = \Lambda(\zeta)$. So we obtain

$$(\lambda^{\widetilde{m}_1}\Lambda_1(\zeta), \ldots, \lambda^{\widetilde{m}_{n-1}}\Lambda_{n-1}(\zeta)) = (\Lambda_1(\kappa_\zeta(\Lambda)), \ldots, \Lambda_{n-1}(\kappa_\zeta(\lambda))). \tag{2.1}$$

Let $1 \leqslant j \leqslant n-1$ be fixed. We write Λ_j as: $\Lambda_j(\zeta) = \Lambda^*(\zeta) + R(\zeta)$ where $\Lambda^*(\zeta) := \sum_{|I|_* = \widetilde{m}} a^*_{i_1,\ldots,i_{n-1}} \zeta_1^{i_1} \ldots \zeta_{n-1}^{i_{n-1}}$ is non identically zero for a smallest integer $\widetilde{m}$ that satisfies this condition: there exists a constant $C > 0$ such that $|R(\kappa_\zeta(\lambda))| \leqslant C|\lambda|^{\widetilde{m}+1}$. From (2.1), we have

$$\lambda^{\widetilde{m}_j}\Lambda_j(\zeta) = \Lambda_j(\kappa_\zeta(\lambda)) = \lambda^{\widetilde{m}}\Lambda^*(\zeta) + R(\kappa_\zeta(\lambda)). \tag{2.2}$$

Now we divide (2.2) by $\lambda^{\widetilde{m}}$. When λ tends to 0 we obtain $\widetilde{m} = \widetilde{m}_j$ and $\Lambda_j(\zeta) = \Lambda^*(\zeta)$ for all $\zeta \in \mathbb{R}^{n-1}$. □

So let the coordinates be chosen such that $\mathbf{X} = \sum_i \widetilde{m}_i \zeta \frac{\partial}{\partial \zeta_i}$. For $\zeta = (\zeta_1, \ldots, \zeta_{n-1})$, $\eta = (\eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^{n-1}$ and λ, $\mu \in \mathbb{R}$, we set $\sigma := \zeta + i.\eta \in \mathbb{C}^{n-1}$, $\kappa_\zeta(\lambda) := (\lambda^{\widetilde{m}_1}\zeta_1, \ldots, \lambda^{\widetilde{m}_{n-1}}\zeta_{n-1})$ and $\kappa_\sigma(\mu, \lambda) := \kappa_\zeta(\mu) + i.\kappa_\eta(\lambda)$. Let ρ be a local defining function of D at $p \in bD$ and $\widetilde{\gamma} : \widetilde{V} \longrightarrow \widetilde{\theta}(\widetilde{V}) := \widetilde{\mathbf{M}}$ be a holomorphic-extension (resp. almost-holomorphic extension) of the parametrization γ of $\mathbf{M}$. In the C^ω-case $\widetilde{\mathbf{M}}$ is a complexification of $\mathbf{M}$ and $\widetilde{V}$ is an open neighborhood of the origin in $\mathbb{C}^{n-1}$. Let M, $K \in \mathbb{N}^*$ be such that $M \leqslant K$ and $m_j := M/\widetilde{m}_j \in \mathbb{N}^*$, $k_j := K/\widetilde{m}_j \in \mathbb{N}^*$. We set $\mathbf{E} = \{\zeta \in \mathbb{R}^{n-1} / \sum_j \zeta_j^{2m_j} = 1\}$. Now, we introduce our second hypothesis:

($\mathcal{H}_2$) There exist constants $\varepsilon > 0$, $0 < c \leqslant C$ such that for every $\sigma = \zeta + i.\eta \in \mathbf{E} + i.\mathbf{E}$, $|\lambda| < \varepsilon$, $|\mu| < \varepsilon$, we have: $c|\lambda|^{2M}(|\mu| + |\lambda|)^{2(K-M)} \leqslant \rho(\widetilde{\gamma}(\kappa_\sigma(\mu, \lambda))) \leqslant C|\lambda|^{2M}(|\mu| + |\lambda|)^{2(K-M)}$.

Definition 2.2. — *If a C^∞ (resp. C^∞)-vector field $\mathbf{X}$ on $\mathbf{M}$ verifies ($\mathcal{H}_1$) and ($\mathcal{H}_2$) we say that $\mathbf{X}$ is peak-admissible of peak-type $(K, M; \widetilde{m}_1, \ldots, \widetilde{m}_{n-1})$ at $p \in \mathbf{M}$ for the class $\mathcal{O}$ (resp. A^∞).*

Remark 2.3. —

1) The hypothesis ($\mathcal{H}_2$) does not depend neither on the choice of the defining function of the boundary bD nor the choice of the almost-holomorphic extension (see Lemma 4.3 in section 4).

2) The geometric meaning of ($\mathcal{H}_2$) will become clear in inequality ($\mathcal{H}$).

3. A sufficient condition for the existence of local peak set for the class $\mathcal{O}$

Theorem 3.1. — *Let D be a pseudoconvex domain in $\mathbb{C}^n$ with C^ω-boundary. Let $\mathbf{M}$ be an $(n-1)$-dimensional C^ω-submanifold in bD that is totally real and complex-tangential at $p \in \mathbf{M}$. We suppose that $\mathbf{M}$ admits a peak-admissible C^ω-vector field $\mathbf{X}$ of peak-type $(K, M; \widetilde{m}_1, \ldots, \widetilde{m}_{n-1})$ at p for $\mathcal{O}$. Then $\mathbf{M}$ is a local peak set at p for the class $\mathcal{O}$.*

Proof. — The proof is based on Propositions 3.2 and 3.4 below after several holomorphic coordinates changes. Also we allow shrinkings of $\mathbf{M}$. □

Proposition 3.2. — *Let D be a domain in $\mathbb{C}^n$ with C^ω (resp. C^∞)-boundary bD. Let $\mathbf{M}$ be an $(n-1)$-dimensional C^ω-submanifold in bD which* *is totally real and complex-tangential near p. Then there exists a holomorphic change (resp. an almost-holomorphic change) of coordinates* (Z,w) *with* $Z = X + i.Y \in \mathbb{C}^{n-1}$ *and* $w = u + iv \in \mathbb{C}$*, such that p corresponds to the origin and in an open neighborhood* $\mathcal{U}$ *of the origin, we have:*

i) $\mathbf{M} = \{(Z,w) \in \mathcal{U}/Y = w = 0\}$ *. Moreover,* $\mathbf{M}$ *is contained in an n-dimensional totally real submanifold* $\mathbf{N} = \{(Z,w) \in \mathcal{U}/Y = u = 0\}$ *of bD.*

ii) For every $c \in \mathbb{R}$*,* $\mathbf{M}_c = \{(Z,w) \in \mathbf{N}/v = c\}$ *is complex-tangential or empty.*

iii) $D \cap \mathcal{U} = \{(Z,w) \in \mathcal{U}/\rho(Z,w) < 0\}$ *with*

$$\rho(Z,w) = u + A(Z) + vB(Z) + v^2R(Z,v).$$

iv) A and B vanish of order $\geqslant 2$ *when* $Y = 0$*.*

Proof. — We give the proof in the C^ω-case. Let γ be a C^ω-parametrization of $\mathbf{M}$ defined on a neighborhood of the origin in $\mathbb{R}^{n-1}$. After a translation and a rotation of the coordinates in $\mathbb{C}^n$ we may assume that p is the origin and the real tangent space at 0 to bD is $T_0(bD) = \mathbb{C}^{n-1} \times i\mathbb{R}$. We set $L(Z,w) = i\mathbf{n}(Z,w)$ where $\mathbf{n}$ is the vector field of the outer exterior normal to bD. Then, for every $(Z,w) \in bD$, there exists a C^ω-integral curve $l_{(Z,w)}(\lambda) \in bD$ of L satisfying $l_{(Z,w)}(0) = (Z,w)$ and $\dfrac{dl_{(Z,w)}}{d\lambda}(\lambda) = L(l_{(Z,w)}(\lambda))$. Now, we consider the map $\theta : (t,\lambda) \longmapsto l_{\gamma(t)}(\lambda)$. It is clear that θ is a C^ω-diffeomorphism from a neighborhood U of the origin in $\mathbb{R}^n$ into an n-dimensional submanifold $N' := \theta(U)$ of bD which is totally real. By complexification of θ in a neighborhood $\mathcal{W}$ of the origin in $\mathbb{C}^n$, we obtain in the new holomorphic coordinates (Z',w'), $M' = \{(Z',w') \in \mathcal{W}/Y' = w' = 0\}$ and $N' = \{(Z',w') \in \mathcal{W}/Y' = v' = 0\}$. We remark that the system $\{\Sigma_q = T_q(N') \cap T_q^{\mathbb{C}}(bD), q \in \mathcal{W}\}$ is C^ω and involutive. By Frobenius theorem [Bo] the leaves $M'_c = \{(Z',w') \in \mathcal{W} \cap N'/v' = c\}_{c\in\mathbb{R}}$ are complex-tangential to bD. Now, we change coordinates again by defining: $Z = Z'$ and $w = iw'$. We obtain in a neighborhood $\mathcal{U}$ of the origin i) and ii). Representing bD as a graph over $\mathbb{C}^{n-1} \times i\mathbb{R}$, we obtain iii). Since $\mathbf{M} \subset bD$ is complex-tangential A vanishes of order $\geqslant 2$ if $Y = 0$. As $\dfrac{\partial}{\partial v}$ is tangent to $\mathbf{N}$ and the complex gradient $\nabla\rho = (0_{\mathbb{C}_{n-1}}, -1)$ is constant along $\mathbf{N}$, we obtain that B vanishes of order $\geqslant 2$ if $Y = 0$. This achieves iv) and the proposition. □

Let the change of coordinates of Proposition 3.2 for the vector field $\mathbf{X}$ which verifies hypothesis $(\mathcal{H}_2)$ be achieved. Now we show the impact of $(\mathcal{H}_2)$. We set $\kappa := K/M = k_j/m_j \geqslant 1$. Since κ is independent of j, we define in a sufficiently small neighborhood $\mathcal{V}$ of the origin in $\mathbb{C}^{n-1}$ the following pseudo-norms of the $Z = (z_1, \ldots, z_{n-1})$ coordinates of Proposition 3.2: $||Y|| = \left(\sum_j y_j^{2m_j}\right)^{1/2M}$ and $||Z||_* = \left(\sum_j |z_j|^{2k_j}\right)^{1/2K}$. We note that $A(Z) = \rho(\widetilde{\gamma}(\kappa_\sigma(\mu, \lambda)))$ where $Z = X + i.Y = \kappa_\sigma(\mu, \lambda)$. Therefore, from now on we may assume that A verifies:

($\mathcal{H}$) There exist two constants $0 < c \leqslant C$ such that, for every $Z = X + iY \in \mathbb{C}^{n-1}$ near the origin, we have:

$$c||Y||_*^{2M}.||Z||_*^{2K-2M} \leqslant A(Z) \leqslant C||Y||_*^{2M}.||Z||_*^{2K-2M}$$

Remark 3.3. —

1) The proof of Proposition 3.2 remains true in the C^∞-case. We indicate the modification in Lemma 4.2 (section 4).

2) If $Z = (z_1, \ldots, z_{n-1}) \in \mathcal{V}$ where $\mathcal{V}$ is a small open neighborhood of the origin in $\mathbb{C}^{n-1}$, then $\sum_j |z_j|^{2(k_j - m_j)} \approx \left(\sum_j |z_j|^{2m_j}\right)^{\kappa - 1}$. Moreover, we may replace k_j by m_j and K by M in the definition of the pseudo-norm $||Z||_*$.

3) If $K = M = \widetilde{m}_1 = \ldots = \widetilde{m}_{n-1} = 1$, we find the property on A for a strongly pseudoconvex boundary.

Proposition 3.4. — *1) If the real hyperplane $H = \mathbb{C}^{n-1} \times \mathbb{R} = \{(Z, iv)/ Z \in \mathbb{C}^{n-1}, v \in \mathbb{R}\}$ lies outside of D in a neighborhood $\mathcal{U}$ of the origin, then there exists a constant $T > 0$ such that $B^2 \leqslant TA$ near the origin.*

2) If there exists a constant $T > 0$ such that $B^2 \leqslant TA$ near the origin, then there exist a sufficiently small neighborhood $\mathcal{U}$ of the origin and a holomorphic function ψ on $\mathcal{U}$ (resp. an almost-holomorphic function with respect to $\mathbf{N} \cap \mathcal{U}$) which satisfies: $\Re\psi < 0$ on $\overline{D} \cap \mathcal{U}$ if $w \neq 0$ and $\psi = 0$ if $w = 0$. Here $\psi = \dfrac{w}{1 - 2K_1 w}$ with a suitable constant $K_1 > 0$.

Proof. — The proof is elementary. See also [B-I]. □

In order to apply Proposition 3.4 2), we should determine the order of vanishing for certain functions on $\mathbf{M}$ at $p = 0 \in \mathbf{M}$. We begin by defining the Z-weights and the Y-weights for polynomial functions.

Definition 3.5. — *Let* $\chi = a_{I,J} z_1^{i_1} \overline{z}_1^{j_1} \dots z_{n-1}^{i_{n-1}} \overline{z}_{n-1}^{j_{n-1}}$, *with* $a_{I,J} \neq 0$, *be a monomial. We define the* Z*-weight* $\mathcal{P}_Z(\chi)$ *of* χ *as :* $P_Z(\chi) = \sum_\nu \widetilde{m}_\nu (i_\nu + j_\nu)$. *If* $g \not\equiv 0$ *is a polynomial function in* Z *and* $\overline{Z}$ *we define the* Z*-weight of* g *as the smallest* Z*-weight in the decomposition of* g *by monomials. If* g *is a sum of monomials which have the same* Z*-weight* L, *we say that* g *is homogeneous with respect to the* Z*-weight. Let* $X \in \mathbb{R}^{n-1}$ *be fixed and* $\Xi = \alpha_{I,J}(X) y_1^{i_1} \dots y_{n-1}^{i_{n-1}}$, *with* $\alpha_{I,J}(X) \not\equiv 0$, *be a monomial at* Y. *We define the* Y*-weight* $P_Y(\Xi)$ *of* χ *as* $\sum_\nu \widetilde{m}_\nu i_\nu$. *If* $h \not\equiv 0$ *is a polynomial function in* Y *we define the* Y*-weight of* h *to be the smallest* Y*-weight in the decomposition of* h. *If* h *is a sum of monomials which have the same* Y*-weight* L', *we say that* h *is homogeneous with respect to the* Y*-weight of order* L'.

Lemma 3.6. — *Let* R, $S \in \mathbb{N}$, $R \geqslant S$ *and* $F(X,Y) = \sum_{I,J} F_{I,J} Y^I X^J$ *be a* C^ω*-function on an open neighborhood of the origin of* $\mathbb{C}^{n-1}$ *such that, for all multi-indices* $I = (i_1, \dots, i_{n-1})$, $J = (j_1, \dots, j_{n1})$ *in* $\mathbb{N}^{n-1}$, $F_{I,J} = 0$ *or* $\mathcal{P}_Y(F_{I,J} Y^I X^J) \geqslant S$ *and* $\mathcal{P}_Z(F_{I,J} Y^I X^J) \geqslant R \geqslant S$. *Then, there exists a constant* $C > 0$ *such that,* $|F(Z)| \leqslant C ||Y||_*^S . ||Z||_*^{R-S}$, $\forall\, Z = X + i.Y$ *near the origin.*

Proof. — This can be seen by Taylor expansion and standard arguments. □

Lemma 3.7. — *With the notations of Lemma 3.6, if* $S \geqslant M$ *and* $R \geqslant K = \kappa M$, *then* $\dfrac{|F|^2}{A}$ *is uniformly bounded on a sufficiently small neighborhood of the origin.*

Proof. — This follows immediately from Lemma 3.6 and inequality $(\mathcal{H})$. □

In order to know the weights of A and B we analyze the restrictions which are imposed on the functions A and B by the pseudoconvexity of bD. We assume that $B \not\equiv 0$ and we set $(\mathcal{P}_Y(B), \mathcal{P}_Z(B)) = (S, R)$. From $(\mathcal{H})$ we have $(\mathcal{P}_Y(A), \mathcal{P}_Z(A)) = (2M, 2K)$. Next, a simple computation of the Levi form at a point near the origin to bD for $t = \sum_\nu \widetilde{m}_\nu y_\nu \chi_\nu \in T^{\mathbb{C}}(bD)$, with $\chi_\nu = i \left[\dfrac{\partial}{\partial z_\nu} - \dfrac{i}{\eta} \dfrac{\partial \rho}{\partial z_\nu} \dfrac{\partial}{\partial w} \right]$ and $\eta = \dfrac{1}{2} \left(i + B + 2vR + v^2 \dfrac{\partial R}{\partial v} \right)$, gives $\mathcal{L}ev\rho[t] = \mathcal{A}(Z) + v\mathcal{B}(Z) + v^2 \mathcal{R}(v, Z)$, Z varying on $\widetilde{\mathbf{M}}$, the complexification of $\mathbf{M}$. By pseudoconvexity of bD and Proposition 3.4 1) there exists a positive constant $T^* > 0$ such that

$$\mathcal{B} \geqslant T^* \mathcal{A}. \tag{3.1}$$

It remains to study the Z-weight and Y-weight of $\mathcal{A}$ and $\mathcal{B}$ and their relationship with the weights of A and B and finally to show $S \geqslant M$ and $R \geqslant K$. Some necessary auxiliaries results are given in Lemmas 3.8 and 3.9 below. We denote by $\partial^2_{\nu\overline{\mu}}$ the partial derivative $\frac{\partial^2}{\partial z_\nu \partial \overline{z}_\mu}$ and $O_Y(L)$ (resp. $O_Z(L)$) is the set of functions that admit an Y-weight (resp. a Z-weight) $\geqslant L$ ($L \in \mathbb{N}$).

• Suppose that $S < M$.

The expressions of $\mathcal{A}$ and $\mathcal{B}$ are:

$$\begin{aligned}\mathcal{A} &= \sum_{\nu,\mu} \partial^2_{\nu\overline{\mu}} A \widetilde{m}_\nu \widetilde{m}_\mu y_\nu y_\mu + O_Y(2M+1)\\ \mathcal{B} &= \sum_{\nu,\mu} \partial^2_{\nu\overline{\mu}} A \widetilde{m}_\nu \widetilde{m}_\mu y_\nu y_\mu + O_Y(2S).\end{aligned}$$

By Lemma 3.8 $A = A_{2M} + \widetilde{A}$ with $\mathcal{P}_Y(A_{2M}) = 2M$ and every term of $\widetilde{A}$ has an Y-weight $> 2M$. We put $\mathcal{A}_{2M} := \sum_{\nu,\mu} \partial^2_{\nu\overline{\mu}} A_{2M} \widetilde{m}_\nu \widetilde{m}_\mu y_\nu y_\mu$. By Lemma 3.9 we obtain $\mathcal{A}_{2M} \not\equiv 0$ and $\mathcal{P}_Y(\mathcal{A}_{2M}) = 2M$. Similary, we have B $= \mathrm{B}_S + \widetilde{B}_S$ where every term of $\widetilde{B}_S$ has an Y-weight $> 2M$. We put $\mathcal{B}_S := \sum_{\nu,\mu} \partial^2_{\nu\overline{\mu}} B_S \widetilde{m}_\nu \widetilde{m}_\mu y_\nu y_\mu$. We obtain $\mathcal{B}_S \not\equiv 0$ and $\mathcal{P}_Y(\mathcal{B}_S) = S$. Inequality (3.1) becomes:

$$(\mathcal{B}_S + O_Y(S+1))^2 \leqslant T^*(\mathcal{A}_{2M} + O_Y(2M+1)). \tag{3.2}$$

Since $\mathcal{B}_S \not\equiv 0$ there exists $Z_0 = X_0 + i.Y_0$ with $Y_0 = (y_{0,1}, \ldots, y_{0,n-1}) \not\equiv 0$ such that $\mathcal{B}_S(Z_0) \neq 0$. Since every term in the decomposition of $\mathcal{B}_S$ has an Y-weight S, we consider for $\lambda > 0$, $\phi_{Y_0}(\lambda) = (\lambda^{\widetilde{m}_1} y_{0,1}, \ldots, \lambda^{\widetilde{m}_{n-1}} y_{0,n-1})$. Then $\mathcal{B}_S(X_0 + i.\phi_{Y_0}(\lambda))$ becomes an homogeneous polynomial in λ of degree S (i.e. $\mathcal{B}_S(X_0 + i.\phi_{Y_0}(\lambda)) = \lambda^S \mathcal{B}_S(X_0 + i.Y_0)$). Therefore, we obtain $\lim\limits_{\lambda \to 0^+} \dfrac{1}{\lambda^S} \mathcal{B}_S(X_0 + i.\phi_{Y_0}(\lambda)) \neq 0$. Now we replace Z by $X_0 + i.\phi_{Y_0}(\lambda)$ in inequality (3.2) and divide by λ^{2S}. We obtain $\mathcal{B}^2_S(X_0 + i.\phi_{Y_0}) \leqslant 0$ when λ tends to 0^+. So $\mathcal{B}_S(X_0 + i.Y_0) = 0$ which is a contradiction. Thus, $S \geqslant M$.

• The case $R < K$ can be falsified in an analogous way by using Lemma 3.9. Now Lemma 3.7 shows that $\dfrac{|B|^2}{A}$ is uniformly bounded. Then Proposition 3.4 implies the theorem. □

Lemma 3.8. — *Let $X = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ be fixed and $P_X \in \mathbb{R}[y_1, \ldots, y_{n-1}]$ be homogeneous with respect to the Y-weight L. Then we have the following equations:*

1) $\displaystyle\sum_{\nu=1}^{n-1} \frac{\partial P_X}{\partial y_\nu}(y_1,\ldots,y_{n-1})\widetilde{m}_\nu y_\nu = LP_X(y_1,\ldots,y_{n1}).$

2) $\displaystyle\sum_{\nu,\mu} \frac{\partial^2 P_X}{\partial y_\nu \partial y_\mu}(y_1,\ldots,y_{n-1})\widetilde{m}_\nu\widetilde{m}_\mu y_\nu y_\mu + \sum_{\nu=1}^{n-1} \frac{\partial P_X}{\partial y_\nu}(y_1,\ldots,y_{n-1})\widetilde{m}_\nu^2 y_\nu =$
$L^2 P_X(y_1,\ldots,y_{n-1}).$

Proof. — For $1 \leqslant \nu \leqslant n-1$, we set $y_\nu = \widetilde{y}_\nu^{\widetilde{m}_\nu}$. Now, we consider the polynomial Q_X defined by : $Q_X(\widetilde{y}_1,\ldots,\widetilde{y}_{n-1}) = P_X(\widetilde{y}_1^{\widetilde{m}_1},\ldots,\widetilde{y}_{n-1}^{\widetilde{m}_{n-1}})$. Q_X is an homogeneous polynomial at $\widetilde{Y} = (\widetilde{y}_1,\ldots,\widetilde{y}_{n-1})$ in the classic sense, of degree L. Then the result follows from Euler's equation. □

Lemma 3.9. — *If $P_X \not\equiv 0$ is a polynomial in $\mathbb{R}[y_1,\ldots,y_{n-1}]$ not containing neither constant nor linear terms which is homogeneous with respect to the Y-weight $L \geqslant 2$ then* $\displaystyle\sum_{\nu,\mu} \frac{\partial^2 P_X}{\partial y_\nu \partial y_\mu}(y_1,\ldots,y_{n-1})\widetilde{m}_\nu\widetilde{m}_\mu y_\nu y_\mu \not\equiv 0.$

Proof. — Let P_X be a polynomial which depends exactly on $(n-r-1)$-variables, where $0 \leqslant r \leqslant n-2$. By a permutation of variables we may assume that $P_X(y_{r+1},\ldots,y_{n-1}) = \displaystyle\sum_{I=(i_{r+1},\ldots,i_{n-1})} a_I(X) y_{r+1}^{i_{r+1}}\ldots y_{n-1}^{i_n-1}$. We suppose that the assertion of lemma is false. From Lemma 3.8, we have $\displaystyle\sum_{\nu=r+1}^{n-1} \frac{\partial P_X}{\partial y_\nu}\widetilde{m}_\nu^2 y_\nu = L^2 P_X$. Since $\displaystyle\sum_{\nu=r+1}^{n-1} \frac{\partial P_X}{\partial y_\nu}\widetilde{m}_\nu y_\nu = LP_X$ we get, for all $(y_{r+1},\ldots,y_{n-1})$:

$$\sum_{\nu=r+1}^{n-1} \widetilde{m}_\nu(L-\widetilde{m}_\nu)\frac{\partial P_X}{\partial y_\nu}(y_{r+1},\ldots,y_{n-1})y_\nu = 0 \tag{3.3}$$

Now, for every $r+1 \leqslant \nu \leqslant n-1$, we set $\tau_\nu = \widetilde{m}_\nu(L-\widetilde{m}_\nu)$. We have $\tau_\nu > 0$. In fact, let us suppose that $\tau_\mu = 0$ for a μ with $r+1 \leqslant \mu \leqslant n-1$.

For every term of P_X we have: $L = \displaystyle\sum_{\nu=r+1}^{n-1} \widetilde{m}_\nu i_\nu$. Then, two cases are possible for this term:

- $i_\mu = 1$ and $i_\nu = 0$ for all $\nu \neq \mu$.
- $i_\mu = 0$.

Since there are no linear terms, the first case is impossible. So, $i_\mu = 0$ for this term. But, this is also impossible from the choice of variables.

Now we show that P_X vanishes identically. In fact, let $Y \neq 0$ be fixed. We consider $f(\lambda) = P_X(\lambda^{\tau_{r+1}} y_{r+1}, \ldots, \lambda^{\tau_{n-1}} y_{n-1})$, $\lambda > 0$. So, we have:

$$f'(\lambda) = \sum_{j=r+1}^{n-1} \frac{\partial P_X}{\partial y_j}(\lambda^{\tau_{r+1}} y_{r+1}, \ldots, \lambda^{\tau_{n-1}} y_{n-1}) \tau_j \lambda^{\tau_j - 1} y_j.$$

For $r+1 \leqslant j \leqslant n-1$, we set $w_j = \lambda^{\tau_j} y_j$. We get by (3.3):

$$f'(\lambda) = \frac{1}{\lambda} \sum_{j=r+1}^{n-1} \tau_j w_j \frac{\partial P_X}{\partial y_j}(w_{r+1}, \ldots, w_{n-1}) = 0.$$

So, f is constant. As $f(1) = P_X(y_{r+1}, \ldots, y_{n-1}) = \lim\limits_{\lambda \to 0} f(\lambda) = P_X(0) = 0$, P_X vanishes identically. Therefore, we obtain a contradiction. □

4. A sufficient condition for the existence of a local peak sets for the class A^∞

This part was inspired by the article of Hakim and Sibony [H-S]. The following lemma can be shown by standard methods [Na].

Lemma 4.1. — *Let $\widetilde{U}_X$ be a neighborhood of the origin in $\mathbb{R}^n$ and $h : (X,Y) \longmapsto h(X,Y)$ a C^ω-function on $\widetilde{U}_X \times \mathbb{R}^n$. We suppose that h is m-flat where $Y = 0$. Then there exist a neighborhood V_Y of the origin in $\mathbb{R}^n$, a neighborhood $U_X \subset\subset \widetilde{U}_X$ of the origin and a function $g \in C^\infty(U_X \times \mathbb{R}^n)$ which vanishes on $U_X \times V_Y$ and verifies for $\varepsilon > 0$: $||g - h||_m^{U_X} \times \mathbb{R} < \varepsilon$.*

Lemma 4.2. — *Let θ: $\widetilde{U} \longrightarrow \mathbb{C}^n$ be a C^∞-parametrization of the submanifold $\mathbf{N}$ in a neighborhood of the origin in $\mathbb{R}^n$. Then θ has an extension $\widetilde{\theta}$ defined on a neighborhood $\widetilde{\mathcal{U}}$ of the origin in $\mathbb{C}^n$ and which is almost-holomorphic with respect to $\mathbf{N} \cap \widetilde{\mathcal{U}}$.*

Proof. — Let $T_m(X,Y) = \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} D_X^\alpha \theta(X)(iY)^\alpha$ and $U_X \subset\subset \widetilde{U}_X$ be a neighborhood of the origin in $\mathbb{R}^n$. For $k \in \mathbb{N}$ it is clear that $T_{k+1} - T_k$ is k-flat at Y when $Y = 0$. Now we apply the preceding Lemma 4.1 to $T_{k+1} - T_k$.

Then there exist a neighborhood V_Y^k of the origin in $\mathbb{R}^n$ and a C^∞-function $g_k(X,Y)$ which vanishes on $U_X \times V_Y^k$ such that

$$||T_{k+1} - T_k - g_k||_k^{U_X \times \mathbb{R}^n} < 2^{-k}. \tag{4.1}$$

For $m \in \mathbb{N}^*$, we set $\widetilde{T}_m := T_0 + \sum_{k=0}^{m}(T_{k+1} - T_k - g_k) \in C^\infty(U_X \times \mathbb{R}^n)$. By (4.1) $\sum_k (T_{k+1} - T_k - g_k)$ is a normal series for all norms C^l on $U_X \times \mathbb{R}^n$, $l \in \mathbb{N}$. So, the sequence $(\widetilde{T}_m)_m$ converges uniformly to $\widetilde{\theta} \in C^\infty(U_X \times \mathbb{R}^n)$. It is clear that for m and k, $T_m(X,0) = \theta(X)$, $g_k(X,0) = 0$. Hence, $\widetilde{\theta}(X,0) = \lim_{m \to +\infty} \widetilde{T}_m(X,0) = \theta(X)$. So $\widetilde{\theta}$ is an C^∞-extension of θ on $U_X \times \mathbb{R}^n$. That $\widetilde{\theta}$ is almost-holomorphic with respect to $U_X \times \mathbb{R}^n$ can be seen by similar arguments as in [H-S]. □

The following lemma shows that $(\mathcal{H}_2)$ does not depend of the choice of the almost-holomorphic extension.

LEMMA 4.3. — *Let $\widetilde{\gamma} : \widetilde{V} \longrightarrow \mathbb{C}^{n-1}$ be an almost-holomorphic extension of γ with respect to $\widetilde{V} \cap \mathbb{R}^{n-1}$ which satisfies the hypothesis $(\mathcal{H}_2)$ (here γ is the C^∞-parametrization of* **M** *defined in section 2). Let $\widetilde{\phi} : \widetilde{W} \longrightarrow \mathbb{C}^{n-1}$ be an another almost-holomorphic extension of γ with respect to $\widetilde{W} \cap \mathbb{R}^{n-1}$. Then, the hypothesis $(\mathcal{H}_2)$ is satisfied for $\widetilde{\phi}$.*

Proof. — The passage from $\widetilde{\gamma}$ to $\widetilde{\phi}$ is given by the transformation $\widetilde{\psi}$: $\widetilde{W} \longrightarrow \widetilde{V}$ which is almost-holomorphic with respect to $\widetilde{W} \cap \mathbb{R}^{n-1}$. So, we have $\widetilde{\psi}\Big|_{\widetilde{W} \cap \mathbb{R}^{n-1}} = Id$ and $\widetilde{\phi} = \widetilde{\gamma} \circ \widetilde{\psi}$. It is sufficient to prove for every $\sigma \in \widetilde{W}$ and for all $l \in \mathbb{N}$: $|\widetilde{\psi}(\sigma) - \sigma| \lesssim |\Im\sigma|^l$.

Let $\sigma = \zeta + i.\eta$ with $\zeta \in \widetilde{W} \cap \mathbb{R}^{n-1}$ and $l \in \mathbb{N}$ be fixed. Then, we have

$$\widetilde{\psi}(\sigma) = \sum_{|I| \leqslant l} \frac{1}{I!} \frac{\partial^{|I|}\widetilde{\psi}}{\partial \eta^I}(\zeta)\eta^I + O(|\eta|^{l+1}).$$

$\widetilde{\psi}(\sigma) = \zeta + \sum_{1 \leqslant |I| \leqslant l} \frac{1}{I!} \frac{\partial^{|I|}\widetilde{\psi}}{\partial \eta^I}(\zeta)\eta^I + O(|\eta|^{l+1})$. So we can write $\widetilde{\psi}$ as $\widetilde{\psi}(\sigma) = \zeta + \sum_{j=1}^{l} \widetilde{\psi}^{(j)}(\sigma) + O(|\eta|^{l+1})$ with $\widetilde{\psi}^{(j)}(\sigma) = \sum_{|I|=j} \frac{1}{I!} \frac{\partial^j \widetilde{\psi}}{\partial \eta^I}(\zeta)\eta^I$. In particular, we have

$$\widetilde{\psi}(\sigma) = \zeta + \widetilde{\psi}^{(1)}(\sigma) + O(|\eta|^2) = \sum_{i=1}^{n-1} \frac{\partial \widetilde{\psi}}{\partial \eta_i}(\zeta)\eta_i + O(|\eta|^2).$$

Since $\overline{\partial}\widetilde{\psi} = O(|\eta|)$, we have $\delta_{kj} + i\dfrac{\partial \widetilde{\psi}_j}{\partial \eta_k}(\zeta) = O(|\eta|)$, $\forall\, 1 \leqslant k, j \leqslant n-1$. This implies $\widetilde{\psi}^{(1)}(\sigma) = i\eta$. Consequently, $\widetilde{\psi}(\sigma) = \sigma + \displaystyle\sum_{j=2}^{l} \widetilde{\psi}^{(j)}(\sigma) + O(|\eta|^{l+1})$. Let $2 \leqslant j_0 \leqslant l$ be the smallest integer such that $\widetilde{\psi}^{(j_0)}$ is non zero. Then we get: $\widetilde{\psi}(\sigma) = \sigma + \widetilde{\psi}^{(j_0)}(\sigma) + O(|\eta|^{j_0+1})$. Now, $\overline{\partial}\widetilde{\psi} = \overline{\partial}\widetilde{\psi}^{(j_0)} + O(|\eta|^{j_0}) = O(|\eta|^{j_0})$. Thus, for all $1 \leqslant k \leqslant n-1$, we have

$$\frac{\partial \widetilde{\psi}^{(j_0)}}{\partial \overline{\sigma}_k} = -\frac{1}{2i}\left(\frac{\partial \widetilde{\psi}^{(j_0)}}{\partial \eta_k}\right) + O(|\eta|^{j_0}) = O(|\eta|^{j_0}).$$

This implies $\dfrac{\partial \widetilde{\psi}^{(j_0)}}{\partial \eta_k} = O(|\eta|^{j_0})$ for all $1 \leqslant k \leqslant n-1$. As $\dfrac{\partial \widetilde{\psi}^{(j_0)}}{\partial \eta_k}$ is a polynomial with respect to η of degree $(j_0 - 1)$ we get, for all $1 \leqslant k \leqslant n-1$, $\frac{\partial \widetilde{\psi}^{(j_0)}}{\partial \overline{\eta}_k} \equiv 0$. So $\widetilde{\psi}^{(j_0)}$ is independent of η. This contradicts our choice of j_0. Therefore, we obtain $\widetilde{\psi}(\sigma) = \sigma + O(|\eta|^{l+1})$. $\square$

Before stating our theorem for the A^∞-case, we need a condition to guarantee the pseudoconvexity of the boundary under an almost-holomorphic change of coordinates. It is the aim of the following lemma.

Lemma 4.4. — *Suppose that the hypotheses of Proposition 3.2 are fulfilled. We denote by $\widetilde{\psi} : (Z, w) \longmapsto (Z', w')$ the almost-holomorphic change of coordinates. We suppose that there exist two constants $C > 0$ and $L \in \mathbb{N}$ such that, in an open neighborhood $\widetilde{\mathcal{U}}$ of $p \in \mathbf{M}$, we have*

($\mathcal{H}_3$)

$$\mathcal{L}ev\ \rho(q)[t] \leqslant C|t|^2 \mathrm{dist}(q, \mathbf{N})^L,\ \forall\, q \in \widetilde{\mathcal{U}} \cap bD.$$

Then, $D' = \widetilde{\theta}(D \cap \widetilde{\mathcal{U}})$ is a locally pseudoconvex at the origin.

Proof. — We set $N' = \widetilde{\theta}(\mathbf{N})$ and $M' = \widetilde{\theta}(\mathbf{M})$. Since $\widetilde{\theta}$ is a local C^∞-diffeomorphism on an open neighborhood $\widetilde{\mathcal{U}}$ of p, $\widetilde{\theta}$ preserves the distances. In particular, we have: $\mathrm{dist}(q', N') \approx \mathrm{dist}(q, \mathbf{N})$ with $q' = \widetilde{\theta}(q)$ and $q \in \widetilde{\mathcal{U}}$.

Set $\Psi = \widetilde{\theta}^{-1}$, $w = z_n$ and $w' = z'_n$. Since $\widetilde{\theta}$ is an almost-holomorphic change of coordinates the matrix

$$\left\{ \frac{\partial \Psi_i}{\partial z'_j} \right\}_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \quad \text{is nonsingular} \tag{4.2}$$

on a sufficiently small neighborhood of the origin.

For $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} \frac{\partial}{\partial z'_j} &= \sum_{j=1}^{n} \frac{\partial \Psi_j}{\partial z'_i} \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} \frac{\partial \overline{\Psi}_j}{\partial z'_i} \frac{\partial}{\partial \overline{z}_j} \\ &= \sum_{j=1}^{n} \frac{\partial \Psi_j}{\partial z'_i} \frac{\partial}{\partial z_j} + \sum_{j=1}^{n} O\left(\mathrm{dist}(q, \mathbf{N})^{L+1}\right) \frac{\partial}{\partial \overline{z}_j} \end{aligned}$$

The domain D' is defined by $\rho' = \rho \circ \Psi$. Let $t' = (t'_1, \ldots, t'_n) \in T^{\mathbb{C}}_{q'}(bD')$. Thus $\sum_{j=1}^{n} \frac{\partial \rho'(q')}{\partial z'_j} t'_j = 0$. This implies

$$\sum_{i,j=1}^{n} \frac{\partial \rho}{\partial z_i} \frac{\partial \Psi_i}{\partial z'_i} t'_j + O\left(\mathrm{dist}(q, \mathbf{N})^{L+1}\right) = 0.$$

For $1 \leqslant i \leqslant n$ we set $t_i = \sum_{i,j=1}^{n} \frac{\partial \Psi_i}{\partial z'_i} t'_j$.

From (4.2) we get: $\sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i} t_i = O\left(|t'| \mathrm{dist}(q, \mathbf{N})^{L+1}\right) = O\left(|t| \mathrm{dist}(q, \mathbf{N})^{L+1}\right)$.
Now we decompose t into tangential component $t^{\mathcal{H}}$ and a normal component $t^{\mathcal{N}}$. So, $t = t^{\mathcal{H}} + t^{\mathcal{N}}$ with $t^{\mathcal{H}} \in T^{\mathbb{C}}_q(bD)$, $t^{\mathcal{N}} \perp T^{\mathbb{C}}_q(bD)$ and $|t^{\mathcal{H}}| + |t^{\mathcal{N}}| \leqslant 2|t|$. Moreover, $t^{\mathcal{N}} = \kappa(q)\mathbf{n}(q)$ with $\kappa(q) \in \mathbb{C}$ and, for all $1 \leqslant i \leqslant n$, we have $t^{\mathcal{N}}_i = \kappa(q)\frac{\partial \rho(q)}{\partial \overline{z}_i}$. This implies

$$\begin{aligned} \kappa(q) \sum_{i=1}^{n} \left| \frac{\partial \rho(q)}{\partial z_i} \right|^2 &= \sum_{i=1}^{n} \frac{\partial \rho(q)}{\partial z_i} \kappa(q) \frac{\partial \rho(q)}{\partial \overline{z}_i} \\ &= \sum_{i=1}^{n} \frac{\partial \rho(q)}{\partial z_i} t^{\mathcal{N}}_i = \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i} t_i \\ &= O\left(|t| dist(q, \mathbf{N})^{L+1}\right). \end{aligned}$$

Consequently,

$$|t^{\mathcal{N}}| = |\kappa(q)| = O\left(|t| \mathrm{dist}(q, \mathbf{N})^{L+1}\right). \tag{4.3}$$

Now, we compute the Levi form of ρ'. As

$$\frac{\partial \rho'(q')}{\partial z'_i} = \sum_{i=1}^{n} \frac{\partial \rho(q)}{\partial z_\kappa} \frac{\partial \Psi_\kappa(q')}{\partial z'_i} + O\left(\text{dist}(q, \mathbf{N})^{L+1}\right)$$

and by replacing L by $L+1$, we get

$$\frac{\partial^2 \rho'(q')}{\partial z'_i \partial \overline{z'}_j} = \sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial \overline{z}_l} \frac{\partial \Psi_k(q')}{\partial z'_i} \overline{\frac{\partial \Psi_l(q')}{\partial z'_j}} + O\left(\text{dist}(q, \mathbf{N})^{L+1}\right)$$

By (4.3) it follows that

$$\begin{aligned}
\sum_{i,j=1}^{n} \frac{\partial^2 \rho'(q')}{\partial z'_i \partial \overline{z'_j}} t'_i \overline{t'_j} &= \sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial \overline{z_l}} \left(\sum_{i=1}^{n} \frac{\partial \Psi_k(q')}{\partial z'_i} t'_i \right) \left(\sum_{j=1}^{n} \overline{\frac{\partial \Psi_l(q')}{\partial z'_j} t'_j} \right) \\
&+ O\left(dist(q, \mathbf{N})^{L+1}\right) \\
&= \sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial \overline{z}_l} t_i^{\mathcal{H}} \overline{t_l^{\mathcal{H}}} + O\left(|t|^2 dist(q, \mathbf{N})^{L+1}\right).
\end{aligned}$$

From $(\mathcal{H}_3)$ and (4.3) we get:

$$\begin{aligned}
\sum_{k,l=1}^{n} \frac{\partial^2 \rho(q)}{\partial z_k \partial \overline{z}_l} t_i^{\mathcal{H}} \overline{t_l^{\mathcal{H}}} &\geqslant C|t^{\mathcal{H}}|^2 dist(q, \mathbf{N})^L \\
&\geqslant C|t|^2 dist(q, \mathbf{N})^L + O\left(|t|^2 dist(q, \mathbf{N})^{L+1}\right).
\end{aligned}$$

Thus there exists a constant $C' > 0$ such that $\mathcal{L}ev\ \rho'(q')[t'] \geqslant C'|t|^2$ $\text{dist}(q, N)^L$. This means that D' is a locally pseudoconvex at the origin. □

Definition 4.5. — *Let F be a C^∞-function on a neighborhood $\mathcal{V}$ of the origin in $\mathbb{C}^{n-1}$. We say that F has Y-weight $\mathcal{P}_Y(F) \geqslant S$ ($S \in \mathbb{N}$) if there exists a constant $C > 0$ such that $|F(X,Y)| \leqslant C||Y||_*^S$, $\forall\, Z = X + i.Y \in \mathcal{V}$. Also, we say that F has Z-weight $\mathcal{P}_Z(F) \geqslant R \geqslant S$ ($R \in \mathbb{N}$) if there exists a constant $c > 0$ such that $|F(X,Y)| \leqslant c||Z||_*^R$, $\forall\, Z = X + i.Y \in \mathcal{V}$.*

In the sequel we have to take into account the following obvious assertions.

Remark 4.6. —

1) Let F be a polynomial function with respect to Y. Then $\mathcal{P}_Y(F) \geqslant S \Longleftrightarrow F(X,Y) = \sum_{I=(i_1,\ldots,i_{n-1})} F_I(X) Y^I$ with $\sum_{\nu=1}^{n-1} \widetilde{m}_\nu i_\nu \geqslant S$.

2) Let F be a polynomial function with respect to X and Y. Then $\mathcal{P}_Z(F) \geqslant R \Longleftrightarrow F(X,Y) = \sum_{\substack{I=(i_1,\dots,i_{n-1})\\ J=(j_1,\dots,j_{n-1})}} F_{I,J} X^J Y^I$ with $\sum_{\nu=1}^{n-1} \widetilde{m}_\nu(i_\nu + j_\nu) \geqslant R$.

3) If $||Y|| < 1$ then there exists a constant $a > 0$ such that $||Y|| \leqslant a||Y||_*$.

Now, we give a version of Lemma 3.6 in the C^∞-case. Its proof is similar.

LEMMA 4.7. — *Let $R, S \in \mathbb{N}$, $R \geqslant S$ and F be a C^∞-function on an open sufficiently small neighborhood $\mathcal{V}$ of the origin in $\mathbb{C}^{n-1}$. We suppose that F has Y -weight $\mathcal{P}_Y(F) \geqslant S$ and Z-weight $\mathcal{P}_Z(F) \geqslant R$. Then, there exists a constant $C > 0$ such that: $|F(Z)| \leqslant C||Y||_*^S.||Z||_*^{R-S}$, $\forall\, Z = X + i.Y \in \mathcal{V}$.*

THEOREM 4.8. — *Let D be a pseudoconvex domain in $\mathbb{C}^n$ with C^∞-boundary. Let $\mathbf{M}$ be an $(n-1)$-dimensional submanifold of bD which is totally real and complex-tangential in a neighborhood $\mathcal{U}$ of $p \in \mathbf{M}$. We suppose*

- *There exist two positives constants C and L such that*

$(\mathcal{H}_3')$

$$\mathcal{L}ev\ \rho(q)[t] \geqslant C|t|^2 \mathrm{dist}(q, M)^L,\ \forall\, q \in \mathcal{U} \cap bD,\ \forall t \in T_q^{\mathbb{C}}(bD).$$

- *$\mathbf{M}$ admits a peak-admissible C^∞-vector field X of peak-type $(K, M; \widetilde{m}_1, \dots, \widetilde{m}_{n-1})$ at p for A^∞.*

Then,

i) $\mathbf{M}$ is a local peak set at p for the class A^∞.

i) $\mathbf{M}$ is a local interpolation set at p for the class A^∞.

Proof. — i) After an almost-analytic change of coordinates we obtain the following properties: The point $p \in \mathbf{M}$ corresponds to the origin and in an open neighborhood of the origin, we have $M' = \widetilde{\theta}(M) = \{(Z', w')/Y' = w' = 0\}$, $D' = \widetilde{\theta}(D)$ has $\rho'(Z', w') = u' + A(Z') + v'B(Z') + v'^2 R(Z', v')$ as local defining function at the origin. Moreover, M' is locally contained in an n-dimensional submanifold $N' = \{(Z', w')/Y' = 0$ and $u' = 0\}$ of bD' which is totally real. By Lemma 4.4, the condition $(\mathcal{H}_3')$ garantees that D' is a locally pseudoconvex at the origin. Moreover, the hypothesis on $\mathbf{M}$ implies:

($\mathcal{H}$) There exist two constants $0 < c_1' \leqslant c'2$ such that, for every $Z' = X' + i.Y' \in \mathbb{C}^{n-1}$ near the origin, we have:

$$c_1'||Y'||_*^{2M}.||Z'||_*^{2K-2M} \leqslant A(Z') \leqslant c_2'||Y'||_*^{2M}.||Z'||_*^{2K-2M}.$$

From ($\mathcal{H}$) and Lemma 4.7 we get $\dfrac{|B|^2}{A}$ is uniformly bounded in a sufficiently small neighborhood of the origin in $\mathbb{C}^{n-1}$. By Proposition 3.4, there exists an almost-holomorphic function with respect to $N' \cap \mathcal{U}'$, $\widetilde{\psi}(w') = \dfrac{w'}{1-2K_1w'}$ defined on an open neighborhood $\mathcal{U}'$ of the origin in $\mathbb{C}^n$ such that: $\Re\widetilde{\psi} < 0$ on $\overline{D'} \cap \mathcal{U}'$ if $w' \neq 0$ and $\widetilde{\psi} = 0$ if $w' = 0$.

As $|\widetilde{\psi}(w')| \lesssim |w'|$, we have for every $(Z', w') \in \overline{D'} \cap \mathcal{U}'$,

$$\begin{aligned} A(Z') &= \rho'(Z', w') - v'B(Z') - v'^2R(Z', v') - u' \\ &\leqslant -v'B(Z') - v'^2R(Z', v') - u' \lesssim |u'| + |v'| \lesssim |w'|. \end{aligned}$$

Moreover, if $\mathcal{U}'$ is sufficiently small we get:

$$\mathrm{dist}((Z', w'), M') \lesssim ||Y'|| + |w'|. \tag{4.4}$$

Since $||Y'||_*^{2M}||Z'||_*^{2(K-M)} \lesssim A(Z') \lesssim |w'|$ and $||Y'||_* \leqslant ||Z'||_*$ we have $||Y'||_*^{2K} \lesssim |w'|$. By Remark 4.6 inequality (4.4) gives: For every $(Z', w') \in \overline{D'} \cap \mathcal{U}'$: $\mathrm{dist}((Z', w'), M') \lesssim |w'|^{1/2K}$. This has two consequences:

a) $\overline{\partial}' \left(\dfrac{1}{\widetilde{\psi}}\right)$ has a C^∞-extension on $\mathcal{U}' \cap \overline{D'}$.

b) If $F \in C^\infty(\mathcal{U}' \cap D')$ is an almost-holomorphic function with respect to $N' \cap \mathcal{U}'$ then $\dfrac{1}{\widetilde{\psi}}\overline{\partial}' F$ has a C^∞-extension on $\mathcal{U}' \cap \overline{D'}$.

(Here $\overline{\partial}'$ denotes the $\overline{\partial}$-operator on D'. Set $\widetilde{\Psi} := \widetilde{\theta}^{-1}$. If $f' \in C^\infty(\mathcal{U}' \cap D')$ then $\overline{\partial}' f' = \widetilde{\Psi}^*(\overline{\partial}(f' \circ \widetilde{\theta}))$ where $\widetilde{\Psi}^*$ is the pull-back of $\widetilde{\Psi}$).

Proof. —

a) On $\mathcal{U}' \cap D'$ we have $\overline{\partial}' \left(\dfrac{1}{\widetilde{\psi}}\right) = -\left(\dfrac{1-2K_1w'}{w'}\right)^2 \overline{\partial}'\widetilde{\psi}$. As $\widetilde{\psi}$ is an almost-holomorphic function with respect to $N' \cap \mathcal{U}'$ we get for all $L \in \mathbb{N}^*$ and $(Z', w') \in \mathcal{U}' \cap \overline{D'}$,

$$|\overline{\partial}'\widetilde{\psi}(w')| \lesssim \mathrm{dist}((Z', w'), N')^L \lesssim \mathrm{dist}((Z', w'), M')^L \lesssim |w'|^{L/2K}. \tag{4.5}$$

b) With an analogous reasoning, we have for every $(Z',w') \in \mathcal{U}' \cap \overline{D'}$ and for all $L \in \mathbb{N}^*$, $|\overline{\partial}' F(Z',w')| \lesssim \text{dist}((Z',w'),M')^L \lesssim |w'|^{L/2K}$. By (4.5) we see that the $(0,1)$-form $\overline{\partial}'\left(\frac{1}{\tilde{\psi}}\right)$ has a $\overline{\partial}'$-closed C^∞-extension on $\mathcal{U}' \cap \overline{D'}$. We set $\psi = \tilde{\psi} \circ \tilde{\theta}$ and get that $\overline{\partial}\left(\frac{1}{\psi}\right)$ is a $\overline{\partial}$-closed $(0,1)$-form of class C^∞ on $\mathcal{U} \cap \overline{D}$.

Let $0 < \varepsilon \ll 1$ be such that $\overline{B(0,\varepsilon) \subset \mathcal{U}}$ and $bB(0,\varepsilon) \cap bD$ be a transversal intersection. Due to Corollary 2 in [Mi] there exists a function $g \in C^\infty(\overline{B(0,\varepsilon) \cap D})$ such that $\overline{\partial} g = \overline{\partial}\left(\frac{1}{\psi}\right)$ on $\overline{B(0,\varepsilon) \cap D}$. Adding a constant, we may assume that $\Re g > 0$. If ε is sufficiently small, we get $|g\psi| \leqslant \frac{1}{2}$ on $\overline{B(0,\varepsilon) \cap D}$. Now we consider $h = \frac{\psi}{1 - g\psi}$. It is clear that $h \in C^\infty(\overline{B(0,\varepsilon) \cap D})$. As $\overline{\partial} h = -\frac{1}{\left(\frac{1}{\psi} - g\right)^2}\overline{\partial}\left(\frac{1}{\psi} - g\right) = 0$ on $B(0,\varepsilon) \cap D$ we obtain $h \in A^\infty(B(0,\varepsilon) \cap D)$. Moreover, $\psi|_{\mathbf{M}} = 0$ so $h|_{\mathbf{M}} = 0$. For every $(Z,w) \in \overline{B(0,\varepsilon) \cap D} \backslash \mathbf{M}$ we have $\Re h = \Re\left(\frac{1}{\frac{1}{\psi} - g}\right) = \frac{\frac{\Re\psi}{|\psi|^2} - \Re\overline{g}}{\left|\frac{1}{\psi} - g\right|^2} < 0.$

Thus, $\mathbf{M}$ is a local peak set at p for the class A^∞. $\square$

ii) Using the notations as above, let $F \in C^\infty(\overline{\mathbf{M} \cap B(0,\varepsilon_1)})$ with $0 < \varepsilon_1 \leqslant \varepsilon$. Let $\widetilde{F}$ be an almost-holomorphic extension of F on $B(0,\varepsilon_2)$ with respect to $\mathbf{N} \cap B(0,\varepsilon_2)(\varepsilon_2 \leqslant \varepsilon_1$. By b) the $(0,1)$-form $\frac{1}{\psi}\overline{\partial}\widetilde{F}$ has a C^∞-extension on $\overline{B(0,\varepsilon_2) \cap D}$. Since $\frac{1}{h} = (1 - g\psi)\frac{1}{\psi}$, $\frac{1}{h}\overline{\partial}\widetilde{F}$ is $\overline{\partial}$-closed on $B(0,\varepsilon_2) \cap D$. Moreover, $\frac{1}{h}\overline{\partial}\widetilde{F}$ has a C^∞-extension on $\overline{B(0,\varepsilon_2) \cap D}$.

Let $0 < \varepsilon_3 \leqslant \varepsilon_2$ be such that $bB(0,\varepsilon_3) \cap bD$ is a transversal intersection. By Corollary 2 of [Mi] there exists a function $G \in C^\infty(\overline{B(0,\varepsilon_3) \cap D})$ such that $\overline{\partial} G = \frac{1}{h}\overline{\partial}\widetilde{F}$ on $\overline{B(0,\varepsilon_3) \cap D}$. Now we set $f = \widetilde{F} - hG$ on $\overline{B(0,\varepsilon_3) \cap D}$.

It is clear that $f \in C^\infty(\overline{B(0,\varepsilon_3) \cap D})$. Moreover, we have $f\Big|_{\mathbf{M} \cap \overline{B(0,\varepsilon_3)}} = \widetilde{F}\Big|_{\mathbf{M} \cap \overline{B(0,\varepsilon_3)}} = F$ and $\overline{\partial} f = \overline{\partial}\widetilde{F} - h\overline{\partial} G = 0$. The theorem is completely proved. $\square$

5. Some implications from the sufficient hypotheses for the multitype

We want to interpret the sufficient hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ in terms of Catlin's multitype. In this section we first recall various concepts of types and we give the multitype for the points on the submanifold **M**.

Let D be a bounded pseudoconvex in $\mathbb{C}^n$ with C^∞-boundary. Let ρ be a local defining function at a point $p \in bD$. The variety (1-)type $\Delta_1(bD, p)$ (or $\Delta_1(p)$ if no confusion can occur), introduced by D'Angelo [DA], is defined as

$$\Delta_1(bD, p) := \sup_z \left\{ \frac{\nu(z^*\rho)}{\nu(z-p)} \right\},$$

where the supremum is taken over all germs of nontrivial one-dimensional complex curves $z : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^n, p)$ with $z(0) = p$. Here, $\nu(f)$ denotes the vanishing order of the function f at 0 and $z^*\rho \equiv \rho \circ z$.

More generally, one can define the q-type, $\Delta q(bD, p)$ [DA], $1 \leqslant q \leqslant n$,

$$\Delta_q(bD, p) := \inf_z \Delta_1(bD \cap S, p).$$

Here S runs over all $(n-q+1)$-dimensional complex hyperplanes passing through p, and $\Delta_1(bD \cap S, p)$ denotes the 1-type of the domain $D \cap S$ (considered as a domain in S) at p. Note that the q-types are biholomorphic invariants [DA], [Ca].

Next we recall the definition of Catlin's multitype. Let Γ_n denote the set of all n-tuples of numbers $\mu = (\mu_1, \ldots, \mu_n)$ with $1 \leqslant \mu_i \leqslant \infty$ such that

(i) $\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n$;

(ii) For each k, either $\mu_k = \infty$ or there is a set of nonnegative numbers $a_1, \ldots, a_k$, with $a_k > 0$ such that $\displaystyle\sum_{j=1}^{k} a_j/\mu_j = 1$.

An element of Γ_n will be referred to as a weight. The set of weights can be ordered lexicographically, i.e. , if $\mu' = (\mu'_1, \ldots, \mu'_n)$ and $\mu'' = (\mu''_1, \ldots, \mu''_n)$, then $\mu' < \mu''$ if for some k, $\mu'_j = \mu''_j$ for all $j < k$, but $\mu'_k < \mu''_k$. A weight $\mu \in \Gamma_n$ is said to be distinguished if there exist holomorphic coordinates $(z_1, \ldots, z_n)$ about p, with p mapped to the origin, such that

$$\text{If } \sum_i \frac{\alpha_i + \beta_i}{\mu_i} < 1, \text{ then } D^\alpha \overline{D}^\beta \rho(p) = 0. \tag{5.1}$$

Here D^α and $\overline{D}^\beta$ denote the partial differential operators:

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}\dots\partial z_n^{\alpha_n}} \text{ and } \frac{\partial^{|\beta|}}{\partial \overline{z}_1^{\beta_1}\dots\partial \overline{z}_n^{\beta_n}}, \text{ respectively.}$$

Definition 5.1. — *The multitype $\mathcal{M}(bD,p)$ (or $\mathcal{M}(p)$) is defined to be the least weight $\mathcal{M}$ in Γ_n (smallest in the lexicographic sense) such that $\mathcal{M} \geqslant \mu$ for every distinguished weight μ.*

We call a weight μ linearly distinguished if there exist a complex linear change of coordinates about p with p mapped to the origin and such that in the new coordinates (5.1) holds. The linear multitype $\mathcal{L}(bD,p)$ is defined to be the smallest weight $\mathcal{L} = (l_1,\dots,l_n)$ such that $\mathcal{L} \geqslant \mu$ for every linearly distinguished weight μ.

Clearly $\mathcal{L}(bD,p)$ is invariant under linear change of coordinates and we have $\mathcal{L}(bD,p) \leqslant \mathcal{M}(bD,p)$. It is easy to see that the first component of $\mathcal{M}(p)$ is always 1.

Let us $\Delta(p) := (\Delta_n(p),\dots,\Delta_1(p))$ where $\Delta_q(p)$ stands for the q-type. Let the multitype of p be $\mathcal{M}(p) = (\mu_1,\dots,\mu_n)$. By the main theorem (property 4) in [Ca] it is always true that $\mathcal{M}(p) \leqslant \Delta(p)$ in the sense that $\mu_{n-q+1} \leqslant \Delta_q(p)$, for all $q = 1,\dots,n$.

Theorem 5.2. — *Let D be a pseudoconvex domain in $\mathbb{C}^n$ with C^ω-boundary. Let $\mathbf{M}$ be an $(n-1)$-dimensional submanifold of bD which is totally real and complex-tangential in a neighborhood $\mathcal{U}$ of $p \in \mathbf{M}$. We suppose that $\mathbf{M}$ admits a peak-admissible C^ω-vector field $\mathbf{X}$ of peak-type $(K,M;\widetilde{m}_1,\dots,\widetilde{m}_{n-1})$ at p for the class $\mathcal{O}$. Then*

(i) $\mathcal{M}(p) = \Delta(p) = (1,2k_1,\dots,2k_{n-1})$.

(ii) $\mathcal{M}(p') = \Delta(p') = (1,2m_1,\dots,2m_{n-1})$ *for* $p' \in \mathbf{M}\cap\mathcal{U} - \{p\}$.

Here, $m_j = M/\widetilde{m}_j$, $k_j = K/\widetilde{m}_j$ *for all* $1 \leqslant j \leqslant n-1$.

Remark 5.3. — An analogous result holds true in the A^∞-case.

Proof. — i) From Proposition 3.2 we know that there exists a holomorphic coordinates change (denoted θ) such that the point $p \in \mathbf{M}$ corresponds to the origin and in an open neighborhood of the origin in $\mathbb{C}^n$, the defining function ρ' of the boundary of $D' = \theta(D)$ is $\rho' = u' + A + v'B + v'^2R$. By hypothesis inequality $(\mathcal{H})$ holds in the new coordinates. So, we may identify the complexification $\widetilde{\mathbf{M}} = \mathbf{M} + i.\mathbf{M}$ of $\mathbf{M}$ to $\mathbb{C}^{n-1} = T_0^{\mathbb{C}}(bD')$ and we may assume that $\rho' |_{\mathbf{M}} \equiv A$ in a sufficiently small neighborhood of the origin in $\mathbb{C}^{n-1}$. Let $Z'_0 = X'_0 + i.Y'_0 \neq 0$ near the origin in $\mathbb{C}^{n-1}$ be fixed. We consider $f(\lambda) = A(\lambda Z'_0)$, $\lambda \in [0,1]$. We set $m = \max_{1 \leqslant i \leqslant n-1} m_i$, $m' = \min_{1 \leqslant i \leqslant n-1} m_i$ and $\kappa = K/M \geqslant 1$. As

$$f(\lambda = \left(\sum_{i=1}^{n-1} \lambda^{2m_i} y'^{2m_i}_{0,i} \right) \left(\sum_{i=1}^{n-1} \lambda^{2m_i} \left(x'^{2m_i}_{0,i} + y'^{2m_i}_{0,i} \right) \right)^{\kappa - 1},$$

we have $\lambda^{2m\kappa} f(1) \lesssim f(\lambda) \lesssim \lambda^{2m'\kappa} f(1)$. Therefore, we obtain

$$\frac{f(1)}{2m\kappa + 1} \lesssim \int_0^1 f(\lambda)\, d\,\lambda \lesssim \frac{f(1)}{2m'\kappa + 1}.$$

By Remark 4 in [B-S], the 1-type of bD' at 0 is equal to line type in the new system of coordinates. This means that $\Delta_1(bD', 0) = \sup_{v \in \mathbb{C}^n, |v|=1} (\rho' \circ \ell_v)$, where ℓ_v: $\zeta \longmapsto \zeta.v$ is a complex line passing through the origin and having v as direction. Inequality $(\mathcal{H})$ implies $\Delta_1(bD', 0) = 2k_{n-1}$. Now we prove that $\Delta(bD', 0) = (1, 2k_1, \ldots, 2k_{n-1})$ is a linearly distinguished weight at 0. Let F: $Z = (z_1, \ldots, z_n) \longmapsto (z_n, z_1, z_2, \ldots, z_{n-1})$ be a $\mathbb{C}$-linear change of coordinates. We set $\widetilde{Z} = (\widetilde{z}_1, \widetilde{Z}') = F(Z)$ with $\widetilde{Z}' = (\widetilde{z}_2, \ldots, \widetilde{z}_n)$ and $\widetilde{\rho} = \rho' \circ F^{-1}$. As $\widetilde{\rho}(\widetilde{Z}) = \Re(\widetilde{z}_1) + A(\widetilde{Z}') + (\Im\widetilde{z}_1)B(\widetilde{Z}') + (\Im\widetilde{z}_1)^2 R(\widetilde{Z}', \Im\widetilde{z}_1)$, $\dfrac{\partial\widetilde{\rho}}{\partial\widetilde{z}_1}(0) \neq 0$ because $\dfrac{\partial\rho'}{\partial z_n}(0) \neq 0$. This implies that $\alpha_1 = \beta_1 = 0$ for the property (5.1). Thus it is sufficient to verify that:

$$\sum_{i=2}^{n} \frac{\alpha_i + \beta_i}{2k_{i-1}} < 1 \quad \text{implies} \quad D^\alpha \overline{D}^\beta A(0) = 0.$$

In fact, let $\alpha = (\alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_2, \ldots, \beta_n) \in \mathbb{N}^{n-1}$ be such that $\displaystyle\sum_{\nu=2}^{n} \frac{\alpha_\nu + \beta_\nu}{2k_{\nu-1}} < 1$. Then, $\displaystyle\sum_{\nu=2}^{n} \widetilde{m}_{\nu-1}(\alpha_\nu + \beta_\nu) < 2k$. Since A is C^ω on a sufficiently small neighborhood of the origin in $\mathbb{C}^{n-1}$, $A(X,Y) = \displaystyle\sum_{\substack{I=(i_2,\ldots,i_n)\\ J=(j_2\ldots,j_n)}} A_{I,J} X^J Y^I$ with $X = (x_2, \ldots, x_n)$ and $Y = (y_2, \ldots, y_n)$. We know that the Z-weight of A is $\geqslant 2K$. By Remark 4.6, we have $\displaystyle\sum_{\nu=2}^{n} \widetilde{m}_\nu(i_\nu + j_\nu) \geqslant 2K$. Thus,

$$\mathcal{P}_Z(D^\alpha \overline{D}^\beta A) \geqslant \sum_{\nu=2}^{n} \widetilde{m}_{\nu-1}(i_\nu + j_\nu) - \sum_{\nu=2}^{n} \widetilde{m}_{\nu-1}(\alpha_\nu + \beta_\nu) > 0.$$

We obtain $D^\alpha \overline{D}^\beta A(0) = 0$. Therefore $\Delta(bD', 0)$ is linearly distinguished and $\Delta(bD', 0) \leqslant \mathcal{M}(bD', 0)$.

It remains to show that $\mathcal{M}(bD', 0) \leqslant \Delta(bD', 0)$. Setting $\mathcal{M}(bD', 0) = (\mu_1, \dots, \mu_n)$, by property 4 of Catlin in [Ca] we have $\mu_{n+1-q} \leqslant \Delta_q(bD', 0)$ for all $q = 1, \dots, n$.

It is sufficient to prove that $\Delta_q(bD', 0) = 2k_{n-q}$ for all $1 \leqslant q \leqslant n-1$.

- For $q = 1$, we have already shown that $\Delta_1(bD', 0) = 2k_{n-q}$.
- Let $2 \leqslant q \leqslant n-1$ be fixed. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{C}^n$ with $T_0^{\mathbb{C}}(bD') = Span_{\mathbb{C}}\{e_1, \dots, e_{n-1}\}$. Consider $V_q = Span_{\mathbb{C}}\{e_{n-q}, \dots, e_{n-1}\}$ and S an $(n-q+1)$-dimensional complex hyperplane in $\mathbb{C}^n$.

As

$$\begin{aligned} \dim(V_q \cap S) &= \dim V_q + \dim S - \dim(V_q + S) \\ &\geqslant q + n - q + 1 - n = 1, \end{aligned}$$

it follows that there exists a complex line ℓ in $S \cap V_q$ that has order of contact $\geqslant 2k_{n-q}$ with the boundary bD' at 0. Therefore $\Delta_q(bD', 0) = 2k_{n-q}$. Moreover, if we set $\widetilde{S} = Span_{\mathbb{C}}\{e_1, \dots, e_{n-q}, e_n\}$ then $\widetilde{S} \cap V_q = Span_{\mathbb{C}}\{e_{n-q}\}$. So $\Delta_1(\widetilde{S} \cap bD', 0) = 2k_{n-q}$. We therefore obtain $\mathcal{M}(bD', 0) \leqslant \Delta(bD', 0) = (1, 2k_1, \dots, 2k_{n-1})$. With $\Delta(bD', 0) = (1, 2k_1, \dots, 2k_{n-1}) \leqslant \mathcal{M}(bD', 0)$, we find i).

ii) Let $p' \in \mathbf{M} \cap \mathcal{U} - \{p\}$. We work with the preceding system of coordinates and we set $\theta(p') = \widetilde{p}' \neq 0$. $\widetilde{p}'$ is a boundary point of bD' near the origin such that $\Re(\widetilde{p}') \neq 0$. Let $Z_0' = X_0' + i.Y_0' \in \mathbb{C}^{n-1}$ be fixed such $Y_0' \neq 0$. We consider $f(\lambda) = A(\lambda Z_0' + \widetilde{p}')$, $\lambda \in [0,1]$. In this case, there exist two constants $0 < c_1 \leqslant c_2$ which depend only of $\widetilde{p}'$ satisfying:

$$c_1 \sum_{i=1}^{n-1} \lambda^{2m_i} y_{0,i}^{'2m_i} \lesssim f(\lambda) \lesssim c_2 \sum_{i=1}^{n-1} \lambda^{2m_i} y_{0,i}^{'2m_i}.$$

Hence, $\lambda^{2m} f(1) \lesssim f(\lambda) \lesssim f(1)\lambda^{2m'}$. We obtain

$$\frac{f(1)}{2m+1} \lesssim \int_0^1 f(\lambda)\, d\lambda \lesssim \frac{f(1)}{2m'+1}.$$

with constants that depend only of $\widetilde{p}'$. By Remark 4 in [B-S] the 1-type of $\widetilde{p}'$ is equal to line type. So, $\Delta_1(bD', \widetilde{p}') = 2m_{n-1}$. In the same way as before one shows that $\Delta(\widetilde{p}') = (1, 2m_1, \ldots, 2m_{n-1})$ is linearly distinguished weight. Next, we proceed analogously as i) we obtain the equality and ii) holds. $\square$

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