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An elementary proof of the Briançon-Skoda theorem

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Abstract. — We give an elementary proof of the Briançon-Skoda theorem. The theorem gives a criterionfor when a function ϕ belongs to an ideal I of the ring of germs of analytic functions at $0 \in \mathbb{C}^n$; more precisely, the ideal membership is obtained if a function associated with ϕ and I is locally square integrable. If I can be generated by m elements,it follows in particular that $\overline{I^{\min(m,n)}} \subset I$, where $\overline{J}$ denotes the integral closure of an ideal J.

Résumé. — Nous proposons une démonstration élémentaire du théorème de Briançon-Skoda. Ce théorème donne un critère d'appartenance d'une fonction ϕ à un idéal I de l'anneau des germes de fonctions holomorphes en $0 \in \mathbb{C}^n$; plus précisement, l'appartenance est établie sous l'hypothèse qu'une fonction dépendante de ϕ et I soit de carré localement sommable. En partiulier, si I est engendré par m éléments, alors $\overline{I^{\min(m,n)}} \subset I$, où $\overline{J}$ dénote la clôture intégrale d'un idéal J.

1. Introduction

Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$. The integral closure $\overline{I}$ of an ideal I is the set of all $\phi \in \mathcal{O}_n$ such that

$$\phi^N + a_1\phi^{N-1} + \ldots + a_N = 0, \tag{1.1}$$

for some integer $N \geqslant 1$ and some $a_k \in I^k,\ k = 1, \ldots, N$.

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By a simple estimate, (1.1) implies that there exists a constant C such that

$$|\phi| \leqslant C|f|, \tag{1.2}$$

where $|f|$ is defined as $\sum |f_i|$ for any generators f_i of I. It is easy to see that the choice of generators f_i does not affect whether ϕ satisfies (1.2) for some C or not.

Conversely, (1.2) implies that $\phi \in \overline{I}$ (however, we do not need this in the present paper), which is a consequence of Skoda's theorem, [S72] and a well-known determinant trick, see for example [D07], (10.5), Ch. VIII. Another proof is given in (the republication) [LTR08].

Theorem 1.1 (Briançon-Skoda). — *Let I be an ideal of $\mathcal{O}_n$ generated by m germs $f_1, \ldots, f_m$. Then $\overline{I^{\min(m,n)+l-1}} \subset I^l$ for all integers $l \geqslant 1$.*

As noted above, $\phi \in \overline{I^{\min(m,n)+l-1}}$ implies that $|\phi| \leqslant C|f|^{\min(m,n)+l-1}$. Thus it suffices to show that any $\phi \in \mathcal{O}_n$ that satisfies this size condition belongs to I^l, in order to prove Theorem 1.1.

Another ideal that is common to consider is $\hat{I}^{(k)}$ which consists of all $\phi \in \mathcal{O}_n$ such that

$$\int_U |\phi|^2 |f|^{-2(k+\varepsilon)} dV < \infty, \tag{1.3}$$

for some neighbourhood U of $0 \in \mathbb{C}^n$ and some (sufficiently small) $\varepsilon > 0$, where dV is the Lebesgue measure.

Lemma 2.3 implies that $\overline{I^k} \subset \hat{I}^{(k)}$. The following theorem is thus a stronger version of Theorem 1.1:

Theorem 1.2. — *For an ideal I as in Theorem 1.1, we have*

$$\hat{I}^{(\min(m,n)+l-1)} \subset I^l,$$

for all integers $l \geqslant 1$.

In 1974 Briançon and Skoda, [BS74], showed Theorem 1.2 as an immediate consequence of Skoda's L^2-division-theorem, [S72]. Usually Theorem 1.1 is the one referred to as the Briançon-Skoda theorem.

An algebraic proof of Theorem 1.1 was given by Lipman and Tessier in [LT81]. Their paper also contains a historical summary. An account of more recent developments and an elementary algebraic proof of the result is found in Schoutens [Sc03].

Berenstein, Gay, Vidras and Yger [BGVY93] proved Theorem 1.1 for $l = 1$ by finding a representation $\phi = \sum u_i f_i$ with u_i as explicit integrals. However, some of their estimates rely on Hironaka's theorem on resolutions of singularities.

In this paper, we provide a completely elementary proof along these lines. The key point is an L^1-estimate (Proposition 2.1), which will be used in Section 4.

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2. The Main Estimate

In order to state Proposition 2.1, we will first recall the notion of the (standard) norm of a differential form in $\mathbb{C}^n$. If x_i and y_i, $1 \leqslant i \leqslant n$, are standard coordinates for $\mathbb{C}^n = \mathbb{R}^{2n}$, this norm is uniquely determined by demanding that the forms $dx_{i_1} \wedge \ldots \wedge dx_{i_j} \wedge dy_{i_{j+1}} \wedge \ldots \wedge dy_{i_k}$ constitute an orthonormal basis (over $\mathbb{C}$) of $\bigwedge^k T_p^*\mathbb{C}^n$.

Proposition 2.1. — *Let $f_1, f_2, \ldots, f_m$ be generators of an ideal $I \subset \mathcal{O}_n$, and assume that $\phi \in \hat{I}^{(k)}$. Then for any integer $1 \leqslant r \leqslant m$,*

$$\frac{|\phi| \cdot |\partial f_1 \wedge \ldots \wedge \partial f_r|}{|f|^{k+r}}$$

is locally integrable at the origin.

Remark 2.2. — Using a Hironaka resolution, the proof of Proposition 2.1 can be reduced to the case when every f_i is a monomial, and then the proof becomes much easier. We proceed however with elementary arguments.

Lemma 2.3. — *For any ideal $I = (f_1, \ldots, f_m) \neq (0)$, there is a positive number δ such that $1/|f|^\delta$ is locally integrable at the origin.*

Proof. — By considering $F = f_1 \cdot f_2 \cdot \ldots \cdot f_m$ (remove any f_j that are identically zero), it suffices to show that $1/|F|^\delta$ is locally integrable. We can assume that F is a Weierstrass polynomial and we consider the integral of $1/|F|^\delta$ on $\Omega = D \times \Delta$, where D is a disk and $\Delta = D^{n-1}$. By choosing D small enough, Rouché's theorem gives that F has the same number of roots, s, on each slice $S_p = D \times \{p\}$, $p \in \Delta$. We partition S_p into sets E_j^p, one for each root $\alpha_j(p) \in S_p$, such that E_j^p consists of those points which are closer to $\alpha_j(p)$ than to the other roots. We have $F(z,p) = \prod_1^s (z - \alpha_j(p))$, so on E_j^p we get $1/|F|^\delta \leqslant |z - \alpha_j(p)|^{-\delta s}$. If δ is sufficiently small, we thus get a uniform bound for the (one variable) integral of $1/|F|^\delta$ on S_p. Fubini's theorem then gives the integrability on Ω. □

Proof of Proposition 2.1. — We assume for the sake of simplicity that $r = m$, but the proof works for the other cases as well. We begin by applying Hölder's inequality to the product of $|\phi|/|f|^{k+\delta'/2}$ and $|\partial f_1 \wedge \ldots \wedge \partial f_m|/|f|^{m-\delta'/2}$. Assume that δ' is small enough to make the first factor L^2-integrable. It thus suffices to show that

$$F = \frac{|\partial f_1 \wedge \ldots \wedge \partial f_m|^2}{\prod_1^m |f_j|^{2-\delta}}$$

is locally integrable for any $\delta > 0$. We will proceed to show that this is a consequence of the Chern-Levine-Nirenberg inequalities. The special case of these inequalities that is needed here will be proved without explicitly relying on facts about positive forms or plurisubharmonic functions. For a shorter proof of the Chern-Levine-Nirenberg inequalities, which involves these notions, see [D07] (3.3), Ch. III.

Let us first set

$$\beta = \frac{i}{2}\partial\overline{\partial}|\zeta|^2 = \frac{i}{2}\sum d\zeta_j \wedge d\overline{\zeta_j}, \quad \text{and} \quad \beta_k = \frac{\beta^k}{k!}.$$

Then β_n is the Lebesgue measure dV. A simple argument gives that for any $(1,0)$-forms α_j,

$$\frac{i}{2}\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \wedge \frac{i}{2}\alpha_p \wedge \overline{\alpha_p} \wedge \beta_{n-p} = |\alpha_1 \wedge \ldots \wedge \alpha_p|^2 dV. \tag{2.1}$$

Fix a sufficiently small $\delta > 0$ as in Lemma 2.3. We will need at least $\delta < 2$ in the sequel. We now compute

$$\partial\overline{\partial}(|f_j|^2 + \varepsilon)^{\delta/2} = \frac{\delta}{2}\left(1 + \frac{\left(\frac{\delta}{2} - 1\right)|f_j|^2}{|f_j|^2 + \varepsilon}\right)(|f_j|^2 + \varepsilon)^{\delta/2-1}\partial f_j \wedge \overline{\partial f_j},$$

which yields that

$$\frac{i\partial f_j \wedge \overline{\partial f_j}}{(|f_j|^2 + \varepsilon)^{1-\delta/2}} = G_j i\partial\overline{\partial}(|f_j|^2 + \varepsilon)^{\delta/2}, \tag{2.2}$$

where

$$G_j = \frac{2}{\delta}\left[1 + \left(\frac{\delta}{2} - 1\right)\frac{|f_j|^2}{|f_j|^2 + \varepsilon}\right]^{-1}.$$

Observe that

$$\left(\frac{2}{\delta}\right) \leqslant G_j \leqslant \left(\frac{2}{\delta}\right)^2. \tag{2.3}$$

We introduce forms $F_\varepsilon^k dV$ by setting

$$\begin{aligned} F_\varepsilon^k dV &= \frac{|\partial f_k \wedge \ldots \wedge \partial f_m|^2}{\prod_k^m \left(|f_j|^2 + \varepsilon\right)^{1-\delta/2}} dV = \frac{\prod_k^m \left(\frac{i}{2}\partial f_j \wedge \overline{\partial f_j}\right) \wedge \beta_{n+k-m-1}}{\prod_k^m \left(|f_j|^2 + \varepsilon\right)^{1-\delta/2}} \\ &= \prod_k^m G_j \frac{i}{2}\partial\overline{\partial}\left(|f_j|^2 + \varepsilon\right)^{\delta/2} \wedge \beta_{n+k-m-1}. \end{aligned} \tag{2.4}$$

Note that $F_\varepsilon^1 dV$ is a regularization of FdV. From the equality $|w \wedge \overline{w}| = 2^p|w|^2$, that holds for all $(p,0)$-forms w, and 2.2, we get

$$F_\varepsilon^k dV = \frac{\left|\prod_k^m \left(\frac{i}{2}\partial f_j \wedge \overline{\partial f_j}\right)\right| dV}{\prod_k^m \left(|f_j|^2 + \varepsilon\right)^{1-\delta/2}} = \left|\prod_k^m G_j \frac{i}{2}\partial\overline{\partial}\left(|f_j|^2 + \varepsilon\right)^{\delta/2}\right| dV. \tag{2.5}$$

Comparing (2.4) with (2.5), we get

$$H_\varepsilon^k dV := \prod_k^m i\partial\overline{\partial}\left(|f_j|^2 + \varepsilon\right)^{\delta/2} \wedge \beta_{n+k-m-1} = \left|\prod_k^m i\partial\overline{\partial}\left(|f_j|^2 + \varepsilon\right)^{\delta/2}\right| dV. \tag{2.6}$$

Let B be a ball about the origin and let χ_B be a smooth cut-off function supported in a concentric ball of twice the radius. We now use (2.5), (2.6) and (2.3) and integrate by parts (going from the second to the third line below) to see that

$$\begin{aligned} &\int_B F_\varepsilon^1 dV \leqslant C_\delta \int \chi_B \left|i\partial\overline{\partial}\left(|f_1|^2 + \varepsilon\right)^{\frac{\delta}{2}} \wedge \ldots \wedge i\partial\overline{\partial}\left(|f_m|^2 + \varepsilon\right)^{\frac{\delta}{2}}\right| dV \\ &= C_\delta \int \chi_B i\partial\overline{\partial}\left(|f_1|^2 + \varepsilon\right)^{\delta/2} \wedge \ldots \wedge i\partial\overline{\partial}\left(|f_m|^2 + \varepsilon\right)^{\delta/2} \wedge \beta_{n-m} \\ &= C_\delta \left|\int \left(\partial\overline{\partial}\chi_B\right)\left(|f_1|^2 + \varepsilon\right)^{\delta/2} \wedge \ldots \wedge i\partial\overline{\partial}\left(|f_m|^2 + \varepsilon\right)^{\delta/2} \wedge \beta_{n-m}\right| \\ &\leqslant C_1 C_\delta \sup_{2B} |f_1|^\delta \int_{2B} \left|i\partial\overline{\partial}\left(|f_2|^2 + \varepsilon\right)^{\frac{\delta}{2}} \wedge \ldots \wedge i\partial\overline{\partial}\left(|f_m|^2 + \varepsilon\right)^{\frac{\delta}{2}}\right| dV \\ &\leqslant C_1 C_\delta \sup_{2B} |f_1|^\delta \int \chi_{2B} H_\varepsilon^2 dV, \end{aligned}$$

where $C_\delta = 2^m/\delta^{2m}$ and $C_1 = \sup \chi_B$. Should the reader have any doubts about the integration by parts, note that $d(\alpha\wedge\beta\wedge\gamma) = \partial\alpha\wedge\beta\wedge\gamma+\alpha\wedge\partial\beta\wedge\gamma$, for any function α and forms β and γ such that γ is a closed $(n-1,n-1)$-form and β is a $(0,1)$-form. A similar relation holds for the $\overline{\partial}$-operator. Since the second integral on the first line in the calculation above is nothing but $\int \chi_B H^1_\varepsilon dV$, we can proceed by induction over k to obtain

$$\int_B |F_\varepsilon| dV \leqslant \frac{C}{\delta^{2m}} \sup_{2^{m+1}B} |f_1\cdot\ldots\cdot f_m|^\delta < \infty,$$

so if we let ε tend to zero, we get the desired bound. □

Remark 2.4. — It is not hard to see that essentially the same proof gives that $|\partial f_1\wedge\ldots\wedge\partial f_r|/\prod_1^r |f_i|$ is locally integrable.

3. Division by weighted integral formulas

We will use a division formula introduced in [B83],but for convenience, we use the formalism from [A03] to describe it.

Consider a fixed point $z\in\mathbb{C}^n$ and define the operator $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}$, where $\delta_{\zeta-z}$ is contraction with the vector field

$$2\pi i \sum_1^n (\zeta_k - z_k)\frac{\partial}{\partial\zeta_k}.$$

Recall that $\delta_{\zeta-z}$ anti-commutes with $\overline{\partial}$.We allow these operators to act on forms of all bidegrees. In particular, the contraction of a function is zero.

A *weight* with respect to z is a smooth differential form $g = g_{0,0}+g_{1,1}+\ldots+g_{n,n}$ such that $\nabla_{\zeta-z} g = 0$ and $g_{0,0}(z) = 1$. The subscripts denote bidegree.

Let s be any $(1,0)$-form such that $\delta_{\zeta-z}s = 1$ outside of $\{\zeta = z\}$, e.g.,

$$s = \frac{\partial|\zeta|^2}{2\pi i\left(|\zeta|^2 - \overline{\zeta}\cdot z\right)},$$

where the dot sign denotes the pairing given by $a\cdot b = \sum a_i b_i$. Next we set

$$u = s + s\wedge\overline{\partial}s+\ldots+s\wedge(\overline{\partial}s)^{n-1},$$

which is defined whenever s is defined. We note that $\delta_{\zeta-z}\overline{\partial}s = -\overline{\partial}\delta_{\zeta-z}s = -\overline{\partial}1 = 0$. Since $s\wedge(\overline{\partial}s)^n$ must vanish, we have $(\overline{\partial}s)^n = \delta_{\zeta-z}(s\wedge(\overline{\partial}s)^n) = 0$.

The reader may check that $\nabla_{\zeta-z}u = 1$. In fact, this can be seen elegantly by using functional calculus of differential forms; then $u = s/\nabla_{\zeta-z}s = s/(1-\overline{\partial}s) = s \wedge \sum_1^{n-1}(\overline{\partial}s)^k$,and $\nabla_{\zeta-z}u = \nabla s/\nabla s = 1$.

One can construct a weight $g_z(\zeta)$ with respect to z, compactly supported in the ball of radius $r+\varepsilon$, such that $(z,\zeta) \mapsto g_z(\zeta)$ is holomorphic in z in the ball of radius $r-\varepsilon$. This is accomplished by setting

$$g_z(\zeta) = \chi - \overline{\partial}\chi \wedge u,$$

where χ is a cut-off function that is 1 whenever $|\zeta| \leqslant r-\varepsilon$ and 0 whenever $|\zeta| > r+\varepsilon$. Note that u is well-defined on the support of $\overline{\partial}\chi$. We see that g_z is a weight since $\nabla_{\zeta-z}$ is an anti-derivation; $\nabla_{\zeta-z}g_z = -\overline{\partial}\chi + \overline{\partial}\delta_{\zeta-z}\chi \wedge u + \overline{\partial}\chi = 0$ (as χ is a function, we have $\delta_{\zeta-z}\chi = 0$).

Proposition 3.1. — *If g is a weight with respect to z which has compact support, and if ϕ is holomorphic in a neighbourhood of the support of g, then*

$$\phi(z) = \int \phi(\zeta)g(\zeta). \tag{3.1}$$

Proof. — As in the construction of a weight with compact support above, we define forms

$$b = \frac{\partial|\zeta-z|^2}{2\pi i|\zeta-z|^2}$$

and $u = b \wedge \sum \left(\overline{\partial}b\right)^k$ such that $\delta_{\zeta-z}b = 1$ and $\nabla_{\zeta-z}u = 1$ hold outside of $\{\zeta = z\}$. The highest degree term of u is the Bochner-Martinelli kernel. We now want to determine the residue $R = 1 - \nabla_{\zeta-z}u$ (where $\nabla_{\zeta-z}$ is taken in the sense of currents) at $\{\zeta = z\}$. The $(k, k-1)$ bidegree component $u_{k,k-1}$ of u is $\mathcal{O}(|\zeta-z|^{-2k+1})$, so only the highest component, $\overline{\partial}u_{n,n-1} = \overline{\partial}(b \wedge (\overline{\partial}b)^{n-1})$ of $\nabla_{\zeta-z}u$ will contribute to the residue. Using Stokes' theorem, it is easy to check that $R = [z]$, the point evaluation current at z. Clearly $\nabla_{\zeta-z}(\phi g) = 0$, so $\nabla_{\zeta-z}(u \wedge \phi g) = \phi g - [z] \wedge \phi g$. Taking highest order terms, we get

$$d(u \wedge \phi g)_{n,n-1} = \overline{\partial}(u \wedge \phi g)_{n,n-1} = [z] \wedge \phi g_{0,0} - \phi g_{n,n} = [z] \wedge \phi - \phi g_{n,n},$$

so by Stokes's theorem

$$\int \phi(\zeta)g(\zeta) = \int \phi(\zeta)g_{n,n}(\zeta) = [z].\phi = \phi(z).$$

□

4. Finishing the proof of Theorem 1.2

We now begin constructing a weight associated with Berndtsson's division formula for an ideal $I \subset \mathcal{O}_n$. Take $h = (h_i)$ to be an m-tuple of so called Hefer forms with respect to the generators f_i of I; these (germs of) $(1,0)$-forms are holomorphic in $2n$ variables, and satisfy $\delta_{\zeta - z} h_i = f_i(\zeta) - f_i(z)$. To see that h exists, write

$$f_i(\zeta) - f_i(z) = \int_0^1 \frac{d}{dt} f_i(z + t(\zeta - z))dt,$$

and compute the derivative inside the integral. Define $\sigma_i = \bar{f}_i/|f|^2$ and let $\chi_\varepsilon = \chi(|f|/\varepsilon)$ be a smooth cut-off function, where χ is approximatively the characteristic function for $[1, \infty)$. Recall that the dot sign refers to the pairing $a \cdot b = \sum a_i b_i$.We now set

$$\mu = \min(m, n+1)$$

and define the weight

$$\begin{aligned} g_B &= (1 - \nabla_{\zeta - z}(h \cdot \chi_\varepsilon \sigma))^\mu \\ &= \left(1 - \chi_\varepsilon + f(z) \cdot \chi_\varepsilon \sigma + h \cdot \bar{\partial}(\chi_\varepsilon \sigma)\right)^\mu \\ &= f(z) \cdot A_\varepsilon + B_\varepsilon, \end{aligned} \tag{4.1}$$

where

$$A_\varepsilon = \sum_{k=0}^{\mu-1} C_k \chi_\varepsilon \sigma [f(z) \cdot \chi_\varepsilon \sigma]^k \left[1 - \chi_\varepsilon + h \cdot \bar{\partial}(\chi_\varepsilon \sigma)\right]^{\mu - k - 1} \tag{4.2}$$

and

$$B_\varepsilon = \left(1 - \chi_\varepsilon + h \cdot \bar{\partial}(\chi_\varepsilon \sigma)\right)^\mu. \tag{4.3}$$

For convenience, we assume that $l = 0$ in Theorem 1.2. The proof goes through verbatim for general l byjust replacing μ with $\mu + l$ in the definition of g_B.

Let g be any weight with respect to z which has compact support and is holomorphic in z near 0. Substitution of the last line of (4.1) into (3.1) applied to the weight $g_B \wedge g$ yields

$$\phi(z) = f(z) \cdot \int \phi(\zeta) A_\varepsilon \wedge g + \int \phi(\zeta) B_\varepsilon \wedge g. \tag{4.4}$$

To obtain the division we will show two claims:

Claim 4.1. — *The second term in (4.4),*

$$\int \phi(\zeta) B_\varepsilon \wedge g,$$

converges uniformly to zero for small $|z|$.

Claim 4.2. — *If $m \leqslant n$, the tuple of integrals in (4.4),*

$$\int \phi(\zeta) A_\varepsilon \wedge g,$$

converges uniformly as $\varepsilon \to 0$.

We give an argument for the case $m > n$ of Theorem 1.2 at the end of the paper. Letting ε go to zero in (4.4), these claims give that $\phi \in I$.

To prove Claim 4.1, we will soon find a function $F(\zeta)$ integrable near $\zeta = 0$, such that $|\phi(\zeta) B_\varepsilon| \leqslant F$. Now we note that the integrand of Claim 4.1 has support on the set $S_\varepsilon = \{|f| \leqslant 2\varepsilon\}$; outside of S_ε, we have that $\chi_\varepsilon = 1$, so $B_\varepsilon = \big(h \cdot \overline{\partial}\sigma\big)^\mu$, which vanishes regardless of whether $\mu = n+1$ or $\mu = m$. In the latter case apply $\overline{\partial}$ to $f \cdot \sigma = 1$ to see that $\overline{\partial}\sigma$ is linearly dependent. Thus for small $|z|$, we get

$$\lim_{\varepsilon \to 0} \left| \int \phi(\zeta) B_\varepsilon \wedge g \right| \leqslant C \lim_{\varepsilon \to 0} \int_{S_\varepsilon} F = 0,$$

where we used that g is smooth.

The existence of F is a consequence of the main estimate of the previous chapter and a little bookkeeping that we will now carry out. Straightforward calculations, based on the fact that χ' is bounded, give that

$$\overline{\partial}\chi_\varepsilon = \mathcal{O}(1)|f|^{-1} \sum \overline{\partial f_j} \quad \text{and} \quad \overline{\partial}\sigma_i = \mathcal{O}(1)|f|^{-2} \sum \overline{\partial f_j}, \tag{4.5}$$

since $|f| \sim \varepsilon$ on the support of $\overline{\partial}\chi_\varepsilon$. Note also that $|\sigma| = |f|^{-1}$. It is easy to see that $\mathcal{O}(1)$ actually represents a function that does not depend on ε.

Using these facts, as we binomially expand (4.3), we get that $\phi(\zeta) B_\varepsilon$ is a linear combination ofterms that are given by

$$\phi(\zeta)\big(\overline{\partial}\chi_\varepsilon h \cdot \sigma\big)^a \wedge \big(\chi_\varepsilon h \cdot \overline{\partial}\sigma\big)^b (1-\chi_\varepsilon)^c = \phi(\zeta)|f|^{-2(a+b)}\overline{\partial f_J} \wedge \mathcal{O}(1), \tag{4.6}$$

where $a+b+c=\mu$, $J\subset\{1,2\ldots m\}$, $|J|=a+b$ and $\overline{\partial f_J}=\bigwedge_{i\in J}\overline{\partial f_i}$. Since $\overline{\partial f_J}=0$ whenever $a+b>n$ we can assume that $a+b\leqslant\min(m,n)$. We now set F to be the sum of the right hand side of (4.6) over all possible J, i.e.

$$F=\sum_{|J|\leqslant\min(m,n)}\phi(\zeta)|f|^{-2|J|}\overline{\partial f_J}\wedge\mathcal{O}(1). \tag{4.7}$$

Clearly $|\phi(\zeta)B_\varepsilon|\leqslant F$. Applying Proposition 2.1 with $k=\min(m,n)$ to (4.7), it follows that F is indeed locally integrable. □

Before dealing with Claim 4.2, we note that there is a way around it; clearly, the integrals in the claim are holomorphic for each $\varepsilon>0$, so the first termin (4.4) belongs to I for fixed $\varepsilon>0$. Thus, due to Claim 4.1, ϕ is in the closure of I with respect to uniform convergence. All ideals are however closed under uniform convergence, see [H90] Chapter 6, so ϕ belongs to I.

The proof of Claim 4.2 is similar to the proof of Claim 4.1. Since we have assumed $m\leqslant n$, we have $\mu=\min(m,n+1)=m$. Expanding $\phi(\zeta)A_\varepsilon$, displayed in (4.2), we get a linear combination of terms that are given by

$$\phi(\zeta)\sigma(f(z)\cdot\chi_\varepsilon\sigma)^k\left(\overline{\partial}\chi_\varepsilon h\cdot\sigma\right)^a\wedge\left(h\cdot\overline{\partial}\sigma\right)^b=\phi(\zeta)|f|^{-(1+k+2a+2b)}\overline{\partial f_J}\wedge\mathcal{O}(1),$$

where $a+b\leqslant\mu-k-1$, $k\leqslant\mu-1$ and $|J|=a+b$. The sum $1+k+2a+2b$ is at most $2\mu-1$, and this happends when $k=0$ and $a+b=\mu-1$. By an argument almost identical to the one proving that F was integrable, we get an integrable upper bound for ϕA_ε independent of z and ε. This is, of course, an upper bound also for the limit

$$A:=\lim_{\varepsilon\to0}A_\varepsilon=\sum_{k=0}^{\mu-1}C_k\sigma[f(z)\cdot\sigma]^k\left[h\cdot\overline{\partial}\sigma\right]^{\mu-k-1}.$$

As in the beginning of the proof of Claim 4.1, one sees that $\int\phi(\zeta)A_\varepsilon\wedge g$ converges uniformly to $\int\phi(\zeta)A\wedge g$. □

The case $m>n$ presents an additional difficulty as our upper bound fails to be integrable. Also, $\phi A\wedge g$ will not be integrable. A remedy is to consider a reduction of the ideal I, that is, an ideal $\mathfrak{a}\subset I$ generated by n germs such that $\overline{\mathfrak{a}}=\overline{I}$, see for example Lemma 10.3, Ch. VIII in [D07]. If a_i generate $\mathfrak{a}$ we have that $|a|\sim|f|$, so $\hat{\mathfrak{a}}^{(k)}=\hat{I}^{(k)}$ for any integer $k\geqslant1$. Thus we have reduced to the case $m\leqslant n$, which has already been proved. □

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