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SV-Rings and SV-Porings

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Abstract. — SV-rings are commutative rings whose factor rings modulo prime ideals are valuation rings. SV-rings occur most naturally in connection with partially ordered rings (= *porings*) and have been studied only in this context so far. The present note first develops the theory of SV-rings systematically, without assuming the presence of a partial order. Particular attention is paid to the question of axiomatizability (in the sense of model theory). Partially ordered SV-rings (*SV-porings*) are introduced, and some elementary properties are exhibited. Finally, SV-rings are used to study convex subrings and convex extensions of porings, in particular of real closed rings.

Résumé. — Les SV-anneaux sont les anneaux commutatifs dont les quotients modulo leurs idéaux premiers sont des anneaux de valuation. Les SV-anneaux apparaissent de la façon la plus naturelle en connection avec les anneaux partiellement ordonnés (= porings); ils ont été étudiés uniquement dans ce contexte so jusqu'à présent. Dans présent article, pour la première fois nous developpons la théorie des SV-anneaux d'une manière systématique, sans supposer la présence d'un ordre partiel. Une attention particulière est consacrée à la question d'axiomatisabilit́(au sense de la théorie des modèles). Nous introduisons les SV-anneaux partiellement ordonnés (SV-porings) et nous démontrons quelques propriétés élémentaires de ces anneaux. Finalement, SV-anneaux sont utilisés pour étudier les sous-anneaux convexes et les extensions convexes des anneaux partiellement ordonnés et, en particulier, des anneaux réels clos.

An *SV-ring* is a commutative ring whose factor rings modulo prime ideals are always valuation rings. Originally the notion was introduced and studied in connection with rings of continuous functions and f-rings (*cf.* [15], [16], [13], [14]). In the introduction of [14] the authors noted that there is no reason why the study of these rings should be restricted a priori to

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partially ordered rings (p. 195). The present note starts with the study of SV-rings without partial orders. The second part of the paper deals with SV-rings that are also porings.

The prime spectrum of an SV-ring is *completely normal*, which means that the specializations of a prime ideal always form a chain with respect to inclusion (or, specialization). Thus, SV-rings only occur in situations where there are "many" minimal prime ideals. This means: If there is any set $M \subseteq \mathrm{Spec}(A)$ of mutually incomparable prime ideals then there are at least as many distinct minimal prime ideals. Most rings in classical number theory or algebraic geometry, e.g., Noetherian rings, do not have this property. On the other hand, such a property is not uncommon in real algebra, e.g., in rings of continuous functions, real closed rings and f-rings. From this perspective it seems rather natural that SV-rings have been studied exclusively in real algebra so far.

Valuation rings have a long and distinguished history; there are a large number of standard texts about the subject, [7] being the most recent one (where further references can be looked up). Cherlin and Dickmann were first to ask for factor domains of rings of continuous functions that are valuation rings ([4], [5]). Most factor domains of rings of continuous functions are not valuation rings; if they are valuation rings then they are convex subrings of real closed fields. Cherlin and Dickmann called such rings *real closed*; here they will be called *real closed valuation rings*. [35] is a study of real closed valuation rings vs. the larger class of real closed domains.

Henriksen and Wilson continued this line of investigation by asking for topological spaces for which *every* factor domain of the ring of continuous functions is a valuation ring ([15], [16]). Such spaces are called *SV-spaces*; their rings of continuous functions are called *SV-rings*. One class of SV-spaces has been known for a long time: F-spaces ([8], 14.25). These spaces arise naturally in connection with the Stone-Cech compactification. Every zero set of βX that does not meet X is an F-space ([8], 140). There exist other SV-spaces ([16]), but so far they seem to be artificial constructs and it is not clear whether they arise anywhere in a natural way. Later the notion of SV-rings was extended to f-rings ([13]). More studies of SV-rings may be found in [14], [24], [25] and [32].

A completely different class of SV-rings arises in semi-algebraic geometry: The ring of continuous semi-algebraic functions on a real algebraic curve is an SV-ring ([35], Corollary 2.6). This is a significant fact since the Curve Selection Lemma ([6], §12) can be used to reduce many questions about semi-algebraic functions to questions about functions defined on curves. There is a potential for applications of SV-rings in semi-algebraic geometry.

The present note starts with a definition and various examples of SV-rings (without partial orders). Basic properties of SV-rings are explored in section 1 and section 3. Section 3 is mostly devoted to the question whether the class of all SV-rings, or some significant subclasses, is elementary (or axiomatizable) with respect to the language of ring theory (*cf.* [3], p. 173, or [18], p. 34). Reduced SV-rings are characterized by the fact that every pair of elements satisfies some polynomial identity (Theorem 3.4). The identity, in particular its degree, depends on the pair of elements. For axiomatizability of the class of SV-rings one would need a uniform bound on the degree of the identities. The degree of the polynomial identities is closely connected with the *rank* of the ring. The rank is defined to be the supremum of the numbers of minimal prime ideals that are contained in a single maximal ideal. It has been shown in [14] that a ring of continuous functions has finite rank if it is an SV-ring. Section 2 of the present paper studies rings with bounded rank. In particular, it is shown that the class of reduced rings with rank at most k is elementary (Corollary 2.3). In section 3 it is shown that there is no hope for axiomatizability of any reasonable class of SV-rings without bounding the rank of the rings in the class (Corollary 3.7). On the other hand, the class of reduced local SV-rings with finite rank at most k and with infinite residue field is elementary (Corollary 3.9). It remains open whether this result can be extended to rings that are not necessarily local. Partial results are: The class of reduced SV-rings with rank 1 is elementary (Corollary 3.12); the class of real closed rings (*cf.* [29], [30] or [36]) with rank at most k is elementary (Corollary 3.16).

The notion of *SV-porings* is introduced in section 4 (Definition 4.1). These are porings that *have compatible spectra* ([34], Definition 3.1) and are also SV-rings. The condition of compatibility establishes a connection between the partial order and the valuations of the residue domains of the SV-ring. Without such a condition the valuations and the partial order can be completely unrelated. The integers provide a simple example: Any localization of the ring of integers at a prime number is a valuation ring and is totally ordered; but the valuation and the total order do not have anything to do with each other. If the poring is an f-ring then compatibility of the spectra is equivalent to bounded inversion ([34], Proposition 3.5). (Bounded inversion means: If $1 \leqslant a$ then $a \in A^{\times}$, *cf.* [11], where this notion was originally introduced, and [22], Definition 7.1.) It is shown that a reduced poring is an SV-poring if and only if its f-ring reflection ([36], Proposition 6.5) is an SV-f-ring (Corollary 4.6). If this is the case then its real closure ([36], §12) is an SV-ring as well (Proposition 4.8).

Section 5 discusses SV-porings in connection with convex subrings of porings. If (A, A^+) is an SV-poring, if (B, B^+) is a proper convex subring and if both porings have compatible spectra then, for each element $s \in B \cap A^{\times}$, the ring $B/(s)$ is an SV-ring of rank 1 (Proposition 5.4). It follows that the prime spectrum of $B/(s)$ is a disjoint union of specialization chains. The main question studied in section 5 is whether the presence of such factor rings is also a sufficient condition for the existence of a proper convex extension of a given poring. (A *convex extension* is a poring that contains a given poring as a convex subring.) For rings of continuous functions the answer is known to be affirmative: If $C(X)$ has a factor ring $C(X)/(a)$ whose prime spectrum is a disjoint union of specialization chains then there is a proper convex extension of $C(X)$. Example 5.5 shows that the same result is not true for real closed rings. For SV-porings of rank 1 there is a complete answer: There exists a proper convex extension if and only if there is a non-zero divisor that is not a unit (Proposition 5.6). Using this result one finds a sufficient condition also for real closed rings. The real closed ring A has a proper convex extension if there is a non-zero divisor s such that $A/\sqrt{(s)}$ is an SV-poring of rank 1 and has a proper convex extension (Corollary 5.10).

Notation and terminology. — All rings in this note are commutative and have a unit element. The group of units of the ring A is denoted by $A^{\times}$. The set of ideals is $\mathrm{Id}(A)$. If $I \in \mathrm{Id}(A)$ is an ideal then $\pi_I : A \to A/I$ is the canonical map. Ring constructions that occur frequently in this note are: the formation of the quotient ring A_S of A with respect to a multiplicative set S; the formation of the quotient field $\mathrm{qf}(A)$ if A is a domain; the formation of the factor ring $A_{\mathrm{red}} = A/\mathrm{Nil}(A)$.

The prime spectrum of A is denoted by $\mathrm{Spec}(A)$, the set of minimal prime ideals by $\mathrm{Min}(A)$, the set of maximal prime ideals by Max(A). If $a \in A$ then $D(a) = \{p \in \mathrm{Spec}(A) | a \notin p\}$. The sets $D(a)$ are the basis of the Zariski topology. Note that $D(a_1 \cdot \ldots \cdot a_k) = \bigcap_{\kappa=1}^{k} D(a_\kappa)$. The sets $V(S) = \{p \in \mathrm{Spec}(A) | S \subseteq p\}$ are the closed sets of the Zariski topology, where S varies in the set of subsets of A. In addition to the Zariski topology, the set $\mathrm{Spec}(A)$ carries another topology that is important in this note. The topology is generated by all the sets $D(a)$ and $V(a)$, where a varies in the ring. This topology is called the *constructible topology*, or the *patch topology*. The constructible topology makes $\mathrm{Spec}(A)$ a Boolean space. The closed sets with respect to the constructible topology are called *proconstructible sets*. Given $p, q \in \mathrm{Spec}(A)$ we say that q is *a specialization* of p, and p is a *generalization* of q, if $q \in \overline{\{p\}}$. A set is *closed under specialization* if it contains all specializations of all its elements. It is *generically closed*, or *closed under generalization*, if it contains all generalizations of all its elements. The set of generalizations of a subset $M \subseteq \mathrm{Spec}(A)$ is denoted by $\mathrm{Gen}(M)$. If M is proconstructible, then so is $\mathrm{Gen}(M)$. Many other facts about the prime spectrum of a ring can be found, e.g., in [17], or in [37].

A ring A together with a subset $A^+ \subseteq A$ is *partially ordered* if $A^+ + A^+ \subseteq A^+$, $A^+ \cdot A^+ \subseteq A^+$, $A^2 \subseteq A^+$, $A^+ \cap -A^+ = \{0\}$. A ring with a partial order is a *partially ordered ring* (= *poring*). Basic facts about porings, in particular about convex ideals, factor porings and quotient porings, can be found in [1], [22] and [36]. A condition about porings that is particularly important in this note is *bounded inversion*: The poring (A, A^+) has *bounded inversion* if $1 + a \in A^\times$ for all $a \in A^+$. It is equivalent to say that all maximal ideals are convex ([22], section 7). The *real spectrum* of a poring (A, A^+) is denoted $\mathrm{Sper}(A, A^+)$. A description of its points and of its topology, as well as a summary of its most important properties can be found in [36], section 4.

Rings of continuous functions with values in $\mathbb{R}$, as well as rings of continuous semi-algebraic functions defined over some real closed field are examples of *real closed rings*. The notion of real closed rings and many important properties can be found in [27], [30] and [36]. The class of real closed rings is axiomatizable in first order model theory, *cf.* [27].

1. SV-rings

The first definition of SV-rings was given in [15], where rings of continuous functions are studied. Some examples were also included. The definition was extended to f-rings in [13], and in the introduction of [14] the authors remark that the notion can be defined without reference to partial orders. This section contains the definition, some examples and basic properties of SV-rings *without a partial order.*

DEFINITION 1.1. — *A ring A is an SV-ring if all factor rings A/p, $p \in$* $\mathrm{Spec}(A)$*, are valuation rings.*

To check whether a given ring is an SV-ring it is clearly sufficient to consider factor rings modulo minimal prime ideals only.

Here are some examples:

Example 1.2. — *SV-rings and constructions that produce SV-rings*

(a) Valuation rings are SV-rings.

(b) Factor rings of SV-rings are SV-rings.

(c) A ring is an SV-ring if and only if its reduction modulo the nilradical is an SV-ring.

(d) A finite direct product of rings is an SV-ring if and only if each factor is an SV-ring. This equivalence is not true for infinite direct products, *cf.* Proposition 3.8.

(e) Every direct product of valuation rings is an SV-ring: If $A = \prod_{i \in I} A_i$ is a product of valuation rings and if $p \subseteq A$ is a minimal prime ideal then there is an ultrafilter $\mathcal{U}$ on the set I such that A/p is the ultraproduct $A/\mathcal{U}$. The class of valuation rings is elementary; hence an ultraproduct of valuation rings also belongs to the class (*cf.* [3], Theorem 4.1.12; [18], Corollary 9.5.10). Thus A/p is a valuation ring.

(f) Suppose that A and B are SV-rings and that $\varphi : A \to C$ and $\psi : B \to C$ are surjective homomorphisms. Then the fibre product $A \times_C B$ is an SV-ring as well: Set $I = \ker \varphi$ and $J = \ker \psi$. Then

$$A \times_C B = \{(a, b) \in A \times B | \varphi(a) = \psi(b)\},$$

and $I \times \{0\}, \{0\} \times J$, $I \times J \subseteq A \times_C B$ are ideals with factor rings $A \times_C B/I \times \{0\} = B$, $A \times_C B/\{0\} \times J = A$ and $A \times_C B/I \times J = C$. If $p \subseteq A \times_C B$ then $I \times \{0\} \subseteq p$ or $\{0\} \times J \subseteq p$, and $A \times_C B/p$ is a factor ring of A or of B, hence is a valuation ring.

(g) Direct limits of SV-rings are SV-rings: Let $I = (I, \leqslant)$ be a directed partially ordered set, and let $(A_i)_{i \in I}$ be a diagram of SV-rings over I with transition maps $f_{ji} : A_i \to A_j$ for $i \leqslant j$. Let $A = \lim_{\to} A_i$ with canonical maps $f_i : A_i \to A$. Pick any prime ideal $p \subseteq A$ and set $p_i = f_i^{-1}(p)$. The valuation rings A_i/p_i are considered as a directed set of subrings of A/p. As $A/p = \bigcup_{i \in I} A_i/p_i$ one concludes that A/p is a valuation ring as well.

(h) If A is an SV-ring and $S \subseteq A$ is a multiplicative set then the quotient ring A_S is an SV-ring as well.

(i) Rings of continuous semi-algebraic functions on algebraic curves are SV-rings. This has been shown in [35], Corollary 2.6.

(j) The ring of continuous functions of an F-space is an SV-ring ([8], 14.25). Pick a completely regular space X, and let βX be its Stone-Cech compactification ([8], Chapter 6). If $f \in C(\beta X) = C^*(X)$ and if the zero set $Z_{\beta X}(f)$ is contained in the *growth* $\beta X \setminus X$ (*cf.* [38], 1.53), then $Z_{\beta X}(f)$ is an F-space ([8], 140).

Example 1.3. — *(a)* Rings of continuous semi-algebraic functions on semi-algebraic sets of dimension at least 2 have plenty of prime ideals whose residue domains are valuation rings (*cf.* [35], Corollary 2.5), but they are never SV-rings (*cf.* [35], Example 2.9).

(b) Noetherian rings are not SV-rings if their dimension is greater than 1: Suppose that A is an SV-ring with $\dim(A) \geqslant 2$. Then there is a prime ideal p such that $\dim(p) \geqslant 2$. The valuation ring A/p has dimension at least 2 and cannot be Noetherian. Thus, A is not Noetherian either.

If $I \subseteq A$ is an ideal then the map $J \longrightarrow \pi_I^{-1}(J)$ is an isomorphism from the partially ordered set of all ideals of A/I onto the set of ideals of A that contain I. This map restricts to an isomorphism from $\mathrm{Spec}(A/I)$ onto the subset $V(I) \subseteq \mathrm{Spec}(A)$. The set of ideals of a valuation ring is always totally ordered, hence so is its prime spectrum. This proves in particular:

Proposition 1.4. — *If A is an SV-ring then the specializations of any prime ideal $p \subseteq A$ form a chain, i.e., the prime spectrum is* completely normal.

The prime spectrum of an SV-ring is completely normal, hence is also normal. Rings with normal prime spectrum are also called *Gelfand rings* (*cf.* [19], p. 199). Other sources about Gelfand rings are [2], [37]. Normal spectra have a *specialization map* onto the maximal prime spectrum; it sends every prime ideal to the unique maximal ideal that contains it, $\sigma : \mathrm{Spec}(A) \rightarrow \mathrm{Max}(A)$, $p \subseteq \sigma(p)$. This is a continuous and closed map.

Proposition 1.5. — *The ring A is an SV-ring if and only if* $\mathrm{Spec}(A)$ *is normal and every localization A_m at a maximal ideal is an SV-ring.*

Proof. — If A is an SV-ring then the claim follows from Example 1.2 (h) and Proposition 1.4. Now assume that $\mathrm{Spec}(A)$ is normal and that the localizations A_m are SV-rings. Pick any prime ideal $p \subseteq A$. By normality of the spectrum, there is a unique maximal ideal m that contains p. Then $A/p \cong A_m/p \cdot A_m$ is a valuation ring. □

The condition that the prime spectrum is normal cannot be completely omitted in the equivalence of Proposition 1.5: All localizations of the ring $\mathbb{Z}$ of integers at maximal ideals are valuation rings, but $\mathbb{Z}$ is not an SV-ring. (Note that a domain is an SV-ring if and only if it is a valuation ring.)

In real algebra the equivalence of Proposition 1.5 can sometimes be simplified: The prime spectrum of every f-ring with bounded inversion is normal. Thus, an f-ring with bounded inversion is an SV-ring if and only if all localizations A_m at maximal ideals are SV-rings (*cf.* [14]; [32], Proposition 7.2).

Proposition 1.5 suggests that, studying SV-rings, one should consider *local* SV-rings first and then *globalize* the local results. The investigation of local rings is a natural first step in the study of general SV-rings.

2. Counting minimal prime ideals – the rank of a ring

The *rank* of a ring at a prime ideal p, $\mathrm{rk}(A,p)$, is the number of distinct minimal prime ideals that are contained in p (*cf.* [14], Definition 1.7). The rank is either a positive integer or ∞. It is clear that $\mathrm{rk}(A,q) \leqslant \mathrm{rk}(A,p)$ if $q \subseteq p$, i.e., $\mathrm{rk} : \mathrm{Spec}(A) \to \mathbb{N} \cup \{\infty\}$ is monotonic as a map of partially ordered sets. The *rank of the ring A* is

$$\mathrm{rk}(A) = \sup\big\{\mathrm{rk}(A,p)|p \in \mathrm{Spec}(A)\big\} = \sup\big\{\mathrm{rk}(A,m)|m \in \mathrm{Max}(A)\big\}.$$

One concludes easily that $\mathrm{rk}(A,p) = \mathrm{rk}(A_p)$ and $\mathrm{rk}(A) = \mathrm{rk}(A_{\mathrm{red}})$.

It is a remarkable fact that a ring of continuous functions has finite rank if it is an SV-ring, [14], Corollary 4.2.1 (a). Most rings of continuous functions do not have finite rank. So the result says that SV-rings are quite rare in the class of rings of continuous functions.

The following discussion explores algebraic characterizations of rings with finite rank. Particular attention will be paid to the question whether rings with finite rank have a first order axiomatization.

A first and almost trivial observation relates the rank to sequences of mutually orthogonal zero divisors. Two zero divisors are called *orthogonal* if their product is 0.

Proposition 2.1. — *If the ring A is reduced then the rank of a prime ideal p is the supremum of the lengths of sequences of mutually orthogonal zero divisors in p.*

The next result generalizes [37], Proposition 7.4, where reduced rings of rank 1, i.e., reduced rings whose spectrum is normal with respect to the inverse topology, are characterized by the following statement:

$$\forall a,b \quad \exists c : a \cdot b = 0 \longrightarrow c \cdot a = 0 \ \&(1-c) \cdot b = 0.$$

Other characterizations of this class of rings are given in [10], Theorem 4.2.2.

Proposition 2.2. — *The reduced ring A has rank at most k if and only if the following condition holds: For all sequences $a_1, \ldots, a_{k+1}$ of mutually orthogonal zero divisors there are elements $b_1, \ldots, b_{k+1}$, $b_\kappa \in \operatorname{Ann}(a_\kappa)$, with $(b_1, \ldots, b_{k+1}) = A$.*

Proof. — Suppose that A has rank at most k. If $a_1, \ldots, a_{k+1} \in A$ are mutually orthogonal zero divisors then, for all distinct indices κ and λ, $D(a_\kappa) \cap D(a_\lambda) = \varnothing$ and $D(a_\kappa) \neq \varnothing$. Pick a maximal ideal m and let $p_1, \ldots, p_r$, $r \leqslant k$, be all minimal prime ideals that are contained in m. Then there is some λ with $D(a_\lambda) \cap \operatorname{Gen}(m) = \varnothing$, and there is some $d_\lambda \in \operatorname{Ann}(a_\lambda)$ such that $\operatorname{Gen}(m) \subseteq D(d_\lambda)$. Thus, $\bigcup_{\kappa=1}^{k+1} \operatorname{Ann}(a_\kappa) \not\subset m$, hence the ideal generated by the annihilators is the full ring. There are $b_\kappa \in \operatorname{Ann}(a_\kappa)$, $\kappa = 1, \ldots, k+1$, such that $1 = b_1 + \ldots + b_{k+1}$.

Conversely, if there is a maximal ideal m that contains $k+1$ distinct prime ideals $p_1, \ldots, p_{k+1}$ then there are mutually orthogonal zero divisors $a_\lambda \in \bigcap_{\kappa \neq \lambda} p_\kappa \setminus p_\lambda$. Because of $\operatorname{Ann}(a_\lambda) \subseteq p_\lambda$ this implies $\bigcup_{\kappa=1}^{k+1} \operatorname{Ann}(a_\kappa) \subseteq m$. □

The condition of Proposition 2.2 can be expressed as a first order statement in the language of rings. Thus:

Corollary 2.3. — *The class of reduced rings with rank at most k is elementary with respect to the language of rings.*

The next result is of a technical nature. It speaks about local rings with rank at least $k+1$. It will be applied in the next section to show that the class of all SV-rings is not elementary in the language of rings.

Lemma 2.4. — *Suppose that A is a reduced local ring with infinite residue field. If there are $k+1$ distinct minimal prime ideals $p_1, \ldots, p_{k+1}$ then there are elements $a, b \in A$ such that, given any polynomial $Q(X,Y) = \sum_{j=0}^{k} c_j \cdot X^{k-j} \cdot Y^j$ with $(c_0, \ldots, c_k) = A$,*

(a) $Q(a,b) \cdot Q(b,a) \neq 0$, and

(b) $Q(a, x \cdot a + b) \neq 0$ for all $x \in A$.

Proof. — For each $\kappa = 1, \ldots, k+1$ there is an element $a_\kappa \in \bigcap_{\lambda \neq \kappa} p_\lambda \setminus p_\kappa$. Pick units $u_1, \ldots, u_{k+1} \in A^\times$ such that they themselves and their inverses are distinct modulo the maximal ideal m. Define $b_\kappa = u_\kappa \cdot a_\kappa$ and set $a = a_1 + \ldots + a_{k+1}$, $b = b_1 + \ldots + b_{k+1}$. Let Q be a polynomial as in the statement of the Lemma. Define $R(Y) = Q(1, Y)$ and note that the reduction of R modulo the maximal ideal is not the zero polynomial since there is at least one unit among the coefficients. The degree of R modulo m is at most k, hence R has at most k distinct roots in A/m. Both (a) and (b) will be proved by showing that the polynomial R has $k+1$ distinct roots in A/m if one of (a) and (b) is false.

(a) Assume that $Q(a, b) \cdot Q(b, a) = 0$. Then, for each $\kappa = 1, \ldots, k+1$,

$$
\begin{aligned}
0 &= Q(a + p_\kappa, b + p_\kappa) \cdot Q(b + p_\kappa, a + p_\kappa) \\
&= Q(a_\kappa + p_\kappa, b_\kappa + p_\kappa) \cdot Q(b_\kappa + p_\kappa, a_\kappa + p_\kappa) \\
&= \left(\sum_{j=0}^{k} (c_j + p_\kappa) \cdot (a_\kappa + p_\kappa)^{k-j} \cdot (b_\kappa + p_\kappa)^j \right) \\
&\qquad \cdot \left(\sum_{j=0}^{k} (c_j + p_\kappa) \cdot (b_\kappa + p_\kappa)^{k-j} \cdot (a_\kappa + p_\kappa)^j \right) \\
&= (a_\kappa + p_\kappa)^k \cdot \left(\sum_{j=0}^{k} (c_j + p_\kappa) \cdot (u_\kappa + p_\kappa)^j \right) \\
&\qquad \cdot (b_\kappa + p_\kappa)^k \cdot \left(\sum_{j=0}^{k} (c_j + p_\kappa) \cdot (u_\kappa^{-1} + p_\kappa)^j \right)
\end{aligned}
$$

and, as $a_\kappa + p_\kappa, b_\kappa + p_\kappa \neq 0 + p_\kappa$,

$$
\begin{aligned}
0 &= \left(\sum_{j=0}^{k} (c_j + p_\kappa) \cdot (u_\kappa + p_\kappa)^j \right) \cdot \left(\sum_{j=0}^{k} (c_j + p_\kappa) \cdot (u_\kappa^{-1} + p_\kappa)^j \right) \\
&= R(u_\kappa + p_\kappa) \cdot R(u_\kappa^{-1} + p_\kappa).
\end{aligned}
$$

Reduction modulo the maximal ideal yields $R(u_\kappa + m) \cdot R(u_\kappa^{-1} + m) = 0$. The polynomial R has $k+1$ distinct roots in A/m (namely one element from each of the sets $\{u_\kappa + m, u_\kappa^{-1} + m\}$).

(b) Assume that there is some $x \in A$ such that $Q(a, x \cdot a + b) = 0$. Then

$$0 = Q(a + p_\kappa, x \cdot a + b + p_\kappa) = Q(a_\kappa + p_\kappa, x \cdot a_\kappa + b_\kappa + p_\kappa)$$

$$= \sum_{j=0}^{k} (c_j + p_\kappa) \cdot (a_\kappa + p_\kappa)^{k-j} \cdot (x \cdot a_\kappa + b_\kappa + p_\kappa)^j$$

$$= (a_\kappa + p_\kappa)^k \cdot \left(\sum_{j=0}^{k} (c_j + p_\kappa) \cdot (x + u_\kappa + p_\kappa)^j \right)$$

Since $a_\kappa + p_\kappa \neq 0 + p_\kappa$ it follows that $R(x + u_\kappa + p_\kappa) = \sum_{j=0}^{k} (c_j + p_\kappa) \cdot (x + u_\kappa + p_\kappa)^j = 0$. Reduction modulo m yields $R(x + u_\kappa + m) = 0$ for each κ. As the elements $x + u_\kappa$ are all distinct modulo m, the polynomial R has $k + 1$ roots modulo m. □

The behavior of the rank of a ring under some standard ring theoretic constructions should be noted:

The rank of a factor ring A/I can be both larger and smaller than $\mathrm{rk}(A)$. For an example, consider the polynomial ring $A = \mathbb{Q}[X, Y]$, which has rank 1, its factor ring $B = A/(X \cdot Y)$, which has rank 2, and the iterated factor ring $A/(X, Y) = B/((X, Y)/(X \cdot Y))$, which has rank 1 again.

If $S \subseteq A$ is a multiplicative set then $\mathrm{rk}(A_S) \leqslant \mathrm{rk}(A)$.

The next result is an algebraic version of [25], Lemma 2.1:

Proposition 2.5. — *Suppose that $f : A \longrightarrow B$ is a generalizing homomorphism (i.e., the Going-down property holds, cf. [26], 5A) and that $f^{-1}(B^\times) = A^\times$. Then* $\mathrm{rk}(B) \geqslant \mathrm{rk}(A)$.

Proof. — The condition $f^{-1}(B^\times) = A^\times$ means that every maximal ideal of A is the restriction of some prime ideal of B, hence also of a maximal ideal of B. So, pick $m \in \mathrm{Max}(A)$ and $n \in \mathrm{Max}(B)$ with $m = f^{-1}(n)$. It is claimed that $\mathrm{rk}(A, m) \leqslant \mathrm{rk}(B, n)$. If $p \in \mathrm{Min}(A) \cap \mathrm{Gen}(m)$ then there is some $q \in \mathrm{Gen}(n)$ with $p = f^{-1}(q)$ (as f is generalizing). One may replace q by any minimal prime ideal that is contained in q, i.e., one may assume that $q \in \mathrm{Min}(B)$. Thus, the restriction of prime ideals maps $\mathrm{Min}(B) \cap \mathrm{Gen}(n)$ onto $\mathrm{Min}(A) \cap \mathrm{Gen}(m)$. □

Here are some examples of rings with finite rank:

Example 2.6. — *(a)* Domains have rank 1. Rings of any given finite rank can be constructed using fibre products: Let A and B be two local rings with finite ranks k and l. Let $I \subseteq A$ and $J \subseteq B$ be dense ideals (i.e., ideals that are not contained in any minimal prime ideal) with the property that $A/I \cong B/J$. Then the fibre product $A \times_{A/I} B$ is a local ring with rank $k+l$.

(b) The ring $C(X)$ has rank 1 if and only if X is an F-space ([8], 14.25).

(c) The ring of continuous semi-algebraic functions on an algebraic curve (Example 1.2 (i)) is an SV-ring of finite rank. The rank is the maximal number of half branches of the curve that meet in a single point. So, if the curve is non-singular then the rank is 2.

(d) Noetherian rings have finite rank: According to [26], section 7 a Noetherian ring has only finitely many minimal prime ideals.

3. Axiomatizability of SV-rings

This section is devoted to the question whether the class of SV-rings is elementary in the language of rings. The definition of SV-rings uses quantification over the set of (minimal) prime ideals, hence the definition is not in terms of first order statements. However, it is conceivable that the definition can be reformulated using only first order sentences. It will be shown that the problem is closely related to the rank of the SV-rings: The class of reduced local SV-rings with infinite residue field and rank at most k is elementary. This result can be globalized only with additional substantial hypotheses. Moreover, without a bound for the rank, axiomatizability also fails for local rings.

To start with, an example shows that there are local SV-rings with infinite rank. Note that all examples exhibited in the previous sections had finite rank.

Example 3.1. — Local SV-rings with infinite rank

(a) Suppose that $\varinjlim_{i \in I} A_i$ is a direct limit of SV-rings with injective transition maps. Then $\mathrm{rk}(A) = \sup\{\mathrm{rk}(A_i) | i \in I\}$, as one sees easily using the description of rank by the length of sequences of mutually orthogonal zero divisors. If all A_i are local rings and the transition maps are local homomorphisms then A is a local ring as well. Here is a construction that yields such a situation.

Define two linear maps $f_k, g_k : \mathbb{R}^k \to \mathbb{R}^{k+1}$ by $f_k(x_1, \dots, x_k) = (x_1, \dots, x_k, 0)$, $g_k(x_1, \dots, x_k) = (x_1, \dots, x_k, x_1)$. Consider $\mathbb{R}^k \subseteq \mathbb{R}^{k+1}$ as a subspace via f_k. A set of 2^{k-1} lines in $\mathbb{R}^k$ is defined recursively as follows: For $k = 1$, let $l_{1,1} = \mathbb{R}$. If the lines $l_{k,1}, \dots, l_{2^{k-1}} \subseteq \mathbb{R}^k$ have been defined, set $l_{k+1,i} = f_k(l_{k,i})$ and $l_{k+1,2^{k-1}+i} = g_k(l_{k,i})$. The union of the lines, $L_k = \bigcup_i l_{k,i} \subseteq \mathbb{R}^k$, is a curve. Let B_k be the ring of continuous semi-algebraic functions on L_k, A_k the localization at 0. The rings A_k are local SV-rings with rank 2^k (as L_k has 2^k half branches that pass through the origin, *cf.* Example 2.6 (c)). The orthogonal projection $\pi_k : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^k$ induces a local homomorphism $\pi_k^* : A_k \to A_{k+1}$. The direct limit $\varinjlim_{1 \leqslant k} A_k$ is a local SV-ring with infinite rank.

(b) Suppose that V is a valuation ring with maximal ideal m, containing a subfield K that is mapped isomorphically to the residue field; the localization of the polynomial ring $K[X]_{(X)}$ is such a valuation ring. The direct product $A = V^{\mathbb{N}}$ contains the direct sum $m^{(\mathbb{N})}$ as an ideal and contains an isomorphic copy of K via the diagonal map. For each $r \in \mathbb{N}$ the sequence $(\delta_{rs})_{s \in \mathbb{N}}$ is denoted by δ_r (where the δ_{rs} are Kronecker symbols). The zero set and the co-zero set of an element $a \in A$ are denoted by $Z(a)$ and $Coz(a)$. The subring $B = K + m^{(\mathbb{N})} \subseteq A$ is a local SV-ring with infinite rank.

Clearly, $m^{(\mathbb{N})} \subseteq B$ is an ideal, and $B/m^{(\mathbb{N})} \cong K$, hence the ring is local with maximal ideal $m^{(\mathbb{N})}$. Suppose that $p \subseteq B$ is a minimal prime ideal. As $m^{(\mathbb{N})}$ is not a minimal prime ideal, there is some element $x \in m^{(\mathbb{N})} \setminus p$, say $x = \sum_{r=0}^{R} x_r$ with $x_r = x \cdot \delta_r$. Since $x_r \cdot x_s = 0$ for $r \neq s$, all but one of the summands belong to p, say $x_r \notin p$. If $y = (y_n)_{n \in \mathbb{N}}$ with $y_r = 0$ then $y \cdot x_r = 0$, and y also belongs to p. Thus, $p \supseteq q = \{y = (y_n) \in m^{(\mathbb{N})} | y_r = 0\}$. Note that q is a prime ideal since it is the kernel of the projection onto the r-th component. One concludes that $p = q$ since p is minimal. The factor ring modulo q is isomorphic to V. Thus, the minimal prime ideals correspond bijectively with the natural numbers and the factor rings are valuation rings. Altogether, B is an SV-ring with infinite rank. (It is also possible to view this example as a direct limit construction of SV-rings with finite rank.)

Now let $I \subseteq A$ be any ideal with $m^{(\mathbb{N})} \subset I \subseteq m^{\mathbb{N}}$. With an argument as in the proof of [14], Theorem 4.1, it can be shown that $C = K + I$ is *not* an SV-ring if the field K is infinite: Pick an element $x \in I \setminus m^{(\mathbb{N})}$, let $z \in A$ be a sequence that takes only values in $K^{\times}$ and is injective as a map from $\mathbb{N}$ to K, and define $y = z \cdot x \in I$. Pick a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ that contains

$Coz(x)$. Then $p = \{c \in C | Z(c) \in \mathcal{U}\}$ is a prime ideal in C. If the factor ring C/p is a valuation ring then one of the two elements $x+p, y+p \in C/p$ must divide the other one.

Suppose that $y + p = a \cdot x + p$, $a \in C$. The set $Z(y - a \cdot x) \cap Coz(x)$ belongs to $\mathcal{U}$, and $z_n \cdot x_n = y_n = a_n \cdot x_n$, hence $a_n = z_n \notin m$ for all $n \in Z(y - a \cdot x) \cap Coz(x)$. Thus, $a \notin I$, and there is an element $b \in K$ with $b - a \in I$. It follows that $z_n = a_n \equiv b \pmod{m}$ for all $n \in Z(y - a \cdot x) \cap Coz(x)$. Pick two distinct elements k and l in the set $Z(y - a \cdot x) \cap Coz(x)$. Then $z_k \neq z_l$, but $z_k = b = z_l$, a contradiction.

Next assume that $x + p = a \cdot y + p$ with $a \in C$. The set $Z(x - a \cdot y) \cap Coz(x)$ belongs to $\mathcal{U}$, and $x_n = a_n \cdot y_n = a_n \cdot z_n \cdot x_n$, hence $a_n = z_n^{-1} \notin m$ for all $n \in Z(y - a \cdot x) \cap Coz(x)$. From here on the argument is identical to the one in the first case. □

Remark 3.2. — Concerning SV-rings with finite rank, the construction of fibre products exhibited in Example 1.2 (f) is particularly significant. In connection with Example 2.6 (a) one sees that there exist local SV-rings with arbitrary finite rank. Now suppose that A is any reduced SV-ring with finite rank. Then each localization at a maximal ideal m is a factor ring of A (since the prime spectrum is normal, *cf.* [37], Theorem 5.4). The localization A_m is an SV-ring of finite rank as well. If the rank is 1 then the ring is a valuation ring. If the rank is larger, then one partitions the set of minimal prime ideals of A_m into two non-empty subsets, $\mathrm{Min}(A_m) = E \cup F$. With $I = \bigcap E$ and $J = \bigcap F$ the factor rings A_m/I and A_m/J have finite rank $|E|$ and $|F|$. The canonical homomorphisms $\pi_{I+J,I}\colon A_m/I \to A_m/I + J$ and $\pi_{I+J,J}\colon A_m/J \to A_m/I + J$ are surjective. The fibre product of A_m/I and A_m/J over $A_m/I + J$ is an SV-ring and is canonically isomorphic to A_m. One concludes that A_m can be constructed from valuation rings using iterated fibre products.

The remark can be used to give a criterion for an SV-ring to have rank 1. This requires the notion of *valuation ideals*: Suppose A is an SV-ring. An ideal $I \subseteq A$ is called a *valuation ideal* if there are a prime ideal p and an ideal J in the valuation ring A/p such that $I = \pi_p^{-1}(J)$. Clearly, valuation ideals are those ideals that contain some prime ideal.

Proposition 3.3. — Suppose that A is an SV-ring. If A has rank 1 then every ideal of A is an intersection of valuation ideals. If A contains a field K with $|K| > 2$ then the converse is also true.

Proof. — First suppose that $\mathrm{rk}(A) = 1$. Given an ideal I, one forms the ideal $\overline{I} = \bigcap_{p \in \mathrm{Spec}(A)} I + p = \bigcap_{p \in \mathrm{Min}(A)} I + p$, which is the smallest intersection of valuation ideals that contains I. One must prove that $I = \overline{I}$. So, pick an element $a \in \overline{I}$ and write $a = a_p + x_p$ with $a_p \in I$, $x_p \in p$ for $p \in \mathrm{Min}(A)$. The sets $C_p = \{q \in \mathrm{Spec}(A) | a - a_p \in q\}$ are closed constructible and contain $\{\overline{p}\}$. The hypothesis about the rank implies that $\overline{\{p\}}$ is also generically closed. It follows from [37], Corollary 2.4 (iv) that there are open constructible sets U_p such that $\overline{\{p\}} \subseteq U_p \subseteq C_p$. Altogether the open sets U_p cover $\mathrm{Spec}(A)$, hence there is a finite subcover: $\mathrm{Spec}(A) = U_1 \cup \ldots \cup U_r$. By [37], Lemma 4.2 one may assume that $U_i = D(u_i)$ is a basic open set. As the ring A is reduced one obtains equalities $u_i \cdot (a - a_i) = 0$ with $a_i \in I$. As the sets $D(u_i)$ cover the spectrum there is an equality $1 = v_1 \cdot u_1 + \ldots + v_r \cdot u_r$. Now it follows that

$$\begin{aligned} a = 1 \cdot a &= v_1 \cdot (u_1 \cdot a) + \ldots + v_r \cdot (u_r \cdot a) \\ &= v_1 \cdot (u_1 \cdot a_1) + \ldots + v_r \cdot (u_r \cdot a_r) \in I, \end{aligned}$$

hence $I = \overline{I}$.

For the converse, suppose that every ideal is an intersection of valuation ideals. Then each reduced factor ring of A has this property as well. Now assume by way of contradiction that $\mathrm{rk}(A) > 1$. There is a maximal ideal m that contains two distinct minimal prime ideals p and q. The factor ring $A/p \cap q$ is a local SV-ring with rank 2. One may assume that $A = A/p \cap q$. As explained in Remark 3.2, A is canonically isomorphic to the fibre product of A/p and A/q over $A/p+q$. The valuations corresponding to A/p and A/q are denoted by v_p and v_q. Pick any unit $u \in A^\times$, $u + (p+q) \neq 1 + (p+q)$, and two elements $x \in p \backslash q$ and $y \in q \backslash p$. The unit exists because of the assumption about the field $K \subseteq A$. Consider the elements $x + y$ and $u \cdot x + y$. As $v_p(x + y + p) = v_p(u \cdot x + y + p)$ and $v_q(x + y + q) = v_q(u \cdot x + y + q)$ it follows that $u \cdot x + y$ belongs to the smallest intersection of valuation ideals that contains $x + y$. By hypothesis, every ideal is an intersection of valuation ideals; therefore, $u \cdot x + y = z \cdot (x + y)$ for some $z \in A$. One concludes that $u \cdot x + q = z \cdot x + q$, hence $z + q = u + q$, and that $y + p = z \cdot y + p$, hence $z + p = 1 + p$. But then $u + (p+q) = z + (p+q) = 1 + (p+q)$, a contradiction. □

The arguments proving, or disproving, axiomatizability of classes of SV-rings rely mostly on arithmetic properties of SV-rings and involve polynomials. The key result is the following characterization of reduced SV-rings.

Theorem 3.4. — The reduced ring A is an SV-ring if and only if for any two elements $a, b \in A$ there is a polynomial $P(X, Y) = \prod_{i=1}^{r} (X - c_i \cdot Y)$ such that $P(a, b) \cdot P(b, a) = 0$.

Proof. — First suppose that A is an SV-ring, and pick two elements $a, b \in A$. For each prime ideal $p \subseteq A$ there is an element $c_p \in A$ such that $a - c_p \cdot b \in p$ or $b - c_p \cdot a \in p$. The constructible sets $C_p = \{q \in \mathrm{Spec}(A) \mid (a - c_p \cdot b) \cdot (b - c_p \cdot a) \in q\}$ cover the entire spectrum. By compactness there is a finite subcover $\mathrm{Spec}(A) = \bigcup_{i=1}^{r} C_i$. The elements c_i that define the sets C_i are now used to define the polynomial P as in the statement of the Theorem. Then the product $P(a, b) \cdot P(b, a)$ belongs to every prime ideal, hence is 0 by reducedness of A.

Conversely, suppose that for all elements $a, b \in A$ a polynomial P exists as stated. If $p \in \mathrm{Spec}(A)$ then $\prod_{i=1}^{r}(a - c_i \cdot b) \cdot (b - c_i \cdot a) \in p$, and one of the factors belongs to p. This means that a is a multiple of b modulo p or that b is a multiple of a modulo p. Thus A/p is a valuation ring. □

Corollary 3.5. — *The class of reduced SV-rings is pseudo elementary (cf. [37], Definition 3.1) with witnesses* $\phi_r(x_1, x_2, y_1, \ldots, y_r) \equiv \prod_{i=1}^{r}(x_1 - y_i \cdot x_2) \cdot \prod_{i=1}^{r}(x_2 - y_i \cdot x_1) = 0.$

Theorem 3.4 does not imply that the class of SV-rings is elementary. One would need a bound on the degree of the polynomial P, independently from the elements a and b. Indeed, Lemma 2.4 will be used to show that no bound exists if the rank is infinite. Then it follows that the class of reduced local SV-rings with infinite residue field is not an elementary class in the language of rings, which also implies that the class of all SV-rings is not elementary.

The proof of non-axiomatizability uses ultraproducts. Recall that an elementary class is always closed under formation of ultraproducts ([3], Theorem 4.1.12; [18], Corollary 9.5.10).

Example 3.6. — An ultraproduct of reduced local SV-rings that is not an SV-ring

Let $(A_n)_{1 \leqslant n \in \mathbb{N}}$ be a sequence of local SV-rings with infinite residue fields such that $\mathrm{rk}(A_n) \geqslant n+1$. The maximal ideals are denoted by m_n, the residue fields by K_n, and $A = \prod_n A_n$. Let $\mathcal{U}$ be any free ultrafilter on the set of integers $\geqslant 1$. It is claimed that $A/\mathcal{U}$ is not an SV-ring.

First note that the class of reduced rings and the class of local rings with infinite residue field are both elementary. Thus, the ultraproduct is reduced and local with infinite residue field. Let $m_{\mathcal{U}}$ be the maximal ideal of $A/\mathcal{U}$ and $K_{\mathcal{U}}$ its residue field.

If A'_n is a factor ring of A_n then $A' = \prod_n A'_n$ is a factor ring of A as well, and $A'/\mathcal{U}$ is a factor ring of $A/\mathcal{U}$. If $A'/\mathcal{U}$ is not an SV-ring then $A/\mathcal{U}$ is not an SV-ring either (Example 1.2 (b)). For each n, let $p_{n,1}, \dots, p_{n,n+1}$ be distinct minimal prime ideals in A_n and define $A'_n = A_n/p_{n,1} \cap \dots \cap p_{n,n+1}$. This is an SV-ring with exact rank $n+1$. It suffices to show that the ultraproduct of these rings is not an SV-ring. Thus, one may assume that $\mathrm{rk}(A_n) = n+1$ and that $p_{n,1}, \dots, p_{n,n+1}$ are the distinct minimal prime ideals of A_n.

For each n there are elements $a_n, b_n \in A_n$ such that $Q_n(a_n, b_n) \cdot Q_n(b_n, a_n) \neq 0$ for all polynomials

$$Q_n(X, Y) = X^n + \sum_{j=1}^{n} c_{n,j} \cdot X^{n-j} \cdot Y^j \in A_n[X, Y]$$

(Lemma 2.4). Consider the sequences $a = (a_n)_n$, $b = (b_n)_n \in A$; the residue classes in $A/\mathcal{U}$ are denoted by $a_{\mathcal{U}}$ and $b_{\mathcal{U}}$. Assume by way of contradiction that $A/\mathcal{U}$ is an SV-ring. Then there is a polynomial

$$Q_{\mathcal{U}}(X, Y) = X^r + \sum_{j=1}^{r} c_{\mathcal{U},j} \cdot X^{r-j} \cdot Y^j \in A/\mathcal{U}[X, Y]$$

with $Q_{\mathcal{U}}(a_{\mathcal{U}}, b_{\mathcal{U}}) \cdot Q_{\mathcal{U}}(b_{\mathcal{U}}, a_{\mathcal{U}}) = 0$ (Theorem 3.4). There is a set $U \in \mathcal{U}$ such that $Q_n(a_n, b_n) \cdot Q_n(b_n, a_n) = 0$ for all $n \in \mathcal{U}$, where

$$Q_n(X, Y) = X^r + \sum_{j=1}^{r} c_{n,j} \cdot X^{r-j} \cdot Y^j \in A_n[X, Y]$$

and $c_{\mathcal{U},j} = (c_{n,j})_n$. As the set U is infinite it contains numbers $n > r$. For such n the condition that $Q_n(a_n, b_n) \cdot Q_n(b_n, a_n) = 0$ contradicts the choice of the elements a_n and b_n, and the proof is finished.

Corollary 3.7. — *The following classes of rings are not elementary:*

- *all SV-rings;*
- *reduced SV-rings;*
- *local reduced SV-rings with infinite residue field.*

The next result shows that unboundedness of the rank is indeed the essential obstruction for first order axiomatizability of local SV-rings with infinite residue field:

Theorem 3.8. — *Let A be a reduced local ring with maximal ideal m and infinite residue field. Then the following conditions are equivalent:*

(a) A is an SV-ring with finite rank at most k.

(b) For all $a, b \in A$ there is a polynomial $P(X,Y) = \prod_{\kappa=1}^{k}(X - c_\kappa \cdot Y)$ such that $P(a,b) \cdot P(b,a) = 0$.

(c) For all $a, b \in A$ there is an element $u \in A^\times \cup \{0\}$ and a polynomial $P(X,Y) = \prod_{\kappa=1}^{k}(X - c_\kappa \cdot Y)$ such that $P(a, u \cdot a + b) = 0$.

(d) Factor domains modulo minimal prime ideals are integrally closed, and for all $a, b \in A$ there is an element $u \in A^\times \cup \{0\}$ and a polynomial $Q(X,Y) = X^k + \sum_{\kappa=1}^{k} d_\kappa \cdot X^{k-\kappa} \cdot Y^k$ such that $Q(a, u \cdot a + b) = 0$.

Proof. — **(a) → (b)** One may assume that the rank of A is exactly k. Let $p_1, \dots, p_k$ be the distinct minimal prime ideals. For each $\kappa = 1, \dots, k$ pick an element $c_\kappa \in A$ such that $a - c_\kappa \cdot b \in p_\kappa$ or $b - c_\kappa \cdot a \in p_\kappa$. Then $\prod_{\kappa=1}^{k}(a - c_\kappa \cdot b) \cdot (b - c_\kappa \cdot a) \in \bigcap_{\kappa=1}^{k} p_\kappa = \{0\}$, and the polynomial $P(X,Y) = \prod_{\kappa=1}^{k}(X - c_\kappa \cdot Y)$ has the desired properties.

(b) → (a) Theorem 3.4 and Lemma 2.4 prove the claim immediately.

(b) → (c) For each minimal prime ideal p_κ, let $v_\kappa : \mathrm{qf}(A/p_\kappa) \to \Gamma_\kappa \cup \{\infty\}$ be the valuation associated with the valuation ring A/p_κ. Pick $a, b \in A$ and consider their residues modulo the minimal prime ideals: Suppose that there is some $u \in A^\times \cup \{0\}$ with $v_\kappa(a + p_\kappa) \geqslant v_\kappa(u \cdot a + b + p_\kappa)$. Then $a - c_\kappa \cdot (u \cdot a + b) \in p_\kappa$ for all κ and suitable elements c_κ. The polynomial $P(X,Y) = \prod_{\kappa=1}^{k}(X - c_\kappa \cdot Y)$ satisfies (c). It is only necessary to show that there is an element $u \in A^\times \cup \{0\}$ that satisfies $v_\kappa(a + p_\kappa) \geqslant v_\kappa(u \cdot a + b + p_\kappa)$ for all κ. Suppose that $u \in A^\times \cup \{0\}$ *fails* to satisfy this condition. Then there is some κ such that $v_\kappa(a+p_\kappa) < v_\kappa(u \cdot a + b + p_\kappa)$. This implies that the quotient $\dfrac{u \cdot a + b + p_\kappa}{a + p_\kappa}$, which belongs to $\mathrm{qf}(A/p_\kappa)$ to start with, is an element of the valuation ring A/p_κ and belongs to the maximal ideal m/p_κ. Moreover, one also concludes that $v_\kappa(a+p_\kappa) \leqslant v_\kappa(b+p_\kappa)$, hence $u + p_\kappa + \dfrac{b+p_\kappa}{a+p_\kappa} = \dfrac{u \cdot a + b + p_\kappa}{a+p_\kappa} \in m/p_\kappa$, and u is uniquely determined modulo m. As the residue field is infinite, almost all elements modulo m yield units that are suitable.

(c) → (b) Pick $a, b \in A$, and suppose that the element $u \in A^\times \cup \{0\}$ and the polynomial $P(X,Y) = \prod_{\kappa=1}^{k}(X - c_\kappa \cdot Y)$ satisfy (c). For each κ at least one of the elements c_κ and $1 - c_\kappa \cdot u$ is a unit. If $1 - c_\kappa \cdot u \in A^\times$, set $d_\kappa = \dfrac{c_\kappa}{1 - c_\kappa \cdot u}$; otherwise set $d_\kappa = \dfrac{1 - c_\kappa \cdot u}{c_\kappa}$. The polynomial $Q(X,Y) = \prod_{\kappa=1}^{k}(X - d_\kappa \cdot Y)$ satisfies the requirements of (b): Suppose that p is a minimal prime ideal of A. Then there is some index $\kappa(p)$ such that $a - c_{\kappa(p)} \cdot (u \cdot a + b) \in p$, and $(1 - c_{\kappa(p)} \cdot u) \cdot a \equiv c_{\kappa(p)} \cdot b \pmod p$. This implies $a - d_{\kappa(p)} \cdot b \in p$ or $b - d_{k(p)} \cdot a \in p$.

(a) → (d) The factor domains are valuation rings, hence they are integrally closed. Pick a polynomial as in (c) and expand it to bring it into the form needed for (d).

(d) → (c) First note that A has at most k minimal prime ideals (Lemma 2.4 (b)); one may assume that there are exactly k minimal prime ideals, say $p_1, \ldots, p_k$. Pick $a, b \in A$ and choose an element $u \in A^\times \cup \{0\}$ and the polynomial Q as specified in (d). If p_κ is a minimal prime ideal then either $u \cdot a + b \in p_\kappa$, or $\dfrac{a + p_\kappa}{u \cdot a + b + p_\kappa} \in \mathrm{qf}(A/p_\kappa)$ is a root of the monic polynomial $Q(X,1)$. In the first case it follows that $a + p_\kappa = 0 + p_\kappa$, and one defines $c_\kappa = 1$. In the second case one concludes that $\dfrac{a+p_\kappa}{u \cdot a + b + p_\kappa} \in A/p_\kappa$ since A/p_κ is integrally closed, i.e., there is an element $c_\kappa \in A$ with $a - c_\kappa \cdot (u \cdot a + b) \in p_\kappa$. The polynomial $P(X,Y) = \prod_{\kappa=1}^{k}(X - c_\kappa \cdot Y)$ satisfies condition (c). □

Both conditions (b) and (c) of Theorem 3.8 can be phrased as first order statements. Thus:

Corollary 3.9. — *The class of reduced local SV-rings with finite rank at most k and with infinite residue field is elementary.*

Note that, given two elements $a, b \in A$ there are at most k units modulo m that do *not* satisfy the conditions (c) and (d) of Theorem 3.8.

Example 3.6 gave a negative answer to the question whether the class of SV-rings is closed under the formation of direct products. More information about direct products of SV-rings can be obtained using Theorem 3.8:

Proposition 3.10. — *Suppose that $(A_i)_{i\in I}$ is an infinite family of reduced F-algebras, F an infinite field. Consider the following conditions:*

(a) The direct product $A = \prod_{i\in I} A_i$ is an SV-ring.

(b) Each A_i is an SV-ring, and there is some $N \in \mathbb{N}$ such that the set $\{i \in I | \mathrm{rk}(A_i) \geqslant N\}$ is finite.

Condition (a) always implies (b). If all SV-rings are local then (b) also implies (a).

Proof. — **(a) → (b)** Each component A_i of the product is a factor ring of A, hence is an SV-ring. Assume by way of contradiction that all sets $I(n) = \{i \in I | \mathrm{rk}(A_i) \geqslant n\}$ are infinite. Then there is a sequence $(i_n)_{2\leqslant n}$ of distinct elements of I with $i_n \in I(n)$. The ring $A' = \prod_{2\leqslant n} A_{i_n}$ is a factor ring of A, hence is an SV-ring. Let $m_{i_n} \subseteq A_{i_n}$ be a maximal ideal with $\mathrm{rk}(A_{i_n}, m_{i_n}) \geqslant n$. As each ring A_{i_n} has normal spectrum, the localizations at the maximal ideals m_{i_n} are factor rings ([37], Proposition 5.4), which will be denoted by B_n, and are therefore SV-rings. The product $B = \prod_{2\leqslant n} B_n$ is a factor ring of A', hence is an SV-ring. But Example 3.6 shows that such a product cannot be an SV-ring. This contradiction finishes the first part of the proof.

(b) → (a) The product A can be written in the form $\prod_{i\in I(N)} A_i \times \prod_{i\notin I(N)} A_i$ (with the same notation as in the proof of (a) → (b)). This is an SV-ring if and only if both factors are SV-rings. The first factor is a finite direct product of SV-rings, hence is an SV-ring. Thus, it remains to prove that the second factor is an SV-ring. Hence one may assume that all components of the product have rank at most N.

The Boolean algebra $B(A)$ of idempotents is isomorphic to the Boolean algebra 2^I of subsets of I via $e = (e_i)_{i\in I} \to Coz(e) = \{i \in I | e_i = 1\}$. If $p \in \mathrm{Spec}(A)$ then the set $p \cap B(A)$ is a prime (= maximal) ideal of $B(A)$. The set $\mathcal{F}(p) = \{Z(e) | e \in p \cap B(A)\} \subseteq 2^I$ is an ultrafilter (where $Z(e) = I \setminus Coz(e)$). If $p \subseteq q$ are two prime ideals then their ultrafilters coincide. The ideal $(p \cap B(A)) \cdot A \subseteq A$ is contained in p, and the factor ring $A/(p \cap B(A)) \cdot A$ coincides with the ultraproduct $A/\mathcal{F}(p)$. One concludes that $A/(p \cap B(A)) \cdot A$ is an SV-ring (as the class of reduced local SV-rings with infinite residue field and rank at most N is elementary, Corollary 3.9). Thus the factor ring $A/p \cong \big(A/(p \cap B(A)) \cdot A\big)/\big(p/(p \cap B(A)) \cdot A\big)$ is a valuation ring. □

The next few results are concerned with the question whether the equivalence of Theorem 3.8 can be globalized in some form, i.e., whether it can be extended to rings that are not necessarily local. This is indeed possible with regard to the equivalence of conditions (a) and (d); the question remains undecided concerning conditions (b) and (c). The special case of SV-rings of rank 1 is considered first. The following result is an adaptation of [37], Proposition 7.4 (*cf.* the comments before Proposition 2.2) to SV-rings. Note that the prime spectrum of an SV-ring of rank 1 is the disjoint union of a collection of totally ordered sets of prime ideals (by Proposition 1.4). This is the same situation that one meets in rings of continuous functions on F-spaces.

PROPOSITION 3.11. — *If A is a reduced ring then the following conditions are equivalent:*

(a) A is an SV-ring of rank 1.

(b) For all $a, b \in A$ there are elements $c_1, c_2, d_1, d_2 \in A$ such that $1 = c_1 + c_2$ and $c_1 \cdot (a - d_1 \cdot b) = 0$, $c_2 \cdot (b - d_2 \cdot a) = 0$.

Proof. — **(a) → (b)** If $p \in \mathrm{Min}(A)$ then $a+p = x_p \cdot b+p$ or $b+p = y_p \cdot a+p$ with suitable elements $x_p, y_p \in A$. There is a partition $\mathrm{Min}(A) = M_x \cup M_y$ such that $a+p = x_p \cdot b+p$ for $p \in M_x$ and $b+p = y_p \cdot a+p$ for $p \in M_y$. For $p \in \mathrm{Min}(A)$ one defines $Z_p = V(a - x_p \cdot b)$ or $Z_p = V(a - y_b \cdot b)$ according as $p \in M_x$ or $p \in M_y$. The set Z_p is closed and constructible and contains the closed irreducible and generically closed set $\overline{\{p\}}$. There are constructible sets U_p, V_p, W_p such that

- $\overline{\{p\}} \subseteq U_p \subseteq V_p \subseteq W_p \subseteq Z_p$,
- U_p is open and $W_p = D(z_p)$ for some $z_p \in A$,
- V_p is closed.

The sets U_p cover $\mathrm{Spec}(A)$, hence there is a finite subcover $\mathrm{Spec}(A) = \bigcup_{j \in J} U_{p_j}$. One defines $J_x = \{j \in J \mid p_j \in M_x\}$, $J_y = \{j \in J \mid p_j \in M_y\}$ and $U_x = \bigcup_{j \in J_x} U_{p_j}$, $U_y = \bigcup_{j \in J_y} U_{p_j}$. Also define $V_x = \bigcup_{j \in J_x} V_{p_j}$, $W_x = \bigcup_{j \in J_x} W_{p_j}$, $V_y = \bigcup_{j \in J_y} V_{p_j}$, $W_y = \bigcup_{j \in J_y} W_{p_j}$. Since $\mathrm{Spec}(A)$ is normal there are elements $c_1, c_2 \in A$ such that $1 = c_1 + c_2$ and $D(c_1) \subseteq U_x$, $D(c_2) \subseteq U_y$ ([37], Proposition 5.1). The set $\mathrm{Gen}(V_x)$ is closed and generically closed (loc.cit.) and is contained in the open constructible set W_x. Therefore there exists an element z_x with $V_x \subseteq V(z_x) \subseteq W_x$ (loc.cit., Lemma 4.2). Similarly, there is an element z_y with $V_y \subseteq V(z_y) \subseteq W_y$. As $\mathrm{Spec}(A) = D(z_x) \cup W_x$ one can write $1 = \alpha_x \cdot z_x + \sum_{j \in J_x} \alpha_{p_j} \cdot z_{p_j}$. (Recall that W_x is the union of the sets $W_{p_j} = D(z_{p_j})$ with $j \in J_x$.) For each j, $z_{p_j} \cdot (a - x_{p_j} \cdot b) = 0$ and $c_1 \cdot z_x = 0$. This implies

$$0 = c_1 \cdot z_x \cdot \alpha_x \cdot a + c_1 \cdot \sum_{j \in J_x} \alpha_{p_j} \cdot z_{p_j} \cdot (a - x_{p_j} \cdot b)$$

$$= c_1 \cdot \left(\left(\alpha_x \cdot z_x + \sum_{j \in J_x} \alpha_{p_j} \cdot z_{p_j} \right) \cdot a - \left(\sum_{j \in J_x} \alpha_{p_j} \cdot z_{p_j} \cdot x_{p_j} \right) \cdot b \right)$$

$$= c_1 \cdot \left(a - \left(\sum_{j \in J_x} \alpha_{p_j} \cdot z_{p_j} \cdot x_{p_j} \right) \cdot b \right)$$

Now set $d_1 = \sum_{j \in J_x} \alpha_{p_j} \cdot z_{p_j} \cdot x_{p_j}$. The same argument (with y instead of x) produces the element d_2.

(b) → (a) To show that A has rank 1, assume that there are distinct minimal prime ideals p and q that are both contained in the same maximal ideal m. There are elements $a \in p \setminus q$ and $b \in q \setminus p$ with $a \cdot b = 0$. Condition (b) yields elements c_1, c_2, d_1, d_2 with $1 = c_1 + c_2$ and $c_1 \cdot (a - d_1 \cdot b) = 0$, $c_2 \cdot (b - d_2 \cdot a) = 0$. Since $a - d_1 \cdot b \notin q$ and $b - d_2 \cdot a \notin p$ it follows that $c_1 \in q \subseteq m$ and $c_2 \in p \subseteq m$. This implies $1 = c_1 + c_2 \in m$, a contradiction.

It remains to prove that A is an SV-ring: Pick $p \in \mathrm{Min}(A)$ and consider two elements $a + p, b + p \in A/p$. It is claimed that one of them is a multiple of the other. Condition (b) is applied to a and b: There are elements c_1, c_2, d_1, d_2 such that $1 = c_1 + c_2$, $c_1 \cdot (a - d_1 \cdot b) = 0$, $c_2 \cdot (b - d_2 \cdot a) = 0$. Let m be some maximal ideal that contains p, and suppose without loss of generality that $m \in D(c_1)$. Then $c_1 \notin p$, and one concludes that $a - d_1 \cdot b \in p$. Thus, $a + p$ is a multiple of $b + p$. □

Condition (b) of Proposition 3.11 can be expressed as a first order statement, hence:

Corollary 3.12. — *The class of reduced SV-rings of rank 1 is an elementary class.*

If A is an SV-ring with rank 1 then the specialization map from $\mathrm{Min}(A)$ to $\mathrm{Max}(A)$ is a bijection. It is also continuous (since the spectrum is normal), but it is not a homeomorphism in general. However, if $\mathrm{Min}(A)$ is compact then it is homeomorphic to $\mathrm{Max}(A)$. From [12], Theorem 3.4 one obtains

Corollary 3.13. — *The class of reduced SV-rings A with rank 1, with compact minimal prime spectrum and with the property that the set $\{D(a) \cap \mathrm{Min}(A) \,|\, a \in A\}$ is closed under finite unions is an elementary class. An axiomatization is provided by the axioms for reduced rings, condition (b) of Proposition 3.11 and the statement:* $\forall a\ \exists b\ \forall c : a \cdot b = 0\ \&\ \big[(a+b)\cdot c = 0 \to c = 0\big]$.

Now the discussion turns to SV-rings with arbitrary finite rank.

Proposition 3.14. — *Let A be a reduced F-algebra, where F is an infinite field. Pick a set of $k+1$ distinct elements $u_1, \ldots, u_{k+1} \in F^\times$. The following conditions are equivalent:*

(a) A is an SV-ring with rank at most k.

(b) The prime spectrum of A is normal, the factor domains of A modulo minimal prime ideals are integrally closed, and for all elements $a, b \in A$ there are elements $c_1, \ldots, c_{k+1} \in A$ and polynomials $Q_1, \ldots, Q_{k+1}$, $Q_k = X^k + \sum_{j=1}^{k} c_{\kappa,j} X^{k-j} \cdot Y^j$ such that $(c_1, \ldots, c_{k+1}) = A$ and $c_\kappa \cdot Q_\kappa(a, u_\kappa \cdot a + b) = 0$.

Proof. — **(a) → (b)** The prime spectrum of an SV-ring is completely normal, hence normal (Proposition 1.4). Factor rings modulo (minimal) prime ideals are valuation rings, hence they are integrally closed. Now pick elements $a, b \in A$ and some maximal ideal m. The local ring A_m is an SV-ring. Thus Theorem 3.8 shows that there is a unit $v_m \in A_m^\times$ and a polynomial $R_m = X^k + \sum_{j=1}^{k} d_{m,j} \cdot X^{k-j} \cdot Y^j \in A_m[X, Y]$ such that

$R_m\left(\frac{a}{1}, v_m \cdot \frac{a}{1} + \frac{b}{1}\right) = 0$. Note that there are at most k units $v \in A_m^\times$ that do *not* satisfy this condition. Thus, one may choose v_m to be one of the units $u_1, \ldots, u_{k+1}$.

Since Spec(A) is normal, the canonical homomorphism $A \to A_m$ is surjective ([37], Proposition 5.4). The polynomial R_m can be lifted to a polynomial $Q_m = X^k + \sum_{j=1}^{k} c_{m,j} \cdot X^{k-j} \cdot Y^j \in A[X,Y]$. The set $C_m = \{q \in \text{Spec}(A) \mid Q_m(a, v_m \cdot a + b) = 0\}$ is closed constructible and contains the set Gen(m) of generalizations of m. There is a basic open set $D(e_m)$ and another closed constructible set D_m such that $\text{Gen}(m) \subseteq D_m \subseteq D(e_m) \subseteq C_m$. Note that $e_m \cdot Q_m(a, v_m \cdot a + b) = 0$. The sets D_m cover Spec(A), and, by compactness, there is a finite subcover $\text{Spec}(A) = \bigcup_{\sigma=1}^{s} D_{m_\sigma}$. For each $\kappa = 1, \ldots, k+1$, one defines $\Sigma(\kappa) = \left\{\sigma \in \{1, \ldots, s\} | v_{m_\sigma} = u_k\right\}$. The set $G_\kappa = \text{Gen}\left(\bigcup_{\sigma\in\Sigma(\kappa)} D_{m_\sigma}\right) = \bigcup_{\sigma\in\Sigma(\kappa)} \text{Gen}(D_{m_\sigma})$ is closed and generically closed and is contained in the open constructible set $\bigcup_{\sigma\in\Sigma(\kappa)} D(e_{m_\sigma})$. [37], Proposition 4.3 yields elements c_κ, d_κ with $G_\kappa \subseteq D(c_\kappa) \subseteq V(d_\kappa) \subseteq \bigcup_{\sigma\in\Sigma(\kappa)} D(e_{m_\sigma})$, and one concludes that there are x_σ, $\sigma \in \Sigma(\kappa)$, and y_κ such that $1 = y_\kappa \cdot d_\kappa + \sum_{\sigma\in\Sigma(\kappa)} x_\sigma \cdot e_{m_\sigma}$. With

$$Q_\kappa = y_\kappa \cdot d_\kappa \cdot X^k + \sum_{\sigma\in\Sigma(\kappa)} x_\sigma \cdot e_{m_\sigma} \cdot Q_{m_\sigma} = X^k + \sum_{j=1}^{k} c_{\kappa,j} \cdot X^{k-j} \cdot Y^j$$

it follows from $c_\kappa \cdot d_\kappa = 0$ and $e_{m_\sigma} \cdot Q_{m_\sigma}(a, u_\kappa \cdot a + b) = 0$ (with $\sigma \in \Sigma(\kappa)$) that

$$c_\kappa \cdot Q_\kappa(a, u_\kappa \cdot a + u_\kappa \cdot b) = c_\kappa \cdot y_\kappa \cdot d_\kappa \cdot a^k + c_\kappa \cdot \sum_{\sigma\in\Sigma(\kappa)} x_\sigma \cdot e_{m_\sigma} \cdot Q_{m_\sigma}(a, u_\kappa \cdot a + b) = 0$$

Finally note that $\text{Spec}(A) = \bigcup_{\kappa=1}^{k+1} G_\kappa = \bigcup_{\kappa=1}^{k+1} D(c_\kappa)$, hence $(c_1, \ldots, c_{k+1}) = A$.

(b) → (a) The prime spectrum is normal. Therefore A is an SV-ring if and only if every localization at a maximal ideal is an SV-ring (Proposition

1.5). So it suffices to show that every localization A_m, m a maximal ideal, satisfies condition (d) of Theorem 3.8. If $p \subseteq m$ is a minimal prime ideal then A/p is integrally closed by hypothesis. Pick elements $\frac{a}{s}, \frac{b}{t} \in A_m$. The hypothesis is applied to $a{\cdot}t, b{\cdot}s \in A$: There are elements $c_1, \ldots, c_{k+1} \in A$ and polynomials $Q_1, \ldots, Q_{k+1}$ as in condition (b). Since $\operatorname{Spec}(A) = \bigcup_{\kappa=1}^{k+1} D(c_\kappa)$, there is some κ such that $m \in D(c_\kappa)$. Then

$$\frac{c_\kappa}{1} \cdot \left(\left(\frac{a \cdot t}{1}\right)^k + \sum_{j=1}^{k} \frac{c_{\kappa,j}}{1} \cdot \left(\frac{a \cdot t}{1}\right)^{k-j} \cdot \left(u_\kappa \cdot \frac{a \cdot t}{1} + \frac{b \cdot s}{1}\right)^j \right) = 0$$

in A_m. As $\frac{c_\kappa}{1}, \frac{s}{1}, \frac{t}{1} \in A_m^\times$, the equation can be multiplied by $\frac{1}{c_\kappa} \cdot \frac{1}{s^k} \cdot \frac{1}{t^k}$, which yields

$$\left(\frac{a}{s}\right)^k + \sum_{j=1}^{k} \frac{c_{\kappa,j}}{1} \cdot \left(\frac{a}{s}\right)^{k-j} \cdot \left(u_\kappa \cdot \frac{a}{s} + \frac{b}{t}\right)^j = 0. \qquad \square$$

Condition (b) of Proposition 3.14 is not a first order statement. But it is conceivable that it can be transformed into such a statement. Note that normality of the prime spectrum is an axiomatizable property, *cf.* [37], Proposition 4.3. Assume for a moment that the condition about factor domains modulo minimal prime ideals (in condition (b) of Proposition 3.14) is equivalent to a first order statement. Then one can conclude immediately that the class of reduced SV-algebras over infinite fields and with rank at most k is elementary. The same is true if either of the following statements can be expressed in a first order form:

- All factor rings modulo arbitrary prime ideals are integrally closed, or

- all factor rings modulo minimal prime ideals are integrally closed with regard to polynomials of degree at most k (i.e., if $P \in A/p[X]$ is a monic polynomial of degree at most k and if $P(\alpha) = 0$ with $\alpha \in \operatorname{qf}(A/p)$ then $\alpha \in A/p$).

It is highly unlikely, but seems to be unknown, whether it is possible to reformulate any of these statements as first order sentences. However, there is a local result in this direction:

PROPOSITION 3.15. — *Suppose that A is a reduced local ring with finite rank k. Let $p_1, \ldots, p_k$ be the minimal prime ideals. Let $a_1, \ldots, a_k$ be a maximal set of mutually orthogonal zero divisors, $a_\lambda \in \bigcap_{\kappa \notin \lambda} p_\kappa \setminus p_\lambda$. Then the following conditions are equivalent:*

(a) For each $p \in \mathrm{Min}(A)$, A/p is integrally closed with regard to polynomials of degree at most l.

(b) If $a, b \in A$ and if $Q(a,b) = 0$, where $Q(X,Y) = X^l + \sum_{j=1}^{l} c_j \cdot X^{l-j} \cdot Y^j$, then $a \cdot a_\kappa \in (b \cdot a_\kappa)$ for each $\kappa = 1, \dots, k$.

Proof. — **(a)** $\to$ **(b)** Pick $a, b \in A$ and suppose that $Q(a,b) = 0$, where Q is a polynomial as in (b). Then either $b + p_\kappa = 0 + p_\kappa$ (which also implies $a + p_\kappa = 0 + p_\kappa$, hence $a \cdot a_\kappa = 0$ and $b \cdot a_\kappa = 0$), or $b \notin p_\kappa$ and $Q\left(\frac{a+p_\kappa}{b+p_\kappa}, 1\right) = 0$ in $\mathrm{qf}(A/p_\kappa)$. Condition (a) implies that $\frac{a+p_\kappa}{b+p_\kappa} \in A/p_\kappa$, hence there is some $d_\kappa \in A$ with $a - d_\kappa \cdot b \in p_\kappa$. But then $(a - d_\kappa \cdot b) \cdot a_\kappa = 0$, and this proves the claim.

(b) $\to$ **(a)** Suppose that $P(X) = X^l + \sum_{j=1}^{l} (c_j + p_\kappa) \cdot X^{l-j} \in A/p_\kappa[X]$ and let $\frac{a+p_\kappa}{b+p_\kappa} \in \mathrm{qf}(A/p_\kappa)$ be a root of P. Then, setting $Q(X,Y) = X^l + \sum_{j=1}^{l} c_j \cdot X^{l-j} \cdot Y^j \in A[X,Y]$ one sees that $Q(a,b) \in p_\kappa$, which implies $Q(a \cdot a_\kappa, b \cdot a_\kappa) = a_k^l \cdot Q(a,b) = 0$. By (b), $a \cdot a_\kappa^2 = d_\kappa \cdot (b \cdot a_\kappa^2)$ for some $d_\kappa \in A$. Reduction modulo p_κ yields $(a + p_\kappa) \cdot (a_\kappa + p_\kappa)^2 = (d_\kappa + p_\kappa) \cdot (b + p_\kappa) \cdot (a_\kappa + p_\kappa)^2$, hence $\frac{a+p_\kappa}{b+p_\kappa} = d_\kappa + p_\kappa \in A/p_\kappa$. □

Lacking an analogous global result, axiomatizability of the class of SV-rings with bounded rank remains undecided. However, there are elementary classes of rings whose residue domains are integrally closed, e.g., real closed rings (*cf.* [29], [27]). It is an immediate consequence of Proposition 3.14 that, in such a class of rings, the SV-rings with bounded rank form an elementary subclass. The following corollary is a result of this type.

Corollary 3.16. — *The real closed SV-rings with rank at most k form an elementary class.*

Explicitly, an axiomatization of the class of real closed SV-rings with rank at most k is given by the axioms of real closed rings ([27], Theorem 2.1), together with the statement:

For all $a, b \in A$ there are $c_1, \dots, c_{k+1} \in A$ and polynomials $Q_1, \dots, Q_{k+1}$, $Q_\kappa = X^k + \sum_{j=1}^{k} c_{\kappa,j} \cdot X^{k-j} \cdot Y^j \in A[X,Y]$ such that $(c_1, \dots, c_{k+1}) = A$ and $c_\kappa \cdot Q_\kappa(a, a \cdot u_\kappa + b) = 0$.

The same statement singles out the SV-rings of rank at most k among the class of rings of continuous functions.

4. Partially ordered SV-rings

In [13], at the beginning of section 2, the authors define an f-ring to be an *SV-ring* if each factor ring modulo a prime ideal is a valuation ring. They do not assume any connection between the valuations and the partial order. But they conclude that their SV-rings have bounded inversion (loc.cit., Theorem 2.9). This is clearly false: Pick any totally ordered field and a valuation ring that is not convex. Then the maximal ideal is not convex, hence bounded inversion fails. But factor rings of valuation rings modulo prime ideals are always valuation rings. (The mistake in their arguments is in the proof of the implication (b) $\rightarrow$ (c) of Lemma 2.2.)

In the present paper the definition of partially ordered SV-rings includes the condition of *compatibility between the real spectrum and the prime spectrum*. This means: Given $\alpha \in \mathrm{Sper}(A, A^+)$ and a maximal ideal m that contains $\mathrm{supp}(\alpha)$, the support of α, then $m/\mathrm{supp}(\alpha)$ is a convex ideal of the totally ordered ring A/α. This notion is introduced and studied in [34], section 3. Spectral compatibility is stronger than bounded inversion, *cf.* loc.cit. The condition strengthens the connection between the valuations of the residue domains and the order structure.

This section contains basic properties of partially ordered SV-rings. The next section is concerned with applications to convexity.

Definition 4.1. — *A poring* (A, A^+) *is a* partially ordered SV-ring (= SV-poring) *if the underlying ring is an SV-ring and the spectra are compatible. If* (A, A^+) *is an* f*-ring and an SV-poring then it is called an* SV-f-ring.

The class of porings is elementary in the language $\{+, -, \cdot, 0, 1, \leqslant\}$. Therefore, a class of SV-porings is elementary whenever its elements are

- porings of some elementary class,
- SV-rings of an elementary class (e.g., reduced SV-rings of rank 1, Corollary 3.12), and
- if compatibility of the spectra can be expressed with first order statements (e.g., if the compatibility condition is equivalent to bounded inversion).

If the underlying ring of a poring is an SV-ring then the prime spectrum is completely normal (Proposition 1.4). Therefore, given an SV-ring that is also a poring, compatibility of the spectra is equivalent to the condition that the support map from the real spectrum to the prime spectrum restricts to a surjective map from maximal prime cones to maximal ideals ([34], Proposition 3.2). For an f-ring, compatibility and bounded inversion are equivalent conditions ([34], Proposition 3.5). Moreover, if one considers a class of f-rings with bounded inversion (such as real closed rings or rings of continuous functions) then compatibility of the spectra is always satisfied and is not an additional condition.

The following characterization of SV-porings is useful:

Theorem 4.2. — *Suppose that (A, A^+) is a poring. Then the following conditions are equivalent:*

(a) (A, A^+) is an SV-poring.

(b) If $p \in \mathrm{Spec}(A)$ then A/p is convex in $\mathrm{qf}(A/p)$ with respect to every total order defined by a prime cone $\alpha \in \mathrm{Sper}(A, A^+)$ with support p.

If the equivalent conditions hold then the support map supp: $Sper(A, A^+) \to \mathrm{Spec}(A)$ is surjective, i.e., every prime ideal of A is convex.

Proof. — **(a)** $\to$ **(b)** Pick any prime ideal p and any prime cone α with support p. There is a unique maximal ideal m that contains p (Proposition 1.4). Compatibility implies that m is α-convex. The maximal ideal m/p of the valuation ring A/p is convex in $\mathrm{qf}(A/\alpha)$. This is equivalent to convexity of the valuation ring ([20], p. 55, Satz 3).

(b) $\to$ **(a)** Note that every minimal prime ideal of A is convex ([36], Proposition 4.4), hence is the support of some prime cone. Pick any prime ideal p. There is some $q \in \mathrm{Min}(A)$, $q \subseteq p$, and some $\alpha \in \mathrm{Sper}(A, A^+)$ with support q. Being a convex subring of $\mathrm{qf}(A/\alpha)$, A/q is a valuation ring. Then every ideal of A/q is convex, in particular the prime ideal p/q. The factor ring A/p is a valuation ring as well, and one concludes that A is an SV-ring.

The prime ideal p of A is convex; hence it is the support of a prime cone β. According to condition (b), A/p is convex in $\mathrm{qf}(A/\beta)$. Let m be the unique maximal ideal of A that contains p (Proposition 1.4). Then m/p is the maximal ideal of the convex valuation ring A/p, hence it is convex as well, and m is convex with respect to β.

The above arguments also prove surjectivity of the support map. □

It is immediately clear from the definition that a factor poring of an SV-poring modulo a convex ideal is an SV-poring: The SV property is preserved under homomorphic images (Example 1.2 (b)); the same is true for compatibility of the spectra ([34], section 3). In particular, the reduction of an SV-poring modulo the nilradical is an SV-poring. The converse of this implication is true as well: Example 1.2 (c) shows that A is an SV-ring if $A/\mathrm{Nil}(A)$ is an SV-ring. The poring (A, A^+) has compatible spectra if and only if the same is true for $\Big(A/\mathrm{Nil}(A), \pi_{\mathrm{Nil(A)}}(A^+)\Big)$ ([34], section 3). Because of this remark it is frequently possible to assume without loss of generality that SV-porings are reduced.

The next result speaks about quotient rings of SV-porings. Note that, in general, compatibility of spectra is not preserved under the formation of quotient rings ([34], section 3).

Corollary 4.3. — If (A, A^+) is an SV-poring and if $S \subset A$ is a multiplicative set then the quotient poring (A_S, A_S^+) is an SV-poring as well.

Proof. — Quotient rings of SV-rings are SV-rings (Example 1.2 (h)). The questionable condition is compatibility of the spectra. However, condition (b) of Theorem 4.2 is clearly preserved under passage to quotient rings. □

Many rings carry various different partial orders. If (A, A^+) is an SV-poring and if $P \subseteq A^+$ is a weaker partial order, then it may happen that one of the two porings (A, A^+) and (A, P) is an SV-poring and the other one is not an SV-poring. But if $\mathrm{Sper}(A, A^+) = \mathrm{Sper}(A, P)$ then either both porings satisfy condition (b) of Theorem 4.2, or both don't. In particular, in the case of a reduced poring, the saturation $(A, \widehat{A^+})$ is an SV-poring if and only if (A, A^+) is an SV-poring. (The underlying ring of the saturation is A, but the partial order is stronger than A^+, namely $\widehat{A^+} = \bigcap_{\alpha \in \mathrm{Sper}(A,A^+)} \alpha$, *cf.* [22], p. 42, Definition 2. The saturation can be formed also without the assumption that the ring is reduced. However, if the ring is not reduced then $\widehat{A^+}$ is only a preordering, not a partial order.)

Theorem 4.2 shows that all prime ideals in an SV-poring are real. Therefore the sets $D(a) \subseteq \mathrm{Spec}(A)$ form a lattice: It is always true that the set of sets $D(a)$ is closed under finite intersections (namely $D(a_1) \cap \ldots \cap D(a_r) = D(a_1 \cdot \ldots \cdot a_r)$. Moreover, $D(a_1) \cup \ldots \cup D(a_r) = D(a_1^2 + \ldots + a_r^2)$ holds in an SV-poring.

Theorem 4.2 establishes a close connection between the factor valuation rings of an SV-poring on the one hand and the partial order on the other hand. The relationship is not nearly so close for porings that are SV-rings with bounded inversion, but without compatible spectra:

Example 4.4. — Let K be a real field that has two non-Archimedean total orders α and β with natural valuation rings V_α and V_β. Suppose that $V_\alpha \cdot V_\beta = K$. The maximal specialization of β in $\mathrm{Sper}(V_\beta)$ is denoted by β_1. Then $V_\beta^+ = V_\beta \cap \alpha \cap \beta_1$ is a partial order on V_β. The poring (V_β, V_β^+) has bounded inversion and is an SV-ring. The real spectrum has two points, namely $\alpha' = \alpha \cap V_\beta$ and $\beta_1' = \beta_1 \cap V_\beta$. The prime cone α' is completely unrelated to the valuation ring.

The connection between real spectrum and prime spectrum is particularly close in the case of an SV-f-ring. Then the support map is a homeomorphism. For, the support map is always a spectral map. It is injective in the case of f-rings, surjective for SV-porings (Theorem 4.2). To see that it is a homeomorphism one only needs to note that $\alpha \subseteq \beta$ if $\mathrm{supp}(\alpha) \subseteq \mathrm{supp}(\beta)$ (where $\alpha, \beta \in \mathrm{Speer}(A, A^+)$).

The f-ring reflection $\big(\varphi(A), \varphi(A)^+\big)$ of a reduced poring (A, A^+) has been introduced and studied in [36]. According to [34], Proposition 3.2, a reduced poring has compatible spectra if and only if its f-ring reflection has compatible spectra (which is equivalent to bounded inversion). This result is supplemented by

Proposition 4.5. — *Suppose that (A, A^+) is a reduced poring with f-ring reflection $\big(\varphi(A), \varphi(A)^+\big)$. Then A is an SV-ring if and only if $\varphi(A)$ is an SV-ring.*

Proof. — The functorial map $\mathrm{Sper}\big(\varphi(A), \varphi(A)^+\big) \to \mathrm{Sper}(A, A^+)$ is a homeomorphism (because $\big(\varphi(A), \varphi(A)^+\big)$ is contained in the real closure $\big(\rho(A), \rho(A^+)\big)$, and the real closure is the strongest reflector of reduced porings that preserves the real spectrum, *cf.* [36], Theorem 12.12). The real spectra will be identified. If $\alpha \in \mathrm{Sper}(A, A^+)$ then the codomain of the canonical map $\pi_\alpha : (A, A^+) \to A/\alpha$ is a reduced f-ring, hence it factors through $\big(\varphi(A), \varphi(A)^+\big)$, and the totally ordered residue rings A/α and $\varphi(A)/\alpha$ are canonically isomorphic.

If q is a minimal prime ideal of $\varphi(A)$ then q is convex ([36], Proposition 4.4), and $q \cup \varphi(A)^+$ is the unique prime cone with support q.

For the proof of the equivalence, first suppose that A is an SV-ring. If $q \subseteq \varphi(A)$ is a minimal prime ideal then $\beta = p \cup \varphi(A)^+$ is a prime cone; the restriction to A is denoted by α. One concludes that $\varphi(A)/q = \varphi(A)/\beta \cong A/\alpha = A/\mathrm{supp}(\alpha)$ is a valuation ring.

Conversely, let $\varphi(A)$ be an SV-ring. Pick a minimal prime ideal $p \subseteq A$. Since $A \subseteq \varphi(A)$ is an extension of rings there is a minimal prime ideal $q \subseteq \varphi(A)$ with $p = q \cap A$. With the same notation as above, $A/p \cong \varphi(A)/q$ is a valuation ring.

Corollary 4.6. — *The reduced poring (A, A^+) is an SV-poring if and only if the f-ring reflection is an SV-f-ring.*

The next example shows that the rank of an SV-poring and its f-ring reflection are different in general. Either rank can be larger than the other one:

Example 4.7. — *(a)* Let K be a non-trivially valued field with valuation ring V and residue field k. The valuation ideal is denoted by m. Suppose that k carries two different total orders, say $\overline{\alpha}$ and $\overline{\beta}$. Then there are total orders α and β on V that induce $\overline{\alpha}$ and $\overline{\beta}$ ([20], p. 72, Theorem). One defines the ring A to be the fibre product $V \times_k V$. The projections $p_1, p_2 : A \to V$ yield prime cones $\gamma = p_1^{-1}(\alpha)$ and $\delta = p_2^{-1}(\beta)$ of A. The closure $\overline{\{\gamma\}}$ is the image of $\mathrm{Sper}(V, \alpha)$ under $\mathrm{Sper}(p_1)$, and $\overline{\{\delta\}}$ is the image of $\mathrm{Sper}(V, \beta)$ under $\mathrm{Sper}(p_2)$. Setting $A^+ = \gamma \cap \delta$, one checks easily that $\mathrm{Sper}(A, A^+)$ is the disjoint union of $\overline{\{\gamma\}}$ and $\overline{\{\delta\}}$. The ring A is local with maximal ideal $n = m \times m$ and residue field k. If γ_1 and δ_1 are the maximal specializations of γ and δ then $\mathrm{supp}(\gamma_1) = n = \mathrm{supp}(\delta_1)$, and γ_1 and δ_1 define $\overline{\alpha}$ and $\overline{\beta}$ on k.

The description of (A, A^+) shows that this is an SV-poring with rank 2. The f-ring reflection $\big(\varphi(A), \varphi(A)^+\big)$ is an SV-f-ring. Its real spectrum is homeomorphic to $\mathrm{Sper}(A, A^+)$ and also to $\mathrm{Spec}(\varphi(A))$, which implies that $\mathrm{rk}(\varphi(A)) = 1$.

(b) With the same notation as in part (a), suppose that k carries a unique total order, but the set C of total orders of K that are compatible with V is infinite. Define $V^+ = \bigcap_{\alpha \in C} \alpha$. Then (V, V^+) is an SV-poring with $\mathrm{rk}(V) = 1$. The real spectrum has one closed point and infinitely many generic points. The f-ring reflection $\big(\varphi(A), \varphi(A)^+\big)$ is an SV-f-ring. It follows from $\mathrm{Spec}(\varphi(A)) \cong \mathrm{Sper}\big(\varphi(A), \varphi(A)^+\big) \cong \mathrm{Sper}(A, A^+)$ that $\mathrm{rk}(\varphi(A)) = \infty$.

It has been shown above that the f-ring reflection of an SV-poring is an SV-f-ring (Corollary 4.6). The same is true for the real closure reflector:

Proposition 4.8. — *Suppose that (A, A^+) is a reduced SV-poring. If $\rho : \mathbf{PO/N} \to \mathbf{RCR}$ denotes the real closure reflector (cf. [36], section 12), then the reflection $\rho(A, A^+)$ is an SV-poring as well.*

Proof. — The real closure of (A, A^+) coincides with the real closure of $\big(\varphi(A), \varphi(A)^+\big)$ ([36], Proposition 8.1). So one may assume that (A, A^+) is an f-ring. The spectra $\mathrm{Sper}(A, A^+)$, $\mathrm{Sper}\big(\rho(A), \rho(A)^+\big)$, $\mathrm{Spec}(A)$ and $\mathrm{Spec}\big(\rho(A)\big)$ are all homeomorphic to each other via the canonical maps. Let $q \subseteq \rho(A)$ be any prime ideal and write $p = q \cap A$. The real closure reflector is H-closed ([36], Definition 12.2), hence the map $\rho(\pi_p) : \rho(A) \to \rho(A/p)$ is surjective ([36], Proposition 10.8), and $\rho(A)/q \cong \rho(A/p)$. Thus, it suffices to prove that the real closure of a totally ordered convex valuation ring is a valuation ring (necessarily totally ordered and convex). This has been proved in [35], Corollary 1.3. □

The converse of Proposition 4.8 is not true – if the poring (A, A^+) is not an SV-ring it may happen that the real closure is an SV-ring. For example, let (A, A^+) be a totally ordered domain with totally ordered quotient field (K, K^+). Suppose that A is cofinal in K. Then $\rho(A, A^+)$ is the real closure of the totally ordered field (K, K^+). This is an SV-ring, but there are many such porings (A, A^+) that are not SV-rings.

The following example exhibits a reduced SV-poring (A, A^+) and an H-closed monoreflector $r : \mathbf{PO/N} \to \mathbf{C}$ that is weaker than the real closure reflector (*cf.* [36]) such that the reflection $r(A, A^+)$ is not an SV-ring. Thus, not every reflection of an SV-poring by a monoreflector is an SV-ring.

Example 4.9. — In this example, the reflector r of the category $\mathbf{PO/N}$ is the differentiable functions reflector introduced in [35], Theorem 3.9. The field of real algebraic numbers is denoted by R_0. Let A be the valuation ring $R_0[X]_{(X)}$, totally ordered by the condition that $0 < X \ll 1$. Then A is the convex hull of R_0 in its totally ordered quotient field. Let R be the real closure of $\mathrm{qf}(A, A^+)$. The real closure $\rho(A, A^+)$ is the convex hull of R_0 in R. It is a one-dimensional real closed valuation ring. It contains R_0 as a maximal subfield, hence R_0 is a field of representatives ([35], Proposition 2.1). The corresponding valuation $v : R \to \mathbb{Q} \cup \{\infty\}$ is normalized such that $v(X) = 1$. The set $I_1 = \{a \in R \mid v(a) \geqslant 1\}$ is a convex ideal in $\rho(A, A^+)$. Then $R_0 + I_1$ is a proper subring of $\rho(A, A^+)$, which is dominated by $\rho(A, A^+)$ and has the same quotient field. It contains the convex valuation ring (A, A^+) and is contained in the convex valuation ring $\rho(A, A^+)$, but is not a valuation ring itself. The reflection $r(A, A^+)$ coincides with $R_0 + I_1$.

The section ends with some thoughts about SV-porings of rank 1. First there is a connection between convex ideals in SV-porings and valuation ideals:

Lemma 4.10. — *Suppose that (A, A^+) is an SV-poring. If the ideal I is an intersection of valuation ideals then I is convex.*

Proof. — For each minimal prime ideal p the ideal $I + p/p \subseteq A/p$ is convex (as all ideals in the valuation ring A/p are convex, *cf.* Theorem 4.2). Thus the inverse image $I + p \subseteq A$ under the canonical homomorphism $\pi_p : A \to A/p$ is convex as well. Since intersections of convex ideals are convex it follows that $I = \bigcap_{p \in \min(A)} I + p$ is convex. □

The converse of Lemma 4.10 is not true, as the following example shows:

Example 4.11. — A convex ideal in an SV-poring that is not an intersection of valuation ideals

Let V be a proper convex subring in some totally ordered field. The maximal ideal is denoted by m. The fibre product $A = V \times_{V/m} V$ is an SV-ring. If A^+ is the partial order induced by the total order on the two copies of V then (A, A^+) is an SV-poring. Pick any element $0 < x \in m$. Then the principle ideal generated by $(x, -x)$ is convex in A: For, if $(p, q), (r, s) \in A^+$ and if $(p, q) + (r, s) = (\alpha, \beta) \cdot (x, -x)$ then, in the convex valuation ring V, $p + r = \alpha \cdot x$ implies that $p = \alpha' \cdot x$, $r = \alpha'' \cdot x$ for some α' and α''. Moreover, $q + s = \beta \cdot (-x)$ implies that $\beta \leqslant 0$, and there are $\beta', \beta'' \leqslant 0$ such that $q = \beta' \cdot (-x)$, $s = \beta'' \cdot (-x)$. It is clear that $\beta \leqslant \beta'$, $\beta'' \leqslant 0 \leqslant \alpha'$, $\alpha'' \leqslant \alpha$. By definition of the ring A, $\alpha + m = \beta + m$. Since $\alpha \in A^+$ and $\beta \in -A^+$ it follows that $\alpha, \beta \in m$. By convexity of m, all the other factors also belong to m. Therefore $(\alpha', \beta'), (\alpha'', \beta'') \in A$, and $(p, q) = (\alpha', \beta') \cdot (x, -x)$, $(r, s) = (\alpha'', \beta'') \cdot (x, -x)$.

The ideal $A \cdot (x, -x)$ is not an intersection of valuation ideals: The element (x, x) belongs to the smallest intersection of valuation ideals that contains the ideal, but is not a multiple of $(x, -x)$.

Proposition 4.12. — *Suppose that (A, A^+) is a reduced poring with compatible spectra. The following conditions are equivalent:*

(a) (A, A^+) is an SV-poring of rank 1.

(b) Every ideal of A is convex.

(c) (A, A^+) satisfies the 1^st convexity condition (cf. [23]), i.e., if $0 \leqslant a \leqslant b$ then there is some $c \in A$ such that $a = c \cdot b$.

Proof. — **(a)** $\Rightarrow$ **(b)** Proposition 3.3 says that every ideal of A is an intersection of valuation ideals. According to Lemma 4.10 these are always convex. – The equivalence **(b)** $\Leftrightarrow$ **(c)** is trivial.

(b) $\Rightarrow$ (a) If $p \subseteq A$ is a minimal prime ideal then it is convex. Every ideal of the factor ring A/p is convex as well. It follows immediately that A/p is convex in its quotient field (which is partially ordered with positive cone $P = \left\{\frac{a}{s} \mid \exists t \in A^+/p : a \cdot s \cdot t \in A^+/p\right\}$). For, if $0 \leqslant \frac{a}{s} \leqslant b \in A/p$ then there is an element $t \in A^+/p$ such that $0 \leqslant a \cdot s \cdot t \leqslant b \cdot s^2 \cdot t$ in A/p. As the principle ideal $(b \cdot s^2 \cdot t)$ is convex there is some $c \in A/p$ such that $a \cdot s \cdot t = c \cdot b \cdot s^2 \cdot t$, and this implies $\frac{a}{s} = c \cdot b \in A/p$.

The convex subring A/p of $\mathrm{qf}(A/p)$ is the intersection of all P-convex valuation rings that contain it. Thus A/p contains the real holomorphy ring of $\mathrm{qf}(A/p)$, which implies that it is a Prüfer domain ([28], section 1). The domain A/p is local since its prime spectrum is normal ([34], Proposition 3.2). Local Prüfer domains are valuation rings, hence A/p is a valuation ring, and it has been proved that A is an SV-poring.

It remains to prove that $\mathrm{rk}(A) = 1$: If this is false then there is a maximal ideal m that contains two distinct minimal prime ideals p and q. The ideals of $A/p \cap q$ are convex as well. So, to derive a contradiction one may assume that $A = A/p \cap q$. Pick a unit $u \in A^\times$ and two elements $x \in p \setminus q$, $y \in q \setminus p$. Replacing u by $\frac{u^2}{1+u^2}$ (note that (A, A^+) has bounded inversion because of compatibility, *cf.* [34], section 3), x by x^2 and y by y^2 one may assume that $0 < u < 1$, $0 < x$ and $0 < y$. It follows that $0 < u \cdot x + y < x + y$, but $u \cdot x + y \notin (x + y)$ (as in the proof of Proposition 3.3). $\square$

A final remark about first order axiomatizability: The class of SV-f-rings with rank 1 is elementary (*cf.* the remark after Definition 4.1). Referring to Proposition 3.11, the axiom system includes the statement

$$\forall a, b\ \exists c_1, c_2, d_1, d_2 : 1 = c_1 + c_2\ \&\ c_1 \cdot (a - d_1 \cdot b) = 0\ \&\ c_2 \cdot (b - d_2) = 0.$$

This statement can now be replaced by the simpler sentence

$$\forall a, b\ \exists c : 0 \leqslant a \leqslant b \rightarrow a = c \cdot b,$$

which says that 1^{st} convexity holds.

5. SV-porings and convexity

Convex subrings of rings of continuous functions, and some other porings, are a natural source of SV-rings. A proper convex subring of a poring (both porings with compatible spectra) has a factor ring that is a nontrivial SV-ring of rank 1. This section contains a study of those SV-porings that arise in this way. It is asked whether the presence of factor rings that are SV-rings of rank 1 implies the existence of convex extensions of a given poring.

Note about terminology: If (B, B^+) is a convex subring of a poring (A, A^+) then (A, A^+) is called a *convex extension* of the poring (B, B^+). In [34], Example 1.11, it is shown that every poring has a convex extension. In this section the question whether a given poring has a proper convex extension will be considered only for the case that both porings have compatible spectra. To avoid tedious repetition of the hypothesis of compatibility, it will be assumed that all porings have compatible spectra whenever the term "convex extension" is being used.

The question why SV-rings occur in connection with convexity has a simple answer: Convex subrings of fields *with respect to total orders* are valuation rings. Convex subrings of *partially* ordered fields are not quite so simple. They are always intersections of real valuation rings, but very often they are not valuation rings. Every convex subring contains the real holomorphy ring ([28]; [20], p. 155). As the real holomorphy ring is a Prüfer domain, the same is true for the convex subring ([9], Theorem 26.1). Therefore the convex subring is a valuation ring if and only if it is a local ring. With these remarks it is easy to prove the following result:

Proposition 5.1. — *Suppose that (K, K^+) is a partially ordered field and that B is a convex subring. Then the following conditions are equivalent:*

(a) (B, B^+) is an SV-poring.

(b) B is an SV-ring.

(c) For all $\alpha \in \mathrm{Sper}(K, K^+)$, B coincides with its the convex hull in K with respect to α. (In particular, B is a valuation ring.)

(d) (B, B^+) has compatible spectra.

Proof. — One identifies $\mathrm{Sper}(K, K^+)$ with the set of generic points of $\mathrm{Sper}(B, B^+)$. Let V_α be the convex hull of B in K with respect to α. Its maximal ideal is denoted by m_α. The ring B is the intersection of the convex valuation rings V_α.

(a) $\rightarrow$ **(b)** is trivial. **(b)** $\rightarrow$ **(c)** The ring B is a domain. Its prime spectrum is completely normal (Proposition 1.4), hence the ring is local. But then it is a valuation ring and $B = V_\alpha$ for each α. **(c)** $\rightarrow$ **(d)** If $B = V_\alpha$ then the maximal ideal is convex with respect to α. Thus, compatibility of the spectra follows. **(d)** $\rightarrow$ **(a)** Compatibility implies that the domain B

has normal prime spectrum; hence it is a valuation ring, in particular an SV-ring. Compatibility holds by hypothesis, hence (B, B^+) is an SV-poring. $\square$

Now consider a poring (A, A^+) with compatible spectra and a convex subring (B, B^+). Setting $S = A^\times \cap B^+$, the porings (A, A^+) and (B_S, B_S^+) are canonically isomorphic to each other ([22], Theorem 7.2; [34], Corollary 2.6). Therefore one can identify $\mathrm{Spec}(A)$ with a dense and generically closed pro-constructible subspace of $\mathrm{Spec}(B)$, and the same is true about the real spectra of the porings. If $q \in \mathrm{Spec}(B) \setminus \mathrm{Spec}(A)$ then there is a unique maximal ideal $m \subset A$ such that $m \cap B \subseteq q$. The ring $B/m \cap B$ is convex in the partially ordered field $\big(A/m, \pi_m(A^+)\big)$ ([34], Corollary 2.7). This is the situation considered in Proposition 5.1.

The following immediate consequence of Proposition 5.1 supplements [34], Theorem 4.4:

Corollary 5.2. — *Suppose that (A, A^+) is a poring with compatibility and that B is a convex subring. Then the following conditions are equivalent:*

(a) (B, B^+) has compatible spectra.

(b) For every maximal ideal $m \subset A$ the convex subring $B/m \cap B$ of the partially ordered field $\big(A/m, \pi_m(A^+)\big)$ is a valuation ring.

(c) For every maximal ideal $m \subset A$, $\big(B/m \cap B, \pi_{m\cap B}(B^+)\big)$ has compatible spectra.

Proof. — **(a)** $\to$ **(b)** Compatibility of spectra is preserved modulo convex ideals. So the claim follows from Proposition 5.1. **(b)** $\to$ **(c)** is trivial. **(c)** $\to$ **(a)** Let $\beta \in \mathrm{Sper}(B, B^+)$ be a generic point and let $\alpha \in \mathrm{Sper}(A, A^+)$ be the corresponding prime cone. Pick a maximal ideal $n \subset B$ that contains $\mathrm{supp}(\beta)$. There is a unique maximal ideal $m \subset A$ that contains $\mathrm{supp}(\alpha)$. It follows from [34], Theorem 4.1 that $m \cap B \subseteq n$ and that $m \cap B$ is β-convex. Condition (c) says that the maximal ideal $n/m \cap B$ of $B/m \cap B$ is β-convex. But then n is β-convex as well. $\square$

It is clear from the consideration of partially ordered fields (which are trivially SV-porings) that convex subrings of SV-porings are not SV-porings, in general. The condition of compatibility provides a criterion to decide whether convex subrings are SV-rings or not:

Corollary 5.3. — *Suppose that (A, A^+) is an SV-poring. The following conditions are equivalent:*

(a) The convex hull of $\mathbb{Z}$ is an SV-poring.

(b) The convex hull of $\mathbb{Z}$ is an SV-ring

(c) The convex hull of $\mathbb{Z}$ has compatible spectra.

(d) Every convex subring is an SV-poring.

(e) Every convex subring is an SV-ring.

(f) Every convex subring has compatible spectra.

Proof. — **(a)** $\to$ **(b)**, **(a)** $\to$ **(c)**, **(d)** $\to$ **(e)** and **(d)** $\to$ **(f)** are all trivial. The equivalence of **(c)** and **(f)** is shown in [34], Theorem 4.4. In view of Proposition 5.1, the following arguments yield the implications **(b)** $\to$ **(a)**, **(c)** $\to$ **(a)**, **(f)** $\to$ **(d)** and **(f)** $\to$ **(e)**: Let B be a convex subring of (A, A^+). Pick $q \in \mathrm{Min}(B)$, let $p \in \mathrm{Min}(A)$ be the unique prime ideal of A that extends q, and let $\alpha \in \mathrm{Sper}(A, A^+)$ be some prime cone with support p. Then B/q is convex in $\big(A/p, \pi_p(A^+)\big)$, and it is sufficient to deal with a domain. Now A is a convex subring of its quotient field and B is convex in A, both with respect to α, hence B is a convex subring of a totally ordered field as well. □

The equivalent conditions of Corollary 5.3 are satisfied, for example, if the map from the maximal real spectrum of (A, A^+) to the maximal prime spectrum is bijective. This is the case for f-rings with bounded inversion, in particular for rings of continuous functions and for real closed rings.

If X is a completely regular space and if $Z \subseteq \beta X \setminus X$ is a zero set then Z is an F-space and $C(Z)$ is an SV-ring of rank 1 (*cf.* Example 1 (j)). The next result is a generalization:

Proposition 5.4. — *Suppose that (A, A^+) is a poring with compatible spectra, that B is a convex subring and that (B, B^+) has compatible spectra as well. If $I \subseteq B$ is an ideal such that $V_A(I) = \big\{p \in \mathrm{Spec}(A) \,|\, I \subseteq p\big\}$ has dimension at most 0 then B/I is an SV-ring of rank 1. This is the case, for example, if $I = (s)$ where $s \in B \cap A^\times$.*

Proof. — Let $q \in \mathrm{Spec}(B/I) = V_B(I)$. There is a unique maximal ideal $m \subset A$ such that $B \cap m \subseteq q$ ([34], Theorem 4.1). It is claimed that B/q is a valuation ring. The hypotheses imply that $B/m \cap B$ is a valuation ring (Corollary 5.2), hence B/q is a factor ring of a valuation ring. □

The prime spectrum of the convex subring B of Proposition 5.4 has a peculiar structure, which has been described in [34], Theorem 2.12 and

Theorem 2.13: The set of specializations of $\mathrm{Max}(A)$ in $\mathrm{Spec}(B)$ is a union of disjoint chains. The specialization map from the compact space $\mathrm{Max}(A)$ (note that A has normal prime spectrum, [34], Proposition 3.2, therefore the space of maximal points is compact, [2]) to the compact space $\mathrm{Max}(B)$ is bijective and continuous, hence a homeomorphism.

The description of $\mathrm{Spec}(B)$ can be rephrased as follows: There is a continuous section $\tau : \mathrm{Max}(B) \to \mathrm{Spec}(B)$ for the specialization map such that the set of specializations of $\mathrm{im}(\tau)$ is a disjoint union of chains and every point of $\mathrm{Spec}(B)$ is a specialization or a generalization of some point of $\mathrm{im}(\tau)$. (Note that $\mathrm{im}(\tau)$ is a compact subset of $\mathrm{Spec}(B)$, hence $\mathrm{Gen}(\mathrm{im}(\tau))$ is generically closed and proconstructible, *cf.* [37], Proposition 2.3.)

If B belongs to a class of rings with completely normal prime spectrum (e.g., real closed rings) then it suffices to demand that there is a continuous section τ such that every point of $\mathrm{Spec}(B)$ is a specialization or a generalization of some point of $\mathrm{im}(\tau)$. For, suppose $\tau : \mathrm{Max}(B) \to \mathrm{Spec}(B)$ is a continuous section satisfying this condition. If $m, n \in \mathrm{Max}(B)$ and if $p \in \overline{\{\tau(m)\}} \cap \overline{\{\tau(n)\}}$ then m and n are closed points in $\overline{\{p\}}$. As $\overline{\{p\}}$ has only one closed point it follows that $m = n$.

Each poring (A, A^+) that contains the poring (B, B^+) as a convex subring (both with compatible spectra) provides such a section. One may ask whether the existence of a nontrivial section implies the existence of a convex extension of B that has compatible spectra as well. This is indeed the case if B is a ring of continuous functions, [32], section 5. However, the answer is negative for arbitrary real closed rings, as the following example shows:

Example 5.5. — Let B be a real closed domain with maximal ideal m. The prime spectrum is a chain, and, given any element $p \in \mathrm{Spec}(B)$, the map $\tau : \mathrm{Max}(B) \to \mathrm{Spec}(B) : m \to p$ is a section that has the properties described above. But there are real closed domains that do not have any convex extensions. To exhibit such a ring, suppose that V is a convex subring of a real closed field T and that there is a prime ideal $q \subset V$ with $(0) \subset q \subset m_V$, where m_V is the maximal ideal of V. Pick a maximal subfield $R \subset V$ (which is a real closed field that is isomorphic to the residue field of V, *cf.* [20], p. 66, Satz 3). The subring $B = R + q \subset V$ is a real closed ring that does not have any proper convex extension with compatible spectra. Any such extension would have to be a quotient ring of B, say B_S, where $S \subseteq B \setminus B^\times$, i.e. $S \subset q$. But then $B_S = V_S$, and this is a convex subring of T. The convex hull of B in V_S is V, and one concludes that $B = V$, a contradiction.

So, it is not always true that topological properties of the prime spectrum alone are sufficient to decide whether a given poring has a convex extension. In general some additional information is required. The discussion of SV-porings and convex subrings provides stronger necessary conditions, e.g., the existence of factor rings that are SV-porings of rank 1. The question will be addressed now whether such additional algebraic information suffices to prove the existence of convex extensions. A first result in this direction is

PROPOSITION 5.6. — *Suppose that (A, A^+) is an SV-poring of rank 1. Then there is a proper convex extension if and only if there is a non-zero divisor that is not a unit.*

Proof. — If there is a proper convex extension (B, B^+) then $B = A_S$ with some multiplicative set $S \subset A$ (*cf.* [22], Theorem 7.2; [34], Corollary 2.6). Since A is a subring of B, the multiplicative set does not contain zero divisors. As the extension is proper, there must be an element in S that is not a unit. Thus, in A there is a non-zero divisor that is not a unit. Conversely, suppose that $s \in A \setminus A^\times$ is not a zero divisor. One may assume that $0 \leqslant s$ (replacing s by s^2). The poring (A_s, A_s^+) contains A and is an SV-poring (Corollary 4.3). It is claimed that A is convex in (A_s, A_s^+): Suppose that $0 \leqslant \frac{a}{s^k} \leqslant b \in A$. Then there is some $l \in \mathbb{N}$ such that $0 \leqslant a \cdot s^l \leqslant b \cdot s^{k+l}$ in A. Convexity of the principle ideal $(b \cdot s^{k+l})$ implies the existence of an element $c \in A$ such that $a \cdot s^l = c \cdot b \cdot s^{k+l}$ (Proposition 4.12). One concludes that $\frac{a}{s^k} = c \cdot b \in A$. $\square$

The equivalence of Proposition 5.6 raises the question whether every SV-poring of rank 1 contains a non-zero divisor that is not a unit. If A is a von Neumann regular ring then the answer is clearly "no". So suppose that $\dim(A) \geqslant 1$. If the set of minimal prime ideals is compact then the answer is clearly "yes". If the set of minimal prime ideals is not compact then one considers the constructible closure X of $\mathrm{Min}(A)$ and the proconstructible set $\mathrm{Gen}(X)$. In $\mathrm{Spec}(A)$, $\mathrm{Gen}(X) = \bigcap_{s \in S} D(s)$, where S is the set of non-zero divisors. One forms the quotient poring (A_S, A_S^+) (the partially ordered total quotient ring of A). This is an SV-poring (Corollary 4.3) of rank 1, and $\dim(A_S) = \dim(\mathrm{Gen}(X)) \geqslant 1$. In this ring every non-zero divisor is a unit, hence there is no proper convex extension. The next example presents a ring where this situation actually occurs. First note that $\mathrm{Min}(A)$ is proconstructible in $\mathrm{Spec}(A)$ if it is compact. (This is either well-known or can be found in [37], Proposition 8.7.) If it is compact then it is a Boolean space (since the constructible topology on $\mathrm{Min}(A)$ coincides with the Zariski topology). The specialization map to the maximal prime spectrum is bijective and continuous, hence a homeomorphism. Therefore, if the space $\mathrm{Max}(A)$ is not Boolean then the minimal prime spectrum cannot be compact. Bob Raphael pointed out to me the following concrete example:

Example 5.7. — Consider the space $X = \beta\mathbb{R}^{\geqslant} \setminus \mathbb{R}^{\geqslant}$, the growth of the nonnegative reals in the Stone-Cech compactification. According to [8], Theorem 14.27, the space is a compact F-space, therefore $C(X)$ is an SV-poring of rank 1 (loc.cit., Theorem 14.25). As the space is connected (loc.cit., 6.10) it is not a Boolean space. It follows that $\mathrm{Min}(C(X))$ is not compact. The total ring of quotients, $\mathrm{Tot}(C(X))$, is a real closed ring (being a quotient ring of the real closed ring $C(X)$, [36], Proposition 12.6). It is also an SV-poring of rank 1 (Corollary 4.3), $\dim\big(\mathrm{Tot}(C(X))\big) \geqslant 1$, and there is no proper convex extension.

The existence of convex extensions of SV-porings of rank 1 is well understood. Convex extensions of real closed rings will be discussed in the remainder of the paper. Note that a convex extension of a real closed ring A is a quotient ring, hence is also real closed ([22], Theorem 7.2; [33], section 5; [34], Corollary 2.6). A Prüfer extension is an epimorphism in the category of rings ([21], Proposition 3.6). Prüfer extensions of reduced rings are reduced (loc.cit., Theorem 5.2, condition (v)). Hence a Prüfer extension of a reduced ring is an epimorphic extension in the category of reduced rings, and Prüfer extensions of real closed rings are real closed ([33], Theorem 3.8). A subring of a real closed ring is a Prüfer subring if and only if it is a convex subring ([22], Theorem 7.2). So, to ask whether a real closed ring has a proper convex extension is the same as to ask whether it is Prüfer closed, i.e., does not have a proper Prüfer extension.

For real closed *domains* there is a very simple answer to the question about the existence of convex extensions:

Proposition 5.8. — *Suppose that A is a real closed domain. There is a proper convex extension $A \subset B$ if and only if A has a factor ring that is a nontrivial valuation ring (i.e., not a field).*

Proof. — If B is a proper convex extension of A, let $m \subset B$ be the maximal ideal. Then $A/m \cap A \subset B/m$ is a proper convex subring, hence a valuation ring, in the real closed field B/m ([34], Corollary 2.7). Conversely, let $p \subset A$ be a prime ideal such that A/p is a valuation ring. Then A/p is convex in the real closed field $\mathrm{qf}(A/p)$ ([35], Theorem 1.1). Pick $0 < s \in A$ such that $0 < s + p \in A/p$ is not a unit. Then $A_s/A_s \cdot p$ is a proper convex extension of A/p. It is claimed that $A \subset A_s$ is a proper convex extension as well. Only convexity must be checked. So suppose that $0 < \frac{a}{s^k} < b \in$

A. Then there is an element $c \in A$ such that $\frac{a}{s^k} \equiv c \pmod{A_s \cdot p}$, say $s^l \cdot (a - c \cdot s^k) \in p$. Since $s \notin p$ it follows that $a - c \cdot s^k \in p$, and therefore $0 < |a - c \cdot s^k|^m < s$ for every $m \geqslant 1$. Now [31], Satz 1, shows that $a - c \cdot s^k \in (s^n)$ for every $n \in \mathbb{N}$. In particular, $a - c \cdot s^k = d \cdot s^k$, and one concludes that $\frac{a}{s^k} = c + d \in A$. $\square$

The answer is only slightly more complicated if one considers arbitrary real closed rings:

THEOREM 5.9. — *For a real closed ring A, the following conditions are equivalent:*

(a) There is a nontrivial continuous section $\tau : \mathrm{Max}(A) \to \mathrm{Spec}(A)$ such that every prime ideal is either a generalization or a specialization of some element of $\mathrm{im}(\tau)$ and that $A/\tau(m)$ is a convex subring of $\mathrm{qf}(A/\tau(m))$ for each $m \in \mathrm{Max}(A)$.

(b) A has a proper convex extension.

Proof. — **(b) ⇒ (a)** is clear from Proposition 5.4 and the remarks immediately thereafter. **(a) ⇒ (b)** Since the section is nontrivial there is an element $m \in \mathrm{Max}(A)$ with $\tau(m) \subset m$. For each $n \in \mathrm{Max}(A)$ there is an element $0 < s_n \in m \setminus \tau(n)$. The open constructible sets $D(s_n)$ cover the compact set $\mathrm{im}(\tau)$. There is a finite subcover $\mathrm{im}(\tau) \subseteq \bigcup_{i=1}^{r} D(s_i)$. If $s = \sum_i s_i$ then $0 < s \in m$ and $\mathrm{im}(\tau) \subseteq D(s)$. It is clear that s is not a zero divisor and not a unit, hence $A \subset A_s$ is a proper extension. It remains to prove that the extension is convex.

Consider $0 \leqslant \frac{a}{s^k} \leqslant b \in A$. For each minimal prime ideal $p \subset A$ the extension $A/p \subset A_s/A_s \cdot p$ is convex (Proposition 5.8), though there may be prime ideals for which the extension is not proper. For each minimal prime ideal there is an element $c_p \in A$ such that $\frac{a}{s^k} \equiv c_p \pmod{A_s \cdot p}$. The closed constructible sets $K_p = \{q \in \mathrm{Spec}(A) \mid a - c_p \cdot s^k \in q\}$ cover the spectrum, hence there is a finite subcover $\mathrm{Spec}(A) = \bigcup_{i=1}^{r} K_i$, $K_i = \{q \in \mathrm{Spec}(A) \mid a - c_i \cdot s^k \in q\}$. Pick such a cover with r as small as possible. If $r = 1$ then $a - c_1 \cdot s^k = 0$ since it belongs to every prime ideal. But then $\frac{a}{s^k} = c_1 \in A$, and the claim has been proved. Now assume that $r > 1$. It will be shown that the cover can be modified to become shorter. This gives a contradiction and finishes the proof.

Let $I_i = \bigcap_{q \in K_i} q$, $I_{12} = \bigcap_{q \in K_1 \cap K_2} q$. Then $I_1 \cap I_2 = \bigcap_{q \in K_1 \cup K_2} q$, and $I_{12} = I_1 + I_2$ ([29], Corollary 15). Now [31], Satz 3 implies that $A/I_1 \cap I_2; A/I_1 \times_{A/I_1+I_2} A/I_2$. Pick $c_{12} \in A$ with $c_{12} - c_1 \in I_1$ and $c_{12} - c_2 \in I_2$. Then $K_1 \cup K_2 \subseteq K_{12} = \{q \in \mathrm{Spec}(A) \mid a - c_{12} \cdot s^k \in q\}$. One may replace the two sets K_1 and K_2 in the cover by the one set K_{12}. This makes the cover shorter, and the proof is finished. □

The theorem does not speak explicitly about SV-porings. However, it is an immediate consequence that the existence of certain factor rings that are SV-porings of rank 1 implies the existence of proper convex extensions:

Corollary 5.10. — *Let A be a real closed ring. Assume that there is an element s that is not a zero divisor and that the factor ring $A/\sqrt{(s)}$ is an SV-poring of rank 1 that has a proper convex extension. Then A has a proper convex extension.*

Proof. — In $A/\sqrt{(s)}$ there is a non-zero divisor t that is not a unit (Proposition 5.6). One checks immediately that the specialization map $\sigma : D(t) = \mathrm{Spec}(A_t) \to \mathrm{Max}(A)$ restricts to a bijective map $\mathrm{Max}(A_t) \to \mathrm{Max}(A)$. Both spaces are compact and the specialization map is continuous, hence it is a homeomorphism. The inverse map $\tau : \mathrm{Max}(A) \to \mathrm{Max}(A_t) \subseteq \mathrm{Spec}(A)$ is the section that is needed to apply Theorem 5.9. □

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