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Ahlfors' currents in higher dimension

Henry de Thélin[1]

Résumé. — On considère une application holomorphe non dégénérée $f : V \mapsto X$ o (X, ω) est une variété Hermitienne compacte de dimension supérieure ou égale à k et V est une variété complexe, connexe, ouverte de dimension k. Dans cet article, nous donnons des critères qui permettent de construire des courants d'Ahlfors dans X.

Abstract. — We consider a nondegenerate holomorphic map $f : V \mapsto X$ where (X, ω) is a compact Hermitian manifold of dimension larger than or equal to k and V is an open connected complex manifold of dimension k. In this article we give criteria which permit to construct Ahlfors' currents in X.

0. Introduction

Let $f : V \mapsto X$ be a nondegenerate holomorphic map between an open connected complex manifold V (non-compact) of dimension k and a compact Hermitian manifold (X, ω) of dimension larger than or equal to k. We consider an exhaustion function τ on V. This means that (see [14]):

(i) $\tau : V \mapsto [0, +\infty[$ is C^1.

(ii) τ is proper (i.e. $\tau^{-1}(\text{compact}) = \text{compact}$).

(iii) There exists $r_0 > 0$ such that τ has only isolated critical points in $\tau^{-1}([r_0, +\infty[)$.

In this article we will employ the notation $V(r) = \tau^{-1}([0, r[)$.

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The first important example is $V = \mathbb{C}^k$ and $\tau = \|z\|^2$. When $k = 1$ we are studying entire cuves in X. Another example is that of a pseudoconvex domain V in $\mathbb{C}^k$. If τ_0 is its exhaustion function, we can easily transform τ_0 into a function τ which satisfies the previous hypothesis (see [11] p. 63-65).

The goal of this article is to construct Ahlfors' currents in X starting from V and f. By definition, an Ahlfors' current is a **closed** positive current of bidimension (k, k) which is the limit of a sequence $\frac{f_*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ (here $r_n \to +\infty$ and $\text{volume}(f(V(r_n))) := \int_{V(r_n)} f^*\omega^k$ is the volume of $f(V(r_n))$ counted with multiplicity). When $V = \mathbb{C}$ and $\tau = \|z\|^2$, M. McQuillan constructed such currents in [10] (see [1] too). These currents are fundamental tools in the study of the hyperbolicity of X (see for example [6]). When the dimension of V is larger than or equal to 2 it is not always possible to produce Ahlfors' currents. Indeed, for example, there exist domains Ω in $\mathbb{C}^2$ which are biholomorphic to $\mathbb{C}^2$ and such that $\overline{\Omega} \neq \mathbb{C}^2$ (Fatou-Bieberbach domains). As a consequence, to produce Ahlfors' currents it is necessary to add a hypothesis on f.

When the dimension of X is equal to k, there exist criteria which imply that $f(V)$ is dense in X (see [3], [13], [14], [8], [7], [2] and [12]). These criteria use the degrees of f (see [3]) or the growth of the function f.

Our goal is to give criteria which use these degrees in order to produce Ahlfors' currents in X. Of course, in the case where the dimension of X is equal to k, the existence of such currents will automatically imply that $f(V)$ is dense in X. Indeed, $[X]$ is the only positive closed current of bidimension (k, k) in X (up to normalization).

In this article, we will use the following degrees (t_{k-1} will be slightly different from Chern's one):

$$t_k(r) = \int_{V(r)} f^*\omega^k,$$

which is the volume of $f(V(r))$ counted with multiplicity, and

$$t_{k-1}(r) = \int_{V(r)} i\partial\tau \wedge \overline{\partial\tau} \wedge f^*\omega^{k-1}.$$

Let $\mathcal{C}$ be the set of critical values of τ in $[r_0, +\infty[$. V is connected and non-compact so we can suppose that $[r_0, +\infty[\subset \tau(V)$.

The criteria that we will give on t_k and t_{k-1} will strongly use the following inequality:

THEOREM 0.1. — *The functions* t_k *and* t_{k-1} *are* C^1 *on* $]r_0,+\infty[\setminus\mathcal{C}$ *and* C^0 *on* $]r_0,+\infty[$*. If* $r\in]r_0,+\infty[\setminus\mathcal{C}$ *then*

$$\|\partial f_*[V(r)]\|^2 \leqslant K(X)t'_{k-1}(r)t'_k(r).$$

Here $K(X)$ *is a constant which depends only on* (X,ω) *and*

$$\|\partial f_*[V(r)]\| := \sup_{\Psi\in\mathcal{F}(k-1,k)} |\langle \partial f_*[V(r)],\Psi\rangle|$$

where $\mathcal{F}(k-1,k)$ *is the set of smooth* $(k-1,k)$ *forms* Ψ *with* $\|\Psi\| := \max_{x\in X}\|\Psi(x)\|\leqslant 1$.

By using the previous inequality we can prove some criteria which imply the existence of Ahlfors' currents. Indeed, the difficulty for the construction of Ahlfors' currents is the closedness of a limit of $\frac{f_*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ and the previous Theorem gives an estimate for $\|\partial f_*[V(r_n)]\|$. Here we give the following two criteria:

THEOREM 0.2. — *We suppose that* f *is nondegenerate and of finite-type (i.e. there exist* C_1*,* C_2*,* $r_1>0$ *such that volume*$(f(V(r)))\leqslant C_1 r^{C_2}$ *for* $r\geqslant r_1$*).*

If

$$\limsup_{r\to+\infty}\frac{t_{k-1}(r)}{r^2 t_k(r)}=0$$

then there exists a sequence r_n *which goes to infinity such that* $\frac{f_*[V(r_n)]}{\mathit{volume}(f(V(r_n)))}$ *converges to a closed positive current with bidimension* (k,k) *and mass equal to* 1.

When $V=\mathbb{C}$ and $\tau=\|z\|^2$, the finite-type hypothesis holds modulo a Brody renormalization (see for example [9]).

We now give one criterion which does not use this hypothesis.

THEOREM 0.3. — *If* f *is nondegenerate and if there exist* $\varepsilon>0$ *and* $L>0$ *such that:*

$$\limsup_{r\notin\mathcal{C},\ r\to+\infty}\frac{t'_{k-1}(r)}{r t_k(r)^{1-\varepsilon}}\leqslant L$$

then there exists a sequence r_n *which goes to infinity such that* $\frac{f_*[V(r_n)]}{\mathit{volume}(f(V(r_n)))}$ *converges to a closed positive current with bidimension* (k,k) *and mass equal to* 1.

The plan of this article is the following: in the first part we prove the inequality (Theorem 0.1), in the second one we give the proof of both criteria (Theorems 0.2 and 0.3). In the third part, we give a new formulation of the criteria in the special case where $V = \mathbb{C}^k$.

1. Proof of the inequality

Let $\mathcal{C}$ be the set of critical values of τ in $[r_0, +\infty[$. We recall that we can suppose $[r_0, +\infty[\subset \tau(V)$. Notice that point (iii) in the hypothesis on τ implies that $\mathcal{C}$ is discrete. When $r \in]r_0, +\infty[$ and $r \notin \mathcal{C}$ then $\tau : \tau^{-1}(]r-\varepsilon, r+\varepsilon[) \mapsto]r-\varepsilon, r+\varepsilon[$ is a submersion for $\varepsilon > 0$ small enough. In particular, $\tau^{-1}(r)$ is a submanifold of V and $\partial V(r) = \tau^{-1}(r)$. When $r \in \mathcal{C}$, then $\tau^{-1}(r)$ is a compact set which is a submanifold of V outside a neighbourhood of a finite number of points.

We begin now with the following lemma:

Lemma 1.1. — *The functions t_k and t_{k-1} are C^1 on $]r_0, +\infty[\backslash \mathcal{C}$ and C^0 on $]r_0, +\infty[$.*

Proof. — The form $f^*\omega^k$ is positive and smooth and $i\partial\tau \wedge \overline{\partial\tau} \wedge f^*\omega^{k-1}$ is positive and continuous (τ is C^1) so it is enough to show that $t(r) = \int_{V(r)} \Phi$ is C^1 on $]r_0, +\infty[\backslash \mathcal{C}$ and C^0 on $]r_0, +\infty[$ with Φ a positive continuous form of bidegree (k, k).

We take $r \in]r_0, +\infty[\backslash \mathcal{C}$ and $\varepsilon > 0$ such that $\tau : \tau^{-1}(]r-\varepsilon, r+\varepsilon[) \mapsto]r-\varepsilon, r+\varepsilon[$ is a submersion. Now, if $r' \in]r-\varepsilon, r[$, we have:

$$\frac{t(r) - t(r')}{r - r'} = \frac{1}{r - r'} \int_{\tau^{-1}([r', r[)} \Phi = \frac{1}{r - r'} \int_{[r', r[} \tau_* \Phi.$$

The form $\tau_*\Phi$ is continuous so it is equal to $\alpha(s)ds$ with α in $C^0(]r-\varepsilon, r+\varepsilon[)$. We obtain:

$$\frac{t(r) - t(r')}{r - r'} = \frac{1}{r - r'} \int_{r'}^{r} \alpha(s)ds$$

which converges to $\alpha(r)$ when $r' \to r$. The same thing happens when we consider $r' \in]r, r+\varepsilon[$, so the function t is differentiable at r and $t'(r) = \alpha(r)$. In particular t is C^1 on $]r_0, +\infty[\backslash \mathcal{C}$.

Remark 1.2. — Notice that here we did not use that Φ is positive. We will use this remark in the proof of Theorem 0.1.

Now, consider $r \in \mathcal{C}$. If we take $\varepsilon > 0$, then we can find two neighbourhoods $W_\varepsilon \Subset W_{2\varepsilon}$ of the (finite) number of the critical points in $\{\tau = r\}$ such that $\int_{W_{2\varepsilon}} \Phi \leqslant \varepsilon$ (because Φ is continuous). Now, let ψ be a C^∞ function which is equal to 1 in a neighbourhood of $\overline{W_\varepsilon}$ and to 0 outside $W_{2\varepsilon}$ ($0 \leqslant \psi \leqslant 1$). Then, if $r' < r$,

$$t(r) - t(r') = \int_{V(r)\setminus V(r')} \psi\Phi + \int_{V(r)\setminus V(r')} (1-\psi)\Phi \leqslant \varepsilon + \int_{V(r)\setminus V(r')} (1-\psi)\Phi.$$

If $\alpha > 0$ is small then τ is a submersion on $\tau^{-1}(]r-\alpha, r+\alpha[) \cap (V \setminus W_\varepsilon)$. In particular the function

$$r' \mapsto \int_{V(r)\setminus V(r')} (1-\psi)\Phi = \int_{r'}^{r} \tau_*((1-\psi)\Phi)$$

goes to 0 when $r' \to r$. The same thing happens when we take $r' > r$. As a consequence, there exists $\delta > 0$ such that if $|r - r'| < \delta$ then $|t(r) - t(r')| \leqslant 2\varepsilon$, i.e. t is continuous at r. $\square$

We give now the proof of Theorem 0.1.

We take $r \in]r_0, +\infty[\setminus \mathcal{C}$. We have:

$$\|\partial f_*[V(r)]\| = \sup_{\Psi \in \mathcal{F}(k-1,k)} |\langle \partial f_*[V(r)], \Psi \rangle|$$

where $\mathcal{F}(k-1,k)$ is the set of smooth $(k-1,k)$ forms Ψ with $\|\Psi\| = \max_{x\in X} \|\Psi(x)\| \leqslant 1$. If $\Psi \in \mathcal{F}(k-1,k)$ then we can write (see for example [5] chapter III Lemma 1.4)

$$\Psi = \sum_{i=1}^{K(X)} \theta_i \wedge \Omega_i$$

where $K(X)$ is a constant which depends only on X, the θ_i are smooth forms of bidegree $(0,1)$ with $\|\theta_i\| \leqslant 1$ and the Ω_i are (strongly) positive smooth forms of bidegree $(k-1,k-1)$ with $\|\Omega_i\| \leqslant K(X)$. So, to prove the inequality it is sufficient to bound from above $|\langle \partial f_*[V(r)], \theta \wedge \Omega \rangle|^2$ by $K'(X) t'_{k-1}(r) t'_k(r)$ with θ a smooth form of bidegree $(0,1)$ with $\|\theta\| \leqslant 1$, Ω a positive smooth form of bidegree $(k-1,k-1)$ with $\|\Omega\| \leqslant 1$ and $K'(X)$ a constant which depends only on (X,ω).

If $\varepsilon > 0$ is small then $\tau : \tau^{-1}(]r-\varepsilon, r+\varepsilon[) \mapsto]r-\varepsilon, r+\varepsilon[$ is a submersion. Now, if we take $r' \in]r-\varepsilon, r[$, we have:

$$A(r',r) := \left| \frac{1}{r-r'} \int_{r'}^{r} \langle \partial f_*[V(s)], \theta \wedge \Omega \rangle ds \right|$$
$$= \left| \frac{1}{r-r'} \int_{r'}^{r} \langle \partial [V(s)], f^*\theta \wedge f^*\Omega \rangle ds \right|.$$

If we use the Stokes' Theorem, we have:

$$A(r',r) = \left| \frac{1}{r-r'} \int_{r'}^{r} \langle [\partial V(s)], f^*\theta \wedge f^*\Omega \rangle ds \right|$$
$$= \left| \frac{1}{r-r'} \int_{r'}^{r} \langle [\tau = s], f^*\theta \wedge f^*\Omega \rangle \, ds \right|,$$

because for $s \in]r-\varepsilon, r+\varepsilon[$ the boundary of $V(s)$ is $\{\tau = s\}$.

We obtain:

$$A(r',r) = \left| \frac{1}{r-r'} \int_{r'}^{r} \left(\int_{\tau = s} f^*\theta \wedge f^*\Omega \right) ds \right|.$$

Now $\tau : \tau^{-1}(]r-\varepsilon, r+\varepsilon[) \mapsto]r-\varepsilon, r+\varepsilon[$ is a submersion, so by using Fubini's Theorem (see [4] p. 334), we have:

$$A(r',r) = \left| \frac{1}{r-r'} \int_{V(r) \setminus V(r')} d\tau \wedge f^*\theta \wedge f^*\Omega \right|$$
$$= \left| \frac{1}{r-r'} \int_{V(r) \setminus V(r')} \partial\tau \wedge f^*\theta \wedge f^*\Omega \right|.$$

Now, if we consider,

$$\{\phi, \psi\} := \int_{V(r) \setminus V(r')} i\phi \wedge \overline{\psi} \wedge f^*\Omega$$

where ϕ and ψ are continuous forms of bidegree $(1,0)$, then $\{\phi, \phi\} \geqslant 0$ (because Ω is positive) and so by using the proof of the Cauchy-Schwarz's inequality we obtain that:

$$|\{\phi, \psi\}| \leqslant (\{\phi, \phi\})^{1/2} (\{\psi, \psi\})^{1/2}.$$

In particular,

$$A(r',r)^2 \leqslant \left| \frac{1}{r-r'} \int_{V(r)\setminus V(r')} i\partial\tau \wedge \overline{\partial\tau} \wedge f^*\Omega \right|$$
$$\times \left| \frac{1}{r-r'} \int_{V(r)\setminus V(r')} i\overline{f^*\theta} \wedge f^*\theta \wedge f^*\Omega \right|.$$

Now $i\overline{f^*\theta} \wedge f^*\theta \wedge f^*\Omega$ is equal to $f^*(i\overline{\theta} \wedge \theta \wedge \Omega)$ and $i\overline{\theta} \wedge \theta \wedge \Omega \leqslant K'(X)\omega^k$ (which means that $K'(X)\omega^k - i\overline{\theta} \wedge \theta \wedge \Omega$ is a (strongly) positive form). Here $K'(X)$ depends only on (X,ω) because $\|\theta\| \leqslant 1$ and $\|\Omega\| \leqslant 1$.

As a consequence, we have:

$$\left| \frac{1}{r-r'} \int_{V(r)\setminus V(r')} i\overline{f^*\theta} \wedge f^*\theta \wedge f^*\Omega \right| \leqslant K'(X) \left| \frac{1}{r-r'} \int_{V(r)\setminus V(r')} f^*\omega^k \right|$$
$$= K'(X) \left(\frac{t_k(r) - t_k(r')}{r-r'} \right).$$

On the other hand, there exists a constant $K''(X)$ with $\Omega \leqslant K''(X)\omega^{k-1}$ (we use $\|\Omega\| \leqslant 1$). So, we have

$$\left| \frac{1}{r-r'} \int_{V(r)\setminus V(r')} i\partial\tau \wedge \overline{\partial\tau} \wedge f^*\Omega \right| \leqslant K''(X) \left(\frac{t_{k-1}(r) - t_{k-1}(r')}{r-r'} \right).$$

We obtain:

$$A(r',r)^2 \leqslant K(X) \left(\frac{t_{k-1}(r) - t_{k-1}(r')}{r-r'} \right) \left(\frac{t_k(r) - t_k(r')}{r-r'} \right). \tag{1.1}$$

Now, when $r' \to r$

$$A(r',r)^2 \to |\langle \partial f_*[V(r)], \theta \wedge \Omega \rangle|^2$$

because the function $s \mapsto \langle \partial f_*[V(s)], \theta \wedge \Omega \rangle = -\int_{V(s)} \partial f^*(\theta \wedge \Omega)$ is continuous on $]r-\varepsilon, r+\varepsilon[$ (see remark 1.2).

Finally, if we take $r' \to r$ in the inequality (1.1), we have:

$$|\langle \partial f_*[V(r)], \theta \wedge \Omega \rangle|^2 \leqslant K(X) t'_{k-1}(r) t'_k(r)$$

which gives the desired inequality.

2. Proof of Theorems 0.2 and 0.3

2.1. Proof of the first criterion

We begin with this lemma:

LEMMA 2.1. — *If f is nondegenerate and of finite-type then there exists a constant $K > 0$ such that:*

$$\forall r_2 > 0 \ \exists r \geqslant r_2 \text{ with } volume(f(V(2r))) \leqslant K volume(f(V(r))).$$

Proof. — The hypothesis implies that there exist C_1, C_2, $r_1 > 0$ such that $\text{volume}(f(V(r))) \leqslant C_1 r^{C_2}$ for $r \geqslant r_1$.

If the conclusion of the lemma fails then for all $K > 0$ there exists $r_2 > 0$ such that for all $r \geqslant r_2$ we have $\text{volume}(f(V(2r))) \geqslant K\text{volume}(f(V(r)))$.

So, if we take $K >> 2^{C_2}$ then we obtain (if l is large enough):

$$C_1(2^l r_2)^{C_2} \geqslant \text{volume}(f(V(2^l r_2))) \geqslant K^l \text{volume}(f(V(r_2))).$$

As a consequence we have

$$\text{volume}(f(V(r_2))) \leqslant C_1 r_2^{C_2} \left(\frac{2^{C_2}}{K}\right)^l$$

which implies that $\text{volume}(f(V(r_2))) = 0$ when we take $l \to \infty$. It contradicts the fact that f is nondegenerate. □

By using this lemma, we can find a sequence $R_n \to +\infty$ which satisfies

$$\text{volume}(f(V(2R_n))) \leqslant K\text{volume}(f(V(R_n))).$$

Theorem 0.1 gives now that:

$$\int_{R_n}^{2R_n} \|\partial f_*[V(r)]\| dr \leqslant K(X) \int_{R_n}^{2R_n} \sqrt{t'_{k-1}(r)}\sqrt{t'_k(r)} dr.$$

We give the following sense to the integrals: for example, if there is one point a_n of $\mathcal{C}$ in $[R_n, 2R_n]$, we consider $\int_{R_n}^{2R_n} = \lim_{\varepsilon\to 0} \int_{[R_n, a_n-\varepsilon]\cup[a_n+\varepsilon, 2R_n]}$. All the functions that we consider are non negative, so the limit exists in $[0, +\infty]$.

Now, by using the Cauchy-Schwarz's inequality, the last integral is smaller than

$$K(X)\left(\int_{R_n}^{2R_n} t'_{k-1}(r)dr\right)^{1/2}\left(\int_{R_n}^{2R_n} t'_k(r)dr\right)^{1/2} \leqslant K(X)\sqrt{t_{k-1}(2R_n)}\sqrt{t_k(2R_n)}.$$

For the last inequality it is important to use that t_{k-1} and t_k are continuous on $]r_0,+\infty[$ (see Theorem 0.1).

It implies that there exists a sequence $r_n \in [R_n, 2R_n]$ such that:

$$\|\partial f_*[V(r_n)]\| \leqslant \frac{K(X)}{R_n}\sqrt{t_{k-1}(2R_n)}\sqrt{t_k(2R_n)},$$

i.e.

$$\frac{\|\partial f_*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \leqslant 2K(X)\sqrt{\frac{t_{k-1}(2R_n)}{(2R_n)^2 t_k(2R_n)}} \times \frac{t_k(2R_n)}{t_k(r_n)}$$

because $\text{volume}(f(V(r_n))) = t_k(r_n)$.

Now we have

$$\frac{t_k(2R_n)}{t_k(r_n)} \leqslant \frac{t_k(2R_n)}{t_k(R_n)} \leqslant K$$

and by using the hypothesis,

$$\sqrt{\frac{t_{k-1}(2R_n)}{(2R_n)^2 t_k(2R_n)}} \to 0.$$

So, we obtain that

$$\frac{\|\partial f_*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0.$$

The current $T_n := \frac{f_*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ is positive with bidimension (k,k) and mass equal to 1, so there exists a subsequence of (T_n) which converges to a positive current T with bidimension (k,k) and mass 1. Moreover,

$$\|\partial T_n\| = \frac{\|\partial f_*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0,$$

so the limit current T is closed. This proves the first criterion.

2.2. Proof of the second criterion

Take $\varepsilon > 0$ and $L > 0$ such that

$$\limsup_{r \notin \mathcal{C},\ r \to +\infty} \frac{t'_{k-1}(r)}{r t_k(r)^{1-\varepsilon}} \leqslant L.$$

Let R_n be a sequence of positive reals which goes to $+\infty$. By using Theorem 0.1, we have (see the proof of the last criterion for the definition of the integrals):

$$\int_{r_0+1}^{R_n} \frac{\|\partial f_*[V(r)]\|^2}{t'_{k-1}(r) t_k(r)^{1+\varepsilon}} dr \leqslant K(X) \int_{r_0+1}^{R_n} \frac{t'_k(r)}{t_k(r)^{1+\varepsilon}} dr.$$

This last integral is smaller than $\frac{K(X)}{\varepsilon t_k(r_0+1)^{\varepsilon}} \leqslant K'(X, f)$ (here we use the fact that $\frac{1}{t_k(r)}$ is continuous on $]r_0, +\infty[$).

So, we have

$$\int_{r_0+1}^{+\infty} \frac{1}{r} \left(\frac{r \|\partial f_*[V(r)]\|^2}{t'_{k-1}(r) t_k(r)^{1+\varepsilon}} \right) dr \leqslant K'(X, f),$$

and $\int_{r_0+1}^{+\infty} \frac{1}{r} dr = +\infty$ implies that there exists a sequence $r_n \to +\infty$ such that $r_n \notin \mathcal{C}$ and:

$$\varepsilon(n) := \frac{r_n \|\partial f_*[V(r_n)]\|^2}{t'_{k-1}(r_n) t_k(r_n)^{1+\varepsilon}} \to 0.$$

We obtain

$$\left(\frac{\|\partial f_*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \right)^2 = \frac{\varepsilon(n)}{r_n} \frac{t'_{k-1}(r_n)}{t_k(r_n)^{1-\varepsilon}} \leqslant (L+1)\varepsilon(n),$$

by hypothesis (for n large enough).

So,

$$\frac{\|\partial f_*[V(r_n)]\|}{\text{volume}(f(V(r_n)))} \to 0.$$

Now, by using exactly the same argument as in the proof of the previous criterion, we obtain that there exists a subsequence of $T_n := \frac{f_*[V(r_n)]}{\text{volume}(f(V(r_n)))}$ which converges to a closed positive current of bidimension (k, k) and with mass equal to 1.

3. The special case $V = \mathbb{C}^k$

In this paragraph we consider the special case where $V = \mathbb{C}^k$.

Let β be the standard Kähler form in $\mathbb{C}^k$. We want to transform our previous criteria by using β instead of $i\partial\tau\wedge\overline{\partial\tau}$. More precisely, we consider:

$$a_k(r) = \int_{B(0,r)} f^*\omega^k$$

and

$$a_{k-1}(r) = \int_{B(0,r)} \beta\wedge f^*\omega^{k-1}.$$

Then we can prove a new formulation of our three Theorems:

THEOREM 3.1. — *The functions a_k and a_{k-1} are C^1 on $]0,+\infty[$ and for $r>0$ we have*

$$\|\partial f_*[B(0,r)]\|^2 \leqslant K(X)a'_{k-1}(r)a'_k(r).$$

Here $\|.\|$ is the norm in the sense of currents and $K(X)$ is a constant which depends only on (X,ω).

Proof. — We apply Theorem 0.1 with $V=\mathbb{C}^k$ and $\tau=\|z\|^2$ (here we have $\mathcal{C}=\{0\}$) and then for $r>0$:

$$\|\partial f_*[V(r^2)]\|^2 \leqslant K'(X)t'_{k-1}(r^2)t'_k(r^2).$$

Now, $a_k(r)=t_k(r^2)$, so a_k is C^1 in $]0,+\infty[$ and

$$t'_k(r^2) = \frac{a'_k(r)}{2r}.$$

The function $a_{k-1}(r)=t(r^2)$ with $t(r)=\int_{V(r)}\beta\wedge f^*\omega^{k-1}$ so a_{k-1} is C^1 in $]0,+\infty[$ (see proof of Lemma 1.1).

Moreover,

$$t_{k-1}(r^2) = \int_{V(r^2)} i\partial\tau\wedge\overline{\partial\tau}\wedge f^*\omega^{k-1} = \int_{B(0,r)} i\partial\tau\wedge\overline{\partial\tau}\wedge f^*\omega^{k-1},$$

and $i\partial\tau\wedge\overline{\partial\tau} = i\sum_{i,j}\overline{z_i}z_j dz_i\wedge d\overline{z_j}$.

On $B(0,r)$ this last form is smaller than $K(k)\beta r^2$.

If we take $0 < r' < r$ then

$$t_{k-1}(r^2) - t_{k-1}(r'^2) = \int_{B(0,r)\setminus B(0,r')} i\partial\tau \wedge \overline{\partial\tau} \wedge f^*\omega^{k-1}$$

$$\leqslant K(k) r^2 \int_{B(0,r)\setminus B(0,r')} \beta \wedge f^*\omega^{k-1}.$$

If we divide by $r - r'$ and take the limit $r' \to r$, we obtain:

$$2rt'_{k-1}(r^2) \leqslant K(k) r^2 a'_{k-1}(r).$$

Finally, we have:

$$\|\partial f_*[B(0,r)]\|^2 = \|\partial f_*[V(r^2)]\|^2 \leqslant K'(X) t'_{k-1}(r^2) t'_k(r^2) \leqslant K(X) a'_{k-1}(r) a'_k(r),$$

with $K(X) = K(k)K'(X)$ (we recall that the dimension of X is larger than or equal to k). This is the inequality that we were looking for. □

Now if we replace in the proof of Theorems 0.2 and 0.3 the function t_{k-1} by a_{k-1}, the function t_k by a_k and $V(r)$ by $B(0,r)$ then we obtain the two following criteria:

THEOREM 3.2. — *We suppose that f is nondegenerate and with finite-type (i.e. there exist C_1, C_2, $r_1 > 0$ such that volume$(f(B(0,r))) \leqslant C_1 r^{C_2}$ for $r \geqslant r_1$).*

If

$$\limsup_{r\to+\infty} \frac{a_{k-1}(r)}{r^2 a_k(r)} = 0$$

then there exists a sequence r_n which goes to infinity such that $\frac{f_*[B(0,r_n)]}{volume(f(B(0,r_n)))}$ converges to a closed positive current with bidimension (k,k) and mass equal to 1.

THEOREM 3.3. — *If f is nondegenerate and if there exist $\varepsilon > 0$ and $L > 0$ such that:*

$$\limsup_{r\to+\infty} \frac{a'_{k-1}(r)}{r a_k(r)^{1-\varepsilon}} \leqslant L$$

then there exists a sequence r_n which goes to infinity such that $\frac{f_*[B(0,r_n)]}{volume(f(B(0,r_n)))}$ converges to a closed positive current with bidimension (k,k) and mass equal to 1.

Notice that when $k = 1$ then $a_{k-1}(r) = \pi r^2$ and therefore, in this context, the hypothesis of this criterion is always fulfilled if f is nondegenerate.

Bibliography

[1] Brunella (M.). — Courbes entières et feuilletages holomorphes, Enseign. Math., 45, p. 195-216 (1999).

[2] Carlson (J.A.) and Griffiths (P.). — The order functions for entire holomorphic mappings, Proc. Tulane Univ. Program, p. 225-248 (1974).

[3] Chern (S.-S.). — The integrated form of the first main theorem for complex analytic mappings in several complex variables, Ann. of Math. (2), 71, p. 536-551 (1960).

[4] Chirka (E.M.). — Complex analytic sets, Kluwer Academic Publishers (1989).

[5] Demailly (J.-P.). — Complex analytic and algebraic geometry, http://www-fourier.ujf-grenoble.fr/~demailly/books.html, 1997.

[6] Duval (J.). — Singularités des courants d'Ahlfors, Ann. Sci. Ecole Norm. Sup., 39, p. 527-533 (2006).

[7] Griffiths (P.). — Some remarks on Nevanlinna theory, Proc. Tulane Univ. Program, p. 1-11 (1974).

[8] Hirschfelder (J.J.). — The first main theorem of value distribution in several variables, Invent. Math., 8, p. 1-33 (1969).

[9] Lang (S.). — Introduction to complex hyperbolic spaces, Springer-Verlag (1987).

[10] McQuillan (M.). — Diophantine approximations and foliations, Inst. Hautes Etudes Sci. Publ. Math., 87, p. 121-174 (1998).

[11] Range (R.M.). — Holomorphic functions and integral representations in several complex variables, Springer-Verlag (1986).

[12] Sibony (N.) and Wong (P.M.). — Some remarks on the Casorati-Weierstrass theorem, Ann. Polon. Math., 39, p. 165-174 (1981).

[13] Stoll (W.). — A general first main theorem of value distribution, Acta Math., 118, p. 111-191 (1967).

[14] Wu (H.). — Remarks on the first main theorem in equidistribution theory II, J. Differential Geometry, 2, p. 369-384 (1968).