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Analysis on Extended Heisenberg Group

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ABSTRACT. — In this paper we study Markov semigroups generated by Hörmander-Dunkl type operators on Heisenberg group.

RÉSUMÉ. — Dans ce travail, nous étudions des semi-groupes de Markov produit par les opérateurs de type d'Hörmander-Dunkl sur le groupe d'Heisenberg.

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1. Introduction

Given a family of smooth Hörmander fields $\{X_j\}_{j=1,..,n}$ on a differentiable manifold $\mathbb{M}$ one can introduce the corresponding reflection maps $\{\sigma_j\}_{j=1,..,n}$, which by definition satisfy the following conditions

$$X_j(f \circ \sigma_j) = -(X_j f) \circ \sigma_j \qquad (*)$$

and $\sigma_j^2 = id$. Choosing antisymmetric continuous functions, i.e. functions satisfying $x_j \circ \sigma_j = -x_j$, we can introduce the following family of Demazure operators

$$A_j f \equiv \kappa_j \; \frac{f - f \circ \sigma_j}{x_j}$$

with the right hand side well defined whenever $x_j \neq 0$ and otherwise extended by continuity, and $\kappa_j \neq 0$. In this way one can define extended first order operators

$$T_j \equiv X_j + A_j$$

and consider the following T-Laplacian

$$\mathcal{L} \equiv \Sigma_{j=1}^n \; T_j^2$$

In particular one can then study the following Cauchy problem

$$\begin{cases} \partial_t u & = \mathcal{L}u \\ u_{t=0} & = f \end{cases}$$

Below we illustrate how one can implement such a programme in an interesting example provided by the Heisenberg type group. The organisation of the paper is as follows. In Section 2 we briefly recall basic elements of analysis on the Heisenberg group. In Section 3 we introduce a Coxeter group of reflections associated with the Heisenberg generators X, Y, define corresponding Demazure operators and provide fundamentals of algebraic and analytic properties of the underlying theory leading to the associated Markov semigroup generated by the T-Laplacian. In Section 4 we study natural quadratic forms associate to the generator as well as provide L_p setup for our theory; in particular including integration by parts formula for T operators with a natural invariant (with respect to the extended Heisenberg group) measure. In Section 5 we prove basic coercive inequalities necessary to obtain ultracontractivity estimates for the Markov semigroup. In Section 6 we get to the (pointwise off diagonal) heat kernel Gaussian bounds employing suitable adaptation of arguments of [17]; (the classical very nice arguments of [8]-[10] seemed difficult to implement). We conclude in Section 7 with a summary and an outlook. Three Appendices contain some computations and/or some additional facts.

2. Analysis on Heisenberg Group

We consider $\mathbb{H}_1 \simeq \mathbb{R}^3$ with the following group operation

$$w \circ w' \equiv (x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + 2a(yx' - xy')),$$

defined with $a \in \mathbb{R} \setminus 0$. The corresponding left invariant vector fields

$$X = \partial_x + 2ay\partial_z, \qquad Y = \partial_y - 2ax\partial_z$$

satisfy the following basic commutation relation

$$[X, Y] = -4a\partial_z \equiv -4aZ, \qquad [X, Z] = 0 = [Y, Z]$$

With such the fields, we introduce the following Heisenberg laplacian

$$\mathcal{L}_0 \equiv X^2 + Y^2$$

The group is furnished with a dilation operation

$$w \mapsto w_t \equiv (tx, ty, t^2z).$$

with a corresponding generator

$$D \equiv xX + yY + 2zZ$$

In such the structure we have useful classification of the operators introduced above by means of the following relations

$$[X, D] = X, \qquad [Y, D] = Y, \qquad [Z, D] = 2Z, \qquad [\mathcal{L}_0, D] = 2\mathcal{L}_0 \quad (2.1)$$

It is well known for a long time that the Markov semigroup $P_t^{(0)} \equiv e^{t\mathcal{L}_0}$ is ultracontractive and therefore can be represented as an integral operator with strictly positive and normalised (heat) kernel h_t which satisfy a Gaussian sandwich bound, (see e.g. [27] and references therein). More recently such bounds where sharpened, ([4], [23], [13]), with the same Gaussian factor on both sides of the sandwich. As a consequence it was possible to prove the following gradient bounds

$$|\boldsymbol{\nabla} P_t^{(0)} f|^q \leqslant C_t P_t^{(0)} |\boldsymbol{\nabla} f|^q$$

with $\boldsymbol{\nabla} \equiv (X, Y)$ and some constant $C_t \in (0, \infty)$; (see [12] for $q > 1$ and more general groups via stochastic methods, and [23], [3], for Heisenberg group, and [14], [20] for general H-type groups with $q = 1$).

3. Dunkl type operators in Heisenberg Group

We augment the above structure by the following maps on the Heisenberg group

$$\sigma_X w \equiv (-x, y, z - cxy), \qquad \sigma_Y w \equiv (x, -y, z + cxy)$$

with $c \in \mathbb{R} \setminus 0$. These are continuous and bounded maps, (in fact with some constant $C \in (0, \infty)$ one has $\frac{1}{C} d(w) \leqslant d(\sigma.(w)) \leqslant C d(w)$ for any homogeneous norm d, i.e. satisfying $d(w_t) = t d(w)$ for a dilation w_t of w). They do not commute and satisfy

$$\sigma_X \circ \sigma_X = id = \sigma_Y \circ \sigma_Y$$

Thus they generate a Coxeter group of infinite order. It is tempting to call σ_X and σ_Y the reflection operations, but it appears to be justified only when

$$\boxed{c = 4a}$$

when one has the following relations

$$X(f \circ \sigma_X) = -(Xf) \circ \sigma_X, \qquad Y(f \circ \sigma_Y) = -(Yf) \circ \sigma_Y$$

(Remark that each of these relations can be implemented non-uniquely, but this choice has certain advantages and we stick to it later on.) Next we introduce the following operators

$$A_X f \equiv \kappa \, \frac{f - f \circ \sigma_X}{x}, \qquad A_Y f \equiv \kappa \, \frac{f - f \circ \sigma_Y}{y}$$

for $x, y \neq 0$, with some $\kappa \in (0, \infty)$. We set $\mathbf{A} \equiv (A_X, A_Y)$. Using this definition one immediately sees that

$$A_X^2 = 0 = A_Y^2 \qquad (3.1)$$

Thus each of them generate a group

$$e^{itA.} f = (1 + itA.)f, \qquad e^{itA.} e^{isA.} = (1 + i(s + t)A.) = e^{i(s+t)A.}$$

These generators are homogeneous of order one, that is one has the following property.

Lemma 3.1. —

$$[A_X, D] = A_X, \qquad [A_Y, D] = A_Y. \qquad (3.2)$$

(For the proof see Appendix.) Next we notice that

$$\lim_{x \to 0} A_X f(x, y, z) = 2\kappa \, X f(0, y, z), \qquad \lim_{y \to 0} A_Y f(x, y, z) = 2\kappa \, Y f(x, 0, z)$$

Thus in particular one can see that they are unbounded operators. More generally, using suitable interpolation and fundamental theorem of calculus, we have

LEMMA 3.2. —

$$A_X f(w) = 2\kappa \int_0^1 ds \, (Xf)(\gamma_{s,X}^{(w)}), \quad A_Y f(w) = 2\kappa \int_0^1 ds \, (Yf)(\gamma_{s,Y}^{(w)}) \quad (3.3)$$

with

$$\gamma_{s,X}^{(w)} \equiv ((1-2s)x, y, z - 2sx \cdot 2ay), \qquad \gamma_{s,Y}^{(w)} \equiv (x, (1-2s)y, z + 2sy \cdot 2ax)$$

As a consequence, in $\mathbb{L}_2(\lambda)$ with the (reflection invariant) Haar measure λ (which for the Heisenberg group coincides with Lebesgue measure), we obtain

PROPOSITION 3.3. —

$$||A_X f||_{\mathbb{L}_2(\lambda)} \leqslant 4|\kappa| \cdot ||Xf||_{\mathbb{L}_2(\lambda)}, \qquad ||A_Y f||_{\mathbb{L}_2(\lambda)} \leqslant 4|\kappa| \cdot ||Yf||_{\mathbb{L}_2(\lambda)}$$

and

$$||A_X f||_\infty \leqslant 2|\kappa| \cdot ||Xf||_\infty, \qquad ||A_Y f||_\infty \leqslant 2|\kappa| \cdot ||Yf||_\infty$$

Proof. — We have

$$||A_X f||_{\mathbb{L}_2(\lambda)} = ||2\kappa \int_0^1 ds \, Xf(\gamma_{s,X}^{(w)})||_{\mathbb{L}_2(\lambda)} \leqslant 2|\kappa| \int_0^1 ds ||Xf(\gamma_{s,X}^{(w)})||_{\mathbb{L}_2(\lambda)}$$

Changing the integration variable $w \to \gamma_{s,X}^{(w)}$, we have

$$||Xf(\gamma_{s,X}^{(w)})||_{\mathbb{L}_2(\lambda)} = \frac{1}{\sqrt{|1-2s|}} ||Xf||_{\mathbb{L}_2(\lambda)}$$

Hence

$$||A_X f||_{\mathbb{L}_2(\lambda)} \leqslant 4|\kappa| \, ||Xf||_{\mathbb{L}_2(\lambda)}$$

and similarly in case of Y. $\qquad \square$

Thus the following homogeneous operators of order one are well defined on a set of differentiable functions

$$T_X \equiv X + A_X, \qquad T_Y \equiv Y + A_Y.$$

We set $\mathbf{T} \equiv (T_X, T_Y)$. Since one has

$$A_X(f \circ \sigma_X)(w) = -A_X f(w) = -(A_X f)(\sigma_X w)$$

and

$$A_Y(f \circ \sigma_Y)(w) = -A_Y f(w) = -(A_Y f)(\sigma_Y w) ,$$

we obtain the following property.

Proposition 3.4. —

$$T_X(f \circ \sigma_X)(w) = -(T_X f)(\sigma_X w) , \qquad T_Y(f \circ \sigma_Y)(w) = -(T_Y f)(\sigma_Y w)$$

and

$$[T_X, D] = T_X, \qquad [T_Y, D] = T_Y$$

We remark that the homogeneity property follows from (2.1) and (3.1).

In particular the above result means that σ_X and σ_Y are reflection operations in our more general framework. From Proposition 3.3 we have the following corollary.

Proposition 3.5. — *Let d be the Carnot-Caratheodory distance for* $\mathbf{\nabla}$, *i.e. distance satisfying the eikonal equation*

$$|\mathbf{\nabla} d| = 1.$$

Then

$$(1 - 4\kappa) \leqslant |\mathbf{T} d| \leqslant (1 + 4\kappa)$$

where $|\mathbf{T} f|^2 \equiv |T_X f|^2 + |T_Y f|^2$.

As for the algebraic properties of $T.$'s we notice the following

Proposition 3.6. —

$$[T_X, T_Y] f = -4aZf - 4a\kappa \ (Zf) \circ \sigma_X - 4a\kappa \ (Zf) \circ \sigma_Y + [A_X, A_Y] f$$

with

$$[A_X, A_Y] f = \kappa^2 \ \frac{f \circ \sigma_Y \circ \sigma_X - f \circ \sigma_X \circ \sigma_Y}{yx}$$

and

$$[T_X, Z] \ = \ 0 \ = \ [T_Y, Z]$$

One can check that in the limit $x, y \to 0$, also $[A_X, A_Y]$ points out into the Z direction. Thus generally we have similar situation as in the Hörmader theory. In a conclusion to the above, it is now time to introduce the following T-Laplacian.

$$\mathcal{L} \equiv T_X^2 + T_Y^2$$

Let $P_t \equiv e^{t\mathcal{L}}$ denote the corresponding semigroup (which for a moment is only well defined on polynomials).

We have the following property.

THEOREM 3.7. — $\mathcal{L}$ with a domain $\mathcal{C}_o^2(\mathbb{H}_1)$ functions satisfies minimum principle, i.e.

$$f(w) = \min f \implies \mathcal{L}f(w) \geqslant 0$$

Remark. — See e.g. [26] and references therein for a commutative case.

Proof. — Evidently $\mathcal{L}$ vanishes on constants. We show that it also satisfies minimum principle. Suppose that $w_0 = (x_0, y_0, z_0)$ is the minimum point of a function $f \in \mathcal{C}^2$, we will show that

$$\mathcal{L}f(w_0) = (X^2 + Y^2)f(w_0) + \{X, A_X\}f(w_0) + \{Y, A_Y\}f(w_0) \geqslant 0$$

Since one has

$$(X^2 + Y^2)f(w_0) \geqslant 0,$$

we need only to prove the positivity for second part containing the anti-commutators. If $x_0, y_0 \neq 0$, then for the minimum point w_0, we also have

$$\{X, A_X\}f(w_0) = \frac{1}{x_0} 2\kappa X f(w_0) - \frac{1}{x_0} A_X f(w_0) = \frac{\kappa}{x_0^2} \left(f \circ \sigma_X(w_0) - f(w_0) \right) \geqslant 0$$

and similarly in direction y. Suppose next that $x_0 = 0$, (similar arguments can be given in direction y). Using the representation of A_X from Lemma 3.1, (for w with $x \neq 0$)), we have

$$\begin{aligned}
\{X, A_X\}f(w) &= \frac{2}{x}\kappa X f(w) - \frac{1}{x} A_X f(w) \\
&= \frac{1}{x} 2\kappa \left(X f(w) - \int_0^1 ds (Xf)(\gamma_{s,X}^w) \right) \\
&= 4\kappa \int_0^1 ds \int_0^s ds' (X^2 f)(\gamma_{s',X}^w)
\end{aligned}$$

Now passing to the limit with $x \to x_0 = 0$, we obtain

$$\{X, A_X\} f(w) \to 2\kappa X^2 f(w_0) \geqslant 0.$$

□

Suppose the pre-generator $\mathcal{L}$ extends to the Markov generator (denoted later on by the same symbol), and let $P_t \equiv e^{t\mathcal{L}}$ denotes the corresponding semigroup on a space including bounded (uniformly) continuous functions. Then the above property implies that P_t is a Markovian semigroup, i.e. it preserves constants and positivity.

We recall that in Proposition 3.4 one has

$$[T_X, D] = T_X, \qquad [T_Y, D] = T_Y$$

Setting $S_\tau f(x, y, z) \equiv e^{\tau D} f(x, y, z) = f(e^\tau x, e^\tau y, e^{2\tau} z)$, $\tau \geqslant 0$, we get

PROPOSITION 3.8. —
$$[\mathcal{L}, D] = 2\mathcal{L}$$

and so

$$P_t S_\tau = S_\tau P_{e^{2\tau} t} \tag{3.4}$$

(For the second relation (3.4) see e.g. [3].)

4. Square of the T – Form and Quadratic Form Bounds

Define

$$\mathbf{\Gamma}_1(f) \equiv \frac{1}{2} (\mathcal{L} f^2 - 2f \mathcal{L} f)$$

By direct computations we get

$$\mathbf{\Gamma}_1(f) = |\boldsymbol{\nabla} f|^2 + \frac{1}{2\kappa} (\mathbf{A} f)^2$$

where we have set $(\mathbf{A} f)^2 \equiv |A_X f|^2 + |A_Y f|^2$.

Next we introduce the following square of the **T** form as follows

$$|\mathbf{T} f|^2 \equiv |T_X f|^2 + |T_Y f|^2.$$

PROPOSITION 4.1. —

$$|\mathbf{T}f|^2 \leqslant 2\max(1, 2\kappa)\mathbf{\Gamma}_1(f)$$

and one has

$$\mathbf{\Gamma}_1(f) \leqslant (2 + 2\kappa)\left(|\mathbf{T}f|^2 + \frac{1}{2\kappa}(\mathbf{A}f)^2\right)$$

Proof. — Note that

$$(T_X f)^2 = (Xf + A_X f)^2 \leqslant 2|Xf|^2 + 2(A_X f)^2 \leqslant 2|Xf|^2 + 4\kappa \cdot \frac{1}{2\kappa}(A_X f)^2$$

$$\leqslant 2\max(1, 2\kappa)\left(|Xf|^2 + \frac{1}{2\kappa}(A_X f)^2\right)$$

and similarly for the case of Y. Hence

$$|\mathbf{T}f|^2 \equiv |T_X f|^2 + |T_Y f|^2 \leqslant 2\max(1, 2\kappa)\mathbf{\Gamma}_1(f)$$

The second statement follows from the following relation

$$\mathbf{\Gamma}_1(f) = |\mathbf{T}f|^2 - 2T_X f \cdot A_X f - 2T_Y f \cdot A_Y f + (1 + \frac{1}{2\kappa})|\mathbf{A}f|^2$$

This ends the proof of the proposition. $\quad\square$

Using this property and Proposition 3.3, we get the following bounds.

PROPOSITION 4.2. —

$$(1-\varepsilon)(1-\frac{16\kappa^2}{\varepsilon})\|\,|\boldsymbol{\nabla}f|^2\,\|_{\mathbb{L}_1(\lambda)} \leqslant \|\,|\mathbf{T}f|^2\,\|_{\mathbb{L}_1(\lambda)} \leqslant 2\max(1, 2\kappa)\|\,\mathbf{\Gamma}_1(f)\,\|_{\mathbb{L}_1(\lambda)}$$

Proof. — The first inequality follows using

$$(T_X f)^2 = (Xf + A_X f)^2 \geqslant (1 - \varepsilon)|Xf|^2 + (1 - \frac{1}{\varepsilon})(A_X f)^2$$

and the following bound from Proposition 3.3

$$\|(A_X f)^2\|_{\mathbb{L}_1(\lambda)} \leqslant 16\kappa^2\|\,|Xf|^2\,\|_{\mathbb{L}_1(\lambda)}$$

with the similar one for Y. $\quad\square$

For later purposes we need the following result.

THEOREM 4.3 (*Integration by Parts Lemma*). — *For Lipschitz continuous functions f, g and σ_X, σ_Y invariant measure $d\nu \equiv \rho d\lambda$, one has*

$$\int A_X(g) f d\nu = \int g \kappa \frac{f + f \circ \sigma_X}{x} d\nu = \int g x^{-1} A_X(fx) d\nu \equiv \int g A_X^*(f) d\nu$$

and similarly for Y. Moreover, if $\rho(w) = |x|^{2\kappa}|y|^{2\kappa}$, with $\kappa \in (-\frac{1}{2}, \infty)$, then the following integration by parts formula holds

$$\boxed{\int T_X(g) f d\nu = -\int g T_X(f) d\nu} \tag{IP}$$

and similarly for T_Y.

Remark 4.4. — We remark that one gets similar properties choosing different value for κ in case of T_X and T_Y. We also notice that there are other reflection invariant measures, (for which the last formula for integration by parts may fail), which can be defined as nonnegative function of (real part) of quantities $\eta_\xi + \eta_{\xi^{-1}}$, where

$$\eta_\xi \equiv \sum_{n \in \mathbb{Z}} \xi^n \, e^{ip(z + \varepsilon nxy)} \rho(x, y, z + \varepsilon nxy)$$

defined with $p, \varepsilon \in \mathbb{R}$ and $\xi \in \mathbb{C}, |\xi| = 1$ with a conditions that $\frac{4a}{\varepsilon} \in \mathbb{Z}$ and $\xi^{\frac{4a}{\varepsilon}} = 1$, and a suitable function ρ.

Proof. — By shifting a differentiable function g by a constant $g(0)$ if necessary, we can and do assume that it vanishes at $x = 0$. By change of variables of integration by the map σ_X, we have

$$\int (g \circ \sigma_X) \frac{1}{x} f \rho d\lambda = -\int g \frac{1}{x} (f \circ \sigma_X) \rho d\lambda.$$

Hence we have

$$\int A_X(g) f \rho d\lambda = \int g \frac{f + f \circ \sigma_X}{x} \rho d\lambda$$

Combining this relation with the formula for integration by parts for the field X with the measure $d\nu = |x|^{2\kappa}|y|^{2\kappa} d\lambda$, one obtains the integration by parts formula for T_X. The proof for the case of T_Y is similar. $\square$

Using the integration by parts formula and the definition of $\boldsymbol{\Gamma}_1$, we obtain the following

COROLLARY 4.5. — *Let $d\nu \equiv x^{2\kappa}y^{2\kappa}d\lambda$. Then*

$$\int f\mathcal{L}(g)d\nu = -\int \mathbf{T}f \cdot \mathbf{T}g d\nu$$

and hence

$$\int f\mathcal{L}(f)d\nu = -\int \mathbf{\Gamma}_1(f)d\nu$$

Proof. — The first result follows from integration by parts (IP) formula. It implies that $\mathcal{L}$ is symmetric in $\mathbb{L}_2(d\nu)$ and hence ν is invariant measure for this generator. This together with the definition of $\mathbf{\Gamma}_1$ yields the second relation. □

Since

$$\mathbf{\Gamma}_1 = |\boldsymbol{\nabla}f|^2 + \frac{1}{2\kappa}\left(\mathbf{A}f\right)^2$$

one can show directly that the quadratic form

$$\mathcal{E}(f) \equiv \int \mathbf{\Gamma}_1(f)d\nu$$

satisfies for any normal contraction S, (that is a real function vanishing at zero and having Lipschitz norm equal to one), the following bound

$$\mathcal{E}(S(f)) \leqslant \mathcal{E}(f).$$

This is clear for the first part of the form involving subgradient. To see that for the second part, using the fact that the normal contraction has by definition the Lipschitz norm equal to one, we get

$$
\begin{aligned}
\frac{1}{2\kappa}\left(\mathbf{A}_X S(f)\right)^2 &= \frac{\kappa}{2x^2}\left(S(f) - S(f)\circ\sigma_X\right)^2 \\
&= \frac{\kappa}{2x^2}\left(S(f) - S(f\circ\sigma_X)\right)^2 \leqslant \frac{\kappa}{2x^2}\left(f - f\circ\sigma_X\right)^2
\end{aligned}
$$

and similarly in direction y. Thus, using the theory of Dirichlet forms [15] we arrive at the following result.

THEOREM 4.6. — *The closure of the quadratic form*

$$\mathcal{E}(f) \equiv \int \mathbf{\Gamma}_1(f)d\nu$$

defines a (self-adjoint) Markov generator denoted later on by the symbol $\mathcal{L}$ with the corresponding Markov semigroup $P_t = e^{t\mathcal{L}}$.

5. Coercive Bounds

We recall that the following Sobolev inequality holds

$$||f||^2_{\mathbb{L}_q(\lambda)} \leqslant a \int |\boldsymbol{\nabla} f|^2 \, d\lambda + b \int f^2 \, d\lambda$$

with some $q > 2$ and $a, b \in (0, \infty)$ independent of f. Hence, by standard arguments (see e.g. [18]), we have

$$\int f^2 \log \frac{f^2}{\int f^2 d\lambda} d\lambda \leqslant \varepsilon \int |\boldsymbol{\nabla} f|^2 \, d\lambda + \widetilde{C}_\varepsilon \int f^2 d\lambda \qquad (5.1)$$

for any $\varepsilon \in (0, 1)$ with

$$\widetilde{C}_\varepsilon \equiv \frac{q}{2(q-2)} (\log \frac{1}{\varepsilon} + \log \frac{q}{2(q-2)} - 1) + \varepsilon b$$

We prove the following generalisation of this fact.

THEOREM 5.1. — *For any $\varepsilon \in (0, 1)$*

$$\int f^2 \log \frac{f^2}{||f||^2_{\mathbb{L}_2(\nu)}} d\nu \leqslant \varepsilon \int \mathbf{\Gamma}_1(f) \, d\nu + C_\varepsilon \int f^2 d\nu. \qquad (5.2)$$

where $C_\varepsilon \equiv C_1 \log \frac{1}{\varepsilon} + C_2$ with some constants $C_1, C_2 \in (0, \infty)$ independent of f.

Proof. — Applying (5.1) to $f|x|^\kappa |y|^\kappa$, we obtain

$$\int f^2 \log \frac{f^2}{||f||^2_{\mathbb{L}_2(\nu)}} d\nu + \int f^2 \log(|x|^{2\kappa} |y|^{2\kappa}) d\nu \qquad (5.3)$$

$$\leqslant \varepsilon \int |\boldsymbol{\nabla}(f \cdot |x|^\kappa |y|^\kappa)|^2 \, d\lambda + \widetilde{C}_\varepsilon \int f^2 d\nu$$

with $d\nu \equiv |x|^{2\kappa} |y|^{2\kappa} d\lambda$. Hence

$$\int f^2 \log \frac{f^2}{||f||^2_{\mathbb{L}_2(\nu)}} d\nu \leqslant \varepsilon \int |\boldsymbol{\nabla} f|^2 \, d\nu$$

$$+ 2\varepsilon \int \left(f\kappa |x|^{-1} Xf + f\kappa |y|^{-1} Yf\right) \, d\nu$$

$$+ \varepsilon\kappa^2 \int \left(f^2 |x|^{-2} + f^2 |y|^{-2}\right) \, d\nu \qquad (5.4)$$

$$+ \kappa \int f^2 \left(\chi_{\{|x|<1\}} \log \frac{1}{|x|^2} + \chi_{\{|y|<1\}} \log \frac{1}{|y|^2}\right) d\nu$$

$$+ \widetilde{C}_\varepsilon \int f^2 d\nu.$$

To bound the last but one term on the r.h.s., we use the inequality

$$\log t \leqslant \varepsilon t + c_\varepsilon$$

with $\varepsilon \in (0,1)$ and $c_\varepsilon \equiv \log \frac{1}{\varepsilon} - 1$, to get

$$\kappa \int f^2 \left(\chi_{\{|x|<1\}} \log \frac{1}{|x|^2} + \chi_{\{|y|<1\}} \log \frac{1}{|y|^2} \right) d\nu$$
$$\leqslant \kappa \varepsilon \int \left(f^2 |x|^{-2} + f^2 |y|^{-2} \right) \, d\nu + c_\varepsilon \kappa \int f^2 d\nu$$

Using this and (5.4), we obtain

$$\int f^2 \log \frac{f^2}{\|f\|_{\mathbb{L}_2(\nu)}^2} d\nu \leqslant \varepsilon \int |\boldsymbol{\nabla} f|^2 \, d\nu$$
$$+ \; \varepsilon(\kappa^2 + \frac{1}{2}\kappa) \int \left(f^2 |x|^{-2} + f^2 |y|^{-2} \right) \, d\nu \quad (5.5)$$
$$+ \; \hat{C}_\varepsilon \int f^2 d\nu$$

with $\hat{C}_\varepsilon \equiv \widetilde{C}_{\varepsilon/2} + c_{\varepsilon/2}\kappa$. Now we complete the estimates using the following lemma ([22]).

LEMMA 5.2. — $\kappa \in (0,\infty), \kappa \neq \frac{1}{2}$. *There exist constants* $\hat{a}, \hat{b} \in (0,\infty)$ *such that*

$$\int \left(g^2 |x|^{-2} + g^2 |y|^{-2} \right) \, d\nu \leqslant \hat{a} \int \boldsymbol{\Gamma}_1(g) d\nu + \hat{b} \int g^2 d\nu$$

for any function g for which the right hand side is well defined.

Using (5.5) together with Lemma 5.2, we obtain

$$\int f^2 \log \frac{f^2}{\|f\|_{\mathbb{L}_2(\nu)}^2} d\nu \leqslant \varepsilon \int \boldsymbol{\Gamma}_1(f) \, d\nu + C_\varepsilon \int f^2 d\nu. \qquad (5.6)$$

with

$$C_\varepsilon \equiv \hat{C}(\varepsilon[1 + (\kappa^2 + \frac{1}{2}\kappa)\hat{a}]^{-1}) + \varepsilon(\kappa^2 + \frac{1}{2}\kappa)[1 + (\kappa^2 + \frac{1}{2}\kappa)\hat{a}]^{-1}.$$

From this and the definition of $\boldsymbol{\Gamma}_1$ the bound (5.2) follows. □

Proof Lemma 5.2. — We mention that for $\kappa > \frac{1}{2}$ one can use the following arguments based on simple integration by parts

$$\int g^2 |x|^{-2} d\nu_\kappa \equiv \int g^2 |x|^{2\kappa-2} |y|^{2\kappa} d\lambda = \frac{1}{2\kappa-1} \int g^2 X |x|^{2\kappa-1} |y|^{2\kappa} d\lambda$$

$$= -\frac{2}{2\kappa-1} \int g|x|^{-1} X g |x|^{2\kappa} |y|^{2\kappa} d\lambda \qquad (5.7)$$

$$\leqslant \frac{2}{2\kappa-1} a \int |Xg|^2 d\nu_\kappa + \frac{2}{a(2\kappa-1)} \int g^2 |x|^{-2} d\nu_\kappa$$

with a constant $a > 0$ and $d\nu_\kappa \equiv |x|^{2\kappa} |y|^{2\kappa} d\lambda$. From this, a choice of $a > \frac{2}{(2\kappa-1)}$ plus similar arguments for the case with factor $|y|^{-2}$, deliver the desired bound.

To consider the case $\kappa \in (0, \frac{1}{2})$, we also note that on the set $\{x > 0\}$ we have the following property

$$T_X(x^{-1}) = Xx^{-1} + \kappa \frac{x^{-1} - x^{-1} \circ \sigma_X}{x} = (2\kappa - 1)x^{-2}$$

and similarly for $T_Y(y^{-1})$ in $\{y > 0\}$, (as well as in other quadrants modulo change of sign of the variable). Thus for $2\kappa \neq 1$, we have

$$\int \left(g^2 |x|^{-2}\right) d\nu_\kappa = \frac{1}{(1-2\kappa)} \int \left(g^2(-T_X x^{-1})\right) d\nu_\kappa$$

$$= \frac{1}{(1-2\kappa)} \int \left((T_X g^2) x^{-1}\right) d\nu_\kappa$$

Since $\qquad T_X g^2 = 2g T_X g - (g - g \circ \sigma_X) A_X(g) \leqslant 2g T_X g$

for $\kappa \in (0, \frac{1}{2})$, we get with some $a \in (0, \infty)$

$$\int \left(g^2 |x|^{-2}\right) d\nu_\kappa = \frac{1}{(1-2\kappa)} \int \left((T_X g^2) x^{-1}\right) d\nu_\kappa$$

$$\leqslant \frac{1}{(1-2\kappa)} a \int |T_X g|^2 d\nu_\kappa + \frac{1}{(1-2\kappa)} a^{-1} \int g^2 |x|^{-2} d\nu_\kappa$$

$$+ \frac{1}{(1-2\kappa)} a \int |A_X g|^2 d\nu_\kappa + \frac{1}{(1-2\kappa)} 2a^{-1} \int g^2 |x|^{-2} d\nu_\kappa$$

Thus, using the definition of $\Gamma_{1,X}(f) \equiv |Xf|^2 + \frac{1}{2\kappa} |A_X f|^2$, we have

$$\int \left(g^2 |x|^{-2}\right) d\nu_\kappa \leqslant \frac{1}{(1-2\kappa)} 2a \max\left(1, \frac{1}{2\kappa}\right) \int \Gamma_{1,X}(g) \, d\nu_\kappa$$

$$+ \frac{1}{(1-2\kappa)} 3a^{-1} \int g^2 |x|^{-2} \, d\nu_\kappa$$

whence choosing $\frac{1}{(1-2\kappa)} 3a^{-1} < 1$, we arrive at

$$\int \left(g^2 |x|^{-2}\right)\, d\nu_\kappa \leqslant Const \int \Gamma_{1,X}(g)\, d\nu_\kappa$$

with $Const \equiv \frac{1}{(1-2\kappa)} 2a \max(1, \frac{1}{2\kappa})[1 - \frac{1}{(1-2\kappa)} 3a^{-1}]^{-1}$. Similar arguments for y direction and all quadrants complete the proof.

Remark. — To get the bound for small $\kappa > 0$, we recall first that the following Hardy type inequality for half spaces in the Heisenberg group holds with the Haar (Lebesgue) measure λ, (see Theorem 1.1 in [22]).

$$\int \left(g^2 |x|^{-2}\right)\, d\lambda \leqslant 4 \int |\boldsymbol{\nabla} g|^2 d\lambda.$$

and similarly with a singular factor $|y|^{-2}$. Replacing the function g by $g|x|^\kappa |y|^\kappa$, simple arguments (involving integration by parts) yield

$$\int \left(g^2 |x|^{-2} + g^2 |y|^{-2}\right) d\nu_\kappa \leqslant 8 \int |\boldsymbol{\nabla} g|^2 d\nu_\kappa + 8\kappa(1-\kappa) \int \left(g^2 |x|^{-2} + g^2 |y|^{-2}\right) d\nu_\kappa \tag{5.8}$$

This for $8\kappa(1 - \kappa) < 1$ implies the desired result with a constant $\hat{a} \leqslant 8(1 - 8\kappa(1 - \kappa))^{-1}$.

Next suppose that (5.8) holds for some $\kappa_0 \in (0, \infty)$. Applying our current assumption to a function $g|x|^{\kappa' - \kappa_0} |y|^{\kappa' - \kappa_0}$, we get

$$\int \left(g^2 |x|^{-2} + g^2 |y|^{-2}\right)\, d\nu_{\kappa'} \tag{5.9}$$

$$\leqslant 2\hat{a} \int |\boldsymbol{\nabla} g|^2 d\nu_{\kappa'} + 2\hat{a}(\kappa' - \kappa_0)^2 \int \left(g^2 |x|^{-2} + g^2 |y|^{-2}\right)\, d\nu_{\kappa'}$$

with $b_0 \equiv b_0(\kappa_0)$, which implies the following bound

$$\int \left(g^2 |x|^{-2} + g^2 |y|^{-2}\right)\, d\nu_{\kappa'} \leqslant b_{\kappa'} \int |\boldsymbol{\nabla} g|^2 d\nu_{\kappa'} \tag{5.10}$$

where $b_{\kappa'} \equiv 2b_0(1 - 2b_0(\kappa' - \kappa_0)^2)^{-1}$, provided that $2b_0(\kappa' - \kappa_0)^2 < 1$. This perturbative procedure shows that the set of κ's for which a statement with the gradient square on the right hand side holds is open.

Using Theorem 5.1 together with the fact that

$$\int \boldsymbol{\Gamma}_1(f) d\nu = -\int f \mathcal{L} f d\nu$$

is a Dirichlet form, one can now apply arguments based on Gross integration lemma ([16], [8], [11]) to obtain the following corollary.

THEOREM 5.3. — *The semigroup $P_t \equiv e^{t\mathcal{L}}$ is ultracontractive, that is for $t > 0$ there exists $\theta_t \in (0, \infty)$ such that*

$$||P_t f||_{\mathbb{L}_\infty(\nu)} \leqslant \theta_t ||f||_{\mathbb{L}_1(\nu)} \tag{5.11}$$

This means that the semigroup P_t has a kernel h_t with respect to the measure ν satisfying with some $\theta_t \in (0, \infty)$

$$h_t(w, \tilde{w}) \leqslant \theta_t$$

6. Heat Kernel Bounds

We begin from recalling the following arguments due to Aronson [1]. Suppose

$$\partial_t h = \mathcal{L} h.$$

Let ψ be a function satisfying

$$\partial_t \psi + \frac{1}{2} |\mathbf{T}\psi|^2 \leqslant 0$$

Then we have

$$\frac{d}{dt} \int h^2 e^\psi d\nu = \int \left(h^2 \partial_t \psi + 2h\mathcal{L}h \right) e^\psi d\nu$$

$$\leqslant \int \left(-\frac{1}{2} h^2 |\mathbf{T}\psi|^2 e^\psi - \mathbf{T}h \cdot \mathbf{T}(he^\psi) \right) d\nu$$

In our case, since $r^2 \equiv x^2 + y^2$ is invariant with respect to all reflections, one can see that the right hand side can be made nonpositive by a choice

$$\psi = \frac{r^2}{2(\delta + t)}$$

with $\delta \in (0, \infty)$. That is one gets the following simple integrated bound

PROPOSITION 6.1. —

$$\sup_{t > 0} \int h^2 e^{\frac{r^2}{2(\delta + t)}} d\nu < \infty \tag{6.1}$$

To improve this bound in other directions, we note the following property.

PROPOSITION 6.2. — *For any $n \in \mathbb{N}$, we have*

$$\int h_{t=1}^2 d^n \, d\nu < \infty \qquad (6.2)$$

Proof. — First of all we note that for a smooth cutoff $d_c \equiv d\chi(d/L)$ of d, with smooth monotone function χ vanishing inside a unit ball and equal to one outside a ball of radius 2 and some large constant $L > 1$, we have

$$\partial_t \int h_t d_c^n \, d\nu = \int \mathcal{L}(h_t) d_c^n \, d\nu = \int h_t (T^2 d_c^n) d\nu$$

Next we observe that

$$X^2 d_c^n = n(n-1)d_c^{n-2} \left(\chi + \frac{d}{L}\chi' \right)^2 |Xd|^2 \qquad (6.3)$$

$$+ nd_c^{n-1} \left(\left(\frac{2}{L}\chi' + \frac{d}{L^2}\chi'' \right) |Xd|^2 + \left(\chi + \frac{d}{L}\chi' \right) X^2 d \right)$$

By arguments used in the proof of Theorem 6.1 in [19], we have with some $\tilde{K} \in (0, \infty)$

$$X^2 d, Y^2 d \leqslant \tilde{K}.$$

Thus using relation (6.3), we obtain with some constants $\tilde{a}, \tilde{b} \in (0, \infty)$

$$X^2 d_c^n, Y^2 d_c \leqslant n(n-1)\tilde{a} \, d_c^{n-2} + n\tilde{b} \, d_c^{n-1} \qquad (6.4)$$

Since

$$T^2 d_c^n = \Delta d_c^n + 4\kappa \int_0^1 ds \int_0^s d\tau \left((X^2 d_c^n)(\gamma_{\tau,X}) + (Y^2 d_c^n)(\gamma_{\tau,Y}) \right)$$

and

$$d_c(\gamma_{\tau,X}) = d_c(w) + 2\kappa \, x \int_0^\tau ds \left((\chi(d) + \chi'(d))Xd \right)(\gamma_{s,X}) \leqslant d_c(w) + 2\kappa\tau \, |x|$$

$$\leqslant (1 + 2\kappa\tau)d_c(w)$$

we get

$$T^2 d_c^n \leqslant n(n-1)a(1+2\kappa)^{n-2} \, d_c^{n-2} + nb(1+2\kappa)^{n-1} \, d_c^{n-1}$$

with some constants $a, b \in (0, \infty)$. This implies the following inductive inequality

$$\partial_t \int h_t d_c^n \, d\nu \leqslant n(n-1)a(1+2\kappa)^{n-2} \int h_t d_c^{n-2} \, d\nu + nb(1+2\kappa)^{n-1} \int h_t d_c^{n-1} \, d\nu.$$

Similar computations for $n = 1$ show that

$$\partial_t \int h_t d_c d\nu < Const < \infty$$

Using the fact that we also have

$$\int h_t d_c d\nu = const < \infty$$

by inductive arguments, one can show that for any $n \in \mathbb{N}$ and $t \in (0, \infty)$

$$\int h_t d_c^n d\nu = const < \infty$$

This implies the statement of the proposition.   □

We conclude with the following result.

COROLLARY 6.3. — *For any $\delta, \beta \in (0, \infty)$ there exists a positive function $C_t(\cdot)$ such that*

$$h_t(w, w') \leqslant C_t(w)C_t(w')\, e^{-\left(\frac{\alpha}{2(\delta+t)} r^2(w,w') + \beta \log(1 + \frac{1}{\sqrt{t}} d(w,w'))\right)} \qquad (6.5)$$

Proof. — By homogeneity of the generator $\mathcal{L}$ it is sufficient to show the bound at $t = 1$. From Proposition 6.2, for any $\beta \in (0, \infty)$ and $t \in (0, \infty)$ we have

$$\int h_t^q(w, \tilde{w}) e^{\beta \log(1 + d(w,\tilde{w}))} \nu(d\tilde{w}) < \infty$$

for any $q \in [1, \infty)$, since h_t is bounded. Hence via Hölder inequality, with sufficiently small $\alpha > 0$, we also have

$$C_t^2(w) \equiv \int h_t^2(w, \tilde{w}) e^{\alpha r^2(w,\tilde{w}) + \beta \log(1 + d(w,\tilde{w}))} \nu(d\tilde{w}) < \infty \qquad (6.6)$$

Now we can follow an idea of [17] to get an off diagonal heat kernel bound, as follows. We note that because $\log(1 + d(w, \tilde{w}))$ is a metric, we have

$$\frac{\alpha}{2} r^2(w, w') + \beta \log(1 + d(w, w'))$$
$$\leqslant \alpha r^2(w, \tilde{w}) + \beta \log(1 + d(w, \tilde{w})) + \alpha r^2(\tilde{w}, w') + \beta \log(1 + d(\tilde{w}, w')).$$

Hence using Chapman-Kolmogorov property and Hölder inequality, we get

$$h_t(w, w') = \int h_{\frac{t}{2}}(w, \tilde{w}) h_{\frac{t}{2}}(\tilde{w}, w') \nu(d\tilde{w}) \tag{6.7}$$

$$= \int \left(h_{\frac{t}{2}}(w, \tilde{w}) e^{\alpha r^2(w,\tilde{w}) + \beta \log(1 + d(w,\tilde{w}))} \right)$$

$$\left(h_{\frac{t}{2}}(\tilde{w}, w') e^{\alpha r^2(\tilde{w},w') + \beta \log(1 + d(\tilde{w},w'))} \right) \nu(d\tilde{w}) \times e^{-\left(\frac{\alpha}{2} r^2(w,w') + \beta \log(1 + d(w,w')) \right)}$$

$$\leqslant C_t(w) C_t(w') \, e^{-\left(\frac{\alpha}{2} r^2(w,w') + \beta \log(1 + d(w,w')) \right)}$$

$\square$

Instead of directly dealing with expression involving the control distance, next we estimate the moments of the z variable. Later we will show that one can combine that with the Gaussian bound in the Euclidean direction to arrive at the Gaussian bounds with respect to a homogeneous metric on the group.

PROPOSITION 6.4. — *There exists $C \in (0, \infty)$ such that for any $n \in \mathbb{N}$, we have*

$$\int h_{t=1} z^{2n} d\nu \;\leqslant\; C^n \, (2n)! \tag{6.8}$$

Proof. — We have

$$\partial_t \int h_t z^{2n} d\nu = \int \mathcal{L}(h_t) z^{2n} d\nu = \int h_t (T^2 z^{2n}) d\nu \tag{6.9}$$

with

$$T^2 z^{2n} = (T_X^2 + T_Y^2) z^{2n} = \left(X^2 + Y^2 \right) z^{2n} + \left(\{X, A_X\} + \{Y, A_Y\} \right) z^{2n}$$

We have

$$\left(X^2 + Y^2 \right) z^{2n} = 4a^2 r^2 \partial_z^2 z^{2n} = 4a^2 \cdot 2n(2n-1) r^2 z^{2(n-1)} \tag{6.10}$$

On the other hand

$$\{X, A_X\} z^{2n} = 2\kappa \int_0^1 ds \int_0^s ds' (X^2 z^{2n})(\gamma_{s',X}^w)$$

$$= 2\kappa 4a^2 2n(2n-1) \int_0^1 ds \int_0^s ds' (y^2 z^{2(n-1)})(\gamma_{s',X}^w)$$

$$= 8\kappa a^2 2n(2n-1) \int_0^1 ds \int_0^s ds' y^2 (z - 2s'x \cdot 2ay)^{2(n-1)}$$

and similarly

$$\{Y, A_Y\} z^{2n} = 8\kappa a^2 2n(2n-1) \int_0^1 ds \int_0^s ds' x^2 (z + 2s'y \cdot 2ax)^{2(n-1)}$$

Hence

$$(\{X, A_X\} + \{Y, A_Y\})\, z^{2n} =$$
$$= 8\kappa a^2 2n(2n-1) \int_0^1 ds \int_0^s ds' \left(y^2 (z - 2s'x \cdot 2ay)^{2(n-1)} \right.$$
$$\left. + x^2 (z + 2s'y \cdot 2ax)^{2(n-1)} \right)$$
$$\leqslant \kappa (4a)^2 2n(2n-1) r^2 \int_0^1 ds \int_0^s ds' (|z| + s'2ar^2)^{2(n-1)}$$

Combining the above bounds we obtain

$$T^2 z^{2n} = 4a^2 2n(2n-1) r^2 z^{2(n-1)} \tag{6.12}$$
$$+ \kappa (4a)^2 2n(2n-1) r^2 \int_0^1 ds \int_0^s ds' (|z| + s'2ar^2)^{2(n-1)}$$

For the first term on the right hand side, by Young inequality one has

$$4a^2 2n(2n-1) r^2 z^{2(n-1)} \quad \leqslant \frac{2(n-1)}{2n} (2n-1)^{\frac{2n}{2(n-1)}} z^{2n} + \left(4a^2(2n)\right)^n r^{2n}$$
$$\leqslant \bar{C}_1 n\, z^{2n} + \bar{C}_2 (4a^2 e)^n \sqrt{(2n)!}\, r^{2n} \tag{6.13}$$

with some constants $\bar{C}_1, \bar{C}_2 \in (0, \infty)$. For the second term, using binomial expansion and applying suitable Youngs inequality to each term, we have

$$(4a)^2 2n(2n-1) r^2 \int_0^1 ds \int_0^s ds' (|z| + s'2ar^2)^{2(n-1)} \tag{6.14}$$
$$\leqslant \sum_{k=0}^{2(n-1)} 2 \binom{2n}{k+2} |z|^{2n-(k+2)} r^{2(k+1)} (2a)^{k+2}$$
$$\leqslant z^{2n} + \sum_{k=0}^{2(n-1)-1} \binom{2n}{k+2} 2^{\frac{2n}{k+2}} (2a)^{2n} r^{2(k+1)\frac{2n}{k+2}} + r^{2(2n-1)} (4a)^{2n}$$
$$\leqslant z^{2n} + 4^{2n} (4a)^{2n} r^{2 \cdot 2n}$$

Combining (6.9) - (6.14) we arrive at the following Gronwal type relation

$$< z^{2n} >_t \leqslant C_1^n (2n)! + C_2 n \int_0^t ds < z^{2n} >_s \tag{6.15}$$

with suitable constants $C_1, C_2 \in (0, \infty)$ uniformly bounded on compact intervals of t and where

$$< f >_t \equiv \int h_t f d\nu$$

Deriving this bound we have taken advantage of the Gaussian estimate in the xy directions

$$< r^{2k} >_t \leqslant C^k k!$$

which follows from Proposition 6.1. Iteration of (6.14) yields

$$< z^{2n} >_t \leqslant C_1^n e^{C_2 n t} (2n)!$$

$\square$

Using Proposition 6.1, Corollary 6.3, and arguments of [17], we obtain the following pointwise Gaussian bounds result.

THEOREM 6.5. — *There exists $\alpha_0 \in (0, \infty)$ and a positive function $C_t(\cdot)$ such that for any $\alpha \in (0, \alpha_0)$ one has*

$$h_t(w, w') \leqslant C_t(w) C_t(w') e^{-\alpha d^2(w, w')} \tag{6.16}$$

Using extra homogeneity of the theory one can refine a bit these estimates. At the origin one can also obtain some monotonicity properties (see Appendix II).

We remark that using the pointwise upper bound one should be able to obtain the lower Gaussian bounds by the arguments of [5].

7. Summary

In this paper we have introduced a representation of infinite Coxeter group with two generators associated to fundamental fields of $\mathbb{H}_1$ given by explicit maps on $\mathbb{H}_1$. (Incidentally that is the same Coxeter group as the one of the Backlund transformations associated to the Painleve II, [25].) While in more general case explicit formulas for reflection maps may be difficult to obtain, in a class of so called *Kolmogorov*-type and *H*-type groups ([6]) it is an easy matter to get. Thus one gets in this case a solution of the set of differential problem (*) and a representation of a Coxeter group. One may hope that the (*) problem can be solved for any set of generating fields associated to a free nilpotent Lie algebra in Euclidean space, [21]. Conversely, given any Coxeter group, one can find a set of maps on a Euclidean space

representing this group and for which (*) is satisfied for a suitable set of fields (generating some free nilpotent Lie group).

In case of manifolds there exist a geometric notion of reflection, see e.g. [2], which could be complemented in a natural way by using (*) for a fundamental set of fields.

In the bulk of the paper we have developed the basis of a calculus necessary for interesting initial analysis of the heat kernel. But the emerging structures are likely to lead to other natural destinations as for example theory of special functions, non-commutative geometry (involving number of noncommuting boundary operators),..., and physics, which we hope to explore elsewhere.

Appendix I

Proof of reflection property. — One has

$$
\begin{aligned}
X(f \circ \sigma_X) &= (\partial_x + 2ay\partial_z)(f(-x, y, z - 4axy)) \\
&= -(\partial_x f) \circ \sigma_X - 4ay(\partial_z f) \circ \sigma_X + 2ay(\partial_z f) \circ \sigma_X \\
&= -(\partial_x f + 2ay\partial_z f) \circ \sigma_X = -(Xf) \circ \sigma_X
\end{aligned}
$$

and

$$
\begin{aligned}
Y(f \circ \sigma_Y) &= (\partial_y - 2ax\partial_z)(f(x, -y, z + 4axy)) \\
&= -(\partial_y f) \circ \sigma_Y + 4ax(\partial_z f) \circ \sigma_Y + -2ax(\partial_z f) \circ \sigma_Y \\
&= -(\partial_y f - 2ax\partial_z f) \circ \sigma_Y = -(Yf) \circ \sigma_Y
\end{aligned}
$$

□

Proof of Lemma 3.1. — We show the relations

$$
[A_X, D] = A_X, \qquad [A_Y, D] = A_Y.
$$

To show the first relation, we note that $D = xX + yY + 2zZ$ and we compute

$$
\begin{aligned}
A_X D f - D A_X f &= \kappa \frac{Df - (Df) \circ \sigma_X}{x} - \kappa D \frac{f - f \circ \sigma_X}{x} \\
&= \kappa \frac{D(f \circ \sigma_X) - (Df) \circ \sigma_X}{x} + A_X f
\end{aligned}
$$

with

$$\frac{D(f \circ \sigma_X) - (Df) \circ \sigma_X}{x} = \frac{xX(f \circ \sigma_X) - (xXf) \circ \sigma_X}{x}$$
$$+ \frac{yY(f \circ \sigma_X) - (yYf) \circ \sigma_X}{x}$$
$$+ \frac{2zZ(f \circ \sigma_X) - (2zZf) \circ \sigma_X}{x}$$

Now we have

$$\begin{aligned}
xX(f \circ \sigma_X) &= x(\partial_x + 2ay\partial_z)(f(-x, y, z - 4axy)) \\
&= -x(\partial_x f) \circ \sigma_X - 4axy(\partial_z f) \circ \sigma_X + x2ay(\partial_z f) \circ \sigma_X \\
&= -x(\partial_x f) \circ \sigma_X - 2axy(\partial_z f) \circ \sigma_X \\
&= -x(Xf) \circ \sigma_X = (xXf) \circ \sigma_X
\end{aligned}$$

and so

$$\frac{xX(f \circ \sigma_X) - (xXf) \circ \sigma_X}{x} = 0$$

Moreover

$$\begin{aligned}
yY(f \circ \sigma_X) &= y(\partial_y - 2ax\partial_z)(f(-x, y, z - 4axy)) \\
&= y(\partial_y f) \circ \sigma_X - 4axy(\partial_z f) \circ \sigma_X - y2ax(\partial_z f) \circ \sigma_X \\
&= y(\partial_y f) \circ \sigma_X - 6axy(\partial_z f) \circ \sigma_X \\
&= y(Yf) \circ \sigma_X - 8axy(\partial_z f) \circ \sigma_X
\end{aligned}$$

and thus

$$\frac{yYf \circ \sigma_X) - (yYf) \circ \sigma_X}{x} = -8ay(\partial_z f) \circ \sigma_X$$

Finally we have

$$zZ(f \circ \sigma_X) = (zZf) \circ \sigma_X + 4axy(\partial_z f) \circ \sigma_X$$

and hence

$$\frac{2zZ(f \circ \sigma_X) - (2zZf) \circ \sigma_X}{x} = 8ay(\partial_z f) \circ \sigma_X$$

Combining these all we obtain

$$\frac{D(f \circ \sigma_X) - (Df) \circ \sigma_X}{x} = 0$$

which ends the proof of for A_X. The case of A_Y is similar. $\quad\square$

Appendix II

Using homogeneity properties of our theory, we can also show the following bound.

PROPOSITION 7.1. — *For any function $\zeta \equiv \chi(z^2)$ defined with a nonincreasing real function χ and any $\varepsilon \in (0, \infty)$, at the origin $w = 0$ we have*

$$\sup_{t>0} \int h_t e^{\varepsilon \frac{z\chi}{t}} \, d\nu < \infty \tag{7.1}$$

Proof. — We recall that

$$[\mathcal{L}, D] = 2\mathcal{L}$$

and therefore one has the following relation

$$P_t D f = D P_t f + 2t P_t \mathcal{L} f$$

(with the first term on the right hand side vanishing at $w = 0$). Now we do the following calculation at $w = 0$, with a function $\zeta \equiv \chi(z^2)$ defined with a nonincreasing real function χ and any $\varepsilon \in (0, \infty)$

$$
\begin{aligned}
\partial_t \int h_t e^{\varepsilon \frac{z\zeta}{t}} \, d\nu &= \int h_t \mathcal{L} e^{\varepsilon \frac{z\zeta}{t}} \, d\nu - \frac{\varepsilon}{t^2} \int h_t \left(z\zeta e^{\varepsilon \frac{z\zeta}{t}} \right) d\nu \\
&= \frac{1}{2t} \int h_t D e^{\varepsilon \frac{z\zeta}{t}} \, d\nu - \frac{\varepsilon}{t^2} \int h_t \left(z\zeta e^{\varepsilon \frac{z\zeta}{t}} \right) d\nu \\
&= \frac{2\varepsilon}{t^2} \int h_t \left(z^2 \chi'(z^2) e^{\varepsilon \frac{z\zeta}{t}} \right) d\nu
\end{aligned}
$$

We conclude noting that the right hand side is nonpositive for nonincreasing function χ. □

Remark. — In case when the kernel is group covariant, we get a similar bound at any point w.

PROPOSITION 7.2. — *Let $\psi_t \equiv \varepsilon \frac{\tilde{d}^2}{t} \chi(\frac{\tilde{d}^2}{t})$ defined with $t \in (0, \infty)$, a homogeneous almost everywhere differentiable distance $\tilde{d}^4 \equiv r^4 + \alpha z^2$, where $\alpha, \varepsilon \in (0, \infty))$ and $0 \leqslant \chi \leqslant 1$ being a Lipschitz function, which is equal to one on an interval $[0, K]$ and zero on $[K + 1, \infty]$ for some $K \in (0, \infty)$. At $w = 0$ we have*

$$\frac{d}{dt} \int h_t e^{\psi_t} \, d\nu \leqslant 0 \tag{7.2}$$

Proof. — We have

$$\partial_t P_t e^{\psi_t} = P_t \mathcal{L} e^{\psi_t} + P_t \partial_t \psi_t e^{\psi_t}$$

Since at $w = 0$ we have

$$P_t \mathcal{L} f = \frac{1}{2t} P_t D f,$$

so we get

$$\partial_t P_t e^{\psi_t} = P_t \left(\left(\frac{1}{2t} D e^{\psi_t} + \partial_t \psi_t \right) e^{\psi_t} \right)$$

Now using the fact that

$$\partial_t \psi_t = -\varepsilon \frac{\tilde{d}^2}{t^2} \left(\chi \left(\frac{\tilde{d}^2}{t} \right) + \frac{\tilde{d}^2}{t} \chi' \left(\frac{\tilde{d}^2}{t} \right) \right)$$

and, since $D\tilde{d}^2 = 2\tilde{d}^2$,

$$D\psi_t = 2\varepsilon \frac{\tilde{d}^2}{t} \left(\chi \left(\frac{\tilde{d}^2}{t} \right) + \frac{\tilde{d}^2}{t} \chi' \left(\frac{\tilde{d}^2}{t} \right) \right)$$

we conclude that

$$\frac{1}{2t} D\psi_t + \partial_t \psi_t = 0$$

Hence at $w = 0$ we obtain

$$\partial_t P_t e^{\psi_t} = 0$$

□

Appendix III: Representation of Coxeter Group and CCRs

In the text we did not get much into algebraic properties of the T-theory. It is however worth to mention that it offers interesting realisation of CCR with different coefficient on symmetric and antisymmetric subspaces. More precisely on the linear span of the following generalised linear functions

$$x, y, \eta \equiv x + \frac{z}{2ay}, \zeta \equiv y - \frac{z}{2ax}$$

we have a representation of the Coxeter group determined by the following relations

$x \circ \sigma_X = -x$	$y \circ \sigma_X = y$	$\eta \circ \sigma_X = \eta - 4x$	$\zeta \circ \sigma_X = -\zeta$
$x \circ \sigma_Y = x$	$y \circ \sigma_Y = -y$	$\eta \circ \sigma_Y = -\eta$	$\zeta \circ \sigma_Y = \zeta - 4y$

and the following commutation relations

$$[T_X, x]f = (1 + 2\kappa)f_{s,X} + (1 - 2\kappa)f_{a,X}$$
$$[T_Y, y]f = (1 + 2\kappa)f_{s,Y} + (1 - 2\kappa)f_{a,Y}$$
$$[T_X, y]f = 0$$
$$[T_X, \zeta]f = -\frac{(1-2\kappa)}{x}f_{s,X} - \frac{(1+2\kappa)}{x}f_{a,X}$$
$$[T_X, \eta]f = 2(1 + 2\kappa)f_{s,X} + 2(1 - 2\kappa)f_{a,X}$$
$$[T_Y, \zeta]f = 2(1 + 2\kappa)f_{s,Y} + 2(1 - 2\kappa)f_{a,Y}$$
$$[T_Y, x]f = 0$$
$$[T_Y, \eta]f = \frac{(1-2\kappa)}{y}f_{s,Y} + \frac{(1+2\kappa)}{y}f_{a,Y}$$

where $f_{s,\cdot} \equiv \frac{1}{2}(f + f \circ \sigma.)$ and $f_{a,\cdot} \equiv \frac{1}{2}(f - f \circ \sigma.)$. The interesting thing is that on symmetric and antisymmetric subspaces the constants in CCRs are different. (There are also some interesting critical points $\kappa = \pm\frac{1}{2}$.)

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