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Remarks on the equatorial shallow water system

Chloé Mullaert[(1)]

Résumé. — Cet article reprend les résultats démontrés par A. Dutrifoy, A. Majda et S. Schochet dans [1] où ils obtiennent une estimation uniforme du système ainsi que sa convergence vers une solution globale lorsque le nombre de Froude tend vers zéro. Ensuite, un résultat de convergence sous des hypothèses plus faibles sera démontré ainsi que le fait que le temps de vie des solutions tend vers l'infini quand le nombre de Froude tend vers zéro.

Abstract. — This article recalls the results given by A. Dutrifoy, A. Majda and S. Schochet in [1] in which they prove an uniform estimate of the system as well as the convergence to a global solution of the long wave equations as the Froud number tends to zero. Then, we will prove the convergence with weaker hypothesis and show that the life span of the solutions tends to infinity as the Froud number tends to zero.

1. Introduction

When studying the climate of equatorial regions, the following mathematical model of equatorial shallow water is used :

$$(ESW)\left\{\begin{array}{c}\partial_t \overrightarrow{v}_\varepsilon + \overrightarrow{v}_\varepsilon \cdot \nabla \overrightarrow{v}_\varepsilon + \varepsilon^{-1}(y\overrightarrow{v}_\varepsilon^{\perp} + \nabla h_\varepsilon) = 0 \\ \partial_t h_\varepsilon + \overrightarrow{v}_\varepsilon \cdot \nabla h_\varepsilon + h_\varepsilon \operatorname{div} \overrightarrow{v}_\varepsilon + \varepsilon^{-1} \operatorname{div} \overrightarrow{v}_\varepsilon = 0 \\ (\overrightarrow{v}_\varepsilon, h_\varepsilon)(0) = \left(\overrightarrow{v}_{0,\varepsilon}, h_{0,\varepsilon}\right)\end{array}\right.$$

where $h_\varepsilon(x,y,t)$ is the height of the fluid, $\overrightarrow{v}_\varepsilon(x,y,t) = (u_\varepsilon, v_\varepsilon)$ is the horizontal velocity and $\overrightarrow{v}_\varepsilon^{\perp} = (-v_\varepsilon, u_\varepsilon)$.

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These equations are written in the spatial domain $(x, y) \in \mathbb{T} \times \mathbb{R}$ because the longitude x is assumed to be periodic and under the assumption that the positive parameter ε which represents the Froude number and the height fluctuations tends to zero. More details about this model can be found in [2] and [3]. Let us define

$$U_\varepsilon = (\overrightarrow{v}_\varepsilon, h_\varepsilon) \quad \text{and} \quad L = \begin{pmatrix} 0 & -y & \partial_x \\ y & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}$$

the penalisation matrix associated to (ESW).

This paper extends results given by A. Dutrifoy, A. Majda and S. Schochet in [1] in which they use the symmetric hyperbolic structure of a modified system to prove a uniform estimate in ε of the system on some time interval $[0, T]$ as well as the convergence of the solutions (U_ε) under the assumption that the initial data is "well-prepared". This notion has been introduced by S. Klainerman and A. Majda in [5].

The Sobolev spaces in which the results are given are those introduced in [1]. Let us define

$$\mathcal{W}^n = \left\{ w \text{ such as } \widetilde{\| w \|}^2_{\mathcal{W}^n} = \sum_{j+k+m \leqslant n} \| y^j \partial_x^k \partial_y^m w \|^2_{L^2} \text{ is finite} \right\}$$

In this paper, we recall the estimate result proved in [1].

Theorem 1.1. — *Let us consider a family of initial data $(U_{\varepsilon,0})_{0<\varepsilon\leqslant\varepsilon_0}$ bounded in $\mathcal{W}^{2n}$ with $n \geqslant 2$ by a constant M_0. Then a positive time T (depending only of the bound M_0) exists such that the family $(U_\varepsilon)_{0<\varepsilon\leqslant\varepsilon_0}$ of associated solutions is defined on $[0, T]$ and remain uniformly bounded in $\mathcal{W}^{2n}$ on $[0, T]$.*

The convergence result proved in [1] under the assumption $\|LU_{\varepsilon,0}\|_{\mathcal{W}^{2n}} \leqslant C\varepsilon$ will be proved under the assumption that $\|LU_{\varepsilon,0}\|_{\mathcal{W}^{2n}}$ tends to zero as ε tends to zero.

Theorem 1.2. — *Let us consider a family of initial data $(U_{\varepsilon,0})_{0<\varepsilon\leqslant\varepsilon_0}$ converging in $\mathcal{W}^{2n}$ to an element of $Ker(L) \cap \mathcal{W}^{2n+1}$ with $n \geqslant 2$. Then, for every $T > 0$, the family $(U_\varepsilon)_{0<\varepsilon\leqslant\varepsilon_0}$ of associated solutions converge to the limit of the initial data in $\mathcal{C}^0\left([0, T], \mathcal{W}^{2n}\right)$. Moreover, the life span T^*_ε of the solution U_ε tends to infinity as ε tends to zero.*

In a first part, we shall briefly recall the method used by A. Dutrifoy, A. Majda and S. Schochet in [1] to prove the uniform estimate. It consists in a non linear change of unknowns which depends on the parameter ε. In a second part, we shall prove Theorem 1.2 using an appropriated Ansatz on the unknowns.

2. Symmetrization and stabilization of the system

The aim of this part is to symmetrize the system and to find derivation matrices which preserve its structure i.e., to change $(\overrightarrow{v}_\varepsilon, h_\varepsilon)$ into V_ε solution of an equation

$$\partial_t V_\varepsilon + B_1(V_\varepsilon)\partial_x V_\varepsilon + B_2(V_\varepsilon)\partial_y V_\varepsilon + \varepsilon^{-1} N V_\varepsilon = 0$$

where $B_1(V_\varepsilon)$, $B_2(V_\varepsilon)$ are symmetric matrices and where N is a skew-symmetric operator ie $N^* = -N$ such as $[N, \partial_x] = [N, D]$ with D a derivation matrix.

Indeed, for a symmetric system with an skewsymmetric penalisation, it is easy to obtain an uniform L^2 estimate and as we can derive the system without changing its structure, we shall obtain uniform estimate in the Sobolev spaces $\mathcal{W}^{2n}$. One can read [4] to see the advantages of an skewsymmetric penalisation.

The following lemma describe the “raising” and “lowering” operators of the harmonic oscillator Hamiltonian.

Lemma 2.1. — *Given the operators*

$$L_+ = \frac{1}{\sqrt{2}}\left(\partial_y - y\right), \; L_- = \frac{1}{\sqrt{2}}\left(\partial_y + y\right) \; \textit{and } H = L_- L_+ + L_+ L_- = \partial_y^2 - y^2$$

then we have the relations :

$$[L_+, L_-] = I, \; L_+ L_- = \frac{1}{2}\left(H + I\right), \; [H, L_+] = -2L_+ \; \textit{and} \; [H, L_-] = 2L_-$$

The following proposition, proved in [1], explains how to change the unknowns of the system to symmetrize it and gives us derivation matrices which preserve its structure.

Proposition 2.2. — *The vector* V_ε *defined by*

$$V_\varepsilon = \begin{pmatrix} \dfrac{1}{\sqrt{2}}\left(-u_\varepsilon + \dfrac{2\left(\sqrt{1+\varepsilon h_\varepsilon}-1\right)}{\varepsilon}\right) \\ v_\varepsilon \\ \dfrac{1}{\sqrt{2}}\left(u_\varepsilon + \dfrac{2\left(\sqrt{1+\varepsilon h_\varepsilon}-1\right)}{\varepsilon}\right) \end{pmatrix}$$

is a solution of

$$\partial_t V_\varepsilon + B_1(V_\varepsilon)\partial_x V_\varepsilon + B_2(V_\varepsilon)\partial_y V_\varepsilon + \varepsilon^{-1} N V_\varepsilon = 0$$

where $B_1(V_\varepsilon)$ *and* $B_2(V_\varepsilon)$ *are symmetric matrices whose coefficients are linear combinations of the components of* V_ε *and where*

$$N = \begin{pmatrix} -\partial_x & L_- & 0 \\ L_+ & 0 & L_- \\ 0 & L_+ & \partial_x \end{pmatrix}$$

is a skew-symmetric operator.

Moreover, if

$$D = \begin{pmatrix} H-2 & 0 & 0 \\ 0 & H & 0 \\ 0 & 0 & H+2 \end{pmatrix}$$

where $H = \partial_y^2 - y^2$ *is the harmonic oscillator, then* $[N, D] = [N, \partial_x Id] = 0$.

This proposition leads the authors of [1] to the following definition of adapted norms

$$\|W\|^2_{\mathcal{W}^{2n}} = \sum_{k+2p\leqslant 2n} \left\|\partial_x^k D^p W\right\|^2_{L^2}.$$

Proposition 2.3. — *The norms* $\|\,.\,\|_{\mathcal{W}^{2n}}$ *and* $\widetilde{\|\,.\,\|}_{\mathcal{W}^{2n}}$ *are equivalent.*

As the above proposition, the following one is proved in [1].

Proposition 2.4. — *As* V_ε *is solution of*

$$\partial_t V_\varepsilon + B_1(V_\varepsilon)\partial_x V_\varepsilon + B_2(V_\varepsilon)\partial_y V_\varepsilon + \varepsilon^{-1} N V_\varepsilon = 0$$

where $B_1(V_\varepsilon)$ *and* $B_2(V_\varepsilon)$ *are symmetric matrices whose coefficients are linear combinations of the components of* V_ε*, and where* N *is a skew-symmetric matrix operator commuting with* D*, we have*

$$\frac{d}{dt}\|V_\varepsilon\|^2_{\mathcal{W}^{2n}} \leqslant C\|V_\varepsilon\|^3_{\mathcal{W}^{2n}}.$$

This result leads to the theorem 1.1 proved in [1] thanks to the following easy lemmas.

Lemma 2.5. — *If f is a positive fonction which satisfies*

$$\begin{cases} |f'| \leqslant Cf^{3/2} \\ f(0) \leqslant M_0 \end{cases}$$

then, $f \leqslant 2M_0$ on $\left[0, \dfrac{1}{C\sqrt{f(0)}}\right]$.

Lemma 2.6. — *The norms $\|V_\varepsilon\|_{\mathcal{W}^{2n}}$ and $\|U_\varepsilon\|_{\mathcal{W}^{2n}}$ are equivalents when ε tends to zero.*

Proof. — Using the Taylor formula at order two, we get

$$V_\varepsilon = \begin{pmatrix} \dfrac{1}{\sqrt{2}}\left(-u_\varepsilon + \dfrac{2\left(\sqrt{1+\varepsilon h_\varepsilon}-1\right)}{\varepsilon}\right) \\ v_\varepsilon \\ \dfrac{1}{\sqrt{2}}\left(u_\varepsilon + \dfrac{2\left(\sqrt{1+\varepsilon h_\varepsilon}-1\right)}{\varepsilon}\right) \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{2}}(-u_\varepsilon + h_\varepsilon) \\ v_\varepsilon \\ \dfrac{1}{\sqrt{2}}(u_\varepsilon + h_\varepsilon) \end{pmatrix}(1 + G(\varepsilon h_\varepsilon))$$

where G is a regular fonction such that $G(0) = 0$. Then we get the result thanks to Faa di Bruno formula. □

3. Convergence and life span

In this part, we shall give a convergence result with weaker hypothesis than in [1] and prove that the life span of the solutions tends to infinity as ε tends to zero. The purpose of this section is to prove the second theorem.

In order to prove that V_ε exists on a time interval which tends to $\mathbb{R}^+$ as ε tends to zero, we compare it to $V_0(y)$, the limit of the initial data, which is, by definition, global. We make an Ansatz

$$V_\varepsilon = V_0 + \varepsilon V_1 + \overline{V_\varepsilon}$$

As the limit of the initial data $V_0 = (l_0, v_0, r_0)$ belongs to $Ker(N)$, we have

$$\left\{\begin{array}{l} -\partial_x l_0 + L_- v_0 = 0 \\ L_+ l_0 + L_- r_0 = 0 \\ L_+ v_0 + \partial_x r_0 = 0 \end{array}\right.$$

so $v_0 \; = [L_+, L_-]\, v_0 = L_+ \partial_x l_0 + L_- \partial_x r_0 \; = \partial_x \left[L_+ l_0 + L_- r_0\right] = 0.$

Thus, $\partial_x r_0 = 0$ and $\partial_x l_0 = 0$ i.e. r_0 and l_0 are independant of x.

Then, we study the system verified by $\widetilde{V_\varepsilon} = V_\varepsilon - V_0$.

As V_ε is solution of

$$\partial_t V_\varepsilon + B_1(V_\varepsilon)\partial_x V_\varepsilon + B_2(V_\varepsilon)\partial_y V_\varepsilon + \varepsilon^{-1} N V_\varepsilon = 0$$

and V_0 belongs to $Ker(N)$, $\widetilde{V_\varepsilon}$ is solution of

$$\partial_t \widetilde{V_\varepsilon} + B_1(V_\varepsilon)\partial_x \widetilde{V_\varepsilon} + B_2(V_\varepsilon)\partial_y \widetilde{V_\varepsilon} + \varepsilon^{-1} N \widetilde{V_\varepsilon} = -B_2(V_\varepsilon)\partial_y V_0.$$

Moreover, $\| \widetilde{V_\varepsilon}(0) \|_{\mathcal{W}^{2n}}$ tends to zero as ε tends to zero.

The problem is that zero is not solution of this equation because the relation $L_+ l_0 + L_- r_0 = 0$ implies that $\partial_y \left(r_0 + l_0\right) = y \left(l_0 - r_0\right)$ so

$$B_2(V_0)\partial_y V_0 = \begin{pmatrix} 0 \\ \dfrac{1}{4}\left(r_0 + l_0\right)\partial_y \left(r_0 + l_0\right) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{1}{4} y \left(l_0^2 - r_0^2\right) \\ 0 \end{pmatrix} \neq 0 \cdot$$

That is the reason why we search solutions of

$$\partial_t W_\varepsilon + \varepsilon^{-1} N W_\varepsilon = \begin{pmatrix} 0 \\ \dfrac{1}{4} y \left(l_0^2 - r_0^2\right) \\ 0 \end{pmatrix} \cdot$$

If $W_\varepsilon = \begin{pmatrix} a_\varepsilon \\ b_\varepsilon \\ c_\varepsilon \end{pmatrix}$, then we want

$$\partial_t \begin{pmatrix} a_\varepsilon \\ b_\varepsilon \\ c_\varepsilon \end{pmatrix} + \varepsilon^{-1} \begin{pmatrix} -\partial_x a_\varepsilon - L_- b_\varepsilon \\ -L_+ a_\varepsilon + L_- c_\varepsilon \\ L_+ b_\varepsilon + \partial_x c_\varepsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4} y \left(l_0^2 - r_0^2\right) \\ 0 \end{pmatrix} \cdot$$

If we search a solution independant of t and x, then $L_- b_\varepsilon = L_+ b_\varepsilon = 0$ so $b_\varepsilon = 0$ and the system becomes

$$-L_+ a_\varepsilon + L_- c_\varepsilon = \frac{1}{\sqrt{2}} \partial_y \left(c_\varepsilon - a_\varepsilon\right) + \frac{1}{\sqrt{2}} y \left(c_\varepsilon + a_\varepsilon\right) = \frac{\varepsilon}{4} y \left(l_0^2 - r_0^2\right) \cdot$$

We search a_ε and c_ε as fonctions of $l_0^2 - r_0^2$, and we find that

$$c_\varepsilon = a_\varepsilon = \frac{\varepsilon}{4\sqrt{2}} \left(l_0^2 - r_0^2\right)$$

is the solution.

Moreover, the norm of this solution is controled by ε and $W_\varepsilon = \varepsilon W_1$ with W_1 belonging to $\mathcal{W}^{2n+1}$ solution of $NW_1 = B_2(V_0)\partial_y V_0$.

It remains to study the system satisfied by $\overline{V_\varepsilon} = \widetilde{V_\varepsilon} - \varepsilon W_1$.

As $\| \widetilde{V_\varepsilon}(0) \|_{\mathcal{W}^{2n}}$ tends to zero as ε tends to zero, so does $\| \overline{V_\varepsilon}(0) \|_{\mathcal{W}^{2n}}$.

Moreover, $\overline{V_\varepsilon}$ is solution of

$$\partial_t \overline{V_\varepsilon} + B_1(V_\varepsilon)\partial_x \overline{V_\varepsilon} + B_2(V_\varepsilon)\partial_y \overline{V_\varepsilon} + \varepsilon^{-1} N \overline{V_\varepsilon} = \overline{F_\varepsilon}$$

where
$$\begin{aligned}\overline{F_\varepsilon} &= -B_2(V_\varepsilon)\partial_y V_0 - \left(\varepsilon B_2(V_\varepsilon)\partial_y W_1 + NW_1\right) \\ &= -B_2(V_\varepsilon)\partial_y V_0 - \left(\varepsilon B_2(V_\varepsilon)\partial_y W_1 + B_2(V_0)\partial_y V_0\right) \\ &= -B_2(\widetilde{V_\varepsilon})\partial_y V_0 - \varepsilon B_2(V_\varepsilon)\partial_y W_1.\end{aligned}$$

In order to obtain a convergence in $\mathcal{W}^{2n}$, we apply the operator $\partial_x^k D^p$, with $k + 2p \leqslant 2n$ to the equation.

Given $V_\varepsilon^{k,p} = \partial_x^k D^p \overline{V_\varepsilon}$, we have

$$\partial_t V_\varepsilon^{k,p} + B_1(V_\varepsilon)\partial_x V_\varepsilon^{k,p} + B_2(V_\varepsilon)\partial_y V_\varepsilon^{k,p} + \varepsilon^{-1} N V_\varepsilon^{k,p} = \overline{F_\varepsilon^{k,p}}$$

with $\overline{F_\varepsilon^{k,p}} = \partial_x^k D^p \overline{F_\varepsilon} + \left[B_1(V_\varepsilon)\partial_x, \partial_x^k D^p\right] \overline{V_\varepsilon} + \left[B_2(V_\varepsilon)\partial_y, \partial_x^k D^p\right] \overline{V_\varepsilon}\cdot$

So, we have the estimate

$$\frac{1}{2}\frac{d}{dt} \| V_\varepsilon^{k,p} \|_{L^2}^2 \leqslant \| V_\varepsilon^{k,p} \|_{L^2} \| \overline{F_\varepsilon^{k,p}} \|_{L^2} + \frac{1}{2} \| \mathrm{div} B(V_\varepsilon) \|_{L^\infty} \| V_\varepsilon^{k,p} \|_{L^2}^2 \ \cdot$$

Moreover,
$$\begin{aligned}\| \mathrm{div} B(V_\varepsilon) \|_{L^\infty} &\leqslant C \| V_\varepsilon \|_{H^3} \\ &\leqslant C \| \overline{V_\varepsilon} \|_{H^3} + C \| V_0 \|_{H^3} + \varepsilon C \| W_1 \|_{H^3} \cdot\end{aligned}$$

It remains to estimate $\overline{F_\varepsilon^{k,p}}$ in L^2.

The arguments given in the section 4 of [1] prove that, for $n \geqslant 2$, we can estimate $\left[B_1(V_\varepsilon)\partial_x, \partial_x^k D^p\right] \overline{V_\varepsilon}$ and $\left[B_2(V_\varepsilon)\partial_y, \partial_x^k D^p\right] \overline{V_\varepsilon}$ in L^2 by

$$\| \overline{V_\varepsilon} \|_{\mathcal{W}^{2n}} \| V_\varepsilon \|_{\mathcal{W}^{2n}} \leqslant \| \overline{V_\varepsilon} \|_{\mathcal{W}^{2n}} \left(\| \overline{V_\varepsilon} \|_{\mathcal{W}^{2n}} + \| V_0 \|_{\mathcal{W}^{2n}} + \varepsilon \| W_1 \|_{\mathcal{W}^{2n}} \right) \cdot$$

It remains to estimate $\partial_x^k D^p \overline{F_\varepsilon}$ in L^2 with $\overline{F_\varepsilon} = -B_2(\widetilde{V_\varepsilon})\partial_y V_0 - B_2(V_\varepsilon)\partial_y W_\varepsilon \cdot$

This term is estimated by $\| \widetilde{V_\varepsilon} \|_{\mathcal{W}^{2n}} \| \partial_y V_0 \|_{\mathcal{W}^{2n}} + \varepsilon \| \partial_y W_1 \|_{\mathcal{W}^{2n}} \| V_\varepsilon \|_{\mathcal{W}^{2n}} \cdot$

As V_0 belongs to $\mathcal{W}^{2n+1}$, we have

$$\frac{1}{2}\frac{d}{dt} \| \overline{V_\varepsilon} \|^2_{\mathcal{W}^{2n}} \leqslant A_\varepsilon \| \overline{V_\varepsilon} \|_{\mathcal{W}^{2n}} + B_\varepsilon \| \overline{V_\varepsilon} \|^2_{\mathcal{W}^{2n}} + C \| \overline{V_\varepsilon} \|^3_{\mathcal{W}^{2n}}$$

with $A_\varepsilon = C \left(\varepsilon \| W_1 \|_{\mathcal{W}^{2n}} \| V_0 \|_{\mathcal{W}^{2n+1}} + \varepsilon \|W_1\|_{\mathcal{W}^{2n+1}} \left(\|V_0\|_{\mathcal{W}^{2n}} + \varepsilon \|W_1\|_{\mathcal{W}^{2n}} \right) \right)$
$\leqslant \varepsilon C'$

and $B_\varepsilon = C \left(\|V_0\|_{\mathcal{W}^{2n+1}} + \varepsilon \|W_1\|_{\mathcal{W}^{2n+1}} \right) \leqslant C' \cdot$

This can be written

$$\frac{d}{dt} \| \overline{V_\varepsilon} \|^2_{\mathcal{W}^{2n}} \leqslant R \left(\| \overline{V_\varepsilon} \|^3_{\mathcal{W}^{2n}} + \| \overline{V_\varepsilon} \|^2_{\mathcal{W}^{2n}} + \varepsilon \| \overline{V_\varepsilon} \|_{\mathcal{W}^{2n}} \right) \cdot$$

To conclude, we use the following two lemmas.

Lemma 3.1. — *If f is a positive fonction which verifies*

$$\begin{cases} |f'| \leqslant R\left(f^{3/2} + f + \varepsilon f^{1/2}\right) \\ f(0) = f_0 \end{cases}$$

then the life span of f tends to infinity as ε and f_0 tend to zero.

Moreover, for every $T > 0$, we have $\lim\limits_{\varepsilon, f_0 \to 0} \|f\|_{L^\infty[0,T]} = 0$.

Proof. — Let us define $T_\varepsilon = \mathrm{Sup}\,\{M : \forall t \in [0, M]\ ,\ f(t) \leqslant 1\}$.

On $[0, T_\varepsilon]$ we have $|f'| \leqslant R\,(2f + \varepsilon)$, so

$$\frac{f'}{2f + \varepsilon} = \left[\frac{1}{2}\log\left(2f + \varepsilon\right)\right]' \leqslant R.$$

Thus,

$$\log\left(2 + \varepsilon\right) - \log\left(2f_0 + \varepsilon\right) \leqslant 2RT_\varepsilon.$$

The left term tends to infinity as ε and f_0 tend to zero. So does T_ε.

For every $T > 0$ given, we have $T \leqslant T_\varepsilon$ when ε and f_0 are small enough, so for every $t \in [0, T]$,we have $2RT \geqslant \log\big(2f(t)+\varepsilon\big) - \log\big(2f_0+\varepsilon\big) \cdot$

This gives

$$f(t) \leqslant \frac{1}{2}\big((2f_0+\varepsilon)\exp(2RT) - \varepsilon\big) = C_{f_0,\varepsilon}$$

and $C_{f_0,\varepsilon}$ tends to zero as ε and f_0 tend to zero. $\square$

Lemma 3.2. — *The norms $\|NV_\varepsilon\|_{\mathcal{W}^{2n}}$ and $\|LU_\varepsilon\|_{\mathcal{W}^{2n}}$ are equivalents when ε tends to zero.*

Proof. — As in the proof of the lemma 2.6, we have

$$LU_\varepsilon = \begin{pmatrix} -yv_\varepsilon + \partial_x h_\varepsilon \\ yu_\varepsilon + \partial_y h_\varepsilon \\ \partial_x u_\varepsilon + \partial_y v_\varepsilon \end{pmatrix}$$

and using the Taylor formula at order two, we get

$$\begin{aligned} NV_\varepsilon &= \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_x u_\varepsilon + \partial_y v_\varepsilon + yv_\varepsilon - \dfrac{\partial_x h_\varepsilon}{\sqrt{1+\varepsilon h_\varepsilon}} \\ 2\left(yu_\varepsilon + \dfrac{\partial_y h_\varepsilon}{\sqrt{1+\varepsilon h_\varepsilon}}\right) \\ \partial_x u_\varepsilon + \partial_y v_\varepsilon - yv_\varepsilon + \dfrac{\partial_x h_\varepsilon}{\sqrt{1+\varepsilon h_\varepsilon}} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \partial_x u_\varepsilon + \partial_y v_\varepsilon + yv_\varepsilon - \partial_x h_\varepsilon \\ 2\,(yu_\varepsilon + \partial_y h_\varepsilon) \\ \partial_x u_\varepsilon + \partial_y v_\varepsilon - yv_\varepsilon + \partial_x h_\varepsilon \end{pmatrix} \Big(1 + \widetilde{G}(\varepsilon h_\varepsilon)\Big) \end{aligned}$$

where $\widetilde{G}$ is a regular fonction such as $\widetilde{G}(0) = 0$. Then we get the result thanks to Faa di Bruno formula. $\square$

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