ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

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Tome XIX, n° S1 (2010), p. 37-56.

<http://afst.cedram.org/item?id=AFST_2010_6_19_S1_37_0>

Annales de la Faculté des Sciences de Toulouse Vol. XIX, n° Spécial, 2010
pp. 37–56

Extension of the two-variable Pierce-Birkhoff conjecture to generalized polynomials(*)

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In honor of Melvin Henriksen's 80th birthday

ABSTRACT. — Let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous, piecewise-polynomial function. The Pierce-Birkhoff conjecture (1956) is that any such h is representable in the form $\sup_i \inf_j f_{ij}$, for some finite collection of polynomials $f_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$. (A simple example is $h(x_1) = |x_1| = \sup\{x_1, -x_1\}$.) In 1984, L. Mahé and, independently, G. Efroymson, proved this for $n \leqslant 2$; it remains open for $n \geqslant 3$. In this paper we prove an analogous result for "generalized polynomials" (also known as signomials), i.e., where the exponents are allowed to be arbitrary real numbers, and not just natural numbers; in this version, we restrict to the positive orthant, where each $x_i > 0$. As before, our methods work only for $n \leqslant 2$.

RÉSUMÉ. — En 1984, L. Mahé, et indépendammant G. Efroymson, ont prouvé le cas où $n \leqslant 2$ de la conjecture de Pierce-Birkhoff (1956) : une fonction $h : \mathbb{R}^n \to \mathbb{R}$ continue polynomiale par morceaux peut s'écrire comme $\sup_i \inf_j f_{ij}$, pour une collection finie de polynômes $f_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$. (Un exemple simple est $h(x_1) = |x_1| = \sup\{x_1, -x_1\}$.) La conjecture reste ouverte pour $n \geqslant 3$. Dans cet article, nous prouvons (encore pour $n \leqslant 2$) un résultat analogue pour « polynômes généralisés », où les exposants peuvent être des nombres réels arbitraires, et non pas seulement des nombres naturels; dans cette version, nous limitons le domaine à l'orthant positif, où chaque $x_i > 0$.

(*) The results in this paper were first presented at the Conference on Ordered Rings ("Ord007"), at Louisiana State University, Baton Rouge, Louisiana, USA, April 25–28, 2007: `http://www.math.lsu.edu/~madden/Ord007`.

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1. Generalized polynomial functions and generalized semialgebraic sets

We write $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_{++} = (0,\infty)$, endowed with the usual, order topology. And the Cartesian product, $\mathbb{R}^2_{++} := \mathbb{R}_{++} \times \mathbb{R}_{++}$, will be endowed with the usual, Euclidean topology.

Definition 1.1. — *A* generalized polynomial function $a(x,y)$ *of two variables is a function* $a : \mathbb{R}^2_{++} \to \mathbb{R}$ *of the form*

$$a := a(x,y) := c_1 x^{\alpha_{1,1}} y^{\alpha_{1,2}} + c_2 x^{\alpha_{2,1}} y^{\alpha_{2,2}} + \cdots + c_m x^{\alpha_{m,1}} y^{\alpha_{m,2}}, \tag{1.1}$$

where $m \in \mathbb{N} := \{0,1,2,\ldots\}$*, the "coefficients"* c_i *of* a *are nonzero elements of* $\mathbb{R}$*, and the (binary) "exponents"* $\alpha_i := (\alpha_{i,1}, \alpha_{i,2})$ *of* a *are distinct elements of* $\mathbb{R}^2$*. We write* $\mathbb{R}[\mathbb{R}^2]$ *for the ring (actually, it is a group ring) of all generalized polynomial functions* $a : \mathbb{R}^2_{++} \to \mathbb{R}$*.*

Thus, generalized polynomial functions (sometimes called "signomial" functions) of two variables can be defined, roughly, as "real polynomial functions on $\mathbb{R}^2_{++}$ with arbitrary real exponents." A simple example is $a(x,y) = y - x^\pi$.

Generalized polynomial functions of two variables are clearly real analytic on $\mathbb{R}^2_{++}$.

See [Delzell, 2008] for background on the general properties and the history of generalized polynomials (in any number of variables), and some motivation for studying them.

Definition 1.2. — *We call a subset* $A \subseteq \mathbb{R}^2_{++}$ *a* generalized semialgebraic set*, or a* semisignomial set*, if it is of the form* $\bigcup_{j=1}^{J} S_j$*, where* $J \in \mathbb{N}$ *and each* S_j *is a "basic semisignomial" set, i.e., one of the form*

$$S_j = \{\, (x,y) \in \mathbb{R}^2_{++} \mid f_j(x,y) = 0,\; g_{j,1}(x,y) > 0, \ldots, g_{j,K_j}(x,y) > 0 \,\}, \tag{1.2}$$

where each $K_j \in \mathbb{N}$ *and the* f_j *and* g_{jk} *are generalized polynomials.*

(Recall that ordinary semialgebraic subsets of $\mathbb{R}^2$ or $\mathbb{R}^n$ are defined analogously, but with the f_j and g_{jk} being (ordinary) polynomials.)

2. Piecewise generalized polynomial functions

Definition 2.1. — *We call a function* $h(x,y) : \mathbb{R}^2_{++} \to \mathbb{R}$ *a* piecewise generalized polynomial function *of two variables if there exist* $g_1, \ldots, g_l \in \mathbb{R}[\mathbb{R}^2]$ *(Definition 1.1) such that the subsets*

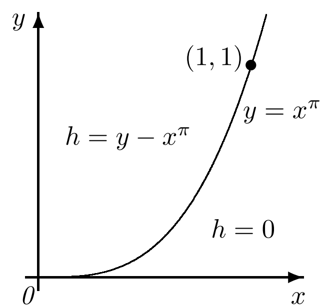
$$A_i := \{\, (x,y) \in \mathbb{R}^2_{++} \mid h(x,y) = g_i(x,y) \,\} \tag{2.1}$$

are generalized semialgebraic and cover $\mathbb{R}^2_{++}$*, i.e.,* $\mathbb{R}^2_{++} = \bigcup_i A_i$*.*

We may, and shall, assume that the g_i *are distinct.*

Example 2.2. —

$$h(x,y) := \begin{cases} y - x^\pi & \text{if } y \geqslant x^\pi, \\ 0 & \text{if } y < x^\pi. \end{cases}$$



The following, technical lemma will not be needed until Proposition 4.8 and Lemma 5.3 below, and can be skipped on a first reading. In it, for any set A in $\mathbb{R}^2_{++}$, we shall write A° for the interior of A.

Lemma 2.3. — *Let* $A_1, \ldots, A_l$ *be as in Definition 2.1.*

(1) $\bigcup_{i=1}^{l} A_i^\circ$ *is dense in* $\mathbb{R}^2_{++}$*.*

(2) $A_i^\circ \cap A_j^\circ = \emptyset$ *for* $i \neq j$*.*

(3) *If* h *is continuous, then each* A_i *is closed, whence* $\overline{A_i^\circ} \subseteq A_i$*.*

(4) *If* h *is continuous, then* $\bigcup_{i=1}^{l} A_i^\circ = \mathbb{R}^2_{++} \setminus \bigcup_{1 \leqslant i < j \leqslant l} \big(\overline{A_i^\circ} \cap \overline{A_j^\circ}\big)$*.*

(5) *Suppose* h *is continuous, and* E *is a connected subset of* $\mathbb{R}^2_{++}$ *such that for each* $(x,y) \in E$*, the* l *values* $g_1(x,y), g_2(x,y), \ldots, g_l(x,y)$ *are distinct. Then there exists an* $i \in \{1, 2, \ldots, l\}$ *such that* $E \subseteq A_i^\circ$ *(in particular, such that* $h = g_i$ *throughout* E*). This* i *is unique in case* $E \neq \emptyset$*.*

Proof. — (1) By Definition 1.2, $\bigcup_i A_i$ is a combined, but still finite, union of suitable basic semisignomial sets S_j as in equation (1.2). Let T be the union of those S_j for which $f_j \not\equiv 0$; thus, $T \subseteq Z(F) := \{\,(x,y) \in \mathbb{R}^2_{++} \mid F(x,y) = 0\,\}$, where F is the product of those f_j's. $\mathbb{R}^2_{++} \setminus Z(F)$ is dense in $\mathbb{R}^2_{++}$, by the identity theorem for real analytic functions. *A fortiori*, $\mathbb{R}^2_{++} \setminus T$ is also dense in $\mathbb{R}^2_{++}$. The union U of the other S_j's (viz., those for which $f_j \equiv 0$) must contain $\mathbb{R}^2_{++} \setminus T$ (since $T \cup U = \bigcup_i A_i = \mathbb{R}^2_{++}$ (2.1)), and so U is also dense in $\mathbb{R}^2_{++}$. But $\bigcup_i A_i^\circ \supseteq U$.[1]

(2) If $A_i^\circ \cap A_j^\circ \neq \emptyset$, then g_i would agree with g_j on a nonempty open set (by equation (2.1)), and hence on all of $\mathbb{R}^2_{++}$ (again by the identity theorem), contradicting the distinctness of the g_i in (2.1).[2]

(3) Obvious.

(4) $\subseteq$. Let $(x,y) \in A_i^\circ$ and suppose $j \neq i$. It is enough to show that $(x,y) \notin \overline{A_j^\circ}$. There exists an open disk in A_i about (x,y). In fact, this disk is in A_i°, and hence is disjoint from A_j°, by (2) above. Therefore $(x,y) \notin \overline{A_j^\circ}$.[3]

$\supseteq$. Suppose $(x,y) \in \mathbb{R}^2_{++} \setminus \bigcup_i A_i^\circ$. For $r \in \mathbb{R}_{++}$ with $r \leqslant \min\{x,y\}$, let B_r denote the open disk in $\mathbb{R}^2_{++}$ of radius $r > 0$ about (x,y), and let $I(r) = \{\, i \in \{1,2,\dots,l\} \mid B_r \cap A_i^\circ \neq \emptyset\,\}$. Then for every r, $|I(r)| \geqslant 1$, by (1) above. In fact, $|I(r)| > 1$. Otherwise, for some i, $A_i^\circ \cap B_r$ would be dense in B_r (by (1) again), whence $B_r = \overline{A_i^\circ} \cap B_r \subseteq A_i \cap B_r$ (by (3)),[4] i.e., $B_r \subseteq A_i$, whence $(x,y) \in A_i^\circ$, contradiction. Now, for any $s \in \mathbb{R}_{++}$ with $s < r$, $I(s) \subseteq I(r)$; i.e., the finite set $I(r)$ decreases monotonically with r, and yet always has cardinality $\geqslant 2$. Thus, there exist at least two indices $i < j$ such that for every $r \in (0, \min\{x,y\})$, B_r meets A_i° and A_j°. Therefore $(x,y) \in \overline{A_i^\circ} \cap \overline{A_j^\circ}$.

(5) The distinctness hypothesis of (5) can be rephrased as

$$E \cap \bigcup_{i<j} (A_i \cap A_j) = \emptyset.$$

A fortiori, $E \cap \bigcup_{i<j} (\overline{A_i^\circ} \cap \overline{A_j^\circ}) = \emptyset$, using (3). By (4), $E \subseteq \bigcup_i A_i^\circ$. The existence of the desired i now follows from (2) and the hypothesis that E is connected. The uniqueness of i in case $E \neq \emptyset$ also follows from (2). □

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(1) In fact, $\bigcup_i A_i^\circ = U$. But we don't need this.

(2) And if g_i agrees with g_j on all of $\mathbb{R}^2_{++}$, then the coefficients of g_i and g_j (i.e., the c's in equation (1.1) above) would agree, too, by [Delzell, 2008, Remark 4.3].

(3) This half of the proof of (4) does not require the hypothesis that h be continuous.

(4) In fact, this inclusion is actually an equality.

Remark 2.4. — In Remark 5.4 below, we shall use (2.3) above to see that when a piecewise generalized polynomial function h is continuous, each A_i in Definition 2.1 can automatically be taken to be a generalized semialgebraic set; it is not necessary to include that condition as a hypothesis in (2.1).

The set of piecewise generalized polynomial functions is closed under differences and products, and so forms a ring; it is also closed under pointwise suprema and infima, and so forms an l-ring under those lattice operations. (This ring is, of course, even an f-ring.) The continuous functions in this f-ring comprise a sub-f-ring. (See, e.g., [Birkhoff, et al., 1956] or [Henriksen, et al., 1962] for background on l-rings and f-rings.)

3. Statement and discussion of the main result

THEOREM 3.1. — (Main Theorem: The Pierce-Birkhoff conjecture for generalized polynomials in two variables) *If $h : \mathbb{R}^2_{++} \to \mathbb{R}$ is continuous and piecewise generalized polynomial, then h is a (pointwise) sup of infs of finitely many generalized polynomial functions; i.e.,*

$$h(x,y) = \sup_j \inf_k f_{jk}(x,y) \quad \textit{on} \quad \mathbb{R}^2_{++}, \tag{3.1}$$

for some finite number of generalized polynomials f_{jk}. (The converse is easy.)

Example 3.2. — For the h in Example 2.2 above, $h(x,y) = \sup\{0, y - x^\pi\}$.

The representation of h in the form of equation (3.1) makes both the continuity and the piecewise generalized polynomial character of h *obvious*.

For ordinary polynomials in $\mathbb{R}[X,Y]$ and ordinary piecewise polynomial functions on $\mathbb{R}^2$, the analog of Theorem 3.1 above was first proved by L. Mahé [Mahé, 1984] and Efroymson (unpublished), independently. The statement and proofs of the Mahé-Efroymson theorem generalize easily to the situation where $\mathbb{R}$ is replaced by an arbitrary real closed field R (furnished with the topology induced by the unique ordering on R). But the fact that then the coefficients of the f_{jk} in the Mahé-Efroymson theorem may be taken to lie in the subfield of R generated by the coefficients of the g_i defining h (in the analog of Definition 2.1), was not trivial, and was proved in [Delzell, 1989].

The extension of the Mahé-Efroymson theorem to functions of three or more variables (like the extension of Theorem 3.1 above) remains unproved and unrefuted; it is known as the Pierce-Birkhoff Conjecture (first formulated in [Birkhoff, et al., 1956]).

In our proof of Theorem 3.1 below, we shall make no attempt to indicate which steps generalize easily to the case where $n > 2$ (though many of those steps do). The first reason for this is that the notation is often simpler when $n = 2$. The second reason is that, considering the many mathematicians who have tried to prove the Pierce-Birkhoff Conjecture for $n > 2$, we now lean toward the opinion that it and Theorem 3.1 are false for $n > 2$.

In 1987 we proved that for all $n \geqslant 1$ and every real closed field R, if $h : R^n \to R$ is "piecewise-rational" (i.e., if there are rational functions $g_1, \ldots, g_l \in R(X)$ such that the sets $A_i := \{\, x \in R^n \mid g_i(x)$ is defined and $h(x) = g_i(x) \,\}$ are s.a. and cover R^n), then there are finitely many $f_{jk} \in R(X)$ and there is a $k \in R[X_1, \ldots, X_n] \setminus \{0\}$ such that for all $x \in R^n$ where $k(x) \neq 0$ (i.e., for "almost all" $x \in R^n$), each $f_{jk}(x)$ is defined and $h(x) = \sup_j \inf_k f_{jk}(x)$; this is true even if h is not continuous. This result was announced in [Delzell, 1989, p. 659], and proved in [Delzell, 1990]. Madden gave an "abstract" version of this result that applies to arbitrary fields (and not just $R(X)$); see [Madden, 1989]. In [Delzell, 2005] we proved an analog of our 1987 result, for "generalized piecewise-rational functions" (i.e., functions that are, piecewise, quotients of generalized polynomial functions).

The rest of this paper will be devoted to the proof of Theorem 3.1. In §4 we shall develop the necessary one-variable machinery; in §5 we shall deal with the additional difficulties arising in the two-variable situation.

4. One-variable methods

We imitate Mahé's proof as much as possible.

We are given a continuous function

$$h(x,y) = \begin{cases} g_1(x,y) & \text{if } (x,y) \in A_1 \\ \vdots & \qquad\vdots \\ g_l(x,y) & \text{if } (x,y) \in A_l, \end{cases} \tag{4.1}$$

where, as in Definition 2.1, the g_i are generalized polynomials and the A_i cover $\mathbb{R}^2_{++}$. (Recall from Remark 2.4 above that the A_i are also, automatically, generalized semialgebraic; but we don't use this.) As before, we assume the g_i are distinct.

Write each $a(x,y) \in \mathbb{R}[\mathbb{R}^2] \setminus \{0\}$ (Definition 1.1) in the form

$$a_1(x)y^{\beta_1} + a_2(x)y^{\beta_2} + \ldots + a_K(x)y^{\beta_K}, \tag{4.2}$$

where $K \geqslant 1$, $\beta_1 < \ldots < \beta_K \in \mathbb{R}$, and each a_i is a nonzero generalized polynomial in x. This representation is unique.

Let $\mathcal{A} = \{ g_i - g_j \mid 1 \leqslant i < j \leqslant l \}$. Let $\mathcal{B}$ be the smallest subset of $\mathbb{R}[\mathbb{R}^2]$ containing $\mathcal{A}$ and closed under the following two operations, for each $a(x, y) \in \mathcal{B}$ for which $K > 1$ in equation (4.2):

$$a \mapsto \begin{cases} a' := \dfrac{\partial a}{\partial y} & \text{if } \beta_1 = 0 \text{ , and} \\ y^{-\beta_1} a(x, y) & \text{if } \beta_1 \neq 0 \text{ }^{5}\text{; and} \end{cases} \tag{4.3}$$

$$a \mapsto \begin{cases} r := r_a(x, y) = a(x, y) - \dfrac{y}{\beta_K} \cdot a'(x, y) & \text{if } \beta_1 = 0,^{6} \text{ and} \\ a & \text{if } \beta_1 \neq 0. \end{cases} \tag{4.4}$$

Remark 4.1. — Suppose no g_i involves the variable x; i.e., each g_i is a function of y alone, and is constant in x. Then the same is, of course, true for each $a \in \mathcal{A}$; in fact, the same is true even for each $a \in \mathcal{B}$, in view of equations (4.3) and (4.4).

LEMMA 4.2. — *For each $a \in \mathcal{B}$ for which $K > 1$ and $\beta_1 = 0$, $a'(x, y)$ and r_a each have exactly $K - 1$ y-terms. Consequently, $\mathcal{B}$ is finite.*

Proof. — This is clear for $a'(x, y)$. For r_a, observe (a) that the K^{th} y-term $a_K(x, y)y^{\beta_K}$ in a (equation (4.2)) is cancelled out by the y-term

$$\frac{y}{\beta_K} \left(\beta_K \, a_K(x, y) \, y^{\beta_K - 1} \right)$$

in

$$\frac{y}{\beta_K} \cdot a'(x, y), \tag{4.5}$$

and (b) that the other y-terms of equation (4.5) involve the y-exponents $\beta_1, \ldots, \beta_{k-1}$, but with coefficients different from those of the corresponding y-terms of a (since for each $i < K$, $\beta_i/\beta_K \neq 1$). $\square$

LEMMA 4.3. — *There exist $L \in \mathbb{N}$ and $\gamma_1 < \gamma_2 < \ldots < \gamma_L \in \mathbb{R}_{++}$ such that, writing $\gamma_0 = 0$ and $\gamma_{L+1} = \infty$, for each $a \in \mathcal{B}$ and for each $p \in \{0, 1, \ldots, L\}$, the zeros of $a(x, y)$ in the pth vertical half strip $H_p := (\gamma_p, \gamma_{p+1}) \times \mathbb{R}_{++}$ are the graphs of continuous, monotonic[7] "generalized*

(5) This trick (of dividing by y^{β_1}) was first used by Sturm [Sturm 1829].
(6) Here we use $\beta_K \neq 0$, which follows from $\beta_1 = 0$ and $K > 1$.
(7) We do not need the monotonicity of the $\xi_{a,p,j}$ in this paper.

semialgebraic"[8] *functions* $y = \xi_{a,p,j}(x)$, $j = 1, 2, \ldots, s$ *(where* $s := s(a,p)$ *satisfies* $0 \leqslant s \leqslant K$[9]*) with*

$$(0 <)\ \xi_{a,p,1} < \cdots < \xi_{a,p,s} \ \text{ on } (\gamma_p, \gamma_{p+1}).$$

Moreover, $\forall a_1, a_2 \in \mathcal{B}$, $\forall p \leqslant L$, $\forall j_1 \leqslant s(a_1,p)$, $\forall j_2 \leqslant s(a_2,p)$, *throughout* $(\gamma_p, \gamma_{p+1}) \subseteq \mathbb{R}_{++}$, *only one of the following three relations holds:*

$$\begin{aligned} \xi_{a_1,p,j_1} &< \xi_{a_2,p,j_2}, \\ \xi_{a_1,p,j_1} &= \xi_{a_2,p,j_2}, \text{ or} \\ \xi_{a_1,p,j_1} &> \xi_{a_2,p,j_2}. \end{aligned} \tag{4.6}$$

Lemma 4.3 and its Corollary 4.5 are illustrated in Figure 1, which also shows the stack of open connected sets $D_{2,1}, D_{2,2}, D_{2,3}$ whose union is a dense open subset of H_2 (looking ahead to (4.5) below).

Proof. — Miller [Miller, 1994] considered a class of functions $f : \mathbb{R}^n \to \mathbb{R}$ that properly contains the class of (extensions by 0 to $\mathbb{R}^n$ of) generalized polynomial functions. Specifically, he considered terms built up (in a formal language) from variable symbols x_1, x_2, and from constants in $\mathbb{R}$ by the usual operation symbols $+$, $-$, and $\cdot$, together with the class of operation symbols $\{\, x_i^r \mid i \geqslant 1,\ r \in \mathbb{R} \,\}$; the symbol x_i^r indicates the function $\mathbb{R} \to \mathbb{R}$ defined by

$$x_i \mapsto \begin{cases} x_i^r & \text{if } x_i > 0 \\ 0 & \text{if } x_i \leqslant 0. \end{cases}$$

He considered the structure

$$\mathbb{R}_{\text{an}}^{\mathbb{R}} := \big(\mathbb{R}, <, +, -, \cdot, 0, 1, (x_i^r)_{r \in \mathbb{R},\, i \geqslant 1}, \ (\tilde{f})_{f \in \mathbb{R}\{X,n\}, n \in \mathbb{N}}\big),$$

where $(\tilde{f})_{f \in \mathbb{R}\{X,n\}, n \in \mathbb{N}}$ denotes a certain class of functions $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ that are analytic on $[-1,1]^n$. He proved that the theory of $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ admits quantifier-elimination and analytic cell-decomposition, and is universally axiomatizable, o-minimal, and polynomially bounded.

The standard properties of o-minimal theories (cf., e.g., [Dries, 1998] or [Miller, 1994]) imply that the zeros in $\mathbb{R}_{++}^2$ of all the various $a \in \mathcal{B}$ consist of

(8) We say that a function is *generalized semialgebraic* if its graph, in the product space, is a generalized semialgebraic set.

(9) Here, K is as in equation (4.2); in fact, s is even bounded by the number of *alternations in sign* in the sequence $a_0(x), \ldots, a_K(x)$, by Sturm's generalization [Sturm, 1829], to one-variable generalized polynomials, of the Fourier-Budan theorem (which contains Descartes' rule of signs as a special case).

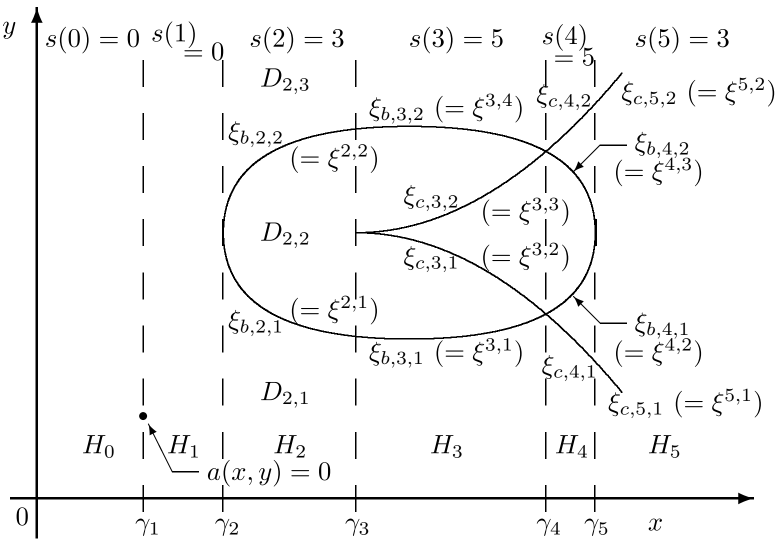

Figure 1. — Illustrating Lemma 4.3 and Corollary 4.5 by showing the zeros in $\mathbb{R}^2_{++}$ of $a, b, c \in \mathcal{B}$: the isolated zero of $a(x, y)$, and the graphs of $y = \xi_{b,p,j}(x)$ and $y = \xi_{c,p,j}(x)$ (which are also the graphs of $y = \xi^{p,k}(x)$, for suitable k). Here, $L = 5$ (the number of γ's).

finitely many isolated points together with the graphs of finitely many continuous, monotonic functions $\xi_{a,p,j} : (\gamma_p, \gamma_{p+1}) \to \mathbb{R}_{++}$ (on suitable intervals $(\gamma_p, \gamma_{p+1}) \subseteq \mathbb{R}_{++}$) satisfying equation (4.6), as stated in the lemma. (That the $\xi_{a,p,j}$ are generalized semialgebraic is just the definition of that term (footnote 8 above), since the $a(x, y)$ are generalized polynomials.) $\square$

Notation 4.4. — *It will be helpful in equation* (4.7) *below if we agree that* $\xi_{a,p,0}(x) = 0$ *and* $\xi_{a,p,s+1}(x) = +\infty$ *for all* $x \in (\gamma_p, \gamma_{p+1})$, *where* $p \in \{0, 1, \ldots, L\}$ *and* $s = s(a, p)$ *is as in Lemma 4.3.*

Corollary 4.5. — *Let* L, $\gamma_0, \ldots, \gamma_{L+1}$, *and* H_p *be as in Lemma 4.3, for some fixed* $p \in \{0, 1, \ldots, L\}$. *Then the zeros in* H_p *of all the* $a \in \mathcal{B}$ *are the graphs of continuous, monotonic, generalized semialgebraic functions* $y = \xi^{p,k}(x)$, $k = 1, 2, \ldots, s(p)$, *where* $s(p)$ *satisfies* $0 \leqslant s(p) \leqslant \sum_{a \in \mathcal{B}} s(a, p)$ (*where* $s(a, p)$ *is as in Lemma 4.3*), *and where, for each* $x \in (\gamma_p, \gamma_{p+1})$,

$$0 =: \xi^{p,0}(x) < \xi^{p,1}(x) < \cdots < \xi^{p,s(p)}(x) < \xi^{p,s(p)+1}(x) := \infty. \tag{4.7}$$

Consequently, the sets

$$D_{p,k} := \{ (x, y) \mid \gamma_p < x < \gamma_{p+1},\ \xi^{p,k}(x) < y < \xi^{p,k+1}(x) \},$$

for $k \in \{0, 1, \ldots, s(p)\}$, are nonempty, pairwise-disjoint, generalized semi-algebraic cells (in particular, they are open and (pathwise) connected), and their union is a dense open subset of H_p. Moreover, the $D_{p,k}$ are "stacked" one upon the other in the y-direction, so that for any $x \in (\gamma_p, \gamma_{p+1})$ and for any $(s(p)+1)$-tuple $y_0, y_1, \ldots, y_{s(p)} \in \mathbb{R}_{++}$ for which each $(x, y_k) \in D_{p,k}$, $y_0 < y_1 < \cdots < y_{s(p)}$.

Proof. — The required sequence $\xi^{p,1}, \xi^{p,2}, \ldots, \xi^{p,s(p)}$ of functions is just a suitable permutation and relabelling of the set of functions $\{\, \xi_{a,p,j} \mid a \in \mathcal{B},\ 1 \leqslant j \leqslant s(a,p) \,\}$. That a permutation of the ξ's satisfying equation (4.7) exists follows from equation (4.6). □

Proposition 4.6. — *The set of suprema of infima of finitely many generalized polynomial functions is closed under subtraction and multiplication, and so is a ring.*

Proof. — This is a special case of a result of Henriksen and Isbell [Henriksen et al. 1962, Corollary 3.4]: If S is a ring of real-valued functions on a set, then the least lattice of functions that contains S is also a ring. Here we may take $S = \mathbb{R}[\mathbb{R}^2]$ (Definition 1.1). For the proof of this corollary, Henriksen and Isbell gave some f-ring identities which, they said, reduce the proof to an exercise; they omitted the details. [Delzell 1989] gave a sketch of a proof. The first complete proof of this fact to appear in print was that of [Hager, et al., 2010, Theorem 2.1(B)]; their proof incorporates some simplifications due to Madden, and their statement is a little more general than the Henriksen-Isbell statement above, in that now S may be an arbitrary subring of an arbitrary f-ring. □

In the next lemma it will be helpful to use the abbreviation $a^+ = \sup\{0, a\}$, for any real-valued function a.

Lemma 4.7. — (Generalized Mahé lemma) *Using the notation of Lemma 4.3 above, for each $p \in \{0, 1, \ldots, L\}$, each $a(x,y) \in \mathcal{B}$, and each $j \in \{0, 1, \ldots, s\}$ (where $s = s(a,p)$ as in Lemma 4.3), there exists a function $c_{a,p,j}(x,y)$ that is a sup of infs of finitely many generalized polynomials, such that for all $x \in (\gamma_p, \gamma_{p+1})$ and for all $y \in \mathbb{R}_{++}$,*

$$c_{a,p,j}(x,y) = \begin{cases} a(x,y) & \text{if } y > \xi_{a,p,j}(x), \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Proof. — Fix any $p \leqslant L$.

We use induction on $K \geqslant 1$, the number of distinct y-exponents occurring in a (recall equation (4.2)). Note that for any $K \geqslant 1$, we may (in fact, we must) take $c_{a,p,0} = a$; this handles the case $K = 1$, i.e., the case where $a(x, y)$ is of the form $a_1(x)y^{\beta_1}$ (which implies $s(a, p) = 0$ for each $p \leqslant L$).

Now assume $K > 1$.

We claim that we may assume

$$\beta_1 = 0. \tag{4.9}$$

If $\beta_1 \neq 0$, then write $b(x, y) = y^{-\beta_1} a(x, y)$. Thus $b \in \mathcal{B}$, by equation (4.3). Note that $b(x, y)$ has the same positive y-roots ξ as $a(x, y)$ has; thus $s(a, p) = s(b, p)$. Therefore, if for each $j \leqslant s(b, p)$ we can construct $c_{b,p,j}$ such that

$$c_{b,p,j}(x, y) = \begin{cases} b(x, y) & \text{if } \ y > \xi_{b,p,j}(x), \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

then we may, for each $j \leqslant s(a, p)$ $(= s(b, p))$, take $c_{a,p,j}(x, y) = y^{\beta_1} c_{b,p,j}(x, y)$; the latter product is a sup of infs of finitely many generalized polynomials, since $c_{b,p,j}$ is, and since $y^{\beta_1} > 0$ for all $y > 0$ (or use Proposition 4.6).

Next, recall that a' (equation (4.3)) and r_a (equation (4.4)) each have exactly $K-1$ y-terms, by Lemma 4.2 and equation (4.9). Thus we assume, by the inductive hypothesis, that for every $k \leqslant s(a', p)$ and $l \leqslant s(r_a, p)$, we can construct $c_{a',p,k}$ and $c_{r_a,p,l}$ satisfying the appropriate analogs of equation (4.8). Note that $c_{a',p,k}$ and $c_{r_a,p,l}$ are, in particular, continuous (either by their form as in equation (4.8), or by the fact that they are sups of infs of finitely many generalized polynomial functions).

Finally, in order to construct $c_{a,p,j}$, we now use induction on $j \in \{0, 1, 2, \ldots, s(a, p)\}$. We have already constructed $c_{a,p,0}$, so now we assume that $j \in \{1, 2, \ldots, s(a, p)\}$ and that $c_{a,p,j-1}$ has already been constructed with the properties stated in Lemma 4.7.

Throughout the rest of this proof, x will range over (γ_p, γ_{p+1}). By the uniform trichotomy in equation (4.6), all order relations involving the various ξ's below will hold uniformly for such x; thus we usually write, e.g., $\xi_{a,p,j}$ instead of $\xi_{a,p,j}(x)$.

Let k be the smallest index such that $\xi_{a,p,j} \leqslant \xi_{a',p,k}$ (then $1 \leqslant k \leqslant 1 + s(a', p)$).

Let l be the smallest index such that $\xi_{a',p,k} \leqslant \xi_{r_a,p,l}$ (then $1 \leqslant l \leqslant 1 + s(r_a, p)$). Then

$$\xi_{a',p,k} < \xi_{a,p,j+1} \quad \text{(unless } \xi_{a',p,k} = \infty\text{), by Rolle's theorem, and} \tag{4.10}$$

$$g(x,y) := \frac{y}{\beta_K} c_{a',p,k}(x,y) + c_{r_a,p,l}(x,y)$$

$$= \begin{cases} 0 & \text{if } 0 < y < \xi_{a',p,k}, \\ \dfrac{y}{\beta_K} a'(x,y) = a(x,y) - r_a(x,y) & \text{if } \xi_{a',p,k} < y < \xi_{r_a,p,l}, \\ \dfrac{y}{\beta_K} a'(x,y) + r_a(x,y) = a(x,y) & \text{if } \xi_{r_a,p,l} < y, \end{cases} \tag{4.11}$$

where equation (4.11) follows from equation (4.4) and from the definitions of $c_{a',p,k}$ and $c_{r_a,p,l}$.[10] This function g is a supremum of infima of finitely many generalized polynomial functions, by Proposition 4.6.

If $a'(x, \xi_{a,p,j}) = 0$, then

$$\begin{aligned} \xi_{a',p,k} &= \xi_{a,p,j} && \text{by the minimality of } k \text{, and} \\ \xi_{r_a,p,l} &= \xi_{a',p,k} && \text{by (4.4) and the minimality of } l. \end{aligned}$$

Thus we may take $c_{a,p,j} = g$, by (4.11).

Now suppose, on the other hand, that

$$a'(x, \xi_{a,p,j}) \neq 0 \tag{4.12}$$

(recall equation (4.6)). (Then

$$\xi_{a,p,j} < \xi_{a',p,k}.) \tag{4.13}$$

We may assume that in fact

$$a'(x, \xi_{a,p,j}) > 0, \tag{4.14}$$

by equation (4.6), by replacing a with $-a$, and by the fact that $-c_{-a,p,j}$ $(= c_{a,p,j})$ will still be a supremum of infima of finitely many generalized polynomial functions if $c_{-a,p,j}$ is, by Proposition 4.6. Then

$$a(x,y) < 0 \quad \text{for} \quad \xi_{a,p,j-1} < y < a_{a,p,j} \quad \text{and} \tag{4.15}$$

$$a(x,y) > 0 \quad \text{for} \quad \xi_{a,p,j} < y < a_{a,p,j+1}, \tag{4.16}$$

by (4.14).

---

(10) In (4.11), the inequalities in the case-distinctions $y < \xi_{a',p,k}$, $\xi_{a',p,k} < y < \xi_{r_a,p,l}$, and $\xi_{r_a,p,l} < y$ are all strict (i.e., they are all $<$, and not $\leqslant$). This strictness is necessary because $\xi_{a',p,k}$ and/or $\xi_{r_a,p,l}$ could be ∞. If either or both of the ξ's are finite, the corresponding inequalities could be relaxed to nonstrict inequalities (with $\leqslant$). But even without such a relaxation, (4.11) still uniquely determines g even when y is $\xi_{a',p,k}$ or $\xi_{r_a,p,l}$, since g is continuous for all $y > 0$.

First suppose $\xi_{a',p,k} = \infty$ (i.e., $k = 1 + s(a',p)$). Then $a'(x,y) > 0$ for all $y > \xi_{a,p,j}$, whence $a(x,y) > 0$ for all $y > \xi_{a,p,j}$. Hence we may take $c_{a,p,j} = \inf\{c^+_{a,p,j-1}, a^+\}$, using also (4.15).

Second, suppose $\xi_{a',p,k} < \infty$ (i.e., $k \leqslant s(a',p)$). Then

$$\begin{aligned} r_a(x,\xi_{a',p,k}) &= a(x,\xi_{a',p,k}) - \frac{\xi_{a',p,k}}{\beta_K} a'(x,\xi_{a',p,k}) \quad \text{(by equation (4.4))} \\ &= a(x,\xi_{a',p,k}) - \frac{\xi_{a',p,k}}{\beta_K} \cdot 0 \qquad (4.17) \\ &= a(x,\xi_{a',p,k}) > 0, \quad \text{by (4.16), (4.10), and (4.13).} \end{aligned}$$

Then for $\xi_{a',p,k} \leqslant y < \xi_{r_a,p,l}$:

$$\begin{aligned} r_a(x,y) &> 0 && \text{by (4.17) and the choice of } l\text{, and} \quad (4.18) \\ g(x,y) &= a(x,y) - r_a(x,y) && \text{by (4.11)} \\ &< a(x,y) && \text{by (4.18).} \qquad (4.19) \end{aligned}$$

Then

$$\sup\{a,g\} = \begin{cases} a^+ & \text{if } 0 < y \leqslant \xi_{a,p,j} \quad \text{by (4.11), and} \\ a & \text{if } y \geqslant \xi_{a,p,j} \quad \text{by (4.11), (4.19), (4.10), and (4.16).} \end{cases}$$

Therefore, we may take $c_{a,p,j} = \inf\{c^+_{a,p,j-1}, \sup\{a,g\}\}$, by (4.15). □

Proposition 4.8. — *Let h, $\mathcal{A}$, and $\mathcal{B}$ be as before Lemma 4.2, and let L and H_p be as in Lemma 4.3, for some fixed $p \in \{0,1,\ldots,L\}$. Then there is a function $d_p : \mathbb{R}^2_{++} \to \mathbb{R}$ that (1) is a supremum of infima of finitely many generalized polynomial functions $\in \mathbb{R}[\mathbb{R}^2]$ and (2) coincides with $h(x,y)$ on H_p.*

Proof. — Let γ_p and γ_{p+1} be as in Lemma 4.3, and let $s(p)$, $\xi^{p,0},\ldots,$ $\xi^{p,s(p)+1}$, and $D_{p,0},\ldots,D_{p,s(p)}$ be as in Corollary 4.5.

For each $k = 0,1,\ldots,s(p)$ there exists a unique $\mu := \mu(p,k) \in \{1,2,\ldots,l\}$ such that $D_{p,k} \subseteq A_\mu$ (hence $h = g_\mu$ on $D_{p,k}$, by equation (4.1)), using Lemma 2.3(5) and the fact that each $g_i - g_j$ is nonzero throughout $D_{p,k}$.

If $s(p) = 0$, we may define the required d_p to be $g_{\mu(p,0)} \in \mathbb{R}[\mathbb{R}^2]$. If $s(p) > 0$, then we shall define d_p as follows. For $k = 0,1,\ldots,s(p)-1$, let $v_{p,k} := g_{\mu(p,k+1)} - g_{\mu(p,k)}$. We have $v_{p,k} = 0$ on $\overline{D_{p,k}} \cap \overline{D_{p,k+1}}$, since h is continuous. We extend the notation $c_{a,p,j}$ of Lemma 4.7 from the case where $a \in \mathcal{B}$ to the case where $a = 0$: for $j = 0,1,\ldots,$ we define the function $c_{0,p,j}$ by $c_{0,p,j}(x,y) = 0 \ \forall (x,y) \in \mathbb{R}^2_{++}$. If $v_{p,k} \neq 0$, then $v_{p,k} \in \mathcal{A} \subset \mathcal{B}$, so by Lemma

4.3 and Corollary 4.5 there exists a unique $j(p,k) \in \{1,2,\dots,s(v_k,p)\}$ such that the graph of $y = \xi_{v_k,p,j}(x)$ over (γ_p, γ_{p+1}) separates $D_{p,k}$ from $D_{p,k+1}$. We may now take

$$d_p = g_{\mu(p,0)} + \sum_{k=0}^{s(p)-1} c(v_{p,k}, p, j(p,k)),$$

by Lemma 4.7 and Proposition 4.6. □

Remark 4.9. — The above proposition proves the one-variable analog of Theorem 3.1. For if the given function h does not involve one of the two variables (say, x), then by Remark 4.1 above, none of the functions that we constructed in the sets $\mathcal{A}$ and $\mathcal{B}$ will involve x, either, whence we would be able to take $L = 0$ (which would mean that H_0 equals all of $\mathbb{R}^2_{++}$) in Lemma 4.3, Notation 4.4, Corollary 4.5, Lemma 4.7, and Proposition 4.8 above.

5. Conclusion of the proof of Theorem 3.1

Recall, after equation (4.1) we defined $\mathcal{A} = \{\, g_i - g_j \mid i < j \,\}$, and we defined $\mathcal{B}$ to be the set obtained from $\mathcal{A}$ by closing under the operations in equations (4.3) and (4.4) with respect to y. We got an $L \geqslant 0$ and certain γ_p on the x-axis such that $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_L < \gamma_{L+1} = \infty$, and for each $p \in \{0,1,,\dots,L\}$ we got (Proposition 4.8) a function $d_p(x,y) : \mathbb{R}^2_{++} \to \mathbb{R}$ that (1) is a supremum of infima of finitely many generalized polynomial functions and (2) agrees with h on H_p $(= (\gamma_p, \gamma_{p+1}) \times \mathbb{R}_{++})$.

Now let $\mathcal{C}$ be the subset of $\mathbb{R}[\mathbb{R}^2]$ obtained from $\mathcal{B} \cup \{\, x - \gamma_p \mid 1 \leqslant p \leqslant L \,\}$ by closing under the "x-analogs" of the operations in equations (4.3) and (4.4); i.e., interchanging x and y in equations (4.2), (4.3), and (4.4). Then we immediately obtain, first, the following x-analog of Lemma 4.3 and its Corollary 4.5:

Lemma 5.1. — *There exist $M \in \mathbb{N}$ and $\eta_1 < \eta_2 < \cdots < \eta_M \in \mathbb{R}_{++}$ such that, writing $\eta_0 = 0$ and $\eta_{M+1} = \infty$, and fixing any $q \in \{0,1,\dots,M\}$, the zeros, in the qth horizontal half-strip $I_q := \mathbb{R}_{++} \times (\eta_q, \eta_{q+1})$, of all the $a \in \mathcal{C}$, are the graphs of continuous, monotonic,[7] generalized semialgebraic functions $x = \zeta^{q,k}(y)$, $k = 1,2,\dots,t(q)$ (for a suitable $t(q) \in \mathbb{N}$). Moreover, for each $y \in (\eta_q, \eta_{q+1})$,*

$$0 =: \zeta^{q,0}(y) < \zeta^{q,1}(y) < \cdots < \zeta^{q,t(q)}(y) < \zeta^{q,t(q)+1}(y) := \infty. \tag{5.1}$$

Consequently, the sets

$$E_{q,k} := \{\, (x,y) \mid \eta_q < y < \eta_{q+1},\ \zeta^{q,k}(y) < x < \zeta^{q,k+1}(y) \,\},$$

for $k \in \{0, 1, \dots, t(q)\}$, are nonempty, pairwise-disjoint, generalized semialgebraic cells (in particular, they are open and (pathwise) connected), and their union is a dense open subset of I_q. Moreover, the $E_{q,k}$ are "stacked" one to the right of the other in the x-direction, so that for any $y \in (\eta_q, \eta_{q+1})$ and for any $(t(q)+1)$-tuple $x_0, x_1, \dots, x_{t(q)} \in \mathbb{R}_{++}$ for which each $(x_k, y) \in E_{q,k}$, $x_0 < x_1 < \cdots < x_{t(q)}$. Finally, for each k, there is a $p \in \{0, 1, \dots, L\}$ such that $E_{q,k} \subseteq H_p$ (since the functions $x - \gamma_1, \dots, x - \gamma_L$ belong to $\mathcal{C}$). □

The second immediate consequence of our choice of $\mathcal{C}$ is the following x-analog of Proposition 4.8:

PROPOSITION 5.2. — *Let h, $\mathcal{A}$, $\mathcal{C}$, M, $\eta_0, \eta_1, \dots, \eta_{M+1}$, q, and I_q be as above. There is a function $e_q : \mathbb{R}^2_{++} \to \mathbb{R}$ that (1) is a supremum of infima of finitely many generalized polynomial functions $\in \mathbb{R}[\mathbb{R}^2]$ and (2) coincides with $h(x,y)$ on I_q.* □

Let

$$Q = \{ (q,k) \mid q \in \{0, 1, \dots, M\},\ k \in \{0, 1, \dots, t(q)\} \},$$

where M and $t(q)$ are as in Lemma 5.1. Then

$$\bigcup_{(q,k) \in Q} E_{q,k} \text{ is a dense open subset of } \mathbb{R}^2_{++}, \tag{5.2}$$

by Lemma 5.1.

LEMMA 5.3. — *There is a function $\nu : Q \to \{1, \dots, l\}$ such that $\forall (q,k) \in Q$, $E_{q,k} \subseteq A^\circ_{\nu(q,k)}$ (in particular, $h = g_{\nu(q,k)}$ on $E_{q,k}$).*

Proof. — This follows from Lemma 2.3(5) and Lemma 5.1. □

Remark 5.4. — (on Definition 2.1) We can now substantiate the statement in Remark 2.4 above, viz., that in the definition of "piecewise generalized polynomial function" (Definition 2.1), it was not necessary to require each A_i to be a generalized semialgebraic set in the case where h is continuous, since in that case we may (by Lemmas 5.3 and 2.1(3)) take each A_i to be the closure of the union of certain $E_{q,k}$, which is automatically generalized semialgebraic.

NOTATION 5.5. — *For $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, let*

$$\Delta(a,b) = \{ (x,y) \in \mathbb{R}^2 \mid xy > 0 \ \&\ a < x + y < b \}.$$

(See Figure 2.)

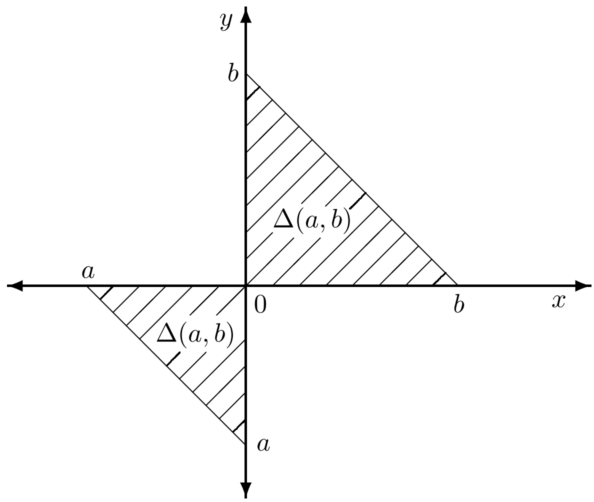

Figure 2. — The "double-triangular" region $\Delta(a, b)$ (5.5). In this figure, $a < 0 < b$.

LEMMA 5.6. — *Let $f(x, y)$ be a real-valued function that is analytic on a neighborhood of $(0, 0)$ in $\mathbb{R}^2$. Write f_x and f_y for $\partial f/\partial x$ and $\partial f/\partial y$, respectively. Suppose $f(0, 0) = 0$, $f_x(0, 0) > 0$, and $f_y(0, 0) > 0$. Then there is an $\epsilon > 0$ such that for all $(x, y) \in \Delta(0, \epsilon)$, $f(x, y) > 0$.*

Proof. — By the Weierstrass Preparation Theorem and the theory of Puiseux series (see, e.g., [Ruiz, 1993, Propositions 3.3 and 4.4, respectively]), the germ at $(0, 0)$ of the zero-set of f consists of finitely many curve germs $(\alpha_1(t), \beta_1(t))$, $(\alpha_2(t), \beta_2(t))$, …, where for each i: α_i and β_i are analytic for $0 \leqslant t < \delta$ (some $\delta > 0$); $\alpha_i(0) = \beta_i(0) = 0$; and

$$\begin{aligned} &\text{either } \alpha_i(t) = t^{m_i} \text{ and } \beta_i'(0) \neq 0, \\ &\quad\text{or } \beta_i(t) = t^{m_i} \text{ and } \alpha_i'(0) \neq 0, \end{aligned} \tag{5.3}$$

for some $m_i \in \{1, 2, \ldots\}$. By the chain rule,

$$0 = \frac{d}{dt} 0 = \frac{d}{dt} f(\alpha_i(t), \beta_i(t))\Big|_0 = f_x(0, 0)\alpha_i'(0) + f_y(0, 0)\beta_i'(0). \tag{5.4}$$

Now we see that we cannot have both $\alpha_i'(0) \geqslant 0$ and $\beta_i'(0) \geqslant 0$, for this, together with equation (5.3) and the hypothesis of the lemma, would make the right hand side of equation (5.4) positive. Thus there is an $\epsilon > 0$ such that for all $(x, y) \in \Delta(\epsilon)$, $f(x, y) \neq 0$. Since $\Delta(\epsilon)$ is connected and f is continuous and nonzero there, f has constant sign (positive or negative) throughout $\Delta(\epsilon)$. This sign must, in fact, be positive, since $\frac{d}{dt} f(t, t)\big|_0 = f_x(0, 0) + f_y(0, 0) > 0$ and $f(0, 0) = 0$. $\square$

Conclusion of the proof of Theorem 3.1. — As in [Mahé, 1984], the idea now is to construct, for each two ordered pairs (q,k) and $(r,m) \in Q$, a function $u_{(q,k),(r,m)}$ that is the supremum of infima of finitely many generalized polynomial functions, and is such that

$$u_{(q,k),(r,m)} \begin{cases} \leqslant g_{\nu(q,k)} & \text{on } E_{q,k} \text{ and} \\ \geqslant g_{\nu(r,m)} & \text{on } E_{r,m}. \end{cases} \tag{5.5}$$

Then we shall be done, since the function

$$u_{(r,m)} := \inf\bigl(\{g_{\nu(r,m)}\} \cup \{\, u_{(q,k),(r,m)} \mid (q,k) \in Q \,\}\bigr)$$

will satisfy

$$\begin{aligned} & u_{(r,m)} = g_{\nu(r,m)} && \text{on } E_{r,m}, \text{ and,} \\ \text{for each } (q,k) \in Q, \quad & u_{(r,m)} \leqslant g_{\nu(q,k)} && \text{on } E_{q,k}; \end{aligned}$$

then $h = \sup_{(r,m)\in Q} u_{(r,m)}$ throughout $\bigcup_{(q,k)\in Q} E_{q,k}$, and hence (by equation (5.2) and the continuity of h) throughout $\mathbb{R}^2_{++}$, as required.

So suppose (q,k) and $(r,m) \in Q$, and let us prepare to construct a $u_{(q,k),(r,m)}$ satisfying equation (5.5). If $E_{\nu(q,k)}$ and $E_{\nu(r,m)}$ are both subsets of the same horizontal half-strip I_q (Lemma 5.1),[11] or of the same vertical half-strip H_p (for some $p \in \{1,2,\ldots,L\}$, using the last sentence of Lemma 5.1), then we may take $u_{(q,k),(r,m)}$ to be either e_q or d_p, respectively, by Proposition 5.2 or 4.8.

The case that makes the proof for two variables harder than the proof for one variable is the case when $E_{\nu(q,k)}$ and $E_{\nu(r,m)}$ do *not* lie in a common half-strip (either horizontal or vertical). We may assume, without loss of generality, that $E_{\nu(q,k)}$ is below and to the left of $E_{\nu(r,m)}$ (i.e., that points in $E_{\nu(q,k)}$ have x- and y-coordinates less than the x- and y-coordinates of points in $E_{\nu(r,m)}$, respectively); the other three possibilities could be handled similarly.

$E_{\nu(q,k)}$ lies in the horizontal half-strip $I_q := \mathbb{R}_{++} \times (\eta_q, \eta_{q+1})$, and in a unique vertical half-strip $H_p := (\xi_p, \xi_{p+1}) \times \mathbb{R}_{++}$, for some p. $E_{\nu(r,m)}$ lies in exactly one of the horizontal half-strips $I_{q+1}, I_{q+2}, \ldots$, and in exactly one of the vertical half-strips $H_{p+1}, H_{p+2}, \ldots$. (See Figure 3, where, for simplicity, $E_{\nu(r,m)}$ is shown lying in I_{q+1} and H_{p+1}.)

For any $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, write

$$\Delta(a,b) + (\xi_{p+1}, \eta_{q+1}) = \{\, (x + \xi_{p+1},\ y + \eta_{q+1}) \mid (x,y) \in \Delta(a,b) \,\}.$$

(11) This will occur if and only if $q = r$.

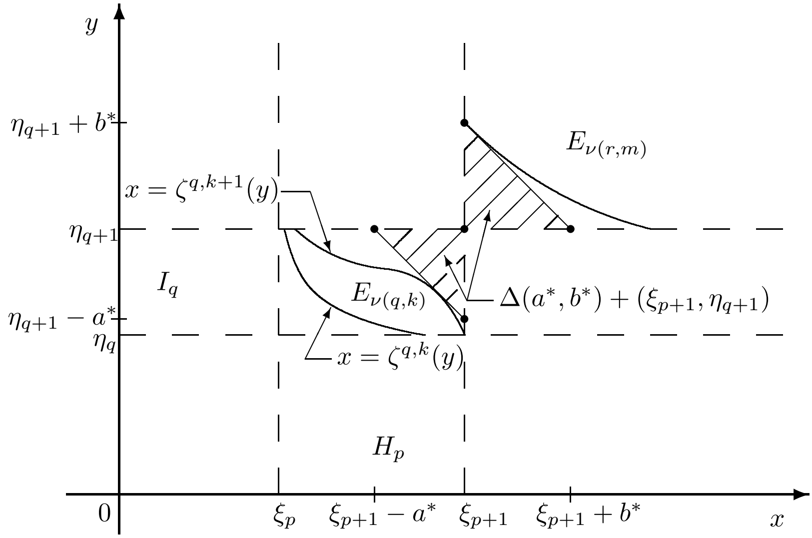

Figure 3. — The case where $E_{\nu(q,k)}$ and $E_{\nu(r,m)}$ do not lie in a common half-strip. (In this illustration, $E_{\nu(r,m)}$ lies in I_{q+1} and H_{p+1}).)

Now let

$$a^* = \min\{\, s \in \mathbb{R} \mid (\Delta(s,0) + (\xi_{p+1}, \eta_{q+1})) \cap E_{\nu(q,k)} = \emptyset \,\} \text{ and} \quad (5.6)$$

$$b^* = \max\{\, t \in \mathbb{R} \mid (\Delta(0,t) + (\xi_{p+1}, \eta_{q+1})) \cap E_{\nu(r,m)} = \emptyset \,\}. \quad (5.7)$$

(Thus, $a^* \leqslant 0 \leqslant b^*$, by the assumptions on $E_{\nu(q,k)}$ and $E_{\nu(r,m)}$ made in the previous paragraph.)

To simplify notation, let

$$g(x,y) = g_{\nu(r,m)}(x,y) - g_{\nu(q,k)}(x,y). \quad (5.8)$$

Pick any $e \in \mathbb{N}$ greater than every x- and y-exponent ($\in \mathbb{R}$) occurring in (the unique representation as in equation (1.1) of) $g(x,y)$. There is a $T \geqslant a^*$ such that for all $(x,y) \in \mathbb{R}^2_{++}$ with $x + y - \xi_{p+1} - \eta_{q+1} \geqslant T$,[12]

$$(x + y - \xi_{p+1} - \eta_{q+1} - a^*)^e \geqslant g(x,y).^{13} \quad (5.9)$$

We may assume that $T > b^*$ (in particular, $T > 0$).

---

(12) In particular, for all $(x,y) \in \Delta(T,\infty) + (\xi_{p+1}, \eta_{q+1})$.
(13) If we had allowed e to be an arbitrary real number (as opposed to an element of $e \in \mathbb{N}$), then $(x + y - \xi_{p+1} - \eta_{q+1} - a^*)^e$ would not necessarily be a signomial function (see [Delzell, 2008, Example 4.7]). Since, in fact, $e \in \mathbb{N}$, $(x + y - \xi_{p+1} - \eta_{q+1} - a^*)^e$ is a signomial function (it is even an ordinary polynomial). We shall need this below.

Case 1: $b^* - a^* > 0$. In this case, there is a $C \in \mathbb{R}$ such that for all $(x, y) \in \Delta(b^*, T) + (\xi_{p+1}, \eta_{q+1})$,

$$C \cdot (x + y - \xi_{p+1} - \eta_{q+1} - a^*)^e \geqslant g(x, y).^{14} \tag{5.10}$$

We may assume that $C \geqslant 1$. Then we may take

$$u_{(q,k),(r,m)} = g_{q,k}(x, y) + C \cdot ((x + y - \xi_{p+1} - \eta_{q+1} - a^*)^+)^e,$$

which satisfies equation (5.5) (using equations (5.6), (5.9), (5.10), and (5.8)), and which is a supremum of infima of finitely many generalized polynomial functions (using Proposition 4.6).

Case 2: $b^* - a^* = 0$ (whence $a^* = 0 = b^*$). In this case, let

$$f(x, y) = g(x + \xi_{p+1},\ y + \eta_{q+1}).^{15}$$

Pick any $D \in \mathbb{R}_{++}$ greater than $\max\{f_x(0, 0),\ f_y(0, 0)\}$. By Lemma 5.6, there is an $\epsilon > 0$ such that $D \cdot (x + y) > f(x, y)$ for all $(x, y) \in \Delta(0, \epsilon)$; equivalently,

$$D \cdot (x + y - \xi_{p+1} - \eta_{q+1}) > g(x, y) \tag{5.11}$$

for all $(x, y) \in \Delta(0, \epsilon) + (\xi_{p+1}, \eta_{q+1})$. We may assume that $\epsilon \leqslant T$.

There is a $C \in \mathbb{R}$ such that for all $(x, y) \in \Delta(\epsilon, T) + (\xi_{p+1}, \eta_{q+1})$,

$$C \cdot (x + y - \xi_{p+1} - \eta_{q+1})^e \geqslant g(x, y).^{16} \tag{5.12}$$

We may assume that $C \geqslant 1$.

Then we may take

$$u_{(q,k),(r,m)}$$
$$= g_{q,k}(x, y) + \sup\{D(x + y - \xi_{p+1} - \eta_{q+1})^+, C((x + y - \xi_{p+1} - \eta_{q+1})^+)^e\},$$

which satisfies equation (5.5) (using equations (5.11), (5.12), (5.9) (with $a^* = 0$), and (5.8)), and which is a supremum of infima of finitely many generalized polynomial functions (using Proposition 4.6).

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(14) Specifically, we may take $C = (\max g(x, y))/\min((x+y-\xi_{p+1}-\eta_{q+1}-a^*)^e)$, where the max and min are taken as (x, y) ranges over the compact set $\overline{\Delta(b^*, T) + (\xi_{p+1}, \eta_{q+1})}$. (Here we need $\min(x + y - \xi_{p+1} - \eta_{q+1} - a^*) > 0$, which follows from our assumption (here in case 1) that $b^* - a^* > 0$.)

(15) In general, f is not a signomial function (again, see [Delzell, 2008, Example 4.7]), but it is, at least, real analytic (for $x > -\xi_{p+1}$ and $y > -\eta_{q+1}$), and this is all we shall need.

(16) Specifically, we may take $C = (\max g(x, y))/\min((x + y - \xi_{p+1} - \eta_{q+1})^e)$, where the max and min are taken as (x, y) ranges over the compact set $\overline{\Delta(\epsilon, T) + (\xi_{p+1}, \eta_{q+1})}$. (Here we need $\min(x + y - \xi_{p+1} - \eta_{q+1}) > 0$, which follows from $\epsilon > 0$.)

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