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Uniqueness and factorization of Coleff-Herrera currents

Mats Andersson[1]

Résumé. — Nous prouvons un résultat d'unicité pour les courants de Coleff-Herrera qui dit en particulier que si $f = (f_1, \ldots, f_n)$ définit une intersection complète, alors le produit de Coleff-Herrera classique associé à f est le seul courant de Coleff-Herrera qui soit cohomologue à 1 pour l'opérateur $\delta_f - \overline{\partial}$, où δ_f est le produit intérieur par f. De ce résultat d'unicité, nous déduisons que tout courant de Coleff-Herrera sur une variété Z est une somme finie de produits de courants résiduels supportés sur Z par des formes holomorphes.

Abstract. — We prove a uniqueness result for Coleff-Herrera currents which in particular means that if $f = (f_1, \ldots, f_m)$ defines a complete intersection, then the classical Coleff-Herrera product associated to f is the unique Coleff-Herrera current that is cohomologous to 1 with respect to the operator $\delta_f - \bar{\partial}$, where δ_f is interior multiplication with f. From the uniqueness result we deduce that any Coleff-Herrera current on a variety Z is a finite sum of products of residue currents with support on Z and holomorphic forms.

1. Introduction

Let X be an n-dimensional complex manifold and let Z be an analytic variety of pure codimension p. The sheaf of Coleff-Herrera currents $\mathcal{CH}_Z$ consists of all $\bar{\partial}$-closed $(*, p)$-currents μ with support on Z such that

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$\bar{\psi}\mu = 0$ for each ψ vanishing on Z, and which in addition fulfill the so-called standard extension property, SEP, see below. Locally, any $\mu \in \mathcal{CH}_Z$ can be realized as the result of an application of a meromorphic differential operator on the current of integration $[Z]$ (combined with contractions with holomorphic vector fields), see, e.g., [4] and [5].

The model case of a Coleff-Herrera current is the Coleff-Herrera product associated to a complete intersection $f = (f_1, \dots, f_p)$,

$$\mu^f = \big[\bar{\partial}\frac{1}{f_p}\wedge \dots \wedge\bar{\partial}\frac{1}{f_1}\big], \tag{1.1}$$

introduced by Coleff and Herrera in [6]. Equivalent definitions are given in [9] and [10]; see also [12]. It was proved in [7] and [9] that the annihilator of μ^f is equal to the ideal $\mathcal{J}(f)$ generated by f. Notice that formally (1.1) is just the pullback under f of the product $\mu^w = \bar{\partial}(1/w_1)\wedge \dots \wedge\bar{\partial}(1/w_p)$. One can also express μ^w as $\bar{\partial}$ of the Bochner-Martinelli form

$$B(w) = \sum_j (-1)^j \bar{w}_j d\bar{w}_1\wedge \dots \wedge d\bar{w}_{j-1}\wedge d\bar{w}_{j+1}\wedge \dots \wedge d\bar{w}_p/|w|^{2p}.$$

In [11], f^*B is defined as a principal value current, and it is proved that $\mu^f_{BM} = \bar{\partial} f^*B$ is indeed equal to μ^f. However the proof is quite involved. An alternative but still quite technical proof appeared in [1]. In this paper we prove a uniqueness result which states that any Coleff-Herrera current that is cohomologous to 1 with respect to the operator $\delta_f - \bar{\partial}$ (see Section 3 for definitions) must be equal to μ^f. In particular this implies that $\mu^f = \mu^f_{BM}$.

It is well-known that any Coleff-Herrera current can be written $\alpha\wedge\mu^f$, where α is a holomorphic $(*,0)$-form and μ^f is a Coleff-Herrera product for a complete intersection f. However, unless Z is a complete intersection itself the support of μ^f is larger than Z. Using the uniqueness result we can prove

Theorem 1.1. — *For any $\mu \in \mathcal{CH}_Z$ (locally) there are residue currents R_I with support on Z and holomorphic $(*,0)$-forms α_I such that*

$$\mu = \sideset{}{'}\sum_{|I|=p} R_I\wedge\alpha_I. \tag{1.2}$$

Here R_I are currents of Bochner-Martinelli type from [11] associated with a not necessarily complete intersection. In particular, it follows that the Lelong current $[Z]$ admits a factorization (1.2).

The SEP goes back to Barlet, [3]. We will use the following definition: *Given any holomorphic h that does not vanish identically on any irreducible component of Z, the function* $|h|^{2\lambda}\mu$*, a priori defined only for* $\mathrm{Re}\,\lambda >> 0$*, has a current-valued analytic extension to* $\mathrm{Re}\,\lambda > -\epsilon$*, and the value at* $\lambda = 0$ *coincides with* μ*.* The reason for this choice is merely practical; for the equivalence to the classical definition, see Section 5. Now, if $\mu \in \mathcal{CH}_Z$ has support on $Z \cap \{h = 0\}$, then $|h|^{2\lambda}\mu$ must vanish if $\mathrm{Re}\,\lambda$ is large enough, and by the uniqueness of analytic continuation thus $\mu = 0$. In particular, $\mu = 0$ identically if $\mu = 0$ on Z_{reg}.

By the uniqueness result we obtain simple proofs of the equivalence of various definitions of the SEP (Section 5) as well as the equivalence of various conditions for the vanishing of a Coleff-Herrera current (Section 6).

2. The Coleff-Herrera product

Let $f_1, \ldots, f_p$ define a complete intersection in X, i.e., $\mathrm{codim}\, Z^f = p$, where $Z^f = \{f = 0\}$. Notice that (1.1) is elementarily defined if each f_j is a power of a coordinate function. The general definition relies on the possibility to resolve singularities: By Hironaka's theorem we can locally find a resolution $\pi: \tilde{\mathcal{U}} \to \mathcal{U}$ such that locally in $\tilde{\mathcal{U}}$, each $\pi^* f_j$ is a monomial times a non-vanishing factor. It turns out that locally μ^f is a sum of terms

$$\sum_\ell \pi_* \tau_\ell \tag{2.1}$$

where each τ_ℓ is of the form

$$\tau_\ell = \bar\partial \frac{1}{t_1^{a_1}} \wedge \ldots \wedge \bar\partial \frac{1}{t_p^{a_p}} \wedge \frac{\alpha}{t_{p+1}^{a_{p+1}} \cdots t_r^{a_r}},$$

t is a suitable local coordinate system in $\tilde{\mathcal{U}}$, and α is a smooth function with compact support. This representation turns out to be very useful; though not explicitly stated, it follows from the definition in [6] as well as from any other reasonable definition of μ^f by taking limits in the resolution manifold; see, e.g., [2] for a further discussion.

It is well-known that μ^f is in $\mathcal{CH}_{Z^f}$ but for further reference we sketch a proof. It follows immediately from the definition that μ^f is a $\bar\partial$-closed $(0, p)$-current with support on Z^f. Given any holomorphic function ψ we may choose the resolution so that also $\pi^*\psi$ is a monomial. Notice that each $|\pi^*\psi|^{2\lambda}\tau_\ell$ has an analytic continuation to $\lambda = 0$ and that the value at 0 is equal to τ_ℓ if none of $t_1, \ldots, t_p$ is a factor in $\pi^*\psi$ and zero otherwise.

According to this let us subdivide the set of τ_ℓ into two groups τ'_ℓ and τ''_ℓ. Notice that $|\psi|^{2\lambda}\mu^f = \sum_\ell \pi_*(|\pi^*\psi|^{2\lambda}\tau_\ell)$ admits an analytic continutation and that the value at $\lambda=0$ is $\sum \pi_*\tau''_\ell$. If $\psi=0$ on Z^f, then $0=|\psi|^{2\lambda}\mu^f$, and hence $\mu^f=\sum_\ell \pi_*\tau'_\ell$; it now follows that $\bar\psi\mu^f = d\bar\psi\wedge\mu^f = 0$. If h is holomorphic and the zero set of h intersects Z^f properly, then $T=\mu^f - |h|^{2\lambda}\mu^f|_{\lambda=0}$ is a current of the type (2.1) with support on $Y=Z^f\cap\{h=0\}$ that has codimension $p+1$. For the same reason as above, $d\bar\psi\wedge T=0$ for each holomorphic ψ that vanishes on Y and by a standard argument it now follows that $T=0$ for degree reasons. Thus μ^f has the SEP and so $\mu^f\in\mathcal{CH}_{Z^f}$. This proof is inspired by a forthcoming joint paper, [2], with Elizabeth Wulcan.

3. The uniqueness result

Let $f=(f_1,\dots,f_m)$ be a holomorphic tuple on some complex manifold X. It is practical to introduce a (trivial) vector bundle $E\to X$ with global frame $e_1,\dots,e_m$ and consider $f=\sum f_j e_j^*$ as a section of the dual bundle E^*, where e_j^* is the dual frame. Then f induces a mapping δ_f, interior multiplication with f, on the exterior algebra ΛE. Let $\mathcal{C}_{0,k}(\Lambda^\ell E)$ be the sheaf of $(0,k)$-currents with values in $\Lambda^\ell E$, considered as as sections of the bundle $\Lambda(E\oplus T^*(X))$; thus a section of $\mathcal{C}_{0,k}(\Lambda^\ell E)$ is given by an expression $v=\sum'_{|I|=\ell} f_I\wedge e_I$ where f_I are $(0,k)$-currents and $d\bar z_j\wedge e_k = -e_k\wedge d\bar z_j$ etc. Notice that both $\bar\partial$ and δ_f act as anti-derivations on these spaces, i.e., $\bar\partial(f\wedge g) = \bar\partial f\wedge g + (-1)^{\deg f} f\wedge\bar\partial g$, if at least one of f and g is smooth, and similarly for δ_f. It is straight forward to check that $\delta_f\bar\partial = -\bar\partial\delta_f$. Therefore, if $\mathcal{L}^k = \oplus_j \mathcal{C}_{0,j+k}(\Lambda^j E)$ and $\nabla_f = \delta_f - \bar\partial$, then $\nabla_f\colon\mathcal{L}^k\to\mathcal{L}^{k+1}$, and $\nabla_f^2=0$. For example, $v\in\mathcal{L}^{-1}$ is of the form $v=v_1+\cdots+v_m$, where v_k is a $(0,k-1)$-current with values in $\Lambda^k E$. Also for a general current the subscript will denote degree in ΛE.

Example 3.1 (The Coleff-Herrera product). — Let $f=(f_1,\dots,f_m)$ be a complete intersection in X. The current

$$V = \Big[\frac{1}{f_1}\Big]e_1 + \Big[\frac{1}{f_2}\bar\partial\frac{1}{f_1}\Big]\wedge e_1\wedge e_2 + \Big[\frac{1}{f_3}\bar\partial\frac{1}{f_2}\wedge\bar\partial\frac{1}{f_1}\Big]\wedge e_1\wedge e_2\wedge e_3 + \cdots \tag{3.1}$$

is in $\mathcal{L}^{-1}$ and solves $\nabla_f V = 1-\mu^f\wedge e$, where μ^f is the Coleff-Herrera product and $e=e_1\wedge\ldots\wedge e_m$. For definition of the coefficients of V and the computational rules used here, see [9]; one can obtain a simple proof of these rules by arguing as in Section 2, see [2].

Example 3.2 (Residues of Bochner-Martinelli type). — Introduce a Hermitian metric on E and let σ be the section of E over $X \setminus Z^f$ with minimal pointwise norm such that $\delta_f \sigma = f \cdot \sigma = 1$. Then $\bar\partial\sigma$ has even total degree (it is in $\mathcal{L}^0$) and we let $(\bar\partial\sigma)^2 = \bar\partial\sigma\wedge\bar\partial\sigma$, etc. Now

$$u = \sigma + \sigma\wedge\bar\partial\sigma + \sigma\wedge(\bar\partial\sigma)^2 + \sigma\wedge(\bar\partial\sigma)^3 \cdots \tag{3.2}$$

is smooth outside Z^f and $\nabla_f u = 1$ there; in fact, since $\delta_f(\bar\partial\sigma) = -\bar\partial\delta_f\sigma = -\bar\partial 1 = 0$ we have that $\delta_f(\sigma\wedge(\bar\partial\sigma)^k) = (\bar\partial\sigma)^k = \bar\partial(\sigma\wedge(\bar\partial\sigma)^{k-1})$, so $\nabla_f u = (\delta_f - \bar\partial)u$ becomes a telescoping sum. (A more elegant way is to notice that (3.2) is equal to $\sigma/\nabla_f\sigma$; then $\nabla_f u = 1$ follows by Leibniz' rule since $\nabla_f^2 = 0$, cf. [1]).

It turns out, see [1], that u has a natural current extension U across Z^f. For instance it can be defined as the value at $\lambda = 0$ of the analytic continuation of $|f|^{2\lambda}u$ from $\operatorname{Re}\lambda >> 0$ (the existence of the analytic continuation is of course nontrivial and requires a resolution of singularities). If $p = \operatorname{codim} Z^f$, then $\nabla_f U = 1 - R^f$, where

$$R^f = R^f_p + \cdots + R^f_m,$$

R^f is the value at $\lambda = 0$ of $\bar\partial|f|^{2\lambda}\wedge u$ and $R^f_k = \sigma\wedge(\bar\partial\sigma)^{k-1}|_{\lambda=0}$. Moreover, these currents have representations like (2.1) so if $\xi \in \mathcal{O}(\Lambda^{m-p}E)$ and $\xi\wedge R^f_p$ is $\bar\partial$-closed, then it is in $\mathcal{CH}^f_Z$ by the arguments given in Section 2. Notice that

$$R^f_k = \sideset{}{'}\sum_{|I|=k} R^f_I \wedge e_{I_1}\wedge\ldots\wedge e_{I_k}. \tag{3.3}$$

If we choose the trivial metric, the coefficients R^f_I are precisely the currents introduced in [11]. In particular, if f is a complete intersection, i.e. $m = p$, then, see [1], $R^f_{1,\ldots,p} = \mu^f_{BM}\wedge e$.

THEOREM 3.3 *(Uniqueness for Coleff-Herrera currents). — Assume that Z^f has pure codimension p. If $\tau \in \mathcal{CH}_{Z^f}$ and there is a solution $V \in \mathcal{L}^{p-m-1}$ to $\nabla_f V = \tau\wedge e$, then $\tau = 0$.*

Remark 3.4. — If Z^f does not have pure codimension, the theorem still holds (with the same proof) with $\mathcal{CH}_{Z^f}$ replaced by $\mathcal{CH}_{Z'}$, where Z' is the irreducible components of Z^f of maximal dimension.

In view of Examples 3.1 and 3.2 we get

COROLLARY 3.5. — *Assume that f is a complete intersection. If $\mu \in \mathcal{CH}_{Z^f}$ and there is a current $U \in \mathcal{L}^{-1}$ such that $\nabla_f U = 1 - \mu\wedge e$, then μ is equal to the Coleff-Herrera product μ^f. In particular, $\mu^f_{BM} = \mu^f$.*

The proof of Theorem 3.3 relies on the following lemma, which is probably known. However, for the reader's convenience we include a proof.

Lemma 3.6. — *If μ is in $\mathcal{CH}_Z$ and for each neighborhood ω of Z there is a current V with support in ω such that $\bar\partial V = \mu$, then $\mu = 0$.*

Proof. — Locally on Z_{reg} we can choose coordinates (z, w) such that $Z = \{w = 0\}$. We claim that there is a natural number M such that

$$\mu = \sum_{|\alpha|\leqslant M-p} a_\alpha(z)\bar\partial\frac{1}{w_1^{\alpha_1+1}}\wedge\ldots\wedge\bar\partial\frac{1}{w_p^{\alpha_p+1}}, \tag{3.4}$$

where a_α are the push-forwards of $\mu\wedge w^\alpha dw/(2\pi i)^p$ under the projection $(z, w)\mapsto z$. In fact, since $\bar w_j\mu = 0$ and $\bar\partial\mu = 0$ it follows that $d\bar w_j\wedge\mu = 0$, $j = 1,\ldots,p$, and hence $\mu = \mu_0 d\bar w_1\wedge\ldots\wedge d\bar w_p$. Therefore it is enough to check (3.4) for test forms of the form $\xi(z, w)dw\wedge d\bar z\wedge dz$. Since $\bar w_j\mu = 0$ we have by a Taylor expansion in w (the sum is finite since μ has finite order) that

$$\begin{aligned}\int_{z,w}\mu\wedge\xi dw\wedge d\bar z\wedge dz &= \sum_\alpha\int_{z,w}\mu\wedge\frac{\partial^\alpha\xi}{\partial w^\alpha}(z,0)\frac{w^\alpha}{\alpha!}dw\wedge d\bar z\wedge dz\\ &= \sum_\alpha\int_z a_\alpha(z)\frac{\partial^\alpha\xi}{\partial w^\alpha}(z,0)dw\wedge d\bar z\wedge dz(2\pi i)^p\\ &= \sum_\alpha\int_z a_\alpha(z)\int_w\bar\partial\frac{1}{w^{\alpha+1}}\wedge\xi(z,w)dw\wedge d\bar z\wedge dz.\end{aligned}$$

Since μ is $\bar\partial$-closed it follows that a_α are holomorphic. Notice that

$$\bar\partial\frac{1}{w_p^{\beta_p}}\wedge\ldots\wedge\bar\partial\frac{1}{w_1^{\beta_1}}\wedge dw_1^{\beta_1}\wedge\ldots\wedge dw_p^{\beta_p}/(2\pi i)^p = \beta_1\cdots\beta_p[w=0],$$

where $[w = 0]$ denote the current of integration over Z_{reg} (locally). Now assume that $\bar\partial\gamma = \mu$ and γ has support close to Z. We have, for $|\beta| = M$, that

$$\bar\partial(\gamma\wedge dw^\beta) = (2\pi i)^p a_{\beta-1}(z)\beta_1\cdots\beta_p[w=0].$$

If ν is the component of $\gamma\wedge dw^\beta$ of bidegree $(p, p-1)$ in w, thus

$$d_w\nu = \bar\partial_w\nu = (2\pi i)^p a_{\beta-1}\beta_1\cdots\beta_p[w=0].$$

Integrating with respect to w we get that $a_{\beta-1}(z) = 0$. By finite induction we can conclude that $\mu = 0$ locally on Z_{reg}. Thus μ vanishes on Z_{reg} and by the SEP it follows that $\mu = 0$. □

Proof. — [Proof of Theorem 3.3] Let ω be any neighborhood of Z and take a cutoff function χ that is 1 in a neighborhood of Z and with support in ω. Let u be any smooth solution to $\nabla_f u = 1$ in $X \setminus Z^f$, cf. Example 3.2. Then $g = \chi - \bar\partial\chi\wedge u$ is a smooth form in $\mathcal{L}^0(\omega)$ and $\nabla_f g = 0$. Moreover, the scalar term g_0 is 1 in a neighborhood of Z^f. Therefore,

$$\nabla_f[g\wedge V] = g\wedge\tau\wedge e = g_0\tau\wedge e = \tau\wedge e,$$

and hence the current coefficient W of the top degree component of $g\wedge V$ is a solution to $\bar\partial W = \tau$ with support in ω. In view of Lemma 3.6 we have that $\tau = 0$. □

4. The factorization

The double sheaf complex $\mathcal{C}_{0,k}(\Lambda^\ell E)$ is exact in the k direction except at $k = 0$, where we have the cohomology $\mathcal{O}(\Lambda^\ell E)$. By a standard argument there are natural isomorphisms

$$\operatorname{Ker}_{\delta_f}\mathcal{O}(\Lambda^\ell E)/\delta_f\mathcal{O}(\Lambda^{\ell+1}) \simeq \operatorname{Ker}_{\nabla_f}\mathcal{L}^{-\ell}/\nabla_f\mathcal{L}^{-\ell-1}. \tag{4.1}$$

When $\ell = 0$ the left hand side is $\mathcal{O}/\mathcal{J}(f)$, where $\mathcal{J}(f)$ is the ideal sheaf generated by f. We have the following factorization result.

Theorem 4.1. — *Assume that Z^f has pure codimension p and let $\mu \in \mathcal{CH}_{Z^f}$ be $(0,p)$ and such that $\mathcal{J}(f)\mu = 0$. Then there is locally $\xi \in \mathcal{O}(\Lambda^{m-p}E)$ such that*

$$\mu\wedge e = R^f_p\wedge\xi. \tag{4.2}$$

Proof. — Since $\nabla_f(\mu\wedge e) = 0$, by (4.1) there is $\xi \in \mathcal{O}(\Lambda^{m-p}E)$ such that $\nabla_f V = \xi - \mu\wedge e$. On the other hand, if U is the current from Example 3.2, then $\nabla_f(U\wedge\xi) = \xi - R^f\wedge\xi = \xi - R^f_p\wedge\xi$. Now (4.2) follows from Theorem 3.3. □

Proof. — [Proof of Theorem 1.1] With no loss of generality we may assume that μ has bidegree $(0,p)$. Let $g = (g_1, \ldots, g_m)$ be a tuple such that $Z^g = Z$. If $f_j = g_j^M$ and M is large enough, then $\mathcal{J}(f)\mu = 0$ and hence by Theorem 4.1 there is a form

$$\xi = \sum_{|J|=m-p}{}' \xi_J\wedge e_J$$

such that (4.2) holds. Then, cf. (3.3), (1.2) holds if $\alpha_I = \pm\xi_{I^c}$, where $I^c = \{1, \ldots, m\} \setminus I$. □

Example 4.2. — Let $[Z]$ be any variety of pure codimension and choose f such that $Z = Z^f$. It is not hard to prove that (each term of) the Lelong current $[Z]$ is in $\mathcal{CH}_Z$, and hence there is a holomorphic form ξ such that $R^f_p\wedge\xi = [Z]\wedge e$. (In fact, one can notice that the proof of Lemma 3.6 works for $\mu = [Z]$ just as well, and then one can obtain fakto for $[Z]$ in the same way as for $\mu \in \mathcal{CH}_Z$. A posteriori it follows that indeed $[Z]$ is in $\mathcal{CH}_Z$.) There are natural ways to regularize the current R^f_p, see, e.g. [12], and thus we get natural regularizations of $[Z]$.

Next we recall the duality principle, [7], [8]: If f is a complete intersection, then

$$\operatorname{ann}\mu^f = \mathcal{J}(f). \tag{4.3}$$

In fact, if $\phi \in \operatorname{ann}\mu$, then $\nabla_f U\phi = \phi - \phi\mu\wedge e = \phi$ and hence $\phi \in \mathcal{J}(f)$ by (4.1). Conversely, if $\phi \in \mathcal{J}(f)$, then there is a holomorphic ψ such that $\phi = \delta_f\psi = \nabla_f\psi$ and hence $\phi\mu = \nabla_f\psi\wedge\mu = \nabla(\psi\wedge\mu) = 0$.

Notice that $\mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}(f), \mathcal{CH}_{Z^f}(\Lambda^p E))$ is the sheaf of currents $\mu\wedge e$ with $\mu \in \mathcal{CH}_{Z^f}$ that are annihilated by $\mathcal{J}(f)$. From (4.3) and Theorem 4.1 we now get

Theorem 4.3. — *If f is a complete intersection, then the sheaf mapping*

$$\mathcal{O}/\mathcal{J}(f) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{O}/\mathcal{J}(f), \mathcal{CH}_Z(\Lambda^p E)), \quad \phi \mapsto \phi\mu^f\wedge e, \tag{4.4}$$

is an isomorphism.

5. The standard extension property

Given the other conditions in the definition of $\mathcal{CH}_Z$ the SEP is automatically fulfilled on Z_{reg}; this is easily seen, e.g. as in the proof of Lemma 3.6 (notice that the SEP is a local property), so the interesting case is when the zero set Y of h contains the singular locus of Z. Classically, cf. [3], [4], and [5], the SEP is expressed as

$$\lim_{\epsilon\to 0}\chi(|h|/\epsilon)\mu = \mu, \tag{5.1}$$

where $Y \supset Z_{sing}$ and h is not vanishing identically on any irreducible component of Z. Here $\chi(t)$ can be either the characteristic function for the interval $[1,\infty)$ or some smooth approximand.

Proposition 5.1. — *Let χ be a fixed function as above. The class of $\bar\partial$-closed $(0,p)$-currents μ with support on Z that are annihilated by $\bar I_Z$ and satisfy (5.1) coincides with our class $\mathcal{CH}_Z$.*

If χ is not smooth the existence of the currents $\chi(|h|/\epsilon)\mu$ in a reasonable sense for small $\epsilon > 0$ is part of the statement.

Proof. — [Sketch of proof] Let f be a tuple such that $Z = Z^f$. We first show that R^f_p satisfies (5.1). From the arguments in Section 2, cf. Example 3.2, we know that R^f_p has a representation (2.1) such that π^*h is a pure monomial (since the possible nonvanishing factor can be incorporated in one of the coordinates) and none of the factors in π^*h occurs among the residue factors in τ_ℓ. Therefore, the existence of the product in (5.1) and the equality follow from the simple observation that

$$\int_{s_1,\dots,s_\mu} \chi(|s_1^{c_1}\cdots s_\mu^{c_\mu}|/\epsilon)\frac{\psi(s)}{s_1^{\gamma_1}\cdots s_\mu^{\gamma_\mu}} \to \int_{s_1,\dots,s_\mu} \frac{\psi(s)}{s_1^{\gamma_1}\cdots s_\mu^{\gamma_\mu}} \tag{5.2}$$

for test forms ψ, where the right hand side is a tensor product of one-variable principal value integrals acting on ψ. Let temporarily $\mathcal{CH}^{cl}_Z$ denote the class of currents defined in the proposition. Since each $\mu \in \mathcal{CH}_Z$ admits the representation (4.2) it follows that $\mu \in \mathcal{CH}^{cl}_Z$. On the other hand, Lemma 3.6 and therefore Theorem 3.3 and (4.2) hold for $\mathcal{CH}^{cl}_Z$ as well (with the same proofs), and thus we get the other inclusion. □

6. Vanishing of Coleff-Herrera currents

We conclude with some equivalent condition for the vanishing of a Coleff-Herrera current. This result is proved by the ideas above, it should be well-known, but we have not seen it in this way in the literature.

THEOREM 6.1. — *Assume that X is Stein and that the subvariety $Z \subset X$ has pure codimension p. If $\mu \in \mathcal{CH}_Z(X)$ and $\bar\partial V = \mu$ in X, then the following are equivalent:*

(i) $\mu = 0$.

(ii) For all $\psi \in \mathcal{D}_{n,n-p}(X)$ such that $\bar\partial\psi = 0$ in some neighborhood of Z we have that

$$\int V\wedge\bar\partial\psi = 0.$$

(iii) There is a solution to $\bar\partial w = V$ in $X \setminus Z$.

(iv) For each neighborhood ω of Z there is a solution to $\bar\partial w = V$ in $X \setminus \bar\omega$.

Proof. — It is easy to check that (i) implies all the other conditions. Assume that (ii) holds. Locally on $Z_{reg} = \{w = 0\}$ we have (3.4), and by choosing $\xi(z,w) = \psi(z)\chi(w)dw^\beta \wedge dz \wedge d\bar{z}$ for a suitable cutoff function χ and test functions ψ, we can conclude from (ii) that $a_\beta = 0$ if $|\beta| = M$. By finite induction it follows that $\mu = 0$ there. Hence $\mu = 0$ globally by the SEP. Clearly (iii) implies (iv). Finally, assume that (iv) holds. Given $\omega \supset Z$ choose $\omega' \subset\subset \omega$ and a solution to $\bar\partial w = V$ in $X \setminus \overline{\omega'}$. If we extend w arbitrarily across ω' the form $U = V - \bar\partial w$ is a solution to $\bar\partial U = \mu$ with support in ω. In view of Lemma 3.6 thus $\mu = 0$. □

Notice that V defines a Dolbeault cohomology class ω^μ in $X \setminus Z$ that only depends on μ, and that conditions (ii)-(iv) are statements about this class. For an interesting application, fix a current $\mu \in \mathcal{CH}_Z$. Then the theorem gives several equivalent ways to express that a given $\phi \in \mathcal{O}$ belongs to the annihilator ideal of μ. In the case when $\mu = \mu^f$ for a complete intersection f, one gets back the equivalent formulations of the duality theorem from [7] and [9].

Remark 6.2. — If μ is an arbitrary $(0,p)$-current with support on Z and $\bar\partial V = \mu$ we get an analogous theorem if condition (i) is replaced by: $\mu = \bar\partial\gamma$ *for some* γ *with support on* Z. This follows from the Dickenstein-Sessa decomposition $\mu = \mu_{CH} + \bar\partial\gamma$, where μ_{CH} is in $\mathcal{CH}_Z$. See [7] for the case Z is a complete intersection and [4] for the general case.

Bibliography

[1] Andersson (M.). — Residue currents and ideals of holomorphic functions Bull. Sci. Math., 128, p. 481-512 (2004).

[2] Andersson (M.) & Wulcan (E.). — Decomposition of residue currents J. Reine Angew. Math. (to appear).

[3] Barlet (D.). — Fonctions de type trace Ann. Inst. Fourier 33, p. 43-76 (1983).

[4] Björk (J-E) Residue calculus and $\mathcal{D}$-modules om complex manifolds Preprint Stockholm (1996).

[5] Björk (J-E). — Residues and $\mathcal{D}$-modules The legacy of Niels Henrik Abel, 605–651, Springer, Berlin, 2004.

[6] Coleff (N.r.) & Herrera (M.e.). — Les courants résiduels associés à une forme méromorphe Lect. Notes in Math. 633, Berlin-Heidelberg-New York (1978).

[7] Dickenstein (A.) & Sessa (C.). — Canonical representatives in moderate cohomology Invent. Math. 80, p. 417-434 (1985).

[8] Passare (M.). — Residues, currents, and their relation to ideals of holomorphic functions Math. Scand. 62, p. 75-152 (1988).

[9] Passare (M.). — A calculus for meromorphic currents. J. Reine Angew. Math. 392, p. 37-56 (1988).

[10] Passare (M.) & Tsikh (A.). — Residue integrals and their Mellin transforms Canad. J. Math. 47, p. 1037-1050 (1995).

[11] Passare (M.) & Tsikh (A.) & Yger (A.). — Residue currents of the Bochner-Martinelli type Publ. Mat. 44, p. 85-117 (2000).

[12] Samuelsson (H.). — Regularizations of products of residue and principal value currents J. Funct. Anal. 239, p. 566-593 (2006).