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Local-global compatibility for $l = p$, I

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Abstract. — We prove the compatibility of the local and global Langlands correspondences at places dividing l for the l-adic Galois representations associated to regular algebraic conjugate self-dual cuspidal automorphic representations of GL_n over an imaginary CM field, under the assumption that the automorphic representations have Iwahori-fixed vectors at places dividing l and have Shin-regular weight.

Résumé. — Nous prouvons la compatibilité entre les correspondances de Langlands locale et globale aux places divisant l pour les représentations galoisiennes l-adiques associèes à des représentations automorphes cuspidales algébriques et régulières de GL_n sur un corps CM qui sont duales de leur conjuguée complexe, sous les hypothèses supplémentaires que ces représentations automorphes ont des vecteurs fixes par un sous-groupe d'Iwahori aux places divisant l et ont un poids régulier au sens de Shin.

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Article proposé par Pierre Colmez.

Introduction

In this paper we prove the compatibility at places dividing l of the local and global Langlands correspondences for the l-adic Galois representations associated to regular algebraic conjugate self-dual cuspidal automorphic representations of GL_n over an imaginary CM field in the special case that the automorphic representations have Iwahori-fixed vectors at places dividing l and have Shin-regular weight. In the sequel to this paper [2] we build on these results to prove the compatibility in general (up to semisimplification in the case of non-Shin-regular weight).

Our main result is as follows (see Theorem 1.2 and Corollary 1.3).

Theorem A. — *Let $m \geqslant 2$ be an integer, l a rational prime and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let F be an imaginary CM field and Π a regular algebraic, conjugate self-dual cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$. If Π has Shin-regular weight and $v|l$ is a place of F such that $\Pi_v^{\mathrm{Iw}_{m,v}} \neq \{0\}$, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

In particular $\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})$ is pure.

(See Section 1 for any unfamiliar terminology.) The proof is essentially an immediate application of the methods of [14], applied in the setting of [13] rather than that of [8], and we refer the reader to the introductions of those papers for the details of the methods that we use. Indeed, if Π is square-integrable at some finite place, then the result is implicit in [14], although it is not explicitly recorded there. For the convenience of the reader, we make an effort to make our proof as self-contained as possible.

Notation and terminology

We write all matrix transposes on the left; so ${}^tA$ is the transpose of A. We let $B_m \subset GL_m$ denote the Borel subgroup of upper triangular matrices and $T_m \subset GL_m$ the diagonal torus. We let I_m denote the identity matrix in GL_m. We will sometimes denote the product $GL_m \times GL_n$ by $GL_{m,n}$.

If M is a field, we let $\overline{M}$ denote a separable closure of M and G_M the absolute Galois group $\mathrm{Gal}\,(\overline{M}/M)$. Let ϵ_l denote the l-adic cyclotomic character

Let p be a rational prime and $K/\mathbb{Q}_p$ a finite extension. We let $\mathcal{O}_K$ denote the ring of integers of K, $\wp_K$ the maximal ideal of $\mathcal{O}_K$, $k(\nu_K)$ the residue field $\mathcal{O}_K/\wp_K$, $\nu_K : K^\times \twoheadrightarrow \mathbb{Z}$ the canonical valuation and $|\ |_K : K^\times \to \mathbb{Q}^\times$

the absolute value given by $|x|_K = \#(k(\nu_K))^{-\nu_K(x)}$. We let $|\ |_K^{1/2} : K^\times \to \mathbb{R}^\times_{>0}$ denote the unique positive unramified square root of $|\ |_K$. If K is clear from the context, we will sometimes write $|\ |$ for $|\ |_K$. We let Frob_K denote the geometric Frobenius element of $G_{k(\nu_K)}$ and I_K the kernel of the natural surjection $G_K \twoheadrightarrow G_{k(\nu_K)}$. We will sometimes abbreviate $\mathrm{Frob}_{\mathbb{Q}_p}$ by Frob_p. We let W_K denote the preimage of $\mathrm{Frob}_K^{\mathbb{Z}}$ under the map $G_K \twoheadrightarrow G_{k(\nu(K))}$, endowed with a topology by decreeing that $I_K \subset W_K$ with its usual topology is an open subgroup of W_K. We let $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ denote the local Artin map, normalized to take uniformizers to lifts of Frob_K.

Let Ω be an algebraically closed field of characteristic 0. A Weil-Deligne representation of W_K over Ω is a triple (V, r, N) where V is a finite dimensional vector space over Ω, $r : W_K \to GL(V)$ is a representation with open kernel and $N : V \to V$ is an endomorphism with $r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_K^{-1}(\sigma)|_K N$. We say that (V, r, N) is Frobenius semisimple if r is semisimple and we let $(V, r, N)^{\mathrm{F-ss}}$ denote the Frobenius semisimplification of (V, r, N) (see for instance Section 1 of [14]) and we let $(V, r, N)^{\mathrm{ss}}$ denote $(V, r^{\mathrm{ss}}, 0)$. If Ω has the same cardinality as $\mathbb{C}$, we have the notions of a Weil-Deligne representation being *pure* or *pure of weight* k – see the paragraph before Lemma 1.4 of [14].

We will let rec_K be the local Langlands correspondence of [8], so that if π is an irreducible complex admissible representation of $GL_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Weil-Deligne representation of the Weil group W_K. We will write rec for rec_K when the choice of K is clear. If ρ is a continuous representation of G_K over $\overline{\mathbb{Q}}_l$ with $l \neq p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil-Deligne representation of W_K. (See for instance Section 1 of [14].)

If $m \geqslant 1$ is an integer, we let $\mathrm{Iw}_{m,K} \subset GL_m(\mathcal{O}_K)$ denote the subgroup of matrices which map to an upper triangular matrix in $GL_m(k(\nu_K))$. If π is an irreducible admissible supercuspidal representation of $GL_m(K)$ and $s \geqslant 1$ is an integer we let $\mathrm{Sp}_s(\pi)$ be the square integrable representation of $GL_{ms}(K)$ defined for instance in Section I.3 of [8]. Similarly, if $r : W_K \to GL_m(\Omega)$ is an irreducible representation with open kernel and π is the supercuspidal representation $\mathrm{rec}_K^{-1}(r)$, we let $\mathrm{Sp}_s(r) = \mathrm{rec}_K(\mathrm{Sp}_s(\pi))$. If K'/K is a finite extension and if π is an irreducible smooth representation of $GL_n(K)$ we will write $\mathrm{BC}_{K'/K}(\pi)$ for the base change of π to K' which is characterized by $\mathrm{rec}_{K'}(\pi_{K'}) = \mathrm{rec}_K(\pi)|_{W_{K'}}$.

If ρ is a continuous de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil-Deligne representation of W_K (its construction, which is due to Fontaine, is recalled in Section 1 of [14]), and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ is a continuous embedding of fields then we will write $\mathrm{HT}_\tau(\rho)$ for the multiset of Hodge-Tate numbers of ρ with respect to τ. Thus $\mathrm{HT}_\tau(\rho)$ is a multiset of $\dim \rho$ integers. In fact, if W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ then the multiset $\mathrm{HT}_\tau(W)$ contains i with multiplicity $\dim_{\overline{\mathbb{Q}}_p}(W \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\epsilon_l) = \{-1\}$.

If F is a number field and v a prime of F, we will often denote Frob_{F_v}, $k(\nu_{F_v})$ and Iw_{m,F_v} by Frob_v, $k(v)$ and $\mathrm{Iw}_{m,v}$. If $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$ or $\mathbb{C}$ is an embedding of fields, then we will write F_σ for the closure of the image of σ. If F'/F is a soluble, finite Galois extension and if π is a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$ we will write $\mathrm{BC}_{F'/F}(\pi)$ for its base change to F', an automorphic representation of $GL_n(\mathbb{A}_{F'})$. If $R : G_F \to GL_m(\overline{\mathbb{Q}}_l)$ is a continuous representation, we say that R is *pure of weight* w if for all but finitely many primes v of F, R is unramified at v and every eigenvalue of $R(\mathrm{Frob}_v)$ is a Weil $(\#k(v))^w$-number. (See Section 1 of [14].) If F is an imaginary CM field, we will denote its maximal totally real subfield by F^+ and let c denote the non-trivial element of $\mathrm{Gal}\,(F/F^+)$.

1. Automorphic Galois representations

We recall some now-standard notation and terminology. Let F be an imaginary CM field with maximal totally real subfield F^+. By a *RACSDC* (regular, algebraic, conjugate self dual, cuspidal) automorphic representation of $GL_m(\mathbb{A}_F)$ we mean that

- Π is a cuspidal automorphic representation of $GL_m(\mathbb{A}_F)$ such that Π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_m,
- and $\Pi^c \cong \Pi^\vee$.

We will say that Π has *level prime to* l (resp. *level potentially prime to* l) if for all $v|l$ the representation Π_v is unramified (resp. becomes unramified after a finite base change).

If Ω is an algebraically closed field of characteristic 0 we will write $(\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega),+}$ for the set of $a = (a_{\tau,i}) \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega)}$ satisfying

$$a_{\tau,1} \geqslant \ldots \geqslant a_{\tau,m}.$$

We will write $(\mathbb{Z}^m)_0^{\mathrm{Hom}\,(F,\Omega)}$ for the subset of elements $a \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega)}$ with

$$a_{\tau,i} + a_{\tau\circ c,m+1-i} = 0.$$

If F'/F is a finite extension we define $a_{F'} \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F',\Omega),+}$ by

$$(a_{F'})_{\tau,i} = a_{\tau|_F,i}.$$

Following [13] we will be interested, *inter alia*, in the case that either m is odd; or that m is even and for some $\tau \in \mathrm{Hom}\,(F,\Omega)$ and for some odd integer i we have $a_{\tau,i} > a_{\tau,i+1}$. If either of these conditions hold then we will say that a is *Shin-regular*. (We warn the reader that this is often referred to as 'slightly regular' in the literature. However as this notion is strictly stronger than 'regularity' we prefer to use the term Shin-regular.)

If $a \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\mathbb{C}),+}$, let Ξ_a denote the irreducible algebraic representation of $GL_m^{\mathrm{Hom}\,(F,\mathbb{C})}$ which is the tensor product over τ of the irreducible representations of GL_m with highest weights a_τ. We will say that a RACSDC automorphic representation Π of $GL_m(\mathbb{A}_F)$ has *weight* a if Π_∞ has the same infinitesimal character as $\Xi_a^\vee$. Note that in this case a must lie in $(\mathbb{Z}^m)_0^{\mathrm{Hom}\,(F,\mathbb{C})}$.

We recall (see Theorem 1.2 of [3]) that to a RACSDC automorphic representation Π of $GL_m(\mathbb{A}_F)$ and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ we can associate a continuous semisimple representation

$$r_{l,\imath}(\Pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_m(\overline{\mathbb{Q}}_l)$$

with the properties described in Theorem 1.2 of [3]. In particular

$$r_{l,\imath}(\Pi)^c \cong r_{l,\imath}(\Pi)^\vee \otimes \epsilon_l^{1-m}.$$

For $v|l$ a place of F, the representation $r_{l,\imath}(\Pi)|_{G_{F_v}}$ is de Rham and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ then

$$\mathrm{HT}_\tau(r_{l,\imath}(\pi)) = \{a_{\imath\tau,1} + m - 1, a_{\imath\tau,2} + m - 2, ..., a_{\imath\tau,m}\}.$$

If $v \not| l$, then the main result of [4] states that

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

We recall the following result which will prove useful.

Proposition 1.1. — *Let Ω be an algebraically closed field of characteristic 0 and of the same cardinality as $\mathbb{C}$.*

1. *Suppose $K/\mathbb{Q}_p$ is a finite extension. Let (V, r, N) and (V', r', N') be pure, Frobenius semisimple Weil-Deligne representations of W_K over Ω. If the representations (V, r^{ss}) and $(V', (r')^{\mathrm{ss}})$ are isomorphic, then $(V, r, N) \cong (V', r', N')$.*

2. *If F is an imaginary CM field and Π is a RACSDC automorphic representation of $GL_n(\mathbb{A}_F)$, then for each $\imath : \Omega \xrightarrow{\sim} \mathbb{C}$ and each finite place v of F, $\imath^{-1}\mathrm{rec}(\Pi_v)$ is pure.*

Proof. — The first part follows from Lemma 1.4(4) of [14]. For the second part, Theorem 1.2 of [4] states that Π_v is tempered for each finite place v of F. If σ is an automorphism of $\mathbb{C}$, then there is a RACSDC automorphic representation $\Pi' = {}^{\sigma}\Pi^{\infty} \otimes \Pi'_{\infty}$ of $GL_n(\mathbb{A}_F)$ (see Théorème 3.13 of [5]) and we deduce that $\sigma\Pi_v$ is tempered. The second part then follows from this and Lemma 1.4(3) of [14]. □

We can now state our main results.

Theorem 1.2. — *Let $m \geqslant 2$ be an integer, l a rational prime and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let L be an imaginary CM field and Π a RACSDC automorphic representation of $GL_m(\mathbb{A}_L)$. If Π has Shin-regular weight and $v|l$ is a place of L such that $\Pi_v^{\mathrm{Iw}_{m,v}} \neq \{0\}$, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{L_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Before turning to the proof, we first record a corollary.

Corollary 1.3. — *Let $m \geqslant 2$ be an integer, l a rational prime and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let L be an imaginary CM field and Π a RACSDC automorphic representation of $GL_m(\mathbb{A}_L)$. If Π has Shin-regular weight and $v|l$ is a place of L then $\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{L_v}})$ is pure.*

Proof. — Choose a finite CM soluble Galois extension F/L such that for each prime $w|v$ of F, $\mathrm{BC}_{F_w/L_v}(\Pi_v)^{\mathrm{Iw}_{m,w}} \neq \{0\}$. Then $\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_w}})$ is pure by Theorem 1.2 and Proposition 1.1. Lemma 1.4 of [14] then implies that $\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{L_v}})$ is pure. □

The rest of this paper will be devoted to the proof of Theorem 1.2.

2. Notation and running assumptions

For the convenience of the reader, we recall here the following notation which appears in [13]:

– If Π is a RACSDC automorphic representation of $GL_m(\mathbb{A}_F)$ for some integer $m \geqslant 2$ and an imaginary CM field F, or if Π is an algebraic Hecke character of $\mathbb{A}_M^{\times}/M^{\times}$ for a number field M, then $R_{l,\imath}(\Pi)$ denotes $r_{l,\imath}(\Pi^{\vee})$.

– If L/F is a finite extension of number fields, then $\mathrm{Ram}_{L/F}$ (resp. $\mathrm{Unr}_{L/F}$, resp. $\mathrm{Spl}_{L/F}$) denotes the set of finite places of F which are ramified (resp. unramified, resp. completely split) in L. We denote by $\mathrm{Spl}_{L/F,\mathbb{Q}}$ the set of rational primes p such that every place of F above p splits completely in L.

– If F is a number field and π is an automorphic representation of $GL_{n/F}$, then $\mathrm{Ram}_{\mathbb{Q}}(\pi)$ denotes the set of rational primes p such that there exists a place $v|p$ of F with π_v ramified.

– If G is a group of the form $H(F)$ for $F/\mathbb{Q}_p$ finite and H/F a reductive group; or $H(\mathbb{A}_F^T)$ for F a number field, H/F a reductive group and T a finite set of places of F containing all infinite places (where as usual $\mathbb{A}_F^T$ is defined in the same way as $\mathbb{A}_F$, except that one takes the restricted direct product over the places not in T); or a product of groups of this form, then we let $\mathrm{Irr}(G)$ (resp. $\mathrm{Irr}_l(G)$) denote the set of isomorphism classes of irreducible admissible representations of G on $\mathbb{C}$-vector spaces (resp. $\overline{\mathbb{Q}}_l$-vector spaces). We let $\mathrm{Groth}\,(G)$ (resp. $\mathrm{Groth}\,_l(G)$) denote the Grothendieck group of the category of admissible $\mathbb{C}$-representations (resp. $\overline{\mathbb{Q}}_l$-representations) of G. (See Section I.2 of [8].)

– $\epsilon : \mathbb{Z} \to \{0, 1\}$ is the unique function such that $\epsilon(n) \equiv n2$.

– Φ_n is the matrix in GL_n with $(\Phi_n)_{ij} = (-1)^{i+1}\delta_{i,n+1-j}$.

– If $R \to S$ is a homomorphism of commutative rings, $R_{S/R}$ denotes the restriction of scalars functor.

– If π is a representation of a group G with a central character, we denote the central character by ψ_π.

We now fix the following notations and assumptions which will be in force from Section 3 to Section 6:

– E is a quadratic imaginary field;
– F^+ is a totally real field with $[F^+ : \mathbb{Q}] \geqslant 2$;
– $F = EF^+$ and $\mathrm{Ram}_{F/\mathbb{Q}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$;
– $\tau : F \hookrightarrow \mathbb{C}$ is an embedding and $\tau_E = \tau|_E$;
– $\Phi_{\mathbb{C}} = \mathrm{Hom}\,(F, \mathbb{C})$ and $\Phi_{\mathbb{C}}^+ = \mathrm{Hom}\,_{E,\tau_E}(F, \mathbb{C})$;
– $n \geqslant 3$ is an odd integer;
– $p \in \mathrm{Spl}_{E/\mathbb{Q}}$ is a rational prime and $u|p$ is a prime of E;

- w is a prime of F above u and $w_1 = w, w_2, \ldots, w_r$ denote all of the primes of F above u;
- $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ is an isomorphism such that $\iota_p^{-1} \circ \tau$ induces the place w;
- l is a rational prime (possibly equal to p) and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$.

Define algebraic groups G_n and $\mathbb{G}_n$ over $\mathbb{Z}$ by setting

$$\begin{aligned} G_n(R) &= \{(\lambda, g) \in R^\times \times GL_n(\mathcal{O}_F \otimes_{\mathbb{Z}} R) : g\Phi_n {}^t g^c = \lambda \Phi_n\}, \text{ and} \\ \mathbb{G}_n(R) &= R_{\mathcal{O}_E/\mathbb{Z}}(G_n \times_{\mathbb{Z}} \mathcal{O}_E)(R) = G_n(\mathcal{O}_E \otimes_{\mathbb{Z}} R) \end{aligned}$$

for any $\mathbb{Z}$-algebra R. Then $G_n \times_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{G}_n \times_{\mathbb{Z}} \mathbb{Q}$ are reductive. We let θ denote the action on $\mathbb{G}_n$ induced by $(1, c)$ on $G_n \times_{\mathbb{Z}} \mathcal{O}_E$.

If R is an E-algebra, then $G_n(R)$ is a subgroup of $R^\times \times GL_n(F \otimes_{\mathbb{Q}} R) = R^\times \times GL_n(F \otimes_E R) \times GL_n(F \otimes_{E,c} R)$ and the projection onto $R^\times \times GL_n(F \otimes_E R)$ defines an isomorphism

$$G_n(R) \cong R^\times \times GL_n(F \otimes_E R).$$

It follows that $G_n \times_{\mathbb{Q}} E \xrightarrow{\sim} \mathbb{G}_m \times R_{F/E}(GL_n)$. [Note that here $\mathbb{G}_m$ is the multiplicative group, rather than $\mathbb{G}_n$ in the case $n = m$.]

If $v \in \mathrm{Unr}_{F/\mathbb{Q}}$, then $K_v := G_n(\mathbb{Z}_v)$ (resp. $\mathbb{K}_v := \mathbb{G}_n(\mathbb{Z}_v)$) is a hyperspecial maximal compact subgroup of $G_n(\mathbb{Q}_v)$ (resp. $\mathbb{G}_n(\mathbb{Q}_v)$). In this case we say that a representation of $G_n(\mathbb{Q}_v)$ (resp. $\mathbb{G}_n(\mathbb{Q}_v)$) is unramified if the space of K_v-invariants (resp. $\mathbb{K}_v$-invariants) is non-zero. Furthermore, we define the unramified Hecke algebras $\mathcal{H}^{\mathrm{ur}}(G_n(\mathbb{Q}_v))$ and $\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{Q}_v))$ with respect to K_v and $\mathbb{K}_v$ respectively, as in Section 1.1 of [13]. (We note that these are $\mathbb{C}$-algebras.) If T is a set of places of $\mathbb{Q}$ with $\{\infty\} \cup \mathrm{Ram}_{F/\mathbb{Q}} \subset T$, we let $K^T = \prod_{v \notin T} K_v \subset G_n(\mathbb{A}^T)$.

We say that a representation Π_v of $\mathbb{G}_n(\mathbb{Q}_v)$ is θ-stable if $\Pi_v \circ \theta \cong \Pi_v$ and we let $\mathrm{Irr}^{\theta-\mathrm{st}}(\mathbb{G}_n(\mathbb{Q}_v)) \subset \mathrm{Irr}(\mathbb{G}_n(\mathbb{Q}_v))$ be the subset of θ-stable representations. For $v \in \mathrm{Unr}_{F/\mathbb{Q}}$, we let $\mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{Q}_v)) \subset \mathrm{Irr}(G(\mathbb{Q}_v))$ (resp. $\mathrm{Irr}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{Q}_v)) \subset \mathrm{Irr}(\mathbb{G}_n(\mathbb{Q}_v))$, resp. $\mathrm{Irr}^{\mathrm{ur},\theta-\mathrm{st}}(\mathbb{G}_n(\mathbb{Q}_v)) \subset \mathbb{G}_n(\mathbb{Q}_v))$ denote the subset consisting of unramified (resp. unramified, resp. unramified, θ-stable) representations.

Let $\# : R_{F/\mathbb{Q}}(GL_n) \to R_{F/\mathbb{Q}}(GL_n)$ denote the map $g \mapsto \Phi_n {}^t g^{-c} \Phi_n^{-1}$. If v is a rational prime, then

$$\mathbb{G}_n(\mathbb{Q}_v) = G_n(E \otimes_{\mathbb{Q}} \mathbb{Q}_v) \cong (E \otimes_{\mathbb{Q}} \mathbb{Q}_v)^\times \times GL_n(F \otimes_{\mathbb{Q}} \mathbb{Q}_v).$$

If $(\lambda, g) \in \mathbb{G}_n(\mathbb{Q}_v)$, then $\theta(\lambda, g) = (\lambda^c, \lambda^c g^\#)$. Let $\Pi_v \in \mathrm{Irr}(\mathbb{G}_n(\mathbb{Q}_v))$ and write $\Pi_v = \psi_v \otimes \Pi_v^1$ with respect to the above decomposition of $\mathbb{G}_n(\mathbb{Q}_v)$.

Then Π_v is θ-stable if and only if $(\Pi_v^1)^\vee \cong \Pi_v^1 \circ c$, and $\psi_{\Pi_v^1}|_{(E\otimes_{\mathbb{Q}}\mathbb{Q}_v)^\times} = \psi_v^c/\psi_v$.

We now recall the existence of local base change maps in the following cases (see Section 4.2 of [13] for details):

- *Case 1:* If $v \in \mathrm{Unr}_{F/\mathbb{Q}}$, we have a map

$$\mathrm{BC}_v : \mathrm{Irr}^{\mathrm{ur}}(G_n(\mathbb{Q}_v)) \to \mathrm{Irr}^{\mathrm{ur},\theta-\mathrm{st}}(\mathbb{G}_n(\mathbb{Q}_v)).$$

  (Note that the assumption $v \notin \mathrm{Ram}_{\mathbb{Q}}(\varpi)$ in Case 1 of Section 4.2 of [13] plays no role there.) This is induced by a homomorphism of $\mathbb{C}$-algebras $\mathrm{BC}_v^* : \mathcal{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{Q}_v)) \to \mathcal{H}^{\mathrm{ur}}(G_n(\mathbb{Q}_v))$.

- *Case 2:* If $v \in \mathrm{Spl}_{F/F^+,\mathbb{Q}}$, we have a map

$$\mathrm{BC}_v : \mathrm{Irr}(G_n(\mathbb{Q}_v)) \to \mathrm{Irr}^{\theta-\mathrm{st}}(\mathbb{G}_n(\mathbb{Q}_v)).$$

  If in addition $v \in \mathrm{Unr}_{F/\mathbb{Q}}$, then this map is compatible with the map in Case 1.

In Case 2, the map BC_v is described explicitly in [13]. We recall the explicit definition here, assuming $v \in \mathrm{Spl}_{E/\mathbb{Q}}$. Let $y|v$ be a place of E, and regard $\mathbb{Q}_v$ as an E-algebra via $\mathbb{Q}_v \xrightarrow{\sim} E_y$. We get an isomorphism

$$G_n(\mathbb{Q}_v) \xrightarrow{\sim} \mathbb{Q}_v^\times \times \prod_{x|y} GL_n(F_x)$$

where the product is over all places of F dividing y. Let $\pi_v \in \mathrm{Irr}(G_n(\mathbb{Q}_v))$ and decompose $\pi_v = \pi_{v,0} \otimes \pi_y$ with respect to the above decomposition of $G_n(\mathbb{Q}_v)$. If we decompose

$$\mathbb{G}_n(\mathbb{Q}_v) = E_y^\times \times E_{y^c}^\times \times \prod_{x|y} GL_n(F_x) \times \prod_{x|y} GL_n(F_{x^c})$$

then $\mathrm{BC}_v(\pi_v) = (\psi_y, \psi_{y^c}, \Pi_y, \Pi_{y^c})$, where

$$(\psi_y, \psi_{y^c}, \Pi_y, \Pi_{y^c}) = (\pi_{p,0}, \pi_{p,0}(\psi_{\pi_y}|_{E_y^\times} \circ c), \pi_y, \pi_y^\#)$$

and $\pi_y^\#(g) := \pi_y(\Phi_n {}^t g^{-c} \Phi_n^{-1})$. (In particular, $\pi_y^\# \cong \pi_y^\vee \circ c$.)

The discussion above can be carried out equally well in the setting of $\overline{\mathbb{Q}}_l$-representations, and we define $\mathrm{Irr}_l^{\theta-\mathrm{st}}(\mathbb{G}_n(\mathbb{Q}_v))$, $\mathrm{Irr}_l^{\mathrm{ur}}(G(\mathbb{Q}_v))$ etc. in the obvious fashion. We also define a base change map BC_v in Case 1 (resp.

Case 2) by setting $\mathrm{BC}_v(\pi) = \imath^{-1}\mathrm{BC}_v(\imath\pi)$ for $\pi \in \mathrm{Irr}_l^{\mathrm{ur}}(G_n(\mathbb{Q}_v))$ (resp. $\pi \in \mathrm{Irr}(G_n(\mathbb{Q}_v))$).

Let $\xi_{\mathbb{C}}$ denote an irreducible algebraic representation of G_n over $\mathbb{C}$. There is an isomorphism $\mathbb{G}_n(\mathbb{C}) = G_n(E \otimes_{\mathbb{Q}} \mathbb{C}) \cong G_n(\mathbb{C}) \times G_n(\mathbb{C})$ induced by the isomorphism $E \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ which sends $e \otimes z$ to $(\tau(e)z, \tau^c(e)z)$. We associate to ξ a θ-stable irreducible algebraic representation Ξ of $\mathbb{G}_n$ over $\mathbb{C}$ by setting $\Xi := \xi \otimes \xi$. Every such Ξ arises in this way.

We also fix the following data:

- $V = F^n$ as an F-vector space;
- $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}$ is a non-degenerate pairing such that $\langle f v_1, v_2 \rangle = \langle v_1, f^c v_2 \rangle$ for all $v_1, v_2 \in V$ and $f \in F$;
- $h : \mathbb{C} \to \mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R}$ is an $\mathbb{R}$-algebra embedding such that the bilinear pairing $(V \otimes_{\mathbb{Q}} \mathbb{R}) \times (V \otimes_{\mathbb{Q}} \mathbb{R}) \to \mathbb{R}; (v_1, v_2) \mapsto \langle v_1, h(i) v_2 \rangle$ is symmetric and positive definite.

Under the natural isomorphism $\mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\sigma \in \Phi_{\mathbb{C}}^+} M_n(\mathbb{C})$ we assume that h sends

$$z \mapsto \left(\begin{pmatrix} z I_{p_\sigma} & 0 \\ 0 & \overline{z} I_{q_\sigma} \end{pmatrix}_{\sigma \in \Phi_{\mathbb{C}}^+} \right)$$

for some $p_\sigma, q_\sigma \in \mathbb{Z}_{\geqslant 0}$ with $p_\sigma + q_\sigma = n$.

Define a reductive algebraic group $G/\mathbb{Q}$ by setting

$$G(R) = \{(\lambda, g) \in R^\times \times GL_n(F \otimes_{\mathbb{Q}} R) : \langle g v_1, g v_2 \rangle = \lambda \langle v_1, v_2 \rangle \text{ for all } v_1, v_2 \in V \otimes_{\mathbb{Q}} R\}$$

for each $\mathbb{Q}$-algebra R. Note that G_n is a quasi-split inner form of G. Let $\nu : G \to \mathbb{G}_m$ denote the homomorphism which sends (λ, g) to λ.

By Lemma 5.1 of [13] we can and do assume that $\langle \cdot, \cdot \rangle$ and h have been chosen so that

- $G_{\mathbb{Q}_v}$ is quasi-split for each rational prime v;
- for each $\sigma \in \Phi_{\mathbb{C}}^+$, we have $(p_\sigma, q_\sigma) = (1, n-1)$ if $\sigma = \tau$ and $(p_\sigma, q_\sigma) = (0, n)$ otherwise.

As a consequence, we can and do fix an isomorphism

$$G \times_{\mathbb{Q}} \mathbb{A}^\infty \cong G_n \times_{\mathbb{Q}} \mathbb{A}^\infty.$$

Using this isomorphism, we will henceforth identify the groups $G_n(\mathbb{Q}_v)$ and $G(\mathbb{Q}_v)$ for all primes v. Let $C_G \in \mathbb{Z}_{>0}$ be the integer $|\ker^1(\mathbb{Q}, G)| \cdot \tau(G)$ in the notation of [13].

Let T be a (possibly infinite) set of places of $\mathbb{Q}$ containing ∞ and let $T_{\mathrm{fin}} = T - \{\infty\}$. Let Γ be a Galois group with its Krull topology, or the Weil group of a local field, or a quotient of such a group. We define an admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^T)\times\Gamma]$-module to be an admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^T)]$-module R with a commuting continuous action of Γ (the continuity condition here means that for each compact open subgroup $U \subset G(\mathbb{A}^T)$, the induced map $\Gamma \to \mathrm{Aut}\,(R^U)$ is continuous for the l-adic topology on R^U). We let $\mathrm{Groth}\,_l(G(\mathbb{A}^T)\times\Gamma)$ denote the Grothendieck group of the category of admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^T)\times\Gamma]$-modules. If R is an admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^T)\times\Gamma]$-module, we let $[R]$ denote its image in $\mathrm{Groth}\,_l(G(\mathbb{A}^T)\times\Gamma)$. We let $\mathrm{Irr}_l(G(\mathbb{A}^T)\times\Gamma)$ denote the set of isomorphism classes of irreducible admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^T)\times\Gamma]$-modules. (See Section I.2 of [8].)

Now suppose that T is finite, that $p \in T$ and let $J/\mathbb{Q}_p$ be a reductive group. Let G' be a topological group which is of the form $G(\mathbb{A}_{T_{\mathrm{fin}}})\times\Gamma$, or $G(\mathbb{A}_{T_{\mathrm{fin}}-\{p\}})\times\Gamma$, or $G(\mathbb{A}_{T_{\mathrm{fin}}-\{p\}})\times J(\mathbb{Q}_p)$. Let $[X]\in\mathrm{Groth}\,_l(G(\mathbb{A}^T)\times G')$ and write $[X]=\sum_{\pi^T,\rho} n(\pi^T\otimes\rho)\cdot[\pi^T\otimes\rho]$ where $n(\pi^T\otimes\rho)\in\mathbb{Z}$ and π^T (resp. ρ) runs through $\mathrm{Irr}_l(G(\mathbb{A}^T))$ (resp. $\mathrm{Irr}_l(G')$). For a given $\pi^T\in\mathrm{Irr}_l(G(\mathbb{A}^T))$, we let

$$[X][\pi^T] = \sum_{\rho} n(\pi^T\otimes\rho)\cdot[\pi^T\otimes\rho]\in\mathrm{Groth}\,_l(G(\mathbb{A}^T)\times G').$$

If $\mathrm{Ram}_{F/\mathbb{Q}}\subset T$ and $\Pi^T\in\mathrm{Irr}(\mathbb{G}_n(\mathbb{A}^T))$ is unramified at all $v\notin T$, then we define

$$[X][\Pi^T] = \sum_{\pi^T}[X][\pi^T]\in\mathrm{Groth}\,_l(G(\mathbb{A}^T)\times G')$$

where the sum is over all $\pi^T\in\mathrm{Irr}_l(G(\mathbb{A}^T))$ with π^T unramified at all $v\notin T$ and $\mathrm{BC}\,^T(\imath\pi^T) := \otimes'_{v\notin T}\mathrm{BC}\,_v(\imath\pi^T_v)\cong\Pi^T$.

Finally, suppose G' of the form $G(\mathbb{A}_{T_{\mathrm{fin}}-\{p\}})\times\Gamma$ or $G(\mathbb{A}_{T_{\mathrm{fin}}})\times\Gamma$ and let R be an admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^T)\times G']$-module. Suppose $\mathrm{Ram}_{F/\mathbb{Q}}\subset T$ and $\Pi^T\in\mathrm{Irr}(\mathbb{G}_n(\mathbb{A}^T))$ is unramified at all $v\notin T$. Let $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}^T)) = \otimes'_{v\notin T}\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_v))$, a commutative polynomial algebra over $\mathbb{C}$ in countably many variables. Similarly, let $\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{A}^T)) = \otimes'_{v\notin T}\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{Q}_v))$. Then Π^T corresponds to a maximal ideal n of $\mathcal{H}^{\mathrm{ur}}(\mathbb{G}_n(\mathbb{A}^T))$ with residue field $\mathbb{C}$. Note that the space of K^T-invariants R^{K^T} is a module over $\iota^{-1}\mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}^T))$. We define

$$R^{K^T}\{\Pi^T\} := \bigoplus_{\mathfrak{m}}(R^{K^T})_{\iota^{-1}\mathfrak{m}}\subset R^{K^T}$$

where $\mathfrak{m}$ runs over the maximal ideals of $\mathcal{H}^{\mathrm{ur}}(G(\mathbb{A}^T))$ with residue field $\mathbb{C}$ and which pull back to n under $\otimes_{v\notin T}\mathrm{BC}\,^*_v$. Then $R^{K^T}\{\Pi^T\}$ is a G'-stable direct summand of R^{K^T}.

3. Shimura varieties

In this section we recall some results from [13]. We begin with some definitions and refer the reader to Section 5 of [13] for more details. Let U be a compact open subgroup of $G(\mathbb{A}^\infty)$ and define a functor $\mathfrak{X}_U$ from the category of pairs (S,s), where is S is a connected locally Noetherian F-scheme and s is a geometric point of S, to the category of sets by sending a pair (S,s) to the set of isogeny classes of quadruples $(A,\lambda,i,\overline{\eta})$ where

- A/S is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n$;
- $\lambda : A \to A^\vee$ is a polarization;
- $i : F \hookrightarrow \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\lambda \circ i(f) = i(f^c)^\vee \circ \lambda$;
- $\overline{\eta}$ is a $\pi_1(S,s)$-invariant U-orbit of isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \xrightarrow{\sim} VA_s$ which take the pairing $\langle \cdot, \cdot \rangle$ on V to a $(\mathbb{A}^\infty)^\times$-multiple of the λ-Weil pairing on $VA_s := H_1(A_s, \mathbb{A}^\infty)$ (see Section 5 of [11]);
- for each $f \in F$ there is an equality of polynomials $\det_{\mathcal{O}_S}(f|\mathrm{Lie}\,A) = \det_E(f|V^1)$ in the sense of Section 5 of [11] (here $V^1 \subset V \otimes_{\mathbb{Q}} E \subset V \otimes_{\mathbb{Q}} \mathbb{C}$ is the E-subspace where $h(\tau_E(e))$ acts by multiplication by $1 \otimes e$ for all $e \in E$);
- two such quadruples $(A,\lambda,i,\overline{\eta})$ and $(A',\lambda',i',\overline{\eta}')$ are isogenous if there exists an isogeny $A \to A'$ taking $\lambda, i, \overline{\eta}$ to $\gamma\lambda', i', \overline{\eta}'$ for some $\gamma \in \mathbb{Q}^\times$.

If s and s' are two geometric points of a connected locally Noetherian F-scheme S then there is a canonical bijection from $\mathfrak{X}_U(S,s)$ to $\mathfrak{X}_U(S,s')$. We may therefore think of $\mathfrak{X}_U$ as a functor from connected locally Noetherian F-schemes to sets and then extend it to a functor from all locally Noetherian F-schemes to sets by setting $\mathfrak{X}_U(\coprod_i S_i) = \prod_i \mathfrak{X}_U(S_i)$. When U is sufficiently small the functor $\mathfrak{X}_U$ is represented by a smooth projective variety X_U/F of dimension $n-1$. The variety X_U is denoted Sh_U in [13]. Let $\mathcal{A}_U$ be the universal abelian variety over X_U.

If U and V are sufficiently small compact open subgroups of $G(\mathbb{A}^\infty)$ and $g \in G(\mathbb{A}^\infty)$ is such that $g^{-1}Vg \subset U$, then we have a map $g : X_V \to X_U$ and a quasi-isogeny $g^* : \mathcal{A}_V \to g^*\mathcal{A}_U$ of abelian varieties over X_V. In this way we get a right action of the group $G(\mathbb{A}^\infty)$ on the inverse system of the X_U which extends to an action by quasi-isogenies on the inverse system of the $\mathcal{A}_U$.

Let l be a rational prime and let ξ be an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$. Then ξ gives rise to a lisse l-adic sheaf $\mathcal{L}_\xi$ on each X_U.

We let

$$H^k(X, \mathcal{L}_\xi) = \varinjlim_U H^k(X_U \times_F \overline{F}, \mathcal{L}_\xi).$$

This is a semisimple admissible representation of $G(\mathbb{A}^\infty)$ with a commuting continuous action of G_F and therefore decomposes as

$$H^k(X, \mathcal{L}_\xi) = \bigoplus_{\pi^\infty} \pi^\infty \otimes R^k_{\xi,l}(\pi^\infty)$$

where π^∞ runs over $\mathrm{Irr}_l(G(\mathbb{A}^\infty))$ and each $R^k_{\xi,l}(\pi^\infty)$ is a finite dimensional continuous representation of G_F over $\overline{\mathbb{Q}}_l$.

We now recall results of Shin on the existence of Galois representations in the cohomology of the Shimura varieties in the following two cases:

3.1. The stable case

Assume we are in the following situation:

- Π^1 is a RACSDC automorphic representation of $GL_n(\mathbb{A}_F)$;
- $\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ is an algebraic Hecke character such that $\psi_{\Pi^1}|_{\mathbb{A}_E^\times} = \psi^c/\psi$;
- $\Pi := \psi \otimes \Pi^1$ (an automorphic representation of $\mathbb{G}_n(\mathbb{A}) \cong GL_1(\mathbb{A}_E) \times GL_n(\mathbb{A}_F)$) is Ξ-cohomological for some irreducible algebraic representation Ξ of $\mathbb{G}_n/\mathbb{C}$;
- $\mathrm{Ram}_{\mathbb{Q}}(\Pi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$.

Then Ξ is θ-stable and so comes from some irreducible algebraic representation $\xi_{\mathbb{C}}$ of G_n over $\mathbb{C}$ as in Section 2. Let $\xi = \imath^{-1}\xi_{\mathbb{C}}$, regarded as a representation of $G/\overline{\mathbb{Q}}_l$. Let $\mathcal{R}_l(\Pi)$ denote the set of $\pi^\infty \in \mathrm{Irr}_l(G(\mathbb{A}^\infty))$ such that

- $R^k_{\xi,l}(\pi^\infty) \neq (0)$ for some k;
- π^∞ is unramified at all $v \notin \mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi)$ and

$$\mathrm{BC}^\infty(\imath\pi^\infty) = \Pi^\infty.$$

(We note that $\mathrm{BC}_v(\imath\pi_v)$ is defined for all $v \nmid \infty$.)

Let $\widetilde{R}^k_{\xi,l}(\Pi) = \bigoplus_{\pi^\infty \in \mathcal{R}_l(\Pi)} R^k_{\xi,l}(\pi^\infty)$.

Now, let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. Note that

$$H^k(X, \mathcal{L}_\xi)^{K^T}\{\Pi^T\} \cong \bigoplus_{\pi^\infty} \pi_{T_{\mathrm{fin}}} \otimes R^k_{\xi,l}(\pi^\infty) \subset H^k(X, \mathcal{L}_\xi)$$

where the sum is over all $\pi^\infty = \pi^T \otimes \pi_{T_{\mathrm{fin}}}$ where π^T is unramified and $\mathrm{BC}\,(\imath\pi^T) \cong \Pi^T$. We then define an admissible $\overline{\mathbb{Q}}_l[\mathbb{G}_n(\mathbb{A}_{T_{\mathrm{fin}}}) \times G_F]$-module

$$\mathrm{BC}_{\,T_{\mathrm{fin}}}(H^k(X, \mathcal{L}_\xi)^{K^T}\{\Pi^T\}) := \bigoplus_{\pi^\infty} \mathrm{BC}_{\,T_{\mathrm{fin}}}(\pi_{T_{\mathrm{fin}}}) \otimes R^k_{\xi,l}(\pi^\infty),$$

where π^∞ runs over the same set.

Theorem 3.1. —

1. *If* $\pi^\infty \in \mathcal{R}_l(\Pi)$ *then* $R^k_{\xi,l}(\pi^\infty) \neq (0)$ *if and only if* $k = n-1$.
2. *We have*
$$\mathrm{BC}_{\,T_{\mathrm{fin}}}(H^{n-1}(X, \mathcal{L}_\xi)^{K^T}\{\Pi^T\}) = (\imath^{-1}\Pi_{T_{\mathrm{fin}}}) \otimes \widetilde{R}^{n-1}_{\xi,l}(\Pi).$$
3. *We have*
$$\widetilde{R}^{n-1}_{\xi,l}(\Pi)^{\mathrm{ss}} \cong R_{l,\imath}(\Pi^1)^{C_G} \otimes R_{l,\imath}(\psi)|_{G_F}.$$

Proof. — The first part follows from Corollary 6.5 of [13]. The second part follows from the proof of Corollary 6.4 of *op. cit.*. The third part follows from the proof of Corollary 6.8 of *op. cit.* (Note that the character $\mathrm{rec}_{l,\imath_l}(\psi)$ which appears in the proof of this corollary is equal to $R_{l,\imath}(\psi^{-1})$.) □

3.2. The endoscopic case

We now assume we are in the following situation:

- m_1, m_2 are positive integers with $m_1 > m_2$ and $m_1 + m_2 = n$;
- for $i = 1, 2$, Π_i is a RACSDC automorphic representation of $GL_{m_i}(\mathbb{A}_F)$ with $\mathrm{Ram}_{\mathbb{Q}}(\Pi_i) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$;
- $\varpi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ is a Hecke character such that
  - $\mathrm{Ram}_{\mathbb{Q}}(\varpi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$;
  - $\varpi|_{\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times}$ is the quadratic character corresponding to the quadratic extension $E/\mathbb{Q}$ by class field theory.
- $\psi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ is an algebraic Hecke character such that

  - $(\psi_{\Pi_1}\psi_{\Pi_2})|_{\mathbb{A}_E^\times} = (\psi\otimes\varpi^{N(m_1,m_2)})^c/(\psi\otimes\varpi^{N(m_1,m_2)})$ where $N(m_1,m_2) = [F^+:\mathbb{Q}](m_1\epsilon(n-m_1)+m_2\epsilon(n-m_1))/2 \in \mathbb{Z}$;
  - $\mathrm{Ram}_{\mathbb{Q}}(\psi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$.
- for $i=1,2$, $\Pi_{M,i} := \Pi_i \otimes (\varpi \circ \mathbf{N}_{F/E} \circ \det)^{\epsilon(n-m_i)}$;
- $\Pi := \psi \otimes \mathrm{n}-\mathrm{Ind}_{GL_{m_1}\times GL_{m_2}}^{GL_n}(\Pi_{M,1}\otimes\Pi_{M,2})$ (an automorphic representation of $\mathbb{G}_n(\mathbb{A})$) is Ξ-cohomological for some irreducible algebraic representation Ξ of $\mathbb{G}_n/\mathbb{C}$. (We note that the normalized induction is irreducible as $\Pi_{M,1}$ and $\Pi_{M,2}$ are unitary.)

As above, we let ξ be the irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$ such that Ξ is associated to $\imath\xi$. Let $\mathcal{R}_l(\Pi)$ denote the set of $\pi^\infty \in \mathrm{Irr}_l(G(\mathbb{A}^\infty))$ such that

- $R^k_{\xi,l}(\pi^\infty) \neq (0)$ for some k;
- π^∞ is unramified at all $v \notin \mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi)$ and
$$\mathrm{BC}^{\infty}(\imath\pi^\infty) = \Pi^\infty.$$

Let $\widetilde{R}^k_{\xi,l}(\Pi) = \bigoplus_{\pi^\infty\in\mathcal{R}_l(\Pi)} R^k_{\xi,l}(\pi^\infty)$.

Let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. We define

$$\mathrm{BC}_{T_{\mathrm{fin}}}(H^k(X,\mathcal{L}_\xi)^{K^T}\{\Pi^T\})$$

exactly as in the previous subsection.

THEOREM 3.2. —

1. *If $\pi^\infty \in \mathcal{R}_l(\Pi)$, then $R^k_{\xi,l}(\pi^\infty) \neq (0)$ if and only if $k = n-1$.*
2. *We have*
$$\mathrm{BC}_{T_{\mathrm{fin}}}(H^k(X,\mathcal{L}_\xi)^{K^T}\{\Pi^T\}) = (\imath^{-1}\Pi_{T_{\mathrm{fin}}}) \otimes \widetilde{R}^{n-1}_{\xi,l}(\Pi).$$
3. *There exists an integer $e_2(\Pi,G) \in \{\pm1\}$ depending on Π and G such that*
   (a) *If $e_2(\Pi,G)=1$ then*
$$\widetilde{R}^{n-1}_{\xi,l}(\Pi)^{\mathrm{ss}} \cong R_{l,\imath}(\Pi_1)^{C_G} \otimes R_{l,\imath}(\psi\varpi^{\epsilon(n-m_1)}|\cdot|^{(n-m_1)/2})|_{G_F}.$$

(b) *If* $e_2(\Pi, G) = -1$ *then*

$$\widetilde{R}^{n-1}_{\xi,l}(\Pi)^{\mathrm{ss}} \cong R_{l,\imath}(\Pi_2)^{C_G} \otimes R_{l,\imath}(\psi\varpi^{\epsilon(n-m_2)}|\cdot|^{(n-m_2)/2})|_{G_F}.$$

Proof. — The first and second parts follow respectively from Corollary 6.5 and the proof of Corollary 6.4 of [13]. The third part follows from the proofs of Corollaries 6.8 and 6.10 of *op. cit.* (Alternative 3.a corresponds to the case when $e_1 = e_2$ in the notation of *op. cit.*, while alternative 3.b corresponds to the case when $e_1 = -e_2$. Note however that by Corollary 6.5 of *op. cit.*, $e_1 = (-1)^{n-1} = 1$. We therefore take $e_2(\Pi, G) = e_2$.) □

4. Integral models

We now proceed to introduce integral models for the varieties X_U and to deduce various results on these models, following the arguments of Section 3 of [14]. Recall that we have fixed an isomorphism $G \times_{\mathbb{Q}} \mathbb{A}^\infty \cong G_n \times_{\mathbb{Q}} \mathbb{A}^\infty$. Since $p \in \mathrm{Spl}_{E/\mathbb{Q}}$, we have an isomorphism

$$G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \prod_{i=1}^r GL_n(F_{w_i})$$

and we decompose $G(\mathbb{A}^\infty)$ as

$$G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,p}) \times \mathbb{Q}_p^\times \times \prod_{i=1}^r GL_n(F_{w_i}).$$

If $m = (m_2, \ldots, m_r) \in \mathbb{Z}_{\geqslant 0}^{r-1}$, set

$$U_p^w(m) = \prod_{i=2}^r \ker(GL_n(\mathcal{O}_{F,w_i}) \to GL_n(\mathcal{O}_{F,w_i}/w_i^{m_i})) \subset \prod_{i=2}^r GL_n(F_{w_i}).$$

We consider the following compact open subgroups of $G(\mathbb{Q}_p)$:

$$\begin{aligned} \mathrm{Ma}(m) &= \mathbb{Z}_p^\times \times GL_n(\mathcal{O}_{F,w}) \times U_p^w(m) \\ \mathrm{Iw}(m) &= \mathbb{Z}_p^\times \times \mathrm{Iw}_{n,w} \times U_p^w(m). \end{aligned}$$

Fix an m as above. If $U^p \subset G(\mathbb{A}^{\infty,p})$ is a compact open subgroup, we let $U_0 = U^p \times \mathrm{Ma}(m)$ and $U = U^p \times \mathrm{Iw}(m)$. For $i = 1, \ldots, r$, let $\Lambda_i \subset V \otimes_F F_{w_i}$ be a $GL_n(\mathcal{O}_{F_{w_i}})$-stable lattice.

For each sufficiently small U^p as above, an integral model of X_{U_0} over $\mathcal{O}_{F,w}$ is constructed in Section 5.2 of [13] (note that X_{U_0} is denoted $\mathrm{Sh}_{U^p(\vec{m})}$

in [13] with $\vec{m} = (0, m_2, \dots, m_r)$). We denote this integral model also by X_{U_0}. It represents a functor $\mathfrak{X}_{U_0}$ from the category of locally Noetherian $\mathcal{O}_{F,w}$-schemes to sets which, as in the characteristic 0 case, is initially defined on the category of pairs (S, s) where S is a connected locally Noetherian $\mathcal{O}_{F,w}$-scheme and s is a geometric point of S. It sends a pair (S, s) to the set of equivalence classes of $(r+3)$-tuples $(A, \lambda, i, \overline{\eta}^p, \{\alpha_i\}_{i=2}^r)$ where

- A/S is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n$;
- $\lambda : A \to A^\vee$ is a prime-to-p polarization;
- $i : \mathcal{O}_F \hookrightarrow \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ such that $\lambda \circ i(f) = i(f^c)^\vee \circ \lambda$;
- $\overline{\eta}^p$ is a $\pi_1(S, s)$-invariant U^p-orbit of isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^p A_s$ which take the pairing $\langle \cdot, \cdot \rangle$ on V to a $(\mathbb{A}^{\infty,p})^\times$-multiple of the λ-Weil pairing on $V^p A_s$;
- for each $f \in \mathcal{O}_F$ there is an equality of polynomials $\det_{\mathcal{O}_S}(f|\mathrm{Lie}\,A) = \det_E(f|V^1)$ in the sense of Section 5 of [11];
- for $2 \leqslant i \leqslant r$, $\alpha_i : (w_i^{-m_i}\Lambda_i/\Lambda_i)_S \xrightarrow{\sim} A[w_i^{m_i}]$ is an isomorphism of S-schemes with $\mathcal{O}_{F,w_i}$-actions;

and

- two such tuples $(A, \lambda, i, \overline{\eta}^p, \{\alpha_i\}_{i=2}^r)$ and $(A', \lambda', i', (\overline{\eta}^p)', \{\alpha_i'\}_{i=2}^r)$ are equivalent if there is a prime-to-p isogeny $A \to A'$ taking λ, i, $\overline{\eta}^p$ and α_i to $\gamma\lambda'$, i', $(\overline{\eta}^p)'$ and α_i' for some $\gamma \in \mathbb{Z}_{(p)}^\times$.

The scheme X_{U_0} is smooth and projective over $\mathcal{O}_{F,w}$. As U^p varies, the inverse system of the X_{U_0}'s has an action of $G(\mathbb{A}^{\infty,p})$.

Given a tuple $(A, \lambda, i, \overline{\eta}^p, \{\alpha_i\}_{i=2}^r)$ over S as above, we let $\mathcal{G}_A = A[w^\infty]$, a Barsotti-Tate $\mathcal{O}_{F,w}$-module over S. If p is locally nilpotent on S, then $\mathcal{G}_A$ has dimension 1 and is compatible (which means that the two actions of $\mathcal{O}_{F,w}$ on $\mathrm{Lie}\,\mathcal{G}_A$ (coming from the structural morphism $S \to \mathrm{Spec}\,\mathcal{O}_{F,w}$ and from $i : \mathcal{O}_F \to \mathrm{End}\,(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$) coincide). We let $\mathcal{A}_{U_0}$ denote the universal abelian scheme over X_{U_0}, and we let $\mathcal{G} = \mathcal{G}_{\mathcal{A}_{U_0}}$.

Let $\overline{X}_{U_0}$ denote the special fibre $X_{U_0} \times_{\mathcal{O}_{F,w}} k(w)$ of X_{U_0}, and for $0 \leqslant h \leqslant n-1$, let $\overline{X}_{U_0}^{[h]}$ denote the reduced closed subscheme of $\overline{X}_{U_0}$ whose closed geometric points s are those for which the maximal étale quotient of $\mathcal{G}_s$ has $\mathcal{O}_{F,w}$-height at most h. Let

$$\overline{X}_{U_0}^{(h)} = \overline{X}_{U_0}^{[h]} - \overline{X}_{U_0}^{[h-1]}$$

(where we set $\overline{X}_{U_0}^{[-1]} = \emptyset$). Then $\overline{X}_{U_0}^{(0)}$ is non-empty. [We exhibit an $\overline{\mathbb{F}}_p$ point of $\overline{X}_{U_0}^{(0)}$. Consider the p-adic type (F, η) over F where $\eta_w = 1/(n[k(w) : \mathbb{F}_p])$ and $\eta_{w_i} = 0$ for $i > 1$. It corresponds to an isogeny class of abelian varieties with F-action over $\overline{\mathbb{F}}_p$. Let $(A, i)/\overline{\mathbb{F}}_p$ be an element of this isogeny class. Then

- A has dimension $[F^+ : \mathbb{Q}]n$;
- $C_0 = \mathrm{End}_F^0(A)$ is the division algebra with centre F which is split outside ww^c and has Hasse invariant $1/n$ at w;
- and the p-divisible group $A[w^\infty]$ has pure slope $1/(n[F_w : \mathbb{Q}_p])$, while $A[w_i^\infty]$ is étale for $i > 1$.

(See section 5.2 of [8].) Just as in the proof of Lemma V.4.1 of [8] one shows that there is a polarization $\lambda_0 : A \to A^\vee$ and an F-vector space W_0 of dimension n together with a non-degenerate alternating form $\langle\ ,\ \rangle_0 : W_0 \times W_0 \to \mathbb{Q}$ such that

- $\lambda_0 \circ i(a) = i(ca)^\vee \circ \lambda_0$ for all $a \in F$;
- $\langle ax, y\rangle_0 = \langle x, (ca)y\rangle_0$ for all $a \in F$ and $x, y \in W_0$;
- $V^pA \cong W_0 \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$ as $\mathbb{A}_F^{\infty,p}$-modules with alternating pairings defined up to $(\mathbb{A}^{\infty,p})^\times$-multiples (the pairing on V^pA being the λ_0-Weil pairing);
- $W_0 \otimes_{\mathbb{Q}} \mathbb{R} \cong V \otimes_{\mathbb{Q}} \mathbb{R}$ as $F \otimes_{\mathbb{Q}} \mathbb{R}$-modules with alternating pairings up to $\mathbb{R}^\times$-multiples.

(In fact W_0 will be the Betti cohomology of a certain lift of (A, i) to characteristic 0.) Let G_0 denote the denote the algebraic group of F-linear automorphisms of W_0 that preserve $\langle\ ,\ \rangle_0$ up to scalar multiples, and let $\phi_0 \in H^1(\mathbb{Q}, G_0)$ represent the difference between $(W_0, \langle\ ,\ \rangle_0)$ and $(V, \langle\ ,\ \rangle)$. So in fact

$$\phi_0 \in \ker(H^1(\mathbb{Q}, G_0) \longrightarrow H^1(\mathbb{R}, G_0)).$$

Let $\ddagger_0$ denote the λ_0 Rosati involution on C_0 and define an algebraic group $H_0^{\mathrm{AV}}/\mathbb{Q}$ by

$$H_0^{\mathrm{AV}}(R) = \{g \in (C_0 \otimes_{\mathbb{Q}} R)^\times : gg^{\ddagger_0} \in R^\times\}.$$

There is a natural isomorphism

$$H_0^{\mathrm{AV}} \times_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} G_0 \times_{\mathbb{Q}} \mathbb{A}^{\infty,p}$$

coming from the isomorphism $V^pA \cong W_0 \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$. As in Lemma V.3.1 of [8] the polarizations of A which induce complex conjugation on $i(F)$ are parametrized by

$$\ker(H^1(\mathbb{Q}, H_0^{\mathrm{AV}}) \longrightarrow H^1(\mathbb{R}, H_0^{\mathrm{AV}})).$$

Let $A(G_0)$ and $A(H_0^{\mathrm{AV}})$ be the groups defined in Section 2.1 of [10], so that there are sequences

$$H^1(\mathbb{Q}, G_0) \longrightarrow H^1(\mathbb{Q}, G_0(\overline{\mathbb{A}})) \longrightarrow A(G_0)$$

and

$$H^1(\mathbb{Q}, H_0^{\mathrm{AV}}) \longrightarrow H^1(\mathbb{Q}, H_0^{\mathrm{AV}}(\overline{\mathbb{A}})) \longrightarrow A(H_0^{\mathrm{AV}}),$$

which are exact in the middle. Note that as all primes of F^+ above p split in F we have $H^1(\mathbb{Q}_p, G_0) = (0)$ and $H^1(\mathbb{Q}_p, H_0^{\mathrm{AV}}) = (0)$. Thus the image $\phi_0^{\infty,p}$ of ϕ_0 in $H^1(\mathbb{Q}, G_0(\overline{\mathbb{A}}^{\infty,p}))$ maps to 0 in $A(G_0)$. By Lemma 2.8 of [11]

$$\begin{array}{ccc} H^1(\mathbb{Q}, H_0^{\mathrm{AV}}(\overline{\mathbb{A}}^{\infty,p})) & \xrightarrow{\sim} & H^1(\mathbb{Q}, G_0(\overline{\mathbb{A}}^{\infty,p})) \\ \downarrow & & \downarrow \\ A(H_0^{\mathrm{AV}}) & = & A(G_0) \end{array}$$

commutes. Thus thinking of $\phi_0^{\infty,p} \in H^1(\mathbb{Q}, H_0^{\mathrm{AV}}(\overline{\mathbb{A}}^{\infty,p}))$ we see that it can be lifted to

$$\phi_0^{\mathrm{AV}} \in \ker(H^1(\mathbb{Q}, H_0^{\mathrm{AV}}) \longrightarrow H^1(\mathbb{R}, H_0^{\mathrm{AV}})).$$

Let λ denote the corresponding polarization of (A, i). There is an isomorphism

$$\eta^p : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \xrightarrow{\sim} V^p A$$

of $\mathbb{A}_F^{\infty,p}$-modules with alternating pairings up to $(\mathbb{A}^{\infty,p})^\times$-multiples. Moreover for $i = 2, ..., r$ the p-divisible group $A[w_i^\infty]$ is étale and so there are isomorphisms $(w_i^{-m_i}\Lambda_i/\Lambda_i)_{\overline{\mathbb{F}}_p} \xrightarrow{\sim} A[w_i^{m_i}]$. Thus

$$(A, \lambda, i, \overline{\eta}^p, \{\alpha_i\}_{i=2}^r) \in \overline{X}_{U_0}^{(0)}(\overline{\mathbb{F}}_p),$$

as desired.]

Just as in Section III.4 of [8], one deduces that each $\overline{X}_{U_0}^{(h)}$ is non-empty and smooth of pure dimension h. Over $\overline{X}_{U_0}^{(h)}$ there is a short exact sequence

$$0 \to \mathcal{G}^0 \to \mathcal{G} \to \mathcal{G}^{\mathrm{et}} \to 0$$

where $\mathcal{G}^0$ is a formal Barsotti-Tate $\mathcal{O}_{F,w}$-module and $\mathcal{G}^{\mathrm{et}}$ is an étale Barsotti-Tate $\mathcal{O}_{F,w}$-module of $\mathcal{O}_{F,w}$-height h.

We define an integral model for X_U over $\mathcal{O}_{F,w}$ (for sufficiently small U^p) as on page 480 of [14]. It represents a functor $\mathfrak{X}_U$ from the category of locally Noetherian $\mathcal{O}_{F,w}$-schemes to sets which, as above, is initially defined on the category of connected locally Noetherian $\mathcal{O}_{F,w}$-schemes with a geometric point. It sends a pair (S, s) to the set of equivalence classes of $(r+4)$-tuples

$(A, \lambda, i, \overline{\eta}^p, \mathcal{C}, \{\alpha_i\}_{i=2}^r)$ where $(A, \lambda, i, \overline{\eta}^p, \{\alpha_i\}_{i=2}^r)$ is as in the definition of $\mathfrak{X}_{U_0}(S, s)$ and $\mathcal{C}$ is a chain of isogenies

$$\mathcal{C} : \mathcal{G}_A = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}_A/\mathcal{G}_A[w]$$

of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each of degree $\#k(w)$ and with composition equal to the canonical map $\mathcal{G}_A \to \mathcal{G}_A/\mathcal{G}_A[w]$. By Lemma 3.2 of [14], which holds equally well in our situation, the functor $\mathfrak{X}_U$ is representable by scheme X_U which is finite over X_{U_0}.

Let U^p be sufficiently small and let $\overline{X}_U = X_U \times_{\mathcal{O}_{F,w}} k(w)$ denote the special fibre of X_U. By parts (1) and (2) of Proposition 3.4 of [14] (whose proof applies in our situation), X_U has pure dimension n, it has semistable reduction over $\mathcal{O}_{F,w}$, it is regular and the natural map $X_U \to X_{U_0}$ is finite and flat. We let $\mathcal{A}_U$ denote the universal abelian variety over X_U.

We say that an isogeny $\mathcal{G} \to \mathcal{G}'$ of one-dimensional compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules of degree $\#k(w)$ over a scheme S of characteristic p has *connected kernel* if it induces the zero map $\operatorname{Lie} \mathcal{G} \to \operatorname{Lie} \mathcal{G}'$. Let $Y_{U,i}$ denote the closed subscheme of $\overline{X}_U$ over which $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel. By part (3) of Proposition 3.4 of [14], each $Y_{U,i}$ is smooth over $\operatorname{Spec} k(w)$ of pure dimension $n-1$, $\overline{X}_U = \cup_{i=1}^n Y_{U,i}$ and for $i \neq j$ the schemes $Y_{U,i}$ and $Y_{U,j}$ have no common connected component. It follows that X_U has strictly semistable reduction.

For each $S \subset \{1, \ldots, n\}$, we let

$$Y_{U,S} = \bigcap_{i \in S} Y_{U,i} \text{ and } Y_{U,S}^0 = Y_{U,S} - \bigcup_{T \supsetneq S} Y_{U,T}.$$

Since X_U has strictly semistable reduction, each $Y_{U,S}$ is smooth over $k(w)$ of pure dimension $n - \#S$ and the $Y_{U,S}^0$ are disjoint for different S.

The inverse systems X_U and X_{U_0}, for varying U^p, have compatible actions of $G(\mathbb{A}^{\infty,p})$. For each $S \subset \{1, \ldots, n\}$, the inverse systems $Y_{U,S}$ and $Y_{U,S}^0$ are stable under this action. As in the characteristic zero case, the actions of $G(\mathbb{A}^{\infty,p})$ extend to actions on the inverse systems of the universal abelian varieties $\mathcal{A}_U$ and $\mathcal{A}_{U_0}$. Here the action is by prime-to-p quasi-isogenies.

Let ξ be an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$. If $l \neq p$, then the sheaf $\mathcal{L}_\xi$ extends to a lisse sheaf on the integral models X_U and X_{U_0}. There exist non-negative integers m_ξ and t_ξ and an idempotent $\varepsilon_\xi \in \overline{\mathbb{Q}}_l[S_{m_\xi} \ltimes F^{m_\xi}]$ (where S_{m_ξ} is the symmetric group on m_ξ-letters) such that

$$\xi \cong \nu^{t_\xi} \otimes \varepsilon_\xi (V^\vee \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_l)^{\otimes m_\xi}.$$

This follows from the discussion on pages 97 and 98 of [8] (applied in our setting). Let $N \geqslant 2$ be prime to p and let

$$\varepsilon(m_\xi, N) = \prod_{x=1}^{m_\xi} \prod_{y \neq 1} \frac{[N]_x - N^y}{N - N^y} \in \mathbb{Q}[(N^{\mathbb{Z}_{\geqslant 0}})^{m_\xi}]$$

where $[N]_x$ is the element of $(N^{\mathbb{Z}_{\geqslant 0}})^{m_\xi}$ with N in the x-th entry and 1 in the other entries and where y runs from 0 to $2[F^+ : \mathbb{Q}]n$ but excludes 1. Thinking of $(N^{\mathbb{Z}_{\geqslant 0}})^{m_\xi} \subset F^{m_\xi}$, we set

$$a_\xi = a_{\xi,N} = \varepsilon_\xi \varepsilon(m_\xi, N)^{2n-1} \in \overline{\mathbb{Q}}_l[S_{m_\xi} \ltimes F^{m_\xi}].$$

Let proj denote the map $\mathcal{A}_U^{m_\xi} \to X_U$ and for $a = 1, \dots, m_\xi$, let $\mathrm{proj}_a : \mathcal{A}_U \to X_U$ denote the composition of the a-th inclusion $\mathcal{A}_U \hookrightarrow \mathcal{A}_U^{m_\xi}$ with proj. Then we have the following (see page 477 of [14]):

- $\varepsilon(m_\xi, N) R^j \mathrm{proj}_* \overline{\mathbb{Q}}_l = \begin{cases} (0) & (j \neq m_\xi) \\ \bigotimes_{a=1}^{m_\xi} R^1 \mathrm{proj}_{a,*} \overline{\mathbb{Q}}_l & (j = m_\xi) \end{cases}$;
- $\varepsilon_\xi \varepsilon(m_\xi, N) R^{m_\xi} \mathrm{proj}_* \overline{\mathbb{Q}}_l = \mathcal{L}_\xi$;
- a_ξ acts as an idempotent on each $H^j(\mathcal{A}_U^{m_\xi} \times_{F_w} \overline{F}_w, \overline{\mathbb{Q}}_l(t_\xi))$ and moreover

$$a_\xi H^j(\mathcal{A}_U^{m_\xi} \times_{\mathcal{O}_{F,w}} \overline{F}_w, \overline{\mathbb{Q}}_l(t_\xi)) = \begin{cases} (0) & (j < m_\xi) \\ H^{j-m_\xi}(X_U \times_{F_w} \overline{F}_w, \mathcal{L}_\xi) & (j \geqslant m_\xi). \end{cases}$$

Let $\mathcal{A}_{U,S}^{m_\xi} = \mathcal{A}_U^{m_\xi} \times_{X_U} Y_{U,S}$. As U^p varies, the inverse system of the $\mathcal{A}_{U,S}^{m_\xi}$ inherits an action of $G(\mathbb{A}^{p,\infty})$ by prime-to-p quasi-isogenies. We now make the following definitions.

- Define admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^{\infty,p}) \times G_F]$-modules:

$$\begin{aligned} H^j(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi) &:= \varinjlim_{U^p} H^j(X_U \times_F \overline{F}, \mathcal{L}_\xi) = H^j(X, \mathcal{L}_\xi)^{\mathrm{Iw}(m)} \\ H^j(\mathcal{A}_{\mathrm{Iw}(m)}^{m_\xi}, \overline{\mathbb{Q}}_l) &:= \varinjlim_{U^p} H^j(\mathcal{A}_U^{m_\xi} \times_F \overline{F}, \overline{\mathbb{Q}}_l). \end{aligned}$$

- If $l \neq p$, we define admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^{p,\infty}) \times \mathrm{Frob}_w^{\mathbb{Z}}]$-modules:

$$\begin{aligned} H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi) &:= \varinjlim_{U^p} H^j(Y_{U,S} \times_{k(w)} \overline{k(w)}, \mathcal{L}_\xi) \\ H_c^j(Y_{\mathrm{Iw}(m),S}^0, \mathcal{L}_\xi) &:= \varinjlim_{U^p} H_c^j(Y_{U,S}^0 \times_{k(w)} \overline{k(w)}, \mathcal{L}_\xi) \\ H^j(\mathcal{A}_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi) &:= \varinjlim_{U^p} H^j(\mathcal{A}_{U,S} \times_{k(w)} \overline{k(w)}, \overline{\mathbb{Q}}_l). \end{aligned}$$

– If $l = p$ and $\sigma : W_0 \hookrightarrow \overline{\mathbb{Q}}_l$ over $\mathbb{Z}_p = \mathbb{Z}_l$, where W_0 is the Witt ring of $k(w)$, then we define the admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}}]$-module
$$H^j(\mathcal{A}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l := \varinjlim_{U^p} H^j(\mathcal{A}_{U,S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l.$$
(Here $H^j(\mathcal{A}_{U,S}/W_0)$ denotes crystalline cohomology and Frob_w acts by the $[k(w) : \mathbb{F}_p]$-power of the crystalline Frobenius.)

We note that if $l \neq p$, then a_ξ acts as an idempotent on $H^j(\mathcal{A}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_l)$ and
$$a_\xi H^j(\mathcal{A}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_l) = \begin{cases} (0) & (j < m_\xi) \\ H^{j-m_\xi}(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi) & (j \geqslant m_\xi). \end{cases}$$
If $l = p$, then a_ξ acts as an idempotent on $H^j(\mathcal{A}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l$. We also note that I_{F_w} acts trivially on $\mathrm{WD}(H^j(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{G_{F_w}})$ and thus the latter can be regarded as a $\mathrm{Frob}_w^{\mathbb{Z}}$-module.

Proposition 4.1. — *Let T be a finite set of places of $\mathbb{Q}$ containing $\{p, \infty\} \cup \mathrm{Ram}_{F/\mathbb{Q}}$ and let $\Pi^T \in \mathrm{Irr}(\mathbb{G}_n(\mathbb{A}^T))$ be unramified at all $v \notin T$. If $l = p$, let $\sigma : W_0 \hookrightarrow \overline{\mathbb{Q}}_l$ over $\mathbb{Z}_l = \mathbb{Z}_p$. Then there is a spectral sequence in the category of admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}_{T_{\mathrm{fin}}-\{p\}}) \times \mathrm{Frob}_w^{\mathbb{Z}}]$-modules*
$$E_1^{i,j}(\mathrm{Iw}(m), \xi)^{K^T}\{\Pi^T\} \Longrightarrow \mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{G_{F_w}})^{K^T}\{\Pi^T\}$$
where $E_1^{i,j}(\mathrm{Iw}(m), \xi) = \bigoplus_{s \geqslant \max(0,-i)} \bigoplus_{\#S=i+2s+1} H^j_{S,s}$, and
$$H^j_{S,s} = \begin{cases} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_l(t_\xi - s)) = H^{j-2s}(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi(-s)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (l \neq p) \\ a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l(t_\xi - s) \qquad\qquad\qquad (l = p). \end{cases}$$
Moreover, the monodromy operator N on $\mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{G_{F_w}})^{K^T}\{\Pi^T\}$ is induced by the identity map
$$N : \bigoplus_{\#S=i+2s+1} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_l(t_\xi - s))$$
$$\xrightarrow{\sim} \bigoplus_{\#S=(i+2)+2(s-1)+1} a_\xi H^{(j-2)+m_\xi-2(s-1)}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_l(t_\xi - (s-1)))$$
in the case when $l \neq p$ (resp.
$$N : \bigoplus_{\#S=i+2s+1} a_\xi H^{j+m_\xi-2s}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l(t_\xi - s)$$
$$\xrightarrow{\sim} \bigoplus_{\#S=(i+2)+2(s-1)+1} a_\xi H^{(j-2)+m_\xi-2(s-1)}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l(t_\xi - (s-1))$$
in the case when $l = p$).

Proof. — The proof of Proposition 3.5 of [14] shows that we have a spectral sequence $E_1^{i,j}(\mathrm{Iw}(m),\xi) \Longrightarrow \mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)|_{G_{F_w}})$ and that the monodromy operator N on $\mathrm{WD}(H^{i+j}(X_{\mathrm{Iw}(m)},\mathcal{L}_\xi)|_{G_{F_w}})$ is induced by the maps above. The result now follows from the fact that $R \mapsto R^{K^T}\{\Pi^T\}$ is an exact functor from the category of admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}}]$-modules to the category of admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}_{T_{\mathrm{fin}}-\{p\}}) \times \mathrm{Frob}_w^{\mathbb{Z}}]$-modules. $\square$

5. Relating the cohomology of $Y_{U,S}$ to the cohomology of Igusa varieties

Let $U^p \subset G(\mathbb{A}^{\infty,p})$ be sufficiently small and let $m \in \mathbb{Z}_{\geqslant 0}^{r-1}$. Following Section 4 of [14], we can relate the cohomology of the open strata $Y_{U,S}^0$ to the cohomology of Igusa varieties of the first kind. For $h = 0,\ldots,n-1$ and $m_1 \in \mathbb{Z}_{\geqslant 0}$, let $I_{U^p,(m_1,m)}^{(h)}/\overline{X}_{U_0}^{(h)}$ denote the Igusa variety of the first kind defined as on page 121 of [8]. It is the moduli space of isomorphisms

$$\alpha_1^{\mathrm{et}} : (w^{-m_1}\mathcal{O}_{F,w}/\mathcal{O}_{F,w})^h_{\overline{X}_{U_0}^{(h)}} \xrightarrow{\sim} \mathcal{G}^{\mathrm{et}}[w^{m_1}].$$

Let $I_U^{(h)}/\overline{X}_{U_0}^{(h)}$ be the Iwahori-Igusa variety of the first kind defined as on page 487 of [14]. It is the moduli space of chains of isogenies

$$\mathcal{G}^{\mathrm{et}} = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_h = \mathcal{G}^{\mathrm{et}}/\mathcal{G}^{\mathrm{et}}[w]$$

of étale Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each of degree $\#k(w)$ and with composition equal to the natural map $\mathcal{G}^{\mathrm{et}} \to \mathcal{G}^{\mathrm{et}}/\mathcal{G}^{\mathrm{et}}[w]$. Then $I_{U^p,(m_1,m)}^{(h)}$ and $I_U^{(h)}$ are finite étale over $\overline{X}_{U_0}^{(h)}$ and the natural map $I_{U^p,(1,m)}^{(h)} \to I_U^{(h)}$ is finite étale and Galois with Galois group $B_h(k(w))$. The inverse systems $I_{U^p,(m_1,m)}^{(h)}$ and $I_U^{(h)}$, for varying U^p, inherit an action of $G(\mathbb{A}^{\infty,p})$. Let ξ be an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$. If $l \neq p$, then ξ gives rise to a lisse sheaf $\mathcal{L}_\xi$ on $I_{U^p,(m_1,m)}^{(h)}$ and $I_U^{(h)}$.

For $S \subset \{1,\ldots,n\}$ and $h = n - \#S$, there is a natural map $\varphi : Y_{U,S}^0 \to I_U^{(h)}$ which is defined by sending the chain of isogenies $\mathcal{C}$ to its étale quotient. By Lemma 4.1 of [14] this map is finite and bijective on geometric points. By Corollary 4.2 of *op. cit.* we have

$$\begin{aligned} H_c^i(Y_{U,S}^0 \times_{k(w)} \overline{k(w)},\mathcal{L}_\xi) &\xrightarrow{\sim} H_c^i(I_U^{(h)} \times_{k(w)} \overline{k(w)},\mathcal{L}_\xi) \\ &\xrightarrow{\sim} H_c^i(I_{U^p,(1,m)}^{(h)} \times_{k(w)} \overline{k(w)},\mathcal{L}_\xi)^{B_h(k(w))} \end{aligned}$$

for each $i \in \mathbb{Z}_{\geqslant 0}$ and these isomorphisms are compatible with the action of $G(\mathbb{A}^{p,\infty})$ as U^p varies.

If $l \neq p$, set

$$H^i_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi) = \varinjlim_{U^p} H^i_c(I^{(h)}_U \times_{k(w)} \overline{k(w)}, \mathcal{L}_\xi).$$

This is an admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}}]$-module. Define

$$\begin{aligned}
[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)] &= \sum_i (-1)^{n-\#S-i} H^i(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi) \\
[H_c(Y^0_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)] &= \sum_i (-1)^{n-\#S-i} H^i_c(Y^0_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi) \\
[H_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)] &= \sum_i (-1)^{h-i} H^i_c(I^{(h)}_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)
\end{aligned}$$

in $\mathrm{Groth}_l(G(\mathbb{A}^{\infty,p}) \times \mathrm{Frob}_w^{\mathbb{Z}})$.

There is, up to isomorphism, a unique one-dimensional compatible formal Barsotti-Tate $\mathcal{O}_{F,w}$-module $\Sigma_{F_w,n-h}$ over $\overline{k(w)}$ of $\mathcal{O}_{F,w}$-height $n-h$. We have $\mathrm{End}_{\mathcal{O}_{F,w}}(\Sigma_{F_w,n-h}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong D_{F_w,n-h}$, the division algebra with centre F_w and Hasse invariant $1/(n-h)$. For $m_1 \in \mathbb{Z}_{\geqslant 0}$, let $\mathrm{Ig}^{(h)}_{U^p,(m_1,m)}/(\overline{X}^{(h)}_{U_0} \times_{k(w)} \overline{k(w)})$ denote the moduli space of $\mathcal{O}_{F,w}$-equivariant isomorphisms

$$\begin{aligned}
j^{\mathrm{et}} : (w^{-m_1}\mathcal{O}_{F,w}/\mathcal{O}_{F,w})^h_{\overline{X}^{(h)}_{U_0} \times_{k(w)} \overline{k(w)}} &\xrightarrow{\sim} \mathcal{G}^{\mathrm{et}}[w^{m_1}] \\
j^0 : (\Sigma_{F_w,n-h}[w^{m_1}])_{\overline{X}^{(h)}_{U_0} \times_{k(w)} \overline{k(w)}} &\xrightarrow{\sim} \mathcal{G}^0[w^{m_1}]
\end{aligned}$$

that extend étale locally to any level $m_1' > m_1$. (In the notation of [13], for each $0 \leqslant h \leqslant n-1$, there is a unique $b \in B(G_{\mathbb{Q}_p}, -\mu)$ corresponding to h (see displayed equation (5.3) of *op. cit.*). If $m = (m_1, \ldots, m_1)$, then $\mathrm{Ig}^{(h)}_{U^p,(m_1,m)}$ is denoted Ig_{b,U^p,m_1} in [13] (see Section 5.2 of *op. cit.* and Section 4 of [12]). We have simply extended the definition to 'non-parallel' $(m_1, m) \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}^{r-1}_{\geqslant 0}$. We also note that the notation $\mathrm{Ig}^{(h)}$ is used in place of Ig_b in Section 7.3 of [13].) If $l \neq p$ and ξ is an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$, then ξ gives rise to a lisse sheaf $\mathcal{L}_\xi$ on each $\mathrm{Ig}^{(h)}_{U^p,(m_1,m)}$. Let

$$H^i_c(\mathrm{Ig}^{(h)}, \mathcal{L}_\xi) = \varinjlim_{U^p,m_1,m} H^i_c(\mathrm{Ig}^{(h)}_{U^p,(m_1,m)}, \mathcal{L}_\xi).$$

This is an admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^{\infty,p}) \times J^{(h)}(\mathbb{Q}_p)]$-module where

$$J^{(h)}(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times (D^\times_{F_w,n-h} \times GL_h(F_w)) \times \prod_{i=2}^r GL_n(F_{w_i})$$

(see Section 5 of [13], where $J^{(h)}(\mathbb{Q}_p)$ is denoted $J_b(\mathbb{Q}_p)$, with b being the element of $B(\mathbb{Q}_p, -\mu)$ corresponding to h; in Section 7.3 of *op. cit.*, J_b is denoted $J^{(h)}$). We have

$$H_c^i(\mathrm{Ig}^{(h)}, \mathcal{L}_\xi)^{\mathbb{Z}_p^\times \times (\mathcal{O}_{D_{F_w,n-h}}^\times \times \mathrm{Iw}_{h,w}) \times U_p^w(m)} \cong H_c^i(I_{\mathrm{Iw}(m)}^{(h)}, \mathcal{L}_\xi),$$

where the latter is regarded as an admissible $\overline{\mathbb{Q}}_l[G(\mathbb{A}^{p,\infty})]$-module. Moreover, the action of Frob_w on the right hand side corresponds to the action of

$$(1, p^{-[k(w):\mathbb{F}_p]}, \varpi_{D_{F_w,n-h}}^{-1}, 1, 1) \in G(\mathbb{A}^{p,\infty}) \times \mathbb{Q}_p^\times \times D_{F_w,n-h}^\times \times GL_h(F_w) \times \prod_{i=2}^r GL_n(F_{w_i})$$

on the left hand side, where $\varpi_{D_{F_w,n-h}}$ is any uniformizer in $D_{F_w,n-h}$. We let

$$[H_c(\mathrm{Ig}^{(h)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h-i} H_c^i(\mathrm{Ig}^{(h)}, \mathcal{L}_\xi)$$

in $\mathrm{Groth}_l(G(\mathbb{A}^{\infty,p}) \times J^{(h)}(\mathbb{Q}_p))$. As on page 489 of [14], we have

$$[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)] = \sum_{T \supset S} (-1)^{(n-\#S)-(n-\#T)} [H_c(I_{\mathrm{Iw}(m)}^{(n-\#T)}, \mathcal{L}_\xi)].$$

As there are $\begin{pmatrix} n-\#S \\ h \end{pmatrix}$ subsets $T \supset S$ with $n - \#T = h$, we deduce the following:

Lemma 5.1. — *Suppose $l \neq p$ and $S \subset \{1, \ldots, n\}$. Then we have an equality*

$$[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)] = \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \begin{pmatrix} n-\#S \\ h \end{pmatrix} [H_c(\mathrm{Ig}^{(h)}, \mathcal{L}_\xi)]^{\mathbb{Z}_p^\times \times (\mathcal{O}_{D_{F_w,n-h}}^\times \times \mathrm{Iw}_{h,w}) \times U_p^w(m)}$$

in $\mathrm{Groth}_l(G(\mathbb{A}^{p,\infty}) \times \mathrm{Frob}_w^{\mathbb{Z}})$.

6. Computing the cohomology of $Y_{U,S}$

In this section we deduce analogues of Proposition 4.4 of [14].

6.1. The stable case

Let Π^1 be a RACSDC automorphic representation of the group $GL_n(\mathbb{A}_F)$. Suppose that Π^1 is Ξ^1-cohomological where Ξ^1 is an irreducible algebraic representation of $R_{F/\mathbb{Q}}GL_n$ over $\mathbb{C}$. Assume that

– $\mathrm{Ram}_{\mathbb{Q}}(\Pi^1) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$.

By Lemma 7.2 of [13], we can and do choose an algebraic Hecke character $\psi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ and an algebraic representation $\xi_{\mathbb{C}}$ of G over $\mathbb{C}$ such that

– $\psi_{\Pi^1}|_{\mathbb{A}_E^\times} = \psi^c/\psi$;

– If Ξ is the representation of $\mathbb{G}_n$ over $\mathbb{C}$ corresponding to $\xi_{\mathbb{C}}$ as in Section 2, then Ξ^1 is isomorphic to the restriction of Ξ to $(R_{F/Q}GL_n) \times_{\mathbb{Q}} \mathbb{C}$;

– $\xi_{\mathbb{C}}|_{E_\infty^\times}^{-1} = \psi_\infty^c$ (see below);

– $\mathrm{Ram}_{\mathbb{Q}}(\psi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$;

– ψ is unramified at u (recall that u is the prime of E below the w_i).

(We note that Lemma 7.2 of [13] does not guarantee that ψ be unramified at u, but the fact that this can be achieved follows from the proof of Lemma VI.2.10 of [8].) In the third bullet point, we consider $E_\infty^\times$ embedded in $G(\mathbb{R}) \subset \mathbb{R}^\times \times GL(V \otimes_{\mathbb{Q}} \mathbb{R})$ via the map $z \mapsto (zz^c, z)$. It then follows from the third bullet point that $R_{l,\imath}(\psi)$ is pure of weight $m_\xi - 2t_\xi$. Set

$$\Pi := \psi \otimes \Pi^1.$$

Then Π is a Ξ-cohomological automorphic representation of $\mathbb{G}_n(\mathbb{A}) \cong GL_1(\mathbb{A}_E) \times GL_n(\mathbb{A}_F)$. Note that Π^1, ψ and Π satisfy the assumptions of Section 3.1. Let $\xi = \imath^{-1}\xi_{\mathbb{C}}$, an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$. Let $\pi_p \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ be such that $\mathrm{BC}_p(\imath\pi_p) \cong \Pi_p$ (note that π_p is unique as p splits in E).

The next result follows from Proposition 7.14 of [13].

Proposition 6.1. — *Suppose that $l \neq p$ and $\pi_p^{\mathrm{Iw}(m)} \neq (0)$. Let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \{p\} \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. Then for every $S \subset \{1, \ldots, n\}$, we have*

$$H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)^{K^T}\{\Pi^T\} = (0)$$

for $j \neq n - \#S$.

The following corollary can be proved in the same way as Corollary 4.5 of [14].

Corollary 6.2. — *Suppose that $l = p$ and $\sigma : W_0 \hookrightarrow \overline{\mathbb{Q}}_l$ over $\mathbb{Z}_p = \mathbb{Z}_l$. Let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \{l\} \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. If $\pi_l^{\mathrm{Iw}(m)} \neq (0)$, then for every $S \subset \{1, \ldots, n\}$, we have*

$$a_\xi(H^{j+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l)^{K^T}\{\Pi^T\} = (0)$$

for $j \neq n - \#S$.

In the next result we place no restriction on the primes l and p.

Corollary 6.3. — *If $\pi_p^{\mathrm{Iw}(m)} \neq 0$, then $\mathrm{WD}(\widetilde{R}^{n-1}_{\xi,l}(\Pi)|_{G_{F_w}})$ is pure of weight $m_\xi - 2t_\xi + n - 1$ and $\mathrm{WD}(R_{l,\imath}(\Pi^1)|_{G_{F_w}})$ is pure of weight $n - 1$.*

Proof. — Let $T = \{\infty\} \cup \mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \{p\}$ and let $D = \dim \pi_p^{\mathrm{Iw}(m)}$. Let $T' = T_{\mathrm{fin}} - \{p\}$. By Theorem 3.1, we have an isomorphism of $\overline{\mathbb{Q}}_l[\mathbb{G}_n(\mathbb{A}_{T'}) \times G_F]$-modules

$$\mathrm{BC}_{T'}(H^{n-1}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)^{K^T}\{\Pi^T\}) \cong ((\imath^{-1}\Pi_{T'}) \otimes \widetilde{R}^{n-1}_{\xi,l}(\Pi))^{\oplus D}.$$

By proposition 4.1, there is a spectral sequence

$$E_1^{i,j}(\mathrm{Iw}(m), \xi)^{K^T}\{\Pi^T\} \Longrightarrow \mathrm{WD}(H^{n-1}(X_{\mathrm{Iw}(m)}, \mathcal{L}_\xi)|_{G_{F_w}})^{K^T}\{\Pi^T\}.$$

Using Proposition 6.1 (when $l \neq p$) and Corollary 6.2 (when $l = p$) we see that $E_1^{i,j}(\mathrm{Iw}(m), \xi)^{K^T}\{\Pi^T\} = (0)$ unless $i + j = n - 1$, and thus the spectral sequence degenerates at E_1. Let $\pi_{T'}$ denote the unique element of $\mathrm{Irr}_l(G(\mathbb{A}_{T'}))$ with $\mathrm{BC}_{T'}(\pi_{T'}) = \imath^{-1}\Pi_{T'}$. Then, for $i + j = n - 1$, $E_1^{i,j}(\mathrm{Iw}(m), \xi)^{K^T}\{\Pi^T\}$ is of the form $\pi_{T'} \otimes R_j$ where R_j is a finite dimensional $\overline{\mathbb{Q}}_l[\mathrm{Frob}_w^{\mathbb{Z}}]$-module which is pure of weight $j + m_\xi - 2t_\xi$ (and possibly zero). The first statement now follows from this and the description of the monodromy operator N in Proposition 4.1. The second statement follows from the first statement together with Theorem 3.1 and Lemma 1.7 of [14]. $\square$

6.2. The endoscopic case

Suppose we are in the following situation:

- Π_1 is a RACSDC automorphic representation of $GL_{n-1}(\mathbb{A}_F)$;
- $\mathrm{Ram}_{\mathbb{Q}}(\Pi_1) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$;
- Π_1 is cohomological for an irreducible algebraic representation Ξ_1 of the group $R_{F/\mathbb{Q}}(GL_{n-1})$ over $\mathbb{C}$;
- Π_1 has Shin-regular weight.

Lemma 6.4. — *We can find*

- *a continuous algebraic character* $\Pi_2 : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ *with* $\Pi_2^{-1} = \Pi_2 \circ c$*;*
- *a continuous algebraic character* $\psi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$*;*
- *a continuous character* $\varpi : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$*; and*
- *an irreducible algebraic representation* $\xi_{\mathbb{C}}$ *of* G *over* $\mathbb{C}$

such that if we set

$$\begin{aligned} \Pi_{M,1} &:= \Pi_1 \otimes (\varpi \circ \mathbf{N}_{F/E} \circ \det) \\ \Pi_{M,2} &:= \Pi_2 \\ \Pi^1 &:= \mathrm{n}-\mathrm{Ind}_{GL_{n-1}\times GL_1}^{GL_n}(\Pi_{M,1} \otimes \Pi_{M,2}) \end{aligned}$$

and let

- Ξ *be the irreducible algebraic representation of* $\mathbb{G}_n$ *over* $\mathbb{C}$ *which corresponds to* $\xi_{\mathbb{C}}$ *as in Section 2;*
- $\Xi^1 := \Xi|_{R_{F/Q}(GL_n)\times_{\mathbb{Q}}\mathbb{C}}$*,*

then

- $\mathrm{Ram}_{\mathbb{Q}}(\Pi_2) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$*;*
- Π_2 *is unramified at* u*;*
- $\mathrm{Ram}_{\mathbb{Q}}(\psi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$*;*
- $\xi_{\mathbb{C}}|_{E_\infty^\times}^{-1} = \psi_\infty^c$*;*
- ψ *is unramified at* u*;*
- $\mathrm{Ram}_{\mathbb{Q}}(\varpi) \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$*;*

– $\varpi|_{\mathbb{A}^\times}$ *factors through* $\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}^\times_{>0}$ *and equals the composite of* $\mathrm{Art}_{\mathbb{Q}}$ *with the surjective character* $G_{\mathbb{Q}}^{\mathrm{ab}} \twoheadrightarrow \mathrm{Gal}\,(E/\mathbb{Q}) \xrightarrow{\sim} \{\pm 1\}$ *(note that this implies* $\varpi^{-1} = \varpi \circ c$*);*

– ϖ *is unramified at* u*;*

– Π^1 *is cohomological for* Ξ^1 *(note that* Π^1 *is irreducible, as* $\Pi_{M,1}$ *and* $\Pi_{M,2}$ *are unitary, and also that* $(\Pi^1)^\vee \cong \Pi^1 \circ c$*);*

– $\psi_{\Pi^1}|_{\mathbb{A}_E^\times} = \psi^c/\psi$ *(recall that* ψ_{Π^1} *denotes the central character of* Π^1*).*

Moreover, if we apply Theorem 3.2 to Π*, then alternative (2)(a) holds. In other words, the integer* $e_2(\Pi, G)$ *equals* 1 *and if* $\xi = \imath^{-1}\xi_{\mathbb{C}}$*, then*

$$\widetilde{R}^{n-1}_{\xi,l}(\Pi)^{\mathrm{ss}} \cong R_{l,\imath}(\Pi_1)^{G_G} \otimes R_{l,\imath}(\psi\varpi|\cdot|^{1/2})|_{G_F}.$$

Proof. — This follows by combining Lemmas 7.1, 7.2 and 7.3 of [13]. (More precisely, we first choose ϖ using Lemma 7.1. The extra condition that ϖ be unramified at u is easily achieved – in the proof of Lemma 7.1 we add the primes u and u^c to the set R and insist that ϖ^0 takes value 1 on $p \in E_u^\times$ and on $p \in E_{u^c}^\times$. We then make two candidate choices χ and χ' for Π_2 with $\chi^{-1} = \chi \circ c$, $\mathrm{Ram}_{\mathbb{Q}}(\chi) \subset \mathrm{Spl}_{F/F^+}(\mathbb{Q})$ and χ unramified at u and with χ' having the same properties. In addition, we assume that the infinity type of χ and χ' are as prescribed in the paragraph before Lemma 7.2 of [13]. (The fact that we can find such characters follows for instance from Lemma 2.2 of [7].) Lemma 7.2 of [13] then tells us that we can choose pairs $(\psi, \xi_{\mathbb{C}})$ and $(\psi', \xi'_{\mathbb{C}})$ corresponding to the choice of $\Pi_2 = \chi$ or $\Pi_2 = \chi'$ and satisfying all the required properties except the requirement that ψ and ψ' be unramified at u and the requirement that the integer $e_2(\Pi, G)$ equal 1. However, the proof of Lemma VI.2.10 of [8] shows that we may choose ψ and ψ' to be unramified at u and Lemma 7.3 of [13] shows that for one of the choices $(\chi, \psi, \xi_{\mathbb{C}})$ or $(\chi', \psi', \xi'_{\mathbb{C}})$, the corresponding integer $e_2(\Pi, G)$ equals 1.) $\square$

Choose Π_2, ψ, ϖ and $\xi_{\mathbb{C}}$ as in the above lemma and keep all additional notation introduced there. Let

$$\Pi = \psi \otimes \Pi^1,$$

an automorphic representation of $\mathbb{G}_n(\mathbb{A})$. Let $\pi_p \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ be the unique representation with $\mathrm{BC}\,(\imath\pi_p) \cong \Pi_p$. Write $\pi_p = \pi_p^0 \otimes \pi_w \otimes (\otimes_{i=2}^r \pi_{w_i})$ corresponding to the decomposition $G(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times GL_n(F_w) \times \prod_{i=2}^r GL_n(F_{w_i})$. Let $\pi_{M,w,1} = \imath^{-1}\Pi_{M,1,w} \in \mathrm{Irr}_l(GL_{n-1}(F_w))$ and $\pi_{M,w,2} = \imath^{-1}\Pi_{M,2,w} \in \mathrm{Irr}_l(GL_1(F_w))$.

For $a, b \in \mathbb{Z}_{\geqslant 0}$, let

$$\mathrm{n}-\mathrm{Red}^{a,b} : \mathrm{Groth}_l(GL_{a+b}(F_w)) \to \mathrm{Groth}_l(D_{F_w,a}^{\times} \times GL_b(F_w))$$

denote the composition

$$\begin{array}{ccc} \mathrm{Groth}_l(GL_{a+b}(F_w)) & \xrightarrow{J_{N_b^{\mathrm{op}}}} & \mathrm{Groth}_l(GL_a(F_w) \times GL_b(F_w)) \\ & \xrightarrow{\mathrm{LJ}_a \otimes \mathrm{id}} & \mathrm{Groth}_l(D_{F_w,a}^{\times} \times GL_b(F_w)) \end{array}$$

where N_b^{op} is unipotent radical of the parabolic subgroup of GL_m consisting of block lower triangular matrices with an $(a \times a)$-block in the upper left corner and a $(b \times b)$-block in the lower right corner;

$$J_{N_b^{\mathrm{op}}} : \mathrm{Groth}_l(GL_m(F_w)) \to \mathrm{Groth}_l(GL_a(F_w) \times GL_b(F_w))$$

is the normalized Jacquet module functor; and

$$\mathrm{LJ}_a : \mathrm{Groth}_l(GL_a(F_w)) \to \mathrm{Groth}_l(D_{F_w,a}^{\times})$$

is the map denoted $\mathbf{LJ}_1$ in Proposition 3.2 of [1]. (See Section 2.4 of [13].)

Let

$$\overline{\delta}_{P_h}^{1/2} : J^{(h)}(\mathbb{Q}_p) \to \mathbb{C}^{\times}$$

denote the character which sends $(g_{p,0}, (d, g), g_i) \in \mathbb{Q}_p^{\times} \times (D_{F_w,n-h}^{\times} \times GL_h(F_w)) \times \prod_{i=2}^{r} GL_n(F_{w_i})$ to $|\det(d)^h \det(g)^{-(n-h)}|_{F_w}^{1/2}$.

THEOREM 6.5. — *Suppose $l \neq p$. Let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \cup \{p\} \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. Then*

$$[H_c(\mathrm{Ig}^{(0)}, \mathcal{L}_{\xi})][\Pi^T] = (0)$$

while for $1 \leqslant h \leqslant n-1$, we have an equality

$$\begin{aligned} &\mathrm{BC}^p([H_c(\mathrm{Ig}^{(h)}, \mathcal{L}_{\xi})][\Pi^T]) = C_G[\imath^{-1}\Pi^{\infty,p}] \times \\ &\left[\left(\pi_{p,0} \otimes \mathrm{n}-\mathrm{Ind}_{GL_{h-1,1}(F_w)}^{GL_h(F_w)}(\mathrm{n}-\mathrm{Red}^{n-h,h-1}(\pi_{M,w,1}) \otimes \pi_{M,w,2}) \otimes (\otimes_{i=2}^{r} \pi_{w_i})\right) \otimes \imath^{-1}\overline{\delta}_{P_h}^{1/2}\right] \end{aligned}$$

in $\mathrm{Groth}_l(\mathbb{G}_n(\mathbb{A}^{\infty,p}) \times J^{(h)}(\mathbb{Q}_p))$.

Proof. — The result is essentially a rewording of part (ii) of Theorem 6.1 of [13]. We freely make use of the notation of *op. cit.* for the rest of this proof. Let $0 \leqslant h \leqslant n-1$ and let $b \in B(\mathbb{Q}_p, -\mu)$ correspond to h (in the sense explained above). The constant e_1 which appears in the statement of Theorem 6.1 of [13] is equal to $(-1)^{n-1} = 1$ by Corollary 6.5(ii) of *op.*

cit. Lemma 6.4 above, and the choices made after it, guarantee that the constant e_2 also equals 1. Applying Theorem 6.1 of [13], we obtain

$$\begin{aligned}&\mathrm{BC}^p([H_c(\mathrm{Ig}^{(h)}, \mathcal{L}_\xi)][\Pi^T]) = \\ &C_G(-1)^h[\imath^{-1}\Pi^{\infty,p}] \times \left[\tfrac{1}{2}\left(\mathrm{Red}_n^b(\pi_p) + \mathrm{Red}_{n-1,1}^b(\pi_{H,p})\right)\right].\end{aligned}$$

(We remark that our definition of $[H_c(\mathrm{Ig}^{(h)}, \mathcal{L}_\xi)]$ differs from Shin's definition of $H_c(\mathrm{Ig}_b, \mathcal{L}_\xi)$ by a factor of $(-1)^h$.) We have $\mathrm{Red}_{n-1,1}^b(\pi_{H,p}) = \mathrm{n}-\mathrm{Red}_{n-1,1}^b(\pi_{H,p}) \otimes \imath^{-1}\overline{\delta}_{P_h}^{1/2}$ (see Section 5.5 of [13]). By Lemma 5.9 of [13] and the discussion immediately preceding it, we have

$$\mathrm{n}-\mathrm{Red}_n^b(\pi_p) + \mathrm{n}-\mathrm{Red}_{n-1,1}^b(\pi_{H,p}) = e_p(J_b)\pi_{p,0} \otimes 2X_1(h, \pi_{H,p}) \otimes (\otimes_{i=2}^r \pi_{w_i})$$

where

$$X_1(h, \pi_{H,p}) = \begin{cases} 0 & (h=0) \\ \mathrm{n}-\mathrm{Ind}_{GL_{h-1,1}(F_w)}^{GL_h(F_w)}(\mathrm{n}-\mathrm{Red}^{n-h,h-1}(\pi_{M,w,1}) \otimes \pi_{M,w,2}) & \\ & (h \neq 0) \end{cases}.$$

Moreover, $e_p(J_b) = (-1)^{n-h-1} = (-1)^h$ (see Case 1 in Section 5.5 of [13]). The result follows. $\square$

In Remark 7.16 of [13], Shin indicates that the following result can proved in the same way as Proposition 7.14 of *op. cit.*. We give a self-contained proof here for the benefit of the reader.

Proposition 6.6. — *Suppose $l \neq p$ and $\pi_p^{\mathrm{Iw}(m)} \neq (0)$. Let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \mathrm{Ram}_{\mathbb{Q}}(\varpi) \cup \{p\} \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. Then for every $S \subset \{1, \ldots, n\}$ we have*

$$H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)^{K^T}\{\Pi^T\} = (0)$$

for $j \neq n - \#S$.

Proof. — Let $D = C_G(\dim(\otimes_{i=2}^r \pi_{w_i})^{U_p^w(m)})$. We deduce from Theorem 6.5 and Lemma 5.1 that

$$\mathrm{BC}^p[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)][\Pi^T] = D[\imath^{-1}\Pi^{\infty,p}] \times \sum_{h=1}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h}$$

$$\left[\pi_{p,0}^{\mathbb{Z}_p^\times} \otimes \left(\mathrm{n}-\mathrm{Ind}_{GL_{h-1,1}(F_w)}^{GL_h(F_w)}(\mathrm{n}-\mathrm{Red}^{n-h,h-1}(\pi_{M,w,1}) \otimes \pi_{M,w,2}) \otimes \imath^{-1}\overline{\delta}_{P_h}^{1/2}\right)^{\mathcal{O}_{D_{F_w,n-h}}^\times \times \mathrm{Iw}_{h,w}}\right]$$

in $\mathrm{Groth}_l(\mathbb{G}_n(\mathbb{A}^{p,\infty}) \times \mathrm{Frob}_w^{\mathbb{Z}})$ (recall that Frob_w acts via $(p^{-[k(w):\mathbb{F}_p]}, (\varpi_{D_{F_w,n-h}}^{-1}, 1))$ in $\mathbb{Q}_p^\times \times (D_{F_w,n-h}^\times \times GL_h(F_w)))$. Since $\pi_p^{\mathrm{Iw}(m)} \neq (0)$, we can write $\pi_{M,w,1} = \mathrm{Sp}_{s_1}(\pi_1) \boxplus \ldots \boxplus \mathrm{Sp}_{s_t}(\pi_t)$ where each π_j is an unramified character $\pi_j : F_w^\times \to \overline{\mathbb{Q}}_l^\times$. As $\Pi_{M,1}$ is generic, we know that $\pi_{M,w,1} = \mathrm{n}-\mathrm{Ind}_{P(F_w)}^{GL_{n-1}(F_w)}(\mathrm{Sp}_{s_1}(\pi_1) \otimes \ldots \otimes \mathrm{Sp}_{s_t}(\pi_t))$ where $P \subset GL_{n-1}$ is an appropriate parabolic subgroup.

Using Lemma I.3.9 of [8] and Théorème 3.1 and Proposition 3.2 of [1], we see that for $1 \leqslant h \leqslant n - \#S$,

$$\begin{aligned}&\mathrm{n}-\mathrm{Ind}_{GL_{h-1,1}(F_w)}^{GL_h(F_w)}(\mathrm{n}-\mathrm{Red}^{n-h,h-1}(\pi_{M,w,1}) \otimes \pi_{M,w,2}) = \\ &\textstyle\sum_i [(\pi_i \cdot \imath^{-1}|\cdot|_{F_w}^{(n-h-1)/2}) \circ \det] \times [\mathrm{n}-\mathrm{Ind}_{P'(F_w)}^{GL_h(F_w)}(\mathrm{Sp}_{s_i-(n-h)}(\pi_i|\cdot|_{F_w}^{n-h}) \otimes \\ &\qquad\qquad (\otimes_{j\neq i}\mathrm{Sp}_{s_j}(\pi_j)) \otimes \pi_{M,w,2})]\end{aligned}$$

in $\mathrm{Groth}_l(D_{F_w,n-h}^\times \times GL_h(F_w))$, where the sum is over all i such that $s_i \geqslant n-h$, and $P' \subset GL_h$ is an appropriate parabolic subgroup. (As pointed out in [14], there is a typo in Lemma I.3.9 of [8] — 'positive integers $h_1, \ldots, h_t$' should be replaced by 'non-negative integers $h_1, \ldots, h_t$'.) Moreover,

$$\begin{aligned}&\dim\left(\mathrm{n}-\mathrm{Ind}_{P'(F_w)}^{GL_h(F_w)}(\mathrm{Sp}_{s_i-(n-h)}(\pi_i|\cdot|_{F_w}^{n-h}) \otimes (\otimes_{j\neq i}\mathrm{Sp}_{s_j}(\pi_j)) \otimes \pi_{M,w,2})\right)^{\mathrm{Iw}_{h,w}} \\ &= \frac{h!}{(s_i-(n-h))!\prod_{j\neq i}s_j!}\end{aligned}$$

(see page 490 of [14]). From this we deduce that

$$\begin{aligned}&\mathrm{BC}^p[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)][\Pi^T] = D[\imath^{-1}\Pi^{\infty,p}] \times \\ &\sum_{h=1}^{n-\#S}(-1)^{n-\#S-h}\binom{n-\#S}{h}\sum_{i:s_i\geqslant n-h}\frac{h!}{(s_i-(n-h))!\prod_{j\neq i}s_j!}[V_i],\end{aligned}$$

where $V_i = \mathrm{rec}(\pi_i^{-1} \cdot \imath^{-1}|\cdot|_{F_w}^{(1-n)/2}(\pi_{p,0} \circ \mathbf{N}_{F_w/E_u})^{-1})$. As on page 490 of [14], it follows that

$$\mathrm{BC}^p[H(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)][\Pi^T] = D[\imath^{-1}\Pi^{\infty,p}] \times \sum_{i:s_i=\#S}\frac{(n-\#S)!}{\prod_{j\neq i}s_j!}[V_i].$$

As $\Pi_{1,w}$ is unitary and tempered (by Corollary 1.3 of [13]) and $\mathrm{rec}(\pi_{p,0}) \cong \imath^{-1}\mathrm{rec}(\psi_u)$ is strictly pure of weight $2t_\xi - m_\xi$ (since $\xi_{\mathbb{C}}|_{E_\infty^\times}^{-1} = \psi_\infty^c$), we see that $\mathrm{rec}(\pi_{M,w,1}^\vee \otimes \imath^{-1}|\cdot|_{F_w}^{(1-n)/2}(\pi_{p,0} \circ \mathbf{N}_{F_w/E_u})^{-1})$ is pure of weight $m_\xi - 2t_\xi + n - 1$. If $s_i = \#S$, it follows that V_i is strictly pure of weight $m_\xi - 2t_\xi + n - 1 - (\#S - 1) = m_\xi - 2t_\xi + n - \#S$. The Weil conjectures now imply that if $j \neq n - \#S$ then

$$[H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)][\Pi^T] = a_\xi[H^{m_\xi+j}(\mathcal{A}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_l(t_\xi))][\Pi^T] = (0)$$

in $\mathrm{Groth}_l(G(\mathbb{A}^{p,\infty}) \times \mathrm{Frob}_w^{\mathbb{Z}})$. Since

$$[H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)^{K^T}\{\Pi^T\}] = ([H^j(Y_{\mathrm{Iw}(m),S}, \mathcal{L}_\xi)][\Pi^T])^{K^T}$$

in $\mathrm{Groth}_l(G(\mathbb{A}_{T_{\mathrm{fin}}-\{p\}}) \times \mathrm{Frob}_w^{\mathbb{Z}})$, the result follows. □

The proof of Corollary 4.5 of [14] allows us to deduce the following.

Corollary 6.7. — *Suppose that $l = p$ and $\sigma : W_0 \hookrightarrow \overline{\mathbb{Q}}_l$ over $\mathbb{Z}_p = \mathbb{Z}_l$. Let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \{p\} \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$. If $\pi_p^{\mathrm{Iw}(m)} \neq (0)$, then for every $S \subset \{1, \dots, n\}$, we have*

$$a_\xi(H^{j+m_\xi}(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l)^{K^T}\{\Pi^T\} = (0)$$

for $j \neq n - \#S$.

The next corollary follows from the previous two results combined with Theorem 3.2 and the proof of Corollary 6.3.

Corollary 6.8. — *If $\pi_p^{\mathrm{Iw}(m)} \neq 0$, then $\mathrm{WD}(\widetilde{R}^{n-1}_{\xi,l}(\Pi)|_{G_{F_w}})$ is pure of weight $m_\xi - 2t_\xi + n - 1$ and $\mathrm{WD}(R_{l,\imath}(\Pi_1)|_{G_{F_w}})$ is pure of weight $n - 2$.*

7. Proof of Theorem 1.2

We now complete the proof of Theorem 1.2. Suppose that $v|l$ is a place of L with $\Pi_v^{\mathrm{Iw}_{m,v}} \neq \{0\}$. Choose a finite CM soluble Galois extension F/L such that

- $[F^+ : \mathbb{Q}]$ is even;
- $F = EF^+$ where E is a quadratic imaginary field in which l splits;
- F splits completely above v;
- $\mathrm{BC}_{F/L}(\Pi)$ is cuspidal;
- $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\mathrm{BC}_{F/L}(\Pi)) \subset \mathrm{Spl}_{\mathbb{Q},F/F^+}$.

(See the argument in the penultimate paragraph of [13].) Let l and $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ as given to us by the statement of Theorem 1.2. Choose another prime $l' \neq l$ and $\imath' : \overline{\mathbb{Q}}_{l'} \xrightarrow{\sim} \mathbb{C}$. Recall that in Section 2 we introduced notation that was then in force from Section 3 to Section 6. We will shortly apply the results of these sections in two scenarios – one where the pair $(l, \imath)$ of

Section 2 is equal to the pair $(l,\imath)$ of the statement of Theorem 1.2 and one where the $(l,\imath)$ of Section 2 is equal to the pair $(l',\imath')$ chosen above. The rest of the notation we fix as follows: we take E, F and F^+ as chosen above. We take $p = l$ and let w be a prime of F lying above the prime v of L. This determines u and $w_1, \ldots, w_r$. We choose some $\tau : F \hookrightarrow \mathbb{C}$ and $\imath_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ such that $\imath_p^{-1} \circ \tau$ induces w. We take $n = m$ if m is odd and $n = m+1$ otherwise. Finally we choose a set of data $(V, \langle\cdot,\cdot\rangle, h)$ satisfying the assumptions of Section 2.

Suppose first of all that m is odd. Denote $\mathrm{BC}_{F/L}(\Pi_L)$ by Π^1. We choose ψ and $\xi_{\mathbb{C}}$ as in Section 6.1 and set $\widetilde{\Pi} = \psi \otimes \Pi^1$, $\xi = \imath^{-1}\xi_{\mathbb{C}}$ and $\xi' = (\imath')^{-1}\xi_{\mathbb{C}}$. Define $\widetilde{R}^{n-1}_{\xi,l}(\widetilde{\Pi})$ and $\widetilde{R}^{n-1}_{\xi',l'}(\widetilde{\Pi})$ as in Section 3.1. Then $\widetilde{R}^{n-1}_{\xi',l'}(\widetilde{\Pi})^{\mathrm{ss}} \cong R_{l',\imath'}(\Pi^1)^{C_G} \otimes R_{l',\imath'}(\psi)|_{G_F}$ by Theorem 3.1 and hence

$$\imath'\mathrm{WD}(\widetilde{R}^{n-1}_{\xi',l'}(\widetilde{\Pi})^{\mathrm{ss}}|_{G_{F_w}})^{\mathrm{F-ss}} \cong \mathrm{rec}((\Pi^1_w)^\vee \otimes |\cdot|^{(1-n)/2} \circ \det)^{C_G} \otimes \mathrm{rec}(\psi_u^{-1} \circ \mathbf{N}_{F_w/E_u})$$

by Theorem 1.2 of [13]. Let $T \supset \{\infty\}$ be a finite set of places of $\mathbb{Q}$ with $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\widetilde{\Pi}) \cup \{p\} \subset T_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F^+,\mathbb{Q}}$ and let $T' = T_{\mathrm{fin}} - \{p\}$. Let $\pi'_{T_{\mathrm{fin}}}$ be the unique element of $\mathrm{Irr}_{l'}(G(\mathbb{A}_{T_{\mathrm{fin}}}))$ with $\mathrm{BC}_{T_{\mathrm{fin}}}(\imath'\pi'_{T_{\mathrm{fin}}}) = \widetilde{\Pi}_{T_{\mathrm{fin}}}$. Choose $m \in \mathbb{Z}^{r-1}$ and a compact open subgroup $U_{T'} \subset G(\mathbb{A}_{T'})$ such that $(\pi'_{T_{\mathrm{fin}}})^{\mathrm{Iw}(m)\times U_{T'}} \neq \{0\}$. Let $(e')^T \in \overline{\mathbb{Q}}_{l'}[K^T \backslash G(\mathbb{A}^T)/K^T]$ be an idempotent with $(e')^T R^{K^T} = R^{K^T}\{\widetilde{\Pi}^T\}$ whenever R is one of $H^j(X_{\mathrm{Iw}(m)}, \mathcal{L}_{\xi'})$ or $a_{\xi'} H^j(\mathcal{A}^{m_{\xi'}}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_{l'})$. Then each of these spaces is $\pi'_{T'}$-isotypic. Let $e = \imath^{-1}\imath' e'$. Then for each $\alpha \in W_{F_w}$, $j \geqslant 0$, $S \subset \{1,\ldots,n\}$ and $\sigma : W_0 \hookrightarrow \overline{\mathbb{Q}}_l$ over $\mathbb{Z}_l$, we have

$$\begin{aligned}&\imath'\mathrm{tr}\,(\alpha e' a_{\xi'}|H^j(\mathcal{A}^{m_\xi}_{\mathrm{Iw}(m),S}, \overline{\mathbb{Q}}_{l'})^{K^T\times U_{T'}}) = \\ &\imath\mathrm{tr}\,(\alpha e a_\xi|(H^j(\mathcal{A}^{m_{\xi'}}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l)^{K^T\times U_{T'}})\end{aligned}$$

by the main results of [9] and [6]. For each $j \geqslant 0$, we have

$$\begin{aligned} e a_\xi (H^j(\mathcal{A}^{m_{\xi'}}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l)^{K^T} &\subset a_\xi (H^j(\mathcal{A}^{m_{\xi'}}_{\mathrm{Iw}(m),S}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l)^{K^T}\{\widetilde{\Pi}^T\} \\ e(H^j(X_{\mathrm{Iw}(m)}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l)^{K^T} &\subset (H^j(X_{\mathrm{Iw}(m)}/W_0) \otimes_{W_0,\sigma} \overline{\mathbb{Q}}_l)^{K^T}\{\widetilde{\Pi}^T\}.\end{aligned}$$

We then deduce from the previous equality of traces together with Proposition 6.1, Corollary 6.2, Proposition 4.1 and Theorem 3.1 that the two inclusions above are equalities (for dimension reasons) and moreover that

$$\imath'\mathrm{tr}\,(\alpha|\mathrm{WD}(\widetilde{R}^{n-1}_{\xi',l'}(\widetilde{\Pi})|_{G_{F_w}})) = \imath\mathrm{tr}\,(\alpha|\mathrm{WD}(\widetilde{R}^{n-1}_{\xi,l}(\widetilde{\Pi})|_{G_{F_w}}))$$

for each $\alpha \in W_{F_w}$ and hence

$$\imath\mathrm{WD}(\widetilde{R}^{n-1}_{\xi,l}(\widetilde{\Pi})|_{G_{F_w}})^{\mathrm{ss}} \cong (\mathrm{rec}((\Pi^1_w)^\vee \otimes |\det|^{(1-n)/2})^{\mathrm{ss}})^{C_G} \otimes \mathrm{rec}(\psi_u^{-1} \circ \mathbf{N}_{F_w/E_u}).$$

Since $\widetilde{R}^{n-1}_{\xi,l}(\widetilde{\Pi})^{\mathrm{ss}} \cong R_{l,\imath}(\Pi^1)^{C_G} \otimes R_{l,\imath}(\psi)|_{G_F}$, by Theorem 3.1, we see that

$$\imath\mathrm{WD}(R_{l,\imath}(\Pi^1)|_{G_{F_w}})^{\mathrm{ss}} \cong \mathrm{rec}((\Pi^1_w)^\vee \otimes |\det|^{(1-n)/2})^{\mathrm{ss}}.$$

By Proposition 1.1, it suffices to show that $\mathrm{WD}(R_{l,\imath}(\Pi^1)|_{G_{F_w}})$ is pure and this is established in Corollary 6.3. As v splits completely in F, we have established Theorem 1.2 in the case when m is odd.

Now suppose that m is even and denote $\mathrm{BC}_{F/L}(\Pi_L)$ by Π_1. We choose ψ, $\xi_{\mathbb{C}}$, ϖ, Π_2 and Π^1 as in Lemma 6.4. Set $\widetilde{\Pi} = \psi \otimes \Pi^1$ and $\xi = \imath^{-1}\xi_{\mathbb{C}}$ and $\xi' = (\imath')^{-1}\xi_{\mathbb{C}}$ and define $\widetilde{R}^{n-1}_{\xi,l}(\widetilde{\Pi})$ and $\widetilde{R}^{n-1}_{\xi',l'}(\widetilde{\Pi})$ as in Section 3.2. The proof now proceeds exactly as in the case where m is odd except that we replace the appeals to Theorem 3.1, Proposition 6.1, Corollary 6.2 and Corollary 6.3 with appeals to Theorem 3.2, Proposition 6.6, Corollary 6.7 and Corollary 6.8 respectively.

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