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SV and related f-rings and spaces

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Abstract. — An f-ring A is an *SV f-ring* if for every minimal prime ℓ-ideal P of A, A/P is a valuation domain. A topological space X is an *SV space* if $C(X)$ is an SV f-ring. SV f-rings and spaces were introduced in [HW1], [HW2]. Since then a number of articles on SV f-rings and spaces and on related f-rings and spaces have appeared. This article surveys what is known about these f-rings and spaces and introduces a number of new results that help to clarify the relationship between SV f-rings and spaces and related f-rings and spaces.

Résumé. — Un f-anneau A est un *SV f-anneau* si pour tout ℓ-idéal premier minimal P de A, A/P est un anneau de valuation. Un espace topologique X est un *SV espace* si $C(X)$ est un SV f-anneau. Les SV f-anneaux et les SV espaces ont été introduits dans [HW1], [HW2]. Depuis lors, plusieurs articles sur les SV f-anneaux et sur les SV espaces ainsi que sur les f-anneaux et sur les f-espaces qui leurs sont apparentés ont paru. Cet article expose les résultats connus sur les f-anneaux et sur les f-espaces et donne des résultats nouveaux qui clarifient la relation entre SV f-anneaux, SV f-espaces et f-anneaux, f-espaces.

1. Introduction

An f-ring A is an *SV f-ring* if for every minimal prime ℓ-ideal P of A, A/P is a valuation domain. A topological space X is an *SV space* if $C(X)$ is an SV f-ring. Mel Henriksen and Richard Wilson initiated the study of SV rings and spaces with their 1992 papers ([HW1], [HW2]). Their work in the area generated interest and inspired a number of authors to join the investigation. This writing is designed to provide a survey of work on issues related to SV f-rings and spaces. In trying to fully understand the landscape

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of SV f-rings and spaces, authors have often looked, on one hand at a little more specialized type of ring and space, the finitely 1-convex f-rings and spaces that are finitely an F-space, and on the other hand, at a little more general type of ring and space, the f-rings of finite rank and the spaces of finite rank. We will investigate these three types of f-rings and spaces, for the most part from the most specialized to the most general. We will see many, but not all, of the results concerning topological spaces have algebraic analogues.

It should be noted that most of the papers currently in the literature simply use the term SV ring to refer to what is here termed an SV f-ring. N. Schwartz has recently introduced the notion of an SV ring as defined on a ring with no lattice ordering, and we wish to be sure it is clear that here, we are referring to the SV property as defined in an f-ring, and not just a ring.

A section on open problems concerning SV f-rings and spaces is included. Other papers that initiate interesting new directions for investigating matters relating to the SV property were presented at the Baton Rouge conference. Henriksen and Banerjee have introduced almost SV f-rings and quasi SV f-rings in [BH]. As mentioned a little earlier, N. Schwartz has introduced SV rings as defined in rings with no lattice ordering, and Robert Redfield has introduced SV ℓ-groups.

We will provide proofs when a result has not appeared elsewhere, or in order to correct an error in the literature. Proofs of results that appear elsewhere will not be included here. A number of new results are included that help to round out the picture.

2. Preliminaries

An f-ring is a lattice ordered ring that is a subdirect product of totally ordered rings. For general information on f-rings see [BKW]. Given an f-ring A, we let $A^+ = \{a \in A : a \geqslant 0\}$, and for an element $a \in A$, we let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. Two distinct positive elements a, b of an f-ring are said to be *disjoint* if $a \wedge b = 0$. If A is an f-ring with identity element, let $A^* = \{a \in A : |a| \leqslant n \cdot 1 \text{ for some positive integers } n\}$. Then A^* is a sub-f-ring of A, and is called the *subring of bounded elements*.

A ring ideal I of an f-ring A is an *l-ideal* if $|a| \leqslant |b|$, and $b \in I$ implies $a \in I$, or equivalently, if it is the kernel of a lattice-preserving homomorphism (ℓ-homomorphism). Given any element a of an f-ring A, there is a smallest ℓ-ideal containing a, and we denote this by $\langle a \rangle$.

Suppose A is an f-ring and I is an ideal of A. The ideal I is *semiprime* (resp. *prime*) if $J^2 \subseteq I$ (resp. $JK \subseteq I$) implies $J \subseteq I$ (resp. $J \subseteq I$ or $K \subseteq I$) for ideals J, K. An ℓ-ideal I of an f-ring is semiprime (resp. prime) if and only if $a^2 \in I$ implies $a \in I$ (resp. $ab \in I$ implies $a \in I$ or $b \in I$). The f-ring A is called semiprime (resp. prime) if $\{0\}$ is semiprime (resp. prime). It is well known that in an f-ring, an ℓ-ideal I is semiprime if and only if it is an intersection of prime ℓ-ideals which are minimal with respect to containing I. If P is a prime ℓ-ideal of the f-ring A, then A/P is a totally ordered prime ring and all ℓ-ideals of A containing P form a chain. In any semiprime f-ring, it is shown in (9.3.2, [BKW]) that a minimal prime ideal is an ℓ-ideal and so in a semiprime f-ring, the collections of minimal prime ideals and minimal prime ℓ-ideals are the same. The following characterization of minimal prime ℓ-ideals in semiprime f-rings is well known.

PROPOSITION 2.1. — *Let A be a semiprime f-ring and P be a prime ℓ-ideal of A. Then P is minimal if and only if for every $p \in P$ there is a $q \notin P$ such that $pq = 0$.*

If P is a proper prime ℓ-ideal of a commutative semiprime f-ring A, we let O_P denote the set $O_P = \{a \in A : \text{there exists } b \notin P \text{ such that } ab = 0\}$. Then O_P is a semiprime ℓ-ideal contained in P and if P is a minimal prime ℓ-ideal, $O_P = P$.

A commutative ring is a *valuation ring* if given any two elements, one divides the other. If A is an f-ring with identity element in which every element $a \geqslant 1$ is invertible, then A is said to be *closed under bounded inversion* or to have *bounded inversion*. A commutative f-ring A is said to satisfy the *1st-convexity condition*, or to be *1-convex* if for any $u, v \in A$ such that $0 \leqslant u \leqslant v$, there is a $w \in A$ such that $u = wv$.

LEMMA 2.2. — *A totally ordered domain with identity element is 1-convex if and only if it is a valuation domain with bounded inversion.*

Proof. — Let A be a totally ordered domain with identity element. Assume first that A is 1-convex. Suppose $a, b \in A$. Then either (i) $a, b \geqslant 0$, (ii) $-a, -b \geqslant 0$, (iii) $-a, b \geqslant 0$, or (iv) $a, -b \geqslant 0$. In the first case, we may assume without loss of generality that $0 \leqslant a \leqslant b$ and it follows immediately from A being 1-convex that b is a divisor of a. In the third case, we may assume without loss of generality that $0 \leqslant -a \leqslant b$ and so there is a $w \in A$ such that $-a = wb$. This implies $a = (-w)b$ and hence b is a divisor of a. The other cases are similar and it follows that A is a valuation domain. If $a \in A$ and $1 \leqslant a$, then the 1st-convexity property implies $a^{-1} \in A$. So A has bounded inversion.

Assume next that A is a valuation domain with the bounded inversion property. Suppose $0 \leqslant a \leqslant b$ in A. By hypothesis, either $a = wb$ or $b = wa$ for some $w \in A$. If $a = wb$, there is nothing we need prove, so suppose $b = wa$. Then $b = wa \vee a = (w \vee 1)a$ and since A has bounded inversion, $(w \vee 1)^{-1} \in A$ and $a = (w \vee 1)^{-1}b$. $\qquad\square$

Let X be a completely regular topological space, and $C(X)$ denote the f-ring of real-valued continuous functions defined on X. Also, let $C^*(X)$ denote the f-ring of all bounded real-valued continuous functions defined on X. Recall that in X, a *zeroset* is a set of the form $\{x \in X : f(x) = 0\}$ for some function $f \in C(X)$, and a *cozeroset* is the complement of a zeroset. Given a function $f \in C(X)$, we let $Z(f)$ (resp. $coz(f)$) denote the zeroset (resp. cozeroset) determined by f. For a completely regular space, we let βX denote the Stone-Čech compactification of X.

It is well known that if X is a compact space, then every maximal ideal of $C(X)$ is of the form $M_x = \{f \in C(X) : f(x) = 0\}$ and the intersection of all of the prime ideals contained in a given maximal ideal M_x is the semiprime ideal $O_x = \{f \in C(X) : Z(f) \text{ is a neighborhood of } p\}$. For a given $x \in X$, the ideal $O_x = O_{M_x}$ as was defined immediately after Proposition 2.1.

A subspace S of X is said to be C^*-*embedded* (resp. C-*embedded*) in X if every function in $C^*(S)$ (resp. $C(S)$) can be extended to a function in $C^*(X)$ (resp. $C(X)$). An *F-space* is a space in which every cozeroset is C^*-embedded. A number of conditions, both topological conditions on X, and algebraic conditions on $C(X)$, are equivalent to X being an F-space and appear in (14.25, [GJ]), (1, [MW]), and (2.4, [L1]). We repeat just a few of these equivalent conditions.

THEOREM 2.3. — *For a completely regular space X the following are equivalent.*

1. *X is an F-space.*
2. *For every $p \in \beta X$, the ℓ-ideal $O^p = \{f \in C(X) : cl_{\beta X} Z(f) \text{ is a neighborhood of } p\}$ is prime.*
3. *Every finitely generated ring ideal of $C(X)$ is principal.*
4. *Every ring ideal of $C(X)$ is an ℓ-ideal.*
5. *βX is an F-space.*
6. *$C(X)$ is 1-convex.*

If X is normal, (1) - (6) above are equivalent to

7. *for every $x \in X$, the maximal ideal M_x contains just one minimal prime ℓ-ideal.*

A point x in the space X is a *P-point* if every G_δ containing x is a neighborhood of x, and x is a *βF-point* if the ℓ-ideal $O^x = \{f \in C(X) : cl_{\beta X} Z(f)$ is a neighborhood of $x\}$ is prime. If x is a P-point or a βF-point, then the maximal ideal $M_x = \{f \in C(X) : f(x) = 0\}$ contains a unique minimal prime ideal.

Throughout, we will assume all spaces are completely regular. When it is necessary to explicitly indicate that we are considering a topological property with respect to the space X, we use "X-" as a prefix to the topological property. For example, when we say a set is X-open, we simply mean that it is an open set with respect to the topology on X.

3. The Initial Henriksen-Wilson Papers and a Little Background

In a 1986 paper ([CD]), G. Cherlin and M. Dickmann call a commutative integral domain D *real-closed* if it (i) is totally ordered, (ii) is closed under taking square roots of positive elements, (iii) has the property that each monic polynomial of odd degree in $D[X]$ has a zero in D, and (iv) has the property that whenever $a, b \in D$, with $0 < a < b$, then a divides b. They show that for a prime ideal P of a $C(X)$, the ring $C(X)/P$ is real-closed if and only if it is a valuation ring. So, they call a prime ideal P *real-closed* if $C(X)/P$ is real-closed. Every maximal ideal of $C(X)$ is a real-closed prime ideal and Cherlin and Dickmann investigate conditions on X in which there are other (nonmaximal) real-closed ideals.

Mel Henriksen and Richard Wilson head in a different direction, taking a more global approach to such matters. The first of their 1992 papers defines a $C(X)$ to be a *survaluation ring* or an *SV ring* if, for each of its prime ideals P, $C(X)/P$ is a valuation domain (and hence P is real-closed) and the space X to be an *SV space* if $C(X)$ is an SV f-ring. They initiate the study of SV rings and spaces and investigate the interplay between topological properties and the algebraically defined notion of an SV space.

An initial observation is made that simplifies the investigation: $C(X)$ is an SV ring if and only if for every minimal prime ideal P of $C(X)$, $C(X)/P$ is a valuation domain. This observation follows from the fact that if P is a prime ideal in a $C(X)$ and Q a proper prime ideal containing P, then $C(X)/Q$ is a homomorphic image of $C(X)/P$ and homomorphic images of valuation domains are valuation domains.

An often used result that is established in [HW1] follows. Its proof depends on the fact that the mapping $P \to P \cap C^*(X)$ is a surjection of the set of minimal prime ideals of $C(X)$ onto the set of minimal prime ideals of $C^*(X)$, and the fact that P is a real-closed ideal of $C(X)$ if and only if

$P \cap C^*(X)$ is a real closed ideal of $C^*(X)$.

THEOREM 3.1 (2.3, [HW1]). — *For any completely regular space X, the following are equivalent.*

1. *X is an SV space*

2. *υX is an SV space*

3. *βX is an SV space.*

The fact that X is an SV space if and only if βX is an SV space is a key result that provides a basis for later work in which we often assume we are dealing with compact spaces.

The initial Henriksen and Wilson paper proves that if a space X is a finite union of C-embedded SV spaces, then X is an SV space. Their proof can be summarized by the following. First, note that it is sufficient to assume that $X = X_1 \cup X_2$ where X_1, X_2 are C-embedded subspaces. Then for a prime ideal P of $C(X)$, if $\{g \in C(X) : g(X_1) = \{0\}\} \subseteq P$ then $P_1 = \{g|_{X_1} : g \in P\}$ is a prime ideal of $C(X_1)$. Then $C(X)/P \cong C(X_1)/P_1$ which, by hypothesis, is a valuation domain. Similarly for a prime ideal P of $C(X)$, if $\{g \in C(X) : g(X_2) = \{0\}\} \subseteq P$ then $C(X)/P$ is a valuation domain. Since every prime ideal of P must contain either $\{g \in C(X) : g(X_1) = \{0\}\}$ or $\{g \in C(X) : g(X_2) = \{0\}\}$, this then says for every prime ideal P of $C(X)$, $C(X)/P$ is a valuation domain and X is an SV space. It is easy to see using Lemma 2.2 and Theorem 2.3 that an F-space must be an SV space and so it follows that

THEOREM 3.2 (2.9, [HW1]). — *A compact space that can be written as a union of finitely many closed F-spaces is an SV space.*

What has become a classic example of an SV space that is not an F-space is now easy to describe.

Example 3.3. — Let X_1, X_2 each denote a copy of $\beta \mathbf{N}$, the Stone-Čech compactification of the natural numbers. Then X_1, X_2 are F-spaces. Let X denote the space obtained by starting with the topological sum of X_1, X_2 and identifying each pair of corresponding points of $\beta \mathbf{N} \backslash \mathbf{N}$. Clearly, X is a compact space that can be written as the union of two closed F-spaces and hence is an SV space by the previous result. It is also useful to see directly why X is an SV space. Recall that every minimal prime ideal of

a compact space is contained in the maximal ideal M_x for some $x \in X$. For any $x \in X$ that was contained in one of the copies of **N**, the point x is isolated and the minimal prime ideal contained in M_x is M_x itself. Then $C(X)/M_x$ is a field and hence a valuation domain. For any $x \in X$ not contained in a copy of **N**, M_x contains just two minimal prime ideals: $P_1 = \{f \in C(X) : f|_{X_1} \in O_{x_1}\}$ and $P_2 = \{f \in C(X) : f|_{X_2} \in O_{x_2}\}$, where for $i = 1, 2$, $O_{x_i} = \{g \in C(X_i) : Z(g)$ is an X_i-neighborhood of $x\}$. Then $C(X)/P_i \cong C(\beta\mathbf{N})/O_x$, and $C(\beta\mathbf{N})/O_x$ is 1-convex and hence a valuation domain.

In their follow-up paper, Henriksen and Wilson consider almost discrete spaces and characterize almost discrete spaces that are SV spaces. A Hausdorff space is an *almost discrete* space if it has a single non-isolated point. When considering almost discrete spaces, we let ∞ denote the non-isolated point. It is useful to note that an almost discrete space is normal and is perfectly normal if and only if ∞ is a G_δ point. A space X is *basically disconnected* if the closure of every cozeroset is open. Every basically disconnected space is an F-space. The characterization of almost discrete spaces that are SV spaces follows.

THEOREM 3.4 (2.3, 3.1, [HW2]). — *If $X = D \cup \{\infty\}$ is an almost discrete space, then the following are equivalent.*

1. *X is an SV space.*

2. *$C(X)/P$ is a valuation domain for each minimal prime ideal P contained in M_∞.*

3. *M_∞ contains only finitely many minimal prime ideals of $C(X)$.*

4. *X is a finite union of closed basically disconnected subspaces.*

Each of the Henriksen and Wilson papers also show that SV spaces share some properties with F-spaces. For example, an infinite SV space contains no nontrivial convergent sequences. They also show that C^*-embedded subspaces of SV spaces are SV spaces and that finite unions of compact SV spaces are SV spaces.

The two Henriksen and Wilson papers sparked further investigation related to SV spaces and SV rings. A number of the ideas that remain central to this work can be seen in these two early papers, though not all were "named" in these papers. We pause now to define several concepts using terminology introduced in later papers, but whose roots can be seen in one of the 1992 Henriksen and Wilson papers.

DEFINITIONS 3.5. —

1. *A space X is finitely an F-space if βX is a union of finitely many closed F-spaces.*

2. *Suppose M is a maximal ℓ-ideal of an f-ring A. The rank of M, is the number of minimal prime ideals contained in M if the set of all such minimal prime ideals is finite, and the rank of M is infinite otherwise.*

3. *If A is an f-ring, then the rank of A is the supremum of the ranks of the maximal ℓ-ideals of A. The f-ring A is said to have finite rank if the rank of A is finite.*

4. *The rank of the space X is the rank of the f-ring C(X). The rank of the point $x \in X$ is the rank of the maximal ℓ-ideal $M_x = \{f \in C(X) : f(x) = 0\}$.*

The reader will note that the concepts of finite rank and (a version of) finitely an F-space appear in Theorem 3.4. Most of the later work that relates to SV matters revolves around the concepts of a space being finitely an F-space, being an SV space, or having finite rank. So, throughout the work that follows, we will investigate three classes of completely regular topologies: spaces that are finitely an F-space, SV spaces, and spaces of finite rank and the corresponding three classes of f-rings.

Because the rank of a maximal ℓ-ideal plays a central role in our dealings with the three classes of spaces to be studied, we conclude this section with a few basics regarding a maximal ℓ-ideal of finite rank. First, it should be noted that if X has finite rank then every point $x \in X$ has finite rank, but the converse need not hold. It is possible for every point in X to have finite rank, while the space X does not have finite rank since not every maximal ℓ-ideal of C(X) has the form M_x. An example where this happens is the space U of Example 4.11. Since for a compact space X, every maximal ℓ-ideal of C(X) is of the form M_x, a compact space X will have finite rank if and only if every $x \in X$ has finite rank.

In a semiprime f-ring, the minimal prime subgroups are exactly the minimal prime ideals (see Theorem 9.3.2 of [BKW]). This, and the Finite Basis Theorem of Conrad (Theorem 46.12 of [D]) imply that the maximal ℓ-ideal M has finite rank n if and only if there is a set of n nonzero pairwise disjoint elements in A/O_M and there is no larger such set. It is straightforward to show that a set of n nonzero pairwise disjoint elements from A/O_M corresponds to a set of n pairwise disjoint elements of A, each contained in M, but not in O_M. The next theorem follows from these facts.

THEOREM 3.6. — *Let A be a commutative semiprime f-ring with identity element. A maximal ℓ-ideal M has finite rank $n \geqslant 2$ if and only if there is a set of n pairwise disjoint elements in $M \backslash O_M$ and there is no larger such set.*

It is straightforward to translate the result of the previous theorem into topological terms as is recorded in the following theorem. This characterization of a point x having rank $n \geqslant 2$ has been an indispensable tool in the literature. Since the following theorem is an application of the previous theorem, no new proof is needed. However, it is instructive to construct a direct proof since it reveals some of the interplay between the algebraic and topological sides of the subject. For that reason, we present an outline of a direct proof.

THEOREM 3.7 (3.1, HLMW). — *Let X be a completely regular space. A point $x \in X$ has finite rank $n \geqslant 2$ if and only if there is a collection of n pairwise disjoint cozerosets such that x is in the closure of each cozeroset, and there is no larger such collection.*

Outline of Proof. — $\Rightarrow$ Suppose x has rank n. Suppose $P_1, P_2, \ldots, P_n$ are n distinct minimal prime ideals contained in M_x. For each $i = 1, 2, \ldots, n$, let $p_i \in (\cap_{j \neq i} P_j) \backslash P_i$. Then for any $i \neq j$, $p_i p_j \in P_1 \cap P_2 \cap \cdots \cap P_n = O_x$. So for each i, j with $i \neq j$, there is a cozeroset neighborhood V_{ij} of x such that $p_i p_j (V_{ij}) = \{0\}$. It is not hard to see the collection of n cozerosets $\{(\cap_{i,j} V_{ij}) \cap coz(p_i)\}$ is pairwise disjoint and x is in the closure of each of these cozerosets. We show there can be no larger such collection of cozerosets indirectly. So suppose $coz(f_1), coz(f_2), \ldots, coz(f_{n+1})$ is a collection of $n+1$ pairwise disjoint cozerosets such that $x \in cl(coz(f_i))$ for each i. Now for each i, $f_i \notin P_j$ for some j, or else f_i would be in $P_1 \cap P_2 \cap \cdots \cap P_n = O_x$, and x would not be in the closure of $coz(f_i)$. By the pigeonhole principle, there is (at least) one of the minimal prime ideals, say P_1, such that two of the f_i, say f_1, f_2, are not in P_1. But then $f_1 f_2 = 0 \in P_1$, while $f_1, f_2 \notin P_1$, and we have a contradiction. So there can be no larger such collection of cozerosets.

$\Leftarrow$ Suppose $f_1, f_2, \ldots, f_n \in C(X)$ and $coz(f_1), coz(f_2), \ldots, coz(f_n)$ is a collection of pairwise disjoint cozerosets such that x is in the closure of each, and there is no larger such collection. For each $i = 1, 2, \ldots, n$, define $P_i = \{f \in C(X) : \exists$ neighborhood V of x such that $coz(f_i) \cap V \subseteq Z(f)\}$. It is straightforward to check and see that P_i is an ℓ-ideal. To see that it is a prime ideal, suppose $g_1, g_2 \in C(X)$ and $g_1 g_2 \in P_i$. Then there exists a cozeroset neighborhood V' of x such that $coz(f_i) \cap V' \subseteq Z(g_1 g_2)$. If $g_1, g_2 \notin P_i$, then $coz(f_1) \cap V', coz(f_2) \cap V', \ldots, coz(f_{i-1}) \cap V', coz(g_1) \cap coz(f_i) \cap V', coz(g_2) \cap coz(f_i) \cap V', coz(f_{i+1}) \cap V', \ldots, coz(f_n) \cap V'$ is a

collection of $n + 1$ pairwise disjoint cozerosets such that x is in the closure of each, contrary to hypothesis. So $g_1 \in P_i$ or $g_2 \in P_i$ and P_i is prime. Now P_i is in fact a minimal prime ideal and so x has rank at least n. The point x cannot have rank greater than n since if it did, the argument in the previous paragraph would show there is a collection of $n+1$ pairwise disjoint cozerosets such that x is in the closure of each cozeroset, a contradiction. □

4. SV Spaces and Related Topological Spaces

In this, and later sections we will not present results as they appeared chronologically; rather in what is (hopefully) a more efficient means. We begin with the topological side – that is, we begin by studying topological spaces and the corresponding ring of continuous functions for spaces that are finitely an F-space, SV spaces, and spaces of finite rank. Along the way we will show that the relationship between these three types of spaces and F-spaces can be summarized by

F-space $\Rightarrow$ finitely an F-space $\Rightarrow$ SV space $\Rightarrow$ space of finite rank,

where the first two arrows cannot be reversed and it is unknown if the third arrow can be reversed.

Topological spaces that are finitely an F-space.

That an F-space is finitely an F-space follows immediately from the definitions and there are many spaces that are finitely an F-space, but not an F-space. Indeed, a common means of constructing a space that is finitely an F-space, but not an F-space is to begin with the disjoint union of n $(n > 1)$ compact F-spaces $X_1, X_2, \ldots, X_n$, closed nowhere dense subsets $A_i \subseteq X_i$ for $i = 1, 2, \ldots, n$, and continuous bijections $g_i : A_1 \to A_i$ for $i = 2, 3, \ldots, n$, and then for each $a \in A_1$, identify all n points $a_1, g_2(a_1), g_3(a_1), \ldots, g_n(a_1)$ as a single point. In the resulting space, every identified point $a \in A$ will have rank at most n, and all other points will have rank 1. Example 3.3 is of this type of construction.

In [A], Aliabad investigates topological spaces X obtained by beginning with a family of topological spaces $\{X_\alpha\}_{\alpha \in \Lambda}$ and specified points $x_\alpha \in X_\alpha$ for each $\alpha \in \Lambda$, and identifying all of the x_α's as a single point σ. Some characterizations of ideals of $C(X)$ are given that rely only on corresponding ideals of the $C(X_\alpha)$'s. If the X_α's are F-spaces and Λ is finite, this construction results in a space that is finitely an F-space, with the identified point σ being the only point that might have rank greater than 1. Then, the point σ

1. (4.1 of [A]) will be a P-point if and only if x_α is a P-point in X_α for each α.

2. (4.1 of [A]) will be a βF-point if and only if x_α is a βF-point in X_α for one α and is a P-point in X_α for all other α.

3. will have rank m for $m \geqslant 2$ if and only if x_α is a βF-point in X_α for m of the α and is a P-point in X_α for all other α.

Suppose X is compact and finitely an F-space, then X can be written as $X = \cup_{i=1}^n X_i$ for some compact F-spaces X_i. Let $Bndry_X(X_i)$ denote the boundary of X_i in X, and let $T = \cup_{i=1}^n Bndry_X(X_i)$. Every point of $X - T$ lies in the interior of a closed F-subspace of X, and so has rank 1. This leads to the following observation:

THEOREM 4.1 (5.16, [HLMW]). — *If X is compact and finitely an F-space, then X has an open and dense subset of points of rank 1.*

By Theorems 3.2 and 3.1, a space that is finitely an F-space is an SV space. In [L3], a construction is given of a type of SV space that is not finitely an F-space. The construction begins with a collection of a certain type of normal SV space for which the Stone-Čech compactification is SV and the collection of points in the Stone-Čech compactification of rank greater than 1 is the closure of a countable set. Then an inverse limit space constructed from this collection is shown to be a compact SV space in which there is a dense set of points of rank $n > 1$ and by the previous theorem cannot be finitely an F-space. This same construction also provided examples showing that the set of points of rank greater than 1 is not necessarily closed in an SV space, however, later constructions provide less complicated examples showing that the set of points of rank greater than 1 is not necessarily closed. One of these is found in [A]. The inverse limit construction still provides the only known type of SV space that is not finitely an F-space.

SV spaces.

We begin our discussion in this subsection by giving characterizations of SV spaces. Since it was known early in the development that a space is SV if and only if its Stone-Čech compactification is SV, much of the search for topological characterizations of an SV space has been restricted to compact spaces.

It is well known that a space is an F-space if and only if every cozeroset of the space is C^*-embedded. That is to say a space is an F-space if and only if every bounded continuous real-valued function defined on a cozeroset can be extended to a continuous function defined on X. Compact SV spaces do not have this property, but compact SV spaces are characterized by continuous real-valued functions defined on cozerosets having a finite number of

continuous "partial extensions" defined on X. The following theorem makes this precise.

THEOREM 4.2 (the equivalence of (1) and (2) is 4.6, [HLMW]). — *Let X be a compact space. The following are equivalent.*

1. *X is an SV space*

2. *For every $x \in X$, there is a $k \in \mathbf{N}$ such that for any cozeroset $Y \subseteq X$ and $l \in C^*(Y)$ there are $l_1, l_2, \ldots, l_k \in C(X)$ and zerosets $Z_1, Z_2, \ldots, Z_k$ such that $l_i|_{Z_i \cap Y} = l|_{Z_i \cap Y}$ for each i and $\cup Z_i$ is a neighborhood of x*

3. *For every cozeroset $Y \subseteq X$ and $l \in C^*(Y)$, there are $l_1, l_2, \ldots, l_n \in C(X)$ such that $(l - l_1|_Y)(l - l_2|_Y) \cdots (l - l_n|_Y) = 0$.*

Proof of 2) $\Rightarrow$ 3). — Suppose $Y \subseteq X$ is a cozeroset and $l \in C^*(Y)$. It follows from (2) that for each $x \in X$, there is $k_x \in \mathbf{N}$, $l_{x1}, l_{x2}, \ldots, l_{xk_x} \in C(X)$ and a neighborhood V_x of x such that $(l|_{V_x \cap Y} - l_{x1}|_{V_x \cap Y})(l|_{V_x \cap Y} - l_{x2}|_{V_x \cap Y}) \cdots (l|_{V_x \cap Y} - l_{xk_x}|_{V_x \cap Y}) = 0$ on $V_x \cap Y$. The collection of all the V_x forms a neighborhood cover of X and since X is compact, there is a finite subcover that we will denote by $V_{x_1}, V_{x_2}, \ldots, V_{x_m}$. Then $\prod_{i=1}^m (l - l_{x_i 1}|_Y)(l - l_{x_i 2}|_Y) \cdots (l - l_{x_i k_{x_i}}|_Y) = 0$.

3) $\Rightarrow$ 1): Let P be a minimal prime ℓ-ideal of $C(X)$. By Lemma 2.2, it will suffice to show that $C(X)/P$ is 1-convex. So suppose $0 \leqslant f + P \leqslant g + P$ in $C(X)/P$. Then there exists $p_1, p_2 \in P$ such that $0 \leqslant f + p_1 \leqslant g + p_2$. Let $Y = coz(g + p_2)$. By (3), there are $l_1, l_2, \ldots, l_n \in C(X)$ such that $(\frac{f+p_1}{g+p_2} - l_1|_Y)(\frac{f+p_1}{g+p_2} - l_2|_Y) \cdots (\frac{f+p_1}{g+p_2} - l_n|_Y) = 0$ on Y. It follows that $(f + p_1 - l_1(g+p_2))(f + p_1 - l_2(g+p_2)) \cdots (f + p_1 - l_n(g+p_2)) = 0$. Since P is a prime ideal, $f + p_1 - l_i(g + p_2) \in P$ for some i. Then $f + P = (l_i + P)(g + P)$ in $C(X)/P$. □

Here, each of the l_i are "partial extensions" of l. The proof of the equivalence of 1) and 2) employs the idea that in a compact space of finite rank, every minimal prime ideal is contained in a maximal ideal of the form M_x for some $x \in X$ and is associated (as in the proof of Theorem 3.7) with a cozeroset U that has x in its closure. Then for any $u + P, v + P$ in $C(X)/P$ with $0 \leqslant u + P \leqslant v + P$, there is a $w + P \in C(X)/P$ such that $(u + P) = (w + P)(v + P)$ if and only if the bounded function $\frac{u}{v}$ in $C(U \cap coz(v))$ can be extended to a function in $C(X)$.

For a bounded continuous function h defined on a cozeroset U of X, we say there is an *h-rift* at the point z if h cannot be extended continuously to $U \cup \{z\}$. It is easy to see that if there is an h-rift at the point z, then

z must be on the boundary of U. To illustrate, consider the space X from Example 3.3. There, X has two copies of $\mathbf{N}$ "attached" to a single copy of $\beta\mathbf{N} - \mathbf{N}$. Let U_1 be the cozeroset consisting of the two copies of $\mathbf{N}$. Define $h \in C^*(U_1)$ by $h(x) = 1$ on the copy of $\mathbf{N}$ in X_1 and $h(x) = 0$ on the copy of $\mathbf{N}$ in X_2. Then $\beta\mathbf{N} - \mathbf{N}$ is the set of points with h-rift. On the other hand if U_2 is the cozeroset consisting only of the copy of $\mathbf{N}$ in X_1 and $k(x) = 1$ on U_2, then there are no points with k-rift. In general, the set of points with h-rift is a subset of the set of points of rank greater than 1 and contains points that could "prevent" a compact space of finite rank from being SV. A compact space of finite rank in which there are finitely many points with h-rift for each such h are SV spaces, so the mere existence of some points with h-rift is not enough to prevent a compact space of finite rank from being SV. Instead, whether or not a compact space of finite rank is SV depends on a characteristic of the closure of the set of points with h-rift for each such h as the next theorem will indicate.

THEOREM 4.3 (2.7, [L4]). — *Let X be a compact space of finite rank. Suppose for a cozeroset U and $h \in C^*(U)$, Y_h denotes the set of all points with h-rift. Then X is an SV space if and only if for every cozeroset U of X and $h \in C^*(U)$, $cl_X Y_h$ contains no points of $cl(U)$-rank 1.*

It can be hard to check whether this h-rift condition holds for a given $C(X)$. However, there are some situations in which this theorem is useful.

In 2.11 [L4], it is shown that for a compact space X of finite rank, if for every cozeroset U and $h \in C^*(U)$, the subspace of points with h-rift is an F-space, then X is an SV space. It is also shown that for a compact SV space X of rank n, a cozeroset U, and $h \in C^*(U)$, the subspace of points with h-rift is normal and has rank at most $n - 1$. In particular, if X is a compact SV space of rank 2, then Y_h, the subspace of points with h-rift, is a normal F-space. So, for compact spaces of rank 2, this produces another characterization of SV spaces.

THEOREM 4.4 (2.12, [L4]). — *A compact space of rank 2 is an SV space if and only if for every cozeroset U and $h \in C^*(U)$, the subspace of points with h-rift is an F-space.*

When the rank of a compact SV space is greater than 2, the subspace of points with h-rift need not be an F-space (see 2.11, [L4]). This represents one of just a few places in the literature where a result holds for a space of a particular rank, but not necessarily for every space of finite rank.

Let A be an f-ring with identity element. A sequence $\{a_n\}$ in A is *uniformly Cauchy* if for each positive integer p, there is a positive integer N_p so that $n, m \geqslant N_p$ implies $p|a_n - a_m| \leqslant 1$. The sequence *converges uniformly* to a if for every positive integer p, there is a positive integer N_p for which $n \geqslant N_p$ implies $p|a_n - a| \leqslant 1$. We say the f-ring A is *uniformly complete* if every uniformly Cauchy sequence converges uniformly to a unique limit. Every uniformly complete f-algebra with identity element is archimedean (i.e. has the property that $0 \leqslant na \leqslant b$ for all positive integers n implies $a = 0$), has bounded inversion, and possesses square roots of positive elements. Certainly then uniformly complete f-algebras are a fairly restricted class of f-rings. However, for every space X, $C(X)$ is a uniformly complete f-algebra with identity element. A few fundamental results have been proven for uniformly complete f-rings, two of which provide the following characterizations of those uniformly complete f-algebras with identity element that are SV f-rings. We state these results in this topological section since their primary applications are for a $C(X)$.

Note that a positive element of an ℓ-group is called *indecomposable* if it is not the sum of a pair of nonzero disjoint elements. A positive element a of a semiprime f-ring A with identity element and bounded inversion is *indecomposable at the maximal ideal M* if $a + O_x$ is indecomposable in A/O_x. For a compact space X and $x \in X$, this means $f \in C(X)$ is indecomposable at the maximal ideal M_x if $f = g + h$ with gh zero on a neighborhood of x implies g or h is zero on a neighborhood of x.

Theorem 4.5 (4.3, [HLMW]). — *Suppose A is a uniformly complete f-algebra with identity element. Then A is an SV algebra if and only if the following conditions hold.*

1. *A has finite rank, and*

2. *for every maximal ideal M and every pair $0 < a \leqslant b$, with b indecomposable at M, there is an $x \in A$ such that $a - xb \in O_M$.*

In particular, note that the preceeding theorem implies every uniformly complete SV algebra with identity element has finite rank and for every SV space X, $C(X)$ has finite rank. The fact that an SV space has finite rank is used extensively in the literature.

An ideal I of an f-ring A is *saturated* if $a + b \in I$ and $ab = 0$ imply a and b are in I. One characterization of an F-space is that every ring ideal of $C(X)$ is an ℓ-ideal. A similar characterization holds for SV spaces.

THEOREM 4.6 (6.2, [HLMW]). — *A uniformly complete f-algebra is an SV algebra if and only if each of its saturated ideals is an ℓ-ideal.*

We call an ideal I *pseudoprime* if $ab = 0$ implies $a \in I$ or $b \in I$. In a uniformly complete f-algebra, every pseudoprime ideal is saturated, but not conversely. We will see in Theorem 5.9, a similar result holds for a commutative semiprime f-ring A with bounded inversion: A is an SV f-ring if and only if each of its pseudoprime ideals is an ℓ-ideal.

Spaces of finite rank.

As we have seen, every SV space has finite rank. Whether or not the converse holds is unknown. In order to better understand SV spaces and to aid in finding an answer to the question of whether a space of finite rank is SV, spaces of finite rank have been studied. The following theorem provides a list of properties relating to rank.

THEOREM 4.7. — *Let X be a completely regular space.*

1. (4.2, [HLMW]) *Suppose every maximal ℓ-ideal of $C(X)$ has finite rank. Then X has finite rank.*

2. (3.4 [L2]) *Suppose Y is a countable discrete subspace of X. If every point of Y has X-rank greater than or equal to n, then every point of $cl_X(Y)$ has X-rank greater than or equal to n.*

3. (3.3, [L2]) *If X is normal, $y \in \beta X - X$, $rk_{\beta X}(y) > 1$, and Y denotes the subspace of points of X of X-rank greater than 1, then $y \in cl_{\beta X}(Y)$. If the set of points of X of rank greater than one is compact, then every point of $\beta X - X$ has βX-rank 1.*

4. (5.6, [HLMW]) *If X is an infinite compact space of finite rank then X contains a copy of $\beta \mathbf{N}$.*

5. (5.7, [HLMW]) *If X is a compact space of finite rank then every infinite closed subspace of X has at least 2^c points.*

6. (3.2, [HLMW]) *X has finite rank if and only if βX has finite rank.*

7. (5.10, [HLMW]) *If X is a normal space, then $rank(\beta X) = sup\{X\text{-}rank(p) : p \in X\}$.*

8. (5.15, [HLMW]) *If X, Y are infinite, compact F-spaces, then $X \times Y$ has infinite rank.*

A few comments about this theorem are in order. Part 1) is true more generally. If A is a uniformly complete f-algebra with identity element in which every maximal ℓ-ideal has finite rank, then A has finite rank. The idea behind the proof of 2) is that because every point $y_i \in Y$ has rank greater than or equal to n, there are n pairwise disjoint cozerosets $U_{i1}, U_{i2}, \ldots, U_{in}$, each having y_i in its closure. Taking care in the choice of these U_i allows one to define $U_j = \cup_{i=1}^{\infty} U_{ij}$ for $j = 1, 2, \ldots, n$ and then every point in $cl_X(Y)$ is in the closure of the n pairwise disjoint cozerosets U_j and hence has rank greater than or equal to n. Part 3) implies that if X is locally compact and the set of points of rank greater than one is compact, then $\beta X - X$ is an F-space. Parts 4) and 5) indicate that SV spaces also have some properties known to hold in F-spaces (see 14N.5 of [GJ]). Part 6) is true more generally - see 3.2 of [HLMW]. The hypothesis that X is normal cannot be left out of part 7) since the space U of Example 4.11 is a space in which every point has rank 1, while some points of βU have rank greater than 1. Part 8) emphasizes the fact that the product of two F-spaces is not an F-space. In fact, the product of two F-spaces is not even of finite rank.

There is something of a gap between known properties of spaces of finite rank and known characterizations of SV spaces. For example, suppose X is a compact space of finite rank, $U \subseteq X$ is a cozeroset, $h \in C^*(U)$, and $x \in X$. If $\mathrm{rank}(x) = n$, then there are n pairwise disjoint cozerosets $C_1, C_2, \ldots, C_n$, each having x in its closure and having the property that h can be extended continuously to $(C_i \cap U) \cup \{x\}$. But this falls short of ensuring the finite number of continuous "partial extensions" required for X to be SV by Theorem 4.2. The proof of Theorem 4.5 requires finite rank and an additional property when showing the space is SV. Still, no known example exists of a space that is of finite rank and is not SV. A few theorems of the form "a space of finite rank and with property P is an SV space" have been proven.

THEOREM 4.8. — *Let X be a normal space.*

1. (3.11, [L2]) *Suppose every point of X has rank less than or equal to some integer m. If the points of X of rank greater than 1 form a countable discrete subspace, then X is an SV space.*

2. *Suppose X is a compact space of finite rank and the subspace consisting of the points of X with rank greater than 1 is a P-space. Then X is an SV space.*

3. *Suppose X has finite rank and the subspace consisting of the points of X with rank greater than 1 is a C^*-embedded P-space, then X is an SV space.*

Proof of 2). — Let U be an X-cozeroset, and $h \in C^*(U)$. The collection of points with h-rift forms a subspace of the subspace consisting of the points with rank greater than 1. Every subspace of a P-space is itself a P-space, and hence an F-space. So the subspace of X of points with h-rift is an F-space and so Theorem 2.11 of [L4] implies X is an SV space.

3): Let Y denote the set of all points of X-rank greater than 1. Since Y is C^*-embedded, $cl_{\beta X} Y = \beta Y$ (see 6.9a, [GJ]). Then because Y is a P-space, $\beta Y = cl_{\beta X} Y$ is a P-space. But by Theorem 4.7 (3), the set of points in βX of βX-rank greater than 1 is contained in $cl_{\beta X} Y$. We now know βX is a compact space of finite rank and the subspace consisting of the points of βX with rank greater than 1 is a P-space. By 2), βX is an SV space. Hence X is an SV space. □

SV and related properties inherited.

The question of when is the SV property or a related property "inherited" by a subspace or an image under a continuous function has been addressed. Note that in a normal F-space, a closed subspace is itself an F-space.

THEOREM 4.9. — *Suppose X is a normal space and Y a closed subspace of X.*

1. *If X is finitely an F-space then Y is also finitely an F-space.*

2. *(2.5, [HW1] and 1.5.1, [HLMW]) If X is an SV space then Y is an SV space.*

3. *(1.8.1, [HLMW]) If X has finite rank, then Y has finite rank and the rank of Y is bounded by the rank of X.*

Proof of (1). — Since X is finitely an F-space, there are finitely many closed F-spaces X_i such that $\beta X = \cup_{i=1}^n X_i$. Since X is normal, Y is C^*-embedded in X and so $\beta Y = cl_{\beta X} Y = \cup_{i=1}^n (X_i \cap cl_{\beta X} Y)$ (see 6.9a, [GJ]). Each $X_i \cap cl_{\beta X} Y$ is an F-space since it is a closed subspace of the normal F-space X_i. This implies βY and Y are finitely an F-space. □

The extent to which the hypothesis of normality is required in the above theorem is not known. However, in [HW2], Henriksen and Wilson present an example of an F-space that under the assumption that the continuum hypothesis holds, has a closed subspace that is not an SV space.

It is well known that if X is an F-space and C a cozeroset of X, then C is itself an F-space. A similar situation holds for the other properties that we have considered.

THEOREM 4.10. — *Let X be a completely regular space and C a cozeroset of X.*

1. (3.2, [L5]) *If X is finitely an F-space, then C is itself finitely an F-space.*

2. (3.3, [L5]) *If X is an SV space, then C is itself an SV space.*

3. *If X is normal and has finite rank, then C also has finite rank.*

Proof of (3). — Suppose the rank of X is n. Since C is a cozeroset, it can be written as the union of countably many closed sets C_i. We show that C is a normal subspace. Let A, B be disjoint C-closed subsets. Then $A = \cup_{i=1}^{\infty}(A \cap C_i)$ and $B = \cup_{i=1}^{\infty}(B \cap C_i)$. Now each $A \cap C_i$ and $B \cap C_i$ is closed in X; hence A, B are F_{σ} sets in X. If $cl_X A \cap B \neq \emptyset$, then since $B \subseteq C$, we have $A \cap B = (cl_X A \cap C) \cap B = cl_X A \cap B \cap C \neq \emptyset$, a contradiction. So $cl_X A \cap B = \emptyset$ and similarly, $A \cap cl_X B = \emptyset$. Thus, A, B are separated sets in X. By Theorem 3 (p. 123) of [K], there are two disjoint open sets that separate A, B. Hence C is a normal subspace. Because C is an X-open set, every $x \in C$ has C-rank at most n. Then by Theorem 4.7(7), βC has rank at most n and by Theorem 4.7(6), C has finite rank. $\square$

As is the case with F-spaces, an open subspace of an SV space does not necessarily inherit the SV property. An example is given in [L5] of a normal F-space in which there is an open subspace whose Stone-Čech compactification does not have finite rank. This serves as an example of a space of finite rank with an open subspace that does not have finite rank, as an example of a space that is finitely an F-space with an open subspace that is not finitely an F-space, and as an example of an SV space with an open subspace that is not SV. We briefly describe this example and the reader is refered to [L5] for the details.

Example 4.11 (3.4, [L5]). — Let L_1 denote the space of all ordinals $\leqslant \omega_1$ under the topology in which neighborhoods of ω_1 are as in the interval topology and all other points are isolated and let L_2 denote the space of all ordinals $\leqslant \omega_2$ under the topology in which neighborhoods of ω_2 are as in the interval topology and all other points are isolated. Let $p \in \beta\mathbf{N}\backslash\mathbf{N}$ and let B denote a copy of the space $\mathbf{N}\cup\{p\}$ under the subspace (of $\beta\mathbf{N}$) topology. Let $X_1 = L_2 \times L_1$ and $X_2 = B \times L_1$. For each $x \in L_1$, identify the two points (ω_2, x) and (p, x), and let X be the resulting topological space. Then X is a normal F-space. Now let $Z = B \times \{\omega_1\}$. Note that Z is closed in X. Now $U = X - Z$ is an open subspace of X. It can be shown that U is non-normal, and not of finite rank.

The image of an F-space (a space that is finitely an F-space, an SV space, a space of finite rank, resp.) under a continuous and open function is not necessarily an F-space (a space that is finitely an F-space, an SV space, a space of finite rank, resp.) as is shown in [L4]. There, an example is given of a normal F-space for which there is an open continuous image that is not of finite rank and hence not an F-space, not an SV space, and not finitely an F-space. However, the SV and related properties are inherited by open images of compact spaces and by the image of a continuous z-open function. A function $f : X \to Y$ is said to be a *z-open function* if for every cozeroset neighborhood H of a zeroset Z in X, the image $f(H)$ is a neighborhood of $cl_Y(f(Z))$ in Y.

THEOREM 4.12 (section 2, [L5]), and (20, [L6]). — *Suppose $f : X \to Y$ is a continuous function mapping X onto Y. If f is open and X is compact, or if f is z-open, then*

1) If X is an F-space, then Y also is an F-space.

2) If X is finitely an F-space, then Y also is finitely an F-space.

3) If X is SV with rank at most 2, then Y also is SV with rank at most 2.

4) If X has finite rank, then Y also has finite rank.

Once more, we see a result about an SV space that may only apply for spaces of rank 2. It is not known if this result holds for spaces of rank n, where n is greater than 2. The proof of this result makes use of Theorem 4.4, which does not give a valid characterization of SV spaces for spaces of rank $n \geqslant 2$. However, we do not know of an example showing that part (3) of the previous result fails to hold for spaces of rank greater than 2.

Finally, we note that there is a borderline of sorts near F-spaces. The F-space property and related, but weaker, properties generally are not preserved by open continuous functions, but are preserved by the stronger continuous z-open functions. On the other hand, properties stronger than the F-space property tend to be preserved by open functions.

5. SV f-Rings and Related f-Rings

We now turn our attention to finitely 1-convex f-rings, SV f-rings, and f-rings of finite rank. These are the f-rings that satisfy an algebraic version of the property that $C(X)$ possesses when X is finitely an F-space, an SV

space, or a space of finite rank respectively. We will see that many, but not all, of the results involving $C(X)$ for X a space that is finitely an F-space, an SV space, or of finite rank have an analogue that holds for f-rings. In particular, we will show f-rings that are finitely 1-convex are analogous to $C(X)$s for X a space that is finitely an F-space and that finitely 1-convex f-rings are SV f-rings and are f-rings of finite rank. However, an SV f-ring is not necessarily of finite rank and an f-ring of finite rank is not necessarily an SV f-ring in contrast to the situation for a $C(X)$, where an SV $C(X)$ must have finite rank and it is not known whether a $C(X)$ of finite rank must be SV. We will be mainly concerned with f-rings that are semiprime, commutative and have bounded inversion.

Finitely 1-convex f-rings.

One might think of a space X that is finitely an F-space as being the union of finitely many F-spaces that have been "glued" together on some closed subspace(s). In a similar way, we define an f-ring to be finitely 1-convex if it is the union of finitely many 1-convex f-rings "glued" together at some semiprime ℓ-ideal. Proofs of the results stated in this subsection will be provided since they do not appear elsewhere in the literature.

Given f-rings A_1, A_2, B and surjective ℓ-homomorphisms $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$, recall that the *fibre product* of A_1 and A_2, denoted $A_1 \times_B A_2$, is the sub-f-ring of $A_1 \times A_2$ given by $A_1 \times_B A_2 = \{(a_1, a_2) : \phi_1(a_1) = \phi_2(a_2)\}$. We say an f-ring is a *finite fibre product of 1-convex f-rings* if it can be constructed in a finite number of steps where every step consists of taking the fibre product of two f-rings, each being either a 1-convex f-ring or a fibre product obtained in an earlier step of the construction.

DEFINITION 5.1. — *An f-ring A is finitely 1-convex if it is either a 1-convex f-ring or can be written as a finite fibre product of 1-convex f-rings.*

Note that for any finitely 1-convex f-ring A, there are a finite number of 1-convex f-rings $A_1, A_2, \ldots, A_n$ such that A is ℓ-isomorphic to a sub-f-ring of $A_1 \times A_2 \times \cdots \times A_n$.

Example 5.2. — Let $\mathbf{R}[x]$ denote the ring of polynomials over the reals in one indeterminate. Totally order $\mathbf{R}[x]$ lexicographically, so that $1 \gg x \gg x^2 \gg \cdots$. Now let $A_1 = \{\frac{p}{q} : p, q \in \mathbf{R}[x], q \geqslant 1\}$ under the usual addition and multiplication of quotients of polynomials and under the order induced by the order on $\mathbf{R}[x]$. That is, $\frac{p_1}{q_1} \leqslant \frac{p_2}{q_2}$ if and only if $p_1 q_2 \leqslant p_2 q_1$. Then A_1 is a totally ordered 1-convex f-ring. Let $A_2 = \{f \in C(\mathbf{N}) : \exists\, n_0 \in \mathbf{N}, r \in \mathbf{R} \text{ such that } f(n) = r \ \forall n \geqslant n_0\}$ under the usual addition, multiplication, and partial order of functions. Then A_2 is also a 1-convex f-ring. Define

$\phi_1 : A_1 \to \mathbf{R}$ by $\phi_1(\frac{p}{q}) = \frac{p(0)}{q(0)}$ and $\phi_2 : A_2 \to \mathbf{R}$ by $\phi_2(f) = r$ where there exists $n_0 \in \mathbf{N}$ such that $f(n) = r$ for all $n \geqslant n_0$. Both ϕ_1, ϕ_2 are surjective ℓ-homomorphisms. Then the f-ring $A_1 \times_{\mathbf{R}} A_2 = \{(\frac{p}{q}, f) \in A_1 \times A_2 : \frac{p(0)}{q(0)} = f(n_0)$, where $f(n) = f(n_0)$ for all $n \geqslant n_0\}$ is finitely 1-convex.

Suppose A' is a 1-convex f-ring and Q a semiprime ℓ-ideal of A'. One type of finitely 1-convex f-ring that is particularly nice to work with can be constructed as the sub-f-ring of $\prod_{i=1}^{n} A'$ given by $A = \{(a_1, a_2, \ldots, a_n) : a_i - a_j \in Q$ for each $i, j\}$. Indeed, the f-ring A could be written as a finite fibre product of n copies of A' in the form $\big([(A' \times_{A'/Q} A') \times_{A'/Q} A'] \cdots \times_{A'/Q} A'\big)$.

Suppose A_1, A_2, B are f-rings and $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ are surjective ℓ-homomorphisms. Let $A = A_1 \times_B A_2$. If we let $\phi_1^* = \phi_1|_{A_1^*}$ and $\phi_2^* = \phi_2|_{A_2^*}$, then ϕ_1^*, ϕ_2^* are surjective ℓ-homomorphisms mapping onto B^* and it is not hard to show that $A^* = A_1^* \times_{B^*} A_2^*$. It is also the case that if A is a 1-convex f-ring then A^* is a 1-convex f-ring. It follows that if A is a finitely 1-convex f-ring, then A^* is also a finitely 1-convex f-ring. We will use this fact when we show that f-rings that are finitely 1-convex are analogous to $C(X)$s where X is finitely an F-space.

THEOREM 5.3. — *For a completely regular space X, if $C(X)$ is finitely 1-convex then X is finitely an F-space. If X is normal, $C(X)$ is finitely 1-convex if and only if X is finitely an F-space.*

Proof. — $\Rightarrow$ Suppose $C(X)$ is finitely 1-convex. We will show X is finitely an F-space. First note that if $C(X)$ is 1-convex, then by Theorem 2.3, X is an F-space, and hence finitely an F-space. We may now assume $C(X)$ is isomorphic to a finite fibre product constructed from n 1-convex f-rings $A_1, A_2, \ldots, A_n$. It follows that $C(X)$ is isomorphic to a sub-f-ring of $\prod_{i=1}^{n} A_i$. Let $\psi : C(X) \to \prod_{i=1}^{n} A_i$ denote the ℓ-embedding. For each i, let π_i denote the projection mapping of $\psi(C(X))$ onto A_i.

Assume first that X is compact. For each $i = 1, 2, \ldots, n$, define $Q_i = \pi_i^{-1}(\{0\})$ and let $Y_i = \cap\{Z(f) : \psi(f) \in Q_i\}$. Since the Y_i are intersections of zerosets, each Y_i is closed in X. We will show $X = \cup_{i=1}^{n} Y_i$. Suppose not; suppose that there is an $x \in X$ such that $x \notin \cup_{i=1}^{n} Y_i$. Then for each i, there is an $f_i \in C(X)$ such that $\psi(f_i) \in Q_i$ and $f_i(x) \neq 0$. Then $f_1 f_2 \cdots f_n(x) \neq 0$ and yet $\psi(f_1 f_2 \cdots f_n) = \psi(f_1)\psi(f_2) \cdots \psi(f_n) \in Q_1 \cdot Q_2 \cdots Q_n = \{0\}$. Since ψ is an ℓ-embedding, this implies that $f_1 f_2 \cdots f_n = 0$, a contradiction. Thus, $X = \cup_{i=1}^{n} Y_i$. Next we will show that Y_1 is an F-space. Suppose $0 \leqslant f \leqslant g$ with $f, g \in C(Y_1)$. Since Y_1 is closed in X, there are extensions $\bar{f}, \bar{g} \in C(X)$ of f, g respectively. We may assume that $0 \leqslant \bar{f} \leqslant \bar{g}$. Suppose $\psi(\bar{f}) = (a_1, a_2, \ldots, a_n)$ and $\psi(\bar{g}) = (b_1, b_2, \ldots, b_n)$. Then $0 \leqslant a_1 \leqslant b_1$

in A_1. Since A_1 is 1-convex, there is a $w_1 \in A_1$ such that $a_1 = w_1 b_1$. Since π_1 is surjective, there is an element $(w_1, w_2, \ldots, w_n) \in \psi(C(X))$ with w_1 being the first coordinate. Now let $\bar{w} \in C(X)$ such that $\psi(\bar{w}) = (w_1, w_2, \ldots, w_n)$. Then $\psi(\bar{f} - \bar{w}\bar{g}) = \psi(\bar{f}) - \psi(\bar{w})\psi(\bar{g}) = (a_1, a_2, \ldots, a_n) - (w_1, w_2, \ldots, w_n)(b_1, b_2, \ldots, b_n) = (0, a_2 - w_2 b_2, a_3 - w_3 b_3, \ldots, a_n - w_n b_n)$. So $\psi(\bar{f} - \bar{w}\bar{g}) \in Q_1$ and hence $Y_1 \subseteq Z(\bar{f} - \bar{w}\bar{g})$. Thus $f - w|_{Y_1} g = 0$ and hence Y_1 is an F-space. Similarly, Y_i is an F-space for all i. We have, $X = \cup_{i=1}^{n} Y_i$, and each Y_i is a closed F-space. Thus X is finitely an F-space.

Now assume that X is not compact and $C(X)$ is finitely 1-convex. The results mentioned just prior to the statement of this theorem imply $C^*(X)$ is finitely 1-convex. Since $C^*(X) \cong C(\beta X)$, then $C(\beta X)$ is finitely 1-convex, and our work in the last paragraph shows that βX is finitely an F-space. This says that X is finitely an F-space.

$\Leftarrow$ Suppose that X is normal and is finitely an F-space. We will show that $C(X)$ is finitely 1-convex. By Theorem 2.3, if X is an F-space, then $C(X)$ is 1-convex and is therefore finitely 1-convex. So now suppose X is finitely an F-space, but not an F-space. Then $\beta X = \cup_{i=1}^{n} Y_i$, where the Y_i are compact F-spaces. First, note that we may assume that for each $j \neq k$, $Y_j \cap Y_k \neq \emptyset$, since if for some $j \neq k$, $Y_j \cap Y_k = \emptyset$, then we could let $Y_j' = Y_j \cup Y_k$ and then write βX as the union of fewer F-spaces: $\beta X = Y_j' \cup (\cup_{i \neq j,k} Y_i)$.

For each i, let $A_i = C(Y_i \cap X)$. First we show that each A_i is 1-convex. So suppose $0 \leqslant u \leqslant v$ in A_i. Since $Y_i \cap X$ is closed in X and X is normal, $Y_i \cap X$ is C^*-embedded in X. So, there exists a $u', v' \in C(X)$ such that $u'|_{Y_i \cap X} = u$ and $v'|_{Y_i \cap X} = v$. We may assume that $0 \leqslant u' \leqslant v'$ in $C(X)$. Then $0 \leqslant u' \wedge 1 \leqslant v' \wedge 1$ and there must exist $u'', v'' \in C(\beta X)$ such that $u''|_X = u' \wedge 1$ and $v''|_X = v' \wedge 1$. Then since $0 \leqslant u''|_{Y_i} \leqslant v''|_{Y_i}$ in Y_i and since $C(Y_i)$ is 1-convex, there exists $w'' \in C(Y_i)$ such that $u''|_{Y_i} = w'' v''_{Y_i}$. We may assume $w'' \leqslant 1$ in $C(Y_i)$. Let $w' \in C(\beta X)$ such that $w'|_{Y_i} = w''$. Since $(v' \vee 1)^{-1} \in C(X)$ and $(v' \vee 1)^{-1}|_{Y_i \cap X} = (v \vee 1)^{-1}$, we have $(v \vee 1)^{-1} \in A_i$. Then

$$
\begin{aligned}
u = u'|_{Y_i \cap X} &= (u' \wedge 1)|_{Y_i \cap X}(u' \vee 1)|_{Y_i \cap X} \\
&= u''|_{Y_i \cap X}(u' \vee 1)|_{Y_i \cap X} \\
&= w''|_{Y_i \cap X} v''|_{Y_i \cap X}(u' \vee 1)|_{Y_i \cap X} \\
&= w''|_{Y_i \cap X}(v \wedge 1)(u' \vee 1)|_{Y_i \cap X} \\
&= w''|_{Y_i \cap X}(v \wedge 1)(v \vee 1)(v \vee 1)^{-1}(u' \vee 1)|_{Y_i \cap X} \\
&= w''|_{Y_i \cap X} v(v \vee 1)^{-1}(u' \vee 1)|_{Y_i \cap X} \\
&= [w''|_{Y_i \cap X}(v \vee 1)^{-1}(u' \vee 1)|_{Y_i \cap X}]v.
\end{aligned}
$$

Thus A_i is 1-convex for each i.

If Y is a normal space and $Y = Y' \cup Y''$ where Y', Y'' are closed sets, then by defining $\phi' : C(Y') \to C(Y' \cap Y'')$ by $\phi'(f) = f|_{Y' \cap Y''}$ and defining $\phi'' : C(Y'') \to C(Y' \cap Y'')$ by $\phi''(f) = f|_{Y' \cap Y''}$, it is not difficult to show that $C(Y) = C(Y') \times_{C(Y' \cap Y'')} C(Y'')$. Using the fact that $X = \cup_{i=1}^{n}(Y_i \cap X)$ and for each i, $Y_i \cap X$ is closed and C^*-embedded in X, a straightforward induction argument will show that $C(X)$ is a finite fibre product constructed from the 1-convex f-rings $A_i = C(Y_i \cap X)$. $\square$

LEMMA 5.4. — *Suppose A is a commutative semiprime finitely 1-convex f-ring with identity element. Suppose A is isomorphic to a finite fibre product constructed from the 1-convex f-rings $A_1, A_2, \ldots, A_n$ and $\psi : A \to A_1 \times A_2 \times \cdots \times A_n$ is an ℓ-embedding. Let $\pi_j : \psi(A) \to A_j$ denote the projection mapping. Define $Q_j = \pi_j^{-1}(\{0\})$. Then Q_j is a semiprime ℓ-ideal of $\psi(A)$ and $\psi(A)/Q_j \cong A_j$ and hence $\psi(A)/Q_j$ is 1-convex.*

Proof. — Without loss of generality, we may assume that $j = 1$. It is easy to see that Q_1 is a semiprime ℓ-ideal. Define $\phi : \psi(A)/Q_1 \to A_1$ by $\phi((a_1, a_2, \ldots, a_n) + Q_1) = a_1$. If $\phi((a_1, a_2, \ldots, a_n) + Q_1) = \phi((b_1, b_2, \ldots, b_n) + Q_1)$ then $a_1 = b_1$ and so $(a_1, a_2, \ldots, a_n) - (b_1, b_2, \ldots, b_n) = (0, a_2 - b_2, a_3 - b_3, \ldots, a_n - b_n) \in Q_1$. So $(a_1, a_2, \ldots, a_n) + Q_1 = (b_1, b_2, \ldots, b_n) + Q_1$, and we can conclude that ϕ is injective. Also, ϕ is surjective since the projection from $\psi(A)$ to A_1 is surjective. It is now easy to show that ψ is an ℓ-isomorphism. $\square$

An algebraic analogue of the topological statement that a space that is finitely an F-space is also an SV space follows.

THEOREM 5.5. — *Suppose A is a commutative semiprime f-ring with identity element. If A is finitely 1-convex then A is an SV f-ring.*

Proof. — Suppose A is finitely 1-convex. Suppose A is isomorphic to a fibre product constructed from the 1-convex f-rings $A_1, A_2, \ldots, A_n$ and $\psi : A \to A_1 \times A_2 \times \cdots \times A_n$ is an ℓ-embedding. Let $\pi_j : \psi(A) \to A_j$ denote the projection mapping and define $Q_j = \pi_j^{-1}(\{0\})$. Then each Q_j is a semiprime ℓ-ideal of $\psi(A)$. Let P be a minimal prime ideal of $\psi(A)$. Note that $Q_1 \cdot Q_2 \cdots Q_n = \{0\} \subseteq P$. Since P is prime, there is a j such that $Q_j \subseteq P$. Now $\psi(A)/P = (\psi(A)/Q_j)/(P/Q_j)$ and by the previous lemma, $\psi(A)/Q_j$ is 1-convex. So $\psi(A)/P$ is the ℓ-homomorphic image of the 1-convex f-ring $\psi(A)/Q_j$. Thus $\psi(A)/P$ is 1-convex and also is a totally ordered domain with identity element and hence it is a valuation domain by Lemma 2.2. $\square$

The reader will recall that an F-space has rank 1. The corresponding algebraic statement for commutative semiprime f-rings with identity element follows.

THEOREM 5.6. — *Suppose A is a commutative semiprime 1-convex f-ring with identity element. Then A has rank 1.*

Proof. — Suppose M is a maximal ℓ-ideal of A and P_1, P_2 are two distinct minimal prime ℓ-ideals contained in M. Then there are elements p_1, p_2 such that $p_1 \in P_1^+\backslash P_2$, $p_2 \in P_2^+\backslash P_1$. Then $0 \leqslant p_1, p_2 \leqslant p_1 + p_2$ and since A is 1-convex, there are $w_1, w_2 \in A$ such that $p_1 = w_1(p_1 + p_2)$ and $p_2 = w_2(p_1+p_2)$. Since P_1, P_2 are prime and $p_1+p_2 \notin P_1 \cup P_2$, we have $w_1 \in P_1$ and $w_2 \in P_2$. Now $p_1 + p_2 = (w_1 + w_2)(p_1 + p_2)$. Then $(1 - (w_1+w_2))(p_1+p_2) = 0 \in P_1$ while $p_1 + p_2 \notin P_1$ implies $1 - (w_1 + w_2) \in P_1 \subseteq P_1 + P_2$. Because $w_1 + w_2 \in P_1 + P_2$, we have $1 \in P_1 + P_2 \subseteq M$, a contradiction. Hence there cannot be two distinct minimal prime ℓ-ideals contained in M. $\qquad\square$

Next we give an algebraic statement for commutative semiprime f-rings with identity element that corresponds to the topological statement that a space that is finitely an F-space has finite rank.

THEOREM 5.7. — *Suppose A is a commutative semiprime finitely 1-convex f-ring with identity element. Then A has finite rank.*

Proof. — Suppose A is a commutative semiprime finitely 1-convex f-ring with identity element. Suppose A is isomorphic to a fibre product constructed from the 1-convex f-rings $A_1, A_2, \ldots, A_n$ and $\psi : A \to A_1 \times A_2 \times \cdots \times A_n$ is an ℓ-embedding. Let $\pi_j : \psi(A) \to A_j$ denote the projection mapping and define $Q_j = \pi_j^{-1}(\{0\})$. Then each Q_j is a semiprime ℓ-ideal of $\psi(A)$. Let M be a maximal ℓ-ideal of $\psi(A)$. We will show there is at most 1 minimal prime ideal contained in M that contains Q_1. Suppose not; suppose there are two minimal prime ideals P_1, P_2 contained in M, each containing Q_1. By Lemma 5.4 and Theorem 5.6, $\psi(A)/Q_1$ is 1-convex and the maximal ℓ-ideal M/Q_1 of $\psi(A)/Q_1$ contains just one minimal prime ideal. Suppose P'/Q_1 is the single minimal prime ideal contained in M/Q_1 in $\psi(A)/Q_1$. Then in $\psi(A)/Q_1$, $P'/Q_1 \subseteq P_1/Q_1$ and $P'/Q_1 \subseteq P_2/Q_1$. Since the prime ideals containing P'/Q_1 are totally ordered (see 14.3(c), [GJ]), $P_1/Q_1 \subseteq P_2/Q_1$ or $P_2/Q_1 \subseteq P_1/Q_1$ in $\psi(A)/Q_1$. Since $Q_1 \subseteq P_1, P_2$, this implies $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ in $\psi(A)$, which is a contradiction to P_1, P_2 being distinct minimal prime ideals. So there is at most 1 minimal prime ideal contained in M that contains Q_1 and similarly for each $j = 2, 3, \ldots, n$, there is at most 1 minimal prime ideal contained in M that contains Q_j.

Now if P is a minimal prime ideal contained in M, then because $Q_1 \cdot Q_2 \cdots Q_n = \{0\} \subseteq P$ and P is prime, there is a j such that $Q_j \subseteq P$. So every minimal prime ideal contained in M contains Q_j for some j, and it follows that there can be at most n minimal prime ideals contained in M. $\square$

SV f-rings.

We begin our look at SV f-rings with a lemma that provides an often used algebraic test for an f-ring being an SV f-ring. If P is a (proper) prime ideal of an f-ring A with identity element and bounded inversion, then A/P is a totally ordered domain with identity element and bounded inversion and so the next lemma follows from Lemma 2.2.

LEMMA 5.8. — *Suppose A is a commutative f-ring with identity element and bounded inversion. Then A is an SV f-ring if and only if for every minimal prime ideal P of A, A/P is 1-convex.*

In [HL], SV f-rings are studied. Parts of [HL] do not assume that an f-ring is commutative and semiprime and the paper contains some errors relating to those cases. Here, we provide a corrected statement of a theorem and its proof. Recall that an ideal I is *pseudoprime* if $ab = 0$ implies $a \in I$ or $b \in I$ and an ideal I is *primal* if it contains a prime ideal.

THEOREM 5.9. — *Suppose A is a commutative f-ring with identity element. Consider the following properties of A.*

1. *Every pseudoprime ideal of A is an ℓ-ideal*

2. *Every primal ideal of A is an ℓ-ideal*

3. *A is an SV f-ring.*

Then (1) implies (2) implies (3). If A is semiprime, (1) and (2) are equivalent. and if A has bounded inversion then (2) and (3) are equivalent. If A is semiprime and has bounded inversion, all three properties are equivalent.

Proof. — $(1) \Rightarrow (2)$: This follows from the fact that every primal ideal is pseudoprime.

$(2) \Rightarrow (3)$: Suppose P is a minimal prime ideal of A and let $a+P, b+P \in A/P$ such that $0 \leqslant a + P \leqslant b + P$. Then $P + (b)$ is a primal ideal and by hypothesis, it is an ℓ-ideal. Then $a \in P + (b)$ which implies $a - rb \in P$ for some $r \in A$. So $a + P = (r + P)(b + P)$ in A/P. Since A/P is a totally ordered domain, Lemma 2.2 implies A is an SV f-ring.

$(2) \Rightarrow (1)$ when A is semiprime: In this case, every pseudoprime ideal is primal.

$(3) \Rightarrow (2)$ when A has bounded inversion: Let Q denote a primal ideal. We first show Q is convex; suppose $0 \leqslant a \leqslant b$ with $b \in Q$. Now Q contains a prime ideal P and by hypothesis, A/P is a valuation domain. By the previous lemma, A/P is 1-convex and hence there is a $w \in A$ such that $a - wb \in P \subseteq Q$. Since $b \in Q$, we have $a \in Q$ and Q is convex. Now for any element $a \in A$, $(a - |a|)(a + |a|) = 0$ and since Q contains a prime ideal, it follows that $a \in Q$ if and only if $|a| \in Q$. $\qquad\square$

In [S2], Schwartz looks at real closed rings and rings of continuous functions in the context of real closed rings. As is pointed out in this paper, elementary algebraic manipulations of a ring of continuous function (such as taking a quotient ring) often lead outside the realm of rings of continuous functions, and, it is desirable to look at a category of rings that contains rings of continuous functions and the rings obtainable from rings of continuous functions by basic algebraic operations. This paper makes the case for considering rings of continuous functions within the category of real closed rings. See [S1] for further background and definitions. In section 7 of [S2], SV f-rings are considered within the context of real closed rings and the following result gives a characterization of local SV f-rings in that context.

THEOREM 5.10 (7.7, [S2]). — *Let A be a local real closed ring, and $1 \leqslant k \in \mathbf{N}$. Then the following statements are equivalent.*

1. *A is an SV f-ring having rank k.*

2. *Every finitely generated ideal of A can be generated by k elements, and there is one ideal which cannot be generated by fewer elements.*

Corollaries follow that give sufficient conditions for a real closed ring to be an SV f-ring.

COROLLARY 5.11 (7.8, 7.9 [S2]). — *Let A be a real closed ring.*

1. *If every finitely generated ℓ-ideal of A is finitely generated as an ideal, then A is an SV f-ring.*

2. *If every finitely generated ideal of A can be generated by k elements $(k \geqslant 1)$, then A is an SV f-ring and every maximal ideal of A has rank at most k.*

f-Rings of finite rank.

In this subsection, we give characterizations of commutative semiprime f-rings with identity element in which every maximal ℓ-ideal has finite rank.

Three examples will then be given showing that (i) an SV f-ring does not necessarily have finite rank, (ii) an f-ring of finite rank is not necessarily an SV f-ring, and (iii) it is possible for commutative semiprime f-rings with identity element to have the property that every maximal ℓ-ideal has finite rank while the f-ring does not have finite rank. All three of these examples reflect a difference in the situation for f-rings and for $C(X)$s.

THEOREM 5.12 (2.2, [L2]). — *Let A be a commutative semiprime f-ring with identity element. The following are equivalent.*

1. *Every maximal ℓ-ideal of A has finite rank.*

2. *For any maximal ℓ-ideal M, O_M is the intersection of a finite number of minimal prime ℓ-ideals.*

3. *Every ℓ-ideal I containing O_M for some maximal ℓ-ideal M is the intersection of a finite number of pseudoprime ℓ-ideals.*

4. *Every semiprime ℓ-ideal I containing O_M for some maximal ℓ-ideal M is the intersection of a finite number of prime ℓ-ideals.*

5. *Given any collection $\{P_i\}_{i=1}^{\infty}$ of distinct minimal prime ℓ-ideals of A, there exists an n such that $\sum_{i=1}^{n} P_i = A$.*

6. *For every collection $\{P_\alpha\}$ of minimal prime ℓ-ideals contained in a given maximal ℓ-ideal M and ideal I, $I \subseteq \cup P_\alpha$ implies $I \subseteq P_\alpha$ for some α.*

In contrast to the case for a $C(X)$, an SV f-ring is not necessarily an f-ring of finite rank as the following example shows.

Example 5.13 (3.7, [HLMW]). — Let $\mathbf{R}[[x]]$ denote the ring of formal power series over the reals, in one indeterminate. Totally order $\mathbf{R}[[x]]$ lexicographically, so that $1 \gg x \gg x^2 \gg \cdots$. Let $\mathbf{R}_0[[x]]$ denote the collection of series with 0 constant term. For each $n \in \mathbf{N}$ let A_n denote a copy of $\mathbf{R}_0[[x]]$ and let A be the direct sum of the A_n with a coordinatewise ordering. Now let $B = \{(a, r) : a \in A, r \in \mathbf{R}\}$, with coordinatewise addition, and multiplication and partial ordering defined as follows:

$$(a, r)(b, s) = (ab + rb + as, rs)$$

and

$$(a, r) > 0 \text{ if } r > 0, \text{ or } r = 0 \text{ and } a > 0.$$

It can be shown that the unique maximal ℓ-ideal of B is $M = \{(a, 0) \in B : a \in A\}$, that the minimal prime ideals of B are $P_n = \{(a, 0) \in B : a_n = 0\}$

for each $n \in \mathbf{N}$, and that $B/P_n \cong \mathbf{R}[[x]]$ for each n. Then B is an SV algebra with bounded inversion and infinite rank.

By part 1) of Theorem 4.7, we know that in $C(X)$, if every maximal ℓ-ideal has finite rank, then there is a positive integer that is an upper bound to the collection of ranks of the maximal ℓ-ideals. The following example shows that this does not hold in general in f-rings.

Example 5.14 (2.3, [L2]). — Let $\alpha \in \beta\mathbf{N}\backslash\mathbf{N}$. For each $n \in \mathbf{N}$ and $i \leqslant n$, let $X_{n,i} = \mathbf{N}\cup\{\alpha\}$ under the relative topology. For each n, let X_n denote the space obtained from $\cup_{i=1}^n X_{n,i}$ by identifying the copies of α. We let α_n denote the identified point in X_n. Let X be the disjoint union of the X_n. Now let $A = \{f \in C(X) : f|_{X_{n,i}}$ is eventually constant for all but finitely many n, i, and there is a $k \in \mathbf{R}$ such that $f(\alpha_i) = k$ for all but finitely many $i\}$. Then A is an f-algebra with identity element. In A, let $M = \{f \in A : f(\alpha_i) = 0$ for all but finitely many $i\}$. For each n, let $M_n = \{f \in A : f(\alpha_n) = 0\}$. Then every maximal ℓ-ideal of A is one of M, M_n, or is simultaneously a maximal ℓ-ideal and a minimal prime ℓ-ideal. The maximal ℓ-ideal M has rank 1, while M_n has rank n for each n. It follows that every maximal ℓ-ideal of A has finite rank, and yet there obviously is no upper bound for the ranks of the maximal ℓ-ideals.

The ring $\mathbf{Z}$ of integers is an f-ring of finite rank that is not an SV f-ring. The next example shows that even an f-ring of finite rank with bounded inversion need not be an SV f-ring.

Example 5.15 (3.6, [HLMW]). — Let $X = \{(0,0)\} \cup \{(1/n, 1/n^n) : n \in \mathbf{N}\}$ with the subspace topology inherited from $\mathbf{R}^2$. Define A to be the sub-f-ring of $C(X)$ where $A = \{f \in C(X) : \exists\, n_0 \in \mathbf{N}, \, p, q \in \mathbf{R}[x,y],$ with $q(0,0) \neq 0$ such that $f((1/n, 1/n^n)) = p((1/n, 1/n^n))/q((1/n, 1/n^n))\,\forall\, n \geqslant n_0\}$. In A, the ℓ-ideal $Q = \{f \in A : \exists\, n_0 \in \mathbf{N}$ such that $f((1/n, 1/n^m)) = 0$ $\forall\, n \geqslant n_0\}$ is the only minimal prime ℓ-ideal of A which is not also maximal. It follows that A has rank 1. However, A is not an SV f-ring since if f, g are defined by $f(a,b) = b$ and $g(a,b) = a$ for all $(a,b) \in X$, then $0 < f + Q < g + Q$ in A/Q, and yet there is no $w + Q \in A/Q$ such that $f + Q = (w + Q)(g + Q)$.

6. Open Problems

The principal classes of spaces whose relationship to SV spaces that have been studied are F-spaces, spaces that are finitely an F-space, and spaces of finite rank. The relationship between these types of spaces can be summarized by

F-space $\Rightarrow$ Finitely an F-space $\Rightarrow$ SV space $\Rightarrow$ Space of finite rank,

where neither of the first two arrows can be reversed. Whether the third arrow can be reversed is the first in our list of open problems. This problem was posed in [HW2]. Recall that there is a known f-ring of finite rank that is not an SV f-ring, but it is not a $C(X)$.

PROBLEM 1. — *If X is a space of finite rank, must X be an SV space?*

A compact space that is finitely an F-space has an open and dense subset of points of rank 1 by Theorem 4.1. Every known SV space contains points of rank 1, but is this necessary? The same can be asked about spaces of finite rank.

PROBLEM 2. — *If X is a (compact) SV space, must it contain a point of rank 1? Must an SV space contain a dense set of points of rank 1?*

A *quasi-F space* is a space in which every dense cozeroset is C^*-embedded. A continuous surjection $f : X \to Y$ is *irreducible* if no proper closed subset of X is mapped by f onto Y. If X is compact, there is an essentially unique quasi-F space $QF(X)$, called the *quasi-F cover* of X, that maps irreducibly onto X and any continuous surjection of a compact quasi-F space factors through $QF(X)$. See [DHH] for details. It is shown in 5.1 of [HLMW] that if a compact space has finite rank, then so does its quasi-F cover. Does the corresponding result hold for SV spaces?

PROBLEM 3. — *If X is a compact SV space, must the quasi-F cover of X be an SV space?*

The quasi-F cover of a compact space X can be realized as an inverse limit space of a collection of spaces, each of which is the Stone-Čech compactification of an intersection of countably many X-cozerosets, as shown in [DHH]. In [L3] it is shown that the inverse limit space of a countably directed collection of SV spaces is itself an SV space. It follows that, to show that the quasi-F cover of a compact SV space X is SV, it would suffice to show that the intersection of countably many X-cozerosets in an SV space is itself an SV space. Of course it may be that the intersection of countably many X-cozerosets in an SV space is not necessarily an SV space.

We know that for a normal space, closed subspaces inherit the three properties we have studied, but do not know if these properties are inherited by closed subspaces in non-normal spaces. In [HW2], an example is given showing that under the assumption that the continuum hypothesis holds, a closed subspace of an SV space is not necessarily an SV space. Is there such an example that does not require assuming the continuum hypothesis?

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PROBLEM 4. — *Is a closed subspace of a space that is finitely an F-space, (SV space, space of finite rank, respectively) finitely an F-space (SV space, space of finite rank, respectively)?*

The rank of a normal space provides an upper bound for the rank for its closed subspaces. Our last open problem asks if this is true for non-normal spaces.

PROBLEM 5. — *Is there a closed subspace of a (non-normal) space of finite rank whose rank is greater than the rank of the space?*

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