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Introduction

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Introduction

1. A brief history of the theory of ordered rings

Numerous mathematical themes with classical roots and enduring mathematical significance come together in the theory of ordered rings. Interestingly, many of them have a common origin in Hilbert's work [Hi] on the foundations of geometry. For example, non-Archimedean ordered fields were first introduced into the mathematical literature in order to show the independence of the Axiom of Archimedes. Another prominence in the landscape of ordered rings is Hilbert's 17th problem, which originated in the context of Hilbert's work on problems concerning geometric constructions—see [PD] for a discussion; and of course it was in working on the 17th Problem that Artin and Schreier developed the theory of real closed fields.

Mathematical themes mix, evolve and transform as the research groups that develop them encounter one another, exchange ideas and collaborate. Since it was the purpose of the 2007 Conference on Ordered Rings ("ORD07") to continue this process, it is meaningful to look back toward the sources and original springs in order to trace the flow and mix to the present. What one views as the main stream is always somewhat subjective. We shall present our thoughts and associations with regard to the historical development, up to the point where currently active research can be identified.

In 1907, H. Hahn [Ha] published path-breaking work on the structure of non-Archimedean ordered abelian groups and fields—the first significant step in studying non-Archimedean ordered systems after Hilbert. In the 1950s, Paul Conrad simplified this work in his dissertation, and later, with his students Harvey and Holland, he generalized the Hahn Representation to lattice-ordered structures. Conrad and his mathematical descendents have been the major contributors to the purely algebraic theory of ordered structures in the U.S.A. Ever expanding their horizons, these researchers have developed detailed structure theories for lattice-ordered groups (both commutative and non-commutative), a great body of knowledge on canonical exten-

sion of Archimedean lattice-ordered groups and numerous results on lattice-ordered rings and fields. There have been significant inputs/exchanges from universal algebra and lattice-theory (areas shaped by the work of Garret Birkhoff) and from logic, and important relationships with Russian research groups working on ordered structures. A good guide to much of this work is Darnel's comprehensive textbook [D]. The conference proceedings [M89], [HM] and [M02] provide a glimpse of the activity in this area in the last 20 years.

A second major source for the theory of ordered rings has come from work on rings of continuous functions, initiated by Hewitt in 1948 and furthered by the influential 1960 textbook of Gilman and Jerison [GJ]. Mel Henriksen is widely recognized as the unnamed third author of this basic reference. During the 1980s, it became clear that certain themes from this work had deep relevance to problems arising in the study of extensions of Archimedean lattice-ordered groups. In this context, Hager developed a major research program of constructing and classifying functorial closure operators of Archimedean lattice-ordered groups with weak unit, using the Yosida representation as an important tool. An early step was to show that every such object has an f-ring canonically associated with it. For the past two decades, Hager and collaborators Ball and Martinez as well as others have been elaborating this work. As a spin-off, Schwartz and Madden transferred the entire program to the context of real algebraic geometry in [SM]. Other important developments have involved the use of frames (i.e., locales or "pointless topologies") in place of traditional spaces for representations, and large parts of already developed theory have been greatly streamlined by this.

The connection between real algebraic geometry and ordered structures comes about as follows. Consider the collection of subsets of real affine space that one would have to consider in attempting to describe the real solutions of a system of real polynomial equations. These would be, in the first place, the zero-sets of polynomials and then, in addition, all sets obtainable from these by means of union, complementation and projection. By Tarski's elimination theorem for real closed fields, this collection coincides precisely with the collection of all sets defined by polynomial inequalities—the so-called semialgebraic sets. Thus, one is forced to consider order relations in order to study the real solutions of systems of polynomial equations. This fundamental idea can be elaborated in an amazing number of ways. In parallel to complex geometry, where classical elimination theory leads to the Nullstellensatz, Tarski's real elimination theory leads to a Real Nullstellensatz, which provides a systematic correspondence between semialgebraic sets in real affine n-space and ordered residue rings of the ring of real polynomials

in n variables. In abstract real algebraic geometry, related ideas lead to the notion of the real spectrum, introduced by Coste and Roy around 1980. (This construction, it turns out, was actually anticipated in certain ways within the theory of representations of ℓ-groups and f-rings by researchers in Ordered Algebraic Systems.) In real algebraic geometry, relations to valuation theory have been very important, since each ordering of the coordinate ring of a real variety gives rise to a real place; see [L]. Mathematical logicians took the idea in a different direction, abstracting the most basic features of the system of sets definable by polynomial inequalities and in this way forming the idea of an o-minimal structure, which today is an exceptionally fertile area of research.

Possibly one of the most interesting unsolved problems in the intersection of the three areas listed is the famous Pierce-Birkhoff Conjecture (PBC), which asserts that every continuous semialgebraic piecewise-polynomial function on real affine n-space can be expressed as the supremum of infima of polynomials from a finite collection. It was first formulated in the way just stated by Henriksen and Isbell in the early 1960s. At present, only the $n = 2$ case has been proved. The difficulty is associated with the singularities of the "pieces", where by "piece" we mean a subset of n-space where the given piecewise polynomial coincides with a single polynomial. The PBC may be the most important "challenge problem" currently being worked on in real algebraic geometry. It is a severe test of the conceptual apparatus we presently possess.

2. A reminiscence of Melvin Henriksen, by James Madden

Mel Henriksen passed away on October 14, 2009, in Albuquerque, New Mexico (USA) at the age of 82, after a brief illness. Mel taught me topology in my first year of graduate school at Wesleyan University, and he has been a guiding figure in my mathematical career ever since. I last saw him in Las Cruces, New Mexico in August 2009 at the Conference on Boolean Algebras, Lattice Theory, Algebra (algebra of logic and universal algebra), Set Theory and Topology (set theoretic and point-free): "BLAST," hosted by New Mexico State University. He was fully engaged in the mathematical activities, attending and making thoughtful comments on all the lectures—or at least those held during the days that I was able to be there—and obviously enjoying the camaraderie of the many mathematicians who shared his passions. Along with his many offspring and friends, both in and out of academia, I will miss him.

3. Overview of this volume

This volume contains 12 carefully refereed articles by participants in the Conference on Ordered Rings (Ord07) at Louisiana State University, in Baton Rouge, Louisiana, USA, April 25–28, 2007. That conference was dedicated to Melvin Henriksen, on the occasion of his 80th birthday.

The 17 (co-)authors who contributed to this volume were among the 39 mathematicians from France, Germany, Tunisia, and the United States who participated in the conference.

This volume contains both expository papers and original research papers on topics such as ℓ-groups, ℓ-rings, f-rings, real holomorphy rings, clean rings, SV-porings, rings generated by units, Hilbert's 17th problem, and the Pierce-Birkhoff conjecture.

4. Participants in the conference on ordered rings

1. **Rick Ball**, University of Denver, USA.
2. **Donald Beken**, University North Carolina, Pembroke, USA.
3. *__Papiya Bhattacharjee__, Bowling Green State University, Ohio, USA.
4. **Karim Boulabiar**, Université Novembre 7 - Carthage, Tunisia.
5. *__Brian Boucher__, University of Florida, Gainesville, USA.
6. **Gerard Buskes**, University of Mississippi, Oxford, USA.
7. **Mike Darnel**, Indiana University South Bend, USA.
8. **Fred Dashiell**, Los Angeles, California, USA.
9. **Charles N. Delzell**, Louisiana State University, Baton Rouge, USA.
10. **Ralph DeMarr**, University of New Mexico, Albuquerque, USA.
11. **Danielle Gondard-Cozette**, Université Paris 6, France.
12. **Anthony W. Hager**, Wesleyan University, Middletown, Connecticut, USA.
13. **Melvin Henriksen**, Harvey Mudd College, Claremont, California, USA.
14. **Wolf Iberkleid**, Bowling Green State University, Ohio, USA.
15. **Don Johnson**, New Mexico State University, Las Cruces, USA.
16. **Klaus Keimel**, Universität Darmstadt, Germany.
17. **Manfred Knebusch**, Universität Regensburg, Germany.
18. **Suzanne Larson**, Loyola Marymount University, Los Angeles, California, USA.
19. **François Lucas**, Université d'Angers, France.
20. **Jingjing Ma**, University of Houston - Clear Lake, Texas, USA.
21. **James Madden**, Louisiana State University, Baton Rouge, USA.
22. **Jorge Martinez**, University of Florida, Gainesville, USA.
23. **James McEnerney**, Lawrence Livermore National Lab., California, USA.
24. **Homeira Pajoohesh**, Georgia Southern University, USA.
25. **Alex Prestel**, Universität Konstanz, Germany.
26. **Robert Redfield**, Hamilton College, Clinton, NY, USA.
27. **Daniel Schaub**, Université d'Angers, France.
28. **Niels Schwartz**, Universität Passau, Germany.
29. **Philip Scowcroft**, Wesleyan University, Middletown, Connecticut, USA.
30. **Mark Spivakovsky**, Université Paul Sabatier, Toulouse, France.
31. *__Ashish Kumar Srivastava__, Ohio University, Athens, USA.
32. **Stuart Steinberg**, University of Toledo, Ohio, USA.
33. **Mohammed Tesemma**, Spelman College, Atlanta, Georgia, USA.
34. **Marcus Tressl**, Universität Regensburg, Germany.
35. **Dejan Veluscek**, University of Ljubljana, Slovenia.
36. **Piotr Wojciechowski**, University of Texas at El Paso, USA.
37. **Eric Zenk**, University of Denver, Colorado, USA.

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* Graduate students.

Besides the two organizers, the following two colleagues at LSU (mainly in algebra) also participated:

38. **Augusto Nobile**.
39. **Robert Perlis**.

5. Scientific Program of the Conference

April 25, 2007 (Wednesday):

9:00– 9:10: **Welcome**.

9:15– 9:55: **Rick Ball**, "P-frames."

10:15–10:55: **Daniel Schaub**, "The Pierce-Birkhoff conjecture and approximate roots of a valuation: Part I."

11:15–11:35: **Wolf Iberkleid**, "Classes of clean rings."

1:30– 2:10: **Anthony W. Hager**, "Uniform convergence in archimedean ℓ-groups and f-rings."

3:00– 3:30: **Don Johnson**, "Adjoining an identity element to a reduced Archimedean f-ring II: Algebras."

3:40– 4:00: **Karim Boulabiar**, "The Arens multiplication in a unital f-ring."

4:10– 4:30: **Homeira Pajoohesh**, "(Positive) derivations on (l-)rings of matrices."

4:40– 5:20: **Stuart Steinberg**, "The Henriksen-Isbell-Weinberg proof of McKenna's Theorem on Hilbert's 17th Problem."

April 26 (Thursday):

8:30– 8:50: **Gerard Buskes**, "A detour via vector lattices to the Loomis-Sikorski Theorem in Boolean algebras."

9:00– 9:30: **François Lucas**, "Spectra of ordered groups and rings."

9:40–10:00: **Papiya Bhattacharjee**, "Minimal prime elements of an algebraic frame."

10:10–10:40: **Danielle Gondard-Cozette**, "On Real Holomorphy Ring of Rings" (joint work with Murray Marshall).

10:50–11:30: **Alexander Prestel**, "Positive Elimination in Valued Fields."

1:30– 1:50: **Dejan Veluscek**, "Central extensions of *-ordered skew fields" (joint work with Igor Klep).

2:00– 2:20: **Ashish Kumar Srivastava**, "Sums of units in right self-injective rings."

2:30– 3:10: **Manfred Knebusch**, "Positivity and convexity in rings of fractions.

3:30– 4:10: **Niels Schwartz**, "Convex extensions of partially ordered polynomial rings."

4:30– 4:50: **Ralph DeMarr**, "Strange Inequalities in a Partially Ordered Linear Algebra."

5:00– 5:20: **P. Wojciechowski**, "Application of lattice ordered rings in enumeration of multiplicative bases of matrices."

April 27 (Friday):

8:30– 9:10: **Suzanne Larson**, "Images and Open Subspaces of SV Spaces."

9:30– 9:50: **Melvin Henriksen**, "Open problems on when the ring $C(X)$ contains 'many' prime ideals P such that $C(X)/P$ is a valuation domain."

10:00–10:40: **Marcus Tressl**, "Super real closed rings."

11:00–11:40: **Jorge Martinez**, "Archimedean frames revisited."

April 28 (Saturday):

9:00– 9:20: **Charles N. Delzell**, "The two-variable Pierce-Birkhoff conjecture for continuous, piecewise 'generalized' polynomial functions on the positive orthant."

9:30–10:10: **Eric Zenk**, z-dimension of C(X) revisited."

10:30–11:10: **Mark Spivakovsky**, "The Pierce-Birkhoff conjecture and approximate roots of a valuation: Part II."

11:30–11:50: **Jim McEnerney**, "Applications of the semi-linear spectrum over an ordered field."

12:00–12:40: **Robert Redfield**, "Super valuation groups."

For abstracts of talks, and other information about the 2007 Conference on Ordered Rings, see `http://www.math.lsu.edu/~madden/Ord07/` .

6. Acknowledgment of funding, and disclaimer

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Any opinions, findings, conclusions, or recommendations expressed in this volume are those of the author(s), and do not necessarily reflect the views of the National Science Foundation.

7. Picture from the conference

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