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Rüdiger W. Braun, Reinhold Meise, B. A. Taylor

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A new characterization of the analytic surfaces in $\mathbb{C}^3$ that satisfy the local Phragmén-Lindelöf condition

Rüdiger W. Braun[(1)], Reinhold Meise[(2)]
and B. A. Taylor[(3)]

Dedicated to Nguyen Than Van on the occasion of the colloquium celebrating his contributions to complex analysis

Abstract. — We prove that an analytic surface V in a neighborhood of the origin in $\mathbb{C}^3$ satisfies the local Phragmén-Lindelöf condition $\mathrm{PL}_{\mathrm{loc}}$ at the origin if and only if V satisfies the following two conditions: (1) V is nearly hyperbolic; (2) for each real simple curve γ in $\mathbb{R}^3$ and each $d \geqslant 1$, the (algebraic) limit variety $T_{\gamma,d}V$ satisfies the strong Phragmén-Lindelöf condition. These conditions are also necessary for any pure k-dimensional analytic variety V to satisify $\mathrm{PL}_{\mathrm{loc}}$.

Résumé. — On démontre qu'une surface analytique V dans un voisinage de l'origine dans $\mathbb{C}^3$ satisfait à la condition Phragmén-Lindelöf locale $\mathrm{PL}_{\mathrm{loc}}$ à l'origine si et seulement si V satisfait aux deux conditions suivantes: (1) V is presque hyperbolique; (2) pour chaque courbe réelle simple γ dans $\mathbb{R}^3$ et chaque $d \geqslant 1$, la variété (algebrique) limite $T_{\gamma,d}V$ satisfait à la condition de Phragmén-Lindelöf forte. Ces conditions sont aussi nécessaires que pour toute variété analytique V de dimension pure k vérifie la condition $\mathrm{PL}_{\mathrm{loc}}$.

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(1) Mathematisches Institut, Heinrich-Heine-Universität Universitätsstraße 1, 40225 Düsseldorf, Germany
Ruediger.Braun@uni-duesseldorf.de
(2) Mathematisches Institut, Heinrich-Heine-Universität, Universitätsstraße 1, 40225 Düsseldorf, Germany
meise@math.uni-duesseldorf.de
(3) Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, U.S.A.
taylor@umich.edu

1. Introduction

An analytic variety V in a neighborhood of the origin in $\mathbb{C}^n$ is said to satisfy the *local Phragmén-Lindelöf condition*, $\mathrm{PL}_{\mathrm{loc}}$, at the origin if every function $u(z)$ that is plurisubharmonic on a neighborhood of the origin in V, is bounded above by the constant 1, and is bounded above by 0 on the real points of V must grow linearly, $u(z) \leqslant A|\operatorname{Im} z|$, at all points z in a (perhaps smaller) neighborhood of the origin in V. This condition was introduced by Hörmander [11] who used it in his characterization of the linear constant coefficient differential operators that map the space $\mathcal{A}(\mathbb{R}^n)$ of all real analytic functions on $\mathbb{R}^n$ surjectively onto itself. It was later used by Vogt [16] to characterize the compact real analytic subvarieties V of $\mathbb{R}^n$ for which there exists a continuous linear extension operator $E : \mathcal{A}(V) \to \mathcal{A}(\mathbb{R}^n)$ for real analytic functions on V. It also turns out to be a very useful condition that is needed in the study of other Phragmén-Lindelöf type conditions on algebraic varieties in $\mathbb{C}^n$.

It is classical that subharmonic functions on the unit disk that are bounded above by 1 everywhere and by 0 on the real points must grow linearly in $|\operatorname{Im} z|$; in fact, the subharmonic function

$$u(z) = u(x+iy) = \frac{2}{\pi} \arctan \frac{2y}{1-x^2-y^2} \leqslant \frac{4}{\pi} \frac{|\operatorname{Im} z|}{1-|z|^2}$$

is the maximal function. It is an immediate consequence that a similar estimate holds on polydisks in higher dimension so that if 0 is a regular point of the variety $V \subset \mathbb{C}^n$, then it will satisfy the condition $\mathrm{PL}_{\mathrm{loc}}$. Thus, the question is to determine what kinds of singularities V is allowed to have at 0. It is clear that if plurisubharmonic functions on V are going to be estimated from their values on real points, then V should contain "many" real points, and there are at least two classical interpretations one can check for this condition. The first is in the sense of dimension. Namely, the set of real points in V should have maximal real dimension, or, more precisely, V should be the *complexification* of $V \cap \mathbb{R}^n$; i.e., the smallest analytic variety that contains $V \cap \mathbb{R}^n$ (in some small neighborhood of 0). In fact, this condition was shown by Hörmander to be necessary for $\mathrm{PL}_{\mathrm{loc}}$. Another sense in which a local variety can have many real points is if it is "maximally real", i.e., locally hyperbolic. That is, there is a proper projection $\pi : V \to \Delta$ of V to a polydisk $\Delta \subset \mathbb{C}^k$ about the origin ($k = \dim(V)$) such that all the points of the fiber $\pi^{-1}(x)$ over a real point $x \in \Delta$ are real. Hörmander also observed that the classical estimate for the harmonic measure implies that such varieties satisfy $\mathrm{PL}_{\mathrm{loc}}(0)$ and that this sufficient condition is also necessary for varieties of dimension 1. But, it is no longer a necessary condition for varieties of dimension 2 or more. For example, the two dimensional variety in $\mathbb{C}^3$ given by the equation $z_1^3+z_2^3+z_3^3=0$ is not locally hyperbolic at the origin but it does satisfy the condition $\mathrm{PL}_{\mathrm{loc}}$.

Thus, the condition characterizing when V satisfies $\mathrm{PL}_{\mathrm{loc}}$ is somewhat stronger than the dimension condition but weaker than local hyperbolicity. It turns out that it is not hard to give lots of necessary conditions that must be satisfied if $\mathrm{PL}_{\mathrm{loc}}$ is to hold. For example, it carries over to tangent varieties and even to limit varieties of V along real simple curves (see Section 2). The condition we use here is that all limit varieties of V must satisfy the strong Phragmén-Lindelöf condition (SPL) (see Section 2), a result of Heinrich [9]. The problem is to determine which combination of these necessary conditions is sufficient to prove that $\mathrm{PL}_{\mathrm{loc}}$ holds. For surfaces in $\mathbb{C}^3$, the present authors showed that the characterizing condition for $\mathrm{PL}_{\mathrm{loc}}$ was that V is hyperbolic in conoids [5] (the extension to germs of algebraic varieties in $\mathbb{C}^n$ of dimension 2 was given later by Heinrich [10]). The aim of this paper is to give a different characterization and proof of a more geometric characterizing condition, one we call nearly hyperbolic. Intuitively, we say that a variety is *nearly hyperbolic* if the parts of the variety that are close to real linear subspaces are hyperbolic; see Definition 2.4 for the precise version. Our main result, Theorem 4.1, is that this condition, together with the strong Phragmén-Lindelöf condition (SPL) for all limit varieties, characterizes $\mathrm{PL}_{\mathrm{loc}}$ for surfaces in $\mathbb{C}^3$.

The main idea of the proof is to carry back the Phragmén-Lindelöf estimates that hold on tangent varieties to plurisubharmonic functions on V. One starts with a plurisubharmonic function on V, bounded above by 1 and equal to 0 on the real points of V and tries to improve the estimate to $u(z) \leqslant A|\operatorname{Im} z|$. For example, near smooth points of the tangent cone to V at the origin, one can do this without much difficulty. However, when one is near the singular points of this tangent cone, it is hard to see how one can do this directly. Near these exceptional points we are able to improve the estimate. One then examines the variety on a different scale appropriate to the smaller size of the exceptional set and repeats the procedure. The main difficulty is to see that this procedure comes to an end after finitely many steps, something we accomplish by using suitably chosen coordinates at the smaller scales.

2. Preliminaries

An analytic variety V in $\mathbb{C}^n$ is defined to be a closed analytic subset of some open set G in $\mathbb{C}^n$ (see Chirka [8], 2.1). By V_{sing} (resp. V_{reg}) we denote the set of all singular (resp. regular) points in V. We also consider families $(V_t)_{t\in T}$ of closed analytic subsets of G, where we suppose that the parameter set T is a subset of $\mathbb{R}^m$ for some $m \in \mathbb{N}$ and where $0 \in \overline{T}$. Note that the case of a single variety V is covered by this notation, since one can let $T := \{0\}$ and $V_0 := V$.

We will also use the following notation: $\mathbb{R}_+ := [0, \infty[$, the Euclidean norm on $\mathbb{C}^n$ is denoted by $|\cdot|$, and for $a \in \mathbb{C}^n$, $r > 0$, the ball around a of the radius r is defined as $B^n(a, r) := \{z \in \mathbb{C}^n : |z - a| < r\}$. The superscript n is omitted if it is clear from the context.

To define the Phragmén-Lindelöf properties that we are interested in, we recall the definition of a plurisubharmonic function on an analytic variety.

Definition 2.1. — *Let V be an analytic variety in $\mathbb{C}^n$ and let Ω be an open subset of V. A function $u : \Omega \to [-\infty, \infty[$ is called* plurisubharmonic*, if it is locally bounded above, plurisubharmonic in the usual sense on the set Ω_{reg} of all regular points and satisfies*

$$u(z) = \limsup_{\zeta \in \Omega_{\text{reg}}, \zeta \to z} u(\zeta)$$

at $z \in \Omega_{\text{sing}}$, the set of singular points of V in Ω. By $\mathrm{PSH}(\Omega)$ *we denote the set of all plurisubharmonic functions on Ω.*

Definition 2.2. — *Let $(V_t)_{t \in T}$ be a family of closed analytic subsets of $B(\xi, r_0)$ for some $\xi \in \mathbb{R}^n$. We say that $(V_t)_{t \in T}$ satisfies the local Phragmén-Lindelöf condition* $\mathrm{PL}_{\mathrm{loc}}(\xi)$ *at ξ if there exist positive numbers A, δ and $r_0 \geqslant r_1 \geqslant r_2$ such that, whenever $t < \delta$, then each $u \in \mathrm{PSH}(V_t \cap B(\xi, r_1))$ that satisfies*

(α) $u(z) \leqslant 1,\ z \in V_t \cap B(\xi, r_1)$ *and*

(β) $u(z) \leqslant 0,\ z \in V_t \cap \mathbb{R}^n \cap B(\xi, r_1)$

also satisfies

(γ) $u(z) \leqslant A|\operatorname{Im} z|,\ z \in V_t \cap B(\xi, r_2)$.

Remark 2.3. — (a) The local Phragmén-Lindelöf condition for algebraic varieties in $\mathbb{C}^n$ arose for the first time in the work of Hörmander [11].

(b) For several equivalent formulations of $\mathrm{PL}_{\mathrm{loc}}$ for analytic varieties we refer to Lemma 3.3 in [5].

(c) The condition $\mathrm{PL}_{\mathrm{loc}}(0)$ is clearly invariant under real, orthogonal tranformations of $\mathbb{C}^n$ with no change in the constant A. It is also invariant under real linear changes of coordinates $z = L(w)$, provided the constant A is changed to $A\|L\|$ and the parameter r_2 is changed to $\|L^{-1}\| r_2$.

In order to derive necessary conditions for analytic varieties in $\mathbb{C}^n$ to satisfy $\mathrm{PL}_{\mathrm{loc}}(0)$, we will use the following definition.

Definition 2.4. — *(a) For $\xi \in \mathbb{R}^n$ and $\varrho > 0$ let V be an analytic variety in $B(\xi,\varrho) \subset \mathbb{C}^n$ of pure dimension $k \geqslant 1$. A k-dimensional linear subspace L of $\mathbb{C}^n$ is called* admissible *if it has a basis of real vectors. A (linear) projection π_L of $\mathbb{C}^n$ onto L is called* admissible *if it maps real vectors to real vectors. The variety V is called* nearly hyperbolic at ξ *with parameters δ with $0 < \delta < \varrho/2$ and $\varepsilon_0 : [1,\infty[\to]\,0,1[$ if the following condition is satisfied:*

For each $K \geqslant 1$, each $x \in B(\xi,\delta) \cap \mathbb{R}^n$, each $0 < r < \delta$, each admissible subspace L, and each admissible projection π_L onto L satisfying $\|\mathrm{id}_{\mathbb{C}^n} - \pi_{\mathrm{L}}\| \leqslant \mathrm{K}$ the following holds: If

$$V \cap B(x,r) \subset S(x+L,\varepsilon_0(K)r) := \{w \in \mathbb{C}^n :\ \mathrm{dist}(\mathrm{w},\mathrm{x}+\mathrm{L}) < \varepsilon_0(\mathrm{K})\mathrm{r}\}$$

and $z \in V \cap B(x,r/2)$ satisfies $\pi_L(z) \in \mathbb{R}^n$, then $z \in \mathbb{R}^n$.

(b) Let $(V_t)_{t\in T}$ be a family of analytic varieties in $B(\xi,\varrho)$ for ξ and ϱ as in part (a). Then the family is called nearly hyperbolic *at ξ, if there are $\delta, \delta_1 > 0$ and $\varepsilon_0 :\ [1,\infty[\to]0,1[$ such that for each $t \in T$ with $|t| \leqslant \delta_1$ the variety V_t is nearly hyperbolic at ξ with parameters δ and ε_0.*

See 2.13 for some examples illustrating the key properties of nearly hyperbolic varieties.

A variant of the present definition of nearly hyperbolic varieties was introduced in [6]. Its significance is explained by the following proposition which is an analog of [6], Theorem 4.3.

Proposition 2.5. — *Let V be an analytic variety in $\mathbb{C}^n$ of pure dimension $k \geqslant 1$ that satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Then V is nearly hyperbolic at the origin.*

Proof. — Assume that V is defined in $B(0,r_0)$ for some $0 < r_0 \leqslant 1$ and that it satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ with the constants from Definition 2.2. Then let $\delta := r_2/2$ and for $K \geqslant 1$ given, let $\varepsilon_0 := \frac{1}{4K(A+1)}$. Then fix $0 < r < \delta$, an admissible linear subspace L and an admissible projection π_L from $\mathbb{C}^n$ onto L, which satisfies $\|\mathrm{id}_{\mathbb{C}^n} - \pi_{\mathrm{L}}\| \leqslant \mathrm{K}$. Next fix $x \in B(0,\delta) \cap \mathbb{R}^n$ and assume that $V \cap B(x,r) \subset S(x+L,\varepsilon_0 r)$. Then fix $z_0 \in V \cap B(x,r/2)$ with $\pi_L(z_0) \in \mathbb{R}^n$. In order to show that z_0 is real, define the function φ by

$$\varphi(z) := (A+1)|\,\mathrm{Im}(z-\pi_L(z))| + \frac{r}{2}H(\frac{z-\mathrm{Re}\,z_0}{r/2}),$$

where $H(z) := \frac{1}{2}(|\operatorname{Im} z|^2 - |\operatorname{Re} z|^2)$. To estimate φ at $V \cap \partial B(\operatorname{Re} z_0, r/2)$, note first that the properties of H imply the estimate

$$\frac{r}{2} H(\frac{z - \operatorname{Re} z_0}{r/2}) \leqslant |\operatorname{Im} z| - \frac{r}{4} \text{ if } |z - \operatorname{Re} z_0| = r/2.$$

To estimate the first term in the definition of φ, note that x is real. Hence each $z \in B(\operatorname{Re} z_0, r/2)$ satisfies

$$|z - x| \leqslant |z - \operatorname{Re} z_0| + |\operatorname{Re} z_0 - x| \leqslant r/2 + |z_0 - x| < r.$$

Since $V \cap B(x, r) \subset S(x + L, \varepsilon_0 r)$, this and the properties of π_L imply that $\operatorname{Im}(x - \pi_L(x)) = 0$ and that for $z \in V \cap \partial B(\operatorname{Re} z_0, \frac{r}{2})$ and each $y \in L$ we have

$$\operatorname{Im}(z - \pi_L(z)) = \operatorname{Im}\left(z - (x + y) - \pi_L(z - (x + y))\right),$$

hence

$$|\operatorname{Im}(z - \pi_L(z))| \leqslant \|\mathrm{id}_{\mathbb{C}^n} - \pi_\mathrm{L}\| \, |\mathrm{z} - (\mathrm{x} + \mathrm{y})|, \ \mathrm{y} \in \mathrm{L},$$

and consequently

$$|\operatorname{Im}(z - \pi_L(z))| \leqslant K \mathrm{dist}(\mathrm{z}, \mathrm{x} + \mathrm{L}) < \mathrm{K}\varepsilon_0 \mathrm{r}.$$

By our choice of ε_0 this implies

$$\varphi(z) \leqslant (A + 1) K \varepsilon_0 r - \frac{r}{4} + |\operatorname{Im} z| \leqslant |\operatorname{Im} z|, \ z \in V \cap \partial B(\operatorname{Re} z_0, r).$$

This shows that there exists $u \in \mathrm{PSH}(V)$ which coincides with φ on $V \cap B(\operatorname{Re} z_0, r/2)$ and with $|\operatorname{Im} z|$ on $V \setminus B(\operatorname{Re} z_0, r/2)$. Since $V \subset B(0, r_0) \subset B(0, 1)$, it follows that u satisfies

$$u(z) \leqslant 1, \ z \in V \cap B(0, r_1) \text{ and } u(z) \leqslant 0, \ z \in V \cap \mathbb{R}^n \cap B(0, r_1).$$

As V satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$, we conclude from this that

$$u(z) \leqslant A |\operatorname{Im} z|, \ z \in B(0, r_2) \cap V. \tag{2.1}$$

To apply this inequality, note that

$$|z_0| \leqslant |z_0 - x| + |x| \leqslant r/2 + \delta < \frac{3}{2}\delta < r_2$$

and that

$$|\operatorname{Im} z_0| = |\operatorname{Im}(z_0 - x)| \leqslant |z_0 - x| < r/2.$$

Hence we can evaluate (2.1) at $z = z_0$. By the properties of H and the fact that $\pi_L(z_0)$ is real by hypothesis, we get

$$A|\operatorname{Im} z_0| \geqslant \varphi(z_0) = (A+1)|\operatorname{Im}(z_0 - \pi_L(z_0))| + \frac{r}{2} H\left(\frac{2}{r}\, i \operatorname{Im} z_0\right) \geqslant (A+1)|\operatorname{Im} z_0|.$$

Obviously, this inequality implies $|\operatorname{Im} z_0| = 0$. Hence we proved that $z_0 \in \mathbb{R}^n$ and completed the proof. $\square$

Remark. — The proof of Proposition 2.5 actually shows that $|\operatorname{Im} z| \leqslant (A+1)|\operatorname{Im} \pi_L(z)|$.

To state more necessary conditions for $\mathrm{PL}_{\mathrm{loc}}(0)$ we need to define limit varieties of a given analytic variety along simple curves. For details we refer to our paper [3].

DEFINITION 2.6. — *Let $V \subset \mathbb{C}^n$ be analytic in a neighborhood of $p \in V$. Following Whitney [17], a vector $v \in \mathbb{C}^n$ is called* tangent *to V at p if there are a sequence $(p_j)_j$ in V and a sequence $(a_j)_j$ in $\mathbb{C}$ such that $\lim_{j\to\infty} p_j = p$ and $\lim_{j\to\infty} a_j(p_j - p) = v$. The set of all tangent vectors forms a complex cone. It is called the* tangent cone *of V in p and is denoted by T_pV.*

If f is a holomorphic function in n variables, its localization *in $p \in \mathbb{C}^n$ is defined as the lowest order nonvanishing term of the expansion $f(\zeta+p) = \sum_{j=0}^{\infty}\sum_{\alpha=j} a_\alpha \zeta^\alpha$. It is denoted by f_p.*

The relation between tangents and localization is given by Whitney [17], Chapter 7, Theorem 4D:

$$T_pV = \{z \in \mathbb{C}^n : f_p(z) = 0 \textit{ for all holomorphic functions } f \textit{ vanishing on } V\}.$$

DEFINITION 2.7. — *A* simple curve *γ in $\mathbb{C}^n$ is a map $\gamma :]0,\alpha[\to \mathbb{C}^n$ which for some $\alpha > 0$ and some $q \in \mathbb{N}$ admits a convergent Puiseux series expansion*

$$\gamma(t) = \sum_{j=q}^{\infty} \xi_j t^{j/q} \textit{ with } \xi_q = 1.$$

Then ξ_q is called the tangent vector *to γ in the origin. The trace of γ is defined as* $\mathrm{tr}(\gamma) := \gamma(]0,\alpha[)$*. A* real simple curve *is a simple curve γ satisfying* $\mathrm{tr}(\gamma) \subset \mathbb{R}^{\mathrm{n}}$.

DEFINITION 2.8. — *Let $V \subset \mathbb{C}^n$ be an analytic variety of pure dimension $k \geqslant 1$ which contains the origin, let $\gamma :]0,\alpha[\to \mathbb{C}^n$ be a simple curve, and let $d \geqslant 1$. Then for $t \in]0,\alpha[$ we define*

$$V_{\gamma,t,d} := \{w \in \mathbb{C}^n : \gamma(t) + wt^d \in V\} = \frac{1}{t^d}(V - \gamma(t))$$

and we define the limit variety *$T_{\gamma,d}V$ of V of order d along γ as the set*

$$T_{\gamma,d}V := \{\zeta \in \mathbb{C}^n : \zeta = \lim_{j\to\infty} z_j, \textit{ where } z_j \in V_{\gamma,t_j,d} \textit{ for } j \in \mathbb{N} \textit{ and } (t_j)_{j\in\mathbb{N}} \textit{ is a null-sequence in }]0,\alpha[\}.$$

From [3], Theorem 3.2 and Proposition 4.1 we recall the following results.

Proposition 2.9. — *Let V be an analytic variety of pure dimension $k \geqslant 1$ which contains the origin, let $\gamma :]0, \alpha[\to \mathbb{C}^n$ be a simple curve in $\mathbb{C}^n$ with tangent vector ξ at the origin, and let $d \geqslant 1$ be given. Then the following assertions hold:*

1. *$T_{\gamma,d}V$ is either empty or an algebraic variety of pure dimension k.*
2. *$T_{\gamma,1}V = T_0V - \xi$.*
3. *If $d > 1$ then $w \in T_{\gamma,d}V$ if and only if $w + \lambda\xi \in T_{\gamma,d}V$ for each $\lambda \in \mathbb{C}$.*
4. *For each $R > 0$ there exists $0 < \delta \leqslant \alpha$ such that $V_{\gamma,t,d}$ is a closed analytic set of $B(0, R)$ for $0 < t < \delta$, and for each null-sequence $(t_j)_{j\in\mathbb{N}}$ in $]0, \delta[$ the varieties $(V_{\gamma,t_j,d} \cap B(0, R))_{j\in\mathbb{N}}$ converge to $T_{\gamma,d}V \cap B(0, R)$ in the sense of Meise, Taylor, and Vogt [14], 4.3.*

We also need the following condition of Phragmén-Lindelöf type.

Definition 2.10. — *An algebraic variety V in $\mathbb{C}^n$ satisfies the condition (*SPL*) if there exists a constant $A \geqslant 1$ such that for each $u \in \mathrm{PSH}(V)$ the conditions (α) and (β) imply (γ), where*

(α) $u(z) \leqslant |z| + o(|z|),\ z \in V,$

(β) $u(z) \leqslant 0,\ z \in V \cap \mathbb{R}^n,$

(γ) $u(z) \leqslant A\,|\mathrm{Im}\, z|,\ z \in V.$

Remark 2.11. — If V is an algebraic variety in $\mathbb{C}^n$ that satisfies (SPL), then for each $\xi \in V \cap \mathbb{R}^n$ the variety V satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$, by Meise and Taylor [12], Proposition 4.4. If V is homogeneous, then V satisfies (SPL) if and only if V satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$, by Meise, Taylor, and Vogt [15], Theorem 3.3. The algebraic curves in $\mathbb{C}^2$ and the algebraic surfaces in $\mathbb{C}^3$ that satisfy (SPL) are characterized in [7].

Theorem 2.12. — *Let V be an algebraic variety of pure dimension $k \geqslant 1$ which contains the origin. If V satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ then the following two conditions are satisfied:*

1. *V is nearly hyperbolic at the origin.*
2. *Each limit variety $T_{\gamma,d}V$ satisfies (*SPL*).*

Proof. — Condition (1) follows from Proposition 2.5. The fact that condition (2) is necessary for $\mathrm{PL}_{\mathrm{loc}}(0)$ was shown in Heinrich [9], Theorem 10. □

Example 2.13. — (a) A simple class of examples that violates the nearly hyperbolic condition is given by certain perturbations of a hyperplane of multiplicity two. To show this, let

$$F(z) := z_1^2 - \varphi(z') = 0, \quad z' = (z_2, \dots, z_n)$$

where φ is a holomorphic function defined in some neighborhood of the origin in $\mathbb{C}^{n-1}$ which has real power series coefficients, satisfies $\varphi(z') = O(|z'|^3)$ and for which there exists a null-sequence (x'_j) in $\mathbb{R}^{n-1}$ satisfying $\varphi(x'_j) < 0$ for each $j \in \mathbb{N}$. To argue by contradiction, assume that $V := V(F)$ is nearly hyperbolic at $\xi = 0$ with parameters δ and ε_0. Then choose $C > 0$ such that $|\varphi(z')| \leqslant C|z'|^3$ for $z' \in B^{n-1}(0, \delta)$ and let $k := n-1$, $x_0 := 0$, and $K := 1$. Next we let $r := \varepsilon_0(1)^2/2C$ and $L := \{0\} \times \mathbb{C}^{n-1}$. If we define π_L by $\pi_L(z_1, z') := (0, z')$, then π_L is admissible and $K = \|\mathrm{id}_{\mathbb{C}^n} - \pi_L\| = 1$. Fix a point $(z_1, z') \in V \cap B(0, r)$. By the definition of V, this implies $z_1^2 = \varphi(z')$ and hence

$$|z_1| \leqslant |\varphi(z')|^{1/2} \leqslant C^{1/2}|z'|^{3/2} \leqslant C^{1/2} r^{3/2}.$$

Consequently, we have

$$\mathrm{dist}((z_1, z'), L) \leqslant |(z_1, z') - (0, z')| = |z_1| \leqslant C^{1/2} r^{3/2} = C^{1/2} r \frac{\varepsilon_0(1)}{(2C)^{1/2}} < \varepsilon_0(1) r.$$

Hence, we proved that $V \cap B(0, r) \subset S(L, \varepsilon_0(1)r)$. Since $(x'_j)_{j\in\mathbb{N}}$ is a null sequence, the conditions on φ imply that we can choose $j \in \mathbb{N}$ so large that

$$|(\varphi(x'_j)^{3/2}, x'_j)| < r/2.$$

Hence, the point $\zeta_j = (\varphi(x'_j)^{3/2}, x'_j)$ is in V and $\pi_L(\zeta_j) \in \mathbb{R}^n$. However, $\zeta_j \notin \mathbb{R}^n$ since $\varphi(x'_j)$ is a negative real number by hypothesis. Thus, we arrive at a contradiction to our assumption that V is nearly hyperbolic at the origin.

In general, it appears to be difficult for $V(F)$ to be nearly hyperbolic when F has a repeated irreducible factor in its leading term. It can happen; e.g. $z_1^2 = z_2^4 + z_3^4$. But, the line of argument just given shows that varieties of the form

$$z_1^\nu - \varphi(z') = 0, \quad z' = (z_2, \dots, z_n), \quad \varphi(z') = O(|z'|^{\nu+1})$$

can never be nearly hyperbolic if $\nu \geqslant 3$.

(b) Varieties that are nearly hyperbolic but not locally hyperbolic at the origin include those of the form

$$F(z) = z_1^3 + z_2^3 + z_3^3 + \varphi(z_1, z_2, z_3) = 0$$

where φ is holomorphic in some neighborhood of the origin, is real at real points, and $\varphi(z) = O(|z|^4)$. From [4], Example 1, it then follows that the varieties $V(F)$ satisfy the condition $\mathrm{PL}_{\mathrm{loc}}$ at the origin so it is a consequence of Proposition 2.5 that they are nearly hyperbolic at the origin. However, it is tedious to give a complete, separate proof that they are nearly hyperbolic so we will not include one here. The main point of such an argument is that the part of $V(F)$ that lies in balls $B(x,r)$ that contain the origin, e.g. with $r > (1+\delta)|x|$, cannot be contained in a small strip about any hyperplane, because the three different branches of $V(F)$ over the hyperplane are distance $\delta_1|z|$ apart. Therefore, the only admissible subspaces L and balls $B(x,r)$ with the property that $V(F)\cap B(x,r)$ is contained in a strip of width εr about $x+L$ occur with x in or near the tangent cone $z_1^3+z_2^3+z_3^3=0$ to $V(F)$ at the origin and have $r < \delta_2|x|$. In this case, $V(F)\cap B(x,r)$ is in fact a single sheeted graph over the tangent cone which makes the required hyperbolicity condition valid.

(c) Define $P \in \mathbb{C}[x,y,z]$ by

$$P(x,y,z) := \frac{1}{2}y(x^2-y^2) - (x-y)z + z.$$

Then $V(P) := \{\zeta \in \mathbb{C}^3 : P(\zeta) = 0\}$ satisfies (SPL) by [7], Example 5.5.

3. Auxiliary results

In this section we will prove some auxiliary results that we need in the next section to prove our main theorem.

Lemma 3.1. — *Let $P \in \mathbb{C}[w_1, ..., w_k]$ be homogeneous of degree $\nu \geqslant 1$ and suppose that P depends on all the variables; that is, there is no linear change of coordinates that makes P a function of fewer than k variables. Then for each $1 \leqslant i \leqslant k$, there is a linear, constant coefficient differential operator L_i of pure degree $\nu - 1$ such that $(L_iP)(w) = w_i$ for all $w \in \mathbb{C}^k$.*

Proof. — Suppose that the span of the linear functions $l_\beta = D_w^\beta P$, $|\beta| = \nu - 1$, does not include all the linear functions. Then there is a “dual linear functional”, i.e., a vector $w_0 \neq 0 \in \mathbb{C}^k$, such that $l_\beta(w_0) = 0$ for all $|\beta| = \nu - 1$. It is no loss of generality to assume coordinates to be chosen so that $w = (w_1, w')$ and the vector w_0 is $(1, 0, \dots, 0)$. We will show that this implies $P(w) = P(w')$, contrary to the hypothesis.

To see this, write out

$$P(w_1, w') = \sum_{i=0}^{\nu} w_1^i P_{\nu-i}(w')$$

where P_j is a homogeneous polynomial of degree j. Apply the differential operators of order $\nu-1$, $D^{i,\gamma}_{w_1,w'}$ where the multi-index γ has length $\nu-1-i$. The resulting set of equations is:

$$\begin{aligned}
\frac{1}{(\nu-1)!} D^{\nu-1,0}_{w_1,w'} P(w_1, w') &= \nu w_1 P_0 + P_1(w') \\
\frac{1}{(\nu-2)!\gamma!} D^{\nu-2,\gamma}_{w_1,w'} P(w_1, w') &= (\nu-1) w_1 D^{\gamma}_{w'} P_1(w') + D^{\gamma}_{w'} P_2(w'), \quad |\gamma| = 1 \\
\vdots &= \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\frac{1}{\gamma!} D^{0,\gamma}_{w_1,w'} P(w_1, w') &= w_1 D^{\gamma}_{w'} P_{\nu-1}(w') + D^{\gamma}_{w'} P_{\nu}(w'), \ |\gamma| = \nu - 1
\end{aligned}$$

Evaluate these equations at the point $w_0 = (1, 0, \dots, 0)$. The left hand sides are all 0 by our choice of w_0. On the right hand sides, the second terms all vanish since they are the values at the origin of the $(j-1)$-st derivative of a homogeneous polynomial of degree j. We conclude that the first terms also must vanish. But these are all the possible j-th derivatives of a homogeneous polynomial in w' of degree $j < \nu$. Thus, each of the homogeneous polynomials P_j must vanish for $0 \leqslant j < \nu$; that is,

$$P(w) = P_\nu(w')$$

which contradicts the assumption that P depends on all the variables. $\square$

Next let p_ν be a homogeneous polynomial in n variables of degree ν that depends exactly on l variables, where $1 \leqslant l \leqslant n$. Then we can assume that coordinates are chosen so that p_ν depends only on $w' = (w_1, \dots, w_l)$, while w'' denotes $(w_{l+1}, \dots, w_n)$. By Lemma 3.1, we can choose homogeneous, linear, constant coefficient differential operators $L_1, \dots, L_l$ in the variables $w_1, \dots, w_l$ such that

$$L_i(p_\nu) = w_i, \ 1 \leqslant i \leqslant l. \tag{3.1}$$

Since L_i commutes with $\frac{\partial}{\partial w_j}$ we have

$$L_i\left(\frac{\partial p_\nu}{\partial w_j}\right) = \delta_{i,j}, \ 1 \leqslant i \leqslant l, \ 1 \leqslant j \leqslant n. \tag{3.2}$$

The operators $L_1, \ldots, L_l$ provide a way to decompose polynomials of degree at most $\nu - 1$ into a part in the span of the derivatives $\frac{\partial p_\nu}{\partial w_j}$ and a part that contains no such terms, as we show next.

LEMMA 3.2. — *Let p_ν and $L_1, \ldots, L_l$ be as above and denote by $\mathcal{P}_{\nu-1}$ the subspace of $\mathbb{C}[w_1 \ldots w_n]$ of polynomials of degree at most $\nu - 1$. Then each $Q \in \mathcal{P}_{\nu-1}$ has a unique decomposition in the form*

$$Q(w) = \sum_{i=1}^{l} c_i \frac{\partial p_\nu}{\partial w_i} + q(w)$$

where

$$q \in \mathcal{N}(L_1, \ldots, L_l) := \{q \in \mathcal{P}_{\nu-1} : L_i(q) = 0,\ 1 \leqslant i \leqslant l\}$$

$$\text{and } c_i = L_i(Q),\ 1 \leqslant i \leqslant l.$$

Whenever $q \in \mathcal{N}(L_1, \ldots, L_l)$ is not identically zero, then $p_\nu + q$ has local vanishing of order less than ν at each $\zeta \in \mathbb{C}^n$.

Proof. — From (3.2) it follows that the vectors $\frac{\partial p_\nu}{\partial w_1}, \ldots, \frac{\partial p_\nu}{\partial w_l} \in \mathcal{P}_{\nu-1}$ are linearly independent and that $\{L_1, \ldots, L_l\}$ is the basis dual to $\{\frac{\partial p_\nu}{\partial w_1}, \ldots, \frac{\partial p_\nu}{\partial w_l}\}$. Obviously, these facts imply the first assertion of the Lemma.

To prove the second assertion, we argue as follows. If $Q := p_\nu + q$ vanished to order ν at $\zeta \in \mathbb{C}^n$, then $p_\nu(x) + q(x) = p_\nu(x - \zeta)$ so that $q(x)$ must also be a function of $x_1, \ldots, x_l$ alone. Applying the differential operators L_i to the last equation then gives

$$x_i = L_i(p_\nu) = L_i(p_\nu + q) = L_i(p_\nu(x - \zeta)) = x_i - \zeta_i$$

so that we must have $\zeta_1 = \ldots = \zeta_l = 0$. Then $p_\nu(x) = p_\nu(x - \zeta)$ so that $q(x) \equiv 0$, contrary to the hypothesis. $\square$

DEFINITION 3.3. — *Let Δ_+^m denote the set of $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$ such that $0 < t_i < 1$ for $i = 1, 2, \ldots, m$. For $t \in \Delta_+^m$ and $z \in B^n(0, r)$ fix*

$$F(t, z) = \sum_{k \geqslant \nu} p_k(z) + \sum_{|j|>0, |\beta| \leqslant \nu-1} a_{j,\beta} t^j z^\beta + \sum_{j>0, \beta \geqslant \nu} a_{j,\beta} t^j z^\beta, \tag{3.3}$$

where $j \in \frac{1}{Q}\mathbb{N}_0^m$ for some $Q \in \mathbb{Q}$, $Q \geqslant 0$. Assume that, for fixed $t \in \Delta_+^m$, $F(t, z)$ is analytic in $\{w \in \mathbb{C}^n : |w| \leqslant \varrho(t)\}$, where $\varrho(t)$ tends to infinity as t tends to zero in Δ_+^m . Assume further that p_ν depends exactly on the *variables $z_1, \dots, z_l$ and choose differential operators $L_1, \dots, L_l$ which satisfy the condition (3.1). Then we say that F is in* normal form *if we have*

$$L_i\left(\sum_{|j|>0, |\beta|\leqslant \nu-1} a_{j,\beta} t^j z^\beta\right) = 0.$$

If F is not in normal form, there is a way to transform it into this form in such a way that neither the property $\mathrm{PL}_{\mathrm{loc}}(0)$ nor the nearly hyperbolicity properties of the family $(V_t)_{t\in\Delta_+^m}$ associated with F are changed. To show this we need the following proposition.

PROPOSITION 3.4. — *Let $(\Phi_t)_{t\in T}$ be a family of biholomorphic maps of a neighborhood of the origin in $\mathbb{C}^n$ to a neighborhood of the origin in $\mathbb{C}^n$ such that the following conditions are satisfied.*

(i) There exist $0 < r_1 < r_0$ and $0 < \varrho_1 < \varrho_0$ such that $\Phi_t(B(0, r_0)) \subset B(0, \varrho_0)$ and $\Phi_t^{-1}(B(0, \varrho_1)) \supset \overline{B(0, r_1)}$.

(ii) $\Phi_t(z) = \Phi_0(z) + \sum_{|j|>0} a_{j,\beta} t^j z^\beta$, $a_{j,\beta} \in \mathbb{R}^n$, has a uniformly and absolutely convergent expansion for $t \in T$, $|z| < \varrho_0$, which is a power series in z and a fractional power series in t; i.e., $j = k/Q$ for some fixed integer $Q \geqslant 1$ and $k \in \mathbb{N}_0^m$.

(iii) $\Phi_0(0) = 0$ and Φ_0 maps real points to real points.

(iv) The derivatives $D_z\Phi_t(z)$ and $D_w\Phi_t^{-1}(w)$ are uniformly bounded for $z \in B(0, r_0)$, $t \in T$, and for $w \in B(0, \varrho_1)$, $t \in T$.

Then for each family $(V_t)_{t\in T}$ in $\mathbb{C}^n$ satisfying $0 \in V_t$ for each $t \in T$ the following assertions hold:

(a) $(V_t)_{t\in T}$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ if and only if the family $(\Phi_t V_t)_{t\in T}$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$.

(b) $(V_t)_{t\in T}$ is nearly hyperbolic in some neighborhood of the origin if and only if the family $(\Phi_t V_t)_{t\in T}$ has this property.

Proof. — (a) The conditions (ii) and (iii) on the mappings Φ_t imply that

$$\frac{1}{C}|\operatorname{Im} w| \leqslant |\operatorname{Im} z| \leqslant C|\operatorname{Im} w|$$

for $w = \Phi_t(z)$. Plurisubharmonic functions u_t on a fixed neighborhood of the origin in V_t correspond uniquely to plurisubharmonic functions v_t on a fixed neighborhood of the origin in $\Phi_t V_t = \{w = \Phi_t(z) : \ z \in V_t\}$ by $v_t(w) = u_t(z) = u_t(\Phi_t^{-1}(w))$. And real points of V_t are mapped to real points of $\Phi_t(V_t)$ and vice-versa. Thus, it is clear that the condition $\mathrm{PL}_{\mathrm{loc}}(0)$ is preserved, although there is a change in the parameters that describe the condition.

(b) Since nearly hyperbolicity is invariant under a real linear change of coordinates, we may assume that $D_w\Phi_0 = \mathrm{id}$. Then, also for t and w near the origin, the derivatives $D_w\Phi_t$ are close to the identity. Therefore, when $x \in \mathbb{R}^n$ the small balls $B(x,r)$ used in the definition of nearly hyperbolicity are almost preserved, i.e., for small $\sigma, r > 0$ we have

$$B(\Phi_t(x), (1-\sigma)r) \subset \Phi_t(B(x,r)) \subset B(\Phi_t(x), (1+\sigma)r).$$

Similary, the admissible subspaces and projections used in the definition of nearly hyperbolic varieties are very well approximated by replacing $\Phi_t(L)$ by L and keeping the same projection π_L. We omit the routine but tedious details. $\square$

Proposition 3.5. — *For F as in formula (3.3) assume that the leading term p_ν of F depends exactly on the first l-variables and denote by L_i, $1 \leqslant i \leqslant l$, operators that satisfy (3.1). Then in a neighborhood of $t = 0 \in \overline{\Delta}_+^m$, $w = 0$, the system of equations*

$$L_i(F(t,w)) = 0, \ 1 \leqslant i \leqslant l,$$

has a unique solution

$$w_i = h_i(t, w_{l+1}, \dots, w_n), \ 1 \leqslant i \leqslant l,$$

where h_i is a fractional power series in t, analytic in w, and has real power series coefficients. Further, there are $\varepsilon > 0$ and fractional power series φ_i, $i = 1, \dots, l$, such that the following assertions hold:

- *(a) $h_i(t, w'') = \varphi_i(t) + O(w''^2) + O(t^\varepsilon w'')$,*
- *(b) $F_1(t, w', w'') := F(t, w' + h(t, w''), w'')$ is in normal form, if we let $h(t, w'') := (h_1(t, w''), \dots, h_l(t, w''))$.*
- *(c) The family $(V_t)_{t \in \Delta_+^m}$ defined by F satisfies* $\mathrm{PL}_{\mathrm{loc}}(0)$ *(resp. is nearly hyperbolic at 0) if and only if the corresponding family defined by F_1 satisfies* $\mathrm{PL}_{\mathrm{loc}}(0)$ *(resp. is nearly hyperbolic at 0).*

Proof. — Consider the system of equations

$$L_i(F(t,w)) = 0, \ 1 \leqslant i \leqslant l. \tag{3.4}$$

Using the expansion of F given in (3.3), the notation

$$Q(t,w)=\sum_{|j|>0,|\beta|\leqslant\nu-1} a_{j,\beta}t^j w^\beta,$$

and the fact $L_i(p_\nu)=w_i$ for $1\leqslant i\leqslant l$, the system (3.4) has the form

$$0=w_i+O(|w|^2)+L_i(Q(t,w))+O(t^\varepsilon|w|),\ 1\leqslant i\leqslant l, \tag{3.5}$$

where ε is the smallest positive exponent of t in the series $\sum\limits_{|j|>0,|\beta|\geqslant\nu} a_{j,\beta}t^j w^\beta$. After replacing the Puiseux series terms in t by $t=s^q$, q the denominator in the Puiseux series, the implicit function theorem gives the existence and uniqueness of a solution with the properties as stated in (a). To prove the remaining part of the Proposition, just observe that the change of variables $\hat{w}_i=w_i-h_i(t,w'')$, $\hat{w}''=w''$, transforms the system (3.4) to one in which the unique solution is $h_i(t,w'')\equiv 0$. In this case, the equations (3.5) (evaluated at $(w_1,\dots,w_l)=0$), show that

$$0=L_i(Q(t,w))+O(t^\varepsilon|w|)+O(|w|^2),\ 1\leqslant i\leqslant l.$$

However, the functions $L_i(Q(t,w))$ are functions of t alone since the L_i are differential operators of order $\nu-1$ applied to polynomials of degree $\leqslant\nu-1$. Thus, letting $w''\to 0$ shows that $L_i(Q(t,w))\equiv 0$, $1\leqslant i\leqslant l$; i.e., after the change of variables the function F_1 is in normal form. The assertions in (c) follow from Proposition 3.4 (a) (resp. Proposition 3.4 (b)) applied to the family $(\Phi_t)_{t\in U}$ of biholomorphic maps defined by

$$\Phi_t(w',w'')=(w'+h(t,w''),w''),$$

where U is a suitable neighborhood of $(0,0)\in\overline{\Delta}_+^m\times\mathbb{C}^n$. $\square$

4. Main result

The aim of the present section is the proof of the following theorem.

THEOREM 4.1. — *Let V be an analytic surface in $B^3(0,r)$ for some $r>0$ which contains the origin. Then V satisfies* $\mathrm{PL}_{\mathrm{loc}}(0)$ *if and only if V satisfies the following three conditions:*

(1) V is nearly hyperbolic at 0.

(2) Each limit variety $T_{\gamma,d}V$ satisfies (SPL)*.*

To prove this theorem, we next introduce some notation.

Definition 4.2. — *For $\eta \geqslant 1$, $T > 0$, and $\delta > 0$ we let*

$$\Gamma_\eta(T,\delta) := \{t \in \mathbb{C} : \ 0 < \operatorname{Re} t < T \text{ and } |\operatorname{Im} t| < \delta |t|^\eta\}.$$

Definition 4.3. — *Let $F : \Gamma_\eta(T,\delta_1) \times B^2(0,\delta_2) \to \mathbb{C}$ be holomorphic in all three variables and a fractional power series in the first variable. Assume that F admits an expansion of the form*

$$F(t,z) = p_\nu(z) + \sum_{l=\nu+1}^{\infty} p_l(z) + \sum_{j>0,\beta\in\mathbb{N}_0^2} a_{j,\beta} t^j z^\beta, \tag{4.1}$$

where p_l is a homogeneous polynomial of degree l with real coefficients or identically zero, $p_\nu \not\equiv 0$, and where $a_{j,\beta} \in \mathbb{R}$ for all $j > 0, \beta \in \mathbb{N}_0^2$.

We say that for $\alpha \geqslant 0$ and $\eta \geqslant 1$ the variety

$$V = V(F) := \{(t,z) \in \Gamma_\eta(T,\delta_1) \times B^2(0,\delta_2) : F(t,z) = 0\}$$

satisfies the condition $\mathrm{PL}(\alpha,\eta)$ if the following holds:

There exist $0 < T_0 \leqslant T$, $\delta_3 > 0$, $\delta_4 > 0$, and $A > 0$ such that each $u \in \mathrm{PSH}(V)$ which satisfies

(a) $u(t,z) \leqslant 0$, $(t,z) \in \mathbb{R}^3 \cap V$ and

(b) $u(t,z) \leqslant |t|^\alpha$, $(t,z) \in V$

already satisfies

(c) $u(t,z) \leqslant A|t|^\alpha \Big(|\operatorname{Im} z| + \frac{|\operatorname{Im} t|}{|t|^\eta}\Big)$, $(t,z) \in V \cap (\Gamma_\eta(T_0,\delta_3) \times B^2(0,\delta_4))$.

Lemma 4.4. — *For F as in 4.3 assume that $(V(F(t,\cdot)))_t$ is nearly hyperbolic, that p_ν is a product of real linear forms, and let*

$$a := \min\left\{\frac{j}{\nu - |\beta|} : \ a_{j,\beta} \neq 0, \ |\beta| < \nu\right\},$$

where the numbers $a_{j,\beta}$ are given by (4.1) and where $a := \infty$ if $a_{j,\beta} = 0$ for each $j > 0$ and $|\beta| < \nu$. Then there exist $C > 0$, $A > 0$, $T_0, \delta_3 > 0$, and $\delta_4 > 0$ such that each $u \in \mathrm{PSH}(V(F))$ which satisfies the conditions (a) and (b) of 4.3 also satisfies:

(i) $u(t,z) \leqslant A|t|^\alpha\{|\operatorname{Im} z| + \frac{|\operatorname{Im} t|}{|t|^\eta}\}$, $(t,z) \in V(F) \cap (\Gamma_\eta(T_0,\delta_3) \times B^2(0,\delta_4))$, $|z| > C|t|^a$.

(ii) $u(t,z) \leqslant A|t|^\alpha\{|z| + |t|^a + \frac{|\operatorname{Im} t|}{|t|^\eta}\}$, $(t,z) \in V(F) \cap (\Gamma_\eta(T_0,\delta_3) \cap B^2(0,\delta_4))$ *whenever* $a < \infty$.

Here, the convention $t^\infty = 0$ *is used.*

Proof. — Since a real linear change in the z-variables does not change the number a defined above, it is no loss of generality to assume that in the expansion (4.1) for F the polynomial p_ν has the form

$$p_\nu(z_1,z_2) = \prod_{k=1}^{p}(z_1 - u_k z_2)^{n_k},$$

where $u_k \in \mathbb{R}$ for $1 \leqslant k \leqslant p$ and where $u_j \neq u_i$ for $i \neq j$, which implies that p_ν is regular in z_1. Hence the Weierstrass factorization theorem implies that for suitable $0 < T_1 < T$ and $0 < \delta_4 < \delta_2$

$$F(t,z) = U(t,z)\prod_{j=1}^{\nu}(z_1 - \alpha_j(t,z_2)),\ t \in \Gamma_\eta(T_1,\delta_3),\ z_2 \in B(0,\delta_4),$$

where U is a unit and where $\prod_{j=1}^{\nu}(z_1 - \alpha_j(t,z_2))$ is a Weierstrass polynomial in z_1, z_2, and $t^{1/q}$ for a suitable number $q \in \mathbb{N}$.

Since F is nearly hyperbolic by hypothesis, we claim that there exist $C > 0$ and $\eta_0 > 0$ such that

$$\alpha_j(t,z_2) \text{ is real for each } (t,z_2) \in \mathbb{R}^2 \text{ satisfying } |t| \leqslant \eta_0 \text{ and } |z_2| \geqslant C|t|^a. \tag{4.2}$$

Since p_ν has the form given above, a standard application of the theorem of Rouché shows that for each sufficiently small number $\varepsilon > 0$ there exist $\eta_0 > 0$ and $C_1 > 1$ such that for each t satisfying $0 < t < \eta_0$ and each z_2 satisfying $|z_2| \geqslant C_1|t|^a$ there are exactly n_k solutions $z_1(t,z_2)$ of the equation $F(t,z_1,z_2) = 0$ which satisfy

$$|z_1(t,z_2) - u_k z_2| \leqslant \varepsilon|z_2|. \tag{4.3}$$

Next we show that they all have to be real for real z_2, due to the hypothesis that $(V(F(t,\cdot)))_t$ is nearly hyperbolic. To do so let $L_k := \{(u_k z_2, z_2) : z_2 \in \mathbb{C}\}$ and define $\pi_k : \mathbb{C}^2 \to L_k$, $\pi_k(z_1,z_2) := (u_k z_2, z_2)$. Since $u_1,\ldots,u_p$ are real, L_k and π_k are admissible in the sense of Definition 2.4. Next choose $\delta > 0$ and the function ε_0 according to the definition of nearly hyperbolicity and let

$$\varepsilon_1 := \min_{1\leqslant k\leqslant p} \varepsilon_0(||\mathrm{id}_{\mathbb{C}^n} - \pi_k||).$$

Since the numbers $u_1, \ldots, u_p$ are pairwise different, we have

$$\sigma := \min\{|u_j - u_k| : 1 \leqslant j \neq k \leqslant p\} > 0.$$

Next choose $M > 1$ such that for $1 \leqslant k \leqslant p$ the following estimate holds:

$$\mathrm{diam}(((\mathrm{L_k} + \mathrm{B}(0, \mathrm{r})) \cap \{(\mathrm{z_1}, 1) : |\mathrm{z_1}| < \mathrm{r}\}) \leqslant \mathrm{Mr}.$$

Then choose $0 < \lambda < \sigma/(2M)$ and $0 < \varepsilon < \varepsilon_1 \lambda/(1+\lambda)$. For this choice of ε we now take $\eta_0 > 0$ and $C_1 \geqslant 1$ such that the statement about the zeros of F above holds. Next we let $C := C_1(1+\lambda)/\lambda$ and we fix $0 < t < \eta_0$ and $x_2 \in \mathbb{R}$ satisfying $|x_2| \geqslant Ct^a$. Then we remark that our choices imply the following: If we fix $x := (u_k x_2, x_2) \in L_k$ and let $r := \lambda|x_2|$ then for each branch α_j of the zero set of F which stays in the ε-cone near L_k we have for $|z_2 - x_2| < r$ that

$$\begin{aligned} |(\alpha_j(t, z_2), z_2) - (u_k z_2, z_2)| &= |\alpha_j(t, z_2) - u_k z_2| < \varepsilon|z_2| \leqslant \varepsilon(1+\lambda)|x_2|| \\ &\leqslant \varepsilon_1 \lambda |x_2 = \varepsilon_1 r. \end{aligned}$$

From this estimate and the choice of λ and M together with $x + L_k = L_k$ we now get

$$V(F(t, \cdot)) \cap B(x, r) \subset S(L_k + x, \varepsilon_0 r) \subset S(L_k, \varepsilon_0(||\mathrm{id}_{\mathbb{C}^n} - \pi_\mathrm{k}||)\mathrm{r}).$$

Since V is nearly hyperbolic by hypothesis, it follows from this that $\alpha_j(t, \xi_2)$ is real whenever $|\xi_2 - x_2| < r/2 = \lambda|x_2|/2$, which implies the statement of our claim (4.2).

Assume now that $a < \infty$. Then let $u \in \mathrm{PSH}(V(F))$ be given and assume that u satisfies the conditions (a) and (b) of 4.3. Then we define

$$\varphi : \ \Gamma_\eta(T_1, \delta_3) \times B(0, \delta_4) \to [-\infty, \infty[,$$

$$\varphi(t, z_2) := \max\{u(t, \alpha_j(t, z_2), z_2) : \ 1 \leqslant j \leqslant \nu\}.$$

Note that φ is plurisubharmonic on $\Gamma_\eta(T_1, \delta_3) \times B(0, \delta_4)$. By the properties of u and (4.2) it follows that $\varphi(t, z_2) \leqslant 0$ whenever $(t, z_2) \in (\Gamma_\eta(T_1, \delta_3) \times B(0, \delta_4)) \cap \mathbb{R}^2$ satisfies $|z_2| \geqslant Ct^a$ and that φ satisfies

$$\varphi(t, z_2) \leqslant t^\alpha, \ 0 < t < T_1, \ z_2 \in B(0, \delta_4). \tag{4.4}$$

Hence for $0 < t < T_1$ the function $z_2 \mapsto \varphi(t, z_2)$ is subharmonic in the disk $B(0, \delta_4)$ and is non-positive for $z_2 \in]-\delta_4, \delta_4[\setminus [-Ct^a, Ct^a]$. By [1], Lemma 5.8, this implies the existence of $C_0 > 0$ such that

$$\frac{\varphi(t, z_2)}{t^\alpha} \leqslant C_0 \left| \mathrm{Im} \sqrt{z_2^2 - (Ct^a)^2} \right|, \ |z_2| \leqslant \delta_4/2.$$

This estimate implies (see, e.g., [1], Lemma 5.7) the existence of $A_1 > 0$ and $A_2 > 0$ such that for $0 < t < T_1$ we have

$$\varphi(t, z_2) \leqslant A_1 t^\alpha \, |\mathrm{Im}\, z_2| \, , \ |z_2| \geqslant 2Ct^a \tag{4.5}$$

$$\varphi(t, z_2) \leqslant A_2 t^\alpha (|z_2| + t^a), \ z_2 \in B(0, \delta_4/2). \tag{4.6}$$

Next fix $0 < t_0 < T_1/2$, $z_2 \in B(0, \delta_4/2)$ satisfying $|z_2| \geqslant 2C(1+T_1)^a t_0^a$, let $A_1' := (1+\delta_3)^\alpha A_1$, and consider the function

$$\psi : \ B(t_0, \delta_3 t_0^\eta) \to [-\infty, \infty[, \ \psi(t) := \varphi(t, z_2) - A_1' t_0^\alpha |\operatorname{Im} z_2|.$$

The choice of A_1' together with the estimate

$$(1-\delta_3)t_0 \leqslant |t| \leqslant (1+\delta_3)t_0, \ t \in B(t_0, \delta_3 t_0^\eta), \tag{4.7}$$

implies $\psi(t) \leqslant 0$ for $t \in B(t_0, \delta_3 t_0^\eta) \cap \mathbb{R}$. From (4.4) it follows that

$$\psi(t) \leqslant (t_0 + \delta_3 t_0^\eta)^\alpha \leqslant (1+\delta_3)^\alpha t_0^\alpha, \ t \in B(t_0, \delta_3 t_0^\eta). \tag{4.8}$$

Using the harmonic measure for the half disk we get the existence of a universal constant $C_1 > 0$ such that

$$\psi(t) \leqslant C_1 \frac{(1+\delta_3)^\alpha t_0^\alpha}{\delta_3 t_0^\eta} |\operatorname{Im} t|, \ t \in B(t_0, \delta_3 t_0^\eta/2). \tag{4.9}$$

This estimate, together with the definition of ψ and φ, implies

$$u(t, z_1, z_2) \leqslant \varphi(t, z_2) \leqslant A_1(1+\delta_3)^\alpha t_0^\alpha \Big(|\operatorname{Im} z_2| + \frac{C_1}{A_1 \delta_3 t_0^\eta} |\operatorname{Im} t| \Big)$$

for $t \in B(t_0, \delta_3 t_0^\eta/2)$, $2C(1+\delta_3)^a t_0^a \leqslant |z_2| \leqslant \delta_4/2$. Because of (4.7) we now get the existence of $A_3 > 0$ such that for $(t, z_1, z_2) \in V(F) \cap \Gamma_\eta(T_1/2, \delta_3/2) \times B^2(0, \delta_4/2)$ and $|z_2| \geqslant 2C \Big(\frac{1+\delta_3}{1-\delta_3} \Big)^a |t|^a$ we have

$$u(t, z_1, z_2) \leqslant A_3 |t|^\alpha \Big(|\operatorname{Im} z_2| + \frac{|\operatorname{Im} t|}{|t|^\eta} \Big).$$

Because of the special form of F, this estimate implies (i), provided that we change the constants appropriately.

Note that for $a = \infty$ the proof of the same estimate is easier, since this implies that F is locally hyperbolic.

To prove part (ii) assume that $a < \infty$, fix $0 < t_0 < T_1/2$ and $z_2 \in B(0, \delta_4/2)$, and consider the function

$$\psi : B(t_0, \delta_3 t_0^\eta) \to [-\infty, \infty[, \ \psi(t) := \varphi(t, z_2) - A_2(1+\delta_3)^\alpha t_0^\alpha(|z_2| + (1+\delta_3)^a t_0^a).$$

For $t \in B(t_0, \delta_3 t_0^\eta) \cap \mathbb{R}$ the estimate (4.6) together with the properties of φ implies $\psi(t) \leqslant 0$, while (4.4) implies that (4.8) holds. Therefore, the same arguments that we used to prove (i) imply that ψ satisfies the estimate (4.9). From (4.8) and the definition of ψ and φ we now get

$$u(t, z_1, z_2) \leqslant \varphi(t, z_2) \leqslant A_2(1 + \delta_3)^\alpha t_0^\alpha \Big(|z_2| + (1 + \delta_3)^a t_0^a + \frac{C_1}{A_2 \delta_3 t_0^\eta} |\operatorname{Im} t| \Big)$$

for $t \in B(t_0, \delta_3 t_0^\eta/2)$, $z_2 \in B(0, \delta_4/2)$, and $0 < t_0 < T_1/2$. Using (4.7) we get the existence of $A_4 > 0$ such that

$$u(t, z_1, z_2) \leqslant A_4 |t|^\alpha \Big(|z_2| + |t|^a + \frac{|\operatorname{Im} t|}{|t|^\eta} \Big),$$

$$(t, z) \in V(F) \cap (\Gamma_\eta(T_1/2, \delta_3/2) \times B^2(0, \delta_4/2)).$$

Hence (ii) holds. $\square$

Lemma 4.5. — *For F as in 4.3 assume that $(V(F(t,\cdot)))_t$ is nearly hyperbolic and that $\nu = 1$. Then $V(F)$ satisfies* PL(α, η) *for each $\alpha \geqslant 0$ and each $\eta \geqslant 1$.*

Proof. — The present hypotheses imply that p_1 is a real linear form. After a real linear change in the variables z_1, z_2, we may therefore assume that $p_1(z_1, z_2) = z_1$. Hence F has the form

$$F(t, z) = z_1 + \sum_{l=2}^{\infty} p_l(z) + \sum_{j>0, \beta \in \mathbb{N}_0^2} a_{j,\beta} t^j z^\beta.$$

Moreover, $F(t, z)$ is real for real (t, z). Since F is a fractional power series in t and analytic in z, an application of the real and the complex implicit function theorem shows that there exist $T_1 < T, \delta_3 < \delta_1, \delta_4 < \delta_2$ such that

$$F(t, z) = z_1 - \alpha(t, z_2), (t, z) \in \Gamma_1(T_1, \delta_3) \times B(0, \delta_4),$$

where $\alpha(t, z_2)$, is Puiseux-analytic in t, analytic in z, and real for (t, z) real. Next let $\alpha \geqslant 0$ and $\eta \geqslant 1$ be given and fix $u \in \mathrm{PSH}(V(F) \cap (\Gamma_\eta(T, \delta_1) \times B^2(0, \delta_2)))$ satisfying the conditions (a) and (b) of PL(α, η). Then we define

$$\varphi : \Gamma_\eta(T_1, \delta_3) \times B(0, \delta_4) \to [-\infty, \infty[, \ \varphi(t, z_2) := u(t, \alpha(t, z_2), z_2).$$

Note that φ is plurisubharmonic and that the properties of u imply $\varphi(t, z_2) \leqslant 0$ whenever (t, z_2) are real and

$$\varphi(t, z_2) \leqslant t^\alpha, 0 < t < T_1, z_2 \in B(0, \delta_4). \tag{4.10}$$

Using the harmonic measure for the half disk and these properties of φ it follows that there exists a universal constant C_1 such that the estimates above imply

$$\varphi(t, z_2) \leqslant C_1 |\operatorname{Im} z_2| t^\alpha, 0 < t < T_1, z_2 \in B(0, \delta_4/2). \tag{4.11}$$

Now note that the proof can be completed by the arguments that we used to prove part (i) of Lemma 4.4. □

Remark 4.6. — If the estimate 4.3 (c) holds for some $\delta > 0$ and $\delta_4 > 0$ in $V(F) \cap (\Gamma_\eta(T, \delta) \times B^2(0, \delta_4))$ then it also holds in $V(F) \cap (\Gamma_\eta(T, C) \times B^2(0, \delta_4))$ for each $C \geqslant \delta$, provided that A is replaced by $A' = A'(A, \delta)$. To see this, note that by 4.3 (b) we have for each $t \in \Gamma_\eta(T, C) \setminus \Gamma_\eta(t, \delta)$

$$u(t, z) \leqslant |t|^\alpha = \frac{1}{\delta}|t|^\alpha \delta \leqslant \frac{1}{\delta}|t|^\alpha \frac{|\operatorname{Im} t|}{|t|^\eta} \leqslant \max(A, \frac{1}{\delta})|t|^\alpha \{|\operatorname{Im} z| + \frac{|\operatorname{Im} z|}{|t|^\eta}\}.$$

PROPOSITION 4.7. — *For F as in 4.3 assume that $(V(F(t, \cdot)))_t$ is nearly hyperbolic and that $T_0(\lim_{t\to 0} V(F(t, \cdot)))$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Moreover, assume that for each $\gamma : [0, \delta[\to \mathbb{R}^2$ which is defined by a Puiseux series, and for each $d > 0$ the family*

$$W_t := \{w \in \mathbb{C}^2 : F(t, \gamma(t) + t^d w) = 0\}$$

has a limit variety $\lim_{t\to 0} W_t$ that satisfies (SPL). *Then $V(F)$ satisfies $\mathrm{PL}(\alpha, \eta)$ for each $\alpha \geqslant 0$ and each $\eta \geqslant 1$.*

Proof. — To prove this by induction on ν, note first that by Lemma 4.5 the assertion holds for $\nu = 1$. To prove the induction step, we assume that the assertion holds for all F with $\nu(F) < \nu$. Then we fix F as in the Proposition as well as $\alpha \geqslant 0$ and $\eta \geqslant 1$.

Case 1: p_ν depends on both variables z_1 and z_2.

Then we choose differential operators L_1 and L_2 for p_ν, according to Lemma 3.1. It is no restriction to assume that F is in normal form with respect to L_1 and L_2 because of the following argument: The transformation that puts $F(t, z)$ into normal form is a shear transformation in the z variable, $z \leftarrow z + \gamma(t)$ where $\gamma(t)$ is a Puiseux series in t with real coefficients. Neither the hypotheses of the condition $\mathrm{PL}(\alpha, \eta)$ nor its conclusion is changed by such a transformation, because $|\operatorname{Im}\gamma(t)| \leqslant |\operatorname{Im} t|/|t|^\eta$ since $\eta \geqslant 1$. Next we let

$$a := \min\left\{\frac{j}{\nu - |\beta|} \,:\, |\beta| < \nu,\ a_{j,\beta} \neq 0\right\},$$

where we use the notation from (4.1).

If $a < \infty$ then for each $\xi \in \mathbb{R}^2$ we define the local reduction H_ξ of F by

$$H_\xi(t, w) := \frac{F(t, t^a(\xi + w))}{t^{a\nu}}.$$

To show that $\nu(H_\xi) < \nu$ for each $\xi \in \mathbb{R}^2$, we let

$$Q(t, z) := \sum_{j>0, |\beta|\leqslant\nu-1} a_{j,\beta} t^j z^\beta = \sum_{|\beta|\leqslant\nu-1} f_\beta(t) z^\beta.$$

As F is in normal form with respect to L_1 and L_2, we have $L_i(Q(t, z)) = 0$ for $i = 1, 2$. Since

$$\frac{1}{t^{a\nu}} Q(t, t^a w) = \sum_{j>0, |\beta|\leqslant\nu-1} a_{j,\beta} t^{j+a|\beta|-a\nu} w^\beta,$$

the definition of the number a implies that

$$\lim_{t\to 0} \frac{1}{t^{a\nu}} Q(t, t^a w) = \sum_{j=(\nu-|\beta|)a} a_{j,\beta} w^\beta =: q_\xi(w),$$

where q is not identically zero and $\deg(q) < \nu$. Since

$$0 \equiv L_i\left(\frac{1}{t^{a\nu}} Q(t, t^a w)\right) = \sum_{|\beta|=\nu-1} f_\beta(t) t^a L_i(w),\ i = 1, 2,$$

for all t, it follows that

$$0 = \lim_{t\to 0} L_i\left(\frac{1}{t^{a\nu}} Q(t, t^a w)\right) = L_i(q_\xi).$$

Hence we get from Lemma 3.2 that $p_\nu + q$ has local vanishing order less than ν at each $\xi \in \mathbb{C}^2$, which implies $\nu(H_\xi) < \nu$ for each $\xi \in \mathbb{R}^2$. Note that $\lim_{t\to 0} V(H_\xi(t, \cdot)) = V(p_\nu + q_\xi)$. By hypothesis, this variety satisfies (SPL). Hence it follows from Remark 2.11 that $V(p_\nu + q_\xi)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$, whenever $0 \in V(p_\nu + q_\xi)$. By [5], Proposition 3.5, this implies that $T_0 V(p_\nu + q_\xi) = T_0(\lim_{t\to 0} V(H_\xi(t, \cdot)))$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$.
Note also that for any Puiseux series $\sigma(t)$ with values in $\mathbb{R}^2$ and any $\delta > 0$ we have

$$H_\xi(t, \sigma(t) + t^\delta w) = \frac{1}{t^{a\nu}} F(t, t^a(\xi + \sigma(t) + t^\delta w)) = \frac{1}{t^{a\nu}} F(t, t^a\xi + t^a\sigma(t) + t^{a+\delta} w).$$

From this identity, the considerations above, and Proposition 3.4 (b) it follows easily that $H_\xi(t,w)$ satisfies the hypotheses of the present lemma.

Next fix $u \in \mathrm{PSH}(V(F) \cap (\Gamma_\eta(T,\delta_1) \times B^2(0,\delta_2)))$ and assume that u satisfies the conditions (a) and (b) of 4.3. Since

$$T_0(\lim_{t\to 0} V(F(t,\cdot))) = T_0\Big(\sum_{k=\nu}^{\infty} p_\nu\Big) = V(p_\nu)$$

satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ by hypothesis, it follows from the homogeneity of p_ν that $V(p_\nu)$ satisfies (SPL) and hence $\mathrm{PL}(\mathbb{R}^2, \log)$, by [1], Lemma 2.5. By Meise, Taylor, and Vogt [13], Theorem 4.11, p_ν is hyperbolic with respect to each non-characteristic direction. Since p_ν has real coefficients, this implies that p_ν is a product of real linear forms. Therefore, we can apply Lemma 4.4 to get the existence of positive numbers C, A, T_0, δ_3, and δ_4 so that the assertions (i) and (ii) of 4.4 hold. Because of 4.4 (i), the estimate 4.3 (c) of $\mathrm{PL}(\alpha,\eta)$ holds for those $(t,z) \in V(F) \cap (\Gamma_\eta(T_0,\delta_3) \times B^2(0,\delta_4))$ which in addition satisfy $|z| > C|t|^a$. To show that it also holds for such points which satisfy $|z| \leqslant C|t|^a$, we note first that for $(t,z) \in V(F) \cap (\Gamma_\eta(T_0,\delta_3) \times B^2(0,\delta_4))$ satisfying

$$|z| \leqslant C|t|^a \leqslant \frac{|\operatorname{Im} t|}{|t|^\eta}$$

or equivalently

$$|z| \leqslant C|t|^a \text{ and } t \in \Gamma_\eta(T_0,\delta_3) \setminus \Gamma_{\eta+a}(T_0,C)$$

the estimate 4.3 (ii) implies (assuming w.l.o.g. $C \geqslant 1$)

$$\begin{aligned} u(t,z) &\leqslant A|t|^\alpha \left(|z| + |t|^a + \frac{|\operatorname{Im} t|}{|t|^\eta}\right) \leqslant A|t|^\alpha \left((C+1)|t|^a + \frac{|\operatorname{Im} t|}{|t|^\eta}\right) \\ &\leqslant 3A|t|^\alpha \frac{|\operatorname{Im} t|}{|t|^\eta} \leqslant 3A|t|^\alpha \left(|\operatorname{Im} z| + \frac{|\operatorname{Im} t|}{|t|^\eta}\right). \end{aligned}$$

Hence we only have to show that the estimate 4.3 (c) holds for those $(t,z) \in V(F) \cap (\Gamma_\eta(T_0,\delta_3) \times B^2(0,\delta_4))$ which satisfy

$$|z| \leqslant C|t|^a \text{ and } t \in \Gamma_{\eta+a}(T_0,C). \tag{4.12}$$

For such points (t,z) we get from 4.4 (ii) that

$$u(t,z) \leqslant A|t|^\alpha \left(C|t|^a + |t|^a + C|t|^a\right) = (2C+1)A|t|^{\alpha+a}. \tag{4.13}$$

Next fix $\xi \in B^2(0,\delta_4/2) \cap \mathbb{R}^2$, $w \in B^2(0,\delta_4/2)$, and $t \in \Gamma_{\eta+a}(T_0,C)$ and assume that

$$0 = H_\xi(t,w) = \frac{1}{t^{a\nu}} F(t, t^a(\xi + w)).$$

Then the function

$$v: \ V(H_\xi)\cap(\Gamma_{\eta+a}(T_0,C)\cap B^2(0,\delta_4/2)) \to [-\infty,\infty[, \ v(t,w) := u(t,t^a(\xi+w))$$

is plurisubharmonic. Since u satisfies the condition 4.3 (a), we have $v(t,w) \leqslant 0$ when t and w are real. From (4.13) it follows that

$$\frac{1}{(2C+1)A} v(t,w) \leqslant |t|^{\alpha+a}.$$

Now note that $V(H_\xi(t,\cdot))_t$ is nearly hyperbolic by Proposition 3.4. Moreover, if the expansion for H_ξ has q_l as lowest degree homogeneous term in w, then $V(q_l)$ satisfies (SPL) by hypothesis, since it is a limit variety of $V(F)$. Since we have shown above that $\nu(H_\xi) = \deg q_l < \nu$, our induction hypothesis implies that $V(H_\xi)$ satisfies $\mathrm{PL}(\alpha+a,\eta+a)$. Consequently, there exist positive numbers $A_\xi, T_\xi, \delta_{3,\xi}$, and $\delta_{4,\xi}$, so that

$$\begin{aligned} \frac{1}{(2C+1)A} v(t,w) &\leqslant A_\xi |t|^{\alpha+a}\left(|\operatorname{Im} w| + \frac{|\operatorname{Im} t|}{|t|^{\eta+a}}\right) \\ &= A_\xi |t|^{\alpha}\left(|t|^a |\operatorname{Im} w| + \frac{|\operatorname{Im} t|}{|t|^{\eta}}\right) \end{aligned} \tag{4.14}$$

for $(t,w) \in V(H_\xi) \cap (\Gamma_{\eta+a}(T_\xi,\delta_{3,\xi}) \times B^2(0,\delta_{4,\xi}))$. Next note that for T_ξ small enough there exists a constant $D > 0$ such that

$$|t|^a \ |\operatorname{Im} w| \leqslant D\Big(|\operatorname{Im}(t^a(\xi+w))| + \frac{|\operatorname{Im} t|}{|t|^\eta}\Big) \tag{4.15}$$

for $t \in \Gamma_{\eta+a}(T_\xi,\delta_{3,\xi})$, $w \in B^2(0,\delta_{4,\xi})$. Hence we get from (4.14) and (4.15) in connection with Remark 4.6 the existence of $B_\xi > 0$ such that

$$u(t,t^a(\xi+w)) \leqslant B_\xi |t|^\alpha \left(|\operatorname{Im}(t^a(\xi+w))| + \frac{|\operatorname{Im} t|}{|t|^\eta}\right), \tag{4.16}$$

for $(t,t^a(\xi+w)) \in V(F)\cap(\Gamma_{\eta+a}(T_\xi,C)\times B^2(0,\delta_{4,\xi}))$, $|\xi+w| \leqslant C$. Next we use a compactness argument to cover $\overline{B^2(0,C)}\cap\mathbb{R}^2$ by $\bigcup_{j=1}^m(\xi_j+B^2(0,\delta_{4,\xi_j}))$. Then there exists $\sigma > 0$ such that $|\operatorname{Im} w| \geqslant \sigma$ for each $w \in \overline{B^2(0,C)} \setminus \bigcup_{j=1}^m(\xi_j + B^2(0,\delta_{4,\xi_j}))$. Finally, set $T_2 := \min\{T_{\xi_j} : 1 \leqslant j \leqslant m\}$.

To complete the proof of case 1, fix $(t,z) \in V(F)\cap(\Gamma_\eta(T_2,\delta_3)\times B^2(0,\delta_4))$ satisfying (4.12). Since we are only interested in $t \neq 0$, we can set $\zeta := t^{-a}z \in \overline{B^2(0,C)}$.

If $\zeta \notin \bigcup_{j=1}^m(\xi_j + B^2(0,\delta_{4,\xi_j}))$, then $\operatorname{Im}\zeta \geqslant \sigma$. Hence $\operatorname{Im} z \geqslant \frac{1}{2}\sigma t^a$, provided T_2 was chosen sufficiently small. Together with the estimate 4.4 (ii) this implies

$$u(t,z) \leqslant A|t|^{\alpha}\left((C+1)|t|^{a} + \frac{|\operatorname{Im} t|}{|t|^{\eta}}\right) \leqslant A|t|^{\alpha}\left(\frac{2C+2}{\sigma}|\operatorname{Im} z| + \frac{|\operatorname{Im} t|}{|t|^{\eta}}\right),$$

which is estimate (c) of $\mathrm{PL}(\alpha,\eta)$.

Otherwise, there are $j \in \{1,\ldots,l\}$ and $w \in B^2(0,\delta_{4,\xi_j})$ with $\zeta = \xi_j + w$. Then, equation (4.16) yields estimate (c) of $\mathrm{PL}(\alpha,\eta)$.

If $a = +\infty$ then there is nothing to prove, since $V(F)$ is hyperbolic near the origin.

Case 2: p_ν depends only on one variable.

Then it is no restriction to assume that $p_\nu(z_1,z_2) = z_1^\nu$. From this it follows as in the proof of (4.2) that for fixed small values of $t \in \Gamma_\eta(T,\delta)$ and all $z_2 \in B(0,\delta_2)$ the solutions of the equation $F(t,z_1,z_2) = 0$ satisfy the estimate (4.3). Using the hypothesis that $(V(F(t,\cdot))_t$ is nearly hyperbolic, it follows as in the proof of (4.2) that all these solutions are real when (t,z_2) is real. Hence it follows as in the proof of part (i) of Lemma 4.4 that $V(F)$ satisfies $\mathrm{PL}(\alpha,\eta)$ for each $\alpha \geqslant 0$ and each $\eta \geqslant 1$. $\square$

In the proof of Theorem 4.1, which we will give next, we will use the following definition.

Definition 4.8. — *For $\xi \in \mathbb{R}^n$ and $r_0 > 0$ let V be an analytic variety in $B(\xi,r)$ which contains ξ. We say that V satisfies the condition* $\mathrm{RPL}_{\mathrm{loc}}(\xi)$ *if there exist positive numbers A and $r_0 \geqslant r_1 \geqslant r_2$ such that each $u \in$* $\mathrm{PSH}(V \cap B(0,r_1))$ *satisfying*

(α) $u(z) \leqslant 1,\ z \in V \cap B(0,r_1)$ and

(β) $u(z) \leqslant 0,\ z \in V \cap \mathbb{R}^n \cap B(0,r_1)$

also satisfies

(γ) $u(z) \leqslant Az - \xi,\ z \in V \cap B(0,r_2)$.

Proof of Theorem 4.1. — Because of Theorem 2.12, we only have to show that the conditions (1) and (2) in Theorem 4.1 are sufficient. To do so, we fix $F \in H(B^n(0,r))$ and assume that $V = V(F)$ satisfies conditions (1) and (2) of 4.1. (Later we will switch to $n = 3$.) Then we can expand

$$F(z) = \sum_{k=m}^{\infty} q_k(z),$$

where q_k is either a homogeneous polynomial of degree k or identically zero, and $q_m \neq 0$. For $\xi = (0, \dots, 0, 1)$ and $w = (w', w_n)$ we have

$$F(t(\xi + w)) = \sum_{k \geqslant m} q_k(tw', t(1 + w_n)) = \sum_{k \geqslant m} (t(1 + w_n))^k \, q_k\Big(\frac{w'}{1 + w_n}, 1\Big). \tag{4.17}$$

For $(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^{n-1}$ we let

$$F_0(\tau, \zeta) := \sum_{k \geqslant m} \tau^{k-m} q_k(\zeta, 1) = \sum_{l=\nu}^{m} p_l(\zeta) + \sum_{j>0, \beta \in \mathbb{N}_0^{n-1}} \tau^j a_{j,\beta} \zeta^\beta,$$

where $p_\nu \neq 0$. From (4.17) we get

$$F(t(\xi + w)) = (t(1 + w_n))^m \, F_0\Big(t(1 + w_n), \frac{w'}{1 + w_n}\Big). \tag{4.18}$$

Next fix $(\tau, \zeta) \in (\Gamma_1(T, \delta) \times B^{n-1}(0, \varrho)) \cap V(F_0)$ and let

$$t := \operatorname{Re} \tau, \ \lambda := \frac{i}{t} \operatorname{Im} \tau, \tag{4.19}$$

so that $\tau = t(1 + \lambda)$, where $|\lambda| < \delta$. Then for $w := ((1 + \lambda)\zeta, \lambda)$ we have

$$(\tau\zeta, \tau) = (t(1 + \lambda)\zeta, \ t(1 + \lambda)) = t(\xi + w)$$

and

$$|w| \leqslant \varrho(1 + \delta) + \delta.$$

From (4.18) we now get

$$F(\tau\zeta, \tau) = F(t(\xi + w)) = (t(1 + w_n))^m \, F_0\Big(t(1 + w_n), \frac{w'}{1 + w_n}\Big) = \tau^m F_0(\tau, \zeta). \tag{4.20}$$

Hence $(\tau\zeta, \tau) \in V(F)$.

We claim that $V(F)$ satisfies $\mathrm{RPL}_{\mathrm{loc}}(0)$. To prove this, note first that $V(q_m)$ satisfies (SPL). Hence so does each irreducible branch W of $V(q_m)$. Consequently, we can pick a regular point η of V with $\eta \in \mathbb{R}^n \setminus \{0\}$. It is no restriction to assume that $\pi: V \to \mathbb{C}^2$, $\pi: z \mapsto (z_1, z_2)$, is proper in a neighborhood of 0. Next apply the nearly hyperbolicity hypothesis at points $x = t\eta$, $t \in [-\varepsilon, \varepsilon] \setminus \{0\}$, in the ball $B(x, \delta x)$, $\delta > 0$ fixed sufficiently small, and for the subspace $L = T_\eta W$ to see that $V(F)$ is 1-hyperbolic in the sense of Definition 9 in [2]. Hence Theorem 10 in [2] shows that $V(F)$ satisfies $\mathrm{RPL}_{\mathrm{loc}}(0)$.

From now on assume that $n = 3$, fix $u \in \mathrm{PSH}(V \cap B^3(0,r))$, and assume that u satisfies the conditions (α) and (β) of $\mathrm{PL}_{\mathrm{loc}}(0)$ (see Definition 2.2). Since $V(F)$ satisfies $\mathrm{RPL}_{\mathrm{loc}}(0)$, there exist $A_1 > 0$ and $0 < r_1 < r$ such that

$$u(z) \leqslant A_1|z|,\ z \in V(F) \cap B^3(0, r_1).$$

Replacing u by u/A_1 it is no restriction to assume that $A_1 = 1$. For an arbitrarily chosen $\xi \in T_0V(F) \cap \mathbb{R}^3$ with $\xi = 1$ let

$$\Gamma(\xi, \eta, r_1) := \bigcup_{0<t<r_1} t(\xi + B^3(0, \eta))$$

and restrict u to the set $V(F) \cap \Gamma(\xi, \eta, r_1)$. By an appropriate change of coordinates we can arrange $\xi = (0, 0, 1)$.

If we choose $0 < \delta < \eta/2$, $\varrho < \eta/4$, and $T := r_1$ then $(\tau, \zeta) \in \Gamma_1(T, \delta) \times B^2(0, \varrho)$ implies that for t and λ as in (4.19) we have that $w := ((1+\lambda)\zeta, \lambda)$ satisfies $|w| < \varrho(1+\delta) + \delta < \eta/4 \cdot 2 + \eta/2 < \eta$, so that $t(\xi + w) = (\tau\zeta, \tau) \in V(F) \cap \Gamma(\xi, \eta, r_1)$. Then the function

$$v :\ V(F_0) \cap (\Gamma_1(T, \delta) \times B^2(0, \varrho)) \to [-\infty, \infty[,\ v(\tau, \zeta) := u(\tau\zeta, \tau)$$

is plurisubharmonic on this set and satisfies

$$v(\tau, \zeta) \leqslant |(\tau\zeta, \tau)| \leqslant |\tau|(1 + \varrho)$$

and

$$v(\tau, \zeta) \leqslant 0,\ (\tau, \zeta) \in (\Gamma_1(T, \delta) \times B^2(0, \varrho)) \cap \mathbb{R}^3.$$

To show that F_0 satisfies the hypotheses of Proposition 4.7, note first that by 4.1 (2) the variety $T_0V(F) = V(q_m)$ satisfies (SPL). Since q_m is homogeneous, it follows from Remark 2.11 and [5], Lemma 6.1, that because of $q_m(\zeta, 1) = \sum_{k=\nu}^{m} p_\nu(\zeta)$ the variety $V\left(\sum_{\nu=m}^{m} p_\nu(\zeta)\right)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. By [5], Proposition 3.5, this implies that $V(p_\nu) = T_0(\lim_{t\to 0} V(F_0(t, \cdot)))$ also satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. Next let $\gamma :]0, \sigma[\to \mathbb{R}^2$ be defined by a convergent Puiseux series and define for $0 < t < \sigma$ the variety

$$W_t := \{w \in \mathbb{C}^2 : z := \gamma(t) + t^d w \in V(F_0(t, \cdot))\}.$$

Then it follows from (4.20) that we have

$$0 = t^m F_0(t, \gamma(t) + t^d w) = F(t(\gamma + t^d w), t) + F(\sigma(t) + t^{d+1}(w, 0)),$$

where $\sigma(t) = (t\gamma(t), t)$ is a real simple curve in $\mathbb{C}^3$. Since the tangent vector to σ at the origin is $\xi = (0, 0, 1)$, it follows from Proposition 2.9 (3), that the limit variety $T_{\sigma,d+1}V(F)$ does not depend on the last variable. Hence we have $T_{\sigma,d+1}V(F) = W_0 \times \mathbb{C}$, for an algebraic variety W_0 in $\mathbb{C}^2$ and it follows that

$$\lim_{t\to 0} W_t = W_0.$$

Since $T_{\sigma,d+1}V(F)$ satisfies (SPL), this implies that $\lim_{t\to 0} W_t$ satisfies (SPL). Moreover, Proposition 3.4 implies that the varieties $(V(F_0(t,\cdot)))_t$ are nearly hyperbolic. Hence F_0 satisfies the hypotheses of Proposition 4.7. Therefore, $V(F_0)$ satisfies PL(1, 1) by Proposition 4.7. Consequently, there exist $B > 0$, $T_0 > 0$, $\delta_3 > 0$, and $\delta_4 > 0$ such that

$$u(\tau\zeta,\tau) = v(\tau,\zeta) \leqslant B|\tau| \left(|\operatorname{Im}\zeta| + \frac{|\operatorname{Im}\tau|}{|\tau|} \right) = B(|\tau|\,|\operatorname{Im}\zeta| + |\operatorname{Im}\tau|),$$

for $(\tau,\zeta) \in V(F_0) \cap \Gamma_1(T_0,\delta_3) \times B^2(0,\delta_4)$. Since there exists $D > 1$ such that

$$|\tau|\,|\operatorname{Im}\zeta| \leqslant D(|\operatorname{Im}\tau\zeta| + |\operatorname{Im}\tau|),\ (\tau,\zeta) \in \Gamma_1(T_0,\delta_3) \times B^2(0,\delta_4)$$

this implies

$$u(\tau\zeta,\tau) \leqslant 2BD|\operatorname{Im}(\tau\zeta,\tau)|. \tag{4.21}$$

Now choose $0 < \sigma < 1/2$ such that $2\sigma < \min(\delta_3,\delta_4)$ and $0 < r < T_0/(1+\sigma)$. Then for $w \in B^3(0,\sigma)$ and $t \in]0,r[$ we let

$$\zeta := \frac{(w_1,w_2)}{1+w_3},\ \tau := t(1+w_3)$$

and we note that

$$|\zeta| \leqslant \frac{\sigma}{|1+w_3|} \leqslant \frac{\sigma}{1-\sigma} \leqslant 2\sigma < \delta_3$$

and

$$\operatorname{Re}\tau = t(1+\operatorname{Re} w) \leqslant t(1+\sigma) < r(1+\sigma) < T_0.$$

These estimates show that $(\tau,\zeta) \in V(F_0) \cap (\Gamma_1(T_0,\delta_3) \times B^2(0,\delta_4))$. Since $(\tau\zeta,\tau) = (tw_1, tw_2, t(1+w_3)) = t(\xi+w)$ we now get from (4.21) that

$$u(t(\xi+w)) = u(\zeta\tau,\tau) = v(\tau,\zeta) \leqslant 2BD|\operatorname{Im}(t(\xi+w))|.$$

This shows that $V(F)$ satisfies the estimate (γ) of $\mathrm{PL}_{\mathrm{loc}}(0)$ in the cone $\Gamma(\xi,\sigma,r)$. Since this holds for each $\xi \in T_0V(F) \cap \mathbb{R}^3$, $|\xi| = 1$, it follows from [5], Lemma 5.13, that $V(F)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. $\square$

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