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Macdonald formula for spherical functions on affine buildings

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Abstract. — In this paper we explicitly determine the Macdonald formula for spherical functions on any locally finite, regular and affine Bruhat-Tits building, by constructing the finite difference equations that must be satisfied and explaining how they arise, by only using the geometric properties of the building.

Résumé. — On détermine explicitement la formule de Macdonald pour les fonctions sphériques sur tout immeuble de Bruhat-Tits localement fini, régulier et affine en construisant d'une manière motivée les équations aux différences finies qu'elles doivent satisfaire, n'utilisant que les propriétés géométriques de l'immeuble.

1. Introduction

Let Δ be a locally finite, regular, irreducible affine building and let $\mathcal{H}(\Delta)$ be the vertex set Hecke algebra of the building, spanned by all averaging operators acting on the space of all complex valued functions defined on all special vertices of Δ. The spherical functions of the building are the eigenfunctions of the algebra $\mathcal{H}(\Delta)$, whose values on any special vertex x of Δ depend only on the position of x, with respect to a fixed vertex e, in any sector based on e and containing x. In the particular case when Δ

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is the linear building associated to a group G of p-adic type, with a maximal compact subgroup K, then the subalgebra $\mathcal{H}_0(\Delta)$ of $\mathcal{H}(\Delta)$, spanned by averaging operators acting on the space of all complex valued functions defined on type 0 special vertices, is isomorphic to the commutative convolution algebra $\mathcal{L}(G, K)$; moreover the restriction of the spherical functions to all type 0 vertices of Δ are the zonal spherical functions on G relative to K considered by Macdonald in [5]. In this paper the author first present the so-called Macdonald formula for the spherical functions. The presentation is clear and accurate and everything is explained in terms of p-adic matrix groups. This omits the case of two dimensional exotic buildings and to some minor extent obscures the essentially building theoretic nature of all the arguments.

Besides the original derivation of Macdonald, there are many proofs of the Macdonald formula, each with different approaches. There are also many algebraic proofs which are more general than both the group approaches and the building approaches because they make sense for an arbitrary spherical Hecke algebra in the sense of Ram [12] (see for example the paper [9] of Opdam). In this context the Macdonald formula is an explicit formula for the image of certain natural basis elements of the spherical Hecke algebra under the Satake isomorphism.

In this paper we provide a systematic derivation of the Macdonald formula for spherical functions on an arbitrary (locally finite, regular, irreducible) affine building, taking more advantage of the geometrical properties of the building. This makes the arguments group independent. In fact the aim of the paper is to present a proof of the results which puts the geometry of the building front and center. We must confess that we have never deeply understood the motivation for the calculations of Macdonald's presentation in his Sections 4.3, 4.4 and 4.5. For this reason we wished to explain the results in such a way that the reader can understand how he might have calculated and proved them himself.

We give a proof in the fashion of Macdonald's original proof, in the sense that we start with a formula for the spherical functions as an integral over the maximal boundary Ω of Δ of the Poisson kernel associated to a convenient character and convert it into a sum of rational functions, obtaining the Macdonald formula. More precisely, we express the spherical function corresponding to a non-singular character χ as a linear combination of the functions $\chi^{\mathbf{w}}, \mathbf{w} \in \mathbf{W}$. The basic tool used to obtain this formula is the construction of a diagonalizable triangular matrix T^λ, of order $d = |\mathbf{W}|$, for every dominant coroot λ, which is diagonalizable by a unique matrix, independent of λ. This matrix codifies the algorithm for computing the values of the spherical function on vertices $x \in \mathcal{V}_\lambda(e)$, through the explicit knowledge of the entries on the principal diagonal of the matrix.

The next step is to investigate in depth the geometric structure of the building and use it to work out convenient finite difference equations which enable one to calculate exactly the coefficients in the above linear combination. The main idea to construct and solve the finite difference equations is the introduction of the α-boundary and the decomposition of the maximal boundary in terms of this α-boundary and the boundary of a tree at infinity; this reduces things to the case of a tree.

For everything that concerns notation and basic facts that we use in this paper, about both the geometric structure of affine buildings and their boundary and about the operator algebra and its eigenvalues, we refer to [8], where these subjects are developed in full detail (see also [10]).

In [11] J. Parkinson independently proves the formula in the general context, using a different approach. His approach, which makes crucial use of [6], is just as geometric as ours. We cannot reasonably claim that our approach, arrived at independently, will be easier to follow for all readers, but it may be easier for some of them.

In [7] and in [1] the formula for the spherical functions has been calculated in the case of an affine building of type $\widetilde{A_2}$. In [2] D. Cartwright generalizes the formula for an affine building of type $\widetilde{A_n}$.

The approach presented here was suggested to us by Tim Steger, who also provided assistance with a few of the details.

2. Notations and definitions

In this section we collect all basic definitions and notation about affine buildings and its boundaries. We refer the reader to [8] for an exhaustive exposition of this argument. We also notice the paper [10] for a similar presentation.

2.1. Affine buildings

We denote by Δ a Bruhat-Tits affine building of rank $n+1$, $n \geqslant 1$. Then Δ is a simplicial complex of rank n which can be expressed as the union of subcomplexes $\mathcal{A}$, called apartments, such that

- each apartment is isomorphic to the Coxeter complex of an affine reflection group W;
- for any two simplices of Δ there is an apartment containing both of them;
- any pair of apartments containing two simplices are isomorphic, through an isomorphism fixing the two simplices.

The vertices are the complexes of rank 1 and the chambers are the complexes of maximal rank n. Δ is a labelled chamber complex through a labelling which assign to each vertex of a chamber an element i of a finite set $I = \{0, 1, 2, \dots, n\}$, called the type of the vertex; the panel of cotype i of a chamber is the maximal subcomplex not containing the vertex of type i.

We always assume that Δ is irreducible, locally finite and regular; for every $i \in I$, we denote by q_i the number of the chambers sharing a panel of cotype i. We refer to the set $\{q_i,\ i \in I\}$ as the parameter system of the building.

We assume, without loss of generality, that W is the affine Weyl group of a root system R; this means that W is the group generated by all affine reflections $s_\alpha^k,\ \alpha \in R,\ k \in \mathbb{Z}$, with respect to the affine hyperplane H_α^k of the vector space $\mathbb{V}$ of dimension n associated with R. Actually it can be proved that the group W can be generated by a finite set $S = \{s_i,\ i \in I\}$, where $s_0 = s^1_{\alpha_0}$ is the affine reflection with respect to the hyperplane $H^1_{\alpha_0}$ associated to the highest root α_0 and $s_i = s^0_{\alpha_i}$, for every $i \in I_0 = \{1, 2, \dots, n\}$, being $B = \{\alpha_i,\ i \in I_0\}$ a basis for R. If $\mathbf{W}$ is the finite Weyl group associated with R, then $W = \mathbf{W} \ltimes L$, if L denotes the coroot lattice of $\mathbb{V}$. Following standard notation, we denote by $\widehat{L}$ the coweight lattice of $\mathbb{V}$. The extended Weyl group of the building is the group $\widehat{W} = \mathbf{W} \ltimes \widehat{L}$. We notice that $\mathbf{W}$ stabilizes the fundamental vertex 0 and W preserves the type of vertices. We denote by G the finite abelian group $G \cong \widehat{L}/L$ which stabilizes in $\widehat{W}$ the fundamental chamber

$$C_0 = \{v \in \mathbb{V}\ :\ \langle v, \alpha_i \rangle > 0,\ \forall i \in I_0, \quad \langle v, \alpha_0 \rangle < 1\}.$$

The linear hyperplanes H_α^0 split up $\mathbb{V}$ into finitely many regions; the connected components of $\mathbb{V} \setminus \bigcup_\alpha H_\alpha^0$ are (open) sectors based at 0, called the (open) Weyl chambers of $\mathbb{V}$ (with respect to R). We denote by $\mathbf{w}_0$ the longest element of $\mathbf{W}$. The so called fundamental Weyl chamber or fundamental sector based at 0 (with respect to the basis B) is the Weyl chamber

$$\mathbb{Q}_0 = \{v \in \mathbb{V}\ :\ \langle v, \alpha_i \rangle > 0,\ i \in I_0\}.$$

We set $\widehat{L}^+ = \{\lambda \in L \ : \ \lambda \in \overline{\mathbb{Q}_0}\}$ and $\widehat{L}^{++} = \{\lambda \in L \ : \ \lambda \in \mathbb{Q}_0\}$.

The building Δ is reduced or not if R is so. When Δ is non-reduced then $\widehat{L} = L$ and G is trivial. If Δ is reduced at most two root lengths occur in R and all roots of a given length are conjugate under $\mathbf{W}$; when there are in R two distinct root lengths, we speak of long and short roots and the highest root α_0 is long.

The root system R and its Weyl group can be characterized by its Dynkin diagram D, which is the usual Coxeter graph of $\mathbf{W}$, where we add an arrow pointing to the shorter of the two roots. For every $n \geqslant 1$ there is exactly one irreducible non-reduced root system of rank n denoted by BC_n. In general different root systems have different affine Weyl group and hence generate different affine buildings. The only exception to this rule are the root systems of type C_n and BC_n, which have the same Weyl group. So, when the group W associated to the building is the affine Weyl group of the root systems of type C_n and BC_n, we have to choose the root system. We assume to operate this choice according to the parameter system of the building. Actually, we choose R to ensure that in each case the group $Aut_{tr}(D)$ of all type-rotating automorphisms of D preserves the parameter system of the building, that is in order to have, for each $\sigma \in Aut_{tr}(D)$, $q_{\sigma(i)} = q_i$, for all $i \in I$. Actually, in both cases $R = C_n$ or BC_n, $q_1 = q_2 = \cdots = q_{n-1}$, but in general $q_0 \neq q_1 \neq q_n$ and q_0 and q_n can have different values only when $R = BC_n$. According to the classification of the root systems and keeping in mind the above choice, we say that Δ has type

1. $\widetilde{X}_n$, if R has type X_n, for $X_n = A_n (n \geqslant 2),\ B_n (n \geqslant 3),\ D_n (n \geqslant 4),\ E_n (n = 6, 7, 8),\ F_4,\ G_2$;
2. $\widetilde{A}_1$, associated to a root system of type A_1, if $q_0 = q_1$ (homogeneous tree);
3. $\widetilde{BC}_1$, associated to a root system of type BC_1, if $q_0 \neq q_1$ (semi-homogeneous tree);
4. $\widetilde{C}_n (n \geqslant 2)$ if $q_0 = q_n$;
5. $\widetilde{BC}_n (n \geqslant 2)$ if $q_0 \neq q_n$.

We refer to Appendix of [10] for the classification of all irreducible, locally finite, regular affine buildings, in terms of diagram and parameter system.

Let Δ be an affine building of type $\widetilde{X}_n$. We denote by $\mathbb{A}$ the simplicial complex of rank $n+1$, realized as a tessellation of the Euclidean space $\mathbb{V}$ of dimension n by the family of hyperplanes $\mathcal{H} = \{H_\alpha^k,\ \alpha \in R^+,\ k \in \mathbb{Z}\}$, in which the chambers are the open connected components of $\mathbb{V} \setminus \bigcup_{\alpha,k} H_\alpha^k$. Thus $\mathbb{A}$ may be regarded as the geometric realization of the Coxeter complex of Δ and it is called the fundamental apartment of the building. The extreme points of the closure of any chamber are the vertices and the 1-codimension facets of any chamber are the panels. If $X_0^0 = 0, X_1^0, \dots, X_n^0$ are the vertices of the fundamental chamber C_0, we declare $\tau(X_i^0) = i$, for every $i = 0, \dots, n$; more generally we declare that a vertex X has type i if $X = w(X_i^0)$, for some $w \in W$. We denote by $\mathcal{C}(\mathbb{A})$ and $\mathcal{V}(\mathbb{A})$ the set of chambers and the set of vertices of $\mathbb{A}$ respectively. Moreover $\mathcal{V}_i(\mathbb{A})$ is, for every $i \in I$, the set of all vertices of type i, $\mathcal{V}_{sp}(\mathbb{A})$ is the set of all special vertices and $\widehat{\mathcal{V}}(\mathbb{A})$ is the set of all special vertices of $\mathbb{A}$ belonging to $\widehat{L}$. We denote by $\widehat{I}$ the set of types of the vertices in $\widehat{L}$. Given any pair of special vertices X, Y, there exists a unique $\widehat{w} \in \widehat{W}$ such that $\widehat{w}(X) = 0$ and $\widehat{w}(Y)$ belongs to $\widehat{L}^+$; we call shape of Y with respect to X the element $\sigma(X, Y) = \widehat{w}(Y)$.

Each apartment $\mathcal{A}$ of Δ is isomorphic to $\mathbb{A}$ and hence it can be regarded as a Euclidean space, tessellated by a family of affine hyperplanes $\mathcal{H}(\mathcal{A})$ isomorphic to $\mathcal{H}$. If $\psi : \mathcal{A} \to \mathbb{A}$ is any type-rotating isomorphism, we set $h = h_\alpha^k$, if $\psi(h) = H_\alpha^k$. We denote by $\mathcal{C}(\Delta)$, $\mathcal{V}(\Delta)$ the set of chambers and the set of vertices of the building respectively. Moreover $\mathcal{V}_i(\mathbb{A})$ denotes, for every $i \in I$, the set of all vertices of type i of the building. There is a natural way to extend to Δ the definition of special vertices given in $\mathbb{A}$; we call special every vertex of the building whose image on $\mathbb{A}$ under a type-preserving isomorphism is special. Thus $\mathcal{V}_{sp}(\Delta)$ is the set of all special vertices of the building and $\widehat{\mathcal{V}}(\Delta)$ is the set of all special vertices of type $i \in \widehat{I}$.

For every pair of chambers $c, d \in \mathcal{C}(\Delta)$, there exists a minimal gallery $\gamma(c, d)$ from c to d. If $f = i_1 \cdots i_k$, is a reduced word in the free monoid on I and $w_f = s_{i_1} \cdots s_{i_k}$ is such that $d = w_f(c)$, we set $\delta(c, d) = w_f$ and write $d = c \cdot \delta(c, d)$; moreover f is said the type of $\gamma(c, d)$. More generally, for every vertex $x \in \widehat{\mathcal{V}}(\Delta)$ and every chamber d, there exists a minimal gallery $\gamma(x, d)$ from x to d and $\gamma(x, d) = \gamma(c, d)$, if c is the chamber of $\gamma(x, d)$ containing x. Finally given two vertices $x, y \in \widehat{\mathcal{V}}(\Delta)$, there exists a minimal gallery $\gamma(x, y)$ from x to y, lying on any apartment $\mathcal{A}(x, y)$ containing x and y; if c and d are the chambers of $\gamma(x, y)$ containing x and y respectively, and $\delta(c, d) = w_f$, then the type of this gallery is $f = i_1 \cdots i_k$.

For every $x \in \widehat{\mathcal{V}}(\Delta)$ and every $w \in W$, we set

$$\mathcal{C}_w(x) = \{d \in \mathcal{C}(\Delta) \ : \ \delta(x, d) = w\}.$$

If $\mathcal{C}_x$ is, for every $x \in \widehat{\mathcal{V}}(\Delta)$, the set of all chambers containing x, then $\mathcal{C}_w(x) = \bigcup_{c \in \mathcal{C}_x} \mathcal{C}_w(c)$, as a disjoint union. Independently of the type of the vertex x, the cardinality of $\mathcal{C}_x$ is the Poincaré polynomial $\mathbf{W}(q)$ of $\mathbf{W}$. Moreover, for every $w \in W$, with $w = s_{i_1} \cdots s_{i_k}$,

$$|\mathcal{C}_w(x)| = \mathbf{W}(q)\ q_w,$$

if $q_w = q_{i_1} \cdots q_{i_k}$.

For any pair of facets $\mathcal{F}_1, \mathcal{F}_2$ of the building, the convex hull of $\{\mathcal{F}_1, \mathcal{F}_2\}$ is the minimal convex region $[\mathcal{F}_1, \mathcal{F}_2]$ of any apartment $\mathcal{A}(\mathcal{F}_1, \mathcal{F}_2)$ containing them, which is delimited by hyperplanes of the apartment and contains $\{\mathcal{F}_1, \mathcal{F}_2\}$.

Finally, given two vertices $x, y \in \widehat{\mathcal{V}}(\Delta)$, the shape of y with respect to x is defined as $\sigma(x,y) = \sigma(X,Y)$, if $X = \varphi(x),\ Y = \varphi(y)$, for a type-preserving isomorphism φ from $\mathcal{A}(x,y)$ onto $\mathbb{A}$. Hence $\sigma(x,y)$ is an element of $\widehat{L}^+$ and, if $\sigma(x,y) = \lambda$, there exists a type-rotating isomorphism $\psi : \mathcal{A}(x,y) \to \mathbb{A}$ such that $\psi(x) = 0$ and $\psi(y) = \lambda$. For every vertex $x \in \widehat{\mathcal{V}}(\Delta)$ and every $\lambda \in \widehat{L}^+$, we define

$$\mathcal{V}_\lambda(x) = \{y \in \widehat{\mathcal{V}}(\Delta)\ :\ \sigma(x,y) = \lambda\}.$$

For every $x \in \widehat{\mathcal{V}}(\Delta)$, we have $\widehat{\mathcal{V}}(\Delta) = \bigcup_{\lambda \in \widehat{L}^+} \mathcal{V}_\lambda(x)$ as a disjoint union. The cardinality $|\mathcal{V}_\lambda(x)|$ does not depend on x; so we set $N_\lambda = |\mathcal{V}_\lambda(x)|$. It can be proved (see for instance [8, Proposition 2.15.1 and Proposition 2.15.2]) that, if $\tau(x) = i,\ \tau(X_\lambda) = l$ and $j = \sigma_i(l)$, then

$$N_\lambda \ = \ \frac{1}{\mathbf{W}(q)} \sum_{w \in \mathbf{W} w_\lambda \mathbf{W}_j} q_w \ = \ \frac{\mathbf{W}(q)}{\mathbf{W}_\lambda(q)}\ q_{w_\lambda} = \frac{\mathbf{W}(q^{-1})}{\mathbf{W}_\lambda(q^{-1})} q_{t_\lambda} \tag{2.1}$$

where $\mathbf{W}_\lambda = \{\mathbf{w} \in \mathbf{W}\ :\ \mathbf{w}\lambda = \lambda\}$, $\mathbf{W}_j$ is the stabilizer in W of the vertex X_j^0 of C_0, w_λ is the unique element of W such that $C_\lambda = w_\lambda(C_0)$, if C_λ is the chamber of $\mathbb{Q}_0$ containing X_λ and nearest to C_0, and t_λ is the translation $v \mapsto v + \lambda$ on $\mathbb{A}$. In particular, if $\lambda \in L^{++}$, then

$$N_\lambda = \mathbf{W}(q)\ q_{w_\lambda} = \mathbf{W}(q^{-1}) q_{t_\lambda}.$$

It will be useful to define, for every hyperplane h on any apartment of the building, $q_h = q_i$, if h contains a panel of cotype i; then, for every $\alpha \in R^+$ and every $k \in \mathbb{Z}$, we can define $q_{\alpha,k} = q_h$, if $\psi(h) = H_\alpha^k$, for any type-rotating isomorphism ψ mapping any apartment containing h onto $\mathbb{A}$. When Δ is reduced, $q_{\alpha,k} = q_{\alpha,0}$, for every $k \in \mathbb{Z}$ and we set $q_\alpha = q_{\alpha,k}$, for every $k \in \mathbb{Z}$; then $q_{\alpha_i} = q_i, \forall i \in I$, and $q_\alpha = q_{\alpha_i}$, if $\alpha = \mathbf{w}\alpha_i$, for some $\mathbf{w} \in \mathbf{W}$. It turns out that $q_\alpha = q$, for every $\alpha \in R^+$, if all roots have the same length, and, if R contains long and short roots, $q_\alpha = q$ for all long α and $q_\beta = p$ for all short β. When Δ is non-reduced,

$$q_\alpha = \begin{cases} q_0 & \forall \alpha \in R_0, \\ q_1 & \forall \alpha \in R_1, \\ q_n & \forall \alpha \in R_2, \end{cases}$$

if $R_0 = \{\alpha \in R : \alpha/2, 2\alpha \notin R\}$, $R_1 = \{\alpha \in R : \alpha/2 \in R, 2\alpha \notin R\}$ and $R_2 = \{\alpha \in R : \alpha/2 \notin R, 2\alpha \in R\}$. For ease of notation, we set $q_1 = p,\ q_0 = q,\ q_n = r$. If we extend the definition of q_α by setting $q_\alpha = 1$ when $\alpha \notin R$, then $q_\alpha = p,\ q_{\alpha/2} = r$ if $\alpha \in R_1$, and $q_\alpha = q,\ q_{\alpha/2} = 1$ if $\alpha \in R_0$.

Taking in account these definitions, we notice that, for every $\lambda \in \widehat{L}^+$,

$$q_{t_\lambda} = \prod_{\alpha \in R^+} q_\alpha^{\langle \lambda, \alpha \rangle}\ q_{2\alpha}^{-\langle \lambda, \alpha \rangle}$$

and hence, for every $\lambda \in \widehat{L}^+$,

$$N_\lambda = \frac{\mathbf{W}(q^{-1})}{\mathbf{W}_\lambda(q^{-1})} \prod_{\alpha \in R^+} q_\alpha^{\langle \lambda, \alpha \rangle}\ q_{2\alpha}^{-\langle \lambda, \alpha \rangle}. \tag{2.2}$$

In particular, for every $\lambda \in \widehat{L}^{++}$,

$$N_\lambda = \mathbf{W}(q^{-1}) \prod_{\alpha \in R^+} q_\alpha^{\langle \lambda, \alpha \rangle}\ q_{2\alpha}^{-\langle \lambda, \alpha \rangle}.$$

We notice that the building is chamber regular, that is, for every triple $w_0, w_1, w_2 \in W$ and every pair of chambers c_1, c_2, such that $\delta(c_1, c_2) = w_0$, the cardinality of the set

$$\{c' \in \mathcal{C}(\Delta)\ :\ \delta(c_1, c') = w_1,\ \delta(c_2, c') = w_2\}$$

does not depend on the choice of the chambers but only depends on w_0, w_1, w_2.

Moreover the building is vertex regular, that is, for every triple $\lambda, \mu, \nu \in \widehat{L}$ and every pair $x, y \in \widehat{\mathcal{V}}(\Delta)$ such that $\sigma(x, y) = \lambda$, the cardinality of the set

$$\{z \in \widehat{\mathcal{V}}(\Delta)\ :\ \sigma(x, z) = \mu,\ \sigma(y, z) = \nu\}.$$

does not depend on the choice of the vertices , but only depends on λ, μ, ν. Moreover

$$|\{z \in \widehat{\mathcal{V}}(\Delta) \;:\; \sigma(x,z) = \mu,\ \sigma(y,z) = \nu\}|$$
$$= |\{z \in \widehat{\mathcal{V}}(\Delta) \;:\; \sigma(x,z) = \nu^\star,\ \sigma(y,z) = \mu^\star\}|.$$

We set

$$N(\lambda,\mu,\nu) = |\{z \in \widehat{\mathcal{V}}(\Delta) \;:\; \sigma(x,z) = \mu,\ \sigma(y,z) = \nu\}| = N(\lambda,\nu^\star,\mu^\star), \quad \text{if} \quad \sigma(x,y) = \lambda, \tag{2.3}$$

where $\mu^\star = \iota(\mu)$ and $\nu^\star = \iota(\nu)$, if ι denotes the map

$$\iota(\mu) = -\mathbf{w}_0(\mu), \quad \forall \mu \in \widehat{L}.$$

There is a partial order on $\widehat{L}$, defined as follows

$$\mu \preceq \lambda, \quad \text{if} \quad \lambda - \mu \in L^+.$$

Since $\widehat{\mathcal{V}}(\mathbb{A})$ may be identified with the coweight lattice $\widehat{L}$, the partial ordering defined on $\widehat{L}$ applies to $\widehat{\mathcal{V}}(\mathbb{A})$. For every $\lambda \in \widehat{L}^+$, we define

$$\Pi_\lambda = \{\mathbf{w}\mu \;:\; \mu \in \widehat{L}^+,\ \mu \preceq \lambda,\ \mathbf{w} \in \mathbf{W}\}.$$

This set is saturated: for every $\eta \in \Pi_\lambda$ and every $\alpha \in R$, then $\eta - j\alpha^\vee \in \Pi_\lambda$, for every $0 \leqslant j \leqslant \langle \eta, \alpha \rangle$. Hence it is stable under $\mathbf{W}$. Moreover λ is the highest coweight of Π_λ. It is easy to prove that $\Pi_\lambda + \Pi_\mu \subset \Pi_{\lambda+\mu}$, for every $\lambda, \mu \in \widehat{L}^+$. We recall that W is endowed with the Bruhat ordering, defined as follows (see [4]). We declare $w_1 < w_2$ if there exists a sequence $w_1 = u_0 \to u_1, \ldots, u_{k-1} \to u_k = w_2$, where $u_j \to u_{j+1}$ means that $u_{j+1} = u_j s$, for some $s \in S$, and $|u_j| < |u_{j+1}|$. This defines a partial order on W that can be extended to $\widehat{W}$ by setting $\widehat{w}_1 \leqslant \widehat{w}_2$, if $\widehat{w}_1 = w_1 g_1$ and $\widehat{w}_2 = w_2 g_2$ with $w_1 < w_2$. We remark that $w_1 \leqslant w_2$ if and only if w_1 can be obtained as a sub-expression $s_{i_{k_1}} \cdots s_{i_{k_m}}$ of any reduced expression $s_{i_1} \cdots s_{i_r}$ for w_2. We notice that, for every $\lambda \in \widehat{L}^+$, if $\widehat{w}(0) \in \Pi_\lambda$, then $\widehat{w}'(0) \in \Pi_\lambda$, for each $\widehat{w}' \leqslant \widehat{w}$.

We may also define a partial ordering on $\mathcal{C}(\mathbb{A})$, in the following way. Given two chambers C_1, C_2 consider all the hyperplanes H_α^k separating C_1 and C_2. We declare $C_1 \prec C_2$, if C_2 belongs to the positive half-space determined by each of these hyperplanes. It is clear that the resulting relation $C_1 \preceq C_2$ is a partial ordering of $\mathcal{C}(\mathbb{A})$. We notice that, by definition of $\mathbb{Q}_0$, we have $C_0 \prec C$ if and only if $C \subset \mathbb{Q}_0$. Moreover, if C is any chamber and $s = s_\alpha^k$ is the affine reflection with respect to the hyperplane containing a panel of C, then $C \prec s(C)$ or $s(C) \prec C$, since C and $s(C)$ are adjacent. Since $\mathcal{C}(\mathbb{A})$ may be identified with W, the previous definition induces a partial ordering on W. We point out that this ordering is different from the Bruhat order. Nevertheless, if $w_1(C_0)$ and $w_2(C_0)$ belong to $\mathbb{Q}_0$, then $w_1(C_0) \prec w_2(C_0)$ if and only if $w_1 < w_2$. Moreover, on $\mathbf{W}$, we have

$$\mathbf{w}_1(C_0) \prec \mathbf{w}_2(C_0) \quad \text{if and only if} \quad \mathbf{w}_1 > \mathbf{w}_2.$$

It may be proved that, if C is any chamber of $\mathbb{A}$ such that $C \prec s(C)$, where $s = s_\alpha^k$ is the affine reflection with respect to the hyperplane H_α^k containing a panel of C, then, for every $w \in W$,

(i) if $w(C) \prec ws(C)$, then $\mathbf{w} < \mathbf{ws}$,

(ii) if $ws(C) \prec w(C)$, then $\mathbf{ws} < \mathbf{w}$,

if $w = \mathbf{w}t_\lambda$ for some $\mathbf{w} \in \mathbf{W}$, $\lambda \in L$ and $\mathbf{s} = s_\alpha^0$. See [8] for the proof of this property.

It will be useful to define, for every $x \in \widehat{\mathcal{V}}(\Delta)$, the retraction ρ_x of the building onto its fundamental apartment. For every $x \in \widehat{\mathcal{V}}(\Delta)$ and $c \in \mathcal{C}(\Delta)$, we denote by $proj_x(c)$ the chamber containing x in any minimal gallery $\gamma(x, c)$. In particular we write $proj_0(c)$ when x is the fundamental vertex e. We note that $proj_x(c)$ does not depend on the minimal gallery we consider. In the fundamental apartment $\mathbb{A}$, let $\mathbb{Q}_0^- = \mathbf{w}_0(\mathbb{Q}_0)$ and C_0^- the base chamber of $\mathbb{Q}_0^-$.

Definition 2.1. — *Let x be any special vertex of Δ (say $\tau(x) = i$). For every $c \in \mathcal{C}(\Delta)$, the retraction of c with respect to x is defined as*

$$\rho_x(c) = C_0^- \cdot \delta_i(proj_x(c), c),$$

if, for every pair c, d of chambers, we set $\delta_i(c, d) = w_{\sigma_i^{-1}(f)}$ when $\delta(c, d) = w_f$. In particular, if $\tau(x) = 0$,

$$\rho_x(c) = C_0^- \cdot \delta(proj_x(c), c).$$

Obviously, $\rho_x(c)$ belongs to $\mathbb{Q}_0^-$, for every c. We extend the previous definition to all special vertices. For every $y \in \mathcal{V}_{sp}(\Delta)$, say $\tau(y) = j$, we set

$$\rho_x(y) = v_l(\rho_x(c)),$$

if c is any chamber containing y, and $l = \sigma_i^{-1}(j)$. Actually this definition does not depend on the choice of the chamber containing the vertices y. In particular, we denote by ρ_0 the retraction with respect to the fundamental vertex e. It will be useful to remark that, if $\lambda \in \widehat{L}^+$, and $t_\lambda = u_\lambda g_l$, then, for every c such that $\delta(proj_0(c), c) = u_\lambda$, we have $\rho_0(c) = \mathbf{w}_0 u_\lambda(C_0)$. Therefore, if $\sigma(e, x) = \lambda$, then $\rho_0(x) = \mathbf{w}_0\lambda$.

We recall that the action of $\widehat{W}$ on the set $\mathcal{C}(\mathbb{A})$ is transitive but not simply transitive; actually, if $\widehat{w}_i = wg_i$, then $\widehat{w}_i(C_0) = w(C_0)$, for every $w \in W$ and for every $i \in \widehat{I}$. Nevertheless, the action of the elements $\widehat{w}_i$ on the special vertices $v_j(C_0)$ of C_0 depends on i, because

$$\widehat{w}_i(v_j(C_0)) = v_{\sigma_i(j)}(w(C_0)).$$

This suggest to enlarge the set $\mathcal{C}(\mathbb{A})$ in the following way. We call extended chamber of $\mathbb{A}$ a pair $\widehat{C} = (C, \sigma)$, for every $C \in \mathcal{C}(\mathbb{A})$ and for every $\sigma \in Aut_{tr}(D)$; we denote by $\widehat{\mathcal{C}}(\mathbb{A})$ the set of all extended chambers. A straightforward consequence of this definition is that $\widehat{W}$ acts simply transitively on $\widehat{\mathcal{C}}(\mathbb{A})$: for every pair of extended chambers $\widehat{C}_1 = (C_1, \sigma_{i_1})$ and $\widehat{C}_2 = (C_2, \sigma_{i_2})$, there exists a unique element $\widehat{w} \in \widehat{W}$ such that $\widehat{C}_2 = \widehat{w}(\widehat{C}_1)$. Actually, if $C_2 = w(C_1)$, $g = g_{i_2} g_{i_1}^{-1}$ and σ is the automorphism of D corresponding to g, then $\widehat{w} = wg = g\sigma(w)$. In particular, for every $\widehat{C} = (C, \sigma_i)$, then $\widehat{w} = wg_i = g_i\sigma_i(w)$ is the unique element of $\widehat{W}$ such that $\widehat{w}(C_0) = \widehat{C}$, if $C = w(C_0)$. In the same way we enlarge the set $\mathcal{C}(\Delta)$ and we define

$$\widehat{\mathcal{C}}(\Delta) = \{\widehat{c} = (c, \sigma_i),\ c \in \mathcal{C}(\Delta),\ i \in \widehat{I}\}.$$

We notice that for every $c \in \mathcal{C}(\Delta)$ and $i \in \widehat{I}$, $\widehat{c} = (c, \sigma_i)$ is the unique extended chamber such that $v_i(c) = v_0(\widehat{c})$. The W-distance on $\mathcal{C}(\Delta)$ can be extended to a $\widehat{W}$-distance on $\widehat{\mathcal{C}}(\Delta)$ in the following way: for every pair of extended chambers $\widehat{c}_1 = (c_1, \sigma_{i_1})$ and $\widehat{c}_2 = (c_2, \sigma_{i_2})$, we set

$$\widehat{\delta}(\widehat{c}_1, \widehat{c}_2) = \delta(c_1, c_2) g_{i_2} g_{i_1}^{-1}.$$

For every $\lambda \in \widehat{L}^+$, with $\tau(\lambda) = l$, $t_\lambda(C_0) = (u_\lambda(C_0), g_l)$ and $v_0(t_\lambda(C_0)) = v_l(u_\lambda(C_0))$, if $t_\lambda = u_\lambda g_l$.

2.2. Maximal boundary

For every $X \in \widehat{\mathcal{V}}(\mathbb{A})$, let $\mathcal{H}_X$ be the collection of all hyperplanes of $\mathbb{A}$ sharing X; a sector of $\mathbb{A}$, based at X, is any connected component Q_X of $\mathbb{V} \setminus \bigcup_{H_\alpha^k \in \mathcal{H}_X} H_\alpha^k$. For every chamber C containing X, $Q_X(C)$ denotes the sector based at X, of base chamber C. We remark that, for every $X \in \widehat{\mathcal{V}}(\mathbb{A})$, and every C containing X, there exists a unique $\widehat{w} \in \widehat{W}$, such that $Q_X(C) = \widehat{w}\ \mathbb{Q}_0$.

More generally, for any $x \in \widehat{\mathcal{V}}(\Delta)$, a sector of Δ, with base vertex x, is a subcomplex Q_x of any apartment $\mathcal{A}$ of the building, such that $\psi_{tp}(Q_x) = Q_X$, if X is any special vertex such that $\tau(X) = \tau(x)$, and $\psi_{tp} : \mathcal{A} \to \mathbb{A}$ is a type-preserving isomorphism mapping x to X. We note that, given any apartment $\mathcal{A}$ of the building, for every sector $Q_x \subset \mathcal{A}$ there exists a unique type-rotating isomorphism $\psi_{tr} : \mathcal{A} \to \mathbb{A}$ mapping Q_x to $\mathbb{Q}_0$. We say that a sector Q_y is a subsector of a sector Q_x if $Q_y \subset Q_x$. Two sectors Q_x and Q_y are said to be equivalent if they share a subsector Q_z. Each equivalence class of sectors is called a boundary point of the building and it is denoted by ω; the set of all equivalence classes of sectors is called the maximal boundary of the building and it is denoted by Ω. We denote by $Q_x(\omega)$ the unique sector in the class ω, based at x. For every $x \in \widehat{\mathcal{V}}(\Delta)$ and every $\omega \in \Omega$, there exists an apartment $\mathcal{A}(x,\omega)$ containing x and ω (in fact containing $Q_x(\omega)$). Analogously, for every chamber c and every $\omega \in \Omega$, there exists an apartment $\mathcal{A}(c,\omega)$ containing c and ω, that is c and a sector in the class ω. On this apartment we denote by $Q_c(\omega)$ the intersection of all sectors in the class ω containing c.

For every $x \in \widehat{\mathcal{V}}(\Delta)$ and every chamber $c \in \mathcal{C}(\Delta)$, we define on the maximal boundary Ω the set

$$\Omega(x,c) = \{\omega \in \Omega \ : \ Q_x(\omega) \supset c\}.$$

Analogously, for every pair of special vertices x, y, we can define the set $\Omega(x,y)$ of Ω given by

$$\Omega(x,y) = \{\omega \in \Omega \ : \ y \in Q_x(\omega)\}.$$

We note that , for every x,

$$\begin{aligned}
&\Omega(x,c'),\ \Omega(x,z) \quad \supset \Omega(x,c), \quad \text{forevery} \quad c',\ z \quad \text{in the convex hull of } \{x,c\},\\
&\Omega(x,c'),\ \Omega(x,z) \quad \supset \Omega(x,y), \quad \text{forevery} \quad c',\ z \quad \text{in the convex hull of } \{x,y\}.
\end{aligned}$$

Let $\omega \in \Omega$ and $x \in \widehat{\mathcal{V}}(\Delta)$; for every apartment $\mathcal{A} = \mathcal{A}(x,\omega)$ containing ω and x, there exists a unique type-rotating isomorphism $\psi_{tr} : \mathcal{A} \to \mathbb{A}$, such that $\psi_{tr}(Q_x(\omega)) = \mathbb{Q}_0$. On the other hand, if $\mathcal{A}'$ contains a subsector $Q_y(\omega)$ of $Q_x(\omega)$, but not x, then there exists a type-preserving isomorphism $\phi : \mathcal{A}' \to \mathcal{A}(x,\omega)$ fixing $Q_y(\omega)$; hence it is well defined the type-rotating isomorphism $\psi'_{tr} = \psi_{tr}\ \phi : \mathcal{A}' \to \mathbb{A}$. Since every facet $\mathcal{F}$ of the building lies on an apartment $\mathcal{A}'$ containing a subsector $Q_y(\omega)$ of $Q_x(\omega)$ (possibly

$Q_x(\omega)$), then, according to previous notation, $\mathcal{F}$ maps uniquely on the facet $\mathbf{F} = \psi'_{tr}(\mathcal{F})$ of $\mathbb{A}$.

Definition 2.2. — *We call retraction of* Δ *on* $\mathbb{A}$, *with respect to* ω *and of center* x, *the map*

$$\rho^x_\omega : \Delta \to \mathbb{A},$$

such that, for every apartment $\mathcal{A}'$ *and for every facet* $\mathcal{F} \in \mathcal{A}'$, $\rho^x_\omega(\mathcal{F}) = \mathbf{F} = \psi'_{tr}(\mathcal{F})$.

In particular $\rho^x_\omega(x) = 0$ and, if we denote by c^x_ω the base chamber of $Q_x(\omega)$, then $\rho^x_\omega(c^x_\omega) = C_0$. Moreover, for every chamber $c \in Q_x(\omega)$ and for every special vertex $y \in Q_x(\omega)$, then

$$\rho^x_\omega(c) = C_0 \cdot \delta(c^x_\omega, c), \quad \text{and} \quad \rho^x_\omega(y) = X_\mu,$$

if X_μ is the special vertex associated with $\mu = \sigma(x, y)$. For ease of notation, we simply set $\rho^x_\omega(z) = \mu$ to mean that $\rho^x_\omega(y) = X_\mu$. In the case $x = e$, we set $\rho_\omega = \rho^e_\omega$.

We collect here, without any proof, the fundamental results concerning the retraction ρ^x_ω. We refer to [8, Section 3] for the proof.

Proposition 2.3. — *Let* $x, y \in \widehat{\mathcal{V}}(\Delta)$, $c \in \mathcal{C}(\Delta)$ *and* $\omega \in \Omega$.

(i) *If* $d \subset Q_x(\omega) \cap Q_c(\omega)$, *then* $\delta(x, d)\ \delta(d, c)$ *is independent of* d *and* $\rho^x_\omega(c) = C_0 \cdot \delta(x, d)\ \delta(d, c)$.

(ii) *If* $z \in Q_x(\omega) \cap Q_y(\omega)$, $\sigma(x, z) - \sigma(y, z)$ *is independent of* z *and* $\rho^x_\omega(y) = X_{\sigma(x,z)} - X_{\sigma(y,z)}$.

The previous proposition implies that, for all x, y, z in $\widehat{\mathcal{V}}(\Delta)$ and for each $\omega \in \Omega$, $\rho^y_\omega(z) = \rho^x_\omega(z) - \rho^x_\omega(y)$. In particular, if $z = x$, then $\rho^y_\omega(x) = -\rho^x_\omega(y)$ and, for all x, y and for each $\omega \in \Omega$, $\rho^x_\omega(y) = \rho_\omega(y) - \rho_\omega(x)$. We point out that in fact this formula is independent of the choice of the fundamental vertex e.

Theorem 2.4. — *Let* $y \in \mathcal{V}_\lambda(x)$ *and* $z \in \mathcal{V}_\mu(x)$. *If* μ *is large enough with respect to* λ, *then* $\Omega(x, z) \subset \Omega(y, z)$. *Moreover, for all* $\omega \in \Omega(x, z)$, $\rho^x_\omega(y) = X_\mu - X_\nu$, *if* $\sigma(y, z) = \nu$.

It can be proved that, for every $x \in \widehat{\mathcal{V}}(\Delta)$ and for every $\lambda \in \widehat{L}^+$, if $y \in \mathcal{V}_\lambda(x)$, then $\rho^x_\omega(y) \in \Pi_\lambda$, for every $\omega \in \Omega$. In particular $\rho^x_\omega(y) = X_\lambda$, if $y \in Q_x(\omega)$.

THEOREM 2.5. — *Let $x \in \mathcal{V}_\lambda(x)$ and $\omega \in \Omega$.*

(i) *For $w, w_1 \in W$, then $|\{c \in \mathcal{C}(\Delta) \ : \ \delta(x,c) = w_1,\ \rho^x_\omega(c) = C_0 \cdot w\}|$ is independent of x and ω.*

(ii) *For every $\mu \in \Pi_\lambda$, $|\{y \in \mathcal{V}_\lambda(x) \ : \ \rho^x_\omega(y) = X_\mu\}|$ is independent of x and ω.*

As a consequence of this theorem we set, for every $x \in \mathcal{V}_\lambda(x)$ and $\omega \in \Omega$,

$$N(\lambda,\mu) = |\{y \in \mathcal{V}_\lambda(x) \ : \ \rho^x_\omega(y) = X_\mu\}|. \tag{2.4}$$

It will be useful to compare the retraction ρ^x_ω with the retraction ρ_x with respect to x.

LEMMA 2.6. — *Let c be any chamber and let $y \in \widehat{\mathcal{V}}(\Delta)$.*

(i) *If c (respectively y) lies on the sector $Q^-_x(\omega)$ opposite to $Q_x(\omega)$, in any apartment $\mathcal{A}(x,\omega)$, then $\rho^x_\omega(c) = \rho_x(c)$,* (respectively $\rho^x_\omega(y) = \rho_x(y)$).

(ii) *If c (respectively y) belongs to the sector $(Q^\alpha_x)^-(\omega)$, $\alpha-$adjacent to $Q^-_x(\omega)$, in any apartment containing c and $Q_x(\omega)$, then $\rho^x_\omega(c) = s_\alpha\rho_x(c)$,* (respectively $\rho^x_\omega(y) = s_\alpha\rho_x(y)$).

The maximal boundary Ω may be endowed with a totally disconnected compact Hausdorff topology in the following way. Fix a special vertex $x \in \widehat{\mathcal{V}}(\Delta)$, say of type $i = \tau(x)$; consider the family

$$\mathcal{B}_x = \{\ \Omega(x,c),\ c \in \mathcal{C}\}.$$

Then $\mathcal{B}_x$ generates a totally disconnected compact Hausdorff topology on Ω; for every $\omega \in \Omega$, a local base at ω is given by

$$\mathcal{B}_{x,\omega} = \{\ \Omega(x,c),\ c \subset Q_x(\omega)\}.$$

We observe that it suffices to consider, as a local base at ω, only the chambers c lying on $Q_x(\omega)$ such that, for some $\lambda \in \widehat{L}^+$, $\delta(c_x(\omega), c) = \sigma_i(t_\lambda)$, if $c_x(\omega)$ is the base chamber of the sector $Q_x(\omega)$, and $i = \tau(x)$. The topology on Ω does not depend on the particular $x \in \widehat{\mathcal{V}}(\Delta)$.

For each x of $\widehat{\mathcal{V}}(\Delta)$, we define a regular Borel probability measure ν_x on Ω by setting

$$\nu_x(\Omega(x,y)) = N_\lambda^{-1} = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R^+} q_\alpha^{-\langle\lambda,\alpha\rangle}\ q_{2\alpha}^{\langle\lambda,\alpha\rangle}, \quad \text{if} \quad y \in \mathcal{V}_\lambda(x).$$

The measure ν_x has the following property.

THEOREM 2.7. — *Let $x \in \widehat{\mathcal{V}}(\Delta)$.*

(i) *Let $w, w_0 \in W$; for each $c \in \mathcal{C}(\Delta)$ such that $\delta(x,c) = w_0$, $\nu_x(\{\omega \in \Omega \, : \, \rho^x_\omega(c) = C_0 \cdot w\})$ is independent of x and c.*

(ii) *Let $\lambda \in \widehat{L}^+$ and $\mu \in \Pi_\lambda$; for each $y \in \widehat{\mathcal{V}}(\Delta)$ such that $\sigma(x,y) = \lambda$, $\nu_x(\{\omega \in \Omega \, : \, \rho^x_\omega(y) = \mu\})$ is independent of x and y.*

2.3. α-boundary and trees at infinity

For every $i \in I_0$, the i-type wall $H_{0,i}$ of the fundamental sector $\mathbb{Q}_0$ of $\mathbb{A}$ is the intersection with $\overline{\mathbb{Q}_0}$ of the hyperplane $H_i = H_{\alpha_i}$, that is the wall of $\mathbb{Q}_0$ containing the cotype i panel of C_0. We extend this definition to each sector of $\mathbb{A}$ by declaring that, for every special vertex X_λ in $\mathbb{A}$, and for every chamber C sharing X_λ, the i-type wall of the sector $Q_\lambda(C)$ based at X_λ is the intersection with $\overline{Q_\lambda(C)}$ of the affine hyperplane H^k_α, $\alpha \in R^+$, $k \in \mathbb{Z}$, which is a wall of the chamber C such that there is a type-preserving isomorphism on $\mathbb{A}$ mapping the wall on the affine hyperplane $H^k_i = H^k_{\alpha_i}$, for some $k \in \mathbb{Z}$.

The definition of wall can be extended to each sector of the building; if $Q_x(c)$ is any sector of Δ, and $\mathcal{A}$ is any apartment containing $Q_x(c)$, then the walls of $Q_x(c)$ are the inverse images of the walls of the sector $Q_\lambda(C) = \psi_{tp}(Q_x(c))$, under a type-preserving isomorphism $\psi_{tp} : \mathcal{A} \to \mathbb{A}$. Moreover, for every $i \in I_0$, a wall of $Q_x(c)$ has type i, if its image in $\mathbb{A}$ has type i. This definition does not depend on the choice of the apartment $\mathcal{A}$ containing the sector and on the type-preserving isomorphism $\psi_{tp} : \mathcal{A} \to \mathbb{A}$. For every sector $Q_x(c)$ and for every $i \in I_0$, we denote by $h_{x,i}(c) = h_{x,i}(Q_x(c))$ the type i wall of the sector. If ω is any element of the maximal boundary Ω, then, for every $x \in \widehat{\mathcal{V}}(\Delta)$ and for every $i \in I_0$, we simply denote by $h_{x,i}(\omega)$ the wall of type i of the sector $Q_x(\omega)$. If α is a simple root, that is $\alpha = \alpha_i$, for some $i \in I_0$, for every special vertex x of Δ, and for every $\omega \in \Omega$, we shall denote by $h_{x,\alpha}(\omega)$ the wall of $Q_x(\omega)$ of type i and we simply call it the α-wall of $Q_x(\omega)$. In general, for every simple root α, we shall denote by $h_{x,\alpha}$ the α-wall of any sector based at x.

DEFINITION 2.8. — *Let $\omega, \omega' \in \Omega$. We say that ω is α-equivalent to ω', and we write $\omega \sim_\alpha \omega'$, if, for some x, $h_{\alpha,x}(\omega) = h_{\alpha,x}(\omega')$.*

Since it can be proved that, if there exists a vertex $x \in \widehat{\mathcal{V}}(\Delta)$ such that $h_{x,\alpha}(\omega_1) = h_{x,\alpha}(\omega_2)$, then $h_{y,\alpha}(\omega_1) = h_{y,\alpha}(\omega_2)$, for every $y \in \widehat{\mathcal{V}}(\Delta)$, the definition of α-equivalence does not depend on the vertex x such that $h_{\alpha,x}(\omega) = h_{\alpha,x}(\omega')$. Moreover, if ω is α-equivalent to ω' and $\mathcal{A} = \mathcal{A}(\omega, \omega')$

denotes any apartment having ω and ω' as boundary points, then for every $x \in \mathcal{A}$, the sectors $Q_x(\omega)$ and $Q_x(\omega')$ are α-adjacent, that is there exists a type rotating isomorphism $\psi_{tr} : \mathcal{A} \to \mathbb{A}$ mapping $Q_x(\omega)$ onto $\mathbb{Q}_0$ and $Q_x(\omega')$ onto $s_\alpha \mathbb{Q}_0$. On the contrary, if x does not lie on any $\mathcal{A}(\omega, \omega')$, then $Q_x(\omega) \cap Q_x(\omega')$ contains properly their common α-wall.

Definition 2.9. — *We call α-boundary of the building Δ the set $\Omega_\alpha = \Omega/\sim_\alpha$, consisting of all equivalence classes $\eta_\alpha = [\omega]_\alpha$ of boundary points.*

For every $\omega \in \Omega$, consider the set $\mathcal{H}_\alpha(\omega) = \{h_{x,\alpha}(\omega),\ x \in \widehat{\mathcal{V}}(\Delta)\}$. If $\omega' \sim_\alpha \omega$, then, for every x, $h_{x,\alpha}(\omega') = h_{x,\alpha}(\omega)$ and hence $\mathcal{H}_\alpha(\omega) = \mathcal{H}_\alpha(\omega')$. Therefore the set $\mathcal{H}_\alpha(\omega)$ only depends on the equivalence class $\eta_\alpha = [\omega]_\alpha$ represented by ω and we shall denote $\mathcal{H}_\alpha(\eta_\alpha) = \mathcal{H}_\alpha(\omega)$, if $\omega \in \eta_\alpha$. Moreover, if $\omega \not\sim_\alpha \omega'$, then, for every $x \in \widehat{\mathcal{V}}(\Delta)$, $h_{x,\alpha}(\omega) \neq h_{x,\alpha}(\omega')$ and hence $\mathcal{H}_\alpha(\omega) \cap \mathcal{H}_\alpha(\omega') = \emptyset$. This implies that the map

$$\eta_\alpha \to \mathcal{H}_\alpha(\eta_\alpha)$$

is a bijection between the α-boundary Ω_α and the set $\{\mathcal{H}_\alpha(\eta_\alpha)\}$. In particular, for every $x \in \widehat{\mathcal{V}}(\Delta)$, each element η_α of Ω_α determines one α-wall based at x; we shall denote this wall by $h_x(\eta_\alpha)$. Of course, $h_x(\eta_\alpha) = h_{x,\alpha}(\omega)$, for every $\omega \in \eta_\alpha$.

If we examine in details, for any class η_α, the set $\mathcal{H}_\alpha(\eta_\alpha)$, we prove that the set $\mathcal{H}_\alpha(\eta_\alpha)$ determines a tree. We need the following definition.

Definition 2.10. — *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$, $x \neq y$; let $h_{x,\alpha}$ and $h_{y,\alpha}$ be α-walls, based at x and y respectively.*

(i) *The walls $h_{x,\alpha}$ and $h_{y,\alpha}$ are said to be equivalent if they definitely coincide, that is there is $h_{z,\alpha}$ such that $h_{z,\alpha} \subset h_{x,\alpha} \cap h_{y,\alpha}$.*

(ii) *The walls $h_{x,\alpha}$ and $h_{y,\alpha}$ are said to be parallel if they are not equivalent , but there is an apartment containing them and, through any type-preserving isomorphism ψ_{tp} of this apartment onto $\mathbb{A}$, they correspond to walls of $\mathbb{A}$ lying on parallel affine α-hyperplanes H_α^k, H_α^j, for some $k, j \in \mathbb{Z}$.*

(iii) *The walls $h_{x,\alpha}$ and $h_{y,\alpha}$ are said to be definitely parallel if there exist $h_{x',\alpha} \subset h_{x,\alpha}$ and $h_{y',\alpha} \subset h_{y,\alpha}$ which are parallel. If $h_{x,\alpha}$ and $h_{y,\alpha}$ are definitely parallel, we call distance between the two walls the usual distance between the two hyperplanes of $\mathbb{A}$ containing the images of their parallel subwalls, that is the positive integer number $|j - k|$, if $\psi_{tp}(h_{x,\alpha}) = H_\alpha^k$ and $\psi_{tr}(h_{y,\alpha}) = H_\alpha^j$.*

We remark that if $h_{x,\alpha}$ and $h_{y,\alpha}$ are definitely parallel, there exists an apartment containing, say, $h_{x,\alpha}$ and a subwall of $h_{y,\alpha}$.

For every $\omega \in \Omega$ and for every pair of special vertices $x, y \in \widehat{\mathcal{V}}(\Delta)$, the walls $h_{x,\alpha}(\omega)$ and $h_{y,\alpha}(\omega)$ are equivalent or definitely parallel; therefore, for every $\eta_\alpha \in \Omega_\alpha$, the set $\mathcal{H}_\alpha(\eta_\alpha)$ consists of walls equivalent or definitely parallel. For every $\eta_\alpha \in \Omega_\alpha$ and for every $x \in \widehat{\mathcal{V}}(\Delta)$, we denote by $\mathbf{x}$ the equivalence class represented by the wall $h_x(\eta_\alpha)$. Obviously, $\mathbf{x} = \mathbf{y}$ if and only if $h_x(\eta_\alpha)$ and $h_y(\eta_\alpha)$ are equivalent. We denote by $T_\alpha(\eta_\alpha)$ the graph having as vertices the classes $\mathbf{x}$ of equivalent walls associated to η_α and as edges the pairs $[\mathbf{x}, \mathbf{y}]$ of equivalence classes represented by (definitely parallel) walls $h_x(\eta_\alpha)$ and $h_y(\eta_\alpha)$ at distance one. It can be proved the following fundamental result (see [8, Section 4]).

PROPOSITION 2.11. — *For every simple root α, and for every $\eta_\alpha \in \Omega_\alpha$, the graph $T_\alpha(\eta_\alpha)$ is a tree.*

(i) *If $\alpha \in R_0$, the tree is homogeneous, with homogeneity q_α.*

(ii) *If $\alpha \in R_2$, the tree is labelled and semi-homogeneous; each vertex of type 0 shares $q_{2\alpha} = p$ edges and each vertex of type 1 shares $q_\alpha = r$ edges.*

We recall that the simple root α belongs to R_2 if and only if R is not reduced and $\alpha = \alpha_n = e_n$. In this particular case, for every $k \in \mathbb{Z}$, we have $H_\alpha^k = H_{2\alpha}^{2k}$; hence the parallel hyperplanes of $\mathbb{A}$ orthogonal to α are the hyperplanes $H_{2\alpha}^h$, for all $h \in \mathbb{Z}$. Moreover, for every $k \in \mathbb{Z}$, $q_{2\alpha,2k} = q_{\alpha,k} = q_\alpha = r$ and $q_{2\alpha,2k+1} = q_{2\alpha} = p$. In all other cases, that is for all simple root of a reduced building or for all simple root $\alpha_i, i \neq n$, for a building of type $\widetilde{BC}_n$, we always have $\alpha \in R_0$, and hence $q_{\alpha,k} = q_\alpha, \quad \text{forevery} \quad k \in \mathbb{Z}$.

On the fundamental apartment $\mathbb{A}$, for every $k \in \mathbb{Z}$, we simply denote by $\mathbf{X}_k$ the class of all α-walls of sectors Q_X equivalent to $\mathbb{Q}_0$, lying on H_α^k, and we set

$$\Gamma_0 = \{\mathbf{X}_k,\ k \in \mathbb{Z}\}.$$

For every apartment $\mathcal{A}$ of the building, the equivalence class $\mathbf{x}$ represented by the walls of $\mathcal{H}_\alpha(\eta_\alpha)$ lying on $\mathcal{A}$, maps to an element $\mathbf{X}_k$ of Γ_0, for some $k \in \mathbb{Z}$, by a type-preserving isomorphism $\psi_{tp} : \mathcal{A} \to \mathbb{A}$. If the root system R has type C_n or BC_n, and $\alpha = \alpha_n$, then, for every $j \in \mathbb{Z}$, H_α^{2j} only contains special vertices of type 0 and H_α^{2j+1} only contains special vertices of type n. (The same is true if R has type B_n and $\alpha = \alpha_i, i < n$). Hence in this case it is natural to endow the set Γ_0 with a labelling in the following way: we say that $\mathbf{X}_k$ has type 0, if $k = 2j$ and has type 1, if $k = 2j + 1$, for $j \in \mathbb{Z}$. This labelling can be extended to all equivalence classes $\mathbf{x}$ represented by walls of $\mathcal{H}_\alpha(\eta_\alpha)$ lying on any apartment $\mathcal{A}$, and hence to all walls of the buildings; we say that $\mathbf{x}$ has type 0 if (through any type-preserving isomorphism) it maps to some $\mathbf{X}_{2j}$, and has type 1, if it maps to some $\mathbf{X}_{2j+1}$.

For every apartment $\mathcal{A}$, the walls $h_{x,\alpha}(\omega)$ of $\mathcal{H}(\eta_\alpha)$ lying on $\mathcal{A}$ determine a geodesic $\gamma(\eta_\alpha)$ of the tree $T(\eta_\alpha)$, consisting of all vertices $\mathbf{x}$ associated to these walls and of all edges connecting each pair of adjacent vertices $\mathbf{x}, \mathbf{y}$. The set Γ_0 can be seen as the fundamental geodesic of the tree, since each geodesic $\gamma(\eta_\alpha)$ of the building is isomorphic to Γ_0 through any type-preserving isomorphism $\psi_{tp} : \mathcal{A} \to \mathbb{A}$, if $\mathcal{A}$ denotes any apartment containing $\gamma(\eta_\alpha)$.

The tree $T(\eta_\alpha)$ is labelled and semi-homogeneous only when R is not reduced and $\alpha = \alpha_n = e_n$, that is only when the building has type $\widetilde{BC}_n$; in this case $\widehat{\mathcal{V}}(\Delta)$ consists only of vertices of type 0. Therefore for such a tree it is straightforward to restrict to consider only its vertices of type 0. Hence, if $\mathbf{x}, \mathbf{y}$ are vertices of type 0, then the geodesic $[\mathbf{x}, \mathbf{y}]$ has length $2n$, for some $n \in \mathbb{N}$. Moreover on the fundamental geodesic Γ_0 we consider only the vertices X_{2n}, for $n \in \mathbb{N}$.

Obviously, for every $\eta_\alpha \in \Omega_\alpha$, we may identify the set $\mathcal{H}_\alpha(\eta_\alpha)$ with the tree $T_\alpha(\eta_\alpha)$. Moreover trees $T_\alpha(\eta_{\alpha,1})$, $T_\alpha(\eta_{\alpha,2})$ associated to any two $\eta_{\alpha,1}$, $\eta_{\alpha,2}$ in Ω_α are isomorphic. For every $x \in \widehat{\mathcal{V}}(\Delta)$, the vertex $\mathbf{x}$ can be seen as the projection of x onto the tree $T_\alpha(\eta_\alpha)$. In this sense we can refer to $T_\alpha(\eta_\alpha)$ as to the tree at infinity associated to the element η_α of the α-boundary.

For every $\eta_\alpha \in \Omega_\alpha$, the set $\{\omega \in \Omega : \omega \in \eta_\alpha\}$ can be identified with the boundary $\partial T_\alpha(\eta_\alpha)$ of the tree $T_\alpha(\eta_\alpha)$. Moreover, for every pair $\eta_{\alpha,1}$, $\eta_{\alpha,2}$ in Ω_α, $\partial T_\alpha(\eta_{\alpha,1})$, $\partial T_\alpha(\eta_{\alpha,2})$ are isomorphic. We denote by T_α an abstract tree such that

$$T_\alpha(\eta_\alpha) \sim T_\alpha, \quad \forall \eta_\alpha \in \Omega_\alpha;$$

moreover we denote by $\mathbf{t}$ any element of T_α and by $\mathbf{b}$ any element of its boundary ∂T_α.

We may conclude that the maximal boundary Ω of the building can be decomposed as a disjoint union of boundaries of trees, one for each equivalence class $\eta_\alpha = [\omega]_\alpha$:

$$\Omega = \bigcup_{\eta_\alpha \in \Omega_\alpha} \partial T(\eta_\alpha).$$

According to this decomposition, each boundary point ω of the building can be seen as a pair $(\eta_\alpha, \mathbf{b})$ of $\Omega_\alpha \times \partial T_\alpha$, where η_α is the equivalence class $[\omega]_\alpha$ containing ω and $\mathbf{b}$ is the boundary point of T_α corresponding on $\partial T(\eta_\alpha)$ to ω. In this sense we may write, up to isomorphism,

$$\Omega = \Omega_\alpha \times \partial T_\alpha.$$

For every simple root a, define, for every $v \in \mathbb{V}$,

$$P_\alpha(v) = \frac{v - s_\alpha v}{2}, \qquad Q_\alpha(v) = \frac{v + s_\alpha v}{2},$$

where s_α is the reflection with respect to the linear hyperplane H_α. By definition, $P_\alpha(v) + Q_\alpha(v) = v$ and $Q_\alpha(v) - P_\alpha(v) = s_\alpha v$. Moreover $P_\alpha(s_\alpha v) = -P_\alpha(v)$ and $Q_\alpha(s_\alpha v) = Q_\alpha(v)$. We observe that, for every v, $Q_\alpha(v)$ lies on H_α and $P_\alpha(v)$ is the component of the vector v in the direction orthogonal to the hyperplane H_α, that is in the direction of the vector α. The α-equivalence of two boundary points implies the following result (see [8, Section 4]).

Proposition 2.12. — *Let ω_1, ω_2 be α-equivalent. Then, for every $x, y \in \widehat{\mathcal{V}}(\Delta)$,*

$$Q_\alpha(\rho_{\omega_2}(y) - \rho_{\omega_2}(x)) = Q_\alpha(\rho_{\omega_1}(y) - \rho_{\omega_1}(x)).$$

If x, y belong to an apartment containing both the boundary points ω_1, ω_2, then

$$P_\alpha(\rho_{\omega_2}(y) - \rho_{\omega_2}(x)) = -P_\alpha(\rho_{\omega_1}(y) - \rho_{\omega_1}(x)).$$

As the maximal boundary, also each α-boundary Ω_α may be endowed with a totally disconnected compact Hausdorff topology and a regular Borel measure for every x of $\widehat{\mathcal{V}}(\Delta)$. For every pair $x, y \in \widehat{\mathcal{V}}(\Delta)$, define a set of Ω_α in the following way:

$$\Omega_\alpha(x, y) = \{\eta_\alpha = [\omega]_\alpha,\ \omega \in \Omega(x, y)\}.$$

We observe that there exists a α-wall based at x containing y, if and only if $y \in \mathcal{V}_\lambda(x)$, with $\lambda \in H_{0,\alpha}$. Then, for every pair of vertices $x, y \in \widehat{\mathcal{V}}(\Delta)$ such that $y \in \mathcal{V}_\lambda(x)$, with $\lambda \in H_{0,\alpha}$, we have

$$\Omega_\alpha(x, y) = \{\eta_\alpha \in \Omega_\alpha \ :\ y \in h^x_\alpha(\eta_\alpha)\}.$$

The family

$$\mathcal{B}^x_\alpha = \{\ \Omega_\alpha(x, y),\ y \in \mathcal{V}_\lambda(\Delta),\ \lambda \in H_{0,\alpha}\}$$

generates a totally disconnected compact Hausdorff topology on Ω_α and, for every $\eta_\alpha \in \Omega_\alpha$, a local base at η_α is given by

$$\mathcal{B}_{x,\eta_\alpha} = \{\, \Omega_\alpha(x,y),\ y \subset h_x(\eta_\alpha)\}.$$

By the same argument used for the maximal boundary, we can prove that the topology on Ω_α does not depend on the particular $x \in \widehat{\mathcal{V}}(\Delta)$.

For every x of $\widehat{\mathcal{V}}(\Delta)$, we define a regular Borel measure ν_x^α on Ω_α, by setting, for every $y \in \mathcal{V}_\lambda(\Delta)$,

$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \frac{N^\alpha_{P_\alpha\lambda}}{N_\lambda},$$

if $N^\alpha_{P_\alpha\lambda} = |\{\mathbf{z} : \sigma(\mathbf{x},\mathbf{z}) = P_\alpha\lambda\}|$, where $\mathbf{x}$ and $\mathbf{y}$ are the projection of x and y on the tree at infinity associated with any $\omega \in \Omega(x,y)$ and $\sigma(\mathbf{x},\mathbf{y}) = P_\alpha\lambda$. We notice that if $\lambda \in H_{0,\alpha}$, then $\mathbf{y} = \mathbf{x}$ and then $P_\alpha\lambda = \lambda$. Therefore in this case $\nu_x^\alpha(\Omega_\alpha(x,y)) = \nu_x(\Omega(x,y))$. Define

$$R_\alpha^+ = \{\beta \in R^+,\ \beta \neq \alpha, 2\alpha\};$$

then, recalling formula (2.1), we have

$$\nu_x^\alpha(\Omega_\alpha(x,y)) = \begin{cases} \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})} \prod_{\beta\in R_\alpha^+} q_\beta^{-<\lambda,\beta>}\ q_{2\beta}^{<\lambda,\beta>} & \text{if } \lambda \in H_{0,\alpha}, \\ \frac{\mathbf{W}_\lambda(q^{-1})(1+q_\alpha^{-1})}{\mathbf{W}(q^{-1})} \prod_{\beta\in R_\alpha^+} q_\beta^{-<\lambda,\beta>}\ q_{2\beta}^{<\lambda,\beta>} & \text{otherwise.} \end{cases}$$

For every $x \in \widehat{\mathcal{V}}(\Delta)$, let $\mathbf{x}$ be its projection on the tree $T(\eta_\alpha)$ associated with an assigned $\omega \in \Omega$ and let $\mathbf{t}$ be the element of the abstract tree T_α, which corresponds to the vertex $\mathbf{x}$; for ease of notation, from now on, we identify $\mathbf{t}$ with $\mathbf{x}$. According to this notation, if we identify the maximal boundary Ω with $\Omega_\alpha \times \partial T_\alpha$, then, if $\omega \in \Omega(x,y)$ and $\omega = (\eta_\alpha, \mathbf{b})$, we have $\Omega(x,y) = \Omega_\alpha(x,y) \times B(\mathbf{x},\mathbf{y})$, where $B(\mathbf{x},\mathbf{y}) = \{\mathbf{b} \in \partial T_\alpha\ :\ \mathbf{y} \in \gamma(\mathbf{x},\mathbf{b})\}$. Therefore each probability measure ν_x splits as product of the probability measure ν_x^α on the α-boundary Ω_α and the canonical probability measure $\mu_{\mathbf{x}}$ on the boundary of the tree T_α :

$$\nu_x = \nu_x^\alpha \times \mu_{\mathbf{x}}.$$

3. Characters and Poisson kernels

3.1. Characters

We call character of $\mathbb{A}$ any multiplicative complex-valued function χ acting on $\widehat{L}$:

$$\chi(\lambda_1 + \lambda_2) = \chi(\lambda_1)\,\chi(\lambda_2), \quad \forall \lambda_1, \lambda_2 \in \widehat{L}.$$

We assume, without loss of generality, that a character of $\mathbb{A}$ is the restriction to $\widehat{L}$ of a multiplicative complex-valued function acting on $\mathbb{V}$. We denote by $\mathbf{X}(\widehat{L})$ the group of all characters of $\mathbb{A}$. If $n = \dim \mathbb{V}$, then $\mathbf{X}(\widehat{L}) \cong (\mathbb{C}^\times)^n$ and the group $\mathbf{X}(\widehat{L})$ can be endowed with the weak topology and the usual measure of $\mathbb{C}^n$. The Weyl group $\mathbf{W}$ acts on $\mathbf{X}(\widehat{L})$ in the following way: for every $\mathbf{w} \in \mathbf{W}$ and for every $\chi \in \mathbf{X}(\widehat{L})$,

$$(\mathbf{w}\chi)(\lambda) = \chi(\mathbf{w}^{-1}(\lambda)), \qquad \text{for all} \quad \lambda \in \widehat{L}. \tag{3.1}$$

It is immediate to observe that $\mathbf{w}\chi$ is a character and we simply denote $\chi^{\mathbf{w}} = \mathbf{w}\chi$.

We shall give some definitions.

Definition 3.1. — *Let χ be a character of $\mathbb{A}$.*

1. *χ is singular if there exists a root α such that $\chi(\alpha^\vee) = 1$;*
2. *χ is non-singular if $\chi(\alpha^\vee) \neq 1$, for each root α;*
3. *χ is good if there exists $\lambda_0 \in \widehat{L}^+$, with $\tau(\lambda_0) = 0$, such that*
$$\chi^{\mathbf{w}_1}(\lambda_0) \neq \chi^{\mathbf{w}_2}(\lambda_0), \ \text{ for } \ \mathbf{w}_1 \neq \mathbf{w}_2;$$
4. *χ is bad for λ if there exist $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}, \mathbf{w}_1 \neq \mathbf{w}_2$, such that $\chi^{\mathbf{w}_1}(\lambda) = \chi^{\mathbf{w}_2}(\lambda)$;*
5. *χ is bad if χ is bad for every λ, that is for every $\lambda \in \widehat{L}$ there exist $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}, \mathbf{w}_1 \neq \mathbf{w}_2$, such that $\chi^{\mathbf{w}_1}(\lambda) = \chi^{\mathbf{w}_2}(\lambda)$.*

We shall denote by $\mathbf{X}_{NS}(\widehat{L})$ the space of all non-singular characters and by $\mathbf{X}_g(\widehat{L})$ the space of all good characters.

Lemma 3.2. — $\mathbf{X}_g(\widehat{L}) \subset \mathbf{X}_{NS}(\widehat{L})$.

Proof. — We prove that every singular character χ is bad. Let χ be singular and let α be a root such that $\chi(\alpha^\vee) = 1$. For every $\lambda \in \widehat{L}$, $s_\alpha\lambda - \lambda$ is orthogonal to the hyperplane H^0_α; moreover $s_\alpha\lambda - \lambda \in L$. This implies that $s_\alpha\lambda - \lambda = k\alpha^\vee$, for some integer k. Thus $\chi(s_\alpha\lambda - \lambda) = 1$, and then $\chi(s_\alpha\lambda) = \chi(\lambda)$. This means that χ is bad. $\square$

We shall prove that in fact the set of all bad characters is negligible in the space of all characters. We notice at first that, for every $\lambda \in \widehat{L}$, the set of all character which are bad for λ is given by

$$\bigcup_{\mathbf{w}_1 \neq \mathbf{w}_2} \mathbf{X}(\widehat{L}_{/<\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)>}),$$

if $<\mathbf{w}_1(\lambda) - \mathbf{w}_2(\lambda)>$ denotes the sublattice of $\widehat{L}$ generated by the element $\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)$. Actually, if χ is bad for λ and $\chi^{\mathbf{w}_1}(\lambda) = \chi^{\mathbf{w}_2}(\lambda)$, then $\chi(\mu) = 1$ for each $\mu \in <\mathbf{w}_1(\lambda) - \mathbf{w}_2(\lambda)>$. The following lemma characterizes each set $\mathbf{X}(\widehat{L}_{/<\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)>})$.

Lemma 3.3. — *Let $\lambda \in \widehat{L}$, and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}, \mathbf{w}_1 \neq \mathbf{w}_2$. Then, there exists an integer m, such that*

$$\mathbf{X}(\widehat{L}_{/<\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)>}) \cong (\mathbb{C}^\times)^{n-1} \times (\mathbb{Z}/m).$$

Proof. — We can choose a basis $\{\eta_1, \ldots, \eta_n\}$ for $\mathbb{V}$, such that $\mathbf{w}_1(\lambda) - \mathbf{w}_2(\lambda) = m\eta_1$, for some integer m. Hence, if χ belongs to $\mathbf{X}(\widehat{L}_{/<\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)>})$ and $\lambda = \sum_i k_i\eta_i$, then we can write

$$\chi(\lambda) = \chi(\eta_1)^j \chi(\eta_2)^{k_2} \cdots \chi(\eta_n)^{k_n},$$

if j is the element of the finite group $\mathbb{Z}/m$ represented by k_1. Therefore χ may be identified with an element of $(\mathbb{C}^\times)^{n-1} \times (\mathbb{Z}/m)$. $\square$

Proposition 3.4. — *The set of all bad characters of $\mathbb{A}$ has measure zero.*

Proof. — Lemma 3.3 implies that $\mathbf{X}(\widehat{L}_{/<\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)>})$ is a subset of $\mathbf{X}(\widehat{L})$ having measure zero, for every $\lambda \in \widehat{L}$ and $\mathbf{w}_1 \neq \mathbf{w}_2$; then also the set $\bigcup_{\mathbf{w}_1 \neq \mathbf{w}_2} \mathbf{X}(\widehat{L}_{/<\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)>})$ of all characters bad for λ has measure zero. Since a characters of $\mathbb{A}$ is bad if it is bad for every λ, the set of all bad characters of $\mathbb{A}$ is the intersection of the sets $\bigcup_{\mathbf{w}_1 \neq \mathbf{w}_2} \mathbf{X}(\widehat{L}_{/<\mathbf{w}_1(\lambda)-\mathbf{w}_2(\lambda)>})$, for all $\lambda \in \widehat{L}$, and then it has measure zero. $\square$

Corollary 3.5. — *The space* $\mathbf{X}_g(\widehat{L})$ *is dense in* $\mathbf{X}_{NS}(\widehat{L})$ *and* $\mathbf{X}_{NS}(\widehat{L})$ *is dense in* $\mathbf{X}(\widehat{L})$ *with respect to the weak topology.*

In Section 8 we shall need to consider particular good characters, named α-good, with respect to a simple root α. Let α be a simple root and consider the linear hyperplane H^0_α; the restriction χ_α of a character χ to the hyperplane H^0_α is a character on $\widehat{L}\cap H^0_\alpha$.

Definition 3.6. — *A character* χ *is* α*-good if*

$$(\chi^{\mathbf{w}_1})_\alpha = (\chi^{\mathbf{w}_2})_\alpha \quad \textit{if and only if} \ \mathbf{w}_1 = \mathbf{w}_2 \text{ or } \mathbf{w}_1 = s_\alpha \mathbf{w}_2.$$

We denote by $\mathbf{X}_{gg}(\widehat{L})$ *the subspace of* $\mathbf{X}(\widehat{L})$ *consisting of all characters of* $\mathbb{A}$ *which are* α*-good for every simple root* α.

We observe that a good character is not necessarily α-good for some simple root α. Nevertheless, we can prove that the good characters, which are α-good for a simple root α, are dense in the space of all good characters, with respect to the weak topology. In order to prove this property we consider , for any $\mathbf{w}\in\mathbf{W}$, the set $M^\alpha_{\mathbf{w}} =< \mathbf{w}(\lambda)-\lambda,\ \lambda\in H^0_\alpha >$ and the quotient space $\widehat{L}_{/M^\alpha_{\mathbf{w}}}$. We notice that, if $(\chi_{\mathbf{w}})_\alpha = \chi_\alpha$, then $\chi(M^\alpha_{\mathbf{w}}) = 1$ and hence χ belongs to $\mathbf{X}(\widehat{L}_{/M^\alpha_{\mathbf{w}}})$.

Proposition 3.7. — *For every simple root* α, $\dim\ \mathbf{X}(\widehat{L}_{/M^\alpha_{\mathbf{w}}}) = \dim\ \mathbf{X}(\widehat{L})$ *if and only if* $\mathbf{w}=e$ *or* $\mathbf{w}=s_\alpha$.

Proof. — If M is a subgroup of the additive group $\widehat{L}$, then the group $\mathbf{X}(\widehat{L}_{/M})$ is total in the group $\mathbf{X}(\widehat{L})$ if and only if $M=\{0\}$. Therefore $\mathbf{X}(\widehat{L}_{/M^\alpha_{\mathbf{w}}})$ is total in the group $\mathbf{X}(\widehat{L})$ if and only if $M^\alpha_{\mathbf{w}}=\{0\}$. So we conclude, because $M^\alpha_{\mathbf{w}}=\{0\}$ if and only if $\mathbf{w}=e$, or $\mathbf{w}=s_\alpha$. □

Corollary 3.8. — $\mathbf{X}_{gg}(\widehat{L})$ *is dense in* $\mathbf{X}(\widehat{L})$.

Proof. — We prove that, for every simple root α, the space of all characters χ of $\mathbb{A}$ such that $(\chi^{\mathbf{w}_1})_\alpha = (\chi^{\mathbf{w}_2})_\alpha$, for $\mathbf{w}_1\neq\mathbf{w}_2, s_\alpha\mathbf{w}_2$, is a subspace of measure zero of $\mathbf{X}(\widehat{L})$. Without loss of generality, we can assume $\mathbf{w}_2=e$ and $\mathbf{w}_1\neq e, s_\alpha$. We know that $\mathbf{X}(\widehat{L})\cong(\mathbb{C}^\times)^n$, if $\dim\mathbb{V}=n$. On the other hand, $\dim\ \mathbf{X}(\widehat{L}/M^\alpha_{\mathbf{w}_1})<n$; then $\mathbf{X}(\widehat{L}/M^\alpha_{\mathbf{w}_1})\cong(\mathbb{C}^\times)^{n'}$, with $n'<n$, and this proves that $\mathbf{X}(\widehat{L}/M^\alpha_{\mathbf{w}_1})$ has measure zero in $\mathbf{X}(\widehat{L})$. □

3.2. Fundamental character and probability measures on the boundaries

We call fundamental character of $\mathbb{A}$ the multiplicative function χ_0 on $\widehat{L}$ defined as follows:

$$\chi_0(\lambda) = \prod_{\alpha \in R^+} q_\alpha^{\langle \lambda, \alpha \rangle} \; q_{2\alpha}^{-\langle \lambda, \alpha \rangle}, \quad \forall \lambda \in \widehat{L}. \tag{3.2}$$

We notice that $\chi_0(\lambda) > 1$, for all $\lambda \in \widehat{L}^+$. We write χ_0 according to the type of the building.

1. If R is reduced and all roots have the same length, that is for buildings of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ and $\widetilde{E}_8$, then $q_\alpha = q$, for every $\alpha \in R^+$; hence by setting $\delta = \frac{1}{2}(\sum_{\alpha \in R^+} \alpha)$, we write
$$\chi_0(\lambda) = q^{\sum_{\alpha \in R^+} \langle \lambda, \alpha \rangle} = q^{2\langle \lambda, \delta \rangle}.$$
2. If R is reduced, but it contains long and short roots, then, denoting by α any long root and by β any short root, we write $q_\alpha = q$ and $q_\beta = p$; hence by setting $\delta_l = \frac{1}{2}(\sum \alpha)$, $\delta_s = \frac{1}{2}(\sum \beta)$, we have
$$\chi_0(\lambda) = q^{2\langle \lambda, \delta_l \rangle} \; p^{2\langle \lambda, \delta_s \rangle}.$$
This happens for buildings of type $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{F}_4$ and $\widetilde{G}_2$.
3. If R is non-reduced, that is the building is of type $\widetilde{(BC)}_n$, we denote by α, β and γ any root of R_0, R_1 and R_2 respectively; then keeping in mind that $R_2 = \{\beta/2, \ \beta \in R_1\}$, it follows that
$$\chi_0(\lambda) = q^{2\langle \lambda, \delta_0 \rangle} (pr)^{\langle \lambda, \delta_1 \rangle}$$
if $\delta_0 = \frac{1}{2}(\sum \alpha)$, $\delta_1 = \frac{1}{2} \sum \beta$.

We notice that, for every $\lambda \in \widehat{L}^+$, $\chi_0(\lambda) = q_{t_\lambda}$. More generally, if λ is any element of $\widehat{L}$ and $t_\lambda = u_\lambda g_l$, with $u_\lambda = s_{i_1} \cdots s_{i_r}$, then the same argument used in Proposition 2.16.1 of [8] shows that,

$$\chi_0(\lambda) = \prod_{j \in J^+} q_{i_j} \cdot \prod_{j \in J^-} q_{i_j}^{-1}, \tag{3.3}$$

where

$$J^+ = \{j \ : \ s_{i_1} \cdots s_{i_{j-1}}(C_0) \prec s_{i_1} \cdots s_{i_j}(C_0)\}$$
$$\text{and} \quad J^- = \{j \ : \ s_{i_1} \cdots s_{i_j}(C_0) \prec s_{i_1} \cdots s_{i_{j-1}}(C_0)\}.$$

We can easily compute the fundamental character in each simple coroot $\alpha^\vee$.

1. If R is reduced, then, for every simple root α, $\chi_0(\alpha^\vee) = q_\alpha^2$.
2. If R is non-reduced, then

   (i) $\chi_0(\alpha^\vee) = q^2$, for every $\alpha = e_i - e_{i+1}$, $i = 1, \dots, n-1$;

   (ii) $\chi_0(\beta^\vee) = pr$, for $\beta = 2e_n$.

For every simple root α we define, for every $\lambda \in \widehat{L}$,

$$\chi_0^\alpha(\lambda) = \prod_{\beta \in R_\alpha^+} q_\beta^{\langle \lambda, \beta \rangle} \ q_{2\beta}^{-\langle \lambda, \beta \rangle}. \tag{3.4}$$

Obviously χ_0^α is a character of $\mathbb{A}$ such that $\chi_0^\alpha(\lambda) = \chi_0(\lambda)$, for $\lambda \in H_{0,\alpha}$.

If T_α is the abstract tree isomorphic to each tree at infinity $T_\alpha(\eta_\alpha)$, Γ_0 is its fundamental apartment and Γ_0^+ is the fundamental geodesic based at 0, then $\overline{\chi}_0$ is the following character on Γ_0 :

1. $\overline{\chi}_0(X_n) = q_\alpha^n$, if X_n is the vertex of Γ_0^+ at distance n from 0, in the homogeneous case;
2. $\overline{\chi}_0(X_{2n}) = (pr)^n$, if X_{2n} is the vertex of Γ_0^+ at distance $2n$ from 0, otherwise.

The characters χ_0, χ_0^α and $\overline{\chi}_0$ are related through the operators P_α and Q_α, as the following lemma shows (see [8, Section 4]).

Lemma 3.9. — *Let $\lambda \in \widehat{L}$; assume $\lambda \in H_{n,\alpha}$, if $\alpha \in R_0$, and $\lambda \in H_{2n,\alpha}$, if $\alpha \in R_2$. Then*

(i) $\chi_0(Q_\alpha(\lambda)) = \chi_0^\alpha(Q_\alpha(\lambda)) = \chi_0^\alpha(\lambda)$,

(ii) $\chi_0(P_\alpha(\lambda)) = \begin{cases} \overline{\chi}_0(\mathbf{X}_n) = q_\alpha^n & \text{if } \alpha \in R_0, \\ \overline{\chi}_0(\mathbf{X}_{2n}) = (pr)^n & \text{if } \alpha \in R_2. \end{cases}$

For every $\lambda \in \widehat{L}$, $\chi_0(\lambda) = \chi_0^\alpha(Q_\alpha(\lambda)) \ \overline{\chi}_0(\mathbf{X}_\lambda)$, if $\mathbf{X}_\lambda$ is the vertex of Γ_0 corresponding to $P_\alpha(\lambda)$.

Proposition 3.10. — *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$. Let $\omega = (\eta_\alpha, \mathbf{b})$ and let $\mathbf{x}$, $\mathbf{y}$ be the projection on the tree at infinity $T_\alpha(\eta_\alpha)$ of x, y respectively. If $\mathbf{e}$ denotes the fundamental vertex of the tree and $\rho_{\mathbf{b}}$ is the retraction of the tree on Γ_0, with respect to $\mathbf{b}$, such that $\rho_{\mathbf{b}}(\gamma(\mathbf{e}, \mathbf{b})) = \Gamma_0^+$, then*

(i) $\chi_0(Q_\alpha(\rho_\omega(y)-\rho_\omega(x))=\chi_0^\alpha(\rho_\omega(y)-\rho_\omega(x))$,

(ii) $\chi_0(P_\alpha(\rho_\omega(y)-\rho_\omega(x))=\overline{\chi}_0(\rho_{\mathbf{b}}(\mathbf{y})-\rho_{\mathbf{b}}(\mathbf{x}))$.

The measure ν_x defined on the maximal boundary Ω and the measure ν_x^α defined on the α-boundary can be characterized in terms of the character χ_0 and χ_0^α respectively.

PROPOSITION 3.11. — *Let $\lambda\in\hat{L}^+$, and $y\in\mathcal{V}_\lambda(x)$; let α be a simple root; then*

(i) *for every $\omega\in\Omega(x,y)$,*

$$\nu_x(\Omega(x,y))=\frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\,\chi_0^{-1}(\rho_\omega^x(y))=\frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\,\chi_0^{-1}(\rho_\omega(y)-\rho_\omega(x));$$

(ii) *for every $\eta_\alpha\in\Omega_\alpha(x,y)$ and for every ω in the class η_α,*

$$\nu_x^\alpha(\Omega_\alpha(x,y))=\begin{cases}\frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\,(\chi_0^\alpha)^{-1}(\rho_\omega(y)-\rho_\omega(x)) & \textit{if }\ \lambda\in H_{0,\alpha},\\ \frac{\mathbf{W}_\lambda(q^{-1})(1+q_\alpha^{-1})}{\mathbf{W}(q^{-1})}\,(\chi_0^\alpha)^{-1}(\rho_\omega(y)-\rho_\omega(x)) & \textit{otherwise.}\end{cases}$$

Taking in account Proposition 3.10, the measures ν_x^α and $\mu_{\mathbf{x}}$ can be expressed in terms of the character χ_0 and the operators P_α and Q_α.

COROLLARY 3.12. — *Let $x,y\in\widehat{\mathcal{V}}(\Delta)$ and $y\in\mathcal{V}_\lambda(x)$; let α be a simple root. Let $\mathbf{x}$ and $\mathbf{y}$ be the projection of x and y on the tree at infinity $T_\alpha(\eta_\alpha)$ associated with any $\omega\in\Omega(x,y)$. Then*

$$\nu_x^\alpha(\Omega_\alpha(x,y))=\begin{cases}\frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\,\chi_0^{-1}(\rho_\omega(y)-\rho_\omega(x)) & \textit{if }\ \lambda\in H_{0,\alpha},\\ \frac{\mathbf{W}_\lambda(q^{-1})(1+q_\alpha^{-1})}{\mathbf{W}(q^{-1})}\,\chi_0^{-1}(Q_\alpha(\rho_\omega(y)-\rho_\omega(x))) & \textit{otherwise.}\end{cases}$$

Moreover

$$\mu_{\mathbf{x}}(B(\mathbf{x},\mathbf{y}))=\begin{cases}1 & \textit{if }\ \lambda\in H_{0,\alpha},\\ \frac{q_\alpha}{1+q_\alpha}\chi_0^{-1}(P_\alpha(\rho_\omega(y)-\rho_\omega(x))) & \textit{otherwise.}\end{cases}$$

Therefore the decomposition of the measure ν_x for the maximal boundary is a direct consequence of the orthogonal decomposition
$\chi_0(\lambda)=\chi_0(P_\alpha(\lambda))\,\chi_0(Q_\alpha(\lambda))$.

Proposition 3.13. — *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and let α be a simple root.*

(i) *The measures ν_x, ν_y are mutually absolutely continuous and*

$$\frac{d\nu_y}{d\nu_x}(\omega) = \chi_0(\rho_\omega^x(y)) = \chi_0(\rho_\omega(y) - \rho_\omega(x)), \quad \forall \omega \in \Omega.$$

(ii) *The measures ν_x^α, ν_y^α are mutually absolutely continuous and*

$$\frac{d\nu_y^\alpha}{d\nu_x^\alpha}(\eta_\alpha) = \chi_0^\alpha(\rho_\omega(y) - \rho_\omega(x)), \quad \forall \omega \in \eta_\alpha, \quad \forall \eta_\alpha \in \Omega_\alpha.$$

Definition 3.14. — *We call Poisson kernel of the building Δ the function*

$$P(x, y, \omega) = \chi_0(\rho_\omega(y) - \rho_\omega(x)) = \chi_0(\rho_\omega^x(y)) = \frac{d\nu_y}{d\nu_x}(\omega),$$
$$\forall x, y \in \widehat{\mathcal{V}}(\Delta), \quad \forall \omega \in \Omega. \tag{3.5}$$

We call generalized Poisson kernel of the building Δ associated with the character χ the function

$$P^\chi(x, y, \omega) = \chi(\rho_\omega(y) - \rho_\omega(x)), \quad \forall x, y \in \widehat{\mathcal{V}}(\Delta), \quad \forall \omega \in \Omega. \tag{3.6}$$

Let $x_0 \in \widehat{\mathcal{V}}(\Delta)$ and let χ be a character on $\mathbb{A}$; for any complex valued function f on Ω, we call generalized Poisson transform of f of initial point x_0, associated with the character χ, the function on $\widehat{\mathcal{V}}(\Delta)$ defined by

$$\mathcal{P}_{x_0}^\chi f(x) = \int_\Omega P^\chi(x_0, x, \omega) f(\omega) d\nu_x(\omega) = \int_\Omega \chi(\rho_\omega(x) - \rho_\omega(x_0)) f(\omega) d\nu_{x_0}(\omega),$$
$$\forall x \in \widehat{\mathcal{V}}(\Delta), \tag{3.7}$$

whenever the integral exists.

These definitions do not depend on the choice of the special vertex e. We simply denote $P(x, y, \omega) = P^{\chi_0}(x, y, \omega)$, $\mathcal{P}_{x_0} = \mathcal{P}_{x_0}^{\chi_0}$ and $\mathcal{P} = \mathcal{P}_e$.

It is well known that, for every pair of vertices $\mathbf{t}, \mathbf{t}'$ in $\widehat{\mathcal{V}}(T_\alpha)$, the measure $\mu_{\mathbf{t}'}$ is absolutely continuous with respect to $\mu_{\mathbf{t}}$ and the Radon-Nikodym derivative $d\mu_{\mathbf{t}'}/d\mu_{\mathbf{t}}(\mathbf{b})$ is the Poisson kernel $P(\mathbf{t}, \mathbf{t}', \mathbf{b})$, where

$$P(\mathbf{t}, \mathbf{t}', \mathbf{b}) = \begin{cases} q_\alpha^{n-1} & \text{if } d(\mathbf{t}, \mathbf{t}') = n, \quad \text{in the homogeneous case,} \\ (pr)^{n-1} & \text{if } d(\mathbf{t}, \mathbf{t}') = 2n, \quad \text{in the semi-homogeneous case.} \end{cases}$$

In both cases, as a straightforward consequence of the definition,

$$P(\mathbf{t}, \mathbf{t}', \mathbf{b}) = \overline{\chi}_0(\rho_{\mathbf{b}}(\mathbf{t}') - \rho_{\mathbf{b}}(\mathbf{t})),\ \forall \mathbf{b} \in \partial T_\alpha.$$

Let us denote, for every $x, y \in \widehat{\mathcal{V}}(\Delta)$ and for every $\eta_\alpha \in \Omega_\alpha$,

$$P^\alpha(x, y, \eta_\alpha) = \frac{d\nu_y^\alpha}{d\nu_x^\alpha}(\eta_\alpha) = \chi_0^\alpha(\rho_\omega(y) - \rho_\omega(x)), \quad \forall \omega \in \eta_\alpha.$$

Corollary 3.15. — *Let $x, y \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$. If $\omega = (\eta_\alpha, \mathbf{b})$ and $\mathbf{x}$, $\mathbf{y}$ are the projection of x, y on the tree at infinity $T_\alpha(\eta_\alpha)$ respectively, then*

$$P(x, y, \omega) = P^\alpha(x, y, \eta_\alpha)\ P(\mathbf{x}, \mathbf{y}, \mathbf{b}).$$

The following proposition shows the properties of $P^\chi(x, y, \omega)$.

Proposition 3.16. — *Let χ be a character on $\mathbb{A}$; then,*

(i) *$P^\chi(x, x, \omega) = 1$, for every x and every ω; moreover, for every x, y and every ω,*

$$P^\chi(y, x, \omega) = (P^\chi(x, y, \omega))^{-1} = P^{\chi^{-1}}(x, y, \omega);$$

(ii) *for every x and every ω, the function $P^\chi(x, \cdot, \omega)$ is constant on*

$$\{y \in \widehat{\mathcal{V}}(\Delta) : \sigma(x, y) = \lambda,\ \rho_\omega^x(y) = \mu\}, \quad \forall \lambda \in \widehat{L}^+,\ \mu \in \Pi_\lambda;$$

(iii) *for every x, y, the function $P^\chi(x, y, \cdot)$ is locally constant on Ω and $P^\chi(x, y, \omega) = \chi(\lambda)$, for all $y \in \mathcal{V}_\lambda(x)$ and $\omega \in \Omega(x, y)$.*

4. Vertex set algebra and its eigenvalues

For every $\lambda \in \widehat{L}^+$, we define a linear operator A_λ acting on the space of complex valued functions f on $\widehat{\mathcal{V}}(\Delta)$ by

$$(A_\lambda f)(x) = \sum_{y \in \mathcal{V}_\lambda(x)} f(y) = \sum_{y \in \widehat{\mathcal{V}}(\Delta)} \mathbb{1}_{\mathcal{V}_\lambda(x)}(y) f(y), \quad \forall x \in \widehat{\mathcal{V}}(\Delta).$$

The operators $\{A_\lambda, \lambda \in \widehat{L}^+\}$ are linearly independent; we denote by $\mathcal{H}(\Delta)$ the linear span of $\{A_\lambda, \lambda \in \hat{L}^+\}$ over $\mathbb{C}$. It can be proved that $\mathcal{H}(\Delta)$ is a commutative $\mathbb{C}$-algebra. We notice that, for each y, the coefficient $1\!\!1_{\mathcal{V}_\lambda(x)}(y)$ only depends on λ.

Let χ be a character on $\mathbb{A}$; for every $\lambda \in \widehat{L}^+$, we define

$$\Lambda^\chi(\lambda) = \sum_{\mu \in \Pi_\lambda} N(\lambda, \mu)\chi(\mu). \tag{4.1}$$

PROPOSITION 4.1. — *For every $\lambda \in \widehat{L}^+$, $\Lambda^\chi(\lambda)$ is an eigenvalue of the operator A_λ and, for every $x \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$, the function $P^\chi(x, \cdot, \omega)$ is an eigenfunction of A_λ associated with $\Lambda^\chi(\lambda)$:*

$$A_\lambda P^\chi(x, \cdot, \omega) = \Lambda^\chi(\lambda)\; P^\chi(x, \cdot, \omega).$$

Since $\{A_\lambda,\ \lambda \in \widehat{L}^+\}$ generates $\mathcal{H}(\Delta)$, then $\{\Lambda^\chi(\lambda),\ \lambda \in \widehat{L}^+\}$ generates an algebra homomorphism Λ^χ from $\mathcal{H}(\Delta)$ to $\mathbb{C}$, such that $\Lambda^\chi(A_\lambda) = \Lambda^\chi(\lambda)$, for every $\lambda \in \widehat{L}^+$. Moreover, for every $x \in \widehat{\mathcal{V}}(\Delta)$ and $\omega \in \Omega$, the function $P^\chi(x, \cdot, \omega)$ is an eigenfunction of $\mathcal{H}(\Delta)$ associated with the eigenvalue Λ^χ.

COROLLARY 4.2. — *For every $f \in L^1(\Omega, \nu_x)$, the Poisson transform $\mathcal{P}^\chi_x(f)$ of f is an eigenfunction of the algebra $\mathcal{H}(\Delta)$, associated with the eigenvalue Λ^χ.*

In the particular case when $\chi = \chi_0$, then, for every $x \in \widehat{\mathcal{V}}(\Delta)$ and for every $\omega \in \Omega$, the Poisson kernel $P(x, \cdot, \omega)$ is an eigenfunction of all operators A_λ with associated eigenvalue $\Lambda^{\chi_0}(\lambda)$. Since $P(x, y, \omega)$ is the Radon-Nikodym derivative of the measure ν_y with respect to the measure ν_x, this implies that

$$\sum_{y \in V_\lambda(x)} \nu_y = \Lambda^{\chi_0}(\lambda)\; \nu_x.$$

On the other hand, since ν_y and ν_x are probability measures on Ω, then

$$\sum_{y \in V_\lambda(x)} \nu_y = \Big|\{y \in \widehat{\mathcal{V}}(\Delta)\ :\ \sigma(x, y) = \lambda\}\Big|\; \nu_x = N_\lambda\; \nu_x.$$

This implies that $\Lambda^{\chi_0}(\lambda) = \sum_{\mu \in \Pi_\lambda} N(\lambda, \mu)\; \chi_0(\mu) = N_\lambda$.

Since the Weyl group $\mathbf{W}$ acts on the characters χ according to (3.1), then $\mathbf{W}$ acts also on the eigenvalues Λ^χ of the algebra $\mathcal{H}(\Delta)$. It can be proved (see [8, Section 6]) that in fact these eigenvalues are invariant with respect to the action of $\mathbf{W}$, in the sense that, for every character χ,

$$\Lambda^{\chi\chi_0^{1/2}} = \Lambda^{\chi^{\mathbf{w}}\chi_0^{1/2}}, \ \ \forall\, \mathbf{w} \in \mathbf{W}. \tag{4.2}$$

Moreover, it can be proved through the Satake isomorphism constructed in [8, Section 6], that, for every eigenvalue Λ of the algebra $\mathcal{H}(\Delta)$, there exists a character χ of $\mathbb{A}$ such that $\Lambda = \Lambda^{\chi\chi_0^{1/2}}$.

5. Spherical functions

We recall the following definition.

Definition 5.1. — *We call spherical function of the building Δ a function ϕ on $\widehat{\mathcal{V}}(\Delta)$ such that*

(i) $\phi(e) = 1$;

(ii) $\phi(x) = \phi(y)$, *if* $\sigma(e,x) = \sigma(e,y)$;

(iii) ϕ *is an eigenfunction of the algebra* $\mathcal{H}(\Delta)$.

If we fix a sector Q_e and, for every $\lambda \in \widehat{L}^+$, we denote by x_λ the unique vertex of Q_e such that $\sigma(e, x_\lambda) = \lambda$, then $\phi(x) = \phi(x_\lambda)$, for all $x \in \mathcal{V}_\lambda(e)$. If $\Lambda \in Hom(\mathcal{H}(\Delta), \mathbb{C})$ denotes the eigenvalue of the algebra $\mathcal{H}(\Delta)$ associated with the eigenfunction ϕ, then, for every $\lambda \in \widehat{L}^+$, $A_\lambda\phi(x) = \Lambda(\lambda)\phi(x)$, for all $x \in \widehat{\mathcal{V}}(\Delta)$; in particular, choosing $x = e$, we get $|\mathcal{V}_\lambda(e)|\ \phi(x_\lambda) = \Lambda(\lambda)$. Therefore, for every $\lambda \in \widehat{L}^+$ and every $x \in \mathcal{V}_\lambda(e)$,

$$\phi(x) = \phi(x_\lambda) = \frac{1}{N_\lambda}\ \Lambda(\lambda). \tag{5.1}$$

For every character χ, let us define $\varphi_\chi = \mathcal{P}_e^\chi \mathbf{1}$, where $\mathbf{1}$ denotes the function on Ω such that $\mathbf{1}(\omega) = 1$, for every $\omega \in \Omega$.

Proposition 5.2. — φ_χ *is a spherical function, for every character* χ.

We refer to [11] for the proof of this proposition. Let ω be a fixed boundary point; since $\Lambda^\chi(\lambda) = \sum_{y \in \mathcal{V}_\lambda(e)} \chi(\rho_\omega(y) - \rho_\omega(e))$, for every $\lambda \in \widehat{L}^+$, then, by (5.1),

$$\varphi_\chi(x) = \frac{1}{N_\lambda} \sum_{y \in \mathcal{V}_\lambda(e)} \chi(\rho_\omega(y) - \rho_\omega(e)), \ \text{ for all } \ x \in \mathcal{V}_\lambda(e).$$

Recalling (2.2) and (4.1) we get, for all $x \in \mathcal{V}_\lambda(e)$,

$$\varphi_\chi(x) = \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})} \chi_0^{-1}(\lambda) \sum_{y \in \mathcal{V}_\lambda(e)} \chi(\rho_\omega(y) - \rho_\omega(e)) = \frac{1}{N_\lambda} \sum_{\mu \in \Pi_\lambda} N(\lambda,\mu)\chi(\mu).$$

THEOREM 5.3. — *For every spherical function* ϕ, *there exists a character* χ *such that*

$$\phi = \varphi_{\chi\chi_0^{1/2}};$$

moreover, for every character χ, *then*

$$\varphi_{\chi\chi_0^{1/2}} = \varphi_{(\chi^{\mathbf{w}})\chi_0^{1/2}}, \quad \forall \mathbf{w} \in \mathbf{W}.$$

Proof. — Let ϕ be a spherical function. We proved in [8, Corollary 7.5.1] that there exists a character χ on $\widehat{L}$ such that $\Lambda = \Lambda^{\chi\chi_0^{1/2}}$; so (5.1) implies that $\phi = \varphi_{\chi\chi_0^{1/2}}$. Moreover (4.2) implies the $\mathbf{W}$-invariance of the function $\varphi_{\chi\chi_0^{1/2}}$. □

From now on we refer to $\varphi_{\chi\chi_0^{1/2}}$ as to the spherical function associated with the character χ.

Let $\lambda \in \widehat{L}$; according to [10], t_λ decomposes as $t_\lambda = u_\lambda g$, with $u_\lambda \in W$ and $g \in G$. Then, if $\tau(\lambda) = j$, $j \in I$, the set $\mathcal{V}_\lambda(e)$ consists of all vertices of type j of the chambers $d \in \mathcal{C}_{u_\lambda}(c)$, for every $c \in \mathcal{C}_e$. For every chamber c and for every $j \in I$, let us denote by $v_j(c)$ the vertex of type j of c; then $\rho_\omega(v_j(c)) = v_j(\rho_\omega(c))$, for every $\omega \in \Omega$. The following proposition exhibits an alternative formula for $\varphi_{\chi\chi_0^{1/2}}(x)$ in terms of the chambers d of $\mathcal{C}_{u_\lambda}(c)$, for every $c \in \mathcal{C}_e$.

PROPOSITION 5.4. — *Let* $\lambda \in \widehat{L}^+$, $\tau(\lambda) = j$, $j \in I$. *For every* $x \in \mathcal{V}_\lambda(e)$,

$$\varphi_{\chi\chi_0^{1/2}}(x) = \frac{1}{\mathbf{W}(q)}\chi_0^{-1}(\lambda) \sum_{c\in\mathcal{C}_e} \sum_{d\in\mathcal{C}_{u_\lambda}(c)} \chi\chi_0^{1/2}(v_j(\rho_\omega(d))). \tag{5.2}$$

Proof. — For every $y \in \mathcal{V}_\lambda(e)$ and every $c \in \mathcal{C}_e$, we denote by $\gamma(e, y)$ and by $\gamma(c, y)$ any gallery connecting y to e and y to c respectively. For each $c \in \mathcal{C}_e$ lying on some $\gamma(e, y)$, we denote by $c_\lambda(y)$ the chamber of $\gamma(c, y)$, containing y. This chamber belongs to $\mathcal{C}_{w_\lambda}(c)$. According to notation of [11], the number of chambers $c \in \mathcal{C}_e$, lying on some $\gamma(e, y)$, is $\mathbf{W}_\lambda(q)$; then there exist exactly $\mathbf{W}_\lambda(q)$ chambers $c_\lambda(y)$. Moreover, for each $c_\lambda(y) \in \mathcal{C}_{w_\lambda}(c)$, the number of chambers $d \in \mathcal{C}_{u_\lambda}(c)$, such that $y = v_j(d)$, is $q_{\mathbf{w}_0}/q_{\mathbf{w}_0^\lambda}$. This implies that

$$\begin{aligned}\sum_{c\in\mathcal{C}_e} \sum_{d\in\mathcal{C}_{u_\lambda}(c)} \chi\chi_0^{1/2}(v_j(\rho_\omega(d))) &= \sum_{c\in\mathcal{C}_e} \sum_{d\in\mathcal{C}_{u_\lambda}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_j(d))) \\ &= \frac{q_{\mathbf{w}_0}}{q_{\mathbf{w}_0^\lambda}} \sum_{c\in\mathcal{C}_e} \sum_{d\in\mathcal{C}_{w_\lambda}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_j(d)))\end{aligned}$$

$$= \quad \mathbf{W}_\lambda(q)\, \frac{q_{\mathbf{w}_0}}{q_{\mathbf{w}_0^\lambda}} \sum_{y\in\mathcal{V}_\lambda(e)} \chi\chi_0^{1/2}(\rho_\omega(y)).$$

Therefore

$$\begin{aligned}
\varphi_{\chi\chi_0^{1/2}}(x) &= \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\chi_0^{-1}(\lambda) \sum_{y\in\mathcal{V}_\lambda(e)} \chi\chi_0^{1/2}(\rho_\omega(y)-\rho_\omega(e)) \\
&= \frac{\mathbf{W}_\lambda(q^{-1})}{\mathbf{W}(q^{-1})}\chi_0^{-1}(\lambda)\, \frac{1}{\mathbf{W}_\lambda(q)}\, \frac{q_{\mathbf{w}_0^\lambda}}{q_{\mathbf{w}_0}} \sum_{c\in\mathcal{C}_e} \sum_{d\in\mathcal{C}_{u_\lambda}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_j(d))) \\
&= \frac{1}{\mathbf{W}(q)}\chi_0^{-1}(\lambda) \sum_{c\in\mathcal{C}_e} \sum_{d\in\mathcal{C}_{u_\lambda}(c)} \chi\chi_0^{1/2}(v_j(\rho_\omega(d))).
\end{aligned}$$

□

In particular, when $\tau(\lambda) = 0$, then $t_\lambda = u_\lambda$ and therefore, for every $x \in \mathcal{V}_\lambda(e)$,

$$\varphi_{\chi\chi_0^{1/2}}(x) = \frac{1}{\mathbf{W}(q)}\chi_0^{-1}(\lambda) \sum_{c\in\mathcal{C}_e} \sum_{d\in\mathcal{C}_{t_\lambda}(c)} \chi\chi_0^{1/2}(v_0(\rho_\omega(d))).$$

6. Macdonald formula for spherical functions on vertices of type 0

6.1. The vector $V^{w_1}(c)$

In this section we fix a boundary point ω and a chamber c. For every $w_1 \in W$, we consider the set $\mathcal{C}_{w_1}(c) = \{c' \in \mathcal{C}(\Delta),\ \delta(c,c') = w_1\}$. For every $c' \in \mathcal{C}_{w_1}(c)$ there exists a unique $w' \in W$, such that $\rho_\omega(c') = w'(w_1(C_0))$; we shall denote by $w_{c'}$ this element. According to [8, Section 2.8], we write $w \in \mathbf{w}$ to mean that $w = t_\lambda \mathbf{w}$, for some $\lambda \in L$; we define, for every $\mathbf{w} \in \mathbf{W}$,

$$\begin{aligned}
\mathcal{C}^{w_1}_{\mathbf{w}}(c) &= \{c' \in \mathcal{C}_{w_1}(c)\ :\ w_{c'} \in \mathbf{w}\} \\
&= \{c' \in \mathcal{C}_{w_1}(c)\ :\ \rho_\omega(c') = w'(w_1(C_0)),\ w' \in \mathbf{w}\}.
\end{aligned}$$

Then $\mathcal{C}_{w_1}(c) = \bigcup_{\mathbf{w}\in\mathbf{W}} \mathcal{C}^{w_1}_{\mathbf{w}}(c)$, as disjoint union.

Definition 6.1. — *We define*

$$V^{w_1}(c) = (V^{w_1}_{\mathbf{w}}(c))_{\mathbf{w}\in\mathbf{W}},$$

where, for every $\mathbf{w} \in \mathbf{W}$,

$$V_{\mathbf{w}}^{w_1}(c) = \sum_{c' \in \mathcal{C}_{\mathbf{w}}^{w_1}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_0(c'))).$$

We notice that $V^{w_1}(c)$ is a vector with respect to the partial ordering induced on the finite group $\mathbf{W}$ by the length of its elements in terms of the generators $\{s_i,\ i \in I_0\} : \mathbf{w}_1 < \mathbf{w}_2 \quad \text{if} \quad |\mathbf{w}_1| < |\mathbf{w}_2|$.

As an immediate consequence of Definition 6.1, if we choose $w_1 = t_\lambda$, for $\lambda \in \widehat{L}^+,\ \tau(\lambda) = 0$, then

$$\sum_{\delta(c,c')=t_\lambda} \chi\chi_0^{1/2}(\rho_\omega(v_0(c'))) = \sum_{\mathbf{w}\in\mathbf{W}} \sum_{c' \in \mathcal{C}_{\mathbf{w}}^{t_\lambda}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_0(c'))) = \sum_{\mathbf{w}\in\mathbf{W}} V_{\mathbf{w}}^{t_\lambda}(c).$$

Therefore, for every vertex $x \in \mathcal{V}_\lambda(e),\ \tau(\lambda) = 0$,

$$\varphi_{\chi\chi_0^{1/2}}(x) = \frac{1}{\mathbf{W}(q)}\ \chi_0^{-1}(\lambda) \sum_{c\in\mathcal{C}_e} \sum_{\mathbf{w}\in\mathbf{W}} V_{\mathbf{w}}^{t_\lambda}(c) = \frac{1}{\mathbf{W}(q)}\ \chi_0^{-1}(\lambda) \sum_{c\in\mathcal{C}_e} I\, V^{t_\lambda}(c), \tag{6.1}$$

if $I = (I_{\mathbf{w}})_{\mathbf{w}\in\mathbf{W}}$, with $I_{\mathbf{w}} = 1$, for every $\mathbf{w}$.

6.2. The matrix $T^{(w_1 s, w_1)}$

In this section we fix ω and c as in Section 6.1.

Let $w_1 \in W$; from now on we assume that $C_0 \prec w_1(C_0)$. This is the case when either $w_1 = t_\lambda$, for $\lambda \in \widehat{L}^+,\ \tau(\lambda) = 0$, or $w_1 = u_\lambda$, for $\lambda \in \widehat{L}^+,\ \tau(\lambda) = j \neq 0$.

Let s be a generator of W, such that $|w_1 s| = |w_1| + 1$, that is $C_0 \prec w_1(C_0) \prec w_1 s(C_0)$.

We consider the vectors $V^{w_1}(c)$ and $V^{w_1 s}(c)$; in this section we construct a matrix, denoted by $T^{(w_1 s, w_1)}$, depending on w_1, s, and $w_1 s$, but not on c, such that $V^{w_1 s}(c) = T^{(w_1 s, w_1)}\ V^{w_1}(c)$.

We consider, for each $\mathbf{w} \in \mathbf{W}$,

$$\mathcal{C}_{\mathbf{w}}^{w_1 s}(c) = \{c'' \in \mathcal{C}^{w_1 s}(c)\ :\ \rho_\omega(c'') = w''(w_1 s)(C_0),\ w'' \in \mathbf{w}\}.$$

For every $c'' \in \mathcal{C}_{\mathbf{w}}^{w_1 s}(c)$, there exists a unique $c' \in \mathcal{C}_{w_1}(c)$, such that $\delta(c', c'') = s$. This fact suggests to define, for each $c' \in \mathcal{C}_{w_1}(c)$,

$$\begin{aligned} \mathcal{C}_{\mathbf{w}}^{w_1 s, s}(c') &= \{c'' \in \mathcal{C}_{\mathbf{w}}^{w_1 s}(c) : \delta(c', c'') = s\} \\ &= \{c'' \in \mathcal{C}_{w_1 s}(c) : \delta(c', c'') = s,\ \rho_\omega(c'') = w''(w_1 s)(C_0),\ w'' \in \mathbf{w}\}. \end{aligned}$$

Taking in account the decomposition $\mathcal{C}_{w_1}(c) = \bigcup_{\mathbf{u}\in\mathbf{W}} \mathcal{C}_{\mathbf{u}}^{w_1}(c)$, this definition implies the following decomposition of the set $\mathcal{C}_{\mathbf{w}}^{w_1 s}(c)$ as disjoint union:

$$\mathcal{C}_{\mathbf{w}}^{w_1 s}(c) = \bigcup_{c'\in\mathcal{C}_{w_1}(c)} \mathcal{C}_{\mathbf{w}}^{w_1 s,s}(c') = \bigcup_{\mathbf{u}\in\mathbf{W}} \bigcup_{c'\in\mathcal{C}_{\mathbf{u}}^{w_1}(c)} \mathcal{C}_{\mathbf{w}}^{w_1 s,s}(c'). \tag{6.2}$$

Lemma 6.2. — *Let $w_1 \in W$, $s \in S$ and $r = w_1 s w_1^{-1}$. Assume $C_0 \prec w_1(C_0) \prec w_1 s(C_0)$. For every $\mathbf{u} \in \mathbf{W}$, let c' be a chamber in $\mathcal{C}_{\mathbf{u}}^{w_1}(c)$. Then, for every $\mathbf{w} \in \mathbf{W}$,*

$$\sum_{c''\in\mathcal{C}_{\mathbf{w}}^{w_1 s,s}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(c''))) = \chi\chi_0^{1/2}(w_{c'} w_1(0))\, T_{\mathbf{w},\mathbf{u}}^{(w_1 s,w_1)}, \tag{6.3}$$

where

$$T_{\mathbf{w},\mathbf{u}}^{(w_1 s,w_1)} = \begin{cases} q_s\ \chi\chi_0^{1/2}(\mathbf{u}(w_1 s(0) - w_1(0))) & \textit{if } \mathbf{w}=\mathbf{u}, \mathbf{u} > \mathbf{ur}, \\ \chi\chi_0^{1/2}(\mathbf{u}(w_1 s(0) - w_1(0))) & \textit{if } \mathbf{w}=\mathbf{u}, \mathbf{u} < \mathbf{u}\ \mathbf{r}, \\ q_s - 1 & \textit{if } \mathbf{w}=\mathbf{ur} > \mathbf{u}, \\ 0 & \textit{otherwise.} \end{cases} \tag{6.4}$$

Moreover $T_{\mathbf{w},\mathbf{u}}^{(w_1 s,w_1)}$ only depends on the choice of w_1, s and $\mathbf{w}, \mathbf{u}$, but it doesn't depend on the choice of the chamber c' in the set $\mathcal{C}_{\mathbf{u}}^{w_1}(c)$, nor on the choice of c.

Proof. — Since $c' \in \mathcal{C}_{\mathbf{u}}^{w_1}(c)$, then $\delta(c,c') = w_1$ and $\rho_\omega(c') = w'(w_1(C_0))$, for some $w' = w_{c'} \in \mathbf{u}$. Moreover the chamber $w_1(C_0)$ is s-adjacent to $(w_1 s)(C_0)$, since $|w_1 s| = |w_1| + 1$, and hence also the chamber $w'(w_1(C_0))$ is s-adjacent to $w'(w_1 s(C_0))$. So there are two possibilities, according to the choice of c' and consequently of w'.

(I) $w'(w_1 s(C_0)) \prec w'(w_1(C_0))$. In this case, $w'(w_1 s(C_0))$ is s-adjacent to $\rho_\omega(c')$; moreover we have $w'(w_1 s(C_0)) \prec \rho_\omega(c')$, since $\rho_\omega(c') = w'(w_1(C_0))$ and hence $\rho_\omega(c'') = w'(w_1 s(C_0))$, for every chamber c'' such that $\delta(c',c'') = s$. Therefore $\mathcal{C}_{\mathbf{w}}^{w_1 s,s}(c') = \emptyset$, if $\mathbf{w} \neq \mathbf{u}$, while, if $\mathbf{w} = \mathbf{u}$, then $\mathcal{C}_{\mathbf{w}}^{w_1 s,s}(c') = \{c'' : \delta(c', c") = s\}$ and $w_{c''} = w_{c'}$, for each c'' in this set. Thus, keeping in mind that $|\{c'' : \delta(c', c") = s\}| = q_s$, and that $v_0(C_0) = 0$, we get the following formula

$$\begin{aligned} &\sum_{c''\in\mathcal{C}_{\mathbf{w}}^{w_1 s,s}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(c''))) \\ &= \begin{cases} q_s\ \chi\chi_0^{1/2}(w_{c'}(w_1 s(v_0(C_0))) = q_s\ \chi\chi_0^{1/2}(w_{c'} w_1 s(0)) & \text{if } \mathbf{w}=\mathbf{u}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \chi\chi_0^{1/2}(w_{c'} w_1(0)) \begin{cases} q_s\ \chi\chi_0^{1/2}(w_{c'} w_1 s(0) - w_{c'} w_1(0)) & \text{if } \mathbf{w}=\mathbf{u}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(II) $w'(w_1(C_0)) \prec w'(w_1s(C_0))$. In this case, $\rho_\omega(c')$ is s-adjacent to $w_{c'}(w_1s(C_0))$; moreover we have $\rho_\omega(c') \prec w_{c'}(w_1s(C_0))$, since $\rho_\omega(c') = w_{c'}(w_1(C_0))$. Hence, among the q_s chambers c'' such that $\delta(c', c'') = s$, only one, say $\overline{c''}$, retracts on $w_{c'}(w_1s(C_0))$, while all the others retract on $w_{c'}(w_1(C_0))$. If we consider the affine reflection $r = w_1sw_1^{-1}$, then $w_{c'}(w_1(C_0)) = (w_{c'}r)(rw_1(C_0)) = (w_{c'}r)(w_1s(C_0))$, where $w_{c'} \in \mathbf{u}, r \in \mathbf{r}$ and consequently $w_{c'}r \in \mathbf{u}\ \mathbf{r}$. Therefore the chamber $\overline{c''}$ retracts on $w_{c'}(w_1s(C_0))$ and then $w_{\overline{c''}} = w_{c'}$, that is $\mathbf{w} = \mathbf{u}$; instead the $q_s - 1$ chambers $c'' \neq \overline{c''}$ retract on $w_{c'}r((w_1s)(C_0))$, with $w_{c'}r \in \mathbf{u}\ \mathbf{r}$, and then $w_{c''} = w_{c'}r$, that is $\mathbf{w} = \mathbf{ur}$. So we have to distinguish 3 cases for the set $\mathcal{C}_{\mathbf{w}}^{w_1s,s}(c')$.

(a) $\mathbf{w} = \mathbf{u}$; the set $\mathcal{C}_{\mathbf{w}}^{w_1s,s}(c')$ contains only the chamber $\overline{c''}$ and $\rho_\omega(\overline{c''}) = w_{c'}((w_1s)(C_0))$;

(b) $\mathbf{w} = \mathbf{ur}$; the set $\mathcal{C}_{\mathbf{w}}^{w_1s,s}(c')$ contains $q_s - 1$ chambers c'' which are $s-$adjacent to c' and they all satisfy the relation $\rho_\omega(c'') = (w_{c'}r)((w_1s)(C_0)) = w_{c'}(w_1(C_0))$;

(c) $\mathbf{w} \neq \mathbf{u}, \mathbf{ur}$; no chamber c'' can retract over a chamber $w_{c''}((w_1s)(C_0))$, with $w_{c''} \in \mathbf{w}$; hence the set $\mathcal{C}_{\mathbf{w}}^{w_1s,s}(c')$ is empty.

Therefore in case (II) we can conclude that

$$\sum_{c'' \in \mathcal{C}_{\mathbf{w}}^{w_1s,s}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(c'')))$$

$$= \begin{cases} \chi\chi_0^{1/2}(w_{c'}(w_1s)(v_0(C_0)) = \chi\chi_0^{1/2}(w_{c'}(w_1s)(0) & \text{if } \mathbf{w} = \mathbf{u}, \\ (q_s - 1)\chi\chi_0^{1/2}(w_{c'}(w_1(v_0(C_0)))) = (q_s - 1)\chi\chi_0^{1/2}(w_{c'}(w_1(0))) & \text{if } \mathbf{w} = \mathbf{ur}, \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi\chi_0^{1/2}(w_{c'}w_1(0)) \begin{cases} \chi\chi_0^{1/2}(w_{c'}w_1s(0) - w_{c'}w_1(0)) & \text{if } \mathbf{w} = \mathbf{u}, \\ (q_s - 1)\chi\chi_0^{1/2}(w_{c'}w_1(0) - w_{c'}w_1(0)) & \text{if } \mathbf{w} = \mathbf{ur}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider in both cases (I) and (II) the vector $w_{c'}w_1s(0) - w_{c'}w_1(0)$; since $w_{c'} \in \mathbf{u}$, we can write $w_{c'} = t_\lambda\mathbf{u}$, for some $\lambda \in L$. Therefore

$$w_{c'}w_1(0) = (t_\lambda\mathbf{u})(w_1(0)) = \lambda + \mathbf{u}(w_1(0)) = \lambda$$

$$\text{and } \ w_{c'}w_1s(0) = (t_\lambda\mathbf{u})(w_1s(0)) = \lambda + \mathbf{u}(w_1s(0)),$$

and thus $w_{c'}w_1s(0) - w_{c'}w_1(0) = \lambda + \mathbf{u}(w_1s(0)) - \lambda - \mathbf{u}(w_1(0)) = \mathbf{u}(w_1s(0)) - \mathbf{u}(w_1(0))$. This equality shows that the vector is independent of the choice of $w_{c'} \in \mathbf{u}$, but only depends on $\mathbf{u}$. Going back to the previous formulas, we can write

$$\sum_{c'' \in \mathcal{C}_{\mathbf{w}}^{w_1s,s}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(c''))) = \chi\chi_0^{1/2}(w_{c'}w_1(0))\, T_{\mathbf{w},\mathbf{u}}^{(w_1s,w_1)},$$

where

$$T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}} = \begin{cases} q_s \ \chi\chi_0^{1/2}(\mathbf{u}(w_1 s(0) - w_1(0))) & \text{if } \mathbf{w} = \mathbf{u}, \\ 0 & \text{otherwise} \end{cases} \qquad \text{in case (I)}$$

$$T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}} = \begin{cases} \chi\chi_0^{1/2}(\mathbf{u}(w_1 s(0) - w_1(0))) & \text{if } \ \mathbf{w} = \mathbf{u}, \\ q_s - 1 & \text{if } \ \mathbf{w} = \mathbf{u}\,\mathbf{r}, \\ 0 & \text{otherwise} \end{cases} \qquad \text{in case (II).}$$

If we put $w_1(C_0) = C_1$, then, according to [8, Proposition 2.16.1], we have

$$w_{c'}(C_1) \prec (w_{c'} r)(C_1) \iff \mathbf{u} < \mathbf{ur};$$

hence we can resume in the following way the formulas for $T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}}$.

$$T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}} = \begin{cases} q_s \ \chi\chi_0^{1/2}(\mathbf{u}(w_1 s(0) - w_1(0))) & \text{if } \mathbf{w} = \mathbf{u}, \mathbf{u} > \mathbf{ur}, \\ \chi\chi_0^{1/2}(\mathbf{u}(w_1 s(0) - w_1(0))) & \text{if } \mathbf{w} = \mathbf{u}, \mathbf{u} < \mathbf{u}\,\mathbf{r}, \\ q_s - 1 & \text{if } \mathbf{w} = \mathbf{ur} > \mathbf{u}, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}}$ only depends on the choice of w_1, s and $\mathbf{w}, \mathbf{u}$, but it doesn't depends on the choice of the chamber c', nor of the choice of the initial chamber c. $\square$

THEOREM 6.3. — *Let $w_1 \in W$, $s \in S$ and $r = w_1 s w_1^{-1}$. Assume $C_0 \prec w_1(C_0) \prec w_1 s(C_0)$. Then, for every chamber c,*

$$V^{w_1 s}(c) = T^{(w_1 s, w_1)} V^{w_1}(c), \tag{6.5}$$

where $T^{(w_1 s, w_1)}$ is the sub-triangular non-singular matrix of order $d = |\mathbf{W}|$

$$T^{(w_1 s, w_1)} = (T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}})_{\mathbf{w},\mathbf{u} \in \mathbf{W}}.$$

Proof. — By Definition 6.1, (6.2) implies that, for every $\mathbf{w} \in \mathbf{W}$,

$$\begin{aligned} V^{w_1 s}_{\mathbf{w}}(c) &= \sum_{c'' \in \mathcal{C}^{w_1 s}_{\mathbf{w}}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_0(c''))) \\ &= \sum_{\mathbf{u} \in \mathbf{W}} \sum_{c' \in \mathcal{C}^{w_1}_{\mathbf{u}}(c)} \sum_{c'' \in \mathcal{C}^{w_1 s, s}_{\mathbf{w}}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(c''))). \end{aligned}$$

Using (6.3) and (6.4), we can write, for every $\mathbf{w} \in \mathbf{W}$,

$$\begin{aligned} V^{w_1 s}_{\mathbf{w}}(c) &= \sum_{\mathbf{u} \in \mathbf{W}} \sum_{c' \in \mathcal{C}^{w_1}_{\mathbf{u}}(c)} \sum_{c'' \in \mathcal{C}^{w_1 s, s}_{\mathbf{w}}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(c''))) \\ &= \sum_{\mathbf{u} \in \mathbf{W}} \sum_{c' \in \mathcal{C}^{w_1}_{\mathbf{u}}(c)} \chi\chi_0^{1/2}(w_{c'} w_1(0))\, T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}} \\ &= \sum_{\mathbf{u} \in \mathbf{W}} T^{(w_1 s, w_1)}_{\mathbf{w},\mathbf{u}} \sum_{c' \in \mathcal{C}^{w_1}_{\mathbf{u}}(c)} \chi\chi_0^{1/2}(w_{c'} w_1(0)). \end{aligned}$$

Keeping in mind that $V_{\mathbf{u}}^{w_1}(c) = \sum_{c' \in \mathcal{C}_{\mathbf{u}}^{w_1}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_0(c')))$
$= \sum_{c' \in \mathcal{C}_{\mathbf{u}}^{w_1}(c)} \chi\chi_0^{1/2}(w_{c'} w_1(0))$, we conclude that

$$V_{\mathbf{w}}^{w_1 s}(c) = \sum_{\mathbf{u} \in \mathbf{W}} T_{\mathbf{w},\mathbf{u}}^{(w_1 s, w_1)} V_{\mathbf{u}}^{w_1}(c).$$

Therefore (6.5) follows by setting $T^{(w_1 s, w_1)} = (T_{\mathbf{w},\mathbf{u}}^{(w_1 s, w_1)})_{\mathbf{w},\mathbf{u} \in \mathbf{W}}$. $\square$

Proposition 6.4. — *Let $w_1 \in W, s \in S$ and $r = w_1 s w_1^{-1}$. The matrix $T^{(w_1 s, w_1)}$ only depends on $\mathbf{w}_1$ and s and, for every $\lambda \in L^+$, $\tau(\lambda) = 0$,*

$$T^{(t_\lambda w_1 s, t_\lambda w_1)} = T^{(w_1 s, w_1)}. \tag{6.6}$$

Proof. — Write $w_1 = t_{\lambda_1} \mathbf{w}_1$; then

$$\begin{aligned} w_1 s(0) - w_1(0) &= t_{\lambda_1} \mathbf{w}_1 s(0) - t_{\lambda_1} \mathbf{w}_1(0) = \lambda_1 + \mathbf{w}_1 s(0) - \lambda_1 - \mathbf{w}_1(0) \\ &= \mathbf{w}_1 s(0) - \mathbf{w}_1(0). \end{aligned}$$

This proves that the matrix $T^{(w_1 s, w_1)}$ does not depend on the choice of w_1 in the class $\mathbf{w}_1$. Moreover, if $w_2 = t_\lambda w_1$, for $\lambda \in L^+$, $\tau(\lambda) = 0$, then $C_0 \prec t_\lambda w_1(C_0)$, and therefore $C_0 \prec t_\lambda w_1(C_0) \prec t_\lambda w_1 s(C_0)$. This implies that the matrix $T^{(w_2 s, w_2)}$ is well defined. Since $\mathbf{w}_2 = \mathbf{w}_1$, we conclude that $T^{(w_2 s, w_2)} = T^{(w_1 s, w_1)}$. $\square$

6.3. The matrix T^w

We fix ω and c as in Sections 6.1 and 6.2. Let $w \in W$; assume that $w = s_1 \cdots s_n$ and set $w_1 = s_1, \quad w_2 = w_1 s_2 = s_1 s_2, \quad \ldots, \quad w_n = s_1 \cdots s_n = w.$

If $C_0 \prec w_1(C_0) \prec \cdots \prec w(C_0)$, then $s_1 = s_{\alpha_0}$ and $|w_j| = |w_{j-1}| + 1 = j$, for every $j = 1, \ldots, n$. We notice that, if $w \in W$ has length $|w| = n$, and $C_0 \prec w(C_0)$, we may always write $w = s_1 \cdots s_n$, in such a way that $C_0 \prec s_1(C_0) \prec \cdots \prec s_1 \cdots s_n(C_0) = w(C_0)$.

By Proposition 6.4, $V^{w_j}(c) = T^{(w_j, w_{j-1})} V^{w_{j-1}}(c)$, for each $j = 1, \ldots, n$; so, for every $j = 1, \ldots, n$,

$$V^w(c) = T^{(w, w_{n-1})} \cdots T^{(w_j, w_{j-1})}\, V^{w_{j-1}}(c) = T^{(w, w_{n-1})} \cdots T^{(w_1, e)}\, V^e(c).$$

The following lemma shows that, for every $j = 1, \ldots, n$, the product $T^{(w, w_{n-1})} \cdots T^{(w_j, w_{j-1})}$ does not depend on the representation of the element w as reduced word.

Lemma 6.5. — *Let $w \in W$, $|w| = n$. If $w = s_1 \cdots s_n = s'_1 \cdots s'_n$, where $s_1, \ldots, s_n$, $s'_1, \ldots, s'_n \in S$ and $C_0 \prec s_1(C_0) \prec \cdots \prec s_1 \cdots s_n(C_0)$, $C_0 \prec s'_1(C_0) \prec \cdots \prec s'_1 \cdots s'_n(C_0)$, then, , for every $j = 1, \ldots, n$,*

$$T^{(s_1\cdots s_n, s_1\cdots s_{n-1})} \ldots T^{(s_1\cdots s_j, s_1\cdots s_{j-1})} = T^{(s'_1\cdots s'_n, s'_1\cdots s'_{n-1})} \ldots T^{(s'_1\cdots s'_j, s'_1\cdots s'_{j-1})}.$$

Proof. — Assume $j = 1$. We know that

$$\begin{aligned} V^w(c) &= T^{(s_1\cdots s_n, s_1\cdots s_{n-1})} \ldots T^{(s_1,e)}\ V^e(c) \\ &= T^{(s'_1\cdots s'_n, s'_1\cdots s'_{n-1})} \ldots T^{(s'_1,e)}\ V^e(c). \end{aligned}$$

In order to compare the matrices $T^{(s_1\cdots s_n, s_1\cdots s_{n-1})} \ldots T^{(s_1,e)}$ and $T^{(s'_1\cdots s'_n, s'_1\cdots s'_{n-1})} \ldots T^{(s'_1,e)}$, we choose the chamber c in an appropriate way. Fix $\mathbf{u} \in \mathbf{W}$ and consider a chamber c containing e, such that $\rho_\omega(c) = \mathbf{u}(C_0)$. According to this choice, $\mathcal{C}_e(c) = c$ and $\mathcal{C}^e_{\mathbf{w}}(c) = \{c\}$ if $\mathbf{w} = \mathbf{u}$, while $\mathcal{C}^e_{\mathbf{w}}(c) = \emptyset$ if $\mathbf{w} \neq \mathbf{u}$. Hence $V^e(c) = \delta_{\mathbf{u}}$, because

$$V^{(e)}_{\mathbf{w}}(c) = \begin{cases} \chi\chi_0^{1/2}(\rho_\omega(0)) = 1 & \text{if } \mathbf{w} = \mathbf{u}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $T^{(s_1\cdots s_n, s_1\cdots s_{n-1})} \ldots T^{(s_1,e)}\ \delta_{\mathbf{u}} = T^{(s'_1\cdots' s_n, s'_1\cdots s'_{n-1})} \ldots T^{(s'_1,e)}\ \delta_{\mathbf{u}}$, that is

$$[T^{(s_1\cdots s_n, s_1\cdots s_{n-1})} \ldots T^{(s_1,e)}]_{\mathbf{w},\mathbf{u}} = [T^{(s'_1\cdots s'_n, s'_1\cdots s'_{n-1})} \ldots T^{(s'_1,e)}]_{\mathbf{w},\mathbf{u}},\ \forall \mathbf{u}, \mathbf{w} \in \mathbf{W}.$$

So we conclude that, for every $j > 0$,

$$\begin{aligned} &T^{(s_1\cdots s_n, s_1\cdots s_{n-1})} \ldots T^{(s_1\cdots s_j, s_1\cdots s_{j-1})} \\ &= \left[T^{(s_1\cdots s_n, s_1\cdots s_{n-1})} \ldots T^{(s_1,e)}\right] \left[T^{(s_1\cdots s_{j-1}, s_1\cdots s_{j-2})} \ldots T^{(s_1,e)}\right]^{-1} \\ &= \left[T^{(s'_1\cdots s'_n, s'_1\cdots s'_{n-1})} \ldots T^{(s'_1,e)}\right] \left[T^{(s'_1\cdots s'_{j-1}, s'_1\cdots s'_{j-2})} \ldots T^{(s'_1,e)}\right]^{-1} \\ &= T^{(s'_1\cdots s'_n, s'_1\cdots s'_{n-1})} \ldots T^{(s'_1\cdots s'_j, s'_1\cdots s'_{j-1})}. \end{aligned}$$

□

Lemma 6.5 suggests the following definition.

Definition 6.6. — *For every $w \in W$, $|w| = n$, such that $C_0 \prec w_1(C_0) \prec \cdots \prec w(C_0)$, we denote by T^w the following sub-triangular non-singular matrix of order $d = |W|$*

$$T^w = T^{(w,w_{n-1})} \cdots T^{(w_1,e)}. \tag{6.7}$$

Moreover, for every $w_0 \in W$ such that $C_0 \prec w_0(C_0) \prec w_0w_1(C_0) \prec \cdots \prec w_0w(C_0)$, we define

$$T^{(w_0w,w_0)} = T^{(w_0w,w_0w_{n-1})}\ T^{(w_0w_{n-1},w_0w_{n-2})} \cdots T^{(w_0w_1,w_0)}. \tag{6.8}$$

We notice that, if $|w_0 w| = |w_0| + |w| = |w_0| + n$ and $C_0 \prec w_0(C_0) \prec w_0 w(C_0)$, we may always write $w = s_1 \cdots s_n$ in such a way that $C_0 \prec w_0(C_0) \prec w_0 s_1(C_0) \prec \cdots \prec w_0 s_1 \cdots s_n(C_0)$.

COROLLARY 6.7. — *Let $w_0, w \in W$, with $|w| = n$; assume $C_0 \prec w_0(C_0) \prec w_0 w(C_0)$, Then*

$$T^{w_0 w} = T^{(w_0 w_n, w_0 w_{n-1})}\ T^{(w_0 w_{n-1}, w_0 w_{n-2})} \cdots T^{(w_0 w_1, w_0)}\ T^{w_0}; \tag{6.9}$$

moreover, for every chamber c,

$$V^w(c) = T^w\ V^e(c) \quad \text{and} \quad V^{w_0 w}(c) = T^{w_0 w}\ V^{w_0}(c). \tag{6.10}$$

From Proposition 6.4 and Definition 6.6 it follows immediately the following corollary.

COROLLARY 6.8. — *Let $w_0, w \in W$ such that $C_0 \prec w_0(C_0) \prec w_0 w(C_0)$. Then, for every $\lambda \in L^+$, $\tau(\lambda) = 0$,*

$$T^{(t_\lambda w_1, t_\lambda w_0)} = T^{(w_1, w_0)}. \tag{6.11}$$

6.4. The matrix T^λ

Let $\lambda \in \widehat{L}^+, \tau(\lambda) = 0$; then $t_\lambda \in W$ and $C_0 \prec t_\lambda(C_0)$. According to Definition 6.6, we can define the non-singular sub-triangular matrix $T^\lambda = T^{t_\lambda}$ such that $V^{t_\lambda}(c) = T^\lambda\ V^e(c)$, for every chamber c.

THEOREM 6.9. — *Let $\lambda \in \widehat{L}^+$, $\tau(\lambda) = 0$; for every $\mathbf{u} \in \mathbf{W}$,*

$$T^\lambda_{\mathbf{u},\mathbf{u}} = \chi_0^{1/2}(\lambda)\chi^{\mathbf{u}}(\lambda). \tag{6.12}$$

Proof. — Let $t_\lambda = s_1 \cdots s_n$; we set, as usual,

$$w_j = w_{j-1} s_j,\ j = 1, \ldots, n-1,\ w_n = t_\lambda,\ r_j = w_{j-1} s_j w_{j-1}^{-1},$$

$$C_j = w_j(C_0),\ j = 1, \ldots, n.$$

Since $\lambda \in \widehat{L}^+$, then $C_0 \prec C_1 \prec \cdots \prec C_{n-1} \prec C_n = t_\lambda(C_0)$; moreover, for every $\mathbf{u} \in \mathbf{W}$,

$$T^\lambda_{\mathbf{u},\mathbf{u}} = T^{(w_{n-1}s_n, w_{n-1})}_{\mathbf{u},\mathbf{u}} \cdots T^{(s_1,e)}_{\mathbf{u},\mathbf{u}}.$$

Therefore, by Theorem 6.3,

$$\begin{aligned} T^\lambda_{\mathbf{u},\mathbf{u}} &= \chi\chi_0^{1/2}(\mathbf{u}(w_n(0) - w_{n-1}(0)))\ \chi\chi_0^{1/2}(\mathbf{u}(w_{n-1}(0) - w_{n-2}(0))) \\ &\qquad \cdots \chi\chi_0^{1/2}(\mathbf{u}(w_1(0)))\ (\Pi_{j\in J_{\mathbf{u}}} q_j) \\ &= \chi\chi_0^{1/2}(\mathbf{u}(\lambda))\ (\Pi_{j\in J_{\mathbf{u}}} q_j) = [(\Pi_{j\in J_{\mathbf{u}}} q_j)\ (\chi_0^{\mathbf{u}})^{1/2}(\lambda)]\ \chi^{\mathbf{u}}(\lambda), \end{aligned}$$

where $J_{\mathbf{u}} = \{j = 1, \dots, n \ : \ \mathbf{u}\, \mathbf{r}_j < \mathbf{u}\}$ and $q_j = q_{s_j}$, for every j. We notice that $J_{\mathbf{e}} = \emptyset$ and $J_{\mathbf{w}_0} = \{1, \dots, n\}$, while, for $\mathbf{u} \neq \mathbf{e}, \mathbf{w}_0$, $J_{\mathbf{u}}$ is non trivial. Since $C_{j-1} \prec r_j(C_{j-1})$ for every j, then it can be proved (see [8, Proposition 2.16.1]) that, for $w' \in \mathbf{u}$,

$$\begin{aligned} \mathbf{u}\, \mathbf{r}_j < \mathbf{u} \quad &\Longleftrightarrow \quad w'(C_j) = w'(r_j(C_{j-1})) \prec w'(C_{j-1}) \\ &\Longleftrightarrow \quad w'w_j(C_0)) \prec w'(w_{j-1}(C_0)). \end{aligned}$$

Therefore $J_{\mathbf{u}} = \{j = 1, \dots, n, \ : \ w'w_j(C_0)) \prec w'(w_{j-1}(C_0), \ w' \in \mathbf{u}\}$.

On the other hand, by (3.3), for every $\mathbf{u} \in \mathbf{W}$ and $w' \in \mathbf{u}$,

$$\chi_0(\mathbf{u}(\lambda)) = \prod_{j \in J_{\mathbf{u}}^+} q_j \cdot \prod_{j \in J_{\mathbf{u}}^-} q_j^{-1},$$

where $J_{\mathbf{u}}^+ = \{j \ : \ \mathbf{u}s_1 \cdots s_{j-1}(C_0) \prec \mathbf{u}s_1 \cdots s_j(C_0)\}$ and $J_{\mathbf{u}}^- = \{j \ : \ \mathbf{u}s_1 \cdots s_j(C_0) \prec \mathbf{u}s_1 \cdots s_{j-1}(C_0)\}$. Since $J_{\mathbf{u}}^+ = \{j \ : \ w'w_{j-1}(C_0) \prec w'w_j(C_0)\}$, for any $w' \in \mathbf{u}$, and so $J_{\mathbf{u}}^+ = J_{\mathbf{u}}$, we conclude that

$$\begin{aligned} (\Pi_{j \in J_{\mathbf{u}}}\, q_j)(\chi_0^{\mathbf{u}})^{1/2}(\lambda) \quad &= \quad (\Pi_{j \in J_{\mathbf{u}}^+}\, q_j)\, (\Pi_{j \in J_{\mathbf{u}}^+}\, q_j^{1/2})\, (\Pi_{j \in J_{\mathbf{u}}^-}\, q_j^{-1/2}) \\ &= \quad (\Pi_{j \in J_{\mathbf{u}}^+}\, q_j^{1/2})\, (\Pi_{j \in J_{\mathbf{u}}^-}\, q_j^{1/2}) = \Pi_{j=1}^n\, q_j^{1/2} \\ &= \quad \chi_0^{1/2}(\lambda) \end{aligned}$$

$\square$

Proposition 6.10. — *Let $\lambda_1, \lambda_2 \in \widehat{L}^+$, $\tau(\lambda_1) = \tau(\lambda_2) = 0$. Then*

$$T^{\lambda_1}\, T^{\lambda_2} = T^{\lambda_2}\, T^{\lambda_1}. \tag{6.13}$$

Proof. — If $\lambda_1, \lambda_2 \in \widehat{L}^+$ and $\tau(\lambda_1) = \tau(\lambda_2) = 0$, it is easy to see that $|t_{\lambda_1+\lambda_2}| = |t_{\lambda_1}| + |t_{\lambda_2}|$. Actually, if c and c'' are two chambers such that $\delta(c, c'') = t_{\lambda_1+\lambda_2}$, then there exist unique c_1, c_2 such that

$$\delta(c, c_1) = t_{\lambda_1},\ \delta(c_1, c'') = t_{\lambda_2} \ \text{ and } \ \delta(c, c_2) = t_{\lambda_2},\ \delta(c_2, c'') = t_{\lambda_1}.$$

Therefore, according to Definition 6.6, $T^{\lambda_1+\lambda_2} = T^{t_{\lambda_2}t_{\lambda_1}} = T^{(t_{\lambda_2}t_{\lambda_1}, t_{\lambda_2})}\, T^{t_{\lambda_1}}$. On the other hand, Corollary 6.8 assures that $T^{(t_{\lambda_2}t_{\lambda_1}, t_{\lambda_2})} = T^{(t_{\lambda_1}, e)} = T^{\lambda_1}$ and then $T^{\lambda_1+\lambda_2} = T^{\lambda_2}\, T^{\lambda_1}$. In an analogous way we prove that $T^{\lambda_2+\lambda_1} = T^{\lambda_1}\, T^{\lambda_2}$. Since $T^{\lambda_1+\lambda_2} = T^{\lambda_2+\lambda_1}$, (6.13) is proved. $\square$

Theorem 6.11. — *If χ is a good character, there exists a unique matrix C independent of λ, such that $D^\lambda = CT^\lambda C^{-1}$ is diagonal for all $\lambda \in \widehat{L}^+, \tau(\lambda) = 0$; moreover*

$$D^\lambda_{\mathbf{u},\mathbf{u}} = \chi_0^{1/2}(\lambda)\ \chi^{\mathbf{u}}(\lambda), \quad \forall \mathbf{u} \in \mathbf{W}. \tag{6.14}$$

Proof. — Let $\chi^{\mathbf{w}_1}(\lambda_0) \neq \chi^{\mathbf{w}_2}(\lambda_0)$, for $\mathbf{w}_1 \neq \mathbf{w}_2$; then (6.12) implies that the matrix T^{λ_0} has distinct eigenvalues and so there exists a matrix C such that $D^{\lambda_0} = CT^{\lambda_0}C^{-1}$ is diagonal. By (6.13), the matrix C diagonalizes T^{λ}, for every $\lambda \in \widehat{L}^+, \tau(\lambda) = 0$; moreover (6.14) follows from (6.12). $\square$

COROLLARY 6.12. — *If χ is a good character, then there exists a matrix C, independent of λ, such that for every $\lambda \in \widehat{L}^+, \tau(\lambda) = 0$, and for every chamber c,*

$$V^{t_\lambda}(c) = (C^{-1}D^{\lambda}C)\ V^e(c), \tag{6.15}$$

where $D^{\lambda}_{\mathbf{u},\mathbf{u}} = \chi_0^{1/2}(\lambda)\ \chi^{\mathbf{u}}(\lambda)$, for every $\mathbf{u} \in \mathbf{W}$.

6.5. Macdonald formula for spherical functions on vertices of type 0

We can state the following matricial formula for the spherical function $\varphi_{\chi\chi_0^{1/2}}$ associated to a character χ.

PROPOSITION 6.13. — *For every $x \in \mathcal{V}_\lambda(e),\ \tau(\lambda) = 0$, we have*

$$\varphi_{\chi\chi_0^{1/2}}(x) = \chi_0^{-1}(\lambda)\ I\ T^{\lambda}\ V_0^e, \tag{6.16}$$

if $V_0^e = (V^e_{0,\mathbf{u}})_{\mathbf{u}\in\mathbf{W}}$ and

$$V^e_{0,\mathbf{u}} = \frac{q_{\mathbf{u}}}{\mathbf{W}(q)}, \quad \forall\ \mathbf{u} \in \mathbf{W}. \tag{6.17}$$

Proof. — Since, for every chamber c, $V^{t_\lambda}(c) = T^{\lambda}V^e(c)$, (6.1) implies that, for every $x \in \mathcal{V}_\lambda(e),\ \tau(\lambda) = 0$,

$$\varphi_{\chi\chi_0{}^{1/2}}(x) = \frac{1}{\mathbf{W}(q)}\ \chi_0^{-1}(\lambda)\, I \sum_{c\in\mathcal{C}_e} T^{\lambda}\ V^e(c) = \frac{1}{\mathbf{W}(q)}\ \chi_0^{-1}(\lambda)\, I\, T^{\lambda} \sum_{c\in\mathcal{C}_e} V^e(c).$$

On the other hand, according to Lemma 6.5, $V^e(c) = \delta_{\mathbf{u}}$, if c contains e and $\rho_\omega(c) = \mathbf{u}(C_0)$. Thus

$$\sum_{c\in\mathcal{C}_e} V^e(c) = \sum_{\mathbf{u}\in\mathbf{W}} \left(\sum_{c\in\mathcal{C}_e:\rho_\omega(c)=\mathbf{u}(C_0)} V^e(c) \right) = \sum_{\mathbf{u}\in\mathbf{W}} q_{\mathbf{u}}\ \delta_{\mathbf{u}} = \mathbf{W}(q)V_0^e,$$

if $V_0^e = (V^e_{0,\mathbf{u}})_{\mathbf{u}\in\mathbf{W}}$ and $(V_0^e)_{\mathbf{u}} = q_{\mathbf{u}}(\mathbf{W}(q))^{-1}$, for every $\mathbf{u} \in \mathbf{W}$. Therefore (6.16) is proved. $\square$

We notice that if $x = e$, we get $\varphi_{\chi\chi_0{}^{1/2}}(x) =\ I\ V_0^e = 1$, since T^0 is the identity matrix.

Theorem 6.14. — *For every good χ, there exist constants $\{c_{\mathbf{w}}(\chi), \mathbf{w} \in \mathbf{W}\}$, $\sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi) = 1$, such that*

$$\varphi_{\chi{\chi_0}^{1/2}}(x) = \chi_0^{-1/2}(\lambda) \sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi)\ \chi^{\mathbf{w}}(\lambda), \tag{6.18}$$

for every $x \in \mathcal{V}_\lambda(e)$, with $\tau(\lambda) = 0$.

Proof. — Assume at first that $x \neq e$. By (6.16) and (6.15), we can write, for every $x \in \mathcal{V}_\lambda(e)$, with $\tau(\lambda) = 0$,

$$\varphi_{\chi\chi_0^{1/2}}(x) = \frac{1}{\mathbf{W}(q)}\ \chi_0^{-1}(\lambda)\ I\ C^{-1} D^\lambda C\ V_0^e.$$

Therefore, by setting $U_1(\chi) = I\ C^{-1}$ and $U_2(\chi) = C\ V_0^e$, we have

$$\begin{aligned}
\varphi_{\chi\chi_0^{1/2}}(x) &= \chi_0^{-1}(\lambda)\ U_1(\chi)\ D^\lambda\ U_2(\chi)\\
&= \chi_0^{-1}(\lambda) \sum_{\mathbf{w}\in\mathbf{W}} (U_2(\chi))_{\mathbf{w}}\ \chi_0^{1/2}(\lambda)\ \chi^{\mathbf{w}}(\lambda)\ (U_2(\chi))_{\mathbf{w}}\\
&= \chi_0^{-1/2}(\lambda) \sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi)\ \chi^{\mathbf{w}}(\lambda),
\end{aligned}$$

if we set $c_{\mathbf{w}}(\chi) = (U_1(\chi))_{\mathbf{w}}\ (U_2(\chi))_{\mathbf{w}}$, for every $\mathbf{w} \in \mathbf{W}$. Moreover

$$\begin{aligned}
\sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi) &= \sum_{\mathbf{w}\in\mathbf{W}} (U_1(\chi))_{\mathbf{w}}\ (U_2(\chi))_{\mathbf{w}}\\
&= (I\ C^{-1})\ (CV_0^e) = I\ (C^{-1}C)V_0^e = I\ V_0^e\\
&= \sum_{\mathbf{w}\in\mathbf{W}} q_{\mathbf{w}} = 1.
\end{aligned}$$

Since $\varphi_{\chi\chi_0^{1/2}}(e) = 1$ and $\sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi)\ \chi^{\mathbf{w}}(0) = 1$, the required formula holds also for $x = e$. $\square$

It will be useful to express formula (6.18) in terms of the retraction $\rho_0(x)$.

Corollary 6.15. — *For every good character χ and for every $x \in \mathcal{V}_0(\Delta)$*

$$\varphi_{\chi{\chi_0}^{1/2}}(x) = \chi_0^{1/2}(\rho_0(x)) \sum_{\mathbf{w}\in\mathbf{W}} d_{\mathbf{w}}(\chi)\ \chi^{\mathbf{w}}(\rho_0(x)), \tag{6.19}$$

if $d_{\mathbf{w}}(\chi) = c_{\mathbf{w}\mathbf{w}_0}(\chi)$, for every $\mathbf{w} \in \mathbf{W}$.

Proof. — Since $\rho_0(x) = \mathbf{w}_0(\lambda)$, for every $x \in \mathcal{V}_\lambda(e)$, then $\chi_0^{-1/2}(\lambda) = \chi_0^{-1/2}(\mathbf{w}_0\rho_0(x)) = \chi_0^{1/2}(\rho_0(x))$. If we set $\mathbf{u} = \mathbf{w}\mathbf{w}_0$, for each $\mathbf{w} \in \mathbf{W}$, and $d_{\mathbf{u}}(\chi) = c_{\mathbf{u}\mathbf{w}_0}(\chi)$, for every $\mathbf{u} \in \mathbf{W}$, we can write

$$\begin{aligned}
\sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi)\chi^{\mathbf{w}}(\mathbf{w}_0\rho_0(x)) &= \sum_{\mathbf{u}\in\mathbf{W}} c_{\mathbf{u}\mathbf{w}_0}(\chi)\chi^{\mathbf{u}\mathbf{w}_0}(\mathbf{w}_0\rho_0(x)) \\
&= \sum_{\mathbf{u}\in\mathbf{W}} c_{\mathbf{u}\mathbf{w}_0}(\chi)\ \chi^{\mathbf{u}}(\rho_0(x)) \\
&= \sum_{\mathbf{u}\in\mathbf{W}} d_{\mathbf{u}}(\chi)\chi^{\mathbf{u}}(\rho_0(x)).
\end{aligned}$$

□

Proposition 6.16. — *If χ is a good character, then for every $\mathbf{u} \in \mathbf{W}$,*

$$c_{\mathbf{u}}(\chi) = c_{\mathbf{e}}(\chi^{\mathbf{u}}) \quad \text{and} \quad d_{\mathbf{u}}(\chi) = d_{\mathbf{e}}(\chi^{\mathbf{u}}). \tag{6.20}$$

Proof. — It is enough to prove that, for every $\mathbf{u} \in \mathbf{W}$, $c_{\mathbf{u}}(\chi) = c_{\mathbf{e}}(\chi^{\mathbf{u}})$, if χ is a good character. Fix $\mathbf{u} \in \mathbf{W}$. Since $\varphi_{\chi\chi_0^{1/2}} = \varphi_{(\chi^{\mathbf{u}})\chi_0^{1/2}}$, then (6.18) implies that, for every $\lambda \in \widehat{L}$, $\tau(\lambda) = 0$,

$$\begin{aligned}
\sum_{\mathbf{v}\in\mathbf{W}} c_{\mathbf{v}}(\chi)\ \chi^{\mathbf{v}}(\lambda) &= \sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi^{\mathbf{u}})\ (\chi^{\mathbf{u}})^{\mathbf{w}}(\lambda) \\
&= \sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi^{\mathbf{u}})\ \chi^{\mathbf{w}\mathbf{u}}(\lambda) \\
&= \sum_{\mathbf{v}\in\mathbf{W}} c_{\mathbf{v}\mathbf{u}^{-1}}(\chi^{\mathbf{u}})\ \chi^{\mathbf{v}}(\lambda),
\end{aligned}$$

by setting $\mathbf{w}\mathbf{u} = \mathbf{v}$. Therefore $c_{\mathbf{v}\mathbf{u}^{-1}}(\chi^{\mathbf{u}}) = c_{\mathbf{v}}(\chi)$, for every $\mathbf{v} \in \mathbf{W}$. In particular, if we choose $\mathbf{v} = \mathbf{u}$, we find the required identity $c_{\mathbf{e}}(\chi^{\mathbf{u}}) = c_{\mathbf{u}}(\chi)$. □

The formulas (6.18) and (6.19) proved for good characters can in fact be extended to all non-singular characters (see Section 8).

7. Macdonal formula for spherical functions on all special vertices of a reduced building

In this section we assume that Δ is a reduced building such that $\widehat{\mathcal{V}}(\Delta) \neq \mathcal{V}_0(\Delta)$ and we extend to all vertices of $\widehat{\mathcal{V}}(\Delta)$ the formula (6.18) provided by

Theorem 6.14. We start by considering the formula (5.2) proved by Proposition 5.4. The vertex of type j of any chamber d can be seen as the vertex of type 0 of the generalized chamber $\widehat{d} = (d, \sigma)$, if σ is the automorphism of the set I such that $\sigma(j) = 0$, that is $v_j(d) = v_0(\widehat{d})$. This remark suggests that, in order to provide the required generalization, we must extend the arguments of sections 6.1, 6.2, 6.3 and 6.4, replacing the affine Weyl group W with the extended affine Weyl group $\widehat{W}$ and the set $\mathcal{C}(\Delta)$ with the set $\widehat{\mathcal{C}}(\Delta)$of all extended chambers of Δ.

In the whole section we fix a boundary point ω and an extended chamber $\widehat{c} = (c, \sigma)$, for some chamber c and some $\sigma \in Aut_{tr}(D)$.

7.1. The vector $V^{\widehat{w}_1}(\widehat{c})$

For every $\widehat{w}_1 \in \widehat{W}$, $\widehat{w}_1 = w_1 g_1$, we define

$$\widehat{\mathcal{C}}_{\widehat{w}_1}(\widehat{c}) = \{\widehat{c}' \in \widehat{\mathcal{C}}(\Delta),\ \widehat{\delta}(\widehat{c}, \widehat{c}') = \widehat{w}_1\},$$

where $\widehat{\delta}(\widehat{c}, \widehat{c}') = \delta(c, c')g^{-1}g'$, if $\widehat{c}' = (c', \sigma')$ and g, g' are the elements of the group G corresponding to σ, σ' respectively. We notice that

$$\widehat{\mathcal{C}}_{\widehat{w}_1}(\widehat{c}) = \{\widehat{c}' = (c', \sigma'),\ c' \in \mathcal{C}_{w_1}(c),\ \sigma' = \sigma_1\sigma\};$$

in particular, for every chamber c,

$$\widehat{\mathcal{C}}_{\widehat{w}_1}(c) = \{\widehat{c}' \in \widehat{\mathcal{C}}(\Delta),\ \widehat{\delta}(c, \widehat{c}') = \widehat{w}_1\} = \{\widehat{c}' = (c', \sigma_1),\ c' \in \mathcal{C}_{w_1}(c)\}.$$

We extend the definition of retraction ρ_ω to all extended chambers, by setting, for every $\widehat{c} = (c, \sigma)$,

$$\rho_\omega(\widehat{c}) = (\rho_\omega(c), \sigma).$$

Then $\rho_\omega(\widehat{c}) = \widehat{w}(C_0)$, $\widehat{w} = wg$, if and only if $\rho_\omega(c) = w(C_0)$ and g is the element of G corresponding to σ.

If $\widehat{c}' = (c', \sigma')$ belongs to $\mathcal{C}_{\widehat{w}_1}(\widehat{c})$, then there exists a unique $w_{c'} \in W$, such that $\rho_\omega(c') = w_{c'}(w_1(C_0))$. Therefore, being $\sigma' = \sigma_1\sigma$,

$$\rho_\omega(\widehat{c}') = (\rho_\omega(c'), \sigma') = (w_{c'}(w_1(C_0)), \sigma') = (w_{c'}(w_1(C_0)), \sigma_1\sigma)$$

$$= w_{c'}(w_1 g_1 g(C_0)) = w_{c'}(\widehat{w}_1(C_0, \sigma)).$$

In particular, for every chamber c, $\rho_\omega(\widehat{c}') = w_{c'}(\widehat{w}_1(C_0))$, if $\widehat{c}' \in \widehat{\mathcal{C}}_{\widehat{w}_1}(c)$. So in this case we have $\rho_\omega(\widehat{c}') = w'(\widehat{w}_1(C_0)) = w'w_1g_1(C_0)$ if and only if $\rho_\omega(c') = w'w_1(C_0)$ and $\sigma' = \sigma_1$.

We define, for each $\mathbf{w} \in \mathbf{W}$,

$$\begin{aligned}\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1}(\widehat{c}) &= \{\widehat{c}' = (c', \sigma') \in \widehat{\mathcal{C}}_{\widehat{w}_1}(\widehat{c}) \ : \ w_{c'} \in \mathbf{w}\} \\ &= \{\widehat{c}' = (c', \sigma') \in \widehat{\mathcal{C}}_{\widehat{w}_1}(\widehat{c}) \ : \ \rho_\omega(\widehat{c}') = w'(\widehat{w}_1(C_0, \sigma)), \ w' \in \mathbf{w}\}.\end{aligned}$$

In particular, when $\widehat{c} = c$,

$$\begin{aligned}\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1}(c) &= \{\widehat{c}' = (c', \sigma') \in \widehat{\mathcal{C}}_{\widehat{w}_1}(c) \ : \ w_{c'} \in \mathbf{w}\} \\ &= \{\widehat{c}' = (c', \sigma') \in \widehat{\mathcal{C}}_{\widehat{w}_1}(c) \ : \ \rho_\omega(\widehat{c}') = w'(\widehat{w}_1(C_0)), \ w' \in \mathbf{w}\}.\end{aligned}$$

Then $\widehat{\mathcal{C}}_{\widehat{w}_1}(\widehat{c}) \ = \ \bigcup_{\mathbf{w}\in\mathbf{W}} \ \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1}(\widehat{c})$, as disjoint union.

Since $v_0(\widehat{c}) = v_{\sigma(0)}(c)$, for every $\widehat{c} = (c, \sigma)$, and $\rho_\omega(v_j(c)) = v_j(\rho_\omega(c))$, for every $j \in I$, then

$$v_0(\rho_\omega(\widehat{c})) = v_0(\rho_\omega(c), \sigma) = v_{\sigma(0)}(\rho_\omega(c)) = \rho_\omega(v_{\sigma(0)}(c)) = \rho_\omega(v_0(\widehat{c})).$$

Definition 7.1. — *We define*

$$V^{\widehat{w}_1}(\widehat{c}) \ = (V_{\mathbf{w}}^{\widehat{w}_1}(\widehat{c}))_{\mathbf{w}\in\mathbf{W}},$$

where, for every $\mathbf{w} \in \mathbf{W}$,

$$V_{\mathbf{w}}^{\widehat{w}_1}(\widehat{c}) = \sum_{\widehat{c}' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1}(\widehat{c})} \chi\chi_0^{1/2}(\rho_\omega(v_0(\widehat{c}'))).$$

$V^{\widehat{w}_1}(\widehat{c})$ is a vector with respect to the usual ordering of $\mathbf{W}$ and, if $\widehat{w}_1 = w_1 g_1$ and $j = \sigma_1^{-1}(0)$,

$$V_{\mathbf{w}}^{\widehat{w}_1}(c) = \sum_{c' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{w_1}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_0(c', \sigma_1))) = \sum_{c' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{w_1}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_j(c'))), \ \forall \mathbf{w} \in \mathbf{W}.$$

In particular if $\widehat{w}_1 = t_\lambda = u_\lambda g_\lambda$, for some $\lambda \in \widehat{L}^+$, and σ_λ is the automorphism of D associated to g_λ, then $\sigma_\lambda(0) = j$, if $j = \tau(\lambda)$, and, for every chamber c,

$$\sum_{\mathbf{w}\in\mathbf{W}} V_{\mathbf{w}}^{t_\lambda}(c) = \sum_{\widehat{c}' \in \mathcal{C}_{\mathbf{w}}^{t_\lambda}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_0(\widehat{c}'))) = \sum_{c' \in \mathcal{C}_{\mathbf{w}}^{u_\lambda}(c)} \chi\chi_0^{1/2}(\rho_\omega(v_j(c'))).$$

Therefore, for every $x \in \mathcal{V}_\lambda(e)$,

$$\varphi_{\chi\chi_0^{1/2}}(x) = \frac{1}{\mathbf{W}(q)} \chi_0^{-1}(\lambda) \sum_{c\in\mathcal{C}_e} \sum_{\mathbf{w}\in\mathbf{W}} V_{\mathbf{w}}^{t_\lambda}(c) = \frac{1}{\mathbf{W}(q)} \chi_0^{-1}(\lambda) \sum_{c\in\mathcal{C}_e} IV^{t_\lambda}(c). \tag{7.1}$$

7.2. The matrix $T^{(\widehat{w}_1 s, \widehat{w}_1)}$

Let $\widehat{w}_1 \in \widehat{W}$, $\widehat{w}_1 = w_1 g_1$; from now on we assume that $C_0 \prec \widehat{w}_1(C_0)$, that is $C_0 \prec w_1(C_0)$; this is the case, in particular, when $\widehat{w}_1 = t_\lambda$, for $\lambda \in \widehat{L}^+$. Let s be any generator of W and consider $\widehat{w}_1 s$; then $\widehat{w}_1 s = w_1 g_1 s = w_1 s_1 g_1$, if $s_1 = g_1 s g_1^{-1}$. We shall assume $|\widehat{w}s| = |\widehat{w}| + 1$; since $|\widehat{w}| = |w|$, if $\widehat{w} = wg$, this means in fact $|w_1 s_1| = |w_1| + 1$; thus $w_1(C_0) \prec w_1 s_1(C_0)$ and then $\widehat{w}_1(C_0) \prec \widehat{w}_1 s(C_0)$. We consider $V^{\widehat{w}_1}(c)$ and $V^{\widehat{w}_1 s}(c)$; then we construct a matrix, denoted $T^{(\widehat{w}_1 s, \widehat{w}_1)}$, depending on $\widehat{w}_1, s$ and $\widehat{w}_1 s$, but not on c, such that $V^{\widehat{w}_1 s}(c) = T^{(\widehat{w}_1 s, \widehat{w}_1)} V^{\widehat{w}_1}(c)$, so extending from W to $\widehat{W}$ the content of Section 6.2.

We consider, for each $\mathbf{w} \in \mathbf{W}$, the set

$$\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s}(c) = \{\widehat{c}'' \in \widehat{\mathcal{C}}^{\widehat{w}_1 s}(c) \ : \ \rho_\omega(\widehat{c}'') = w_{c''}(\widehat{w}_1 s)(C_0), \ w_{c''} \in \mathbf{w}\}.$$

For every $\widehat{c}'' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s}(c)$, there exists a unique $\widehat{c}' \in \widehat{\mathcal{C}}_{\widehat{w}_1}(c)$, such that $\widehat{\delta}(\widehat{c}', \widehat{c}'') = \delta(c', c'') = s$; actually, if $\widehat{c}'' = (c'', \sigma'')$, we can choose $\widehat{c}' = (c', \sigma')$ with $\sigma' = \sigma''$. This fact suggests to define, for each $\widehat{c}' \in \widehat{\mathcal{C}}_{\widehat{w}_1}(c)$,

$$\begin{aligned}\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s, s}(\widehat{c}') &= \{\widehat{c}'' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s}(c) : \widehat{\delta}(\widehat{c}', \widehat{c}'') = s\} \\ &= \{\widehat{c}'' \in \widehat{\mathcal{C}}^{\widehat{w}_1 s}(c) : \widehat{\delta}(\widehat{c}', \widehat{c}'') = s, \ \rho_\omega(\widehat{c}'') = w_{c''}(\widehat{w}_1 s)(C_0), \ w_{c''} \in \mathbf{w}\}.\end{aligned}$$

Taking in account the decomposition $\widehat{\mathcal{C}}_{\widehat{w}_1}(c) = \bigcup_{\mathbf{u} \in \mathbf{W}} \widehat{\mathcal{C}}_{\mathbf{u}}^{\widehat{w}_1}(c)$, this definition implies the following decomposition of the set $\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s}(c)$ as disjoint union:

$$\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s}(c) = \bigcup_{\widehat{c}' \in \widehat{\mathcal{C}}_{\widehat{w}_1}(c)} \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s, s}(\widehat{c}') = \bigcup_{\mathbf{u} \in \mathbf{W}} \ \bigcup_{\widehat{c}' \in \widehat{\mathcal{C}}_{\mathbf{u}}^{\widehat{w}_1}(c)} \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s, s}(\widehat{c}'), \tag{7.2}$$

Lemma 7.2. — *Let $\widehat{w}_1 \in \widehat{W}$, $\widehat{w}_1 = w_1 g_1$, $s \in S$ and $r = \widehat{w}_1 s \widehat{w}_1^{-1}$. Assume $C_0 \prec \widehat{w}_1(C_0)$ and $|\widehat{w}s| = |\widehat{w}| + 1$. For every $\mathbf{u} \in \mathbf{W}$, let $\widehat{c}' = (c', \sigma') \in \widehat{\mathcal{C}}_{\mathbf{u}}^{\widehat{w}_1}(c)$. Then, for every $\mathbf{w} \in \mathbf{W}$,*

$$\sum_{\widehat{c}'' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s, s}(\widehat{c}')} \chi\chi_0^{1/2}(\rho_\omega(v_0(\widehat{c}''))) = \chi\chi_0^{1/2}(w_{c'} \widehat{w}_1(0)) \, T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 s, \widehat{w}_1)}, \tag{7.3}$$

where

$$T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 s, \widehat{w}_1)} = \begin{cases} q_s \ \chi\chi_0^{1/2}(\mathbf{u}(\widehat{w}_1 s(0) - \widehat{w}_1(0))) & \text{if } \mathbf{w} = \mathbf{u}, \mathbf{u} > \mathbf{ur}, \\ \chi\chi_0^{1/2}(\mathbf{u}(\widehat{w}_1 s(0) - \widehat{w}_1(0))) & \text{if } \mathbf{w} = \mathbf{u}, \mathbf{u} < \mathbf{u}\,\mathbf{r}, \\ q_s - 1 & \text{if } \mathbf{w} = \mathbf{ur} > \mathbf{u}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

Moreover $T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 s,\widehat{w}_1)}$ *only depends on the choice of* $\widehat{w}_1, s$ *and* $\mathbf{w}, \mathbf{u}$, *but it doesn't depend on the choice of the extended chamber* $\widehat{c}'$ *in the set* $\widehat{\mathcal{C}}_{\mathbf{u}}^{\widehat{w}_1}(c)$, *nor on* c.

Proof. — Let $\widehat{c}' = (c', \sigma')$. Since $\widehat{c}' \in \widehat{\mathcal{C}}_{\mathbf{u}}^{\widehat{w}_1}(c)$, then $\widehat{\delta}(c, \widehat{c}') = \widehat{w}_1 = w_1 g_1$ and $\rho_\omega(\widehat{c}') = w_{c'}(\widehat{w}_1(C_0))$, for some $w_{c'} \in \mathbf{u}$. Hence, if we consider the chamber c', then $\rho_\omega(c') = w_{c'}(w_1(C_0))$. Moreover, since $|w_1 s_1 g_1| = |\widehat{w}_1 s| = |\widehat{w}_1| + 1 = |w_1 g_1| + 1$, the chamber $w_1(C_0)$ is s_1-adjacent to $(w_1 s_1)(C_0)$, and hence also the chamber $w_{c'}(w_1(C_0))$ is s_1-adjacent to $w_{c'}(w_1 s_1(C_0))$. So we can apply the argument of Lemma 6.2 to characterize the set $\mathcal{C}_{\mathbf{w}}^{w_1 s_1, s_1}(c')$. On the other hand, an extended chamber $\widehat{c}''$ belongs to the set $\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s, s}(\widehat{c}')$ if and only if $\widehat{c}'' = (c'', \sigma'')$, with $\sigma'' = \sigma' = \sigma_1$ and $c'' \in \mathcal{C}_{\mathbf{w}}^{w_1 s_1, s_1}(c')$. So we distinguish two cases as in Lemma 6.2.

(I) If $w'(\widehat{w}_1 s(C_0)) \prec w'(\widehat{w}_1(C_0))$, then $w'(w_1 s_1(C_0)) \prec w'(w_1(C_0))$ and so, by Lemma 6.2,

$$\sum_{c'' \in \mathcal{C}_{\mathbf{w}}^{w_1 s_1, s_1}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(c''))) = \begin{cases} q_s\ \chi\chi_0^{1/2}(w_{c'}(w_1 s(v_0(C_0))) & \text{if } \mathbf{w} = \mathbf{u}, \\ 0 & \text{otherwise.} \end{cases}$$

In the same way, for each $j \in I$,

$$\sum_{c'' \in \mathcal{C}_{\mathbf{w}}^{w_1 s, s}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_j(c''))) = \begin{cases} q_s\ \chi\chi_0^{1/2}(w_{c'}(w_1 s(v_j(C_0))) & \text{if } \mathbf{w} = \mathbf{u}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, recalling that $v_0(\widehat{c}) = v_{\sigma(0)}(c)$, if $\widehat{c} = (c, \sigma)$,

$$\begin{aligned} \sum_{\widehat{c}'' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 s, s}(\widehat{c}')} \chi\chi_0^{1/2}(\rho_\omega(v_0(\widehat{c}''))) &= \sum_{c'' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{w_1 s_1, s_1}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_0(\widehat{c}''))) \\ &= \sum_{c'' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{w_1 s_1, s_1}(c')} \chi\chi_0^{1/2}(\rho_\omega(v_{\sigma_1(0)} c'')) \\ &= \begin{cases} q_s\ \chi\chi_0^{1/2}(w_{c'} w_1 s_1(v_{\sigma_1(0)}(C_0)) = q_s\ \chi\chi_0^{1/2}(w_{c'} w_1 s_1 g_1(0) & \text{if } \mathbf{w} = \mathbf{u}, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} q_s\ \chi\chi_0^{1/2}(w_{c'} w_1 s_1(v_{\sigma_1(0)}(C_0)) = q_s\ \chi\chi_0^{1/2}(w_{c'} \widehat{w}_1 s(0) & \text{if } \mathbf{w} = \mathbf{u}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(II) In the same way, if $w'(\widehat{w}_1(C_0)) \prec w'(\widehat{w}_1 s(C_0))$, we get the following formula:

$$\sum_{\widehat{c''}\in\widehat{\mathcal{C}}_{\mathbf{w}}^{w_1 s,s}(\widehat{c'})} \chi\chi_0^{1/2}(\rho_\omega(v_0(\widehat{c''})))$$

$$= \begin{cases} \chi\chi_0^{1/2}(w_{c'}(w_1 s_1)(v_{\sigma_1(0)}(C_0)) = \chi\chi_0^{1/2}(w_{c'}(\widehat{w}_1 s)(0) & \text{if } \mathbf{w}=\mathbf{u}, \\ (q_s-1)\chi\chi_0^{1/2}(w_{c'}(w_1(v_{\sigma_1(0)}(C_0)))) & \text{if } \mathbf{w}=\mathbf{ur}, \\ \qquad = (q_s-1)\chi\chi_0^{1/2}(w_{c'}(\widehat{w}_1(0))) & \\ 0 & \text{otherwise,} \end{cases}$$

since $r=\widehat{w}_1 s\widehat{w}^{-1}=w_1 s_1 w^{-1}$.

Consider, in both cases I and II, the vector $w_{c'}\widehat{w}_1 s(0)-w_{c'}\widehat{w}_1(0)$; since $w_{c'}\in\mathbf{u}$, we can write $w_{c'}=t_\lambda\mathbf{u}$, for some $\lambda\in L$. Therefore, as in Lemma 6.2 we have

$$w_{c'}\widehat{w}_1 s(0)-w_{c'}\widehat{w}_1(0)=\lambda+\mathbf{u}(\widehat{w}_1 s(0))-\lambda-\mathbf{u}(\widehat{w}_1(0))=\mathbf{u}(\widehat{w}_1 s(0))-\mathbf{u}(\widehat{w}_1(0)).$$

This equality shows that the vector is independent of the choice of $w_{c'}\in\mathbf{u}$, but only depends on $\mathbf{u}$. Going back to the previous formulas, we can write

$$\sum_{\widehat{c''}\in\widehat{\mathcal{C}}_{\mathbf{w}}^{w_1 s,s}(\widehat{c'})} \chi\chi_0^{1/2}(\rho_\omega(v_0(\widehat{c''})))=\chi\chi_0^{1/2}(w_{c'}\widehat{w}_1(0))\,T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 s,\widehat{w}_1)},$$

where

$$T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 s,\widehat{w}_1)}=\begin{cases} q_s\ \chi\chi_0^{1/2}(\mathbf{u}(\widehat{w}_1 s(0)-\widehat{w}_1(0))) & \text{if } \mathbf{w}=\mathbf{u},\mathbf{u}>\mathbf{ur}, \\ \chi\chi_0^{1/2}(\mathbf{u}(\widehat{w}_1 s(0)-\widehat{w}_1(0))) & \text{if } \mathbf{w}=\mathbf{u},\mathbf{u}<\mathbf{u}\,\mathbf{r}, \\ q_s-1 & \text{if } \mathbf{w}=\mathbf{ur}>\mathbf{u}, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 s,\widehat{w}_1)}$ only depends on the choice of $\widehat{w}_1, s$ and $\mathbf{w},\mathbf{u}$, but it doesn't depends on the choice of the extended chamber $\widehat{c}'$, nor of the choice of the initial chamber c. □

Theorem 7.3. — *Let $\widehat{w}_1\in\widehat{W}, \widehat{w}_1=w_1g_1$, and let $s=s_i$, for some $i\in I$; assume $C_0\prec\widehat{w}_1(C_0)$ and $|\widehat{w}_1 s|=|\widehat{w}_1|+1$. Denote $r=\widehat{w}_1 s\widehat{w}_1^{-1}$. Then, for every chamber c,*

$$V^{\widehat{w}_1 s}(c)=T^{(\widehat{w}_1 s,\widehat{w}_1)}V^{\widehat{w}_1}(c), \tag{7.5}$$

where $T^{(\widehat{w}_1 s,\widehat{w}_1)}$ is the sub-triangular non-singular matrix of order $d=|\mathbf{W}|$

$$T^{(w_1 s,w_1)}=(T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 s,\widehat{w}_1)})_{\mathbf{w},\mathbf{u}\in\mathbf{W}}.$$

Proof. — The required statement follows from Lemma 7.2 in the same way as Theorem 6.3 follows from Lemma 6.2. □

Proposition 7.4. — *Let $\widehat{w}_1 \in \widehat{W}, s \in S$ and $r = \widehat{w}_1 s \widehat{w}_1^{-1}$. The matrix $T^{(\widehat{w}_1 s, \widehat{w}_1)}$ only depends on $\mathbf{w}_1$ and s and, for every $\lambda \in \widehat{L}$,*

$$T^{(t_\lambda \widehat{w}_1 s, t_\lambda \widehat{w}_1)} = T^{(\widehat{w}_1 s, \widehat{w}_1)}.$$

Proof. — Write $\widehat{w}_1 = t_{\lambda_1}\mathbf{w}_1$; then

$$\begin{aligned}\widehat{w}_1 s(0) - \widehat{w}_1(0) &= t_{\lambda_1}\mathbf{w}_1 s(0) - t_{\lambda_1}\mathbf{w}_1(0) = \lambda_1 + \mathbf{w}_1 s(0) - \lambda_1 - \mathbf{w}_1(0)\\ &= \mathbf{w}_1 s(0) - \mathbf{w}_1(0).\end{aligned}$$

This proves that the matrix $T^{(\widehat{w}_1 s, \widehat{w}_1)}$ does not depend on the choice of $\widehat{w}_1$ in the class $\mathbf{w}_1$. Moreover, if $\widehat{w}_2 = t_\lambda \widehat{w}_1$, for any $\lambda \in \widehat{L}$, then $\mathbf{w}_2 = \mathbf{w}_1$, and so we conclude that $T^{(\widehat{w}_2 s, \widehat{w}_2)} = T^{(\widehat{w}_1 s, \widehat{w}_1)}$. □

7.3. The matrix $T^{(\widehat{w}_1 g, \widehat{w}_1)}$

Let $\widehat{w}_1 \in \widehat{W}$, say $\widehat{w}_1 = w_1 g_1$, and assume $C_0 \prec \widehat{w}_1(C_0)$; this is the case in particular when $\widehat{w}_1 = t_\lambda$, for $\lambda \in \widehat{L}^+$. Fix $g \in G$. Then $\widehat{w}_1 g = w_1 g_1 g = w_1 g_2$, if $g_2 = g_1 g$ and therefore $|\widehat{w}_1 g| = |\widehat{w}_1| = |w_1|$, because $|\widehat{w}| = |w|$ if $\widehat{w} = wg$.

If we consider the vectors $V^{\widehat{w}_1}(c)$ and $V^{\widehat{w}_1 g}(c)$, we shall construct a matrix $T^{(\widehat{w}_1 g, \widehat{w}_1)}$, depending on $\widehat{w}_1, g$, and $\widehat{w}_1 s$, but not on c, such that $V^{\widehat{w}_1 g}(c) = T^{(\widehat{w}_1 g, \widehat{w}_1)} V^{\widehat{w}_1}(c)$.

Theorem 7.5. — *Let $\widehat{w}_1 \in \widehat{W}$, $\widehat{w}_1 = w_1 g_1$, and $g \in G$; assume $C_0 \prec \widehat{w}_1(C_0)$. Then, for every chamber c,*

$$V^{\widehat{w}_1 g}(c) = T^{(\widehat{w}_1 g, \widehat{w}_1)} V^{\widehat{w}_1}(c), \tag{7.6}$$

where $T^{(\widehat{w}_1 g, \widehat{w}_1)} = (T^{(\widehat{w}_1 g, \widehat{w}_1)}_{\mathbf{w},\mathbf{u}})_{\mathbf{w},\mathbf{u}\in\mathbf{W}}$, and, for every $\mathbf{w}, \mathbf{u} \in \mathbf{W}$,

$$T^{(\widehat{w}_1 g, \widehat{w}_1)}_{\mathbf{w},\mathbf{u}} = \begin{cases} \chi\chi_0^{1/2}(\mathbf{u}(\widehat{w}_1 g(0) - \widehat{w}_1(0)) & \textit{if } \mathbf{w} = \mathbf{u},\\ 0 & \textit{otherwise.}\end{cases} \tag{7.7}$$

Proof. — We consider, for each $\mathbf{w} \in \mathbf{W}$, the set $\widehat{\mathcal{C}}^{\widehat{w}_1 g}_{\mathbf{w}}(c)$, defined in Section 7.1:

$$\widehat{\mathcal{C}}^{\widehat{w}_1 g}_{\mathbf{w}}(c) = \{\widehat{c}'' \in \mathcal{C}^{\widehat{w}_1 g}(c) \ : \ \rho_\omega(\widehat{c}'') = w_{c''}(\widehat{w}_1 g)(C_0),\ w_{c''} \in \mathbf{w}\}.$$

Since $\widehat{w}_1 g = w_1 g_1'$, for $g_1' = g_1 g$, then $\widehat{c}'' = (c'', \sigma'')$ belongs to the set $\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 g}(c)$ if and only if $c'' \in \mathcal{C}_{\mathbf{w}}^{w_1}(c)$ and $\sigma'' = \sigma_1 \sigma$. On the other hand $\widehat{c}' = (c', \sigma')$ belongs to the set $\widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1}(c)$ if and only if $c' \in \mathcal{C}_{\mathbf{w}}^{w_1}(c)$ and $\sigma' = \sigma_1$. Therefore if, for each $\widehat{c}'' = (c'', \sigma'') \in \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1 g}(c)$, we consider the chamber $\widehat{c}' = (c', \sigma')$ such that $c' = c''$ and $\sigma' = \sigma'' \sigma^{-1}$, then $\widehat{c}' \in \widehat{\mathcal{C}}_{\mathbf{w}}^{\widehat{w}_1}(c)$, but $v_0(w_{c''}(\widehat{w}_1 g(C_0)) = v_{\sigma(0)}(\widehat{w}_1(C_0))$. So, for each $\mathbf{w} \in \mathbf{W}$,

$$
\begin{aligned}
V_{\mathbf{w}}^{\widehat{w}_1 g}(c) &= \sum_{c' \in \mathcal{C}_{\mathbf{w}}^{w_1}(c)} \chi\chi_0^{1/2}(w_{c'} w_1 g_1 g(0)) \\
&= \sum_{c' \in \mathcal{C}_{\mathbf{w}}^{w_1}(c)} \chi\chi_0^{1/2}(w_{c'} \widehat{w}_1(0))\ \chi\chi_0^{1/2}(\mathbf{u}(\widehat{w}_1 g(v_0(C_0)) - \widehat{w}_1(v_0(C_0)))
\end{aligned}
$$

and we conclude that $V^{\widehat{w}_1 g}(c) = T^{(\widehat{w}_1 g, \widehat{w}_1)} V^{\widehat{w}_1}(c)$, if

$$
T_{\mathbf{w},\mathbf{u}}^{(\widehat{w}_1 g, \widehat{w}_1)} = \begin{cases} \chi\chi_0^{1/2}(\mathbf{u}(\widehat{w}_1 g(v_0(C_0)) - \widehat{w}_1(v_0(C_0))) & \text{if } \mathbf{w} = \mathbf{u}, \\ 0 & \text{otherwise.} \end{cases}
$$

□

7.4. The matrix $T^{\widehat{w}}$

Let $\widehat{w}_0 \in \widehat{W}$; we can split $\widehat{w}_0$ as $\widehat{w}_0 = w_0 g_0$, for some $w_0 \in W$ and $g_0 \in G$. Define, for $s_1, \ldots, s_n \in S$,

$$
\widehat{w}_1 = \widehat{w}_0 s_1, \quad \widehat{w}_2 = \widehat{w}_1 s_2 = \widehat{w}_0 s_1 s_2, \quad \cdots \quad \widehat{w}_n = \widehat{w}_{n-1} s_n g = \widehat{w}_0 s_1 \cdots s_n.
$$

We notice that if we set $s_j' = g_0^{-1} s_j g_0$ and $w_j = w_0 s_1' \cdots s_j'$, then $\widehat{w}_j = w_0 s_1' \cdots s_j' g_0 = w_j g_0$, for every $j = 1, \ldots, n$. We assume $\widehat{w}_0(C_0) \prec \widehat{w}_1(C_0) \prec \cdots \prec \widehat{w}_n(C_0)$, that is $w_0(C_0) \prec w_1(C_0) \prec \cdots \prec w_n(C_0)$. This implies that $|w_j| = |w_{j-1}| + 1$, that is $|\widehat{w}_j| = |\widehat{w}_{j-1}| + 1 = |\widehat{w}_0| + j$, for every $j = 1, \ldots, n$. In particular, when $\widehat{w}_0 = g_0$, then $s_1 = s_{\alpha_0}$ and $|w_j| = j$ for $j = 1, \ldots, n$. Theorem 7.3 and Theorem 7.5 assure that $V^{\widehat{w}_j}(c) = T^{(\widehat{w}_j, \widehat{w}_{j-1})}\ V^{\widehat{w}_{j-1}}(c)$, for each $j = 1, \ldots, n$, and $V^{\widehat{w}_0}(c) = T^{(\widehat{w}_0, w_0)}\ V^{w_0}(c)$. Consequently

$$
\begin{aligned}
V^{\widehat{w}_n}(c) &= T^{(\widehat{w}_n, \widehat{w}_{n-1})} \cdots T^{(\widehat{w}_1, \widehat{w}_0)}\ V^{\widehat{w}_0}(c) \\
&= T^{(\widehat{w}_n, \widehat{w}_{n-1})} \cdots T^{(\widehat{w}_1, \widehat{w}_0)}\ T^{(\widehat{w}_0, w_0)}\ V^{w_0}(c).
\end{aligned}
$$

In particular, if $\widehat{w}_0 = g_0$,

$$
\begin{aligned}
V^{\widehat{w}_n}(c) &= T^{(g_0 s_1 \cdots s_n, g_0 s_1 \cdots s_{n-1})} \cdots T^{(g_0 s_1, g_0)} V^{g_0}(c) \\
&= T^{(g_0 s_1 \cdots s_n, g_0 s_1 \cdots s_{n-1})} \cdots T^{(g_0 s_1, g_0)}\ T^{(g_0, e)} V^{e}(c).
\end{aligned}
$$

Proposition 7.6. — *Let $\widehat{w} \in \widehat{W}$, with $\widehat{w} = wg$ and $w = s_1 \cdots s_n$.*

(i) Let $g_0 \in G$; if $g_0(C_0) \prec g_0 s_1(C_0) \prec \cdots \prec g_0 s_1 \cdots s_n(C_0)$, then the matrix

$$T^{(g_0 s_1 \cdots s_n, g_0 s_1 \cdots s_{n-1})} \cdots T^{(g_0 s_1, g_0)} \; T^{(g_0, e)}$$

only depends on w, but is independent of $s_1, \cdots, s_n$.

(ii) Let $\widehat{w}_0 \in \widehat{W}$, $\widehat{w}_0 = w_0 g_0$ if $|\widehat{w}_0 \widehat{w}| = |\widehat{w}_0| + n$ and $C_0 \prec \widehat{w}_0(C_0) \prec \widehat{w}_0 \widehat{w}(C_0)$, then the matrix

$$T^{(\widehat{w_0}\widehat{w}, \widehat{w_0}\widehat{w}_{n-1})} \; T^{(\widehat{w_0}\widehat{w}_{n-1}, \widehat{w_0}\widehat{w}_{n-2})} \cdots T^{(\widehat{w_0}\widehat{w}_1, \widehat{w_0})}$$

only depends on $\widehat{w}_0$ and $\widehat{w}$, but is independent of their representation as reduced words.

The proof of this proposition is the same as that of Lemma 6.5 and we omit it.

Definition 7.7. — *Let $\widehat{w} \in \widehat{W}$, such that $C_0 \prec \widehat{w}(C_0)$. Assume $\widehat{w} = wg = gw'$, being $w' = \sigma^{-1}(w)$ and $w = s_1 \cdots s_n$. We denote by $T^{\widehat{w}}$ the following sub-triangular matrix of order $d = |W|$*

$$\begin{aligned} T^{\widehat{w}} &= T^{(gs'_1 \cdots s'_n, gs'_1 \cdots s'_{n-1})} \cdots T^{(gs'_1, g)} \; T^{(g,e)} \\ &= T^{(s_1 \cdots s_n g, s_1 \cdots s_{n-1} g)} \; \cdots \; T^{(s_1 g, g)} \; T^{(g,e)}. \end{aligned}$$

Moreover, for every $\widehat{w}_0 \in \widehat{W}$, $\widehat{w}_0 = w_0 g_0$, such that $|\widehat{w}_0 \widehat{w}| = |\widehat{w}_0| + n$ and $C_0 \prec \widehat{w}_0(C_0) \prec \widehat{w}_0 \widehat{w}(C_0)$, we set

$$T^{(\widehat{w_0}\widehat{w}, \widehat{w_0})} = T^{(\widehat{w_0}\widehat{w}, \widehat{w_0}\widehat{w}_{n-1})} \; T^{(\widehat{w_0}\widehat{w}_{n-1}, \widehat{w_0}\widehat{w}_{n-2})} \cdots T^{(\widehat{w_0}\widehat{w}_1, \widehat{w_0})}. \tag{7.8}$$

Corollary 7.8. — *Let $\widehat{w}_0, \widehat{w} \in \widehat{W}$, such that $\widehat{w}_0 = w_0 g_0$ and $\widehat{w} = wg$; assume $|\widehat{w}_0 \widehat{w}| = |\widehat{w}_0| + n$ and $C_0 \prec \widehat{w}_0(C_0) \prec \widehat{w}_0 \widehat{w}(C_0)$; then*

$$T^{\widehat{w_0}\widehat{w}} = T^{(\widehat{w_0}\widehat{w}, \widehat{w_0})} \; T^{\widehat{w_0}}; \tag{7.9}$$

moreover, if $C_0 \prec \widehat{w}(C_0)$, then, for every chamber c,

$$V^{\widehat{w}}(c) = T^{\widehat{w}} \; V^e(c). \tag{7.10}$$

Proof. — We notice that $\widehat{w}_0 \widehat{w} = w_0 g_0 w g = w_0 g_0 g w' = g_0 g w'_0 w'$, for suitable $w', w'_0 \in W$. Hence, if we set $\widehat{w}_1 = \widehat{w}_0 \widehat{w}$ then $\widehat{w}_1 = g_1 w_1$, with $w_1 = w'_0 w'$ and $g_1 = g_0 g$. Assume at first $g = e$; in this case $\widehat{w}_1 = \widehat{w}_0 w$ and then $V^{\widehat{w}_1}(c) = [T^{(\widehat{w}_1,\widehat{w}_0)}]\ V^{\widehat{w}_0}(c) = [T^{(\widehat{w}_1,\widehat{w}_0)}\ T^{\widehat{w}_0}]\ V^e(c)$, and this proves the required identity. Moreover the matrix $T^{(\widehat{w}_1,\widehat{w}_0)}$ is independent of the representation of $\widehat{w}_1$ and $\widehat{w}_0$ as reduced words, because this is true for the matrices $T^{\widehat{w}_1}$ and $T^{\widehat{w}_0}$. Otherwise, if $g \neq e$, then $\widehat{w}_1 = \widehat{w}_0 wg$, and hence $V^{\widehat{w}_1}(c) = [T^{(\widehat{w}_1,\widehat{w}_0 w)}]\ V^{\widehat{w}_0 w}(c) = [T^{(\widehat{w}_1,\widehat{w}_0 w)}\ T^{(\widehat{w}_0 w,\widehat{w}_0)}\ T^{\widehat{w}_0}]\ V^e(c)$, so the proposition is proved. □

The following corollary is a direct consequence of Proposition 7.4 and Definition 7.7.

Corollary 7.9. — *Let $\widehat{w}_0, \widehat{w} \in \widehat{W}$ and $\widehat{w}_1 = \widehat{w}_0\widehat{w}$. Assume $|\widehat{w}_1| = |\widehat{w}_0| + |\widehat{w}|$ and $C_0 \prec \widehat{w}_0(C_0) \prec \widehat{w}_1(C_0)$. Then, for every $\lambda \in \widehat{L}$,*

$$T^{(t_\lambda\widehat{w}_1, t_\lambda\widehat{w}_0)} = T^{(\widehat{w}_1,\widehat{w}_0)}.$$

7.5. The matrix T^λ for $\tau(\lambda) = j,\ j \in \widehat{I}$

Let $\lambda \in \widehat{L}^+$ and let $\tau(\lambda) = j$, for some $j \in \widehat{I}$. Then $t_\lambda \in \widehat{W}$, and $C_0 \prec t_\lambda(C_0)$. According to the Definition 7.7, we can define the non-singular sub-triangular matrix $T^\lambda = T^{t_\lambda}$, such that $V^{t_\lambda}(c) = T^\lambda V^e(c)$, for every chamber c.

Theorem 7.10. — *Let $\lambda \in \widehat{L}^+$ with $\tau(\lambda) = j,\ j \in \widehat{I}$; then, for every $\mathbf{u} \in \mathbf{W}$,*

$$T^\lambda_{\mathbf{u},\mathbf{u}} = \chi_0^{1/2}(\lambda)\chi^{\mathbf{u}}(\lambda). \tag{7.11}$$

Proof. — Let $t_\lambda = u_\lambda g_l$, where $u_\lambda = s_1 \cdots s_n$; for $j = 1, \dots, n$ we set, as usual,

$$\widehat{w}_1 = s_1 g_l,\ \ \widehat{w}_2 = \widehat{w}_1 s_2 g_l = s_1 s_2 g_l,\ \dots,\ t_\lambda = \widehat{w}_n = \widehat{w}_{n-1} s_n g_l = s_1 \cdots s_n g_l,$$

and $C_j = \widehat{w}_j(C_0)$. Since $\lambda \in \widehat{L}^+$, then $C_0 \prec C_1, \dots, C_{n-1} \prec C_n = t_\lambda(C_0)$. Moreover, for every $\mathbf{u} \in \mathbf{W}$,

$$T^{t_\lambda}_{\mathbf{u},\mathbf{u}} = T^{(\widehat{w}_{n-1}s_n,\widehat{w}_{n-1})}_{\mathbf{u},\mathbf{u}} \cdots T^{(s_1 g_l, g_l)}_{\mathbf{u},\mathbf{u}}\ T^{(g_l,e)}_{\mathbf{u},\mathbf{u}};$$

therefore, using Theorem 7.3 and Theorem 7.5 and keeping in mind (3.2), we prove (7.11) by the same argument used in Theorem 6.9. □

Proposition 7.11. — *Let $\lambda_1, \lambda_2 \in \widehat{L}^+$. Then*

$$T^{\lambda_1}\ T^{\lambda_2} = T^{\lambda_2}\ T^{\lambda_1}. \tag{7.12}$$

Proof. — It is easy to see that $|t_{\lambda_1+\lambda_2}| = |t_{\lambda_1}| + |t_{\lambda_2}|$, for every $\lambda_1, \lambda_2 \in \widehat{L}^+$. Therefore (7.12) follows from Corollaries 7.8 and 7.9, by the same argument used in Proposition 6.10. $\square$

Taking in account the proposition above, Theorem 6.11 and Corollary 6.12 can easily extended to every $\lambda \in \widehat{L}^+$.

COROLLARY 7.12. — *If χ is a good character, there exists a unique matrix C, independent of λ, such that $D^\lambda = CT^\lambda C^{-1}$ is diagonal, for every $\lambda \in \widehat{L}^+$, and for every chamber c,*

$$V^{t_\lambda}(c) = (C^{-1}D^\lambda C)\ V^e(c), \tag{7.13}$$

where $D^\lambda_{\mathbf{u},\mathbf{u}} = \chi_0^{1/2}(\lambda)\ \chi^{\mathbf{u}}(\lambda)$, for every $\mathbf{u} \in \mathbf{W}$.

7.6. Macdonald formula for spherical functions on all special vertices of a reduced building

We can extend to all vertices of $\widehat{\mathcal{V}}(\Delta)$ the formulas (6.18), (6.19) proved for the spherical function $\varphi_{\chi\chi_0^{1/2}}$ on vertices of type 0, when c is good.

THEOREM 7.13. — *Let Δ be a reduced building such that $\widehat{\mathcal{V}}(\Delta) \neq \mathcal{V}_0(\Delta)$. For every good character χ on $\widehat{L}$, there exist constants $\{c_{\mathbf{w}}(\chi), \mathbf{w} \in \mathbf{W}\}$, $\sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi) = 1$, such that*

$$\varphi_{\chi{\chi_0}^{1/2}}(x) = \chi_0^{-1/2}(\lambda) \sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}(\chi)\ \chi^{\mathbf{w}}(\lambda), \quad \forall x \in \mathcal{V}_\lambda(e),\ \forall \lambda \in \widehat{L}. \tag{7.14}$$

Moreover, for every $x \in \widehat{\mathcal{V}}(\Delta)$,

$$\varphi_{\chi{\chi_0}^{1/2}}(x) = \chi_0^{1/2}(\rho_0(x)) \sum_{\mathbf{w}\in\mathbf{W}} d_{\mathbf{w}}(\chi)\ \chi^{\mathbf{w}}(\rho_0(x)), \tag{7.15}$$

if $d_{\mathbf{w}}(\chi) = c_{\mathbf{w}\mathbf{w}_0}(\chi)$, for every $\mathbf{w} \in \mathbf{W}$.

Proof. — The proof is the same of Theorem 6.14 and Corollary 6.15. $\square$

Formulas (7.14) and (7.15) can be extended to all non-singular characters (see Section 8).

8. Computation of the coefficients of Macdonald formula

This section is devoted to the explicit computation of the coefficients $c_{\mathbf{w}}(\chi)$ and $d_{\mathbf{w}}(\chi)$ that appear in formulas (7.14) and (7.15) or equivalently in formulas (6.16) and (6.17) for the spherical function $\varphi_{\chi\chi_0^{1/2}}$ associated to a good character χ. Since, for every $\mathbf{w} \in \mathbf{W}$, $d_{\mathbf{w}}(\chi) = c_{\mathbf{w}\mathbf{w}_0}(\chi)$, we shall determinate $d_{\mathbf{w}}(\chi), \mathbf{w} \in \mathbf{W}$, and we shall deduce $c_{\mathbf{w}}(\chi)$ from them.

8.1. Preliminary results

For ease of notation, we shall denote by $X, Y, \ldots$ the vertices of $\mathcal{V}_0(\mathbb{A})$, while, as usual, we shall denote by $\lambda, \mu, \ldots$ the shape of any vertex Y with respect to any X.

We fix a simple root α; if the root system is non-reduced we assume that $\alpha \in R_1 \cup R_0$. Let q_α be the parameter associated to α; in particular $q_\alpha = q_{\alpha,1}$, if $\alpha \in R_1$. We consider, in the the fundamental apartment $\mathbb{A}$, the hyperplane H_α^1. Depending on the type of the building, such hyperplane may or may not contain vertices of type 0. We distinguish two cases.

(i) If H_α^1 contains vertices of type 0, also H_α^{-1} does. Choose in $H_\alpha^{-1} \cap \mathbb{Q}_0^-$ a vertex X of type 0 and define $Y_\alpha = s_\alpha X$. Then $Y_\alpha \in H_\alpha^1$ and $\tau(Y_\alpha) = 0$. Since X and Y_α are symmetric with respect to the hyperplane H_α^0, any minimal gallery $\Gamma(X, Y_\alpha)$ connecting them has an even number $2m_0$ of chambers and in particular the chambers C_{m_0} and C_{m_0+1} of this gallery have a common panel lying on H_α^0.

(ii) If H_α^1 does not contain vertices of type 0, surely H_α^2 does. Choose in $H_\alpha^0 \cap \mathbb{Q}_0^-$ a vertex X of type 0 and define $Y_\alpha = r_\alpha X$, if $r_\alpha = s_\alpha^1$. Then $Y_\alpha \in H_\alpha^2$ and $\tau(Y_\alpha) = 0$. Since X and Y_α are symmetric with respect to the hyperplane H_α^1, any minimal gallery $\Gamma(X, Y_\alpha)$ connecting them has an even number $2m_0$ of chambers and in particular the chambers C_{m_0} and C_{m_0+1} of this gallery have a common panel lying on H_α^1.

We set in both cases $\lambda = \sigma(X, Y_\alpha)$. For every $Y \in \mathcal{V}_\lambda(X)$, a minimal gallery $\Gamma(X, Y)$ has the same length and the same type as $\Gamma(X, Y_\alpha)$; moreover we can choose X far enough from 0, so that $\Pi_\lambda(X) \subset \mathbb{Q}_0^- \cup s_\alpha \mathbb{Q}_0^-$. This choice implies that Y_α is the unique element of $\mathcal{V}_\lambda(X)$ which does not belong to $\mathbb{Q}_0^-$, but lies on the interior part of the sector $s_\alpha \mathbb{Q}_0^-$.

Let ω be a fixed boundary point; we assign X and λ as above and we choose a vertex x in a sector $Q_e^-(\omega)$ opposite to $Q_e(\omega)$ in such a way that $\rho_0(x) = \rho_\omega(x) = X$. Let

$$N_\alpha = |\{y \in \mathcal{V}_\lambda(x),\ \rho_\omega(y) = Y_\alpha\}|. \tag{8.1}$$

We compare the retractions $\rho_0(y)$ and $\rho_\omega(y)$ of each $y \in \mathcal{V}_\lambda(x)$, separately in case (i) and (ii).

PROPOSITION 8.1. — *Assume that H^1_α contains vertices of type* 0. *Let $y \in \mathcal{V}_\lambda(x)$. Then*

$$\rho_0(y) = \begin{cases} s_\alpha \rho_\omega(y) = X & \text{if } \rho_\omega(y) = Y_\alpha, \\ \rho_\omega(y) & \text{otherwise.} \end{cases} \tag{8.2}$$

Proof. — For every $y \in \mathcal{V}_\lambda(x)$, let $\gamma(x,y) = \{c_1, ..., c_m = c\}$ be a minimal gallery from x to y and let $\delta(x,y) = \delta(c_1, c)$ be the element of W such that $c_1 \cdot \delta(x,y) = c$. There exists an isomorphism between any apartment containing x, y (and thus $\gamma(x,y)$) and $\mathbb{A}$, sending $\gamma(x,y)$ onto $\Gamma(X, Y_\alpha)$; then it occurs that $|\gamma(x,y)| = |\Gamma(X, Y_\alpha)| = 2m_0$ and $\delta(x,y) = \delta(X, Y_\alpha)$. Since $\rho_\omega(y) = v_0(\rho_\omega(c))$ and $\rho_0(y) = v_0(\rho_0(c))$, it is enough to compare $\rho_\omega(c)$ and $\rho_0(c)$, for $c \in \mathcal{C}_w(c_1)$ and $w = \delta(c_1, c)$.

Let $\mathcal{A}_1$ be an apartment containing c_1 and ω. In this apartment, $c_1 \in Q_e^-(\omega)$. Let k be the biggest integer such that the chamber c_k of $\gamma(x,y)$ lies on $Q_e^-(\omega)$.

1. If $k = 2m_0$, the whole gallery $\gamma(x,y)$ lies on $Q_e^-(\omega)$, and hence $y \in Q_e^-(\omega)$.

2. If $k < 2m_0$, then $c_1, \ldots, c_k \in Q_e^-(\omega)$, but $c_{k+1} \notin Q_e^-(\omega)$. We have to distinguish two cases.

   (a) The panel shared by c_k and c_{k+1} does not lie on the hyperplane h^0_α of $\mathcal{A}_1$. In this case, for a convenient apartment $\mathcal{A}_{k+1}$ containing c_{k+1} and $Q_e(\omega)$, the chamber c_{k+1} belongs to the sector Q_e^- of $\mathcal{A}_{k+1}$, opposite to $Q_e(\omega)$. We repeat the same reasoning for all other chambers $c_{k+2}, ..., c_{2m_0} = c$ and we get that, in a convenient apartment $\mathcal{A}_{2m_0}$ containing c and $Q_e(\omega)$, the chamber c belongs to the sector opposite to $Q_e(\omega)$.

   (b) The panel shared by c_k and c_{k+1} lies on the hyperplane h^0_α of $\mathcal{A}_1$. In this case $k = m_0$ and we can choose an apartment $\mathcal{A}_{m_0+1}$ containing c_{m_0+1} and $Q_e(\omega)$. We may have the following possibilities.

      (i) On $\mathcal{A}_{m_0+1}$, the chamber c_{m_0+1} belongs to the sector $Q_e^-(\omega)$. The same holds for the next chambers and then, in a convenient apartment $\mathcal{A}_{2m_0}$ containing c and $Q_e(\omega)$, the chamber c belongs to the sector $Q_e^-(\omega)$.

(ii) On $\mathcal{A}_{m_0+1}$, the chamber c_{m_0+1} belongs to the sector $(Q_e^\alpha)^-(\omega)$, $\alpha-$adjacent to $Q_e^-(\omega)$. This can happen only if there exists an apartment $\mathcal{A}_1$ containing $c_1, ..., c_{m_0+1}$, that is $\mathcal{A}_{m_0+1} = \mathcal{A}_1$. In this case $proj_0(c_{m_0+1})$ is the base chamber of $(Q_e^\alpha)^-(\omega)$ in the apartment $\mathcal{A}_1$ and the same happens for all the chambers $c_{m_0+2}, ..., c_{2m_0} = c$; therefore $proj_0(c) = proj_0(c_{m_0+1})$ and c belongs to the sector $(Q_e^\alpha)^-(\omega)$, in a convenient apartment containing c and $Q_e(\omega)$.

In cases (1), (2 a) and (2 b i) we can conclude, by Lemma 2.6 (i), that $\rho_\omega(c) = \rho_0(c)$ and then $\rho_\omega(y) = \rho_0(y)$, since c and y belong to a sector $Q_e^-(\omega)$ opposite to $Q_e(\omega)$.

Consider now the case (2 b ii); since c belongs to the sector $(Q_e^\alpha)^-(\omega)$, in a convenient apartment containing c and $Q_e(\omega)$, then $\rho_\omega(y)$ belongs to $(\mathbb{Q}_0^\alpha)^-$. As we remarked in Section 2, $\rho_\omega(y) \in \Pi_\lambda(X)$; moreover the choice of X implies that Y_α is the unique element of $\Pi_\lambda(X)$ in the interior part of $(Q_e^\alpha)^-$. Hence, in this case, either $\rho_\omega(y) = Y_\alpha$, or $\rho_\omega(y) \in H_\alpha^0$. On the other hand, Lemma 2.6 (ii) assures that $\rho_0(c) = s_\alpha \rho_\omega(c)$ and hence $\rho_0(y) = s_\alpha \rho_\omega(y)$. Therefore we conclude that, if $\rho_\omega(y) = Y_\alpha$, then $\rho_0(y) = X$, while, if $\rho_\omega(y) \in H_\alpha^0$, then $\rho_\omega(y) = s_\alpha \rho_\omega(y) = \rho_0(y)$. So (8.2) is proved. □

Proposition 8.2. — *Assume that H_α^1 does not contain vertices of type 0. Then there exist $(q_\alpha - 1)N_\alpha$ vertices $y \in \mathcal{V}_\lambda(x)$, such that $\rho_\omega(y) = X$. Moreover, if $y \in \mathcal{V}_\lambda(x)$,*

$$\rho_0(y) = \begin{cases} s_\alpha Y_\alpha & \text{if } \rho_\omega(y) = Y_\alpha \text{ or } \rho_\omega(y) = X, \\ \rho_\omega(y) & \text{otherwise.} \end{cases} \tag{8.3}$$

Proof. — As in Proposition 8.1, we call $\mathcal{A}_1$ any apartment containing c_1 and ω. In this apartment either $c_1 \in Q_e^-(\omega)$ or $c_1 \in (Q_e^\alpha)^-(\omega)$.

1. Assume that $c_1 \in Q_e^-(\omega)$. In this case, if $k < 2m_0$, the panel shared by c_k and c_{k+1} cannot lie on the hyperplane h_α^0 of $\mathcal{A}_1$. Hence we can repeat the argument of case (1) or case (2, a) of Proposition 8.1 and we conclude that c belongs to the sector $Q_e^-(\omega)$.

2. Assume now that $c_1 \in (Q_e^\alpha)^-(\omega)$. In this case, c_1 lies between the hyperplanes h_α^0 and h_α^1 of $\mathcal{A}_1$. We distinguish three cases.
   (a) If the whole gallery $\gamma(x, y)$ lies between the hyperplanes h_α^0 and h_α^1, then $c_{2m_0} = c$ does and so $y \in h_\alpha^0$, because $y \notin h_\alpha^1$.

(b) If, for some $1 < k < 2m_0$, the chamber c_{k+1} does not lie between h_α^0 and h_α^1 and the panel shared by c_k and c_{k+1} is not on h_α^1, there exists a convenient apartment containing c_{k+1} and $Q_e(\omega)$, where c_{k+1} lies between h_α^0 and h_α^1. We can repeat the reasoning for $c_{k+2}, ..., c$ and we get that, in a convenient apartment, $c \in (Q_e^\alpha)^-(\omega)$ lies between h_α^0 and h_α^1. Then we conclude that $y \in h_\alpha^0$.

(c) Assume finally that, for some $1 < k < 2m_0$, the chamber c_{k+1} does not lie between h_α^0 and h_α^1, but the panel shared by c_k and c_{k+1} lies on h_α^1; this means that $k = m_0$. In this third case we have the following two possibilities.

  (i) c_{m_0+1} lies on the sector $(Q_e^\alpha)^-(\omega)$; then the same is true for all next chambers and then also for c.

  (ii) c_{m_0+1} does not belong to $\mathcal{A}_1$. In this case the apartment $\mathcal{A}_{m_0+1}$ does not contain $Q_e(\omega)$, because it does not contain neither e nor c_ω; moreover, on this apartment, c_{m_0+1} lies between h_α^0 and h_α^1. Then c cannot lie on an apartment containing $Q_e(\omega)$. Again $y \in h_\alpha^0$, but c_{m_0+1} does not belong to a sector of type $(Q_e^\alpha)^-(\omega)$.

In case (1), according to Lemma 2.6 (i), $\rho_\omega(c) = \rho_0(c)$ and then $\rho_\omega(y) = \rho_0(y)$, since c and y belong to a sector opposite to $Q_e(\omega)$.

In both cases (2, a) and (2, b), the chamber c belongs to $(Q_e^\alpha)^-(\omega)$; hence $\rho_\omega(c) = s_\alpha \rho_0(c)$, by Lemma 2.6 (ii), but $\rho_\omega(y) = s_\alpha \rho_0(y) = \rho_0(y)$, because y belongs to the wall of the sector shared by $(Q_e)^-(\omega)$.

In case (2, c, i) Lemma 2.6 (ii) implies, as above, that $\rho_\omega(c) = s_\alpha \rho_0(c)$ and also $\rho_\omega(y) = s_\alpha \rho_0(y)$, because c and y belong to the interior part of $(Q_e^\alpha)^-(\omega)$. Since Y_α is the unique element of $\Pi_\lambda(X)$ which lies into the interior part of $(Q_e^\alpha)^-(\omega)$, and $\rho_\omega(y)$ belongs to $\Pi_\lambda(X)$, then $\rho_\omega(y) = Y_\alpha$. Moreover $\rho_0(y) = s_\alpha Y_\alpha = X$.

Finally we examine more closely the last case (2, c, ii), to find the relationship between $\rho_\omega(y)$ and $\rho_0(y)$. Let $\widetilde{\mathcal{A}}$ be an apartment containing c and ω. Thus $\widetilde{\mathcal{A}}$ and $\mathcal{A}_1$ share the hyperplane h_α^1 and h_α^j, for $j > 1$, but $\widetilde{\mathcal{A}}$ does not contain c_ω nor $proj_0(c)$. Let i be the isomorphism between $\widetilde{\mathcal{A}}$ and $\mathcal{A}_1$ who fixes $\widetilde{\mathcal{A}} \cap \mathcal{A}_1$. We have $c_{m_0+1} = i(c_{m_0}),\ c_{m_0+2} = i(c_{m_0-1}), \dots, c = c_{2m_0} = i(c_1)$. Therefore

$$\rho_\omega(c_{m_0+1}) = \rho_\omega(c_{m_0}),\ \rho_\omega(c_{m_0+2}) = \rho_\omega(c_{m_0-1}), \dots, \rho_\omega(c) = \rho_\omega(c_1).$$

Call $\widetilde{c}$ the chamber of $\widetilde{\mathcal{A}} \cap \mathcal{A}_1$ such that $\gamma(c_1, \widetilde{c}) = \gamma(c_1, c)$. Then $\delta(c_1, \widetilde{c}) = \delta(c_1, c)$ and $\rho_0(\widetilde{c}) = \rho_0(c)$, since $proj_0(\widetilde{c}) = proj_0(c)$. Moreover

$$\delta(proj_0(\widetilde{c}), \widetilde{c}) = \delta(proj_0(\widetilde{c}), c_1)\delta(c_1, \widetilde{c}) = \delta(proj_0(c), c_1)\delta(c_1, c) = \delta(proj_0(c), c).$$

We have $\rho_0(\widetilde{c}) = s_\alpha \rho_\omega(\widetilde{c})$ and $\rho_\omega(\widetilde{c}) = r_\alpha \rho_\omega(c_1)$, if r_α is the reflexion with respect to H^1_α. Hence $\rho_0(\widetilde{c}) = s_\alpha r_\alpha \rho_\omega(c_1)$ and $\rho_0(c) = s_\alpha r_\alpha \rho_\omega(c)$, or equivalently $\rho_\omega(c) = r_\alpha s_\alpha \rho_0(c)$. Therefore $\rho_\omega(y) = r_\alpha s_\alpha \rho_0(y)$. Since $r_\alpha s_\alpha$ is a translation, mapping $s_\alpha Y_\alpha$ to X, we conclude that

$$\rho_\omega(y) = X \;\text{ and } \rho_0(y) = s_\alpha Y_\alpha.$$

Finally, we observe that $|\{y : \rho_\omega(y) = X, \rho_0(y) = s_\alpha Y_\alpha\}| = (q_\alpha - 1)N_\alpha$. In fact, if $c_1 \in (Q^\alpha_e)^-(\omega)$, then $c_{m_0+1} \in \mathcal{A}_1$, in the case (2, c, i), while $c_{m_0+1} \notin \mathcal{A}_1$, in the case (2, c, ii). Since, fixed $c_1, \ldots, c_{m_0}$, there is a unique c_{m_0+1} with property (2, c, i), while all the others $q_\alpha - 1$ chambers adjacent to c_{m_0} have property (2, c, ii) , we conclude that there exist $(q_\alpha - 1)N_\alpha$ vertices in $\mathcal{V}_\lambda(x)$ such that $\rho_\omega(y) = X$ and $\rho_0(y) = s_\alpha Y_\alpha$. $\square$

8.2. Fundamental equations for $d_{\mathbf{w}}$, $\mathbf{w} \in \mathbf{W}$

Let χ be a good character. Fix $\omega \in \Omega$.

Definition 8.3. — *For every $x \in \mathcal{V}_0(\Delta)$, we define*

$$\tau_{\chi\chi_0^{1/2}}(x) = \frac{1}{\mathbf{W}(q)} \; {\chi_0}^{1/2}(\rho_\omega(x)) \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}}(\chi)\chi^{\mathbf{w}}(\rho_\omega(x)). \tag{8.4}$$

Since $\chi^{\mathbf{w}}{\chi_0}^{1/2}(\rho_\omega(x)) = P^{\chi^{\mathbf{w}}{\chi_0}^{1/2}}(x, \omega)$ and $P^{\chi^{\mathbf{w}}{\chi_0}^{1/2}}(\cdot, \omega)$ is an eigenfunction of the operator algebra $\mathcal{H}(\Delta)$ associated to the eigenvalue $\Lambda_{\chi^{\mathbf{w}}{\chi_0}^{1/2}} = \Lambda_{\chi{\chi_0}^{1/2}}$, for every $\mathbf{w} \in \mathbf{W}$, then the function $\tau_{\chi\chi_0^{1/2}}$, as well as the spherical function $\varphi_{\chi\chi_0^{1/2}}$, belongs to the eigenspace of the algebra $\mathcal{H}(\Delta)$ associated to the eigenvalue $\Lambda_{\chi{\chi_0}^{1/2}}$. Moreover we notice that $\tau_{\chi\chi_0^{1/2}}(x) = \varphi_{\chi\chi_0^{1/2}}(x)$, for all x lying on any sector $Q^-_e(\omega)$, because, by Lemma 2.6, $\rho_\omega(x) = \rho_0(x)$, for all $x \in Q^-_e(\omega)$. Hence, for every $\lambda \in \widehat{L}^+$ and for every $x \in Q^-_e(\omega)$, $A_\lambda \; \tau_{\chi\chi_0^{1/2}} = \Lambda_{\chi{\chi_0}^{1/2}}(\lambda) \; \tau_{\chi\chi_0^{1/2}} = \Lambda_{\chi{\chi_0}^{1/2}}(\lambda) \; \varphi_{\chi\chi_0^{1/2}} = A_\lambda \; \varphi_{\chi\chi_0^{1/2}}$. This implies that

$$\sum_{y \in \mathcal{V}_\lambda(x)} \left[\varphi_{\chi\chi_0^{1/2}}(y) - \tau_{\chi\chi_0^{1/2}}(y)\right] = 0$$

and, by (8.4) and (7.15),

$$\sum_{y \in \mathcal{V}_\lambda(x)} \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}}(\chi) \left[\chi^{\mathbf{w}}{\chi_0}^{1/2}(\rho_0(y)) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(\rho_\omega(y))\right] = 0. \tag{8.5}$$

For each $x \in Q_e^-(\omega)$ and each $\lambda \in \widehat{L}^+$, formula (8.5) can be interpreted as a linear equation for the coefficients $d_{\mathbf{w}}(\chi), \mathbf{w} \in \mathbf{W}$. We shall choose finitely many x and λ in such a way that the expression of the associated equation (8.5) becomes easy enough to manage it, so producing a finite number of equations which we shall be able to solve, so giving an explicit computation of $d_{\mathbf{w}}(\chi), \mathbf{w} \in \mathbf{W}$. For ease of notation we shall simply denote $d_{\mathbf{w}} = d_{\mathbf{w}}(\chi)$.

For each simple root α, we assign X and λ as in Section 8.1 and we choose in a sector $Q_e^-(\omega)$ opposite to $Q_e(\omega)$ a vertex x such that $\rho_0(x) = \rho_\omega(x) = X$.

Lemma 8.4. — *Let α be a simple root.*

(i) Assume that H_α^1 contains vertices of type 0. *Then*

$$\sum_{y \in \mathcal{V}_\lambda(x)} \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}} [\chi^{\mathbf{w}} {\chi_0}^{1/2}(\rho_0(y)) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(\rho_\omega(y))]$$

$$= N_\alpha \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}} [\chi^{\mathbf{w}} {\chi_0}^{1/2}(X) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha)].$$

(ii) Assume that H_α^1 does not contain vertices of type 0. *Then*

$$\sum_{y \in \mathcal{V}_\lambda(x)} \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}} [\chi^{\mathbf{w}} {\chi_0}^{1/2}(\rho_0(y)) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(\rho_\omega(y))]$$

$$= N_\alpha \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}} [\chi^{\mathbf{w}} {\chi_0}^{1/2}(s_\alpha Y_\alpha) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha)]$$

$$+ (q_\alpha - 1) N_\alpha \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}} [\chi^{\mathbf{w}} {\chi_0}^{1/2}(s_\alpha Y_\alpha) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(X)].$$

Proof. — Proposition 8.1 and Proposition 8.2 imply (i) and (ii) respectively. □

For each simple root α, we set

$$\mathbf{W}_\alpha^+ = \{\mathbf{w} \in \mathbf{W} \ : \ |\mathbf{w} s_\alpha| = |\mathbf{w}| + 1\}, \quad \mathbf{W}_\alpha^- = \{\mathbf{w} \in \mathbf{W} \ : \ |\mathbf{w} s_\alpha| = |\mathbf{w}| - 1\}.$$

Then $\mathbf{W} = \mathbf{W}_\alpha^+ \cup \mathbf{W}_\alpha^-$ as disjoint union. We observe that $\mathbf{w}' = \mathbf{w} s_\alpha \in \mathbf{W}_\alpha^-$, for all $\mathbf{w} \in \mathbf{W}_\alpha^+$, and, for every $\mathbf{w}' \in \mathbf{W}_\alpha^-$, there exists a unique $\mathbf{w} \in \mathbf{W}_\alpha^+$, such that $\mathbf{w}' = \mathbf{w} s_\alpha$.

PROPOSITION 8.5. — *Let α be a simple root.*

(i) If H^1_α contains vertices of type 0*, then, for all $Z \in H^0_\alpha \cap \mathbb{Q}_0^-$, $\tau(Z) = 0$,*

$$\sum_{\mathbf{w}\in\mathbf{W}^+_\alpha} \{ \; d_{\mathbf{w}} \; [\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y^0_\alpha) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(X_0)]$$

$$+ d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y^0_\alpha) - \chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X_0)] \; \}\chi^{\mathbf{w}}(Z) = 0.$$

(ii) If H^1_α does not contain vertices of type 0*, then, for all $Z \in H^0_\alpha \cap \mathbb{Q}_0^-$, $\tau(Z) = 0$,*

$$\sum_{\mathbf{w}\in\mathbf{W}^+_\alpha} \{ \; d_{\mathbf{w}} \; [\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y^0_\alpha) + (q_\alpha - 1)\chi^{\mathbf{w}}{\chi_0}^{1/2}(X_0) - q_\alpha\chi^{\mathbf{w}}{\chi_0}^{1/2}(s_\alpha Y^0_\alpha)]$$
$$+ d_{\mathbf{w}s_\alpha} \; [\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y^0_\alpha) + (q_\alpha - 1)\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X_0) - q_\alpha\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(s_\alpha Y^0_\alpha)] \; \}$$
$$\chi^{\mathbf{w}}(Z) = 0.$$

Proof. — We prove separately (i) and (ii).

(i) According to Proposition 8.1,

$$\begin{aligned}
&\sum_{\mathbf{w}\in\mathbf{W}} d_{\mathbf{w}}[\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(X)] \\
&\quad = \sum_{\mathbf{w}\in\mathbf{W}^+_\alpha} d_{\mathbf{w}}[\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(X)] \\
&\qquad + \sum_{\mathbf{w}'\in\mathbf{W}^-_\alpha} d_{\mathbf{w}'}[\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(X)] \\
&\quad = \sum_{\mathbf{w}\in\mathbf{W}^+_\alpha} \{d_{\mathbf{w}}[\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(X)] \\
&\qquad + d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y_\alpha) - \chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X)]\},
\end{aligned}$$

where $\mathbf{w}' = \mathbf{w}s_\alpha$. Hence, by Lemma 8.4 (i),

$$\sum_{\mathbf{w}\in\mathbf{W}^+_\alpha} \{d_{\mathbf{w}}[\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(X)]$$

$$+ d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y_\alpha) - \chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X)]\} = 0.$$

Fix $X_0 \in H^{-1}_\alpha \cap \mathbb{Q}_0^-$, $\tau(X_0) = 0$. We may choose X_0 far enough from the origin so to have $\Pi_\lambda(X_0) \subset \mathbb{Q}_0^- \cup s_\alpha\mathbb{Q}_0^-$. Consider $X = X_0 + Z$, with $Z \in H_\alpha^0 \cap \mathbb{Q}_0^-$. Then X belongs to $H_\alpha^{-1} \cap \mathbb{Q}_0^-$ and is far enough from the origin so to have $\Pi_\lambda(X) \subset \mathbb{Q}_0^- \cup s_\alpha \mathbb{Q}_0^-$. Moreover

$$Y_\alpha = s_\alpha X = s_\alpha(X_0 + Z) = s_\alpha X_0 + s_\alpha Z = Y_0^\alpha + Z.$$

Choosing X this way, our formula becomes, for all $Z \in H_\alpha^0 \cap \mathbb{Q}_0^-$,

$$\sum_{\mathbf{w} \in \mathbf{W}_\alpha^+} \{d_{\mathbf{w}} {\chi_0}^{1/2}(Z)[\chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha^0) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(X_0)]\chi^{\mathbf{w}}(Z)$$

$$+ d_{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(Z)[\chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(Y_\alpha^0) - \chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(X_0)]\chi^{\mathbf{w}s_\alpha}(Z)\} = 0.$$

Since $Z = s_\alpha Z$, we get, simplifying by ${\chi_0}^{1/2}(Z)$,

$$\sum_{\mathbf{w} \in \mathbf{W}_\alpha^+} \{d_{\mathbf{w}}[\chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha^0) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(X_0)]$$

$$+ d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(Y_\alpha^0) - \chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(X_0)]\}\chi^{\mathbf{w}}(Z) = 0.$$

(ii) According to Proposition 8.2,

$$\sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}}[\chi^{\mathbf{w}} {\chi_0}^{1/2}(s_\alpha Y_\alpha) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha)]$$

$$+ (q_\alpha - 1) \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}}[\chi^{\mathbf{w}} {\chi_0}^{1/2}(s_\alpha Y_\alpha) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(X)]$$

$$= \sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}}[q_\alpha \chi^{\mathbf{w}} {\chi_0}^{1/2}(s_\alpha Y_\alpha) - (q_\alpha - 1)\chi^{\mathbf{w}} {\chi_0}^{1/2}(X) - \chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha)]$$

$$= - \sum_{\mathbf{w} \in \mathbf{W}_\alpha^+} \{d_{\mathbf{w}}[\chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha) + (q_\alpha - 1)\chi^{\mathbf{w}} {\chi_0}^{1/2}(X) - q_\alpha \chi^{\mathbf{w}} {\chi_0}^{1/2}(s_\alpha Y_\alpha)]$$

$$+ d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(Y_\alpha) + (q_\alpha - 1)\chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(X) - q_\alpha \chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(s_\alpha Y_\alpha)]\}.$$

Hence, by Lemma 8.4 (ii),

$$\sum_{\mathbf{w} \in \mathbf{W}_\alpha^+} \{d_{\mathbf{w}}[\chi^{\mathbf{w}} {\chi_0}^{1/2}(Y_\alpha) + (q_\alpha - 1)\chi^{\mathbf{w}} {\chi_0}^{1/2}(X) - q_\alpha \chi^{\mathbf{w}} {\chi_0}^{1/2}(s_\alpha Y_\alpha)]$$

$$+ d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(Y_\alpha) + (q_\alpha - 1)\chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(X) - q_\alpha \chi^{\mathbf{w}s_\alpha} {\chi_0}^{1/2}(s_\alpha Y_\alpha)]\}$$

$$= 0.$$

As in case (i), fix $X_0 \in H_\alpha^{-1} \cap \mathbb{Q}_0^-$, far enough from the origin so that $\Pi_\lambda(X_0) \subset \mathbb{Q}_0^- \cup s_\alpha \mathbb{Q}_0^-$, and consider $X = X_0 + Z$, with $Z \in H_\alpha^0 \cap \mathbb{Q}_0^-$. Then X belongs to $H_\alpha^{-1} \cap \mathbb{Q}_0^-$ and is far enough from the origin so to have $\Pi_\lambda(X) \subset \mathbb{Q}_0^- \cup s_\alpha \mathbb{Q}_0^-$. If we set $Y_0^\alpha = r_\alpha X_0$, then $Y_\alpha - X = Y_0^\alpha - X_0$; so $Y_\alpha - Y_0^\alpha = X - X_0 = Z$ and therefore $Y_\alpha = Y_0^\alpha + Z$; moreover $s_\alpha Y_\alpha = s_\alpha Y_0^\alpha + Z$. Choosing X this way, our formula becomes, for all $Z \in H_\alpha^0 \cap \mathbb{Q}_0^-$,

$$\sum_{\mathbf{w}\in\mathbf{W}_\alpha^+} \{d_\mathbf{w} {\chi_0}^{1/2}(Z)\chi^\mathbf{w}(Z)[\chi^\mathbf{w}{\chi_0}^{1/2}(Y_\alpha^0)$$

$$+(q_\alpha - 1)\chi^\mathbf{w}{\chi_0}^{1/2}(X_0) - q_\alpha\chi^\mathbf{w}{\chi_0}^{1/2}(s_\alpha Y_\alpha^0)]$$
$$+d_{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Z)\chi^\mathbf{w}(Z)[\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y_\alpha^0)$$
$$+(q_\alpha - 1)\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X_0) - q_\alpha\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(s_\alpha Y_\alpha^0)]\} = 0.$$

Hence, simplifying by ${\chi_0}^{1/2}(Z)$, we get

$$\sum_{\mathbf{w}\in\mathbf{W}_\alpha^+} \{d_\mathbf{w}[\chi^\mathbf{w}{\chi_0}^{1/2}(Y_\alpha^0) + (q_\alpha - 1)\chi^\mathbf{w}{\chi_0}^{1/2}(X_0) - q_\alpha\chi^\mathbf{w}{\chi_0}^{1/2}(s_\alpha Y_\alpha^0)]$$

$$+d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y_\alpha^0)+(q_\alpha-1)\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X_0)-q_\alpha\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(s_\alpha Y_\alpha^0)]\}$$
$$\chi^\mathbf{w}(Z) = 0.$$

□

We assume now that χ is α-good; according to Definition 3.6, there exists a vertex Z on the hyperplane H_α^0, such that $\chi^{\mathbf{w}_1}(Z) = \chi^{\mathbf{w}_2}(Z)$ if and only if $\mathbf{w}_2 = \mathbf{w}_1$ or $\mathbf{w}_2 = \mathbf{w}_1 s_\alpha$. We suppose, without loss of generality that Z belongs to $H_\alpha^0 \cap \mathbb{Q}_0^-$.

Theorem 8.6. — *Let α be a simple root. Let χ be an α-good character.*

(i) If the hyperplane H_α^1 contains vertices of type 0, then, for every $\mathbf{w} \in \mathbf{W}_\alpha^+$,

$$d_\mathbf{w}\left[\chi^\mathbf{w}{\chi_0}^{1/2}(\alpha^\vee) - 1\right] + d_{\mathbf{w}s_\alpha}\left[\chi_0^{1/2}(\alpha^\vee) - \chi^\mathbf{w}(\alpha^\vee)\right] = 0.$$

(ii) If the hyperplane H_α^1 does not contain vertices of type 0, then, for every $\mathbf{w} \in \mathbf{W}_\alpha^+$,

$$d_\mathbf{w}\left[\left(\chi^\mathbf{w}{\chi_0}^{1/2}(\alpha^\vee) - 1\right)\left(\chi^\mathbf{w}{\chi_0}^{1/2}(\alpha^\vee) + q_\alpha\right)\right]$$
$$+d_{\mathbf{w}s_\alpha}\left[\left({\chi_0}^{1/2}(\alpha^\vee) - \chi^\mathbf{w}(\alpha^\vee)\right)\left({\chi_0}^{1/2}(\alpha^\vee) + q_\alpha\chi^\mathbf{w}(\alpha^\vee)\right)\right] = 0.$$

Proof. — The assumption that χ is α-good assures that the functions $\chi^{\mathbf{w}}$, for $\mathbf{w} \in \mathbf{W}_\alpha^+$, when restricted to the hyperplane H_α^0, are linearly independent (see [5, Lemma (4.5.7)]); therefore $\sum_{\mathbf{w}\in\mathbf{W}_\alpha^+} k_{\mathbf{w}}\chi^{\mathbf{w}}(Z) = 0$ implies that $k_{\mathbf{w}} = 0$, for every $\mathbf{w} \in \mathbf{W}_\alpha^+$. Applying this argument to the sums in Proposition 8.5, we get the following results according to case (i) or case (ii).

(i) If the hyperplane H_α^1 contains vertices of type 0, then, for every $\mathbf{w} \in \mathbf{W}_\alpha^+$,

$$d_{\mathbf{w}}\ [\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha^0) - \chi^{\mathbf{w}}{\chi_0}^{1/2}(X_0)]$$

$$+d_{\mathbf{w}s_\alpha}[\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y_\alpha^0) - \chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X_0)] = 0$$

and

$$d_{\mathbf{w}}\left[\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha^0 - X_0) - 1\right] + d_{\mathbf{w}s_\alpha}\left[\chi_0^{1/2}(Y_\alpha^0 - X_0) - \chi^{\mathbf{w}}(Y_\alpha^0 - X_0)\right] = 0.$$

Since $Y_\alpha^0 - X_0 = \alpha^\vee$, we get the result.

(ii) If the hyperplane H_α^1 does not contain vertices of type 0, then, for every $\mathbf{w} \in \mathbf{W}_\alpha^+$,

$$d_{\mathbf{w}}\ [\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha^0) + (q_\alpha - 1)\chi^{\mathbf{w}}{\chi_0}^{1/2}(X_0) - q_\alpha\chi^{\mathbf{w}}{\chi_0}^{1/2}(s_\alpha Y_\alpha^0)]$$

$$+d_{\mathbf{w}s_\alpha}\ [\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(Y_\alpha^0) + (q_\alpha - 1)\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(X_0) - q_\alpha\chi^{\mathbf{w}s_\alpha}{\chi_0}^{1/2}(s_\alpha Y_\alpha^0)]$$

$$= 0$$

and

$$d_{\mathbf{w}}\left[\chi^{\mathbf{w}}{\chi_0}^{1/2}(Y_\alpha^0 - X_0) + (q_\alpha - 1) - q_\alpha\chi^{\mathbf{w}}{\chi_0}^{1/2}(s_\alpha(Y_\alpha^0 - X_0))\right]$$

$$+d_{\mathbf{w}s_\alpha}\left[\chi^{\mathbf{w}}(s_\alpha(Y_0^\alpha - X_0)){\chi_0}^{1/2}(Y_\alpha^0 - X_0) + (q_\alpha - 1)\right.$$

$$\left.-q_\alpha\chi^{\mathbf{w}}(Y_0^\alpha - X_0){\chi_0}^{1/2}(s_\alpha(Y_\alpha^0 - X_0)\right) = 0.$$

Since $Y_0^\alpha - X_0 = \alpha^\vee$ and $s_\alpha(Y_0^\alpha - X_0) = -\alpha^\vee$, we obtain the result.

□

To make clear the exposition, we distinguish from now on between reduced and non-reduced case.

8.3. Computation of $d_{\mathbf{w}}(\chi), \mathbf{w} \in \mathbf{W}$, for good characters; reduced case

In this section we assume that Δ is reduced. This assumption implies that all roots have the same length or there are short roots β and long roots α. Moreover in the first case $q_\alpha = q$, for every positive root α, while, in the second case, $q_\alpha = q$, for every positive long root α and $q_\beta = p$, for every positive short root β. We denote, as usual,

$$2\delta^\vee = \sum_{\alpha \in R^+} \alpha^\vee. \tag{8.6}$$

To determine the coefficients $d_{\mathbf{w}}(\chi)$, for a good character χ, we assume at first that χ is α-good for every α.

Corollary 8.7. — *Let $\chi \in \mathbf{X}_{gg}(\widehat{L})$. For every simple root α and for every $\mathbf{w} \in \mathbf{W}_\alpha^+$,*

$$d_{\mathbf{w}} \left[\chi^{\mathbf{w}} {\chi_0}^{1/2}(\alpha^\vee) - 1\right] + d_{\mathbf{w}s_\alpha} \left[\chi_0^{1/2}(\alpha^\vee) - \chi^{\mathbf{w}}(\alpha^\vee)\right] = 0. \tag{8.7}$$

Proof. — If the hyperplane H_α^1 contains vertices of type 0, then (8.7) follows from Theorem 8.6 (i). If the hyperplane H_α^1 does not contain vertices of type 0, then (8.7) follows from Theorem 8.6 (ii), by noting that $\chi^{\mathbf{w}} {\chi_0}^{1/2}(\alpha^\vee) + q_\alpha = {\chi_0}^{1/2}(\alpha^\vee) + q_\alpha \chi^{\mathbf{w}}(\alpha^\vee)$, because $\chi_0(\alpha^\vee) = q_\alpha^2$. $\square$

We shall denote by $X_{gg}^0(\widehat{L})$ the space of all characters of $X_{gg}(\widehat{L})$ satisfying

$$\chi(\mathbf{w}\alpha^\vee) \neq q_\alpha, \quad \forall \alpha \in R^+, \ \forall \mathbf{w} \in \mathbf{W}. \tag{8.8}$$

This space is dense in $X_{gg}(\widehat{L})$ with respect to the weak topology. From now on we assume that χ belongs to $X_{gg}^0(\widehat{L})$.

Lemma 8.8. — *Let χ be a character of $\mathbf{X}_{gg}^0(\widehat{L})$. Then $d_{\mathbf{w}}(\chi) \neq 0$, for every $\mathbf{w} \in \mathbf{W}$.*

Proof. — By (8.8), $\chi_0^{1/2}(\alpha^\vee) - \chi^{\mathbf{w}}(\alpha^\vee) \neq 0$ and $(\chi^{\mathbf{w}} {\chi_0}^{1/2})(\alpha^\vee) - 1 \neq 0$, for every $\mathbf{w} \in \mathbf{W}$ and every $\alpha \in R^+$. Hence (8.7) implies that $d_{\mathbf{w}}(\chi) = 0$ if and only if $d_{\mathbf{w}s_\alpha}(\chi) = 0$. Therefore either $d_{\mathbf{w}}(\chi) = 0$ for every $\mathbf{w} \in \mathbf{W}$ or $d_{\mathbf{w}}(\chi) \neq 0$ for every $\mathbf{w} \in \mathbf{W}$. Since $\sum_{\mathbf{w} \in \mathbf{W}} d_{\mathbf{w}}(\chi) = 1$, the lemma is proved. $\square$

Corollary 8.9. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. For every simple root α and every $\mathbf{w} \in \mathbf{W}$,*

$$\frac{d_{\mathbf{w}s_\alpha}(\chi)}{d_{\mathbf{w}}(\chi)} = -\chi^{\mathbf{w}}(\alpha^\vee)\frac{1-\chi_0^{-1/2}(\alpha^\vee)\chi^{\mathbf{w}s_\alpha}(\alpha^\vee)}{1-\chi_0^{-1/2}(\alpha^\vee)\chi^{\mathbf{w}}(\alpha^\vee)} = -\chi^{\mathbf{w}}(\alpha^\vee)\frac{1-q_\alpha^{-1}\chi^{\mathbf{w}s_\alpha}(\alpha^\vee)}{1-q_\alpha^{-1}\chi^{\mathbf{w}}(\alpha^\vee)}. \tag{8.9}$$

Proof. — By (8.8), $\chi^{\mathbf{w}}(\alpha^\vee) \neq \chi_0^{1/2}(\alpha^\vee)$ and $\chi^{\mathbf{w}}{\chi_0}^{1/2}(\alpha^\vee) \neq 1$. Moreover $d_{\mathbf{w}} \neq 0$, by Lemma 8.8. Since $\chi^{\mathbf{w}}(-\alpha^\vee) = \chi^{\mathbf{w}s_\alpha}(\alpha^\vee)$, Corollary 8.7 implies that

$$\frac{d_{\mathbf{w}s_\alpha}(\chi)}{d_{\mathbf{w}}(\chi)} = -\frac{\chi^{\mathbf{w}}{\chi_0}^{1/2}(\alpha^\vee)-1}{\chi_0^{1/2}(\alpha^\vee)-\chi^{\mathbf{w}}(\alpha^\vee)} = -\chi^{\mathbf{w}}(\alpha^\vee)\ \frac{1-\chi_0^{-1/2}(\alpha^\vee)\chi^{\mathbf{w}s_\alpha}(\alpha^\vee)}{1-\chi_0^{-1/2}(\alpha^\vee)\chi^{\mathbf{w}}(\alpha^\vee)}.$$

□

Let $\mathbf{w}_0 \in \mathbf{W}$ be the unique element of maximal length and write $\mathbf{w}_0 = s_1 s_2 \cdots s_N$, as reduced word, where $s_j = s_{\alpha_j}$, with $\alpha_j \in B$, for every $j = 1, \ldots, N$. Define

$$\mathbf{w}^0_0 = \mathbf{e} \quad \text{and} \quad \mathbf{w}^0_k = s_1 s_2 \cdots s_k = \mathbf{w}^0_{k-1} s_k, \ \text{ for } \ k = 1, 2, \ldots, N.$$

Proposition 8.10. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. Then*

$$\frac{d_{\mathbf{w}_0}(\chi)}{d_{\mathbf{e}}(\chi)} = (-1)^{|\mathbf{w}_0|}\ \frac{\chi^{\mathbf{w}_0}(-\delta^\vee)}{\chi(-\delta^\vee)} \prod_{\alpha \in R^+}\left(\frac{1-q_\alpha^{-1}\chi^{\mathbf{w}_0}(\alpha^\vee)}{1-q_\alpha^{-1}\chi(\alpha^\vee)}\right). \tag{8.10}$$

Proof. — Property (8.8) assures that $d_{\mathbf{e}}(\chi), d_{\mathbf{w}^0_1}(\chi), \ldots, d_{\mathbf{w}^0_N}(\chi) \neq 0$; then

$$\frac{d_{\mathbf{w}_0}(\chi)}{d_e(\chi)} = \frac{d_{\mathbf{w}_0}(\chi)}{d_{\mathbf{w}^0_{N-1}}(\chi)}\ \frac{d_{\mathbf{w}^0_{N-1}}(\chi)}{d_{\mathbf{w}^0_{N-2}}(\chi)} \cdots \frac{d_{\mathbf{w}^0_2}(\chi)}{d_{\mathbf{w}^0_1}(\chi)}\ \frac{d_{\mathbf{w}^0_1}(\chi)}{d_e(\chi)}.$$

By Corollary 8.9 we get

$$\frac{d_{\mathbf{w}_0}(\chi)}{d_e(\chi)} = (-1)^N \chi(\alpha_1^\vee)\ \chi^{\mathbf{w}^0_1}(\alpha_2^\vee)\cdots$$

$$\chi^{\mathbf{w}^0_{N-1}}(\alpha_N^\vee)\ \frac{1-q_{\alpha_1}^{-1}\chi^{\mathbf{w}^0_1 s_1}(\alpha_1^\vee)}{1-q_{\alpha_1}^{-1}\chi^{\mathbf{w}^0_1}(\alpha_1^\vee)} \cdots \frac{1-q_{\alpha_N}^{-1}\chi^{\mathbf{w}^0_{N-1}s_N}(\alpha_N^\vee)}{1-q_{\alpha_N}^{-1}\chi^{\mathbf{w}^0_{N-1}}(\alpha_N^\vee)}$$

$$= (-1)^N \chi(\alpha_1^\vee + \mathbf{w}_1^0(\alpha_2^\vee) + \cdots$$
$$+\mathbf{w}_{N-1}^0(\alpha_N^\vee)) \; \frac{1 - q_{\alpha_1}^{-1}\chi^{\mathbf{w}_1^0 s_1}(\alpha_1^\vee)}{1 - q_{\alpha_1}^{-1}\chi^{\mathbf{w}_1^0}(\alpha_1^\vee)} \cdots \; \frac{1 - q_{\alpha_N}^{-1}\chi^{\mathbf{w}_{N-1}^0 s_N}(\alpha_N^\vee)}{1 - q_{\alpha_N}^{-1}\chi^{\mathbf{w}_{N-1}^0}(\alpha_N^\vee)}.$$

We consider $\alpha_1, \mathbf{w}_1^0(\alpha_2^\vee), \mathbf{w}_2^0(\alpha_3^\vee), \dots, \mathbf{w}_{N-1}^0(\alpha_N^\vee)$. For every $k = 1, \dots, N-1$, $\mathbf{w}_k^0(\alpha_{k+1}^\vee) > 0$. In fact, if we assume that $\mathbf{w}_k^0(\alpha_{k+1}^\vee) < 0$, for some k, then, by Lemma in [4, Section (1.6)], should occurs that $|\mathbf{w}_k^0 s_{k+1}| = |\mathbf{w}_k^0| - 1$, while, by definition, $|\mathbf{w}_k^0 s_{k+1}| = |\mathbf{w}_k^0| + 1$.

Moreover $\mathbf{w}_k^0(\alpha_{k+1}^\vee) \neq \mathbf{w}_{k'}^0(\alpha_{k'+1}^\vee)$, for $k \neq k'$. In fact, if $\mathbf{w}_k^0(\alpha_{k+1}^\vee) = \mathbf{w}_{k'}^0(\alpha_{k'+1}^\vee)$ for some $k < k'$, we could write $\mathbf{w}_{k'}^0 = \mathbf{w}_k^0 s_{k+1} \cdots s_{k'}$ and $\mathbf{w}_k^0(\alpha_{k+1}^\vee) = \mathbf{w}_k^0 s_{k+1} \cdots s_{k'}(\alpha_{k'+1}^\vee)$. This should imply that $\alpha_{k+1}^\vee = s_{k+1} \cdots s_{k'}(\alpha_{k'+1}^\vee)$ and then $s_{k+1}(\alpha_{k+1}^\vee) = s_{k+2} \cdots s_{k'}(\alpha_{k'+1}^\vee)$. This is absurd, because $s_{k+1}(\alpha_{k+1}^\vee) < 0$, while $s_{k+2} \cdots s_{k'}(\alpha_{k'+1}^\vee)$ cannot be negative. In fact, $s_{k+2} \cdots s_{k'}(\alpha_{k'+1}) < 0$ should imply, by Lemma in [4, Section (1.6)], $|s_{k+2} \cdots s_{k'}| = |s_{k+2} \cdots s_{k'}| - 1$, in contradiction with the hypothesis that $|s_{k+2} \cdots s_{k'}| = |s_{k+2} \cdots s_{k'}| + 1$.

Therefore we deduce that $\{\alpha_1, \mathbf{w}_1^0(\alpha_2^\vee), \mathbf{w}_2^0(\alpha_3^\vee), \dots, \mathbf{w}_{N-1}^0(\alpha_N^\vee)\} = R^+$ and then

$$\sum_{k=1}^{N} \mathbf{w}_{k-1}^0(\alpha_k^\vee) = \sum_{\alpha \in R^+} \alpha^\vee = 2\delta^\vee.$$

Since α and $\mathbf{w}\alpha$ have the same length, for every α and every $\mathbf{w}$, our formula can be written

$$\begin{aligned} \frac{d_{\mathbf{w}_0}(\chi)}{d_e(\chi)} &= (-1)^{|\mathbf{w}_0|}\, \chi(2\delta^\vee) \prod_{\alpha \in R^+} \left(\frac{1 - q_\alpha^{-1}\chi^{\mathbf{w}_0}(\alpha^\vee)}{1 - q_\alpha^{-1}\chi(\alpha^\vee)} \right) \\ &= (-1)^{|\mathbf{w}_0|}\, \frac{\chi^{\mathbf{w}_0}(-\delta^\vee)}{\chi(-\delta^\vee)} \prod_{\alpha \in R^+} \left(\frac{1 - q_\alpha^{-1}\chi^{\mathbf{w}_0}(\alpha^\vee)}{1 - q_\alpha^{-1}\chi(\alpha^\vee)} \right). \end{aligned}$$

□

For ease of notation, we define, for every character $\chi \in \mathbf{X}_{gg}^0(\widehat{L})$,

$$D(\chi) = \chi(-\delta^\vee) \prod_{\alpha \in R^+} (1 - q_\alpha^{-1}\chi(\alpha^\vee)), \tag{8.11}$$

and, for every $\mathbf{w} \in \mathbf{W}$,

$$D_{\mathbf{w}}(\chi) = D(\chi^{\mathbf{w}}) = \chi^{\mathbf{w}}(-\delta^\vee) \prod_{\alpha \in R^+} \left(1 - q_\alpha^{-1}\chi^{\mathbf{w}}(\alpha^\vee)\right). \tag{8.12}$$

Property (8.8) assures that $D_{\mathbf{w}}(\chi) \neq 0$, for every $\mathbf{w} \in \mathbf{W}$; then we set

$$K(\chi) = \frac{d_{\mathbf{e}}(\chi)}{D_{\mathbf{e}}(\chi)}. \tag{8.13}$$

We note that $K(\chi) \neq 0$; thus Proposition 8.10 states that

$$d_{\mathbf{w}_0}(\chi) = (-1)^{|\mathbf{w}_0|} \ \ K(\chi) \ \ D_{\mathbf{w}_0}(\chi). \tag{8.14}$$

The following proposition extends this result to $d_{s_i}(\chi)$, for every $i \in I_0$.

Proposition 8.11. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. Then, for every $i \in I_0$,*

$$d_{s_i}(\chi) = -K(\chi) D_{s_i}(\chi). \tag{8.15}$$

Proof. — Let $\mathbf{w} = s_i$, for some $i \in I_0$. Since s_i permutes the positives roots different from α_i,

$$\prod_{\alpha \neq \alpha_i} \left(1 - q_\alpha^{-1} \chi^{s_i}(\alpha^\vee)\right) = \prod_{\alpha \neq \alpha_i} \left(1 - q_\alpha^{-1} \chi(\alpha^\vee)\right).$$

Hence, by (8.13) and noting that $\chi(\alpha_i^\vee - \delta^\vee) = \chi^{s_i} \ (-\delta^\vee)$, we can write

$$\begin{aligned}
d_{s_i}(\chi) &= -\ \chi(\alpha_i^\vee) \ \ \frac{1 - q_i^{-1} \chi^{s_i}(\alpha_i^\vee)}{1 - q_i^{-1} \chi(\alpha_i^\vee)} \ d_{\mathbf{e}}(\chi) \\
&= -\ \chi(\alpha_i^\vee) \ \chi(-\delta^\vee) \ K(\chi) \ D_{\mathbf{e}}(\chi) \ \frac{1 - q_i^{-1} \chi^{s_i}(\alpha_i^\vee)}{1 - q_i^{-1} \chi(\alpha_i^\vee)} \\
&= -\ \chi(\alpha_i^\vee - \delta^\vee) \ K(\chi) \ \frac{1 - q_i^{-1} \chi^{s_i}(\alpha_i^\vee)}{1 - q_i^{-1} \chi(\alpha_i^\vee)} \ \prod_{\alpha \in R^+} \left(1 - q_\alpha^{-1} \chi(\alpha_i^\vee)\right) \\
&= -\ \chi^{s_i} \ (-\delta^\vee) \ K(\chi) \ \left(1 - q_i^{-1} \chi^{s_i}(\alpha_i^\vee)\right) \ \prod_{\alpha \neq \alpha_i} \left(1 - q_\alpha^{-1} \chi(\alpha_i^\vee)\right) \\
&= -\ \chi^{s_i} \ (-\delta^\vee) \ K(\chi) \prod_{\alpha \in R^+} \left(1 - q_\alpha^{-1} \chi^{s_i}(\alpha_i^\vee)\right) = (-1)^{|s_i|} \ K(\chi) \ D_{s_i}(\chi).
\end{aligned}$$

$\square$

Formula (8.15) can be extended to every $\mathbf{w} \in \mathbf{W}$. We need the following lemma.

Lemma 8.12. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. For every $\mathbf{w} \in \mathbf{W}$,*

$$K(\chi^{\mathbf{w}}) = (-1)^{|\mathbf{w}|} \ K(\chi). \tag{8.16}$$

Proof. — We proceed by induction on the length of $\mathbf{w}$. If $\mathbf{w}$ has length one, say $\mathbf{w} = s_i$ for some $i \in I_0$, (8.16) follows from Proposition 8.11, because $K(\chi^{s_i}) = -K(\chi) = (-1)^{|s_i|}\ K(\chi)$. We suppose now that (8.16) is true for every $\mathbf{w} \in \mathbf{W}$ of length $|\mathbf{w}| \leqslant j$. Let $\mathbf{w}' = \mathbf{w}s_i$, such that $|\mathbf{w}'| = j+1$; then

$$K(\chi^{\mathbf{w}'}) = K(\chi^{\mathbf{w}s_i}) = -\ K(\chi^{\mathbf{w}}).$$

Hence $K(\chi^{\mathbf{w}'}) = -\ (-1)^{|\mathbf{w}|}\ K(\chi) = (-1)^{|\mathbf{w}'|}\ K(\chi)$ and the lemma is proved. □

Proposition 8.13. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. For every $\mathbf{w} \in \mathbf{W}$,*

$$d_{\mathbf{w}}(\chi) = K(\chi^{\mathbf{w}})\ D_{\mathbf{w}}(\chi) = (-1)^{|\mathbf{w}|}\ K(\chi)\ D_{\mathbf{w}}(\chi). \tag{8.17}$$

Proof. — By definition, $d_{\mathbf{w}}(\chi) = d_{\mathbf{e}}(\chi^{\mathbf{w}}) = D(\chi^{\mathbf{w}})\ K(\chi^{\mathbf{w}})$; so (8.17) follows from Lemma 8.12. □

Our next goal is to determinate $K(\chi)$. We renumerate the N positive roots by setting $R^+ = \{\alpha_i,\ i = 1, \dots, N\}$ and we set, as usual, $q_i = q_{\alpha_i}$, for every $i = 1, \dots, N$. Since $\sum_{\mathbf{w}\in\mathbf{W}} d_{\mathbf{w}} = 1$, then, by Proposition 8.13, $K(\chi)\ \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\ D_{\mathbf{w}}(\chi) = 1$. Therefore,

$$\frac{1}{K(\chi)} = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\ D_{\mathbf{w}}(\chi) = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\ \chi^{\mathbf{w}}(-\delta^\vee) \prod_{i=1}^{N} \left(1 - q_i^{-1}\chi^{\mathbf{w}}(\alpha_i)\right). \tag{8.18}$$

We need some preliminary results, to simplify the calculation of the previous sum.

Lemma 8.14. — *Let $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$. Let $\mu^\vee_k = \sum_{j=1}^{k} \alpha^\vee_{i_j}$, for $i_1 < \cdots < i_k$ and $1 \leqslant k \leqslant N$. If there exists $\mathbf{w}_1 \in \mathbf{W}$ such that $\mathbf{w}_1(\delta^\vee - \mu^\vee_k) = \delta^\vee$, then*

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\ \chi^{\mathbf{w}}(-\delta^\vee + \mu^\vee_k) = (-1)^{|\mathbf{w}_1|} \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\ \chi^{\mathbf{w}}(-\delta^\vee);$$

otherwise

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\ \chi^{\mathbf{w}}(-\delta^\vee + \mu^\vee_k) = 0.$$

Proof. — Consider $\Pi_{\delta^\vee} = \Pi_{\delta^\vee}(0)$. According to [3, (13.1)], $\delta^\vee$ is the unique element of $\Pi_{\delta^\vee}$ lying on $(\mathbb{Q}_0)^0$, because $\delta^\vee \in \mathbb{Q}_0$ and $\delta^\vee \in H^1_i$, for every $i = 1, \dots, n$.

Fix $k \in \{1, \ldots, N\}$, $i_1 < \cdots < i_k$ and consider $\mu_k^\vee = \sum_{j=1}^k \alpha_{i_j}^\vee$. We shall prove that

$$\delta^\vee - \mu_k^\vee \in \Pi_{\delta^\vee}.$$

Actually, $\delta^\vee - \mu_k^\vee \preceq \delta^\vee$ (with respect to the Bruhat ordering of $\mathbf{W}$); moreover there exists $\mathbf{w}_1 \in \mathbf{W}$ such that $\mathbf{w}_1(\delta^\vee - \mu_k^\vee) \in \mathbb{Q}_0$. By comparing the explicit formulas for $\mu_k^\vee$ and $\mathbf{w}_1(\delta^\vee - \mu_k^\vee)$ as sums of $\alpha_i^\vee$, we deduce that $\mathbf{w}_1(\delta^\vee - \mu_k^\vee) \preceq \delta^\vee$. So $\mathbf{w}_1(\delta^\vee - \mu_k^\vee) \in \Pi_{\delta^\vee}$, by [3, Lemma B]. Since a saturated set is stable under $\mathbf{W}$, also $\delta^\vee - \mu_k^\vee \in \Pi_{\delta^\vee}$.

Moreover, if $\mathbf{w}_1(\delta^\vee - \mu_k^\vee) \neq \delta^\vee$, we must have $\mathbf{w}_1(\delta^\vee - \mu_k^\vee) \in \partial(\mathbb{Q}_0)$, that is $\mathbf{w}_1(\delta^\vee - \mu_k^\vee) \in H_i^0$ for some $i = 1, \ldots, n$. Let $\lambda_k = \mathbf{w}_1(\delta^\vee - \mu_k^\vee)$.

(i) Suppose that $\lambda_k = \delta^\vee$. In this case

$$\begin{aligned}
\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\ \chi^{\mathbf{w}}(-\delta^\vee + \mu_k^\vee) &= \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\mathbf{w}_1^{-1}(\lambda_k)) \\
&= \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\mathbf{w}_1^{-1}(\delta^\vee)) \\
&= \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}\mathbf{w}_1^{-1}}(-\delta^\vee).
\end{aligned}$$

Then, if we set $\mathbf{u} = \mathbf{w}\mathbf{w}_1^{-1}$ and write $|\mathbf{u}\mathbf{w}_1^{-1}| = |\mathbf{u}| + |\mathbf{w}_1| - 2K$, for a convenient K, we get

$$\begin{aligned}
\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\mathbf{w}_1^{-1}(\lambda_k)) &= \sum_{\mathbf{u}\in\mathbf{W}} (-1)^{|\mathbf{u}|+|\mathbf{w}_1|-2K}\ \chi^{\mathbf{u}}(-\delta^\vee) \\
&= (-1)^{|\mathbf{w}_1|} \sum_{\mathbf{u}\in\mathbf{W}} (-1)^{\mathbf{u}}\ \chi^{\mathbf{u}}(-\delta^\vee).
\end{aligned}$$

(ii) Suppose now that there does not exist any $\mathbf{w} \in \mathbf{W}$, such that $\mathbf{w}(\delta^\vee - \mu^\vee) = \delta^\vee$. In this case, λ_k lies on H_i^0, for some $i = 1, \ldots, n$, and hence $s_i(\lambda_k^\vee) = \lambda_k^\vee$. So, by using the same change of variables as in (i), we can write

$$\begin{aligned}
\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_k^\vee) &= \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}\mathbf{w}^{-1}}(-\lambda_k^\vee) \\
&= (-1)^{|\mathbf{w}|} \sum_{\mathbf{u}\in\mathbf{W}} (-1)^{\mathbf{u}}\chi^{\mathbf{u}}(-\lambda_k^\vee).
\end{aligned}$$

We split this sum into two sums over $\mathbf{W}_i^+ = \mathbf{W}_{\alpha_i}^+$ and $\mathbf{W}_i^- = \mathbf{W}_{\alpha_i}^-$. We recall that, for every $\mathbf{u} \in \mathbf{W}_i^-$, we can write $\mathbf{u} = \mathbf{u}' s_i$ with

$\mathbf{u}' \in \mathbf{W}_i^+$ and $|\mathbf{u}| = |\mathbf{u}'| + 1$. So we get

$$\begin{aligned}
&\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_k^\vee) \\
&= (-1)^{|\mathbf{w}|}\left[\sum_{\mathbf{u}\in\mathbf{W}_1^+} (-1)^{|\mathbf{u}|}\chi^{\mathbf{u}}(-\lambda_k^\vee) + \sum_{\mathbf{u}\in\mathbf{W}_1^-} (-1)^{|\mathbf{u}|}\chi^{\mathbf{u}}(-\lambda_k^\vee)\right] \\
&= (-1)^{|\mathbf{w}|}\left[\sum_{\mathbf{u}\in\mathbf{W}_1^+} (-1)^{|\mathbf{u}|}\chi^{\mathbf{u}}(-\lambda_k^\vee) + \sum_{\mathbf{u}'\in\mathbf{W}_1^+} (-1)^{|\mathbf{u}'|+1}\chi^{\mathbf{u}' s_1}(-\lambda_k^\vee)\right] \\
&= (-1)^{|\mathbf{w}|}\left[\sum_{\mathbf{u}\in\mathbf{W}_1^+} (-1)^{|\mathbf{u}|}\chi^{\mathbf{u}}(-\lambda_k^\vee) - \sum_{\mathbf{u}'\in\mathbf{W}_1^+} (-1)^{|\mathbf{u}'|}\chi^{\mathbf{u}'}(-s_1(\lambda_k^\vee))\right] \\
&= (-1)^{|\mathbf{w}|}\sum_{\mathbf{u}\in\mathbf{W}_1^+} (-1)^{|\mathbf{u}|}\left(\chi^{\mathbf{u}}(-\lambda_k^\vee) - \chi^{\mathbf{u}}(-\lambda_k^\vee)\right) = 0.
\end{aligned}$$

□

Let $N \geqslant 1$. For every $k = 1, \ldots, N$, define

$$\begin{aligned}
\mathcal{I}_k^N &= \{(i_1, \ldots, i_k) \ : \ i_1, \ldots, i_k \in \{1, \ldots, N\},\ i_1 < \cdots < i_k\}, \\
\mathcal{I}_k^- &= \{(i_1, \ldots, i_k) \in \mathcal{I}_k^N \ : \ \exists\ \mathbf{w}, |\mathbf{w}| = k, \text{ such that } \mathbf{w}(\alpha_{i_j}^\vee) < 0,\ \forall j = 1, \ldots, k\}
\end{aligned}$$

and $\mathcal{I}_0^N = \mathcal{I}_0^- = \emptyset$.

Lemma 8.15. — *Let $X_1, X_2, \ldots, X_N$ be N variables and $\Pi_N^1 = \prod_{i=1}^N (1-X_i)$, for $N \geqslant 1$. Then*

$$\Pi_N^1 = \sum_{k=0}^N (-1)^k \left(\sum_{(i_1,\ldots,i_k)\in\mathcal{I}_k^N} X_{i_1}\cdots X_{i_k}\right),$$

where $\sum_{\mathcal{I}_0^N} X_{i_1}\cdots X_{i_k} = 1$.

Proof. — The proof follows easily by induction on N. □

Proposition 8.16. — *For every $\chi \in \mathbf{X}_{gg}^0(\widehat{L})$,*

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|} D_{\mathbf{w}}(\chi) = \left(\sum_{k=0}^N \sum_{\mathcal{I}_k^-} q_{i_1}^{-1}\cdots q_{i_k}^{-1}\right) \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee),$$

where $\sum_{\mathcal{I}_0^-} q_{i_1}^{-1}\cdots q_{i_k}^{-1} = 1$.

Proof. — By Lemma 8.15

$$\begin{aligned}
D_{\mathbf{w}}(\chi) &= \chi^{\mathbf{w}}(-\delta^\vee) \prod_{i=1}^{N} \big(1 - q_i^{-1}\chi^{\mathbf{w}}(\alpha_i)\big) \\
&= \chi^{\mathbf{w}}(-\delta^\vee) \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} (q_{i_1}^{-1}\cdots q_{i_k}^{-1})\ \chi^{\mathbf{w}}(\alpha_{i_1}^\vee)\cdots\chi^{\mathbf{w}}(\alpha_{i_k}^\vee) \\
&= \chi^{\mathbf{w}}(-\delta^\vee) \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} (q_{i_1}^{-1}\cdots q_{i_k}^{-1})\ \chi^{\mathbf{w}}(\alpha_{i_1}^\vee+\cdots+\alpha_{i_k}^\vee) \\
&= \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} (q_{i_1}^{-1}\cdots q_{i_k}^{-1})\ \chi^{\mathbf{w}}(-\delta^\vee+\alpha_{i_1}^\vee+\cdots+\alpha_{i_k}^\vee).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|} D_{\mathbf{w}}(\chi) \\
&\qquad = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|} \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} (q_{i_1}^{-1}\cdots q_{i_k}^{-1})\ \chi^{\mathbf{w}}(-\delta^\vee+\alpha_{i_1}^\vee+\cdots+\alpha_{i_k}^\vee) \\
&\qquad = \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} q_{i_1}^{-1}\cdots q_{i_k}^{-1} \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee+\alpha_{i_1}^\vee+\cdots+\alpha_{i_k}^\vee).
\end{aligned}$$

By Lemma 8.14, if there exists $\mathbf{w}_1 \in \mathbf{W}$ such that $\mathbf{w}_1(-\delta^\vee+\alpha_{i_1}^\vee+\cdots+\alpha_{i_k}^\vee) = -\delta^\vee$,

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee+\alpha_{i_1}^\vee+\cdots+\alpha_{i_k}^\vee) = (-1)^{|\mathbf{w}_1|} \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee).$$

Otherwise the sum on the left is zero.

Since k is the number of positive roots mapped by $\mathbf{w}_1$ to negative roots, $|\mathbf{w}_1| = k$. Then, by definition of $\mathcal{I}_k^-$,

$$\begin{aligned}
&\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|} D_{\mathbf{w}}(\chi) \\
&\qquad = \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} (q_{i_1}^{-1}\cdots q_{i_k}^{-1}) \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}_1|}\chi^{\mathbf{w}}(-\delta^\vee) \\
&\qquad = \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^-} (q_{i_1}^{-1}\cdots q_{i_k}^{-1}) \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}_1|}(-1)^k\chi^{\mathbf{w}}(-\delta^\vee)
\end{aligned}$$

$$= \left(\sum_{k=0}^{N} \sum_{\mathcal{I}_k^-} q_{i_1}^{-1} \cdots q_{i_k}^{-1} \right) \sum_{\mathbf{w} \in \mathbf{W}} (-1)^{|\mathbf{w}|} \chi^{\mathbf{w}}(-\delta^\vee).$$

□

By Proposition 8.16, to compute $\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}D_{\mathbf{w}}(\chi)$ we calculate separately $\sum_{k=0}^{N} \sum_{\mathcal{I}_k^-} q_{i_1}^{-1} \cdots q_{i_k}^{-1}$ and $\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee)$.

Lemma 8.17. — *We have*

$$\sum_{k=0}^{N} \sum_{\mathcal{I}_k^-} q_{i_1}^{-1} \cdots q_{i_k}^{-1} = \mathbf{W}(q^{-1}).$$

Proof. — We distinguish two cases.

1. All the roots have the same length. Then $q_\alpha = q$, for every α, and therefore
$$\sum_{k=0}^{N} \sum_{\mathcal{I}_k^-} q_{i_1}^{-1} \cdots q_{i_k}^{-1} = \sum_{k=0}^{N} |\{\mathbf{w} \in \mathbf{W} \ ,$$
$$|\mathbf{w}| = k\}| q^{-k} = \sum_{k=0}^{N} \sum_{|\mathbf{w}|=k} q_{\mathbf{w}}^{-1} = \sum_{\mathbf{w}\in\mathbf{W}} q_{\mathbf{w}}^{-1} = \mathbf{W}(q^{-1}).$$

2. Assume now that there are long roots α and short roots β. In this case, if $|\mathbf{w}| = k$, we suppose that $\mathbf{w}$ is the product of h generators associated with simple long roots and $(k-h)$ generators associated with simple short roots. Then $\mathbf{w}$ changes sign to h long roots and $(k-h)$ short roots. We set
$$\mathbf{W}_{k,h} = \{\mathbf{w} \in \mathbf{W} : \ \mathbf{w} = s_{i_1} \cdots s_{i_k}, \ \text{ where } h \text{ of the roots } \alpha_{i_1} \cdots \alpha_{i_k} \text{ are long}\}.$$
Then
$$\begin{aligned}\sum_{k=0}^{N} \sum_{\mathcal{I}_k^-} q_{i_1}^{-1} \cdots q_{i_k}^{-1} &= \sum_{k=0}^{N} \sum_{h=0}^{k} |\mathbf{W}_{k,h}| q_\alpha^{-h} q_\beta^{-k+h} \\ &= \sum_{k=0}^{N} \sum_{h=0}^{k} \sum_{\mathbf{w}\in\mathbf{W}_{k,h}} q_{\mathbf{w}}^{-1} = \sum_{\mathbf{w}\in\mathbf{W}} q_{\mathbf{w}}^{-1} = \mathbf{W}(q^{-1}).\end{aligned}$$

□

Lemma 8.18. — *For every* $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$,

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee) = \chi(-\delta^\vee) \prod_{\alpha\in R^+} (1-\chi(\alpha^\vee))\,.$$

Proof. — Let $|\mathbf{w}| = k$. We denote by $\alpha_{i_1},\dots,\alpha_{i_k}$ the k positive roots such that $\mathbf{w}(\alpha_{i_j}) < 0$. Hence $\mathbf{w}(\delta^\vee) = \delta^\vee - \alpha^\vee_{i_j} - \cdots - \alpha^\vee_{i_k}$ and we get

$$\begin{aligned}\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee) &= \sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}^-_k} \chi(-\delta^\vee + \alpha^\vee_{i_j} + \cdots + \alpha^\vee_{i_k})\\ &= \chi(-\delta^\vee)\sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}^-_k} \chi(\alpha^\vee_{i_j} + \cdots + \alpha^\vee_{i_k}).\end{aligned}$$

We denote by $\nu^\vee$ any element of $\Pi_{\delta^\vee}$, lying on $\mathbb{Q}_0$ but different from $\delta^\vee$. By Lemma 8.14, $\nu^\vee \in \partial\mathbb{Q}_0$ and

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\nu^\vee) = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi(-\mathbf{w}\nu^\vee) = 0.$$

On the other hand $\delta^\vee - \mathbf{w}\nu^\vee \in \mathbb{Q}_0$, and therefore $\delta^\vee - \mathbf{w}\nu^\vee$ is sum of k positive coroots $\alpha^\vee_{j_1},\dots,\alpha^\vee_{j_k}$. Define

$$\mathcal{I}_{k,\nu^\vee} = \{(j_1,\dots,j_k) \in \mathcal{I}^N_k \;:\; \exists\, \mathbf{w},\;\; |\mathbf{w}| = k,\quad \alpha^\vee_{j_1} + \cdots + \alpha^\vee_{j_k} = \delta^\vee - \mathbf{w}\nu^\vee\}.$$

We have

$$\begin{aligned} 0 &= \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|} \sum_{\mathcal{I}_{k,\nu^\vee}} \chi(-\delta^\vee + \alpha^\vee_{j_1} + \cdots + \alpha^\vee_{j_k})\\ &= \chi(-\delta^\vee) \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|} \sum_{\mathcal{I}_{k,\nu^\vee}} \chi(\alpha^\vee_{j_1} + \cdots + \alpha^\vee_{j_k}).\end{aligned}$$

Finally, putting all terms together, we get

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee) = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee) + \sum_{\nu^\vee}\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\nu^\vee)$$

$$= \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\left[\chi^{\mathbf{w}}(-\delta^\vee) + \sum_{\nu^\vee}\chi^{\mathbf{w}}(-\nu^\vee)\right]$$

$$= \chi(-\delta^\vee)\sum_{k=0}^{N}(-1)^k\left[\sum_{\mathcal{I}^N_k}\chi(\alpha^\vee_{i_1} + \cdots + \alpha^\vee_{i_k}) + \sum_{\nu^\vee}\sum_{\mathcal{I}_{k,\nu^\vee}}\chi^{\mathbf{w}}(\alpha^\vee_{j_1} + \cdots + \alpha^\vee_{j_k})\right]$$

$$= \chi(-\delta^\vee)\sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} \chi(\alpha_{i_1}^\vee + \cdots + \alpha_{i_k}^\vee)$$

$$= \chi(-\delta^\vee)\sum_{k=0}^{N}(-1)^k \sum_{\mathcal{I}_k^N} \chi(\alpha_{i_1}^\vee)\cdots\chi(\alpha_{i_k}^\vee)$$

$$= \chi(-\delta^\vee)\prod_{i=1}^{N}(1-\chi(\alpha_i^\vee)).$$

□

Proposition 8.19. — *Let χ be a character of $\mathbf{X}_{gg}^0(\widehat{L})$. Then,*

$$K(\chi) = \frac{\chi(\delta^\vee)}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R^+} \frac{1}{1-\chi(\alpha^\vee)}. \tag{8.19}$$

Proof. — Lemma 8.17 and Lemma 8.18 imply that

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|} D_{\mathbf{w}}(\chi) = \mathbf{W}(q^{-1})\ \chi(-\delta^\vee) \prod_{\alpha\in R^+} (1-\chi(\alpha^\vee))\,.$$

Hence, by (8.13),

$$\frac{1}{K(\chi)} = \mathbf{W}(q^{-1})\chi(-\delta^\vee) \prod_{\alpha\in R^+} (1-\chi(\alpha^\vee)).$$

Since χ is non-singular, then $\prod_{\alpha\in R^+}(1-\chi(\alpha^\vee)) \neq 0$. So the required formula (8.19) is proved. □

The following theorem exhibits the expression of each coefficient $d_{\mathbf{w}}(\chi)$, for every character $\chi \in \mathbf{X}_{gg}^0(\widehat{L})$.

Theorem 8.20. — *Let $\chi \in \mathbf{X}_{gg}^0(\widehat{L})$. Then, for every $\mathbf{w}\in\mathbf{W}$,*

$$d_{\mathbf{w}}(\chi) = \frac{1}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R^+} \frac{1-q_\alpha^{-1}\chi^{\mathbf{w}}(\alpha^\vee)}{1-\chi^{\mathbf{w}}(\alpha^\vee)}. \tag{8.20}$$

Proof. — Let $\mathbf{w}=\mathbf{e}$. In this case

$$d_{\mathbf{e}}(\chi) = K(\chi)D(\chi) = K(\chi)\ \chi(-\delta^\vee) \prod_{\alpha\in R^+} \left(1-q_\alpha^{-1}\chi^{\mathbf{w}}(\alpha^\vee)\right)$$

and, by Lemma 8.18,

$$\begin{aligned} d_{\mathbf{e}}(\chi) &= \frac{\chi(\delta^\vee)}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R^+} \frac{1}{1-\chi(\alpha^\vee)}\, \chi(-\delta^\vee) \prod_{\alpha\in R^+} (1-q_\alpha^{-1}\chi(\alpha^\vee)) \\ &= \frac{1}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R^+} \frac{1-q_\alpha^{-1}\chi(\alpha^\vee)}{1-\chi(\alpha^\vee)}. \end{aligned}$$

On the other hand, if $\mathbf{w} \neq \mathbf{e}$, then $d_{\mathbf{w}}(\chi) = d_{\mathbf{e}}(\chi^{\mathbf{w}})$; therefore formula (8.20) is proved. $\square$

Remark 8.21. — For every $\mathbf{w} \in \mathbf{W}$, the coefficient $d_{\mathbf{w}}(\chi)$, as well as the function on the right side of (8.20), is defined on the space $\mathbf{X}_g(\widehat{L})$ and depends continuously on the character χ with respect to the weak topology. Therefore the formula (8.20) can be extended to all characters of $\mathbf{X}_g(\widehat{L})$ by a standard argument of continuity, taking in account that $\mathbf{X}^0_{gg}(\widehat{L})$ is dense in $\mathbf{X}_{gg}(\widehat{L})$ and then in $\mathbf{X}_g(\widehat{L})$.

8.4. Computation of $d_{\mathbf{w}}(\chi), \mathbf{w} \in \mathbf{W}$, for good characters; non-reduced case

In this section we assume that Δ is a non-reduced building of type $(\widetilde{BC})_n$. According to notation of Section 3.2, we shall denote by α, β, γ any root of R_0, R_1, R_2 respectively; moreover we set $q_\alpha = q$, $q_\beta = p$ and $q_\gamma = r$. We define $\delta_0^\vee = \frac{1}{2}\sum_{\alpha\in R_0^+} \alpha^\vee$, $\delta_1^\vee = \sum_{\beta\in R_1^+} \beta^\vee$ and $\delta^\vee = \delta_0^\vee + \delta_1^\vee$. Since $\alpha^\vee = \alpha$, for every $\alpha \in R_0^+$, and $\beta^\vee = \frac{1}{2}\beta$, for every $\beta \in R_1^+$, then $\delta_0^\vee = \delta_0$ and $\delta_1^\vee = \frac{1}{2}\sum_{\beta\in R_1} \beta = \frac{1}{2}\sum_{\gamma\in R_2} \gamma^\vee$. Thus $\delta^\vee$ corresponds to a reduced root system of type B_n. Moreover

$$\delta^\vee = \delta_0^\vee + \delta_1^\vee = \frac{1}{2}\sum_{\alpha\in R_0^+} \alpha + \frac{1}{2}\sum_{\beta\in R_1^+} \beta = \delta, \tag{8.21}$$

if δ corresponds to a reduced root system of type C_n. Moreover $\delta_0^\vee$ and $\delta_1^\vee$ belong to the lattice $\widehat{L} = L$ and lie on $\mathbb{Q}_0$; moreover $\delta_0^\vee, \delta_1^\vee \in H_i^1 \cup H_i^0$, for every i, because $\langle \delta_0^\vee, \alpha_i\rangle = 0$ or 1 and $\langle \delta_1^\vee, \alpha_i\rangle = 0$ or 1; furthermore $\langle \delta^\vee, \alpha_i\rangle = 1$, because $\delta_0^\vee \in H_i^0$ when $\delta_1^\vee \in H_i^1$ and viceversa. Finally we recall that $\mathbf{W}$ can be seen as the Weyl group of a reduced root system of type C_n.

We shall denote by $X^0_{gg}(\widehat{L})$ the space of all characters of $X_{gg}(\widehat{L})$ satisfying

$$\begin{cases} \chi(\mathbf{w}\alpha^\vee) \neq q & \forall\alpha \in R_0^+,\ \forall\mathbf{w}\in\mathbf{W}, \\ \chi(\mathbf{w}\beta^\vee) \neq \sqrt{pr},\ -\sqrt{\frac{r}{p}} & \forall\beta \in R_1^+,\ \forall\mathbf{w}\in\mathbf{W}. \end{cases} \tag{8.22}$$

This space is dense in $X_{gg}(\widehat{L})$ with respect to the weak topology. Analogously to the reduced case, from now on we assume that χ belongs to $X^0_{gg}(\widehat{L})$. This assumption is motivated by the following lemma, analogous to Lemma 8.8 stated in the reduced case.

Lemma 8.22. — *Let $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$. Then $d_{\mathbf{w}}(\chi) \neq 0$, for every $\mathbf{w}\in\mathbf{W}$.*

Proof. — According to Theorem 8.6,

(i) if $\alpha = e_i - e_{i+1},\ \ i = 1,\dots,n-1$, and $\mathbf{w}\in\mathbf{W}^+_\alpha$,

$$d_{\mathbf{w}}\left[\chi^{\mathbf{w}}{\chi_0}^{1/2}(\alpha^\vee) - 1\right] + d_{\mathbf{w}s_\alpha}\left[\chi_0^{1/2}(\alpha^\vee) - \chi^{\mathbf{w}}(\alpha^\vee)\right] = 0;$$

(ii) if $\beta = 2e_n$ and $\mathbf{w}\in\mathbf{W}^+_\beta$

$$d_{\mathbf{w}}\left[(\chi^{\mathbf{w}}{\chi_0}^{1/2}(\beta^\vee) - 1)(\chi^{\mathbf{w}}{\chi_0}^{1/2}(\beta^\vee) + q_\beta\right]$$
$$+d_{\mathbf{w}s_\beta}\left[({\chi_0}^{1/2}(\beta^\vee) - \chi^{\mathbf{w}}(\beta^\vee))({\chi_0}^{1/2}(\beta^\vee) + q_\beta\chi^{\mathbf{w}}(\beta^\vee))\right] = 0.$$

If we assume $\chi \in X^0_{gg}(\widehat{L})$, then (i) and (ii) imply that $d_{\mathbf{e}}(\chi) = 0$ if and only if $d_{s_\alpha}(\chi) = 0$ for every $\alpha = e_i - e_{i+1}$ and $d_{s_\beta}(\chi) = 0$ for $\beta = 2e_n$. Therefore either $d_{\mathbf{w}}(\chi) = 0$ for every $\mathbf{w}\in\mathbf{W}$ or $d_{\mathbf{w}}(\chi) \neq 0$ for every $\mathbf{w}\in\mathbf{W}$. Since $\sum_{\mathbf{w}\in\mathbf{W}} d_{\mathbf{w}}(\chi) = 1$, the lemma is proved. □

Corollary 8.23. — *Let $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$. Then, for every $\mathbf{w}\in\mathbf{W}$,*

$$\frac{d_{\mathbf{w}s_\alpha}(\chi)}{d_{\mathbf{w}}(\chi)} = -\chi^{\mathbf{w}}(\alpha^\vee)\,\frac{1 - \chi_0^{-1/2}(\alpha^\vee)\chi^{\mathbf{w}s_\alpha}(\alpha^\vee)}{1 - \chi_0^{-1/2}(\alpha^\vee)\chi^{\mathbf{w}}(\alpha^\vee)}, \tag{8.23}$$

for every $\alpha = e_i - e_{i+1},\ i = 1,\dots,n-1$, while, for $\beta = 2e_n$,

$$\frac{d_{\mathbf{w}s_\beta}(\chi)}{d_{\mathbf{w}}(\chi)} =$$
$$-\chi^{\mathbf{w}}(2\beta^\vee)\,\frac{\left(1 - \chi_0^{-1/2}(\beta^\vee)\chi^{\mathbf{w}s_\alpha}(\beta^\vee)\right)\left(1 + q_\beta\ \chi_0^{-1/2}(\beta^\vee)\chi^{\mathbf{w}s_\alpha}(\beta^\vee)\right)}{\left(1 - \chi_0^{-1/2}(\beta^\vee)\chi^{\mathbf{w}}(\beta^\vee)\right)\left(1 + q_\beta\ \chi_0^{-1/2}(\beta^\vee)\chi^{\mathbf{w}}(\beta^\vee)\right)}. \tag{8.24}$$

Proof. — If $\alpha = e_i - e_{i+1}$, for some $i = 1, \dots, n-1$, then the hyperplane H^1_α contains vertices of type 0 and (8.23) follows from (8.5) by the same argument as in the reduced case.

Assume now $\beta = 2e_n$, and set $\alpha = e_n$; then $\beta^\vee = \alpha$ and $\alpha^\vee = \beta$. In this case, the hyperplane H^1_β does not contain vertices of type 0; hence, by Theorem 8.6 (i),

$$d_{\mathbf{w}} \left[(\chi^{\mathbf{w}} {\chi_0}^{1/2}(\beta^\vee) - 1)(\chi^{\mathbf{w}} {\chi_0}^{1/2}(\beta^\vee) + q_\beta \right]$$

$$+ d_{\mathbf{w}s_\beta} \left[({\chi_0}^{1/2}(\beta^\vee) - \chi^{\mathbf{w}}(\beta^\vee))({\chi_0}^{1/2}(\beta^\vee) + q_\beta \chi^{\mathbf{w}}(\beta^\vee)) \right] = 0,$$

which implies, by Lemma 8.22, the required formula (8.24). $\square$

According to Section 3.2, $\chi_0(\alpha^\vee) = q^2$, for $\alpha = e_i - e_{i+1}$, $i = 1, \dots, n-1$, and $\chi_0(\beta^\vee) = pr$, for $\beta = 2e_n$.

Corollary 8.24. — *Let $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$. Then, for every $\mathbf{w} \in \mathbf{W}$,*

$$\frac{d_{\mathbf{w}s_\alpha}(\chi)}{d_{\mathbf{w}}(\chi)} = -\, \chi^{\mathbf{w}}(\alpha^\vee)\, \frac{1 - q^{-1}\chi^{\mathbf{w}s_\alpha}(\alpha^\vee)}{1 - q^{-1}\chi^{\mathbf{w}}(\alpha^\vee)},\ \text{if } \alpha = e_i - e_{i+1},\ i = 1, \dots, n-1; \tag{8.25}$$

$$\frac{d_{\mathbf{w}s_\beta}(\chi)}{d_{\mathbf{w}}(\chi)} = -\, \chi^{\mathbf{w}}(2\beta^\vee)\, \frac{\left(1 - \frac{1}{\sqrt{pr}}\chi^{\mathbf{w}s_\beta}(\beta^\vee)\right)\left(1 + \sqrt{\frac{p}{r}}\chi^{\mathbf{w}s_\beta}(\beta^\vee)\right)}{\left(1 - \frac{1}{\sqrt{pr}}\chi^{\mathbf{w}}(\beta^\vee)\right)\left(1 + \sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee)\right)},\ \text{if } \beta = 2e_n. \tag{8.26}$$

Let $\mathbf{w}_0 \in \mathbf{W}$ be the unique element of maximal length and write $\mathbf{w}_0 = s_1 s_2 \cdots s_N$, as reduced word, where $s_j = s_{\alpha_j}$, with $\alpha_{i_j} = e_{i_j} - e_{i_j+1}$, if $i_j = 1, \dots, n-1$, and $\alpha_{i_j} = \beta = 2e_n$, if $i_j = n$. Define

$$\mathbf{w}^0_0 = \mathbf{e} \quad \text{and} \quad \mathbf{w}^0_k = s_1 s_2 \cdots s_k = \mathbf{w}^0_{k-1} s_k, \ \text{ for } \ k = 1, 2, \dots, N.$$

Proposition 8.25. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. Then*

$$\frac{d_{\mathbf{w}_0}(\chi)}{d_{\mathbf{e}}(\chi)} = (-1)^{|\mathbf{w}_0|} \frac{\chi^{\mathbf{w}_0}(-\delta^\vee)}{\chi(-\delta^\vee)} \prod_{\alpha \in R_0^+} \frac{1 - q^{-1}\chi^{\mathbf{w}_0}(\alpha^\vee)}{1 - q^{-1}\chi(\alpha^\vee)}$$

$$\prod_{\beta \in R_1^+} \frac{(1 - \frac{1}{\sqrt{pr}}\chi^{\mathbf{w}_0}(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi^{\mathbf{w}_0}(\beta^\vee))}{(1 - \frac{1}{\sqrt{pr}}\chi(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi(\beta^\vee))}. \tag{8.27}$$

Proof. — Property (8.22) assures that $d_{\mathbf{e}}(\chi), d_{\mathbf{w}_1^0}(\chi), \ldots, d_{\mathbf{w}_N^0}(\chi) \neq 0$; then

$$\frac{d_{\mathbf{w}_0}(\chi)}{d_e(\chi)} = \frac{d_{\mathbf{w}_0}(\chi)}{d_{\mathbf{w}^0_{(N-1)}}(\chi)} \; \frac{d_{\mathbf{w}^0_{(N-1)}}(\chi)}{d_{\mathbf{w}^0_{(N-2)}}(\chi)} \; \cdots \frac{d_{\mathbf{w}_2^0}(\chi)}{d_{\mathbf{w}_1^0}(\chi)} \; \frac{d_{\mathbf{w}_1^0}(\chi)}{d_{\mathbf{e}}(\chi)}.$$

By (8.25) and (8.26),

$$\frac{d_{\mathbf{w}_k^0}(\chi)}{d_{\mathbf{w}^0_{(k-1)}}(\chi)} = \begin{cases} -\chi^{\mathbf{w}}(\alpha_{i_k}^\vee)\frac{1-q^{-1}\chi^{\mathbf{w}s_\alpha}(\alpha_{i_k}^\vee)}{1-q^{-1}\chi^{\mathbf{w}}(\alpha_{i_k}^\vee)} & \text{if } \ i_k < n, \\ -\chi^{\mathbf{w}}(2\beta^\vee)\frac{(1-\frac{1}{\sqrt{pr}}\chi^{\mathbf{w}s_\beta}(\beta^\vee))(1+\sqrt{\frac{p}{r}}\chi^{\mathbf{w}s_\beta}(\beta^\vee))}{(1-\frac{1}{\sqrt{pr}}\chi^{\mathbf{w}}(\beta^\vee))(1+\sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee))} & \text{if } i_k = n. \end{cases}$$

Since $\mathbf{W}$ can be seen as the Weyl group of the reduced building of type $\widetilde{C}_n$, we can repeat the argument used in the proof of Proposition 8.10 and write

$$\frac{d_{\mathbf{w}_0}(\chi)}{d_{\mathbf{e}}(\chi)} = -\chi(2\delta_0^\vee + 2\delta_1^\vee) \prod_{\alpha \in R_0^+} \frac{1 - q^{-1}\chi^{\mathbf{w}_0}(\alpha^\vee)}{1 - q^{-1}\chi(\alpha^\vee)}$$

$$\prod_{\beta \in R_1^+} \frac{(1 - \frac{1}{\sqrt{pr}}\chi^{\mathbf{w}_0}(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi^{\mathbf{w}_0}(\beta^\vee))}{(1 - \frac{1}{\sqrt{pr}}\chi(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi(\beta^\vee))}.$$

Since $\chi(2\delta_0^\vee + 2\delta_1^\vee) = \chi(2\delta^\vee) = \frac{\chi^{\mathbf{w}_0}(-\delta^\vee)}{\chi(-\delta^\vee)}$, the required formula (8.27) is proved. □

For ease of notation, we define, for every $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$,

$$\begin{cases} D_0(\chi) = \chi(-\delta_0^\vee) \prod\limits_{\alpha \in R_0^+} \left(1 - q^{-1}\chi(\alpha^\vee)\right) \\ D_1(\chi) = \chi(-\delta_1^\vee) \prod\limits_{\beta \in R_1^+} \left(1 - \sqrt{pr}^{-1}\chi(\beta^\vee)\right) \left(1 + \sqrt{\frac{p}{r}}\chi(\beta^\vee)\right) \end{cases} \tag{8.28}$$

and $D(\chi) = D_0(\chi)\ D_1(\chi)$. Moreover, we define, for every $\mathbf{w} \in \mathbf{W}$,

$$\begin{cases} D_{0,\mathbf{w}}(\chi) = D_0(\chi^{\mathbf{w}}) = \chi^{\mathbf{w}}(-\delta_0^\vee) \prod\limits_{\alpha \in R_0^+} (1 - q^{-1}\chi^{\mathbf{w}}(\alpha^\vee)) \\ D_{1,\mathbf{w}}(\chi) = D_1(\chi^{\mathbf{w}}) = \chi^{\mathbf{w}}(-\delta_1^\vee) \prod\limits_{\beta \in R_1^+} (1 - \sqrt{pr}^{-1}\chi^{\mathbf{w}}(\beta^\vee))\ (1 + \sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee)) \end{cases} \tag{8.29}$$

and $D_{\mathbf{w}}(\chi) = D_{0,\mathbf{w}}(\chi)\ D_{1,\mathbf{w}}(\chi)$. Property (8.22) assures that $D_{\mathbf{w}}(\chi) \neq 0$, for every $\mathbf{w} \in \mathbf{W}$; then we set

$$K(\chi) = \frac{d_{\mathbf{e}}(\chi)}{D(\chi)} = \frac{d_{\mathbf{e}}(\chi)}{D_0(\chi)\ D_1(\chi)}. \tag{8.30}$$

We note that $K(\chi) \neq 0$; thus Proposition 8.25 states that

$$d_{\mathbf{w}_0}(\chi) = (-1)^{|\mathbf{w}_0|}\ K(\chi)\ D_{\mathbf{w}_0}(\chi). \tag{8.31}$$

This formula can extended to each $d_{s_i}(\chi),\ i \in I_0$.

Proposition 8.26. — *Let $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$. Then, for every $i \in I_0$,*

$$d_{s_i}(\chi) = -K(\chi) D_{s_i}(\chi). \tag{8.32}$$

Proof. — Let $\mathbf{w} = s_{\alpha_i}$, for some $i = 1, \dots, n-1$. In this case $\alpha_i \in R_0^+$ and

$$\begin{aligned} d_{s_i}(\chi) &= -\chi(\alpha_i^\vee) \frac{1 - q_1^{-1}\chi^{s_i}(\alpha_i^\vee)}{1 - q_1^{-1}\chi(\alpha_i^\vee)}\ d_{\mathbf{e}}(\chi) \\ &= -\chi(\alpha_i^\vee) \frac{1 - q_1^{-1}\chi^{s_i}(\alpha_i^\vee)}{1 - q_1^{-1}\chi(\alpha_i^\vee)} D_{0,e}(\chi) D_{1,e}(\chi) K(\chi). \end{aligned}$$

We observe that $D_{1,s_i}(\chi) = D_{1,e}(\chi)$, because s_i doesn't change sign of any $\beta \in R_1$. Moreover, as in Proposition 8.11,

$$\begin{aligned} &\chi(\alpha_i^\vee) \frac{1 - q_1^{-1}\chi^{s_i}(\alpha_i^\vee)}{1 - q_1^{-1}\chi(\alpha_i^\vee)} D_{0,e}(\chi) \\ &\quad = \chi(\alpha_i^\vee - \delta_0^\vee)(1 - q_1^{-1}\chi^{s_i}(\alpha_i^\vee)) \prod_{\alpha \neq \alpha_i} (1 - q_1^{-1}\chi(\alpha_i^\vee)) \\ &\quad = \chi^{s_i}(-\delta_0^\vee) \prod_{\alpha \in R_0^+} (1 - q_1^{-1}\chi^{s_i}(\alpha_i^\vee)), \end{aligned}$$

because (see [3, (10.2)]) $\alpha_i^\vee - \delta_0^\vee = \frac{1}{2}\alpha_i^\vee - \frac{1}{2}\sum_{\alpha^\vee \neq \alpha_i^\vee} \alpha^\vee = s_i(-\delta_0^\vee)$. Therefore

$$d_{s_i}(\chi) = -K(\chi) D_{0,s_i}(\chi) D_1(\chi) = -K(\chi) D_{0,s_i}(\chi) D_{1,s_i}(\chi) = -K(\chi) D_{s_i}(\chi).$$

Assume now $\alpha_i = \beta = 2e_n$. In this case ,

$$\begin{aligned} d_{s_i}(\chi) &= -\chi(2\beta^\vee) \frac{(1 - \frac{1}{\sqrt{pr}}\chi^{s_\beta}(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi^{s_\beta}(\beta^\vee))}{(1 - \frac{1}{\sqrt{pr}}\chi(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi(\beta^\vee))}\ d_{\mathbf{e}}(\chi) \\ &= -\chi(2\beta) \frac{(1 - \frac{1}{\sqrt{pr}}\chi^{s_\beta}(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi^{s_\beta}(\beta^\vee))}{(1 - \frac{1}{\sqrt{pr}}\chi(\beta^\vee))(1 + \sqrt{\frac{p}{r}}\chi(\beta^\vee))} D_{0,e}(\chi) D_{1,e}(\chi) K(\chi). \end{aligned}$$

We note that $D_{0,s_\beta}(\chi) = D_{0,e}(\chi)$, because s_β doesn't change sign of any α. Moreover

$$\chi(2\beta^\vee)\frac{(1-\frac{1}{\sqrt{pr}}\chi^{s_\beta}(\beta^\vee))(1+\sqrt{\frac{p}{r}}\chi^{s_\beta}(\beta^\vee))}{(1-\frac{1}{\sqrt{pr}}\chi(\beta^\vee))(1+\sqrt{\frac{p}{r}}\chi(\beta^\vee))}D_{1,e}(\chi) = D_{1,s_\beta}(\chi).$$

Hence

$$d_{s_i}(\chi) = -K(\chi)D_{0,e}(\chi)D_{1,s_i}(\chi) = -K(\chi)D_{s_i}(\chi).$$

□

As in the reduced case, we shall prove that this formula holds for all $\mathbf{w} \in \mathbf{W}$.

Lemma 8.27. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$; then, for every $\mathbf{w} \in \mathbf{W}$,*

$$K(\chi^{\mathbf{w}}) = (-1)^{|\mathbf{w}|}K(\chi).$$

Proof. — We proceed by induction on the length of $\mathbf{w}$, as in Lemma 8.12, using Proposition 8.26. □

Proposition 8.28. — *Let $\chi \in \mathbf{X}^0_{gg}(\widehat{L})$. Then, for every $\mathbf{w} \in \mathbf{W}$,*

$$d_{\mathbf{w}}(\chi) = (-1)^{|\mathbf{w}|}K(\chi)D_{\mathbf{w}}(\chi). \tag{8.33}$$

Proof. — The formula follows from Proposition 8.26 and Lemma 8.27.
□

The next step is to compute the constant $K(\chi)$, recalling that

$$\frac{1}{K(\chi)} = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}D_{\mathbf{w}}(\chi).$$

We compute $\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}D_{\mathbf{w}}(\chi)$ as in the reduced case, changing things in the appropriate way. More precisely, we renumerate the N_0 positive roots in R_0, and the N_1 positive roots in R_1,by setting $R_0^+ = \{\alpha_1,\dots,\alpha_{N_0}\}$ and $R_1^+ = \{\beta_1,\dots,\beta_{N_1}\}$. So we can write

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}D_{\mathbf{w}}(\chi) = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta_0^\vee)\chi^{\mathbf{w}}(-\delta_1^\vee)\prod_{i=1}^{N_0}\left(1-q^{-1}\chi^{\mathbf{w}}(\alpha_i^\vee)\right)$$
$$\prod_{j=1}^{N_1}\left(1-\sqrt{pr}^{-1}\chi^{\mathbf{w}}(\beta_j^\vee)\right)\left(1+\sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta_j^\vee)\right).$$

We need some preliminary results, to simplify the calculation of the previous sum.

Lemma 8.29. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. Consider, for $1 \leqslant r \leqslant N_0$ and $1 \leqslant k \leqslant N_1$,*

$$\mu_0^\vee = \sum_{j=1}^{r} \alpha_{i_j}^\vee, \qquad \nu_1^\vee = \sum_{m=1}^{k} \beta_{i_m}^\vee.$$

If there exists $\mathbf{w}_1 \in \mathbf{W}$ such that $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - 2\nu_1^\vee) = \delta^\vee$, then

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_0^\vee + 2\nu_1^\vee) = (-1)^{|\mathbf{w}_1|} \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee);$$

otherwise

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_0^\vee + 2\nu_1^\vee) = 0.$$

Proof. — Let $\mu_0^\vee = \sum_{j=1}^r \alpha_{i_j}^\vee$, for $r \leqslant N_0$. The argument used in Lemma 8.14 proves that $\delta^\vee - \mu_0^\vee \in \Pi_{\delta^\vee}$.

Let $\nu_1^\vee = \sum_{m=1}^k \beta_{i_m}^\vee$, for $k \leqslant N_1$. Then

$$\delta^\vee - 2\nu_1^\vee = \frac{1}{2}\sum_{\alpha\in R_0^+} \alpha^\vee + \frac{1}{2}\sum_{\beta\neq\beta_{i_m}^\vee} -\frac{1}{2}\sum_{m=1}^{k} \beta_{i_m}^\vee \beta_j^\vee$$

and we can prove as above that $\delta^\vee - 2\nu_1^\vee \in \Pi_{\delta^\vee}$. As in reduced case, there exist choices of $\mu_0^\vee, \nu_1^\vee$, such that $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - 2\nu_1^\vee) = \delta^\vee$. Otherwise, if $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - 2\nu_1^\vee) \neq \delta^\vee$, then there exists $i = 1, \ldots, n$ such that $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - 2\nu_1^\vee) \in H_i^0$. Therefore the required result holds. $\square$

Lemma 8.30. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. For every $1 \leqslant r \leqslant N_0$ and $1 \leqslant h, k \leqslant N_1$, consider*

$$\mu_0^\vee = \sum_{j=1}^{r} \alpha_{i_j}^\vee, \quad \mu_1^\vee = \sum_{l=1}^{h} \beta_{i_l}^\vee, \quad \nu_1^\vee = \sum_{m=1}^{k} \beta_{i_m}^\vee.$$

Assume that $i_l \neq i_m$, for every l and m. For every choice of $\mu_0^\vee, \mu_1^\vee$ and $\nu_1^\vee$,

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_0^\vee + \mu_1^\vee + 2\nu_1^\vee) = 0.$$

Proof. — We have proved in Lemma 8.29 that $\delta^\vee - \mu_0^\vee$ and $\delta^\vee - 2\nu_1^\vee$ belong to $\Pi_{\delta^\vee}$. Since

$$\delta^\vee - \mu_1^\vee = \delta_0 + (\delta_1 - \mu_1^\vee) = \frac{1}{2}\sum_{\alpha\in R_0^+} \alpha^\vee + \sum_{\beta\neq\beta_{i_l}} \beta^\vee,$$

$\delta^\vee - \mu_1^\vee$ is a sum of positive roots and $\delta^\vee - \mu_1^\vee \preceq \delta^\vee$; so $\delta^\vee - \mu_1^\vee \in \Pi_{\delta^\vee}$ and $\delta^\vee - \mu_1^\vee \in \mathbb{Q}_0$.

More generally, if we assume that $i_l \neq i_m$, for every l and m, we can prove, combining previous results, that $\delta^\vee - \mu_0^\vee - \mu_1^\vee - 2\nu_1^\vee$ belong to $\Pi_{\delta^\vee}$. This fact implies that there exists $\mathbf{w}_1 \in \mathbf{W}$, such that $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - \mu_1^\vee - 2\nu_1^\vee) \in \mathbb{Q}_0$. Since, as in the reduced case, $\delta^\vee$ is the only element in $\Pi_{\delta^\vee}$ such that $\Pi_{\delta^\vee} \in (\mathbb{Q}_0)^0$, then we have two possibilities: either $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - \mu_1^\vee - 2\nu_1^\vee) = \delta^\vee$, or $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - \mu_1^\vee - 2\nu_1^\vee)$ belongs to the linear hyperplane H_i^0, for some $i = 1, \dots, n$.

But $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - \mu_1^\vee - 2\nu_1^\vee)$ can not be equal to $\delta^\vee$, since $\delta^\vee - \mu_0^\vee - \mu_1^\vee - 2\nu_1^\vee$ does not contains the roots $\beta_{i_1}^\vee, \dots, \beta_{i_k}^\vee$ such that $\mu_1^\vee = \sum_{j=1}^h \beta_{i_j}^\vee$. Therefore it must belong to some linear hyperplane H_i^0, $i = 1, \dots, n$. This proves the required result. $\square$

The following lemma generalizes Lemma 8.15 given in the reduced case.

Lemma 8.31. — *The following formulas hold.*

(i) Let $Y_1, \dots, Y_M$, $Z_1, \dots, Z_M$ be $2M$ variables and $\Pi_M^2(Y, Z) = \prod_{k=1}^M (1+ Y_k + Z_k)$; then

$$\Pi_M^2(Y, Z) = \sum_{k=0}^{M} \sum_{m=0}^{k} \sum_{\mathcal{I}_{k,m}^M} Y_{i_1} \cdots Y_{i_m} Z_{i_{m+1}} \cdots Z_{i_k},$$

where, for every $k \geqslant 1$ and $0 < m < k$,

$$\mathcal{I}_{k,m}^M = \{(j_1, \dots, j_k),\ \ j_1, \dots, j_k \in \{1, \dots, M\},\ \ j_1 < \cdots < j_m,$$

$$j_{m+1} < \cdots < j_k,\ \textit{such that}\ j_l \neq j_{l'}\ \textit{if}\ 1 \leqslant l \leqslant m\ \textit{and}\ m+1 \leqslant l' \leqslant k\ \},$$

while, for every $k \geqslant 1$,

$$\mathcal{I}_{k,0}^M = \mathcal{I}_{k,k}^M = \{(j_1, \dots, j_k),\ \textit{where}\ j_1, \dots, j_k \in \{1, \dots, M\},\ j_1 < \cdots < j_k\},$$

and

$$\mathcal{I}_{0,0}^M = \emptyset \quad \textit{with} \sum_{\mathcal{I}_{0,0}^M} Y_{i_1} \cdots Y_{i_m} Z_{i_{m+1}} \cdots Z_{i_k} = 1.$$

(ii) Let $X_1,\ldots,X_N,Y_1,\ldots,Y_M,Z_1,\ldots,Z_M$ *be* $N+2M$ *variables and*

$$\Pi^1_N(X)\ \Pi^2_M(Y,Z)=\prod_{h=1}^{N}(1+X_h)\ \prod_{k=1}^{M}(1+Y_k+Z_k);$$

then

$$\Pi^1_N(X)\ \Pi^2_M(Y,Z)=$$

$$\sum_{h=0}^{N}\sum_{k=0}^{M}\sum_{m=0}^{k}\quad\sum_{\mathcal{I}^N_h}\sum_{\mathcal{I}^M_{k,m}}X_{i_1}\cdots X_{i_h}Y_{j_1}\cdots Y_{j_m}Z_{j_{m+1}}\cdots Z_{j_k}.$$

(iii) If a,b,c *are non-zero constants, we set*

$$\Pi^1_N(c)\ \Pi^2_M(a,b)=\prod_{h=1}^{N}(1-cX_h)\ \prod_{k=1}^{M}(1+aY_k-bZ_k).$$

Then

$$\Pi^1_N(c)\ \Pi^2_M(a,b)=$$

$$\sum_{h=0}^{N}\sum_{k=0}^{M}\sum_{m=0}^{k}(-1)^{h+k}\ a^mb^kc^h\ \sum_{\mathcal{I}^N_h}\sum_{\mathcal{I}^M_{k,m}}X_{i_1}\cdots X_{i_h}Y_{j_1}\cdots Y_{j_m}Z_{j_{m+1}}\cdots Z_{i_k}.$$

Proof. — The formula for $\Pi^2_M(Y,Z)$ can be proved by induction; (ii) and (iii) are an immediate consequence of (i) and Lemma 8.15. □

For ease of notation we write $|\mathbf{w}|=h+k$ if $\mathbf{w}$ is a reduced word consisting of h generators $s_{\alpha_{i_t}}$ and k generators $s_{\beta_{j_l}}$. Define

$$\mathbf{W}_{h+k}=\{\mathbf{w}\in\mathbf{W},\ |\mathbf{w}|=h+k\}.$$

Proposition 8.32. — *Let* χ *be a character of* $\mathbf{X}^0_{gg}(\widehat{L})$*. Then,*

$$\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}D_{\mathbf{w}}(\chi)=\left[\sum_{h=0}^{N_0}\sum_{k=0}^{N_1}q^{-h}\ r^{-k}|\mathbf{W}_{h+k}|\right]\ \left[\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee)\right].\tag{8.34}$$

Proof. — By (8.29), for every $\mathbf{w} \in \mathbf{W}$,

$$D_{\mathbf{w}}(\chi) = \chi^{\mathbf{w}}(-\delta_0^\vee)\chi^{\mathbf{w}}(-\delta_1^\vee) \prod_{h=1}^{N_0} \left(1 - q^{-1}\chi^{\mathbf{w}}(\alpha_h^\vee)\right)$$
$$\prod_{k=1}^{N_1} \left(1 - \sqrt{pr}^{-1}\chi^{\mathbf{w}}(\beta_k^\vee)\right)\left(1 + \sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta_k^\vee)\right)$$
$$= \chi^{\mathbf{w}}(-\delta^\vee) \prod_{h=1}^{N_0}(1 - q^{-1}\chi^{\mathbf{w}}(\alpha_h^\vee)) \prod_{k=1}^{N_1} \left(1 + \frac{p-1}{\sqrt{pr}}\chi^{\mathbf{w}}(\beta_k^\vee) - r^{-1}\chi^{\mathbf{w}}(2\beta_k^\vee)\right).$$

For ease of notation, we set $N = N_0$, $M = N_1$ and $a = \frac{p-1}{\sqrt{pr}}$, $b = r^{-1}$, $c = q^{-1}$. Moreover we set $X_h = \chi^{\mathbf{w}}(\alpha_s^\vee)$, for all $h = 1, \dots, N_0$, and $Y_k = \chi^{\mathbf{w}}(\beta_k^\vee)$, $Z_k = \chi^{\mathbf{w}}(2\beta_k^\vee) = Y_k^2$, for all $k = 1, \dots, N_1$. Then we can write

$$D_{\mathbf{w}}(\chi) = \chi^{\mathbf{w}}(-\delta^\vee) \prod_{h=1}^{N}(1 - cX_h) \prod_{k=1}^{M}(1 + aY_k - bZ_k).$$

Therefore Lemma 8.31 implies that

$$\begin{aligned} D_{\mathbf{w}}(\chi) &= \chi^{\mathbf{w}}(-\delta^\vee) \sum_{h=0}^{N}\sum_{k=0}^{M}\sum_{m=0}^{k}(-1)^h(-1)^{k-m}a^m b^{k-m}c^h \\ &\quad \sum_{\mathcal{I}_h^N}\sum_{\mathcal{I}_{k,m}^M} X_{i_1}\cdots X_{i_h}Y_{j_1}\cdots Y_{j_m}Z_{j_{m+1}}\cdots Z_{i_k} \\ &= \sum_{h=0}^{N}\sum_{k=0}^{M}\sum_{m=0}^{k}(-1)^{h+k-m}\left(\frac{p-1}{\sqrt{pr}}\right)^m r^{k-m}q^{-h} \\ &\quad \sum_{\mathcal{I}_h^N}\sum_{\mathcal{I}_{k,m}^M} \chi^{\mathbf{w}}(-\delta^\vee + \mu_0^\vee + \mu_1^\vee + 2\nu_1^\vee), \end{aligned}$$

if we set, according to notation of Lemma 8.30, $\mu_0^\vee = \alpha_{i_1}^\vee + \cdots + \alpha_{i_h}^\vee$, $\mu_1^\vee = \beta_{j_1}^\vee + \cdots + \beta_{j_m}^\vee$, and $\nu_1^\vee = \beta_{j_{m+1}}^\vee + \cdots + \beta_{j_k}^\vee$. By Lemma 8.30, $\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_0^\vee + \mu_1^\vee + 2\nu_1^\vee) = 0$ except when $m = 0$. Moreover, by Lemma 8.29, if $m = 0$ then

$$\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_0^\vee + 2\nu_1^\vee) = (-1)^{|\mathbf{w}_1|}\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee),$$

if $\mathbf{w}_1(\delta^\vee - \mu_0^\vee - 2\nu_1^\vee) = \delta^\vee$, and $\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee + \mu_0^\vee + 2\nu_1^\vee) = 0$ otherwise. To conclude, we choose $m = 0$. Since the only choices that give a non-zero contribution in the previous sum arose from $\mathbf{w}_1 \in \mathbf{W}_{h+k}$, then

$$\begin{aligned}\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}D_{\mathbf{w}}(\chi) &= \sum_{h=0}^{N_0}\sum_{k=0}^{N_1}(-1)^{h+k}\ q^{-h}\ r^{-k}|\mathbf{W}_{h+k}|(-1)^{|\mathbf{w}_1|}\\ &\quad \sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee)\\ &= \left(\sum_{h=0}^{N_0}\sum_{k=0}^{N_1}q^{-h}r^{-k}|\mathbf{W}_{h+k}|\right)\ \sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee).\end{aligned}$$

□

We calculate separately

$$\sum_{h=0}^{N_0}\sum_{k=0}^{N_1}q^{-r}r^{-k}|\mathbf{W}_{h+k}|) \quad \text{and} \quad \sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee).$$

LEMMA 8.33. — *We have*

$$\sum_{h=0}^{N_0}\sum_{k=0}^{N_1}q^{-h}r^{-k}|\mathbf{W}_{h+k}| = \mathbf{W}(q^{-1}). \tag{8.35}$$

Proof. — Since $\mathbf{W}$ can be seen as the Weyl group of a building of type $\widetilde{B_n}$, with parameters q and r, then formula (8.35) follows by definition of $q_{\mathbf{w}}$. □

LEMMA 8.34. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. Then*

$$\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee) = \chi(-\delta^\vee)\ \prod_{\alpha\in R_0^+}(1-\chi(\alpha^\vee))\ \prod_{\beta\in R_1^+}(1-\chi^2(\beta^\vee)). \tag{8.36}$$

Proof. — Since $R_2 \cup R_0$ is a root system of type B_n and $\mathbf{W}$ can be seen as the Weyl group associated to this root system, we can apply Lemma 8.18 to this root system and so we get

$$\begin{aligned}\sum_{\mathbf{w}\in\mathbf{W}}(-1)^{|\mathbf{w}|}\chi^{\mathbf{w}}(-\delta^\vee) &= \chi(-\delta^\vee)\ \prod_{h=1}^{N_0}(1-\chi(\alpha_{i_h}^\vee))\ \prod_{k=1}^{N_1}(1-\chi(\gamma_{j_k}^\vee))\\ &= \chi(-\delta^\vee)\ \prod_{h=1}^{N_0}(1-\chi(\alpha_{i_h}^\vee))\ \prod_{k=1}^{N_1}(1-\chi(2\beta_{j_k}^\vee)).\end{aligned}$$

□

Proposition 8.35. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. Then,*

$$K(\chi) = \frac{\chi(\delta^\vee)}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R_0^+} \left(\frac{1}{1-\chi(\alpha^\vee)}\right) \prod_{\beta\in R_1^+} \left(\frac{1}{1-\chi^2(\beta^\vee)}\right). \qquad (8.37)$$

Proof. — Lemma 8.33 and 8.34 imply that

$$\sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\, D_{\mathbf{w}}(\chi) = \mathbf{W}(q^{-1})\, \chi(-\delta^\vee) \prod_{\alpha\in R_0^+} (1-\chi(\alpha^\vee)) \prod_{\beta\in R_1^+} (1-\chi(2\beta^\vee))\,.$$

Since $K(\chi)^{-1} = \sum_{\mathbf{w}\in\mathbf{W}} (-1)^{|\mathbf{w}|}\, D_{\mathbf{w}}(\chi)$, then $\prod_{\alpha\in R_0^+} (1-\chi(\alpha^\vee)) \prod_{\beta\in R_1^+} (1-\chi(2\beta^\vee)) \neq 0$ and (8.37) is proved. $\square$

We are now able to exhibit, also in the non-reduced case, the explicit expression of each coefficient $d_{\mathbf{w}}(\chi)$, for characters of $\mathbf{X}^0_{gg}(\widehat{L})$.

Theorem 8.36. — *Let χ be a character of $\mathbf{X}^0_{gg}(\widehat{L})$. Then, for every $\mathbf{w}\in\mathbf{W}$,*

$$d_{\mathbf{w}}(\chi) = \frac{1}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R_0^+} \left(\frac{1-q_\alpha^{-1}\chi^{\mathbf{w}}(\alpha^\vee)}{1-\chi^{\mathbf{w}}(\alpha^\vee)}\right)$$

$$\prod_{\beta\in R_1^+} \left(\frac{\left(1-\sqrt{pr}^{-1}\chi^{\mathbf{w}}(\beta^\vee)\right)\ \left(1+\sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee)\right)}{1-\chi^{\mathbf{w}}(\beta^\vee)}\right). \qquad (8.38)$$

Proof. — By Proposition 8.35,

$$d_{\mathbf{e}}(\chi) = K(\chi)D(\chi) = K(\chi)\chi(-\delta^\vee) \prod_{\alpha\in R_0^+} \left(1-q^{-1}\chi(\alpha^\vee)\right)$$

$$\prod_{\beta\in R_1^+} \left(1-\sqrt{pr}^{-1}\chi(\beta^\vee)\right)\left(1+\sqrt{\frac{p}{r}}\chi(\beta^\vee)\right)$$

$$= \frac{1}{\mathbf{W}(q^{-1})} \prod_{\alpha\in R_0^+} \left(\frac{1-q^{-1}\chi(\alpha^\vee)}{1-\chi(\alpha^\vee)}\right)$$

$$\prod_{\beta\in R_1^+} \left(\frac{\left(1-\sqrt{pr}^{-1}\chi(\beta^\vee)\right)\ \left(1+\sqrt{\frac{p}{r}}\chi(\beta^\vee)\right)}{1-\chi^{\mathbf{w}}(\beta^\vee)}\right).$$

Since $d_{\mathbf{w}}(\chi) = d_{\mathbf{e}}(\chi^{\mathbf{w}})$, for every $\mathbf{w}\in\mathbf{W}$, then formula (8.38) is proved. $\square$

Remark 8.37. — Since, for every $\mathbf{w} \in \mathbf{W}$, the coefficient $d_{\mathbf{w}}(\chi)$, as well as the function on the right side in (8.38), is defined on the space $\mathbf{X}_g(\widehat{L})$ and depends continuously on the character χ with respect to the weak topology, formula (8.38) can be extended to all characters of $\mathbf{X}_g(\widehat{L})$ by a standard argument of continuity, taking in account that $\mathbf{X}^0_{gg}(\widehat{L})$ is dense in $\mathbf{X}_{gg}(\widehat{L})$ and then in $\mathbf{X}_g(\widehat{L})$.

8.5. The main theorem

Formulas (8.20) and (8.38) can be expressed in a single formula, if we fix a convenient notation. We set

$$\begin{aligned}
\tau_\alpha &= q_\alpha = q, \ \text{ and } \ \tau_{\alpha/2} = 1, \ \ \forall\, \alpha \in R_0,\\
\tau_\beta &= q_\beta = p, \ \text{ and } \ \tau_{\beta/2} = \frac{q_\gamma}{q_\beta} = \frac{r}{p}, \ \ \forall\, \beta \in R_1,\\
\tau_\gamma &= \frac{q_\gamma}{q_\beta} = \frac{r}{p}, \ \text{ and } \ \tau_{\gamma/2} = 1, \ \ \forall\, \gamma \in R_2.
\end{aligned}$$

THEOREM 8.38. — *Let Δ let an affine building, reduced or non-reduced. Let $\chi \in \mathbf{X}_g(\widehat{L})$. Then, for all $\mathbf{w} \in \mathbf{W}$,*

$$d_{\mathbf{w}}(\chi) = \frac{1}{\mathbf{W}(q^{-1})} \prod_{\alpha \in R^+} \frac{1 - \tau_\alpha^{-1}\tau_{\alpha/2}^{-1/2}\chi^{\mathbf{w}}(\alpha^\vee)}{1 - \tau_{\alpha/2}^{-1/2}\chi^{\mathbf{w}}(\alpha^\vee)}. \tag{8.39}$$

Proof. — If Δ is reduced, then $R = R_0$, and hence the formula is a direct consequence of Theorem 8.20. Assume now that the building is non-reduced. For every $\alpha \in R_0$, we have

$$\frac{1 - q^{-1}\chi^{\mathbf{w}}(\alpha^\vee)}{1 - \chi^{\mathbf{w}}(\alpha^\vee)} = \frac{1 - \tau_\alpha^{-1}\tau_{\frac{\alpha}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\alpha^\vee)}{1 - \tau_{\frac{\alpha}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\alpha^\vee)}.$$

Moreover, for every $\beta \in R_1$,

$$1 + \sqrt{\frac{p}{r}}\,\chi^{\mathbf{w}}(\beta^\vee) = \frac{1 - \frac{p}{r}\chi^{\mathbf{w}}(2\beta^\vee)}{1 - \sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee)};$$

so

$$\frac{\left(1+\sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee)\right)\ \left(1-\frac{1}{\sqrt{pr}}\chi^{\mathbf{w}}(\beta^\vee)\right)}{1-\chi^{\mathbf{w}}(2\beta^\vee)} = \frac{1-\frac{p}{r}\chi^{\mathbf{w}}(2\beta^\vee)}{1-\chi^{\mathbf{w}}(2\beta^\vee)}\ \frac{1-\frac{1}{\sqrt{pr}}\chi^{\mathbf{w}}(\beta^\vee)}{1-\sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee)}$$
$$= \frac{1-\frac{p}{r}\chi^{\mathbf{w}}(\gamma^\vee)}{1-\chi^{\mathbf{w}}(\gamma^\vee)}\ \frac{1-\frac{1}{\sqrt{pr}}\chi^{\mathbf{w}}(\beta^\vee)}{1-\sqrt{\frac{p}{r}}\chi^{\mathbf{w}}(\beta^\vee)}$$
$$= \frac{1-\tau_\beta^{-1}\tau_{\beta/2}^{-1/2}\chi^{\mathbf{w}}(\beta^\vee)}{1-\tau_{\beta/2}^{-1/2}\chi^{\mathbf{w}}(\beta^\vee)}\ \frac{1-\tau_\gamma^{-1}\tau_{\gamma/2}^{-1/2}\chi^{\mathbf{w}}(\gamma^\vee)}{1-\tau_{\gamma/2}^{-1/2}\chi^{\mathbf{w}}(\gamma^\vee)}.$$

Hence

$$\prod_{\beta\in R_1^+}\left(\frac{\left(1-\sqrt{pr}^{-1}\chi(\beta^\vee)\right)\ \left(1+\sqrt{\frac{p}{r}}\chi(\beta^\vee)\right)}{1-\chi^{\mathbf{w}}(\beta^\vee)}\right)$$
$$= \prod_{\beta\in R_1^+}\frac{1-\tau_\beta^{-1}\tau_{\frac{\beta}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\beta^\vee)}{1-\tau_{\frac{\beta}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\beta^\vee)}\ \prod_{\gamma\in R_2^+}\frac{1-\tau_\gamma^{-1}\tau_{\frac{\gamma}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\gamma^\vee)}{1-\tau_{\frac{\gamma}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\gamma^\vee)}.$$

Therefore, by Theorem 8.35, we conclude that

$$\begin{aligned} d_{\mathbf{w}}(\chi) &= \frac{1}{\mathbf{W}(q^{-1})}\prod_{\alpha\in R_0^+}\frac{1-\tau_\alpha^{-1}\tau_{\alpha/2}^{-1/2}\chi^{\mathbf{w}}(\alpha^\vee)}{1-\tau_{\alpha/2}^{-1/2}\chi^{\mathbf{w}}(\alpha^\vee)}\prod_{\beta\in R_1^+}\frac{1-\tau_\beta^{-1}\tau_{\frac{\beta}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\beta^\vee)}{1-\tau_{\frac{\beta}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\beta^\vee)} \\ &\quad \prod_{\gamma\in R_2^+}\frac{1-\tau_\gamma^{-1}\tau_{\frac{\gamma}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\gamma^\vee)}{1-\tau_{\frac{\gamma}{2}}^{-\frac{1}{2}}\chi^{\mathbf{w}}(\gamma^\vee)} \\ &= \frac{1}{\mathbf{W}(q^{-1})}\prod_{\alpha\in R^+}\frac{1-\tau_\alpha^{-1}\tau_{\alpha/2}^{-1/2}\chi^{\mathbf{w}}(\alpha^\vee)}{1-\tau_{\alpha/2}^{-1/2}\chi^{\mathbf{w}}(\alpha^\vee)}, \end{aligned}$$

since $R = R_1\cup R_2\cup R_0$. □

8.6. The formula of the spherical function associated with a non-singular character

We can finally state the explicit formula for the spherical function $\varphi_{\chi{\chi_0}^{1/2}}(x)$ associated with any non-singular character χ, for every affine building Δ.

THEOREM 8.39. — *Let χ be a non-singular character on $\widehat{L}$. For every $\lambda\in\widehat{L}^+$ and every $x\in V_\lambda(e)$,*

$$\varphi_{\chi{\chi_0}^{1/2}}(x) = \frac{\chi_0^{1/2}(\lambda)}{\mathbf{W}(q^{-1})}\sum_{\mathbf{w}\in\mathbf{W}} c_{\mathbf{w}}^0(\chi)\ \chi^{\mathbf{w}}(\lambda), \tag{8.40}$$

where, for every $\mathbf{w} \in \mathbf{W}$,

$$c^0_{\mathbf{w}}(\chi) = \prod_{\alpha \in R^+} \frac{1 - \tau_\alpha^{-1} \tau_{\frac{\alpha}{2}}^{-\frac{1}{2}} \chi^{\mathbf{w}}(-\alpha^\vee)}{1 - \chi^{\mathbf{w}}(-\alpha^\vee)}. \tag{8.41}$$

Proof. — If $\chi \in \mathbf{X}_g(\widehat{L})$, the statement follows from (6.16), (6.17) and (8.39). Assume now that χ is any non-singular character. We notice that, for every $\lambda \in \widehat{L}^+$ and every $x \in V_\lambda(e)$, the function $\mathbf{W}^{-1}(q^{-1})\chi_0^{1/2}(\lambda)\sum_{\mathbf{w}\in\mathbf{W}} c^0_{\mathbf{w}}(\chi)\chi^{\mathbf{w}}(\lambda)$ with coefficients $c^0_{\mathbf{w}}(\chi)$ given by (8.41), as well as $\varphi_{\chi{\chi_0}^{1/2}}(x)$, is defined for all non-singular characters on $\mathbb{A}$ and depends continuously on the character χ, with respect to the weak topology on the space $\mathbf{X}_{NS}(\widehat{L})$. Since the space $\mathbf{X}_g(\widehat{L})$ is dense in $\mathbf{X}_{NS}(\widehat{L})$, we can conclude that the formula (8.40) actually holds for every character of $\mathbf{X}_{NS}(\widehat{L})$. $\square$

8.7. The singular case

If we assume that $\widetilde{\chi}$ is a singular character, that is $\widetilde{\chi}(\alpha^\vee) = 1$, for some positive root α, we can calculate $\varphi_{\widetilde{\chi}\chi^{1/2}}(x)$ as the limit of $\varphi_{\chi{\chi_0}^{1/2}}(x)$, as $\chi \to \widetilde{\chi}$, through non-singular values of χ. This limit can be computed as in [5, Section (4.6)].

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