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Non-axiomatizability of real spectra in $\mathscr{L}_{\infty\lambda}$

Timothy Mellor[1], Marcus Tressl[2]

Abstract. — We show that the property of a spectral space, to be a spectral subspace of the real spectrum of a commutative ring, is not expressible in the infinitary first order language $\mathscr{L}_{\infty\lambda}$ of its defining lattice. This generalises a result of Delzell and Madden which says that not every completely normal spectral space is a real spectrum.

Résumé. — Nous montrons que la propriété d'un espace spectral d'être un sous-espace spectral du spectre réel d'un anneau commutatif n'est pas exprimable dans le langage infinitaire du premier ordre $\mathscr{L}_{\infty\lambda}$ de son treillis de définition. Ceci généralise un résultat de Delzell et Madden qui dit qu'en général, un espace spectral complètement normal n'est pas un spectre réel.

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1. Introduction

For a while it was an open question whether real spectra are precisely the completely normal spectral spaces. In [3], a counterexample is given, hence a completely normal spectral space is constructed, which is not homeomorphic to the real spectrum of any (commutative, unital) ring. We extend this result here in two ways:

1. We show that the spectral space constructed in [3] is also not homeomorphic to any spectral subspace of a real spectrum.
2. We show that there is no infinitary first order description which characterises real spectra.

The second point needs explanation. By Stone duality, the category of spectral spaces and spectral maps is anti-equivalent to the category of bounded distributive lattices and bounded lattice homomorphisms. Hence the question what the topological type of real spectra is, can also be asked on the lattice side: Classify all lattices that correspond to real spectra via Stone duality.

Now the class of all bounded distributive lattices is first order axiomatisable in the language of posets (consisting of one binary relation symbol) and we can rephrase our question in terms of model theory: Is the class of those lattices that correspond to real spectra via Stone duality, first order axiomatizable? It should be mentioned here that this indeed generalises the original question, since the class of all lattices corresponding to completely normal spectral spaces is easily seen to be first-order axiomatizable (by expressing that for each element a of the lattice L, the lattice $\{b \wedge \neg a \mid b \in L\}$ is normal, cf. [8]).

Now our theorem 5.1 also negates this more general question in a strong way: For every cardinal λ, the class of all lattices that correspond to real spectra via Stone duality is not first order axiomatizable in the infinitary language $\mathscr{L}_{\infty\lambda}$ of posets.

In this context it must be mentioned that the (specialisation-)order type of real spectra is know by [5]. Whereas the (specialisation-)order type of arbitrary spectral spaces is still unknown (this is called Kaplansky's problem and asked in [9, chap. 1]). The topological type of Zariski spectra of rings has been determined by Hochster in the first place: these are precisely the spectral spaces (cf. [6]).

For model theoretic terminology see [7]; the definition of $\mathscr{L}_{\infty\lambda}$ can be found in [4, p.65]. For basic properties of spectral spaces we refer to [6], [2]

and [8]; a summary can be found in [15, section 2]). The definition of the real spectrum and the fact that it is indeed a completely normal spectral space can be found in [1].

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2. Spectral spaces from Dedekind complete orders

Recall that a totally ordered set $X = (X, \leqslant)$ is **Dedekind complete** if every subset of X has a supremum in X. In this case, every subset of X has an infimum, X has a largest element, denoted by $\top$ and a smallest element, denoted by $\bot$.

If X is a totally ordered set, then the **Dedekind completion** of X is defined to be "the" Dedekind complete set $\overline{X}$, containing X as an ordered subset, such that X is dense in $\overline{X}$, i.e. for all $y_1, y_2 \in \overline{X}$ with $y_1 < y_2$, if there is no point $x \in X$ with $y_1 < x < y_2$, then $y_1, y_2 \in X$. An explicit description of $\overline{X}$ is

$$\overline{X} := X \cup \{+\infty, -\infty\} \cup \{\text{the non-principal cuts of } X\}$$

together with its natural order. Of course we won't add $\pm\infty$ if X has already a largest or a smallest element. Recall that a cut (L, R) of X is called principal, if the set L has a supremum in $X \cup \{\pm\infty\}$; these are the cuts of the form $\pm\infty$ or of the form x^+ or x^- for some $x \in X$.

We recall a few easy facts related to Dedekind complete (total) orders. Firstly, recall that the interval topology (or order topology) of a totally ordered set $(X, \leqslant)$ is defined to have the subbasis of open sets of the form $(-\infty, x)$ and $(x, +\infty)$, where $x \in X$.

Fact 2.1. — *For every totally ordered set X, the following are equivalent:*

(i) X is compact in the interval topology.

(ii) X is Dedekind complete.

Fact 2.2. — *If X and Y are Dedekind complete (total) orders, then the lexicographical product $X \times Y$ is again Dedekind complete.*

FACT 2.3. — *The following are equivalent for every totally ordered set X:*

(i) *X is compact and connected in the interval topology.*

(ii) *X is Dedekind complete and there is no **jump** in X, i.e. there are no $x < y$ in X such that the open interval (x, y) is empty.*

FACT 2.4. — *The following are equivalent for every totally ordered set X:*

(i) *X is boolean (i.e. X is compact and every connected subset of X is a singleton) in the interval topology.*

(ii) *X is Dedekind complete and **jump dense**, i.e. for all $x < y$ from X there are $x \leqslant a < b \leqslant y$ such that $(a, b) = \emptyset$.*

If this is the case, then the clopen subsets of X are the finite unions of closed intervals $[a, b]$ such that a has an immediate predecessor in $X \cup \{\pm\infty\}$ and b has an immediate successor in $X \cup \{\pm\infty\}$.

LEMMA 2.5. — *Let X be a Dedekind complete totally ordered set and let Y be a Dedekind complete, totally ordered set that is boolean in the order topology and that has at least two elements. Then $X \times Y$ is again Dedekind complete and boolean in the lexicographical order topology.*

Proof. — By 2.2 and 2.4 we only need to show that $X \times Y$ is jump dense. This is obvious. □

Let S be a finite totally order set with at least 2 elements. **We will now consider $X \times S$ lexicographically ordered together with the induced order topology.**

By 2.5 the lexicographic product $X \times S$ is a boolean and Dedekind complete (total) order. It is worth noting that the order topology on $X \times S$ is in general incompatible with the product topology on $X \times S$, i.e. neither refines nor coarsens this topology.

DEFINITION 2.6. — *Let S be an arbitrary boolean space. A partial order $\leqslant$ on S is called a **spectral order** on S if for all $x, y \in X$ with $x \not\leqslant y$ there is a clopen subset C of X such that $x \in C$, $y \notin C$ and such that C is a **final segment of** $\leqslant$, i.e. for all $c, z \in X$, $c \in C$ and $c \leqslant z$ implies $z \in C$. The pair $(S, \leqslant)$ is called a **Priestley space** (cf. [10]).*

A morphism between Priestley spaces $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ is a continuous map $f : X \longrightarrow Y$ which preserves the spectral orders, i.e. $a \leqslant_X b \Rightarrow f(a) \leqslant_Y f(b)$.

We shall use basic properties and terminology of spectral spaces. If X is a spectral space, then $\overline{\mathcal{K}}(X)$ denotes the set of closed and constructible subsets of X. Note that $\overline{\mathcal{K}}(X)$ is the bounded distributive lattice corresponding to X via Stone duality (in our setup, such a lattice L is mapped to the spectral space $\operatorname{Spec} L$ of prime filters of L which has the sets $V_L(a) := \{\in \operatorname{Spec} L \mid a \in L\}$ as a basis of closed sets). We refer to [15, section 2] for more details.

THEOREM 2.7. — *The functor from the category of spectral spaces together with spectral maps to the category of Priestley spaces, which sends X to $(X_{\mathrm{con}}, \rightsquigarrow_X)$ is an isomorphism. Here X_{con} denotes the patch space of X and $\rightsquigarrow_X$ denotes specialisation in X.*

Proof. — This is the content of [11]. □

Example 2.8. — If $X = (X, \leqslant)$ is a totally ordered set and boolean in the order topology, then there is a (unique) spectral topology τ on X such that

(a) The constructible topology of τ is the order topology of X.

(b) For all $x, y \in X$, $x \leqslant y$ X if and only if y is in the closure of x w.r.t. τ.

Hence the category of Dedekind complete, totally ordered and jump dense sets is isomorphic to the category of spectral spaces which have a total specialization order.

PROPOSITION 2.9. — *Let X be a Dedekind complete total order and let S be a finite spectral space. Let $\leqslant$ be any total order of S such that $\bot_S$ and $\top_S$ are minimal points in the spectral topology of S (of course, such a total order only exists if S is a singleton or not irreducible). Let $X \times S$ be equipped with the order topology of the lexicographic order of X and $(S, \leqslant)$. Then the partial order*

$$(x, s) \preccurlyeq (y, t) :\iff x = y \text{ and } s \rightsquigarrow t \text{ in } S \text{ (i.e. } t \in \overline{\{s\}})$$

is a spectral order on the boolean space $X \times S$.

Proof. — $X \times S$ is boolean by 2.5.

Take $\alpha = (x, s), \beta = (y, t) \in (X, S)$ with $\alpha \not\preccurlyeq \beta$. We have to find a clopen subset of $X \times S$ that is a final segment w.r.t. $\preccurlyeq$ such that $\alpha \in C$ and $\beta \notin C$.

Case 1. $x < y$.

If $x < y$ is a jump of X, then take $C := (-\infty, (x, \top_S)] = (-\infty, (y, \bot_S))$. If $x < y$ is not a jump of X, then there is some $z \in X$ with $x < z < y$. Let $v \in S$ be the predecessor of $\top_S$ in the total order of S. Then, as $\top_S$ is minimal in the spectral order of S, the clopen subset $C := (-\infty, (z, v)] = (-\infty, (z, \top_S))$ of $X \times S$ is a final segment w.r.t. $\preccurlyeq$.

Case 2. $y < x$.

If $y < x$ is a jump of X, then take $C := [(x, \bot_S), +\infty) = ((y, \top_S), +\infty)$. If $y < x$ is not a jump of X, then there is some $z \in X$ with $y < z < x$. Let $u \in S$ be the successor of $\bot_S$ in the total order of S. Then, as $\bot_S$ is minimal in the spectral order of S, the clopen subset $C := [(z, u), +\infty) = ((z, \bot_S), +\infty)$ of $X \times S$ is a final segment w.r.t. $\preccurlyeq$.

Case 3. $x = y$. Hence, since $\alpha \not\preccurlyeq \beta$ we know $s \rightsquigarrow t$ in S.

We first claim that for every $r \in S$, there is a clopen set $C_r \subseteq X \times S$ such that $C_r \setminus (\{x\} \times S)$ is closed under $\preccurlyeq$ and such that $C_r \cap (\{x\} \times S) = \{(x, r)\}$:

- If $r \neq \top_S$ and $r \neq \bot_S$, then the point (x, r) is isolated in the order topology of $X \times S$, hence $C_r := \{(x, r)\}$ has the required properties.
- If $r = \top_S$, then let $v \in S$ be the predecessor of $\top_S$ in the total order of S. Then $C_r = [(x, r), +\infty) = ((x, v), +\infty)$ has the required properties.
- If $r = \bot_S$, then then let $u \in S$ be the successor of $\bot_S$ in the total order of S. Then $C_r = (-\infty, (x, r)] = (-\infty, (x, u))$ has the required properties.

So we can find C_r as claimed and we define C as the finite union

$$C := \bigcup_{r \in S, s \rightsquigarrow r} C_r.$$

Then C is clopen in $X \times S$ and $C \setminus (\{x\} \times S)$ is closed under $\preccurlyeq$ (as each C_r has this property). But $C \cap (\{x\} \times S)$ is equal to $\{x\} \times \overline{\{s\}}^S$, which is a final segment w. r. t. $\preccurlyeq$, too. Thus C is a final segment w.r.t. $\preccurlyeq$ such that $\alpha \in C$ and $\beta \notin C$. $\square$

Definition 2.10. — *Let X be Dedekind complete and let S be a finite spectral space which is not irreducible. Let $\leqslant$ be any total order of S such that $\bot_S$ and $\top_S$ are minimal points in the spectral topology of S.*

We define

$$X\#(S,\leqslant)$$

as the spectral space whose patch space is the Dedekind complete total order $X \times S$ and whose specialisation order is the partial order from 2.9. If S is the spectral space consisting of three elements $-1, 0, 1$ with specialization order $-1, 1 \rightsquigarrow 0$ and total order $-1 < 0 < 1$, then we write

$$X^{\#} \text{ instead of } X\#(S,\leqslant).$$

For $x \in X$ we write $x^- := (x,-1)$, $x^+ := (x,1)$ and we identify X with $X \times \{0\}$ in $X^{\#}$.

PROPOSITION 2.11. — *Let $X = (X,\leqslant)$ be Dedekind complete and densely totally ordered. A subset Y of $X^{\#}$ is closed and constructible if and only if Y is a finite union of sets of the form $[(x,0),(x',0)]$, where $x, x' \in X$ and $x \leqslant x'$.*

In particular $\overline{\mathcal{K}}(X^{\#})$ is order isomorphic to the lattice generated by the closed intervals of X in the powerset of X

Proof. — This is clear with the characterization of the constructible subsets of $X^{\#}$ in 2.4: Observe that the constructible subsets of $X^{\#}$ are by definition the clopen subsets of the lexicographic order $X \times \{-1,0,1\}$. □

3. Completely normal spectral spaces not occurring in real spectra

We use standard notation from commutative algebra: Let A be a ring (this means, commutative and unital always). For $f \in A$, we denote by $V(f)$ the set of all prime ideals of A containing f and $D(f) = \operatorname{Spec} A \setminus V(f)$.

Following [3] we shall work with Zariski spectra of real closed rings (in the sense of N. Schwartz) instead of real spectra of commutative rings. Note that every real spectrum is (naturally) homeomorphic to the Zariski spectrum of a real closed ring. We refer to [12] and [13] for real closed rings.

PROPOSITION 3.1. — *Let A be a real closed ring and let $X \subseteq \operatorname{Spec} A$. Suppose we are given*

(I) an open quasi-compact subset U of $\operatorname{Spec} A$ such that $U \cap X$ is connected and such that there are points $x, y \in U \cap X$ which do not have a common specialization in U;

(II) an ordinal λ and for each $\alpha < \lambda$, $f_\alpha \in A$ such that for all $\alpha < \beta < \lambda$ we have

$$f_\alpha(u) > f_\beta(u) > 0 \ (u \in U \cap X).$$

Then there is a collection $(U_\alpha)_{\alpha<\lambda}$ of open, nonempty and disjoint subsets of $U \cap X$.

Proof. — Since A is real closed, there is some $s \in A$ such that $U = D(s)$. Since $x, y \in X$ do not have a common specialization in U, there are $\varphi, \psi \in A$ with $x \in V(\varphi) \cap D(s)$, $y \in V(\psi) \cap D(s)$ and $V(\varphi) \cap V(\psi) \cap D(s) = \emptyset$. Then $\varphi^2 + \psi^2$ is a unit in A_s, hence there is some $h \in A$ such that

$$s^n \cdot (h \cdot (\varphi^2 + \psi^2) - s^{2k}) = 0.$$

Define

$$g := f_0 \cdot \varphi^2 \cdot h.$$

Then $g(x) = 0$ and $g(y) = f_0(y) \cdot s^{2k}(y)$. We take

$$U_\alpha := \{u \in U \cap X \mid s^{2k} \cdot f_\alpha(u) < g(u) < s^{2k} \cdot f_{\alpha^+}(u)\} \ (\alpha < \lambda).$$

Clearly the U_α are open and disjoint subsets of X. It remains to show that each U_α is nonempty.

Otherwise $U \cap X \subseteq \{g \leqslant s^{2k} \cdot f_\alpha\} \cup \{g \geqslant s^{2k} \cdot f_{\alpha^+}\}$ is covered by two closed subsets of $\operatorname{Spec} A$, which have empty intersection in $U \cap X$ by assumption. Since $g(x) = 0 \leqslant s^{2k}(x) f_\alpha(x)$ and $g(y) = s^{2k}(y) \cdot f_0(y) \geqslant s^{2k}(y) \cdot f_{\alpha^+}(y)$ both sets are nonempty. This contradicts the assumption that $U \cap X$ is connected. □

Corollary 3.2. — *Let A be a real closed ring and let $X \subseteq \operatorname{Spec} A$ contain at least two points. Suppose*

(I) X is quasi-compact, connected and there is no specialization inside X.

(II) There are an ordinal λ and for each $\alpha < \lambda$, $f_\alpha \in A$ such that for all $\alpha < \beta < \lambda$ we have

$$f_\alpha(u) > f_\beta(u) > 0 \ (u \in X).$$

Then there is a collection $(U_\alpha)_{\alpha<\lambda}$ of open, nonempty and disjoint subsets of X.

Proof. — Take $x, y \in X$, $x \neq y$ and suppose x, y have a common specialization in $\operatorname{Spec} A$. Let z be the first common specialization of x, y in $\operatorname{Spec} A$.

Then for each $w \in X$ we have $z \rightsquigarrow w$, hence there is an open quasi-compact subset U_w of $\operatorname{Spec} A$ with $w \in U_w$ and $z \notin U_w$. Then $X \subseteq \bigcup_{w \in X} U_w$ and since X is quasi-compact, there is an open quasi-compact subset U of $\operatorname{Spec} A$ containing X and not containing z. Since z is the first common specialization of x, y in $\operatorname{Spec} A$, x, y do not have a common specialization in U. Now we may apply 3.1. □

Recall that for an infinite cardinal κ, a κ-**set** X is a totally ordered set with the property that for all $A, B \subseteq X$ of size strictly less than κ, if $A < B$, then there is some $x \in X$ with $A < x < B$. For example every κ-saturated totally ordered set is a κ-set.

PROPOSITION 3.3. — *Let $D = E \times I$, where E is the Dedekind completion of a κ-set Y, κ an infinite cardinal, and I is Dedekind complete.*

Suppose $D^{\#}$ is a proconstructible subset of $\operatorname{Sper} A$ *for some ring A. Then there are $f_\alpha \in A$ and $x_\alpha, y_\alpha \in Y$ $(\alpha < \kappa)$ such that for all $\alpha < \beta < \kappa$ we have*

$$x_\alpha < x_\beta < y_\beta < y_\alpha \text{ and}$$
$$\text{for all } u \in D \text{ with } (x_\beta, \perp_I) < u < (y_\beta, \top_I) : f_\alpha(u) > f_\beta(u) > 0.$$

Proof. — Let $f_0 = 1$ and suppose $x_\mu < x_\nu < y_\nu < y_\mu$, $f_\mu > f_\nu > 0$ on $[(x_\nu, \perp_I), (y_\nu, \top_I)]$, have already been constructed for $\mu < \nu < \lambda$, $\lambda < \kappa$. We define $x_\nu < x_\lambda < y_\lambda < y_\nu$ and $f_\lambda \in A$ with $f_\nu > f_\lambda > 0$ on $[(x_\lambda, \perp_I), (y_\lambda, \top_I)]$ as follows:

Pick $u, v \in Y$ with $x_\nu < u < v < y_\nu$ $(\nu < \lambda)$. Then the set $[(u, \top_I)^+, (v, \perp_I)^-] \subseteq D^{\#}$ is open and quasi-compact, there is some $f_\lambda \in A$ that is > 0 on that set and 0 on the complement of this set in $D^{\#}$.

Pick $\mu < \lambda$. Then $\{f_\mu > f_\lambda\} \cap D^{\#}$ is open and quasi-compact containing $(u, \top_I)$. Since the open quasi-compact subsets of $D^{\#}$ are finite unions of sets of the form $[d_0^+, d^-]$ with $d_0, d \in D$ there is some $d_\mu \in D$ with $(u, \top_I) < d_\mu$ such that $[(u, \top_I), d_\mu^-) \subseteq \{f_\mu > f_\lambda\}$. From $(u, \top_I) < d_\mu$ we get some $e \in E$ with $u < e_\mu \leqslant d_\mu$. Since $u \in Y$ and E is the Dedekind completion of the dense set Y, there is some $u_\mu \in Y$ with $u < u_\mu < d_\mu$.

Hence $f_\mu > f_\lambda > 0$ on $[(u, \top_I)^+, (u_\mu, \top_I)]$. Since Y is a κ-set, there is some $y_\lambda \in Y$ with $u < y_\lambda < x_\mu$ for all $\mu < \lambda$. Now choose $x_\lambda \in Y$ with $u < x_\lambda < y_\lambda$. □

THEOREM 3.4. — *Let $D = E \times I$, where E is the Dedekind completion of a κ^+-set Y, κ an infinite cardinal, and I is a compact and connected Dedekind complete total order with at least two points, such that there is no collection of nonempty, open and disjoint subsets of I of size κ. Then the completely normal spectral space $D^\#$ is not homeomorphic to any spectral subspace of the real spectrum of any ring.*

The main example here is $I = [0,1] \subseteq$ and $\kappa = \aleph_1$.

Proof. — By 3.3, there are $f_\alpha \in A$ and $x_\alpha, y_\alpha \in Y$ such that for all $\alpha < \beta < \kappa$ we have $x_\alpha < x_\beta < y_\beta < y_\alpha$ and

$$f_\alpha(u) > f_\beta(u) > 0$$

when u ranges in the interval $((x_\beta, \bot_I), (y_\beta, \top_I))$ of D. Since Y is a κ^+-set, there are $x, y \in Y$ with $x_\alpha < x < y < y_\alpha$ for all $\alpha < \beta < \kappa$. Hence

$$f_\alpha(u) > f_\beta(u) > 0 \ (u \in D, (x, \bot_I) < u < (y, \top_I)).$$

Pick $z \in Y$ with $x < z < y$ and let $X := \{z\} \times I$. Then

$$f_\alpha(u) > f_\beta(u) > 0 \ (u \in X).$$

Since X is homeomorphic to I, X is compact and connected by assumption. Hence we may apply 3.2, which gives a collection $(U_\alpha)_{\alpha<\kappa}$ of open, nonempty and disjoint subsets of X. Since X is homeomorphic to I, this contradicts our assumption on I. □

4. Back and forth equivalence of lattices generated by closed intervals

Let $X = (X, \leqslant)$ be a totally ordered set. A **closed interval** of X is any subset of the form $[a,b]$, $[a,+\infty)$, $(-\infty,b]$ or $(-\infty,\infty)$ with $a, b \in X$. The **boundary points** of $[a,b]$ are defined to be a, b if $a \leqslant b$, the only boundary point of $[a,+\infty)$ is defined to be a (provided X has no largest element) and the only boundary point of $(-\infty,b]$ is defined to be b (provided X has no smallest element). The empty set and X (provided X has no smallest and no largest element) do not have boundary points. Observe that a boundary point according to this definition is in general not a boundary point w.r.t. the order topology of X (X might be discrete).

Let $L(X)$ be the set of finite unions of closed intervals of X. Obviously, $L(X)$ is the sublattice of the powerset of X generated by the set of closed intervals.

For $\alpha \in L(X)$, a closed interval of X contained in α and maximal with this property is called a **component of** α; observe again that this is in general not a connected component of α in the sense of the order topology of X. However, it is clear that α is the disjoint union of its components and there are only finitely many of them.

For $\alpha \in L(X)$, a **boundary point of** α is a boundary point of one of its components. We write $\mathrm{bd}(\alpha)$ for the (finite) set of boundary points of α. It is useful to notice the following:

Observation 4.1. — Let $\alpha \in L(X)$.

(i) If $\alpha = I_1 \cup ... \cup I_n$ with closed intervals $I_1, ..., I_n$ of X, then $\mathrm{bd}(\alpha) \subseteq \bigcup_{k=1}^{n} \mathrm{bd}(I_k)$.

(ii) There are $n \in \mathbb{N}_0$ and nonempty closed intervals $I_1, ..., I_n$ of X with $\alpha = I_1 \cup ... \cup I_n$ such that for each $k \in \{1, ..., n-1\}$ there is some $x \in X$ with $I_k < x < I_{k+1}$.

Moreover, whenever α is represented in this form, then $I_1, ..., I_n$ are precisely the components of α.

Given $S \subseteq X$, we define

$$L(X, S) := \{\alpha \in X \mid \mathrm{bd}(\alpha) \subseteq S\}.$$

Remark 4.2. — $L(X, S)$ is a bounded sublattice of $L(X)$ and for every other set $S' \subseteq X$ we have $S \subseteq S' \iff L(X, S) \subseteq L(X, S')$.

Proof. — By 4.1(i) it is clear that $L(X, S)$ is a bounded sublattice of $L(X)$. The equivalence follows with the observation $\{s\} \in L(X, S)$. □

Now let X, Y be totally ordered sets and let $S \subseteq X$, $T \subseteq Y$ be arbitrary sets. Suppose we are given an order isomorphism $f : S \longrightarrow T$. We define a map $F_f : L(X, S) \longrightarrow L(Y, T)$ as follows: For a closed interval α of X we define the closed interval $F_f(\alpha)$ of Y by

$$F_f(\alpha) = \begin{cases} \emptyset & \text{if } \alpha = \emptyset \\ Y & \text{if } \alpha = X \\ [f(s), +\infty) & \text{if } \alpha = [s, +\infty) \text{ and } s \text{ is not the smallest element of } X \\ (-\infty, f(s)] & \text{if } \alpha = (-\infty, s] \text{ and } s \text{ is not the largest element of } X \\ [f(s_1), f(s_2)] & \text{if } s_1 \leqslant s_2,\ \alpha = [s_1, s_2] \text{ and } s_1 \text{ is not the smallest element of } X \text{ and } s_2 \text{ is not the largest element of } X \end{cases}$$

Now we define F_f on all of $L(X,S)$ by

$$F_f(\alpha) = \bigcup\{F_f(\gamma) \mid \gamma \text{ is a component of } \alpha\}.$$

In the situation above, we say that the order isomorphism $f : S \longrightarrow T$ is **faithful** if the following two conditions are satisfied:

(a) For every $s \in S$, s is the smallest or largest element of X if and only if $f(s)$ is the smallest or largest element of Y, respectively.

(b) For all $s_1, s_2 \in S$ we have

$$s_1 < s_2 \text{ is a jump in } X \iff f(s_1) < f(s_2) \text{ is a jump in } Y.$$

PROPOSITION 4.3. — *If f is faithful, then F_f is a lattice isomorphism $L(X,S) \longrightarrow L(Y,T)$ and the inverse is $F_{f^{-1}}$. If $S' \subseteq X$, $T' \subseteq Y$ and $f' : S' \longrightarrow T'$ is another faithful order isomorphism, then f' extends f if and only if $F_{f'}$ extends F_f.*

Proof. — On the level of closed intervals, the map $F_{f^{-1}}$ is inverse to F_f, since f satisfies condition (a) of the definition of "faithful". Using 4.1(ii) and because f satisfies condition (b) of the definition of "faithful", F_f maps a component γ of $\alpha \in L(X,S)$ to the component $F(\gamma)$ of $F(\alpha)$. It is then clear that $F_{f^{-1}}$ is inverse to F_f. Since both maps obviously are monotone, the first assertion follows.

For the equivalence, only $\Leftarrow$ needs a proof. However if $F_{f'}$ extends F_f and $s \in S$ then $\{f(s)\} = F_f(\{s\}) = F_{f'}(\{s\}) = \{f'(s)\}$ and so f' extends f. □

Recall that a **back-and-forth system** of first order structures M and N (in an arbitrary first order language) is a non-empty family $(f_i : M_i \longrightarrow N_i \mid i \in I)$ of isomorphisms f_i between substructures M_i of M and N_i of N, respectively, satisfying the following conditions:

Forth: For all $i \in I$ and $a \in M$ there is $j \in I$ with $a \in M_j$ such that f_j extends f_i; hence also $M_i \subseteq M_j, N_i \subseteq N_j$.

Back: For all $i \in I$ and $b \in N$ there is $j \in I$ with $b \in N_j$ such that f_j extends f_i.

If there is a back and forth system between M and N, then M and N are called **back and forth equivalent**.

If λ is a cardinal, then a back and forth system $(f_i : M_i \longrightarrow N_i \mid i \in I)$ is said to have the **λ-extension property** if for all $J \subseteq I$ of cardinality

$< \lambda$ such that the family $(f_j : M_j \longrightarrow N_j \mid j \in J)$ is totally ordered by extension, there is some $i \in I$ such that f_i extends f_j for all $j \in J$. So trivially, every back and forth system satisfies the $\aleph_0$-extension property.

If there is a back and forth system of M and N that has the λ-extension property, then M and N are called **strongly λ-back and forth equivalent**.

Let X and Y again be totally ordered sets and let $(f_i : S_i \longrightarrow T_i \mid i \in I)$ be a back and forth system of X and Y.

Let $F_i := F_{f_i}$ in the notation introduced before 4.3. Since $(f_i : S_i \longrightarrow T_i \mid i \in I)$ is a back and forth system of X and Y, it is clear that every f_i is faithful. By 4.3,

$$F_i : L(X, S_i) \longrightarrow L(Y, T_i)$$

is a lattice isomorphism and we claim that $(F_i : L(X, S_i) \longrightarrow L(Y, T_i) \mid i \in I)$ is a back and forth system of $L(X)$ and $L(Y)$.

By 4.2 we already know that $L(X, S_i)$ and $L(Y, T_i)$ are sublattices of $L(X), L(Y)$ respectively. By symmetry we then only need to check the "Forth" condition. Let $i \in I$ and $\alpha \in L(X)$. Then $\mathrm{bd}(\alpha)$ is finite and by applying the "Forth" condition of the system $(f_i : S_i \longrightarrow T_i \mid i \in I)$ finitely many times, there is some $j \in I$ such that $\mathrm{bd}(\alpha) \subseteq S_j$. By 4.1(ii), $\alpha \in L(X, S_j)$ and as f_j extends f_i, also F_j extends F_i.

Thus we know that $(F_i : L(X, S_i) \longrightarrow L(Y, T_i) \mid i \in I)$ is a back and forth system of $L(X)$ and $L(Y)$.

Now suppose $(f_i : S_i \longrightarrow T_i \mid i \in I)$ is a back and forth system of X and Y that has the λ-extension property for some cardinal λ.

Then also $(F_i : L(X, S_i) \longrightarrow L(Y, T_i) \mid i \in I)$ has the λ-extension property. To see this, it is enough to recall that F_i extends F_j if and only if f_i extends f_j (cf. 4.3).

THEOREM 4.4 *[4, Thm 5.3.7, p. 316 and the notation before Thm. 5.3.2]. — If M, N are elementary equivalent, λ-saturated, first order structures, then they are strongly λ-back and forth equivalent.*

Moreover, strongly λ-back and forth equivalent structures are elementary equivalent in the infinitary language $\mathscr{L}_{\infty\lambda}$.

Scholium 4.5. — Let λ be a cardinal and let X, Y be λ-saturated totally ordered sets. Suppose X and Y are elementary equivalent (for example if

X and Y are dense and totally ordered with endpoints). Then the lattices $L(X)$ and $L(Y)$ are strongly λ - back and forth equivalent.

Moreover $L(X)$ and $L(Y)$ are elementary equivalent in the infinitary language $\mathscr{L}_{\infty\lambda}$.

Proof. — By 4.4, X and Y are strongly λ - back and forth equivalent. We have shown in this section that in this case, also the lattices $L(X)$ and $L(Y)$ are strongly λ - back and forth equivalent. By 4.4 again, $L(X)$ and $L(Y)$ are elementary equivalent in the infinitary language $\mathscr{L}_{\infty\lambda}$. □

5. Main Theorem

THEOREM 5.1. — *For every cardinal λ, there are bounded distributive lattices L and L', such that*

(i) The lattices L and L' are strongly λ-back and forth equivalent; in particular, they are elementary equivalent in the infinitary language $\mathscr{L}_{\infty\lambda}$.

(ii) $L = \overline{\mathcal{K}}(C)$ for some spectral space C that is not homeomorphic to any proconstructible subset of any real spectrum of a ring.

(iii) $L' = \overline{\mathcal{K}}(\operatorname{Sper} A)$ for some ring A.

Proof. — DEFINITION OF L' AND A:

We choose a λ-saturated real closed field R. Let A be the ring of continuous semi-algebraic functions $[0,1] \longrightarrow R$, where $[0,1]$ is the unit interval in R and define $L' = \overline{\mathcal{K}}(\operatorname{Sper} A)$.

DEFINITION OF L AND C:

Let I be a densely totally ordered, λ-saturated set. Let $\kappa \geqslant \lambda$ be such that there is no collection of nonempty, open and disjoint subsets of I of size κ. Let Y be a κ^+-saturated total order. Now define $L = \overline{\mathcal{K}}((\overline{Y} \times \overline{I})^{\#})$ (recall that $\overline{Y}$, $\overline{I}$ denote the Dedekind completions of Y, I respectively).

(i) We show that $L \cong L(\overline{Y} \times \overline{I})$, $L' \cong L([0,1])$ and verify the assumptions of 4.5 for $\overline{Y} \times \overline{I}$ and $[0,1]$:

- Since I and Y are λ-saturated, also the Dedekind completions $\overline{Y}$ and $\overline{I}$ are λ-saturated. Thus, $\overline{Y} \times \overline{I}$ is λ-saturated, too.
- By 2.11 we have $L \cong L(\overline{Y} \times \overline{I})$.

- Since R is λ-saturated, the totally ordered set $[0,1]$ is λ-saturated, too.
- It is well known that L' is naturally homeomorphic to the lattice $L([0,1])$ of finite unions of closed intervals from $[0,1]$, thus $L' \cong L([0,1])$.
- Both $[0,1]$ and $\overline{Y}\times\overline{I}$ are dense and totally ordered sets with endpoints, so they are elementary equivalent.

Hence all assumptions of 4.5 are satisfied and we obtain (i) from 4.5.

(ii) holds by 3.4 applied to Y, $\overline{I}$ and the choice of C.

(iii) holds by definition of L'. $\square$

A possible answer to the question on the determination of the topological type of real spectra can thus not be formulated in terms of infinitary first order languages; at least not in an obvious way.

An interesting alteration of the question is the following: Is every spectral subspace of the real spectrum of a ring A itself the real spectrum of a ring B? More importantly, can we even construct B in a natural way out of A and the given set?

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