ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

HA HUY KHOAI, VU HOAI AN

Value distribution problem for p-adic meromorphic functions and their derivatives

Tome XX, n^{o} S2 (2011), p. 137-151.

<http://afst.cedram.org/item?id=AFST_2011_6_20_S2_137_0>

Annales de la Faculté des Sciences de Toulouse Vol. XX, n° Spécial, 2011
pp. 137–151

Value distribution problem for p-adic meromorphic functions and their derivatives

Ha Huy Khoai[(1)], Vu Hoai An[(2)]

Abstract. — In this paper we discuss the value distribution problem for p-adic meromorphic functions and their derivatives, and prove a generalized version of the Hayman Conjecture for p-adic meromorphic functions.

Résumé. — Dans cet article on discute le probème de la distribution des valeurs pour des fonctions méromorphes p-adiques et ses dérivés, et démontre une version généralisée de la conjecture de Hayman pour des fonctions méromorphes p-adiques

1. Introduction

In [11] Hayman proved the following well-known result:

Theorem 1.1. — *Let f be a meromorphic function on $\mathbb{C}$. If $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ for some fixed positive integer k and for all $z \in \mathbb{C}$, then f is constant.*

Hayman also proposed the following conjecture (see [12]).

Hayman Conjecture. — *If an entire function f satisfies $f^n(z)f^{'}(z) \neq 1$ for a positive integer n and all $z \in \mathbb{C}$, then f is a constant.*

(*) The work was supported by a NAFOSTED grant

(1) Institute of Mathematics,18 Hoang Quoc Viet, 10307, Hanoi, Viet Nam
hhkhoai@@math.ac.vn
(2) Hai Duong Pedagogical College, Hai Duong, Viet Nam
vuhoaianmai@@yahoo.com

It has been verified for transcendental entire functions by Hayman himself for $n > 1$ ([12]), and by Clunie for $n \geqslant 1$ ([5]). These results and some related problems have become to be known as Hayman's Alternative, and caused increasingly attensions (see [1], [2], [4], [14], [15], [17]).

In recent years the similar problems are investiged for functions in a non-Archimedean fields. In [16] J. Ojeda proved that for a transcendental meromorphic function f in an algebraically closed fields of characteristic zero, complete for a non-Archimedean absolute value $\mathbb{K}$, the function $f^{'} f^n - 1$ has infinitely many zeros, if $n \geqslant 2$.

The aim of this paper is to establish a similar results for a differential monomial of the form $f^n(f^{(k)})^m$, where f is a meromorphic function in $\mathbb{C}_p$. Namely, we prove the following theorem.

THEOREM 1.2 *(A generalized version of the Hayman Conjecture for p-adic meromorphic functions). — Let f be a meromorphic function on $\mathbb{C}_p$, satisfying the condition $f^n(f^{(k)})^m(z) \neq 1$ for all $z \in \mathbb{C}_p$ and for some non-negative integers n, k, m. Then f is a polynomial of degree $< k$ if one of the following conditions holds:*

1. f is an entire function.

2. $k > 0$, and either $m = 1$, $n > \frac{1+\sqrt{1+4k}}{2}$, or $m > 1, n \geqslant 1$.

3. $n \geqslant 0, m > 0, k > 0$, and there are constants C, r_0 such that $|f|_r < C$ for all $r > r_0$.

In the next section we first recall some facts of the p-adic Nevanlinna theory ([6-10], [13]) for later use. Theorem 1.2 is proved in Section 3 by using several Lemmas.

2. Value distribution of p-adic meromorphic functions

Let f be a non-constant holomorphic function on $\mathbb{C}_p$. For every $a \in \mathbb{C}_p$, expanding f around a as $f = \sum P_i(z-a)$ with homogeneous polynomials P_i of degree i, we define

$$v_f(a) = \min\{i : P_i \not\equiv 0\}.$$

For a point $d \in \mathbb{C}_p$ we define the function $v_f^d : \mathbb{C}_p \to \mathbb{N}$ by

$$v_f^d(a) = v_{f-d}(a).$$

Fix a real number ρ with $0 < \rho \leqslant r$. Define

$$N_f(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_f(a,x)}{x} dx,$$

where $n_f(a,x)$, as usually, is the number of the solutions of the equation $f(z) = a$ (counting multiplicity) in the disk $D_x = \{z \in \mathbb{C}_p : |z| \leqslant x\}$.

If $a = 0$, then set $N_f(r) = N_f(0,r)$.

For l a positive integer , set

$$N_{l,f}(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_{l,f}(a,x)}{x} dx,$$

where

$$n_{l,f}(a,r) = \sum_{|z|\leqslant r} \min\left\{v_{f-a}(z), l\right\}.$$

Let k be a positive integer. Define the function $v_f^{\leqslant k}$ from $\mathbb{C}_p$ into $\mathbb{N}$ by

$$v_f^{\leqslant k}(z) = \begin{cases} 0 & \text{if } v_f(z) > k \\ v_f(z) & \text{if } v_f(z) \leqslant k, \end{cases}$$

and

$$n_f^{\leqslant k}(r) = \sum_{|z|\leqslant r} v_f^{\leqslant k}(z), \quad n_f^{\leqslant k}(a,r) = n_{f-a}^{\leqslant k}(r).$$

Define

$$N_f^{\leqslant k}(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_f^{\leqslant k}(a,x)}{x} dx.$$

If $a = 0$, then set $N_f^{\leqslant k}(r) = N_f^{\leqslant k}(0,r)$.

Set

$$N_{l,f}^{\leqslant k}(a,r) = \frac{1}{\ln p} \int\limits_{\rho}^{r} \frac{n_{l,f}^{\leqslant k}(a,x)}{x} dx,$$

where

$$n_{l,f}^{\leqslant k}(a,r) = \sum_{|z|\leqslant r} \min\left\{v_{f-a}^{\leqslant k}(z), l\right\}.$$

In a like manner to used for holomorphic functions we define

$$N_f^{<k}(a,r), N_{l,f}^{<k}(a,r), N_f^{>k}(a,r), N_f^{\geqslant k}(a,r), N_{l,f}^{\geqslant k}(a,r), N_{l,f}^{>k}(a,r).$$

Recall that for a holomorphic function $f(z)$ in $\mathbb{C}_p$, represented by the power series

$$f(z) = \sum_{o}^{\infty} a_n z^n,$$

for each $r > 0$, we define $|f|_r = \max\{|a_n|r^n, 0 \leqslant n < \infty\}$.

Now let $f = \dfrac{f_1}{f_2}$ be a non-constant meromorphic function on $\mathbb{C}_p$, where f_1, f_2 be holomorphic functions on $\mathbb{C}_p$ having no common zeros, we set $|f|_r = \dfrac{|f_1|_r}{|f_2|_r}$. For a point $d \in \mathbb{C}_p \cup \{\infty\}$ we define the function $v_f^d : \mathbb{C}_p \to \mathbb{N}$ by

$$v_f^d(a) = v_{f_1 - df_2}(a)$$

with $d \neq \infty$, and

$$v_f^\infty(a) = v_{f_2}(a).$$

For a point $a \in \mathbb{C}$ define:

$$m_f(\infty, r) = \max\left\{0, \log|f|_r\right\}, m_f(a, r) = m_{\frac{1}{f-a}}(\infty, r),$$

$$N_f(a, r) = N_{f_1 - af_2}(r), N_f(\infty, r) = N_{f_2}(r),$$

$$T_f(r) = \max_{1 \leqslant i \leqslant 2} \log|f_i|_r.$$

In a like manner we define

$$N_{l,f}(a, r), N_f^{\leqslant k}(a, r), N_{l,f}^{\leqslant k}(a, r), N_f^{<k}(a, r), N_{l,f}^{<k}(a, r), N_f^{>k}(a, r),$$

$$N_f^{\geqslant k}(a, r), N_{l,f}^{\geqslant k}(a, r), N_{l,f}^{>k}(a, r),$$

with $a \in \mathbb{C}_p \cup \{\infty\}$.

Then we have (see [11])

$$N_f(a, r) + m_f(a, r) = T_f(r) + O(1)$$

with $a \in \mathbb{C}_p \cup \{\infty\}$,

$$T_f(r) = T_{\frac{1}{f}}(r) + O(1),$$

$$|f^{(k)}|_r \leqslant \frac{|f|_r}{r^k},$$

$$m_{\frac{f^{(k)}}{f}}(\infty, r) = O(1).$$

The following two lemmas were proved in [11] (see also [3], [6]).

Lemma 2.1. — *Let f be a non-constant holomorphic function on $\mathbb{C}_p$. Then*

$$T_f(r) - T_f(\rho) = N_f(r),$$

where $0 < \rho \leqslant r$.

Notices that $N_f(r)$ depends on fixed ρ.

Lemma 2.2. — *Let f be a non-constant meromorphic function on $\mathbb{C}_p$ and let $a_1, a_2, ..., a_q$ be distinct points of $\mathbb{C}_p$. Then*

$$(q-1)T_f(r) \leqslant N_{1,f}(\infty, r) + \sum_{i=1}^{q} N_{1,f}(a_i, r) - N_{0,f'}(r) - \log r + O(1),$$

where $N_{0,f'}(r)$ is the counting function of the zeros of $f^{'}$ which occur at points other than roots of the equations $f(z) = a_i, i = 1, ..., q$, and $0 < \rho \leqslant r$.

3. A Generalized Hayman-Conjecture for p-adic meromorphic functions

We are going to prove Theorem 1.2. We need the following Lemmas.

Lemma 3.1. — *Let f be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \not\equiv 0$ and n, k, m be positive integers. Then*

1. $T_f(r) \leqslant T_{f^n(f^{(k)})^m-1}(r) + O(1)$,

2. $T_f(r) \leqslant T_{f^n(f^{(k)})^m}(r) + O(1)$,

In particular $f^n(f^{(k)})^m$ is non-constant.

Proof. —

1. Set $A = f^n(f^{(k)})^m - 1$. Then we have

$$A + 1 = f^n(f^{(k)})^m,$$

$$N_f(0, r) \leqslant N_{A+1}(0, r),$$

$$\frac{1}{f^{n+m}} = \frac{1}{A+1}\left(\frac{f^{(k)}}{f}\right)^m.$$

Moreover

$$m_{\frac{f^{(k)}}{f}}(\infty, r) = O(1).$$

Therefore

$$m_f(0,r) \leqslant (n+m)m_f(0,r) = m_{f^{n+m}}(0,r) \leqslant m_{A+1}(0,r) + O(1).$$

Thus

$$T_f(r) = N_f(0,r)+m_f(0,r) \leqslant N_{A+1}(0,r)+m_{A+1}(0,r) = T_{f^n(f^{(k)})^m-1}+O(1).$$

2. Since $T_{f^n(f^{(k)})^m}(r) = T_{f^n(f^{(k)})^m-1}(r) + O(1)$ we have

$$T_f(r) \leqslant T_{f^n(f^{(k)})^m}(r) + O(1).$$

From this it follows that $f^n(f^{(k)})^m$ is non-constant.

Lemma 3.1 is proved. □

LEMMA 3.2. — *Let f be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \not\equiv 0$, and let $m, n > 1, k > 0$ be integers, $a \in \mathbb{C}_p$, $a \neq 0$. Then we have:*

1. $\dfrac{n(n-2)+k(mn-m-n)+m(n-1)}{(n+k)(n+m+km)} T_f(r) \leqslant N_{1,f^n(f^{(k)})^m}(a,r) - \log r + O(1)$,

2. If $n^2-n-k>0$,

$$\frac{n^2-n-k-1}{(n+k)(n+1+k)} T_f(r) \leqslant N_{1,f^n(f^{(k)})}(a,r) - \log r + O(1).$$

Proof. —

1. Since $m, n > 1$ we have $n(n-2)+k(mn-m-n)+m(n-1) \geqslant 0$.

Because $f^{(k)} \not\equiv 0$, from Lemma 3.1 it follows that $f^n(f^{(k)})^m$ is not constant.

Applying Lemma 2.2 to $f^n(f^{(k)})^m$ with the values ∞, 0 and a, we obtain

$$\begin{aligned} &T_{f^n(f^{(k)})^m}(r) \\ &\leqslant N_{1,f^n(f^{(k)})^m}(\infty,r) + N_{1,f^n(f^{(k)})^m}(0,r) + N_{1,f^n(f^{(k)})^m}(a,r) - \log r + O(1). \end{aligned}$$

Denote by $N_{f^{(k)}}(0,r; f \neq 0)$ the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity. Then we get

$$\begin{aligned} N_{f^{(k)}}(0,r;f\neq 0) &= N_{\frac{f^{(k)}}{f}}(0,r) \\ &\leqslant N_{\frac{f^{(k)}}{f}}(\infty,r)+m_{\frac{f^{(k)}}{f}}(\infty,r)+O(1) \\ &\leqslant kN_{1,f}(\infty,r)+N_f^{\leqslant k}(0,r)+kN_{1,f}^{>k}(0,r)+O(1). \end{aligned}$$

Therefore,

$$N_{f^{(k)}}(0,r;f\neq 0)\leqslant kN_{1,f}(\infty,r)+N_f^{\leqslant k}(0,r)+kN_{1,f}^{>k}(0,r)+O(1).$$

From this it follows

$$\begin{aligned} N_{1,f^n(f^{(k)})^m}(0,r) &\leqslant N_{1,f}(0,r)+N_{f^{(k)}}(0,r;f\neq 0) \\ &\leqslant kN_{1,f}(\infty,r)+N_f^{\leqslant k}(0,r)+kN_{1,f}^{>k}(0,r)+O(1) \\ &\leqslant (k+1)N_{1,f}(0,r)+kN_{1,f}(\infty,r) \end{aligned} \tag{3.3}$$

Again, we see that

$$\begin{aligned} &N_{f^n(f^{(k)})^m}(0,r)-N_{1,f^n(f^{(k)})^m}(0,r) \\ &\geqslant \big((1+k)n+m-1\big)N_{1,f}^{\geqslant(k+1)}(0,r)+(n-1)N_{1,f}^{\leqslant k}(0,r). \end{aligned} \tag{3.4}$$

On ther other hand,

$$N_{1,f}(0,r)=N_{1,f}^{\leqslant k}(0,r)+N_{1,f}^{\geqslant(k+1)}(0,r).$$

From this and (3.3), (3.4) we obtain

$$\begin{aligned} N_{f^n(f^{(k)})^m}(0,r) &\leqslant (k+1)N_{1,f}^{\geqslant(k+1)}(0,r)+kN_{1,f}(\infty,r) \\ &+ \frac{k+1}{n-1}\big(N_{f^n(f^{(k)})^m}(0,r)-N_{1,f^n(f^{(k)})^m}(0,r) \\ &- ((k+1)n+m-1)N_{1,f}^{\geqslant(k+1)}(0,r)\big)+O(1). \end{aligned}$$

Thus

$$\begin{aligned} \frac{n+k}{n-1}N_{1,f^n(f^{(k)})^m}(0,r) &\leqslant \frac{k+1}{n-1}N_{f^n(f^{(k)})^m}(0,r)+kN_{1,f}(\infty,r) \\ &+ (k+1-\frac{(k+1)((k+1)n+m-1)}{n-1})N_{1,f}^{\geqslant(k+1)}(0,r) \\ &+ O(1). \end{aligned}$$

Note that

$$k+1-\frac{(k+1)((k+1)n+m-1)}{n-1}<0,$$

we have

$$N_{1,f^n(f^{(k)})^m}(0,r) \leqslant \frac{k+1}{n+k} N_{f^n(f^{(k)})^m}(0,r) + \frac{k(n-1)}{n+k} N_{1,f}(\infty,r) + O(1).$$

Moreover if a is a pole of f with multiplicity t then a is a pole of $f^n\left(f^{(k)}\right)^m$ with multiplicity $nt+(t+k)m \geqslant n+(1+k)m$. Thus

$$N_{f^n(f^{(k)})^m}(\infty,r) \geqslant (n+(k+1)m) N_{1,f}(\infty,r),$$

and

$$N_{1,f^n(f^{(k)})^m}(\infty,r) = N_{1,f}(\infty,r).$$

Therefore,

$$\begin{aligned} T_{f^n(f^{(k)})^m}(r) \leqslant\ & \frac{k+1}{n+k} N_{f^n(f^{(k)})^m}(0,r) \\ &+ \left(1+\frac{k(n-1)}{n+k}\right) N_{1,f^n(f^{(k)})^m}(\infty,r) \\ &+ N_{1,f^n(f^{(k)})^m}(a,r) - \log r + O(1), \end{aligned}$$

$$\begin{aligned} T_{f^n(f^{(k)})^m}(r) \leqslant\ & \frac{k+1}{n+k} N_{f^n(f^{(k)})^m}(0,r) \\ &+ \frac{n(k+1)}{(n+k)(n+(k+1)m)} N_{f^n(f^{(k)})^m}(\infty,r) \\ &+ N_{1,f^n(f^{(k)})^m}(a,r) - \log r + O(1). \end{aligned}$$

From this and by Lemma 2.1, we have

$$\begin{aligned} &\frac{n(n-2)+k(mn-m-n)+m(n-1)}{(n+k)(n+m+km)} T_{f^n(f^{(k)})^m}(r) \\ &\leqslant N_{1,f^n(f^{(k)})^m}(a,r) - \log r + O(1). \end{aligned}$$

By Lemma 3.1

$$\frac{n(n-2)+k(mn-m-n)+m(n-1)}{(n+k)(n+m+km)} T_f(r) \leqslant N_{1,f^n(f^{(k)})^m}(a,r) - \log r + O(1).$$

Applying the above arguments to case $m=1$, and using $n^2-n-k>0$, we obtain 2.

Lemma 3.2 is proved. □

For simplicity we denote:

$$B = f\left(f^{(k)}\right)^m, b = f^{(k)}, c = f\left(f^{(k)}\right)^m - 1, v = \frac{1}{\left(f^{(k)}\right)^m},$$

$$a_0 = \frac{v^{(k+1)} - \frac{b'}{b}v^{(k)}}{v}, a_i = \frac{\binom{k+1}{i}v^{(k+1-i)} - \binom{k}{i}\frac{b'}{b}v^{(k-i)}}{v},$$
$$i = 1, 2, ..., k, a_{k+1} = 1.$$

Then we have the following lemma.

LEMMA 3.3. — *Let f be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \not\equiv 0$, and let $k > 0, m > 1$ be integers. Then we have*

$$B^{(k+1)} + a_k B^{(k)} + ... + a_1 B^{(1)} + a_0 B \equiv 0.$$

Proof. — We first prove that

$$(Bv)^{(j)} \equiv \sum_{i=0}^{j} \binom{j}{i} B^{(i)} v^{(j-i)}, \tag{3.5}$$

$j = 1, 2..., k+1$, by induction.

For $j = 1$, we have

$$(Bv)^{(1)} \equiv \sum_{i=0}^{1} \binom{1}{i} B^{(i)} v^{(1-i)}.$$

Assume

$$(Bv)^{(j)} \equiv \sum_{i=0}^{j} \binom{j}{i} B^{(i)} v^{(j-i)},$$

we will prove that

$$(Bv)^{(j+1)} \equiv \sum_{i=0}^{j+1} \binom{j+1}{i} B^{(i)} v^{(j+1-i)}.$$

Indeed, we have

$$(Bv)^{(j+1)} \equiv ((Bv)^{(j)})^{(1)} \equiv \sum_{i=0}^{j} \binom{j}{i} (B^{(i)} v^{(j-i)})^{(1)}$$
$$\equiv \sum_{i=0}^{j} \binom{j}{i} (B^{(i+1)} v^{(j-i)} + B^{(i)} v^{(j+1-i)}) \equiv \sum_{i=0}^{j+1} \binom{j+1}{i} B^{(i)} v^{(j+1-i)}.$$

Returning to the proof of Lemma 3.3, from $b = f^{(k)}$, we have $b' = f^{(k+1)}$. Therefore

$$f^{(k+1)} - \frac{b'}{b} f^{(k)} \equiv 0 \tag{3.6}$$

Because $B = f(f^{(k)})^m$, $v = \frac{1}{(f^{(k)})^m}$, we obtain $f \equiv Bv$. Since (3.6) we have

$$(Bv)^{(k+1)} - \frac{b'}{b}(Bv)^{(k)} \equiv 0 \tag{3.7}$$

From (3.5), (3.7) we obtain

$$\sum_{i=0}^{k+1}\binom{k+1}{i}B^{(i)}v^{(k+1-i)} - \sum_{i=0}^{k}\binom{k}{i}B^{(i)}v^{(k-i)} \equiv 0.$$

Thus

$$\begin{aligned} &Bv^{(k+1)} + \tbinom{k+1}{1}B^{(1)}v^{(k)} + \tbinom{k+1}{2}B^{(2)}v^{(k-1)} + ... + \tbinom{k+1}{k}B^{(k)}v^{(1)} + B^{(k+1)}v \\ &-\frac{b'}{b}\big(Bv^{(k)} + \tbinom{k}{1}B^{(1)}v^{(k-1)} + \tbinom{k}{2}B^{(2)}v^{(k-2)} + ... + \tbinom{k}{k-1}B^{(k-1)}v^{(1)} + B^{(k)}v\big) \\ &\equiv 0. \end{aligned}$$

Dividing the left hand side by v, we get

$$\begin{aligned} &\frac{v^{(k+1)} - \frac{b'}{b}v^{(k)}}{v}B + \frac{\binom{k+1}{1}v^{(k)} - \binom{k}{1}\frac{b'}{b}v^{(k-1)}}{v}B^{(1)} \\ &+ ... + \frac{\binom{k+1}{k}v^{(1)} - \binom{k}{k}\frac{b'}{b}v^{(0)}}{v}B^{(k)} + B^{(k+1)} \equiv 0. \end{aligned}$$

So

$$B^{(k+1)} + a_k B^{(k)} + ... + a_1 B^{(1)} + a_0 B \equiv 0. \tag{3.8}$$

□

LEMMA 3.4. — *Let f be a non-constant meromorphic function on $\mathbb{C}_p$ such that $f^{(k)} \not\equiv 0$, and let $k > 0, m > 1$ be integers. Suppose that f is not a polynomial of degree k. Then we have $a_0 \not\equiv 0$, and*

$$\frac{m^2k + m^2 - 2mk - m - 1}{m(k+1)(mk+m+1)}T_f(r) \leqslant N_{1,f(f^{(k)})^m}(1,r) + O(1).$$

Proof. — Suppose $a_0 \equiv 0$. Because $a_0 = \dfrac{v^{(k+1)} - \dfrac{b'}{b}v^{(k)}}{v}$, we get

$$v^{(k+1)} \equiv \frac{b'}{b}v^{(k)} \tag{3.9}$$

Consider following two cases.

Case 1. $v^{(k)} \equiv 0$. We have $v \equiv h$, a polynomial of degree $< k$, and $h \not\equiv 0$. Thus $\left(f^{(k)}\right)^m h \equiv 1$. If z_0 is a pole of $f^{(k)}$, then z_0 is a pole of f with multiplicity at least $k+1$. So z_0 is a zero of h with multiplicity at least $k+1$, a contradiction. Thus $f^{(k)}$ has no poles, and from $\left(f^{(k)}\right)^m h \equiv 1$ it follows that f is a polynomial of degree k, a contradiction.

Case 2. $v^{(k)} \not\equiv 0$. From (3.8), we have

$$\frac{v^{(k+1)}}{v^{(k)}} \equiv \frac{b^{'}}{b}.$$

So $v^{(k)} \equiv cb \equiv cf^{(k)}, c \neq 0$. Solving this, we get

$$v \equiv c(f+t), t^{(k)} \equiv 0.$$

From this t we see that t is a polynomial of degree $< k$, and $\frac{1}{(f^{(k)})^m} \equiv c(f+t)$. Thus $c(f+t)\left(f^{(k)}\right)^m \equiv 1$. Set $F = f+t$. Then $F^{(k)} \equiv f^{(k)}$ and $cF\left(F^{(k)}\right)^m \equiv 1$. By Lemma 3.1, we get a contradiction, and then $a_0 \not\equiv 0$.

Now we are going to prove the inequality in the lemma. Since k, m are positive integers and $m \geqslant 2$, it is easy to see that $m^2k+m^2-2mk-m-1 \geqslant 0$. From (3.8) and $B \equiv c+1$ we get

$$(c+1)^{(k+1)} + a_k(c+1)^{(k)} + ... + a_1(c+1)^{(1)} + a_0(c+1) \equiv 0,$$

$$c^{(k+1)} + a_k c^{(k)} + ... + a_1 c^{(1)} + a_0(c+1) \equiv 0,$$

$$a_0 c + c^{(k+1)} + a_k c^{(k)} + ... + a_1 c^{(1)} \equiv -a_0, \tag{3.10}$$

$$\frac{1}{a_0}\left(\frac{c^{(k+1)}}{c} + a_k\frac{c^{(k)}}{c} + ... + a_1\frac{c^{(1)}}{c}\right) + \frac{1}{c} + 1 \equiv 0. \tag{3.11}$$

Since $a_0 = \dfrac{v^{(k+1)} - \frac{b^{'}}{b}v^{(k)}}{v}$, we see that any pole of a_0 can occur only at poles or zeros of b, and each pole of a_0 has mutiplicity at most $k+1$. So

$$\begin{aligned} N_{a_0}(\infty, r) &\leqslant (k+1)\big(N_{1,b}(\infty, r) + N_{1,b}(0, r)\big) \\ &\leqslant (k+1)\big(N_{1,f}(\infty, r) + N_{1,b}(0, r)\big). \end{aligned}$$

On the other hand, a zero of b of multiplicity s is a zero of c' of multiplicity at least $ms-1 \geqslant (m-1)s$. Also, $c+1 \neq 0$ at such a zero of b.

$$N_{1,b}(0, r) \quad \leqslant \quad \frac{1}{m-1} N_{\frac{c}{c'}}(\infty, r)$$

$$\begin{aligned}
&\leqslant \frac{1}{m-1}H_{\frac{c}{c'}}(r) = \frac{1}{n-1}H_{\frac{c'}{c}}(r)\\
&= \frac{1}{m-1}\big(N_{\frac{c'}{c}}(\infty,r)+m_{\frac{c'}{c}}(\infty,r)\big)\\
&= \frac{1}{m-1}N_{\frac{c'}{c}}(\infty,r)+O(1)\\
&= \frac{1}{m-1}\big(N_{1,c}(\infty,r)+N_{1,c}(0,r)\big)+O(1)\\
&= \frac{1}{m-1}\big(N_{1,f}(\infty,r)+N_{1,c}(0,r)\big)+O(1).
\end{aligned}$$

Thus

$$\begin{aligned}
N_{a_0}(\infty,r) &\leqslant (k+1)\big(N_{1,b}(\infty,r)+N_{1,b}(0,r)\big)\\
&\leqslant (k+1)\Big(N_{1,f}(\infty,r)+\frac{1}{m-1}\big(N_{1,f}(\infty,r)+N_{1,c}(0,r)\big)\Big)+O(1)\\
&= \frac{m(k+1)}{m-1}N_{1,f}(\infty,r)+\frac{k+1}{m-1}N_{1,c}(0,r).
\end{aligned}$$

Note that $B \equiv c+1 \equiv f\big(f^{(k)}\big)^m$. Therefore a pole of f of multiplicity s is a pole of B of multiplicity $s+(s+k)m \geqslant 1+(1+k)m$. So

$$N_{1,f}(\infty,r) \leqslant \frac{1}{1+m(k+1)}N_B(\infty,r) \leqslant \frac{1}{1+m(k+1)}T_B(r)+O(1).$$

Combining the above inequalities and note that $T_B(r)=T_c(r)+O(1)$ we obtain

$$N_{a_0}(\infty,r) \leqslant \frac{m(k+1)}{(m-1)(1+m(k+1))}T_c(r)+\frac{k+1}{m-1}N_{1,c}(0,r)+O(1).$$

Since (3.10), a zero of c of multiplicity $s>k+1$ is a zero of a_0. From this and (3.11) we have

$$N_c(0,r) \leqslant N_{a_0}(0,r)+(k+1)N_{1,c}(0,r),$$

$$m_c(0,r) \leqslant m_{a_0}(0,r)+O(1).$$

Then (3.8) and Lemma 2.1 give us

$$\begin{aligned}
T_c(r) &= N_c(0,r)+m_c(0,r)+O(1)\\
&\leqslant N_{a_0}(0,r)+(k+1)N_{1,c}(0,r)+m_{a_0}(0,r)+O(1)\\
&\leqslant T_{a_0}(r)+(k+1)N_{1,c}(0,r)+O(1)\\
&= N_{a_0}(\infty,r)+m_{a_0}(\infty,r)+(k+1)N_{1,c}(0,r)+O(1)
\end{aligned}$$

$$\begin{aligned}
&= N_{a_0}(\infty, r) + m_{\frac{B^{(k+1)}+a_k B^{(k)}+\ldots+a_1 B^{(1)}}{-B}}(\infty, r) \\
&+ (k+1)N_{1,c}(0,r) + O(1) \\
&= N_{a_0}(\infty, r) + (k+1)N_{1,c}(0,r) + O(1) \\
&\leqslant \frac{m(k+1)}{(m-1)(1+m(k+1))}T_c(r) + \frac{k+1}{m-1}N_{1,c}(0,r)(0,r) \\
&\quad +(k+1)N_{1,c} + O(1) \\
&\leqslant \frac{m(k+1)}{(m-1)(1+m(k+1))}T_c(r) + \frac{m(k+1)}{m-1}N_{1,c}(0,r) + O(1).
\end{aligned}$$

So

$$(1 - \frac{m(k+1)}{(m-1)(1+m(k+1))})T_c(r) \leqslant \frac{m(k+1)}{m-1}N_{1,c}(0,r) + O(1).$$

From this and Lemma 3.1 we obtain

$$\frac{m^2k + m^2 - 2mk - m - 1}{m(k+1)(mk+m+1)}T_f(r) \leqslant N_{1,f(f^{(k)})^m}(1,r) + O(1).$$

$\square$

Now we use the above Lemmas to prove the main result of the paper.

Proof of Theorem 1.2. — Assume, on the contrary, that f is not a polynomial of degree $< k$.

If f is an entire function, then from Lemma 3.1 it implies that $(f^n(f^{(k)})^m)$ is not constant. Then $(f^n(f^{(k)})^m(z) - 1)$ must have a zero, a contradiction.

Assume $k > 0$. If $m > 1, n > 1$ then the condition 1. in Lemma 3.2 holds, and we see that $(f^n(f^{(k)})^m(z) - 1)$ is not constant, so it must have a zero, a contradiction.

If $m = 1,\ n > \frac{1+\sqrt{1+4k}}{2}$, the condition 2. in Lemma 3.2 is satisfied. Then $(f^n f^{(k)}(z) - 1)$ must have a zero, a contradiction.

Now let $m > 1, n = 1$. Is is easy to see that in this case we have $m^2k + m^2 - 2mk - m - 1 > 0$. If f is a polynomial of degree $> k$, then by Lemma 3.3, we see that $(f(f^{(k)})^m(z) - 1)$ has a zero, a contradiction. On the other hand, if f is a polynomial of degree k, or f is a transcendental function, then it is obviously that $(f(f^{(k)})^m(z) - 1)$ also has a zero, a contradiction.

It remains to consider the case when the condition 3. is satisfied. Then $f^{(k)} \not\equiv 0$. Write $f = \frac{f_1}{f_2}$, where f_1 and f_2 are holomorphic functions, having no common zeros, and $f^{(k)} = \frac{a_k}{f_2^{k+1}}$, where a_k is a polynomial of $f_1, f_2, f_1^{'}, f_2^{'}, ..., f_1^{(k)}, f_2^{(k)}$. If f_2 is constant, then by $|f_1|_r < C|f_2|_r$, we see that f_1 is constant, and therefore, f is constant, a contradiction. Suppose that f_2 is non-constant. Then f_2 has a zero. Let d denote the greatest common divisor of a_k and f_2^{k+1}. Set $h = \frac{a_k}{d}$ and $l = \frac{f_2^{k+1}}{d}$. Let d_1 denote the greatest common divisor of h^m and f_2^n. Set $h_1 = \frac{h^m}{d_1}$ and $l_1 = \frac{f_2^n}{d_1}$.Then

$$f^n\left(f^{(k)}\right)^m - 1 = \frac{f_1^n h^m - f_2^n l^m}{f_2^n l^m} = \frac{f_1^n h_1 - l_1 l^m}{l_1 l^m} \tag{3.11}$$

Note that $f^n\left(f^{(k)}\right)^m(z) \neq 1$ for all $z \in \mathbb{C}_p$. Thus $f^n\left(f^{(k)}\right)^m \not\equiv 1$. If l is constant, then $f^{(k)}$ is an entire function. Thus f is an entire function, a contradiction. So l is non-constant. Therefore, l has a zero.

Next we are going to show by induction that $|f_1^n|_r|a_k^m|_r < |f_2^{n+(k+1)m}|_r$, for all r satisfying $r > R_0, r > r_0$, where R_0 is a some constant. For $k = 1$, we have $a_1 = f_1^{'}f_2 - f_2^{'}f_1$. Since $|f_1^{'}|_r \leqslant \frac{|f_1|_r}{r}$,$|f_2^{'}|_r \leqslant \frac{|f_2|_r}{r}$ and $|f_1|_r < C|f_2|_r$ we get $|f_1^{'}f_2|_r \leqslant \frac{|f_1|_r|f_2|_r}{r}$, $|f_2^{'}f_1|_r \leqslant \frac{|f_1|_r|f_2|_r}{r}$ and $|f_1^n|_r|a_1^m|_r < |f_2|_r^{n+2m}$, for all r satisfying $r > R_1, r > r_0$, where R_1 is a some constant. Assume we have $|f_1^n|_r|a_i^m|_r < |f^{n+(i+1)m}|_r$ for all r satisfying $r > R_i, r > r_0$, where R_i is a some constant. Now for $k = i+1$ we get $a_{i+1} = a_i^{'}f_2 - f_2^{'}(i+1)a_i$. By the induction hypothesis and $|i+1| \leqslant 1$, $|a_i^{'}|_r \leqslant \frac{|a_i|_r}{r}$, $|f_2^{'}|_r \leqslant \frac{|f_2|_r}{r}$ we have $|f_1^n|_r|a_{i+1}^m|_r < |f_2^{n+(i+2)m}|_r$, for all r satisfying $r > R_{i+1}, r > r_0$. So $|f_1^n|_r|a_k^m|_r < |f_2^{n+(k+1)m}|_r$, for all r satisfying $r > R_0, r > r_0$, where R_0 is a some constant. From this and (3.11) it follows $N_{f^n(f^{(k)})^m}(1,r) = N_{f_1^n h_1 - l_1 l^m}(r)$ and $|f_1^n|_r|h|_r^m < |f_2^n|_r|l^m|_r$, $|f_1^n|_r|h_1|_r < |l_1|_r|l^m|_r$.Therefore $|f_1^n h_1 - l_1 l^m|_r = |l_1 l^m|_r$. So $T_{f_1^n h_1 - l_1 l^m}(r) = T_{l_1 l^m}(r)$. By Lemma 2.1 we get $N_{f_1^n h_1 - l_1 l^m}(r) = N_{l_1 l^m}(r) + O(1)$. Because l has a zero. Thus $l_1 l^m$ has a zero. Therefore, $f^n\left(f^{(k)}\right)^m - 1$ has a zero, a contradiction.

Theorem 1.2 is proved.

By taking $k = 1$ we have a differential monomial like in Hayman results, and from Theorem 1.2 it follows the following

Corollary 3.5. — *Let f be a meromorphic function on $\mathbb{C}_p$, satisfying the condition $f^n\left(f^{'}\right)^m(z) \neq 1$ for all $z \in \mathbb{C}_p$ and for some positive integers n, m. Then f is a constant function if one of the following conditions holds:*

1. *f is an entire function,*
2. $\max\{m, n\} > 1$,
3. *There exist constants C, r_0 such that $|f|_r < C$ for all $r > r_0$.*

Remark. — Indeed, in [16], Theorem 3 shows that $f^{'} + f^4$ has at least one zero that is not a zero of f, hence setting $g(x) = \frac{1}{f(x)}$, we can check that $g^2 g^{'}$ takes the value 1 at least one time. So the case $n = 2, m = k = 1$ of Theorem 1.2 has been established in [16].

Bibliography

[1] Alotaibi (A.). — On the Zeros of $af(f^{(k)})^n$ for $n \geqslant 2$, Computational Methods and Function Theory, Vol. 4, No.1, p. 227-235 (2004).

[2] Bergweiler (W.) and Eremenko (A.). — On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoamericana 11, p. 355-373 (1995).

[3] Boutabaa (A.) and Escassut (A.). — Urs and Ursim for p-adic meromorphic functions inside a disk, Proc. Edinburgh Math. Soc., 44, p. 485-504 (2001).

[4] Chen (H. H.) and Fang (M. L.). — On the value distribution of $f^n f^{'}$, Sci. China Ser. A, 38, p. 789-798 (1995).

[5] Clunie (J.). — On a result of Hayman, J. London Math. Soc. 42, p. 389-392 (1967).

[6] Escassut (A.). — Analytic Elements in P-dic Analysis, World Scientific Publishing (1995).

[7] Ha Huy Khoai. — On p-adic meromorphic functions, Duke Math. J. 50, p. 695-711 (1983).

[8] Ha Huy Khoai. — Height of p-adic holomorphic functions and applications, International Symposium "Holomorphic Mappings, Diophantine Geometry and Related Topics" (Kyoto, 1992). Surikaisekikenkyusho Kokyuroku No. 819, p. 96-105 (1993).

[9] Ha Huy Khoai and Vu Hoai An. — Value distribution for p-adic hypersurfaces, Taiwanese Journal of Mathematics, Vol. 7, No.1, p. 51-67 (2003).

[10] Ha Huy Khoai and My Vinh Quang. — On p-adic Nevanlinna theory, Lecture Notes in Math. 1351, p. 146-158 (1988).

[11] Hayman (W. K.). — Picard values of meromorphic functions and their derivativesAnn. Math. (2) 70, p. 9-42 (1959).

[12] Hayman (W. K.). — Research Problems in Function Theory, The Athlone Press University of London, London (1967).

[13] Hu (P.C.) and Yang (C.C.). — Meromorphic functions over Non-Archimedean fields, Kluwer Academic Publishers (2000).

[14] Lahiri (I.) and Dewan (S.). — Value distribution of the product of a meromorphic function and its derivative, Kodai Math. J. 26, 95-100 (2003).

[15] Nevo (Sh.), Pang (X. C.) and Zalcman (L.). — Picard-Hayman behavior of derivatives of meromorphic functions with multiple zeros, Electronic Research Announcements of the American Mathematical Society, Volume 12, p. 37-43 (March 31, 2006).

[16] Ojeda (J.). — Hayman's conjecture in a p-adic field, Taiwanese J. Math. 12, N.9, p. 2295-2313 (2008).

[17] Pang (X. C.), Nevo (Sh.) and Zalcman (L.). — Quasinormal families of meromorphic functions, Rev. Mat. Iberoamericana 21, p. 249-262 (2005).