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OULD AHMED IZID BIH ISSELKOU

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The Lane-Emden Function and Nonlinear Eigenvalues Problems

Isselkou Ould Ahmed Izid Bih[(1)]

Résumé. — Nous considérons un problème aux valeurs propres, semi-linéaire elliptique, sur une boule de $\mathbb{R}^n$ et montrons que ces valeurs et fonctions propres peuvent s'obtenir à partir de la fonction de Lane-Emden.

Abstract. — We consider a semilinear elliptic eigenvalues problem on a ball of $\mathbb{R}^n$ and show that all the eigenfunctions and eigenvalues, can be obtained from the Lane-Emden function.

1. Introduction

We consider the problem

$$(P_\lambda^\alpha)\begin{cases} \Delta u + \lambda(1+u)^\alpha = 0, \ in\ B_1 \\ u > 0, \ in\ B_1 \\ u = 0, \ on\ \partial B_1 \end{cases}$$

where B_1 is the unit ball of $\mathbb{R}^n$, $n \geqslant 3$, $\lambda > 0$ and $\alpha > 1$.

This problem arises in many physical models like the nonlinear heat generation and the theory of gravitational equilibrium of polytropic stars(cf. [2] and [11]). It is well known (cf. [2], [10], [12]) that there exists a critical constant $\lambda^*(\alpha)$, such that (P_λ^α) admits, at least, one solution if $0 < \lambda < \lambda^*(\alpha)$ and no solution if $\lambda > \lambda^*(\alpha)$. We deal here with these critical constants and the corresponding eigenfunctions.

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(1) Faculté des Sciences et Techniques, B.P. 5026 Nouakchott, Mauritanie.
isselkou@univ-nkc.mr

Let ϕ be the Lane-Emden function(cf. [1], [5], [6],[15]) in the n-dimensional space and r_0 the first "zero" of ϕ, we show that

$$\lambda^*(\alpha) = \max_{r\in[0,r_0[} r^2\phi^{\alpha-1}(r).$$

We use this formula to compute $\lambda^*(\alpha)$, when α is the Critical Sobolev Exponent. We also extend, to the subcritical case, an estimate of $\lambda^*(\alpha)$ given in [10] and show qualitative properties of the eigenfunctions.
In the Appendix, we show how to approximate ϕ, so one can use numerical approaches (Maple or Matlab) to get estimates of $\lambda^*(\alpha)$.

2. Scalings of the Lane-Emden function as solutions

When $0 < \lambda \leqslant \lambda^*(\alpha)$, it is known that any regular solution of (P_λ^α) is radial and the minimal one is stable and analytical (cf.[8], [12]).

Proposition 2.1. — *Let u be a regular solution of (P_λ^α), then*

$$u(r) = (1+u(0))\phi\left(\sqrt{\lambda}(1+u(0))^{\frac{\alpha-1}{2}}r\right) - 1, \ \forall\ r \in [0,1]$$

where ϕ is the Lane-Emden function, in the n-dimensional space.

Proof. — The Lane-Emden function(cf. [1], [5], [6], [15]) is the solution of

$$(L-E)\begin{cases} \phi''(r) + \frac{n-1}{r}\phi'(r) + \phi(r)|\phi(r)|^{\alpha-1} = 0, \\ \phi(0) = 1,\ \phi'(0) = 0. \end{cases}$$

The proof of the proposition is quite immediate.

3. The Subcritical Case

Let us consider the problem (P_λ^α), with $1 < \alpha < \frac{n+2}{n-2}$. Let ϕ be the Lane-Emden function.

Proposition 3.1. — *There exists $r_0 > 0$, such that $\phi(r_0) = 0$, $\phi(r) > 0$, $\forall r \in [0, r_0[$ and*

$$\lambda^*(\alpha) = \max_{\rho\in[0,r_0]} \rho^2\phi^{\alpha-1}(\rho).$$

We also have

$$\lambda^*(\alpha) \geqslant \frac{2}{(\alpha-1)^2}\left(\alpha(n-2)-n\right),\ if\ \frac{n}{n-2} < \alpha < \frac{n+2}{n-2}.$$

Proof. — As $\phi(0) > 0$, we infer that $\phi > 0$, on a maximal interval $[0, r_0[$. The problem

$$\begin{cases} \Delta u + u^\alpha = 0, \; in \; \mathbb{R}^n \\ u > 0, \; in \; \mathbb{R}^n \end{cases}$$

does not admit a solution (cf.[4]), so we infer that $r_0 < \infty$ and $\phi(r_0) = 0$.

Let us put

$$\psi_\rho(r) = \frac{\phi(\rho r) - \phi(\rho)}{\phi(\rho)}, \; \forall \; r \in [0, 1],$$

with $0 < \rho < r_0$, then ψ_ρ is a solution of (P^α_λ), with $\lambda = \rho^2 \phi^{\alpha-1}(\rho)$. We infer that

$$\max_{\rho \in [0, r_0]} \rho^2 \phi^{\alpha-1}(\rho) \leqslant \lambda^*(\alpha).$$

Let us suppose that

$$\max_{\rho \in [0, r_0]} \rho^2 \phi^{\alpha-1}(\rho) < \lambda^*(\alpha),$$

if $u_{\lambda^*(\alpha)}$ is the unique solution of $(P^\alpha_{\lambda^*(\alpha)})(cf.[10])$, one can use Proposition 1 to show that

$$u_{\lambda^*(\alpha)}(r) = \left(1 + u_{\lambda^*(\alpha)}(0)\right) \left(\phi\left((\lambda^*(\alpha))^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha-1}{2}} r\right) - \frac{1}{1 + u_{\lambda^*(\alpha)}(0)}\right).$$

Let us put $\rho_{\lambda^*(\alpha)} = (\lambda^*(\alpha))^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha-1}{2}}$. As $u_{\lambda^*(\alpha)} \geqslant 0$, we infer that $\rho_{\lambda^*(\alpha)} < r_0$. As $u_{\lambda^*(\alpha)}(1) = 0$, we infer that

$$\frac{1}{1 + u_{\lambda^*(\alpha)}(0)} = \phi\left((\lambda^*(\alpha))^{\frac{1}{2}} \left(1 + u_{\lambda^*(\alpha)}(0)\right)^{\frac{\alpha-1}{2}}\right).$$

So we get

$$u_{\lambda^*(\alpha)}(r) = \frac{\phi(\rho_{\lambda^*(\alpha)} r) - \phi(\rho_{\lambda^*(\alpha)})}{\phi(\rho_{\lambda^*(\alpha)})} \; and \; \lambda^*(\alpha) = \left(\rho_{\lambda^*(\alpha)}\right)^2 \phi^{\alpha-1}(\rho_{\lambda^*(\alpha)}).$$

The last equality leads to a contradiction.

To prove the last statement, we use the fact that the maximum here is achieved at a unique r_α (see the next lemma). So we get

$$\phi'(r_\alpha) = -\frac{2}{(\alpha-1) r_\alpha} \phi(r_\alpha), \text{ and}$$

$$\phi^{\alpha-3}(r_\alpha) \left(2\phi^2(r_\alpha) + 4 r_\alpha(\alpha-1)\phi(r_\alpha)\phi'(r_\alpha) + (\alpha-1) r_\alpha^2 \left((\alpha-2)\left(\phi'(r_\alpha)\right)^2 + \phi(r_\alpha)\phi''(r_\alpha)\right)\right) \leqslant 0.$$

We first replace $\phi''(r_\alpha)$ by its value from $(L-E)$ and then $\phi'(r_\alpha)$, from the previous equality, to get

$$\phi^{\alpha-1}(r_\alpha)\left(-(\alpha-1)\lambda^*(\alpha)+2(n-4)+4\frac{\alpha-2}{\alpha-1}\right)\leqslant 0.$$

Simplifying, one gets the estimate.

Remark 3.2. — The last statement in Proposition 2 is also true for $\alpha \geqslant \frac{n+2}{n-2}$, with the same proof, provided that $\sup_{r\in\mathbb{R}_+} r^2\phi^{\alpha-1}(r)$ is attained (see the next Proposition 6); this has been proved in [10], using sophisticated arguments.

Lemma 3.3. — *Let us put* $g(r)=r^2\phi^{\alpha-1}(r),\ r\in[0,r_0]$, *there exists* $\rho_0\in]0,r_0[$ *such that* g *is increasing on* $[0,\rho_0]$ *and decreasing on* $[\rho_0,r_0]$.

Proof. — Let ρ be an arbitrary positive constant with $\rho<r_0$, then, as we have already mentioned ψ_ρ is a solution of $\left(P^\alpha_\gamma\right)$, where $\gamma=g(\rho)$. As $g'(r)=r\phi^{\alpha-2}(r)\left(2\phi(r)+(\alpha-1)r\phi'(r)\right)$, we infer that g is increasing on a maximal interval $I_0\subset[0,r_0]$ with $0\in I_0$.

Using Proposition 2, there exists $\rho_0\in]0,r_0[$, such that $g(\rho_0)=\max_{r\in[0,r_0]}g(r)=\lambda^*(\alpha)$. This ρ_0 is unique, otherwise, if there exists $\lambda\in[0,r_0]$, such that $g(\lambda)=\max_{r\in[0,r_0]}g(r)=\lambda^*(\alpha)$, then ψ_{ρ_0} and ψ_λ are both solutions of the problem $\left(P^\alpha_{\lambda^*(\alpha)}\right)$. As ϕ is decreasing on $[0,r_0]$, we infer that $\psi_{\rho_0}(0)=\frac{1-\phi(\rho_0)}{\phi(\rho_0)}\neq\frac{1-\phi(\lambda)}{\phi(\lambda)}=\psi_\lambda(0)$. So we get two different solutions of the problem $\left(P^\alpha_{\lambda^*(\alpha)}\right)$. This leads to a contradiction (cf. [10]). As $g(r_0)=0$, we infer that $I_0\neq[0,r_0]$. Let us put $\delta=\sup I_0$. The function g can't be constant on a nontrivial interval $J\subset[\delta,r_0]$, for if $g(r)=c\ in\ J$, then for every $\lambda\in J$, ψ_λ is a solution of (P^α_c). As $\psi_{\lambda_1}(0)\neq\psi_{\lambda_2}(0)$, if $\lambda_1,\ \lambda_2\ \in J$ and $\lambda_1\neq\lambda_2$, we infer that the problem (P^α_c) admits an infinity of solutions. This leads again to a contradiction (cf. [10]).

So if g is not decreasing on $[\delta,r_0]$, then there exists β_1 and β_2 with $r_0>\beta_2>\beta_1>\delta$, such that g is decreasing on $[\delta,\beta_1]$ and increasing on $[\beta_1,\beta_2]$. Let us put $c_0=min(g(\delta),g(\beta_2))$, then $c_0>g(\beta_1)$. Let us choose $c\in]g(\beta_1),c_0[$, so the problem $g(t)=c$ admits at least three different solutions $\lambda_i\in]0,\beta_2[,\ 1\leqslant i\leqslant 3$. As $\psi_{\lambda_i}(0)\neq\psi_{\lambda_j}(0)$, if $i\neq j,\ 1\leqslant i,j\leqslant 3$, we obtain three solutions for the problem (P^α_c). So we get a contradiction.

We conclude that g is increasing on $[0,\delta]$, decreasing on $[\delta,r_0]$ and $\delta=\rho_0$.

Proposition 3.4. — *If* $\lambda=\lambda^*(\alpha)$, *there exists a unique* $\rho_{\lambda^*(\alpha)}\in]0,r_0[$, *such that*

$\lambda^*(\alpha) = \left(\rho_{\lambda^*(\alpha)}\right)^2 \phi^{\alpha-1}(\rho_{\lambda^*(\alpha)})$ and the unique solution $u_{\lambda^*(\alpha)}$ of $(P^\alpha_{\lambda^*(\alpha)})$ is

$$u_{\lambda^*(\alpha)}(r) = \frac{\phi(\rho_{\lambda^*(\alpha)}r) - \phi(\rho_{\lambda^*(\alpha)})}{\phi(\rho_{\lambda^*(\alpha)})} = \psi_{\rho_{\lambda^*(\alpha)}}(r),\ \forall\ r \in [0,1].$$

When $0 < \lambda < \lambda^*(\alpha)$, there exist exactly two constants r_λ and ρ_λ, such that $0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0$, $\lambda = r_\lambda^2\phi^{\alpha-1}(r_\lambda) = \rho_\lambda^2\phi^{\alpha-1}(\rho_\lambda)$ and the only two solutions of (P^α_λ) are

$$u_\lambda = \psi_{r_\lambda},\ v_\lambda = \psi_{\rho_\lambda};$$

the minimal one(cf.[2]) is u_λ, $\lim_{\lambda\to 0} u_\lambda = 0$ in $C^0\left(\overline{B_1}\right)$ and $\lim_{\lambda\to 0} v_\lambda(r) = \infty,\ \forall\ r \in [0,1[$.

Proof. — Using Proposition 2 and Lemma 1, one infers that the only solution of $(P^\alpha_{\lambda^*(\alpha)})$ is ψ_{ρ_0}. We put $\rho_{\lambda^*(\alpha)} = \rho_0$. If $0 < \lambda < \lambda^*(\alpha)$, using the lemma again, we infer that $g(t) = \lambda$ admits exactly two solutions r_λ and ρ_λ, with $0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0$. Let us put $u_\lambda = \psi_{r_\lambda}$ and $v_\lambda = \psi_{\rho_\lambda}$, $u_\lambda(0) \neq v_\lambda(0)$. These two functions u_λ and v_λ are solutions of the the problem (P^α_λ), which admits only two ones (cf. [10]).

As ϕ is decreasing on $[0, r_0]$, one can verify that $u_\lambda(0) < v_\lambda(0)$, so we infer that the minimal solution (cf.[2]) is u_λ.

As $\lambda = r_\lambda^2\phi^{\alpha-1}(r_\lambda) = \rho_\lambda^2\phi^{\alpha-1}(\rho_\lambda)$, $0 < r_\lambda < \rho_{\lambda^*(\alpha)} < \rho_\lambda < r_0$, we get $\lim_{\lambda\to 0} r_\lambda = 0$, $\lim_{\lambda\to 0}\rho_\lambda = r_0$, $\lim_{\lambda\to 0} u_\lambda(r) = \lim_{r_\lambda\to 0}\frac{\phi(r_\lambda r)}{\phi(r_\lambda)} - 1 = 0$, *and* $lim_{\lambda\to 0} v_\lambda(r) = \lim_{\rho_\lambda\to r_0}\frac{\phi(\rho_\lambda r) - \phi(\rho_\lambda)}{\phi(\rho_\lambda)} = \phi(r_0 r)\left(lim_{\rho_\lambda\to r_0}\frac{1}{\phi(\rho_\lambda)}\right) = \infty,\ \forall\ r \in\ [0,1[$.

4. The Critical Sobolev Exponent Case

In this section, we suppose that $\alpha = \frac{n+2}{n-2}$ and $n \geqslant 3$.

Let us consider the following problem

$$(P^\alpha)\begin{cases} \Delta u + u^\alpha = 0,\ in\ \mathbb{R}^n \\ u > 0,\ in\ \mathbb{R}^n. \end{cases}$$

Remark 4.1. — Every radially symmetrical solution of (P^α) verifies $\lim_{r\to\infty} u(r) = 0$ (cf. [9]).

Following the method of Pohozaev in [14], the problem

$$(Q^\alpha)\begin{cases} u"(r) + \frac{n-1}{r}u'(r) + u^\alpha(r) = 0,\ \forall\ r > 0 \\ u > 0,\ u(0) = 1,\ u'(0) = 0 \end{cases}$$

admits a solution ϕ.

Lemma 4.2. — *Let u be a radially symmetrical regular solution of (P^α), then*

$$u(r) = u(0)\phi\left(u(0)^{\frac{\alpha-1}{2}}r\right).$$

Proof. — This proof is immediate.

Lemma 4.3. — *Let us put $g(r) = r^2\phi^{\alpha-1}(r),\ \ r \in \mathbb{R}_+$, then there exists $r_0 > 0$, such that g is increasing on $[0, r_0]$, decreasing on $[r_0, \infty[$, with $\lim_{r\to\infty} g(r) = 0$.*

Proof. — As we have already mentioned, g is increasing near 0. Let us assume that g is nondecreasing on $[0, \infty[$, then we have two possibilities

$$\lim_{r\to\infty} g(r) = \infty \ \textit{or} \ \lim_{r\to\infty} g(r) = c,\ 0 < c < \infty.$$

For every $\rho > 0$, ψ_ρ is a solution of (P^α_γ), with $\gamma = \rho^2\phi^{\alpha-1}(\rho) = g(\rho)$. We infer (cf. [2], [10]) that $g(r) \leqslant \lambda^*(\alpha),\ \ \forall\ r > 0$, so the first limit becomes impossible.

In the second case, we have two subcases: c is achieved or not.

If c is not achieved, then $\forall\ l$ such that $0 < l < c$, there exists $r_l > 0$ such that $g(r_l) = l$. One can verify that $\forall\ 0 < l < c$, the problem (P^α_l) admits the solution ψ_{r_l}, so we infer that $c \leqslant \lambda^*(\alpha)$. Let u be a radially symmetrical solution (cf. [2], [10] and [3]) of (P^α_c). As in the proof of Proposition 2, one can verify that

$$u = \psi_\rho,\ \ \rho = \sqrt{c}\,(1 + u(0))^{\frac{\alpha-1}{2}} \ \textit{and}\ \frac{1}{1 + u(0)} = \phi(\rho).$$

As $c = \rho^2\phi^{\alpha-1}(\rho) = g(\rho)$, we get a contradiction.

Let us suppose that c is achieved, as g is assumed to be nondecreasing, there exists r_0 such that $g(r) = c,\ \forall\ r \geqslant r_0$. Let us choose, an arbitrary constant $\rho > 0$ such that $\rho \geqslant r_0$. The function ψ_ρ is a solution of the problem $\left(P^\alpha_\gamma\right)$, where $\gamma = \rho^2\phi^{\alpha-1}(\rho) = g(\rho) = c,\ \forall\ \rho \geqslant r_0$. This means that this problem, with such a γ, admits an infinity of solutions ψ_ρ; this leads to a contradiction (cf. [2], [10]). So g is not nondecreasing on $[0, \infty[$. As g can't be constant on a nontrivial interval, we deduce that there exists positive constants r_1 and r_2, such that $r_1 < r_2$, with g is increasing on $[0, r_1]$ and decreasing on a maximal interval $[r_1, r_2[$. Let us suppose that g increases again on $[r_2, r_3]$, with $r_2 < r_3$. If $\gamma \in \,]g(r_2), min\,(g(r_1), g(r_3))\,[$, then $g(r) = \gamma$ admits, at least, three roots, so the problem $\left(P^\alpha_\gamma\right)$ admits, at least, three solutions; this gives again a contradiction (cf. [10]).

Finally, we get the existence of $r_0 > 0$, such that g is increasing on $[0, r_0]$ and decreasing on $[r_0, \infty[$. As $g > 0$, we infer that $\lim_{r\to\infty} g(r) = c_0 \geqslant 0$. If $c_0 > 0$, then for every $c \in]0, c_0[$, there exists a unique $\rho_c \in \mathbb{R}_+$, verifying $g(\rho_c) = c$. As $c < \lambda^*(\alpha)$, the problem (P_c^α) admits exactly two solutions (cf. [10]). One of these two solutions is ψ_{ρ_c}. Let u_c be the other one, then, using Proposition 2 again, we get

$$u_c(r) = \psi_\gamma,\ \gamma = c^{\frac{1}{2}}\left(1 + u_c(0)\right)^{\frac{\alpha-1}{2}} = c^{\frac{1}{2}}\phi^{\frac{1-\alpha}{2}}\ \left(c^{\frac{1}{2}}\left(1 + u_c(0)\right)^{\frac{\alpha-1}{2}}\right).$$

So we infer that $c = g(\gamma)$. As the two solutions are different, $\rho_c \neq \gamma$ and γ is another root of $g(r) = c$. This gives a contradiction and proves that necessarily $c = 0$. This ends the proof of the lemma.

Proposition 4.4. — *Let us assume* $\alpha = \frac{n+2}{n-2},\ n \geqslant 3$, *then*

$$\lambda^*(\alpha) = \max_{r\in\ ]0,\infty[} g(r).$$

Proof. — Let $\gamma = g(\rho) = \rho^2\phi^{\alpha-1}(\rho), \rho \in \mathbb{R}_+^*$, we have seen that ψ_ρ is a solution of $\left(P_\gamma^\alpha\right)$. So we infer that $g(\rho) \leqslant \lambda^*(\alpha),\ \forall\ \rho\ \in \mathbb{R}_+$.

Let us suppose that

$$\max_{r\in\ ]0,\infty[} g(r) < \lambda^*(\alpha)$$

and let u be the unique solution (cf. [10]) of $(P_{\lambda^*(\alpha)}^\alpha)$. As in the proof of Proposition 2, we get that $u = \psi_\rho$ and $\lambda^*(\alpha) = g(\rho)$. This gives a contradiction.

Proposition 4.5. — *We have* $\lambda^*(\alpha) = \frac{n(n-2)}{4}$. *There exists a unique* $r_{\lambda^*(\alpha)} = \sqrt{n(n-2)}$, *such that* $\lambda^*(\alpha) = r_{\lambda^*(\alpha)}^2\phi^{\alpha-1}(r_{\lambda^*(\alpha)})$ *and a unique solution of* $(P_{\lambda^*(\alpha)}^\alpha)$

$$u_{\lambda^*(\alpha)} = \psi_{r_{\lambda^*(\alpha)}}.$$

If $0 < \lambda < \lambda^*(\alpha)$, *there exist exactly two constants*

$$r_\lambda = \frac{\sqrt{1 - \frac{2\lambda}{n(n-2)} - \sqrt{1 - \frac{4\lambda}{n(n-2)}}}}{\left(n(n-2)\right)^{-1}\sqrt{2\lambda}}\ \textit{and}\ \rho_\lambda = \frac{\sqrt{1 - \frac{2\lambda}{n(n-2)} + \sqrt{1 - \frac{4\lambda}{n(n-2)}}}}{\left(n(n-2)\right)^{-1}\sqrt{2\lambda}}$$

such that $0 < r_\lambda < r_{\lambda^*(\alpha)} < \rho_\lambda$, $\lambda = g(r_\lambda) = g(\rho_\lambda)$ *and the only two solutions of* (P_λ^α) *are*

$$u_\lambda = \psi_{r_\lambda}\ \textit{and}\ v_\lambda = \psi_{\rho_\lambda},$$

the minimal one (cf. [2]) is u_λ; $\lim_{\lambda\to 0} u_\lambda = 0$, *in* $C^0(\overline{B_1})$ *and* $\lim_{\lambda\to 0} v_\lambda(r) = r^{2-n} - 1,\ \forall\ r \in]0, 1]$.

Proof. — One can use Lemma 3 to get the existence (and the uniqueness) of $r_{\lambda^*(\alpha)} = r_0$, r_λ and ρ_λ. It is then easy to verify that $\psi_{r_{\lambda^*(\alpha)}}$ is a solution of $(P^\alpha_{\lambda^*(\alpha)})$, $u_\lambda = \psi_{r_\lambda}$ and $v_\lambda = \psi_{\rho_\lambda}$ are solutions of (P^α_λ). The problem (P^α_λ) admits only two solutions (cf. [10]), as ϕ is decreasing on $\mathbb{R}^*_+$, one can verify that $u_\lambda(0) < v_\lambda(0)$, so $u_\lambda \neq v_\lambda$. We conclude that u_λ and v_λ are the only solutions of (P^α_λ) and the minimal one (cf. [2]) is u_λ.

Let us compute the constants $r_{\lambda^*(\alpha)}$, r_λ and ρ_λ.

It is well known (cf. [13]) that, if $\alpha = \frac{n+2}{n-2}$, the problem (Q^α) admits the continuum of spherically symmetrical "instantons"

$$u_\gamma(r) = \gamma^{\frac{n-2}{2}} \left(n(n-2)\right)^{\frac{n-2}{4}} \left(\gamma^2 + r^2\right)^{\frac{2-n}{2}}, \; \gamma > 0.$$

Let us fix $\gamma > 0$, so $u_\gamma(0) = \gamma^{\frac{2-n}{2}} \left(n(n-2)\right)^{\frac{n-2}{4}}$. Using Lemma 2, we get the expression of the Lane-Emden function

$$\phi(r) = \frac{1}{u_\gamma(0)} u_\gamma \left(u_\gamma(0)^{\frac{-2}{n-2}} r\right) = \left(1 + \frac{r^2}{n(n-2)}\right)^{\frac{2-n}{2}}.$$

As $\alpha - 1 = \frac{n+2}{n-2} - 1 = \frac{4}{n-2}$, we infer that

$$g(r) = r^2 \phi^{\alpha-1}(r) = r^2 \left(1 + \frac{r^2}{n(n-2)}\right)^{-2}.$$

Using Proposition 4, a direct calculation gives

$$\lambda^*(\alpha) = \max_{r>0} r^2 \left(1 + \frac{r^2}{n(n-2)}\right)^{-2}$$

$$= r^2 \left(1 + \frac{r^2}{n(n-2)}\right)^{-2}_{|_{r = r_{\lambda^*(\alpha)} = \sqrt{n(n-2)}}} = \frac{n(n-2)}{4}.$$

In [7], the previous constant has been computed, using the Pohozaev Identity. If $0 < \lambda < \lambda^*(\alpha)$, the equation $g(r) = \lambda$ admits two positive roots

$$r_\lambda = \frac{\sqrt{1 - \frac{2\lambda}{n(n-2)} - \sqrt{1 - \frac{4\lambda}{n(n-2)}}}}{\left(n(n-2)\right)^{-1}\sqrt{2\lambda}} \; \textit{and} \; \rho_\lambda = \frac{\sqrt{1 - \frac{2\lambda}{n(n-2)} + \sqrt{1 - \frac{4\lambda}{n(n-2)}}}}{\left(n(n-2)\right)^{-1}\sqrt{2\lambda}}.$$

This gives us $u_\lambda = \psi_{r_\lambda}$ and $v_\lambda = \psi_{\rho_\lambda}$; as $r_\lambda < \rho_\lambda$, we get $u_\lambda(0) < v_\lambda(0)$, so u_λ is the minimal solution.

As $\lambda = r_\lambda^2 \phi^{\alpha-1}(r_\lambda) = \rho_\lambda^2 \phi^{\alpha-1}(\rho_\lambda), 0 < r_\lambda < r_{\lambda^*(\alpha)} < \rho_\lambda < \infty$, one can verify that
$\lim_{\lambda\to 0} r_\lambda = 0,\ \lim_{\lambda\to 0}\rho_\lambda = \infty,\ \lim_{\lambda\to 0} u_\lambda = 0,\ \textit{in } C^0\left(\overline{B_1}\right)\ \textit{and}$
$\lim_{\lambda\to 0} v_\lambda(0) = \lim_{\rho_\lambda\to\infty}\frac{\phi(\rho_\lambda r)}{\phi(\rho_\lambda)} - 1 = r^{2-n} - 1,\ \forall\ r \in]0,1]$.

5. The Supercritical Case

We consider here the case $\alpha > \frac{n+2}{n-2},\ n \geqslant 3$. Let us put

$$f(\alpha) = \frac{4\alpha}{\alpha - 1} + 4\sqrt{\frac{\alpha}{\alpha-1}},\ \forall \alpha > 1.$$

Let's first detail a condition, $f(\alpha) > n-2$, used in [10].

Lemma 5.1. — *If* $\left(3 \leqslant n \leqslant 10 \textit{ and } \alpha > \frac{n+2}{n-2}\right)$
or $\left(n > 10 \textit{ and } \frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4}\right)$,
then $f(\alpha) > n-2$. *If* $n > 10$ *and* $\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leqslant \alpha$, *then* $f(\alpha) \leqslant n-2$.

Proof. — Let us put $p(t) = 4t^2 + 4t$ and $u = \sqrt{\frac{\alpha}{\alpha-1}}$, so we get $f(\alpha) = p(u)$. The only positive root of $p(t) = n-2$, is $t_0 = \frac{\sqrt{n-1}-1}{2}$ and the equation $u = \frac{\sqrt{n-1}-1}{2}$ has the only solution $\alpha_0 = \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4}$. But $\alpha_0 > 0$, if and only if $n > 10$.

For every $\alpha > \frac{n+2}{n-2}$, we have $\alpha > 1$ so we get $\sqrt{\frac{\alpha}{\alpha-1}} > 1 > \frac{\sqrt{n-1}-1}{2}$, if $3 \leqslant n \leqslant 10$. We infer that $f(\alpha) > n-2$, if $3 \leqslant n \leqslant 10$.

If $n > 10$, we have $\alpha_0 > \frac{n+2}{n-2} > 1$, one can verify that if $\frac{n+2}{n-2} < \alpha < \alpha_0$, then $f(\alpha) > n-2$ and $f(\alpha) \leqslant n-2$, if $\alpha \geqslant \alpha_0$.

Proposition 5.2. — *Let us put* $\lambda_s = \frac{2}{(\alpha-1)^2}\left(\alpha(n-2) - n\right)$.
If $\left(3 \leqslant n \leqslant 10 \textit{ and } \frac{n+2}{n-2} < \alpha\right)$ *or* $\left(n > 10 \textit{ and } \frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4}\right)$ *then*

$$\lambda^*(\alpha) = \max_{\mathbb{R}_+^*} g(r),\ \lambda^*(\alpha) > \lambda_s \textit{ and } \phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}},\ \textit{as } r \to \infty.$$

If (ρ_i) *is an increasing sequence of positive reals, such that* (ψ_{ρ_i}) *are solutions of* $(P^\alpha_{\lambda_s})$ *and* $\lim_{i\to\infty}\rho_i = \infty$, *then* $\lim_{i\to\infty}\psi_{\rho_i}(r) = \lambda_s^{\frac{1}{\alpha-1}}(r^{\frac{2}{1-\alpha}} - 1),\ \forall\ r \in]0,1]$.

If $n > 10$ *and* $\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leqslant \alpha$ *then*

$$\lambda^*(\alpha) = \sup_{\mathbb{R}_+^*} g(r) = \lambda_s \text{ and } \phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}}, \text{ as } r \to \infty.$$

If (λ_i) *is an increasing positive sequence such that* $\lim_{i\to\infty} \lambda_i = \lambda_s$ *and* $\forall i$, w_i *is the unique solution of* $(P^\alpha_{\lambda_i})$, *then*
$\lim_{i\to\infty} w_i(r) = \lambda_s^{\frac{1}{\alpha-1}}(r^{\frac{2}{1-\alpha}} - 1),\ \forall\, r \in]0,1]$.

Proof. — As in the proof of Proposition 4, one can verify that $\lambda^*(\alpha) = \sup_{\mathbb{R}_+^*} g(r)$, where $g(r) = r^2\phi^{\alpha-1}(r)$.

If $\left(3 \leqslant n \leqslant 10 \text{ and } \frac{n+2}{n-2} < \alpha\right)$ or $\left(n > 10 \text{ and } \frac{n+2}{n-2} < \alpha < \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4}\right)$, using Lemma 4, we get $f(\alpha) > n-2$. So we can use Theorem 1 in [10] to infer that $\lambda^*(\alpha) > \lambda_s$, $\left(P^\alpha_{\lambda^*(\alpha)}\right)$ admits a unique solution and $\left(P^\alpha_{\lambda_s}\right)$ admits an infinity of solutions. Using the unique solution $u_{\lambda^*(\alpha)}$ of $\left(P^\alpha_{\lambda^*(\alpha)}\right)$, one can deduce from Proposition 1 that $u_{\lambda^*(\alpha)} = \psi_\rho$, where $\rho \in \mathbb{R}_+^*$ and $g(\rho) = \lambda^*(\alpha)$. We conclude that the supremum is achieved and $\lambda^*(\alpha) = \max_{\mathbb{R}_+^*} g(r)$.

Let us suppose that

$$a = \liminf_{r\to\infty} g(r) < A = \limsup_{r\to\infty} g(r).$$

For every $\lambda \in]a, A[$, the equation $g(r) = \lambda$ admits a sequence of roots (r_i), with $\lim_{i\to\infty} r_i = \infty$. As for every i, ψ_{r_i} is a solution of (P^α_λ), we get an infinity of solutions for this problem; but an infinity of solutions exists only when $\lambda = \lambda_s$ (cf. [10]). We get a contradiction and infer that

$$a = A = \lambda_s = \lim_{r\to\infty} g(r), \text{ so } \phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}}, \text{ as } r \to \infty.$$

If (ρ_i) is an increasing sequence of positive constants, such that (ψ_{ρ_i}) are solutions of $(P^\alpha_{\lambda_s})$ and $\lim_{i\to\infty} \rho_i = \infty$, then one can use the previous asymptotic behavior of ϕ to get $\lim_{i\to\infty} \psi_{\rho_i}(r) = \lambda_s^{\frac{1}{\alpha-1}}(r^{\frac{2}{1-\alpha}} - 1),\ \forall\, r \in]0,1]$.

If $n > 10$ and $\frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1}-4} \leqslant \alpha$, we get from Lemma 4 that $f(\alpha) \leqslant n-2$. Using [10] again, we infer that $\lambda^*(\alpha) = \lambda_s$, (P^α_λ) admits a unique solution for every $\lambda \in]0, \lambda^*(\alpha)[$. As the function g is increasing near $r = 0$, we infer that g is increasing on $\mathbb{R}_+^*$. For, on one hand, if g decreases on a nontrivial open interval $I \subset \mathbb{R}_+^*$, then the equation $g(r) = \lambda$ admits at least two roots $r_1 < r_2$, if $\lambda \in]\min_I g(r), \max_I g(r)[$. As ψ_{r_1} and ψ_{r_2} are solutions of (P^α_λ), with $\psi_{r_1}(0) \neq \psi_{r_2}(0)$, this violates the uniqueness result of [10]. On another hand, the function g can't be constant on a nontrivial interval, otherwise we get an infinity of solutions for some λ. One can then see that

$$\lim_{r\to\infty} g(r) = \sup_{\mathbb{R}_+^*} g(r) = \lambda^*(\alpha);\ \ \lambda^*(\alpha) = \lambda_s\ (cf.\ [10]).$$

So $\phi(r) \sim \lambda_s^{\frac{1}{\alpha-1}} r^{\frac{2}{1-\alpha}},\ as\ r \to \infty.$

Using this asymptotic behavior, one can show the last statement of the proposition.

Let us put

$$(Q_\lambda^\alpha)\begin{cases} \Delta u + \lambda(1+u)^\alpha = 0,\ in\ B_{r_0} \\ u > 0,\ in\ B_{r_0} \\ u = 0,\ on\ \partial B_{r_0} \end{cases}$$

where $B_{r_0} = \{x \in \mathbb{R}^n,\ \|x\| < r_0\}$. For every solution u of (Q_λ^α), we put $v(r) = u(r_0 r)$ for every $r \in [0,1]$. Let $\lambda_{r_0}^*(\alpha)$, be the maximal eigenvalue of (Q_λ^α).

Lemma 5.3. — *A function u is a solution of (Q_λ^α), if and only if v is a solution of $(P_{r_0^2\lambda}^\alpha)$. In particular, we get $\lambda_{r_0}^*(\alpha) = r_0^2\lambda^*(\alpha)$.*

Proof. — The proof is easy.

Remark 5.4. — According to the previous lemma, the results obtained here for (P_λ^α) (on the unit ball B_1), can be easily stated for (Q_λ^α) (on any ball B_{r_0}).

6. Appendix

Let S_k^i be the set of all the $(k-i)-$selections of $\{1,...,i\}$ and $s(j)$ the multiplicity of the element j, $1 \leqslant j \leqslant i$. If u is a analytical solution of (P_λ^α), with $u(r) = \Sigma_{k=0}^\infty a_k r^k$ near $r = 0$, r_0 the convergence radius of this series, then

Proposition 6.1. —

$$\forall\ k \geqslant 0,\ a_{2k+1} = 0,\quad a_2 = \frac{\lambda}{n-2}(1+a_0)^\alpha\left(\frac{1}{n} - \frac{1}{2}\right)$$

$$and\ \forall\ k > 1,\ a_{2k} = \frac{\lambda}{n-2}\left(\frac{1}{2k+n-2} - \frac{1}{2k}\right)\times$$

$$\Sigma_{i=1}^{k-1}(1+a_0)^{\alpha-i}\,\frac{1}{i!}\Pi_{p=0}^{i-1}(\alpha-p)\Sigma_{s\in S_{k-1}^i}\Pi_{j=1}^i a_{2(1+s(j))}.$$

Proof. — Let us choose $0 < r \leqslant \rho < r_0$, by standard integrations, we get

$$u(r) - u(\rho) = \frac{\lambda}{n-2}\times$$

$$\left((r^{2-n} - \rho^{2-n}) \int_0^r t^{n-1}(1+u(t))^\alpha dt + \int_r^\rho (t - \rho^{2-n}t^{n-1})(1+u(t))^\alpha dt\right).$$

Let us point out that

$$(1+u(r))^\alpha = (1+u(0)-u(0)+u(r))^\alpha$$

$$= (1+u(0))^\alpha \left(1 + \frac{u(r)-u(0)}{1+u(0)}\right)^\alpha = (1+a_0)^\alpha \left(1 + \Sigma_{i=1}^{\infty} \frac{a_i}{1+a_0} r^i\right)^\alpha, \ u(0) = a_0.$$

By the Maximum Principle, we have $\forall\ r \in]0,1[,\ 0 < u(r) < u(0)$, so we get

$$\left|\frac{u(0)-u(r)}{1+u(0)}\right| < 1, \ \forall\ r \in \ [0,1],$$

we infer that

$$(1+u(r))^\alpha = (1+a_0)^\alpha \left(1 + \Sigma_{j=1}^{\infty} \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} \left(\Sigma_{i=1}^{\infty} \frac{a_i}{1+a_0} r^i\right)^j\right).$$

All these series are uniformly convergent on $[0,\rho]$. If we put $(1+u(r))^\alpha = \Sigma_{j=0}^{\infty} c_j r^j$, we get

$$\begin{aligned}
u(r) &= \frac{\lambda}{n-2}\left((r^{2-n} - \rho^{2-n}) \int_0^r t^{n-1} \Sigma_{j=0}^{\infty} c_j t^j dt + \int_r^\rho (t - \rho^{2-n}t^{n-1}) \Sigma_{j=0}^{\infty} c_j t^j dt\right)\\
&= \frac{\lambda}{n-2}\left(\Sigma_{j=0}^{\infty} c_j \frac{r^{2+j}}{j+n} - \Sigma_{j=0}^{\infty} c_j \frac{\rho^{2-n} r^{j+n}}{j+n} + \Sigma_{j=0}^{\infty} c_j \frac{\rho^{j+2}}{j+2} - \Sigma_{j=0}^{\infty} c_j \frac{\rho^{j+2}}{j+n}\right)\\
&\quad + \frac{\lambda}{n-2}\left(-\Sigma_{j=0}^{\infty} c_j \frac{r^{j+2}}{j+2} + \Sigma_{j=0}^{\infty} c_j \frac{\rho^{2-n} r^{j+n}}{j+n}\right)\\
&= \frac{\lambda}{n-2}\left(\Sigma_{j=2}^{\infty} c_{j-2} \frac{r^j}{j+n-2} + \Sigma_{j=0}^{\infty} c_j \frac{\rho^{j+2}}{j+2} - \Sigma_{j=0}^{\infty} c_j \frac{\rho^{j+2}}{j+n} - \Sigma_{j=2}^{\infty} c_{j-2} \frac{r^j}{j}\right).
\end{aligned}$$

We finally obtain

$$(2) \quad u(r) = \frac{\lambda}{n-2}\left(\Sigma_{j=2}^{\infty} c_{j-2} (\frac{1}{j+n-2} - \frac{1}{j}) r^j + \Sigma_{j=0}^{\infty} c_j \rho^{j+2} (\frac{1}{j+2} - \frac{1}{j+n})\right).$$

Using the previous identity, we obtain

$$a_1 = 0, \ \ \forall\ k > 1, \ a_k = \frac{\lambda}{n-2}\left(\frac{1}{k+n-2} - \frac{1}{k}\right) c_{k-2}.$$

Using (1), we get

$$c_0 = (1+a_0)^{\alpha}\,,\ c_1 = \alpha\,(1+a_0)^{\alpha-1}\,a_1 = 0$$

and

$$\forall\ k>1,\ c_k = (1+a_0)^{\alpha}\,\Sigma_{j=1}^{k}\frac{1}{j!}\Pi_{p=0}^{j-1}(\alpha-p)\frac{1}{(1+a_0)^{j}}\Sigma_{s\in S_k^j}\Pi_{i=1}^{j}a_{1+s(i)}$$

$$=\Sigma_{j=1}^{k}\frac{1}{j!}\Pi_{p=0}^{j-1}(\alpha-p)\,(1+a_0)^{\alpha-j}\,\Sigma_{s\in S_k^j}\Pi_{i=1}^{j}a_{1+s(i)}.$$

Using the previous relation and the fact that $a_1 = 0$, one can verify (by induction) that $a_{2k+1} = 0,\ \forall k > 0$. We then obtain from (2) and the expression of c_k

$$a_{2k} = \frac{\lambda}{n-2}\left(\frac{1}{2k+n-2}-\frac{1}{2k}\right)c_{2k-2}$$

$$=\frac{\lambda}{n-2}\left(\frac{1}{2k+n-2}-\frac{1}{2k}\right)\Sigma_{j=1}^{k-1}\frac{1}{j!}\Pi_{p=0}^{j-1}(\alpha-p)\,(1+a_0)^{\alpha-j}\,\Sigma_{s\in S_{k-1}^j}\Pi_{i=1}^{j}a_{2(1+s(i))}.$$

$\forall\ j\ \in\ [1,k-1],\ Card(S_{k-1}^j) = C_{k-2}^{j-1}$.

Let us put

$$d_2 = \frac{1}{2n}\quad \textit{and}\quad \forall\ k>1,$$

$$d_{2k} = \frac{1}{(2k+n-2)(2k)}\Sigma_{i=1}^{k-1}\frac{1}{i!}\Pi_{p=0}^{i-1}(\alpha-p)\Sigma_{s\in S_{k-1}^i}\Pi_{j=1}^{i}d_{2(1+s(j))},$$

then

Lemma 6.2. — $a_{2k} = (-1)^k\lambda^k\,(1+a_0)^{k(\alpha-1)+1}\,d_{2k},\ \forall\ k>1.$

Proof. —

$$a_4 = \frac{\alpha\lambda^2}{(n-2)^2} = (1+a_0)^{2\alpha-1}\left(\frac{1}{n+2}-\frac{1}{4}\right)\left(\frac{1}{n}-\frac{1}{2}\right)$$

$$=\lambda^2(1+a_0)^{2\alpha-1}\frac{1}{4(n+2)}\frac{\alpha}{2n} = \lambda^2(1+a_0)^{2(\alpha-1)+1}\frac{1}{4(n+2)}\frac{\alpha}{2n}.$$

$$d_4 = \frac{1}{4(n+2)}\Sigma_{i=1}^{1}\frac{1}{i!}\Pi_{p=0}^{i-1}(\alpha-p)\Sigma_{s\in S_1^i}\Pi_{j=1}^{i}d_{2(1+s(j))}$$

$$=\frac{\alpha}{4(n+2)}d_2 = \frac{1}{4(n+2)}\frac{\alpha}{2n},$$

so we infer that the formula is true for $k = 2$. Let us suppose it true for every j, such that $2 \leqslant j \leqslant k$. From Proposition 7, we have

$$a_{2(k+1)}$$

$$= \frac{\lambda}{n-2}\left(\frac{1}{2k+n} - \frac{1}{2(k+1)}\right)\Sigma_{j=1}^{k}\frac{1}{j!}\Pi_{p=0}^{j-1}(\alpha - p)\left(1+a_0\right)^{\alpha-j}\Sigma_{s\in S_k^j}\Pi_{i=1}^{j}a_{2(1+s(i))}$$

$$= \frac{-\lambda}{(2(k+1)+n-2)(2(k+1))}\Sigma_{j=1}^{k}\frac{1}{j!}\Pi_{p=0}^{j-1}(\alpha - p)\left(1+a_0\right)^{\alpha-j}\Sigma_{s\in S_k^j}\Pi_{i=1}^{j}a_{2(1+s(i))}.$$

$$\forall\ j\in[1,k],\ \forall s\ \in S_k^j,\ if\ i\in[1,j],\ then\ 1\leqslant 1+s(i)\leqslant k,$$

so one can use the hypothesis to get $\forall\ i \in [1, j]$,

$$a_{2(1+s(i))} = (-1)^{1+s(i)}\lambda^{1+s(i)}\left(1+a_0\right)^{(s(i)+1)(\alpha-1)+1}d_{2(1+s(i))}.$$

We then obtain

$$\Pi_{i=1}^{j}a_{2(1+s(i))}$$

$$= (-1)^{\Sigma_{i=1}^{j}(1+s(i))}\lambda^{\Sigma_{i=1}^{j}(1+s(i))}\left(1+a_0\right)^{\Sigma_{i=1}^{j}\{(\alpha-1)(s(i)+1)+1\}}\Pi_{i=1}^{j}d_{2(1+s(i))}$$

$$= (-1)^{j+\Sigma_{i=1}^{j}s(i)}\lambda^{j+\Sigma_{i=1}^{j}s(i)}\left(1+a_0\right)^{\alpha j+(\alpha-1)\Sigma_{i=1}^{j}s(i)}\Pi_{i=1}^{j}d_{2(1+s(i))}.$$

But for every $s \in S_k^j$, we have $\Sigma_{i=1}^{j}s(i) = k - j$.

We infer that

$$\Pi_{i=1}^{j}a_{2(1+s(i))} = (-1)^{k}\lambda^{k}\left(1+a_0\right)^{\alpha j+(\alpha-1)(k-j)}\Pi_{i=1}^{j}d_{2(1+s(i))}$$

$$= (-1)^{k}\lambda^{k}\left(1+a_0\right)^{(\alpha-1)k+j}\Pi_{i=1}^{j}d_{2(1+s(i))}.$$

Substituting in the expression of $a_{2(k+1)}$, we obtain

$$a_{2(k+1)} = (-1)^{k+1}\lambda^{k+1}\left(1+a_0\right)^{k(\alpha-1)+\alpha}\frac{1}{(2(k+1)+n-2)(2(k+1))}\times$$

$$\Sigma_{j=1}^{k}\frac{1}{j!}\Pi_{p=0}^{j-1}(\alpha-p)\Sigma_{s\in S_k^j}\Pi_{i=1}^{j}d_{2(1+s(i))}$$

$$= (-1)^{k+1}\lambda^{k+1}\left(1+a_0\right)^{(k+1)(\alpha-1)+1}\frac{1}{(2(k+1)+n-2)(2(k+1))}\times$$

$$\Sigma_{j=1}^{k}\frac{1}{j!}\Pi_{p=0}^{j-1}(\alpha-p)\Sigma_{s\in S_k^j}\Pi_{i=1}^{j}d_{2(1+s(i))}.$$

$$= (-1)^{k+1}\lambda^{k+1}\left(1+a_0\right)^{(k+1)(\alpha-1)+1}d_{2(k+1)}.$$

Let us compute the first terms of the Lane-Emden function, $\phi(r) = \Sigma_{i=0}^{\infty} a_{2i} r^{2i}$, near $r = 0$, where $a_0 = 1$, *and* $a_{2i} = (-1)^i 2^{i(\alpha-1)+1} d_{2i}, \ \forall i > 1.$

$$d_0 = 1;\ d_2 = \frac{1}{2n};\ d_4 = \frac{1}{4(n+2))}\alpha d_2 = \frac{\alpha}{(2n)\,(4(n+2))};$$

$$d_6 = \frac{1}{6(n+4)}\left(\alpha d_4 + \frac{1}{2}\alpha(\alpha-1)d_2^2\right) = \frac{1}{6(n+4)}\left\{\frac{\alpha^2}{(2n)\,(4(n+2))} + \frac{\alpha(\alpha-1)}{2\,(2n)^2}\right\};$$

$$d_8 = \frac{1}{8(n+6)}\left(\alpha d_6 + \alpha(\alpha-1)d_4 d_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}d_2^3\right)$$

$$= \frac{1}{8(n+6)}\left\{\frac{\alpha^3}{(2n)\,(4(n+2))\,(6(n+4))} + \frac{\alpha^2(\alpha-1)}{2(2n)^2\,(6(n+4))} + \frac{\alpha^2(\alpha-1)}{(2n)^2\,(4(n+2))} + \frac{\alpha(\alpha-1)(\alpha-2)}{6(2n)^3}\right\};$$

$$d_{10} = \frac{1}{10(n+8)}\left\{\alpha d_8 + \frac{\alpha(\alpha-1)}{2}\left(2d_2 d_6 + d_4^2\right) + 3\frac{\alpha(\alpha-1)(\alpha-2)}{6}d_2^2 d_4 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24}d_2^4\right\}$$

$$= \frac{1}{10(n+8)}\left\{\frac{\alpha^4}{(2n)\,(4(n+2))\,(6(n+4))\,(8(n+6))} + \frac{\alpha^3(\alpha-1)}{2(2n)^2\,(6(n+4))\,(8(n+6))} + \frac{\alpha^3(\alpha-1)}{(2n)^2\,(4(n+2))\,(8(n+6))} + \frac{\alpha^2(\alpha-1)(\alpha-2)}{6(2n)^3\,(8(n+6))} + \frac{\alpha^3(\alpha-1)}{(2n)^2\,(4(n+2))\,(6(n+4))} + \frac{\alpha^2(\alpha-1)^2}{2(2n)^3\,(6(n+4))} + \frac{\alpha^3(\alpha-1)}{2(2n)^2\,(4(n+2))^2} + \frac{\alpha^2(\alpha-1)(\alpha-2)}{2(2n)^3\,(4(n+2))} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{24(2n)^4}\right\}.$$

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