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Spectral Real Semigroups

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Abstract. — The notion of a *real semigroup* was introduced in [8] to provide a framework for the investigation of the theory of (diagonal) quadratic forms over commutative, unitary, semi-real rings. In this paper we introduce and study an outstanding class of such structures, that we call *spectral real semigroups* (SRS). Our main results are: (i) The existence of a natural functorial duality between the category of SRSs and that of hereditarily normal spectral spaces; (ii) Characterization of the SRSs as the real semigroups whose representation partial order is a distributive lattice; (iii) Determination of all quotients of SRSs, and (iv) Spectrality of the real semigroup associated to any lattice-ordered ring.

Résumé. — Dans [8] nous avons introduit la notion de *semigroupe réel* dans le but de donner un cadre général pour l'étude des formes quadratiques diagonales sur des anneaux commutatifs, unitaires, semi-réels. Dans cet article nous étudions une classe de semigroupes réels avec des propriétés remarquables : les *semigroupes réels spectraux* (SRS). Nos résultats principaux sont : (i) l'existence d'une dualité fonctorielle naturelle entre la catégorie des SRS et celle des espaces spectraux héréditairement normaux ; (ii) la caractérisation des SRS comme étant les semigroupes réels dont l'ordre de représentation est un treillis distributif ; (iii) la détermination des quotients des SRS ; (iv) le caractére spectral des semigroupes réels associés aux anneaux réticulés.

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À François Lucas, en témoignage d'une longue et loyale amitié.

Introduction

In the next few paragraphs we briefly review some basic concepts of the theory of real semigroups, introduced in [8], further investigated in [9], and extensively developed in [10] (unpublished[1]).

The motivation behind the theory of real semigroups is the study of quadratic form theory over rings[2] having a minimum of orderability in their structure. The requirement that -1 is not a sum of squares gives the right level of generality for most purposes. Rings with this property are called *semi-real*. This condition is equivalent to require that the real spectrum of the given ring, A – henceforth denoted by Sper(A) – be non-empty.[3]

With the real spectra of (semi-real) rings as a point of departure, and building on prior ideas of Bröcker, Marshall introduced and investigated in [14], Chs. 6–8, an axiomatic theory – the *abstract real spectra* (ARS) – combining topology and quadratic-form-theoretical structure.[4]

In [8] we introduced a purely algebraic (or, rather, model-theoretic) dual to abstract real spectra, namely the theory of *real semigroups* (abbreviated RS), given by a finite set of simple first-order axioms in the language $\mathcal{L}_{\mathbf{RS}} = \{\cdot, 1, 0, -1, D\}$ consisting of:

– A binary operation $\cdot$ verifying the axioms for commutative semigroups with unit 1 (a.k.a. monoids);

– Constants $1, 0, -1$, satisfying, in addition: $-1 \neq 1$, $x^3 = x$, $x \cdot 0 = 0$, and $(-1) \cdot x = x \Rightarrow x = 0$ for all x. Semigroups with these properties are called *ternary*.

– A ternary relation, written $a \in D(b, c)$, whose intended meaning is "a is *represented* by the binary quadratic form with coefficients b, c".[5] The axioms required for the relation D are expounded in [8], §2, p. 106, and [9], §2, p. 58; the basic theory of RSs is developed in [8].

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(1) Preliminary version available online at http://www.maths.manchester.ac.uk/raag/index.php?preprint=0339.

(2) All rings considered in this paper are commutative and unitary.

(3) For basic information on the real spectrum of rings, cf. [2], §7.1, pp. 133–142; [11], §24; [13], Kap. III.

(4) An equivalent axiom system occurs in [1], Ch. III, under the name *spaces of signs*.

(5) The notion of *representation by a quadratic form*, well understood and extensively studied in the case of fields and preordered fields, is much less straightforward in the general context of rings; see [7], Ch. 2, §§2, 4, and the discussion in [14], §5.5, p. 95. The restriction of the primitive terms to representation by *binary* forms is not an impairment.

The way in which rings give rise to real semigroups is briefly outlined in 9.1.A below; cf. also [9], pp. 50-51, and [14], p. 92.

The aim of this paper is to study an important class of real semigroups, that we call *spectral*. The abstract real spectra dual to these real semigroups were considered by Marshall in §8.8 of [14] under the name "real closed abstract real spectra"[6]; he briefly outlined some of their basic properties. We adopt the name "spectral" in view of the nowadays standard name for the objects of these RSs, namely *spectral maps* (here with values in $\mathbf{3} = \{1, 0, -1\}$ endowed with the *spectral* topology; cf. 1.1.B(4) below).

We shall prove four significant sets of results concerning spectral real semigroups.

(I) The first is a categorical duality (a.k.a. anti-equivalence) between the category **HNSS** of hereditarily normal spectral spaces with spectral maps[7] and **SRS**, the category of spectral real semigroups with RS-homomorphisms; see Theorem 5.4. Specifically, our results show (Theorem 1.7) that the $\mathcal{L}_{\mathrm{RS}}$-structures dual to *arbitrary* spectral spaces verify all axioms for real semigroups with the possible exception of [RS3b] (i.e., $D^t(\cdot,\cdot) \neq \emptyset$), while this axiom is equivalent to the hereditary normality of the space (Theorem 1.8).

The main thrust in sections 3, 4 and 5 is directed at proving this duality, though several other results are obtained as a by-product. Noteworthy among the latter is that any real semigroup has a natural hull in the category **SRS**, with the required functorial properties; cf. section 4 and Theorem 5.3(ii) (the existence of this spectral hull was observed in [14], p. 177). Further,

(i) The operation of forming the spectral hull of a real semigroup is idempotent: iteration does not produce a larger structure (Theorem 4.5 and Corollary 4.6).

(ii) Every RS-character of a real semigroup extends *uniquely* to its spectral hull (Corollary 5.5).

(II) The second set of results deals with the properties of the representation partial order (Definition 1.1) in spectral RSs. Our main theorem here, 6.6, shows that the spectral RSs are exactly the real semigroups for which the representation partial order is a *distributive lattice*; distributivity is the crucial point here. In fact, this property is also equivalent to the assertions that the representation partial order has a lattice structure and that the

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(6) A name presumably motivated by the homonymous terminology in the ring case, that we treat in broader generality in section 9 below.

(7) These notions are defined in 1.1.B(2),(3) below.

RS-characters are lattice homomorphisms (into **3**). We also prove (Theorem 6.2) that any real semigroup generates its spectral hull as a lattice.

An important feature is that the lattice operations $\wedge$ and $\vee$ of a spectral RS are first-order definable in terms of the real semigroup product operation and binary representation relation by *positive-primitive* formulas (Theorem 2.1 and Remark 7.2). A consequence of this is Corollary 2.2, which shows that the RS-characters of – and, more generally, the RS-homomorphisms between – spectral real semigroups are automatically lattice homomorphisms (into **3** with the order $1 < 0 < -1$). This fact plays a key role in the development of the theory presented here. Theorem 2.1 also yields a useful first-order axiomatisation of the class of spectral RSs (Theorem 7.1), having as a corollary that the class of spectral RS is closed under (right-directed) inductive limits, reduced products – in particular, arbitrary products – (Proposition 7.3) and, more significantly, also under quotients modulo arbitrary RS-congruences (Fact 8.2).

(III) In section 8 we deal with quotients of spectral RSs. Theorem 8.5 and Corollary 8.7 elucidate the structure of the RS-congruences[8] of any spectral RS, showing that they are completely determined by a proconstructible subset of its character space. This is an exceptional situation: even though any RS-congruence of a real semigroup gives rise to a proconstructible subset of its character space (see Theorem 8.3(1)), in the absence of additional requirements this set alone *is not enough* to determine the given congruence (an example is given in [10], Ex. II.2.10). As a corollary we obtain that the operation of forming the spectral hull of a real semigroup commutes with that of taking quotients modulo a RS-congruence.

(IV) As a by-product of the preceding theory we prove (Theorem 9.3) that the RS associated to *any* lattice-ordered ring is spectral and that the spectral hull of the RS associated to *any* semi-real ring is canonically isomorphic to the RS of its real closure in the sense of Prestel-Schwartz [16] (Proposition 9.4); cf. also [14], Rmk. (3), p. 178. Thus, spectral RSs occur in profusion amongst the real semigroups associated to rings.

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(8) Cf. Definition 8.1.

1. Spectral real semigroups. Basic theory

1.1. Preliminaries, Reminders and Notation

We include here some basic notions, notation and results concerning real semigroups and spectral spaces used throughout. Further material will be introduced as needed.

(A) Real semigroups.

(1) The axioms for real semigroups (RS) appear in [8], §2, p. 106, and in [9], Def. 2.2, p. 58. Further information on ternary semigroups can be found in [8], pp. 100-105. The axioms for *abstract real spectra* can be found in [14], pp. 99-100. The functorial duality between abstract real spectra and real semigroups is proved in [8], Thm. 4.1. For the main, motivating example of the RSs associated to (commutative, unitary, semi-real) rings, cf. 9.1.A.

(2) The set $\mathbf{3} = \{1, 0, -1\}$ has a *unique* structure of RS, with constants as displayed, the usual (integer) multiplication as product, representation given by

$$D_{\mathbf{3}}(0,0) = \{0\}; \qquad D_{\mathbf{3}}(0,1) = D_{\mathbf{3}}(1,0) = D_{\mathbf{3}}(1,1) = \{0,1\};$$
$$D_{\mathbf{3}}(0,-1) = D_{\mathbf{3}}(-1,0) = D_{\mathbf{3}}(-1,-1) = \{0,-1\};$$
$$D_{\mathbf{3}}(1,-1) = D_{\mathbf{3}}(-1,1) = \mathbf{3};$$

and transversal representation given by:

$$D^t_{\mathbf{3}}(0,0) = \{0\}; \qquad D^t_{\mathbf{3}}(0,1) = D^t_{\mathbf{3}}(1,0) = D^t_{\mathbf{3}}(1,1) = \{1\};$$
$$D^t_{\mathbf{3}}(0,-1) = D^t_{\mathbf{3}}(-1,0) = D^t_{\mathbf{3}}(-1,-1) = \{-1\};$$
$$D^t_{\mathbf{3}}(1,-1) = D^t_{\mathbf{3}}(-1,1) = \mathbf{3}.$$

Cf. [8], Cor. 2.4, p 109, or [9], Ex. 2.3(3), p. 58. Note that, for $x, y, z \in \mathbf{3}$,

$$x \in D_{\mathbf{3}}(y,z) \;\Leftrightarrow x \neq 0 \Rightarrow x = y \vee x = z, \qquad \text{and}$$
$$x \in D^t_{\mathbf{3}}(y,z) \;\Leftrightarrow (x = 0 \Rightarrow y = -z) \vee (x \neq 0 \Rightarrow x = y \vee x = z).$$

(3) (RS-characters) $\mathcal{L}_{\mathrm{RS}}$-homomorphisms of a $\mathcal{L}_{\mathrm{RS}}$-structure with values in $\mathbf{3}$ are called **RS-characters**.

The set of (RS-)characters of a RS, G, denoted by X_G, is endowed with a natural topology having as a basis the family of sets $\bigcap_{i=1}^{n} [\![\, a_i = 1 \,]\!]$, for all finite sequences $a_1, \ldots, a_n \in G$ where, for $a \in G$ and $i \in \{1, 0, -1\}$, $[\![\, a = i \,]\!] = \{h \in X_G \,|\, h(a_i) = i\}$. Endowed with this topology, X_G is a *spectral space* (cf. [11], §1; [13], Kap. III; or [14], Prop. 6.3.3, p. 113). The corresponding constructible topology – denoted $(X_G)_{\mathrm{con}}$ – has a basis the family of sets

$$\bigcap_{i=1}^{n} [\![a_i = 1]\!] \cap \bigcap_{j=1}^{m} [\![b_j = 0]\!],$$

for all finite sequences $a_1, \ldots, a_n, b_1, \ldots, b_m \in G$ (one can always take $m \leqslant 1$).

Specialization in (the spectral topology of) X_G admits the following algebraic characterization ([8], Lemma 1.13, p. 105): for $g, h \in X_G$,

$$g \rightsquigarrow h \text{ (i.e., } h \text{ specializes } g) \Leftrightarrow h^{-1}[1] \subseteq g^{-1}[1] \Leftrightarrow h = h^2 g$$
$$\text{(equivalently, } h^2 = hg).$$

(4) (The representation partial order)

Definition 1.1. — ([10], Def. I.5.2) *Given a RS, G, the binary relation*

$$a \leqslant_G b : \Leftrightarrow a \in D_G(1, b) \text{ and } -b \in D_G(1, -a),$$

is a partial order – called the **representation partial order** *of G –, with the following properties (we omit the subscript G):*

Proposition 1.2. — ([10], Prop. I.5.4, Cor. I.5.5(5) and Prop. I.5.6(2))

(*a*) $a \leqslant b \Leftrightarrow -b \leqslant -a$.

(*b*) *For all $a \in G, 1 \leqslant a \leqslant -1$.*

(*c*) $a \leqslant 0 \Leftrightarrow a = a^2 \in \mathrm{Id}(G)$ (= *the set $\{x^2 \mid x \in G\}$ of idempotents of G*),

$0 \leqslant a \Leftrightarrow a = -a^2 \in -\mathrm{Id}(G)$.

(*d*) *Let X_G be the character space of G. For $a, b \in G$,*

$$\begin{aligned} a \leqslant b \quad & \Leftrightarrow \forall h \in X_G \ (h(a) \leqslant_{\mathbf{3}} h(b)) \Leftrightarrow \\ & \Leftrightarrow \forall h \in X_G \ [(h(b) = 1 \Rightarrow h(a) = 1) \wedge (h(b) = 0 \Rightarrow h(a) \in \{0, 1\})]. \end{aligned}$$

(*e*) *For all $a \in G$, the infimum and the supremum of a and $-a$ for the representation partial order $\leqslant$ exists, and $a \wedge -a = a^2, a \vee -a = -a^2$. In particular,*

(*f*) $a \wedge -a \leqslant 0 \leqslant b \vee -b$ *for all $a, b \in G$* (*called the* Kleene inequality).

(*g*) *With join and meet defined by*

$$a \vee b = a \cdot b,$$
$$a \wedge b = \text{the unique element } c \in D^t(a, b).$$

for $a, b \in \mathrm{Id}(G) = \{x^2 \,|\, x \in G\}$, $\langle\, \mathrm{Id}(G), \wedge, \vee, 1, 0\,\rangle$ *is a distributive lattice with smallest element* 1 *and largest element* 0*, whose order is the restriction to* $\mathrm{Id}(G)$ *of the representation partial order of* G*.*

Remarks. — (i) The representation partial order of a real semigroup is nothing else but the trace of the (lattice) order of its Post hull (cf. [9], §4, pp. 61-65). Fact 1.4 shows that it is also the trace of the (lattice) order of its *spectral* hull.

(ii) Note that the representation partial order of the real semigroup **3** is $1 < 0 < -1$ (1.2(b)), rather than the order inherited from $\mathbb{Z}$.

(B) Spectral spaces.

For general background on spectral spaces the reader is referred to [11], whose notation we shall systematically use; certain results therein will be cited as needed but, due to space limitations, proofs are omitted[9]. See also [13].

(1) If X is a spectral space, the associated constructible topology is denoted by X_{con}.

(2) Recall that a spectral space X is called **hereditarily normal** iff any of the following equivalent conditions hold:

(i) The specialization order of X is a root-system.

(ii) Every proconstructible subset of X endowed with the induced topology is normal.

(iii) Every open, quasi-compact subset of X endowed with the induced topology is normal.

A proof of the equivalence of these conditions can be found in [11], Thm. 20.2.2.

(3) A map $f : X \longrightarrow Y$ between spectral spaces X, Y is called **spectral** iff the preimage of every open and quasi-compact subset of Y under f is, again, open and quasi-compact. Each of the following conditions is equivalent to f being spectral:

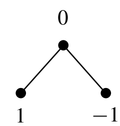
(i) f is continuous (for the spectral topologies) and continuous for the constructible topologies of X and Y ([11], Corol. 3.1.12).

(9) These can be obtained on demand from the first author. The numbering of references from [11] may change.

(ii) f is continuous for the constructible topologies and monotone for the specialization orders, of X and Y ([11], Lemma 5.6.6).

See also [11], Corol. 4.2.3.

(4) We shall consider two different topologies and two different orders on the set $\{1, 0, -1\}$. The first is the discrete topology and the representation partial order $1 < 0 < -1$ (see 1.2(b)), denoted by $\mathbf{3}$. The second is the spectral topology, where the singletons $\{1\}$ and $\{-1\}$ are a basis of opens, endowed with the specialization partial order:



The set $\{1, 0, -1\}$ endowed with this topology (and order) will be denoted by $\mathbf{3}_{\mathrm{sp}}$. Clearly, the singletons $\{\pm 1\}$ are quasi-compact and $\{0\}$ is closed in $\mathbf{3}_{\mathrm{sp}}$. Note that $(\mathbf{3}_{\mathrm{sp}})_{\mathrm{con}}$ is just the discrete topology on $\mathbf{3}$.

Definition 1.3. — *Spectral maps $f : X \longrightarrow \mathbf{3}_{\mathrm{sp}}$ from a spectral space X into $\mathbf{3}_{\mathrm{sp}}$ will be called* **spectral characters**. *The set of spectral characters on X will be denoted by* $\mathrm{Sp}(X)$.

(5) Clearly, $f : X \longrightarrow \mathbf{3}_{\mathrm{sp}}$ is a spectral character iff $f^{-1}[1]$ and $f^{-1}[-1]$ are quasi-compact open in X.

(C) The structure of $\mathrm{Sp}(X)$.

(1) (Product in $\mathrm{Sp}(X)$) $\mathrm{Sp}(X)$ has a product operation: the pointwise defined product of spectral characters h, g is a spectral character; indeed,

$$(hg)^{-1}[1] = (h^{-1}[1] \cap g^{-1}[1]) \cup (h^{-1}[-1] \cap g^{-1}[-1]),$$

$$(hg)^{-1}[-1] = (h^{-1}[-1] \cap g^{-1}[1]) \cup (h^{-1}[1] \cap g^{-1}[-1]);$$

the sets on the right-hand side of these equalities are quasi-compact open.

Obviously, $\mathrm{Sp}(X)$ contains the functions with constant values $1, 0, -1$ (denoted by the same symbols). Thus, $\mathrm{Sp}(X)$ is a commutative semigroup and, since product is pointwise defined, also a ternary semigroup ([8], Def. 1.1, p. 100).

(2) (Representation in $\mathrm{Sp}(X)$) A ternary (representation) relation is pointwise defined: for $h, h_1, h_2 \in \mathrm{Sp}(X)$,

$$h \in D_{\mathbf{Sp}(X)}(h_1, h_2) :\Leftrightarrow \forall x \in X (h(x) \in D_{\mathbf{3}}(h_1(x), h_2(x))).$$

Note that (by the definition of D^t in terms of D), $D^t_{\mathbf{Sp}(X)}$ is also pointwise defined in terms of $D^t_{\mathbf{3}}$. The structure $\langle \mathrm{Sp}(X), \cdot, 1, 0, -1, D_{\mathbf{Sp}(X)}\rangle$ will be denoted by $\mathbf{Sp}(X)$.

(3) (The pointwise partial order of $\mathrm{Sp}(X)$) This order is induced by the total order $1 < 0 < -1$ of $\mathbf{3}$ in the obvious way: for $f, g \in \mathrm{Sp}(X)$,

$$f \leqslant g \Leftrightarrow \forall x \in X (f(x) \leqslant g(x)) \text{ (in } \mathbf{3}).$$

The structure $\mathbf{Sp}(X)$ is also endowed with a binary relation $\leqslant_{\mathbf{Sp}(X)}$ defined as in 1.1: for $f, g \in \mathrm{Sp}(X)$,

$$(\dagger) \qquad f \leqslant_{\mathbf{Sp}(X)} g :\Leftrightarrow f \in D_{\mathbf{Sp}(X)}(1, g) \text{ and } -g \in D_{\mathbf{Sp}(X)}(1, -f).$$

Fact 1.4. — *The relation $\leqslant_{\mathbf{Sp}(X)}$ coincides with the pointwise partial order $\leqslant$.*

Proof. — Since the representation partial order of the RS $\mathbf{3}$ is $1 < 0 < -1$, for $f, g \in \mathrm{Sp}(X)$ we have:

$$\begin{aligned} f \leqslant g &\Leftrightarrow \forall x \in X (f(x) \leqslant_{\mathbf{3}} g(x)) \\ &\Leftrightarrow \forall x \in X [f(x) \in D_{\mathbf{3}}(1, g(x)) \wedge -g(x) \in D_{\mathbf{3}}(1, -f(x))] \\ &\Leftrightarrow f \in D_{\mathbf{Sp}(X)}(1, g) \wedge -g \in D_{\mathbf{Sp}(X)}(1, -f) \\ &\Leftrightarrow f \leqslant_{\mathbf{Sp}(X)} g. \end{aligned}$$

(The equivalences are, respectively, the definition of $\leqslant$ together with the preceding observation, the definition of $\leqslant_{\mathbf{3}}$, the definition of $D_{\mathbf{Sp}(X)}$ (see 1.1.C(1),(2)), and the definition of $\leqslant_{\mathbf{Sp}(X)}$.) $\square$

Remark. — Note that Fact 1.4 does not require $\mathbf{Sp}(X)$ to be a real semigroup. In spite of this equivalence it will be useful to keep the notational distinction between $\leqslant$ and $\leqslant_{\mathbf{Sp}(X)}$.

(4) (Lattice structure of $\mathrm{Sp}(X)$) $\mathrm{Sp}(X)$ has a lattice structure pointwise induced by the total order $1 < 0 < -1$ of $\mathbf{3}$: for $f, g \in \mathrm{Sp}(X)$ and $x \in X$,

$$(f \vee g)(x) := \max\{f(x), g(x)\}, \quad (f \wedge g)(x) := \min\{f(x), g(x)\} \text{ (in } \mathbf{3}).$$

One must check that the maps $f \vee g, f \wedge g$ thus defined are spectral; this is clear as

$$\begin{aligned} &(f \vee g)^{-1}[1] = f^{-1}[1] \cap g^{-1}[1], \quad (f \vee g)^{-1}[-1] = f^{-1}[-1] \cup g^{-1}[-1], \text{ and} \\ &(f \wedge g)^{-1}[1] = f^{-1}[1] \cup g^{-1}[1], \quad (f \wedge g)^{-1}[-1] = f^{-1}[-1] \cap g^{-1}[-1], \end{aligned}$$

are quasi-compact open subsets of X.

Since **3** is a chain, the lattice structure just defined is *distributive*. The functions with constant value 1 or -1 (denoted by the same symbols) are the smallest and largest elements, respectively, of the lattice $\langle \mathrm{Sp}(X), \wedge, \vee \rangle$.

We also note:

(5) The product operation and both representation relations of the $\mathcal{L}_{\mathsf{RS}}$-structure $\mathbf{Sp}(X)$ can be described (in a quantifier-free manner) in terms of the constants together with its lattice order and operations, as follows:

PROPOSITION 1.5. — *With notation as above, for $f, g, h \in \mathrm{Sp}(X)$, we have:*

(*i*) *Product in* $\mathrm{Sp}(X)$ *is identical with symmetric difference* (*defined in terms of the lattice operations and* $-$)*:*

$$g \cdot h = g \,\underline{\triangle}\, h (:= (g \wedge -h) \vee (h \wedge -g)).$$

In particular (*with* $g = -1$), $(-1) \cdot h = -h$.

(*ii*) $f \in D_{\mathbf{Sp}(X)}(g, h) \Leftrightarrow g \wedge h \wedge 0 \leqslant f \leqslant g \vee h \vee 0.$

(*iii*) $f \in D^t_{\mathbf{Sp}(X)}(g, h) \Leftrightarrow g \wedge h \leqslant f \leqslant g \vee h, f \wedge -g \leqslant h \leqslant f \vee -g$ *and* $f \wedge -h \leqslant g \leqslant f \vee -h$.

Proof. — (i) and (ii) follow straightforwardly from the pointwise definition of product and representation in $\mathrm{Sp}(X)$ (cf. 1.1.C(1),(2) above) and the validity of the respective items in **3**.

The absence of 0 in the inequalities in (iii) demands a bit of extra work to prove.

($\Rightarrow$) Noting that $f \in D^t_{\mathbf{Sp}(X)}(g, h)$ implies $h \in D^t_{\mathbf{Sp}(X)}(f, -g)$ and $g \in D^t_{\mathbf{Sp}(X)}(f, -h)$, it suffices to prove

(*) $g(x) \wedge h(x) \leqslant f(x) \leqslant g(x) \vee h(x)$ for all $x \in X$.

Assume $f(x) \in D^t_{\mathbf{3}}(g(x), h(x))$. If $f(x) \neq 0$, by the last equivalence in 1.1.A(2), $f(x) = g(x)$ or $f(x) = h(x)$, from which (*) clearly follows. If $f(x) = 0$, by the same equivalence, $g(x) = -h(x)$, whence (*) reduces in this case to $g(x) \wedge -g(x) \leqslant 0 \leqslant g(x) \vee -g(x)$, a particular instance of Kleene's inequality 1.2(f).

($\Leftarrow$) Assuming the inequalities in the right-hand side of (iii), we prove $f(x) \in D^t_{\mathbf{3}}(g(x), h(x))$ for all $x \in X$. If $f(x) \neq 0$, say $f(x) = 1$, the left inequality in (*) yields $g(x) \wedge h(x) = 1$, whence one of $g(x)$ or $h(x)$ is 1.

A similar argument applies in case $f(x) = -1$, using the right inequality in (*).

Suppose, then, $f(x) = 0$, and argue according to the values of, say, $g(x)$, to prove $g(x) = -h(x)$ (1.1.A(2)). If $g(x) = 0$, then $-g(x) = 0$, and $f(x) \wedge -g(x) = 0 = f(x) \vee -g(x)$. From $f \wedge -g \leqslant h \leqslant f \vee -g$ we conclude $h(x) = 0 = -g(x)$. If $g(x) = 1$, then $f \wedge -h \leqslant g \leqslant f \vee -h$ yields $1 = f(x) \wedge -h(x) = 0 \wedge -h(x)$, whence $-h(x) = 1$, and $h(x) = -g(x)$. A similar argument applies if $g(x) = -1$.

Next, we examine, for an arbitrary spectral space X, which of the axioms for real semigroups are satisfied by $\mathbf{Sp}(X)$, and the requirements to be imposed on X for $\mathbf{Sp}(X)$ to become a real semigroup.

1.6 Reminder. It is proved in [14], Prop. 6.1.1, p. 100 and Thm. 6.2.4, pp. 107-108, that the *strong associativity* axiom for real semigroups, i.e. the statement

[RS3] If $a \in D^t(b, c)$ and $c \in D^t(d, e)$, then there exists $x \in D^t(b, d)$ such that $a \in D^t(x, e)$,

is equivalent to the conjuntion of the *weak associativity* axiom [RS3a] (called AX3a in [14], p. 99), i.e., the same statement with D^t replaced by D, and the axiom

[RS3b] For all $a, b, D^t(a, b) \neq \emptyset$.

THEOREM 1.7. — *For every spectral space X, the structure $\mathbf{Sp}(X)$ satisfies all axioms for real semigroups – including the weak associativity axiom* [RS3a] *– except, possibly, axiom* [RS3b].

Proof. — The validity of the axioms [RSi] for $i \neq 3 (0 \leqslant i \leqslant 8)$ is straightforward, stemming from the following observations:

(i) Product and representation are pointwise defined in $\mathbf{Sp}(X)$, and

(ii) All axioms for real semigroups, except [RS3], are universal statements in the language $\mathcal{L}_{\mathbf{RS}} = \{\cdot, 1, 0, -1, D\}$.

Details are left to the reader.

To prove the weak associativity axiom [RS3a] we use the lattice structure of $\mathrm{Sp}(X)$, see 1.1.C(4).

Let $a, b, c, d, e \in \mathrm{Sp}(X)$ be such that $a \in D_{\mathbf{Sp}(X)}(b, c)$ and $c \in D_{\mathbf{Sp}(X)}(d, e)$. We must find an $f \in \mathrm{Sp}(X)$ so that $f \in D_{\mathbf{Sp}(X)}(b, d)$ and $a \in D_{\mathbf{Sp}(X)}(f, e)$. We claim that

$$f = (a \wedge -e) \vee (b \wedge d) \vee (a \wedge b) \vee (a \wedge d)$$

has the desired property. Indeed, we have:

– $\underline{f \in D_{\mathbf{Sp}(X)}(b, d)}$. By Proposition 1.5(ii) our assumptions amount to:

$$(*)\ b \wedge c \wedge 0 \leqslant a \leqslant b \vee c \vee 0 \qquad \text{and} \qquad (**)\ d \wedge e \wedge 0 \leqslant c \leqslant d \vee e \vee 0,$$

and we must prove $b \wedge d \wedge 0 \leqslant f \leqslant b \vee d \vee 0$. The left-hand side inequality is clear as $b \wedge d$ occurs as a disjunct in f. For the inequality in the right-hand side, the last three disjuncts of f are either $\leqslant b$ or $\leqslant d$, and so $\leqslant b \vee d \vee 0$. It remains to show that $a \wedge -e \leqslant b \vee d \vee 0$. From (*) comes

$$a \wedge -e \leqslant b \vee (c \wedge -e) \vee (0 \wedge -e) \leqslant b \vee (c \wedge -e) \vee 0,$$

and from (**), using the Kleene inequality 1.2(f), we obtain

$$c \wedge -e \leqslant (d \vee e \vee 0) \wedge -e \leqslant d \vee (e \wedge -e) \vee 0 = d \vee 0.$$

Hence, $a \wedge -e \leqslant b \vee d \vee 0$, as required.

– $\underline{a \in D_{\mathbf{Sp}(X)}(f, e)}$. By 1.5(ii) we must now prove $f \wedge e \wedge 0 \leqslant a \leqslant f \vee e \vee 0$. For the left inequality: three of the disjuncts of f are $\leqslant a$. So, it suffices to prove $b \wedge d \wedge e \wedge 0 \leqslant a$, which is clear using successively the left-hand side inequalities in (*) and (**). For the inequality in the right-hand side, we show its validity at each point $x \in X$. This clearly holds if $a(x) \in \{0, 1\}$. So, assume $a(x) = -1$ and prove that either $f(x) = -1$ or $e(x) = -1$. Since $a(x) \wedge z = -1 \wedge z = z$, we have

$$f(x) = -e(x) \vee (b(x) \wedge d(x)) \vee b(x) \vee d(x) = b(x) \vee d(x) \vee -e(x).$$

From the right inequality in (*) we get either $b(x) = -1$ (whence $f(x) = -1$), or $c(x) = -1$. This, together with the right inequality in (**) yields $d(x) = -1$ (and hence $f(x) = -1$) or $e(x) = -1$, as required. □

Concerning the remaining axiom [RS3b], we have:

Theorem 1.8. — *The following are equivalent for every spectral space X:*

(1) *X is hereditarily normal.*

(2) *The structure* $\mathbf{Sp}(X)$ *verifies axiom* [RS3b], *i.e. it is a real semigroup.*

The following known results concerning spectral spaces will be needed in the proof of 1.8. Recall that a subset A of a spectral space X is called **generically closed** if it is downward closed for the specialization order of X: for $x, y \in X$, $x \rightsquigarrow y$ and $y \in A$ imply $x \in A$; cf. [11], 5.1.6.

Proposition 1.9. — *Let X be a spectral space and let D_1, D_2 be disjoint, generically closed, quasi-compact subsets of X. Then,*

(*i*) *D_1, D_2 are contained in disjoint quasi-compact* open *subsets of X.*

(*ii*) *Given quasi-compact opens U_1, U_2 such that $D_i \subseteq U_i (i = 1, 2)$ there are disjoint quasi-compact opens V_1, V_2 so that $D_i \subseteq V_i \subseteq U_i (i = 1, 2)$.*

Remark. — Item (ii) is a consequence of (i): if V_1', V_2' are disjoint quasi-compact opens such that $D_i \subseteq V_i'$, then the sets $V_i = V_i' \cap U_i (i = 1, 2)$ satisfy the conclusion of (ii). Item (i) is Prop. 6.1.14(iii) in [11].

Fact 1.10. — *Let X be a topological space, and $B, C \subseteq X$. If B is quasi-compact and C is closed, then $B \cap C$ is quasi-compact.*

Proof of Theorem 1.8. — (1) $\Rightarrow$ (2). Let $f, g \in \mathrm{Sp}(X)$. We consider the following subsets of X:

$$\begin{aligned} K_1 &= (f^{-1}[0,1] \cap g^{-1}[1]) \cup (g^{-1}[0,1] \cap f^{-1}[1]), \\ K_2 &= (f^{-1}[0,-1] \cap g^{-1}[-1]) \cup (g^{-1}[0,-1] \cap f^{-1}[-1]). \end{aligned}$$

By 1.10 these sets are quasi-compact. Let $\mathrm{Gen}\,(K_i) = \{x \in X \mid \text{ There is } y \in K_i \text{ such that } x \rightsquigarrow y\}$ denote the *generization* of $K_i (i = 1, 2)$ in X, i.e., the downward closure of K_i under the specialization order $\rightsquigarrow$ of X. Since open sets are downward closed under $\rightsquigarrow$, it is easily checked that $\mathrm{Gen}\,(K_i)$ is also quasi-compact. We claim:

Claim 1. — $\mathrm{Gen}\,(K_1) \cap \mathrm{Gen}\,(K_2) = \emptyset$.

Proof of Claim 1. — Assume there is $t \in \mathrm{Gen}\,(K_1) \cap \mathrm{Gen}\,(K_2)$, and let $k_i \in K_i$ be such that $t \rightsquigarrow k_i$ $(i = 1, 2)$. Since X is assumed hereditarily normal, either $k_1 \rightsquigarrow k_2$ or $k_2 \rightsquigarrow k_1$, say the first. Since $k_2 \in K_2 \subseteq f^{-1}[-1] \cup g^{-1}[-1]$ and the latter set is open, we get $k_1 \in f^{-1}[-1] \cup g^{-1}[-1]$, contradicting $k_1 \in K_1$. $\square$

Since $f^{-1}[1] \cup g^{-1}[1]$ is open and contains K_1, we have $\mathrm{Gen}\,(K_1) \subseteq f^{-1}[1] \cup g^{-1}[1]$. Similarly, $\mathrm{Gen}\,(K_2) \subseteq f^{-1}[-1] \cup g^{-1}[-1]$. By Proposition 1.9 there are disjoint quasi-compact opens V_1, V_2 so that $\mathrm{Gen}\,(K_1) \subseteq V_1 \subseteq f^{-1}[1] \cup g^{-1}[1]$ and $\mathrm{Gen}\,(K_2) \subseteq V_2 \subseteq f^{-1}[-1] \cup g^{-1}[-1]$. Let $h : X \longrightarrow \mathbf{3}$ be the map

$$h(x) = \begin{cases} 1 & \text{if } x \in V_1 \\ -1 & \text{if } x \in V_2 \\ 0 & \text{if } x \notin V_1 \cup V_2. \end{cases}$$

Clearly, $h \in \mathrm{Sp}(X)$, and we assert

Claim 2. — $h \in D^t_{\mathbf{Sp}(X)}(f,g)$.

Proof of Claim 2. — By Proposition 1.5(iii) it suffices to show that each of the following pairs of inequalities

(*) $f \wedge g \leqslant h \leqslant f \vee g, h \wedge -f \leqslant g \leqslant h \vee -f$ and $h \wedge -g \leqslant f \leqslant h \vee -g$,

holds at every point $x \in X$. If $h(x) = 1$, then $x \in V_1$, whence $f(x) = 1$ or $g(x) = 1$. This yields $(f \wedge g)(x) = 1$, $f(x) \leqslant -g(x)$ and $g(x) \leqslant -f(x)$, which, together, imply the validity of all inequalities in (*) at the point x. A similar argument works if $h(x) = -1$.

So, assume $h(x) = 0$, i.e., $x \notin V_1 \cup V_2$. Then, $x \notin K_1 \cup K_2$, and this implies

$x \in (g^{-1}[0,1] \cup f^{-1}[1]) \cap (f^{-1}[0,1] \cup g^{-1}[1]) \cap (g^{-1}[0,-1] \cup f^{-1}[-1]) \cap (f^{-1}[0,-1] \cup g^{-1}[-1])$.

If $f(x) = g(x) = -1$, then x does not belong to the first conjunct; hence $(f \wedge g)(x) \leqslant 0$. Likewise, $f(x) = g(x) = 1$ is not possible; so, $(f \wedge g)(x) \geqslant 0$, proving that the first (double) inequality in (*) holds at x. Note that $g(x) = \pm 1 \Leftrightarrow f(x) = \mp 1$, which proves that the two last pairs of inequalities in (*) hold at x. $\square$

(2) $\Rightarrow$ (1). With $\rightsquigarrow$ denoting the specialization order of X, assume there are $x, y, z \in X$ such that $x \rightsquigarrow y, z$, but $y \not\rightsquigarrow z$ and $z \not\rightsquigarrow y$. Thus, $z \notin \overline{\{y\}}$ and $y \notin \overline{\{z\}}$. Then, there are quasi-compact opens U, V such that $z \in U, y \in V, y \notin U$ and $z \notin V$. Let $f_U, f_V : X \longrightarrow \mathbf{3}$ be the spectral maps defined by:

$$f_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}, \qquad f_V(x) = \begin{cases} -1 & \text{if } x \in V \\ 0 & \text{if } x \notin V. \end{cases}$$

Since $\mathbf{Sp}(X) \models$ [RS3b], there is $f \in \mathbf{Sp}(X)$ such that $f \in D^t_{\mathbf{Sp}(X)}(f_U, f_V)$. In particular, for $w \in \{y, z\}$ we have $f(w) \in D^t_{\mathbf{3}}(f_U(w), f_V(w))$. Now, $f_U(y) = 0, f_V(y) = -1$ and $f_U(z) = 1$, $f_V(z) = 0$ imply $f(y) = -1$ and $f(z) = 1$ (cf. 1.1.A(2)). Since f is monotone for the order of $\mathbf{3}_{\rm sp}$, $x \rightsquigarrow y, z$ entails $f(x) = -1$ and $f(x) = 1$, contradiction. $\square$

Definition 1.11. — *A real semigroup is called* **spectral** *if it is of the form* $\mathbf{Sp}(X)$ *for some spectral space* X *(necessarily hereditarily normal by the preceding theorem).*

2. Definability of the lattice structure

In this section we prove a weak converse to Proposition 1.5 by giving an explicit first-order (but not quantifier-free) definition of the lattice operations of any spectral RS in the language $\mathcal{L}_{\mathbf{RS}} = \{\cdot, 1, 0, -1, D\}$ for real semigroups. The specific (logical) form of this definition entails that the RS-characters of spectral RSs are automatically lattice homomorphisms, a key result towards a structural theory of spectral real semigroups and, hence, to many later results in this paper.

Theorem 2.1. — *Let G be a spectral real semigroup. For $a, b, c, d \in G$ we have:*

(i) $a \wedge 0 = c \Leftrightarrow c = c^2, a \cdot c = c$ *and* $-a \in D_G(1, -c)$.

Setting $a^- := a \wedge 0$ *and* $a^+ := -(-a)^- = a \vee 0$, *we have:*

(ii) $a \wedge b = d \Leftrightarrow d \in D_G(a, b), d^+ = -a^+ \cdot b^+$ *and* $d^- \in D^t_G(a^-, b^-)$.

Remark. — These equivalences prove, in particular, that the elements c, d in (i) and (ii) are uniquely determined by a and b.

Proof. — Let $G = \mathbf{Sp}(X)$, X a hereditarily normal spectral space. Owing to Fact 1.4, the lattice operations in G can interchangeably be taken in the pointwise order $\leqslant$ or in the representation partial order $\leqslant_G$ (we shall use both).

(i) ($\Rightarrow$) We check that $c := a \wedge 0$ verifies the three conditions on the right-hand side of (i).

Firstly, $c \leqslant 0$ implies $c = c^2$ (1.2(c)), and $c \leqslant a$ implies $-a \in D_{\mathbf{Sp}(X)}(1, -c)$ (1.1). To check $ac = c$, recall that $ac = a \,\underline{\triangle}\, c$ ($\underline{\triangle}$ = symmetric difference, cf. 1.5(i)). Hence,

$$\begin{aligned} a \,\underline{\triangle}\, (a \wedge 0) &= (a \wedge -(a \wedge 0)) \vee ((a \wedge 0) \wedge -a) \\ &= (a \wedge (-a \vee 0)) \vee (a \wedge -a \wedge 0) \\ &= (a \wedge -a) \vee (a \wedge 0) \vee (a \wedge -a \wedge 0) = (a \wedge -a) \vee (a \wedge 0); \end{aligned}$$

since $a \wedge -a \leqslant 0, a$ (cf. 1.2(f)), the last term equals $a \wedge 0$.

($\Leftarrow$) By 1.2(c), $c = c^2 \leqslant 0$; by [8], Prop. 2.3(5) we have $c = c^2 \in D_G(1, a)$. By assumption we also have $-a \in D_G(1, -c)$, whence $c \leqslant a$.

To prove $c = a \wedge 0$, let $z \in G$ be such that $z \leqslant 0$ and $z \leqslant a$, and show $z \leqslant c$, i.e., $z(x) \leqslant c(x)$ for all $x \in X$. Otherwise, since $z = z^2$, we must have $z(x) = 0$ and $c(x) = 1$ for some $x \in X$. From $ac = c$ we get $a(x) = 1$, contradicting $z(x) \leqslant a(x)$.

(ii) ($\Rightarrow$) Set $d := a \wedge b$. We check the three conditions on the right-hand side of (ii).

a) $\underline{d \in D_G(a, b)}$. This is clear from $a \wedge b \wedge 0 \leqslant d \leqslant a \vee b \vee 0$, using Proposition 1.5(ii).

b) $\underline{(a \wedge b)^+ = -a^+ \cdot b^+}$. We compute the right-hand side using the distributive lattice structure of G and that product in G is symmetric difference (1.5(i)). Recall that $z^+ = z \vee 0$ and $-(z \,\underline{\triangle}\, w) = (-z \vee w) \wedge (-w \vee z)$. We have:

$$\begin{aligned} -a^+ \cdot b^+ &= -(a \vee 0) \,\underline{\triangle}\, (b \vee 0) = (-(a \vee 0) \vee (b \vee 0)) \wedge (-(b \vee 0) \vee (a \vee 0)) \\ &= ((-a \wedge 0) \vee (b \vee 0)) \wedge ((-b \wedge 0) \vee (a \vee 0)) = (b \vee 0) \wedge (a \vee 0) \\ &= (a \wedge b) \vee 0 = (a \wedge b)^+, \end{aligned}$$

as asserted.

c) $\underline{(a \wedge b)^- \in D^t_G(a^-, b^-)}$. By Proposition 1.5(iii) this amounts to proving

(I) $a^- \wedge\ b^- \leqslant (a \wedge b)^- \leqslant a^- \vee\ b^-$,

(II) $(a \wedge b)^- \wedge\ -a^- \leqslant b^- \leqslant (a \wedge b)^- \vee\ -a^-$,

(III) $(a \wedge b)^- \wedge\ -b^- \leqslant a^- \leqslant (a \wedge b)^- \vee\ -b^-$.

Observe that $a^- \wedge\ b^- = (a \wedge 0) \wedge (b \wedge 0) = a \wedge b \wedge 0 = (a \wedge\ b)^-$, which clearly implies (I). For (II) we have

$$\begin{aligned} &(a \wedge b)^- \wedge\ -a^- = a^- \wedge\ -a^- \wedge\ b^- \leqslant b^- \qquad \text{and} \\ &(a \wedge b)^- \vee\ -a^- = (a^- \wedge\ b^-) \vee\ -a^- = (b^- \vee\ -a^-) \wedge (a^- \vee\ -a^-). \end{aligned}$$

Since $z^- \leqslant 0$ for all $z \in G$, invoking Kleene's inequality 1.2(f) we get $b^- \leqslant 0 \leqslant a^- \vee\ -a^-$, and hence $b^- \leqslant (a \wedge b)^- \vee\ -a^-$, as required. Item (III) is equivalent to (II) by symmetry.

($\Leftarrow$) Given $d \in G$, we assume $d \in D_G(a, b), d^+ = -a^+ \cdot b^+, d^- \in D^t_G(a^-, b^-)$, and prove $d = a \wedge\ b$.

a) $\underline{d \leqslant_G a \text{ and } d \leqslant_G b}$. By symmetry it suffices to prove the first inequality. In view of the pointwise definition of $\leqslant_G$ (see (†) in 1.1.C(3)), it suffices, in turn, to prove, for $x \in X$:

$$a(x) = 1 \Rightarrow d(x) = 1 \qquad \text{and} \qquad a(x) = 0 \Rightarrow d(x) \in \{0, 1\}.$$

Note that $z^- = z \wedge 0 \leqslant 0$ clearly implies

(*) $z = 1 \Leftrightarrow z^- = 1$ (equivalently, $z = 0 \Leftrightarrow z^- \in \{0, -1\}$).

For the first implication, $a(x) = 1$ yields $a^-(x) = 1$. From $d^- \in D^t_G(a^-, b^-)$ and $b^-(x) \in \{0, 1\}$ we get $d^-(x) \in D^t_{\mathbf{3}}(a^-(x), b^-(x)) = D^t_{\mathbf{3}}(1, b^-(x)) = \{1\}$, which, by (*), gives $d(x) = 1$, as needed.

For the second implication, if $a(x) = 0$ but $d(x) = -1$, we would have $a^+(x) = a(x) \vee 0 = 0$ and $d^+(x) = d(x) \vee 0 = -1$, contradicting the equality $d^+ = -a^+ \cdot b^+$ at the point x.

b) For all $z \in G, z \leqslant_G a$ and $z \leqslant_G b$ imply $z \leqslant_G d$. We must check, for all $x \in X$:

$$d(x) = 1 \Rightarrow z(x) = 1 \qquad \text{and} \qquad d(x) = 0 \Rightarrow z(x) \in \{0, 1\}.$$

For the first implication, (*) yields $d^-(x) = 1$. On the other hand, since $a^-(x), b^-(x) \in \{0, 1\}$, from $d^- \in D^t_G(a^-, b^-)$ we obtain $1 = d^-(x) \in D^t_{\mathbf{3}}(a^-(x), b^-(x))$. This relation implies that $a^-(x), b^-(x)$ cannot both be 0 (cf. 1.1.A(2)). If, e.g., $a^-(x) = 1$, then $a(x) = 1$, and $z \leqslant_G a$ yields $z(x) = 1$.

For the second implication, suppose $d(x) = 0$; hence $d^+(x) = d(x) \vee 0 = 0$. This and $d^+ = -a^+ \cdot b^+$ imply that one of $a^+(x)$ or $b^+(x)$ is 0, say, e.g., $a^+(x) = 0$. Then, $0 = a^+(x) = a(x) \vee 0$ entails $a(x) \in \{0, 1\}$; this, together with $z \leqslant_G a$ yields $z(x) \in \{0, 1\}$, completing the proof of Theorem 2.1.

Remark. — First-order definability of the lattice structure of $\mathbf{Sp}(X)$ in $\mathcal{L}_{\mathrm{RS}}$ follows also from Fact 1.4: it suffices to express

(i) The definition of $\leqslant_{\mathbf{Sp}(X)}$ in terms of $D_{\mathbf{Sp}(X)}$ (Definition 1.1), and

(ii) The usual definition of the glb ($\wedge$) and the lub ($\vee$) in terms of the order $\leqslant_{\mathbf{Sp}(X)}$.

However, the definition of the lattice operations obtained in this way does not guarantee that the next Corollary holds, while that of Theorem 2.1 does. Though only implicit at this stage, the reason is that the definition in 2.1 is given by a *positive-primitive* $\mathcal{L}_{\mathrm{RS}}$-formula, while that above is not. For more details, see 7.2.

COROLLARY 2.2. — *The RS-characters of a spectral real semigroup are lattice homomorphisms. Further, RS-homomorphisms between spectral RSs are automatically homomorphisms of the corresponding lattice structures.*

Proof. — We do the proof for characters, leaving the second assertion to the reader. To begin with, observe that $\mathbf{3}$ is a spectral RS. Indeed, $\mathbf{3} = \mathrm{Sp}(\mathbf{1})$, where $\mathbf{1}$ the singleton spectral space; the three functions $\mathbf{1} \longrightarrow \mathbf{3}_{\mathrm{sp}}$ map the unique element to $1, 0$ and -1, respectively; clearly, they are pointwise ordered in the correct way.

Let G be a spectral RS, $a, b \in G$ and $\sigma \in X_G$; we show that $\sigma(a \wedge b) = \sigma(a) \wedge \sigma(b)$. The equivalences (i) and (ii) of 2.1 can (and will) be applied to both G and $\mathbf{3}$.

We first treat the case $b = 0$. We know that $c = a \wedge 0$, verifies the conditions in the right-hand side of 2.1(i). Since σ is a RS-homomorphism we get $\sigma(c) \in \mathrm{Id}(\mathbf{3}) = \{0, 1\}$, $\sigma(a)\sigma(c) = \sigma(c)$ and $-\sigma(a) \in D^t_{\mathbf{3}}(1, -\sigma(c))$. Using the implication ($\Leftarrow$) of 2.1(i) in $\mathbf{3}$, gives $\sigma(c) = \sigma(a) \wedge 0$, i.e.,

(†) $\sigma(a \wedge 0) = \sigma(a) \wedge 0$ (equivalently, $\sigma(a^-) = \sigma(a)^-$).

Since $\sigma(-(-a)^-) = -\sigma((-a)^-) = -(\sigma(-a))^- = -(-\sigma(a))^-$, we also get

(††) $\sigma(a^+) = \sigma(a)^+$.

Next, for arbitrary $b \in G$, applying item (ii) of 2.1 with $d = a \wedge b$, taking into account that σ is a RS-homomorphism, and using (†) and (††) above, we get $\sigma(d) \in D_{\mathbf{3}}(\sigma(a), \sigma(b))$, $\sigma(d)^+ = -\sigma(a)^+ \cdot \sigma(b)^+$ and $\sigma(d)^- \in D^t_{\mathbf{3}}(\sigma(a)^-, \sigma(b)^-)$. On the other hand, the element $x = \sigma(a) \wedge \sigma(b)$ exists in $\mathbf{3}$, and verifies $x \in D_{\mathbf{3}}(\sigma(a), \sigma(b)), x^+ = -\sigma(a)^+ \cdot \sigma(b)^+$ and $x^- \in D^t_{\mathbf{3}}(\sigma(a)^-, \sigma(b)^-)$. That is, both x and $\sigma(d)$ verify in $\mathbf{3}$ the conditions of the right-hand side of (ii) in 2.1. This implies $x = \sigma(d)$, i.e., $\sigma(a) \wedge \sigma(b) = \sigma(a \wedge b)$, as asserted.

3. The functor Sp

We begin here the study of the correspondence $X \longmapsto \mathbf{Sp}(X)$ assigning to each hereditarily normal spectral space, X, the real semigroup $\mathbf{Sp}(X)$. In this and the next two sections we set the stage to prove that this correspondence is the object map of a duality between the category **HNSS** of hereditarily normal spectral spaces with spectral maps, and **SRS**, the category of spectral real semigroups with real semigroup morphisms, a goal attained in Theorem 5.4 below.

We now describe the behaviour of our functor on spectral maps.

Definition and Notation 3.1. — *Given a spectral map* $\varphi : X \longrightarrow Y$ *between spectral spaces* X, Y *we define a dual map* $\mathrm{Sp}(\varphi) : \mathrm{Sp}(Y) \longrightarrow \mathrm{Sp}(X)$ *by composition: for* $f \in \mathrm{Sp}(Y)$ *we set,*

$$\mathrm{Sp}(\varphi)(f) := f \circ \varphi.$$

Being a composition of spectral maps, we have $\mathrm{Sp}(\varphi)(f) \in \mathrm{Sp}(X)$.

Proposition 3.2. — *Let $\varphi : X \longrightarrow Y$ be a spectral map between spectral spaces. Then* $\mathrm{Sp}(\varphi)$ *is a homomorphism of* $\mathcal{L}_{\mathrm{RS}}$*-structures.*

The proof is routine verification using the fact that product and representation in both $\mathbf{Sp}(Y)$ and $\mathbf{Sp}(X)$ are pointwise defined. We omit it. Note that it is not required that X, Y be hereditarily normal.

As a beginning step in proving that this functor is an anti-equivalence of categories we show that any hereditarily normal spectral space, X, is isomorphic in the category of spectral spaces to the abstract real spectrum $X_{\mathbf{Sp}(X)}$ of the real semigroup $\mathbf{Sp}(X)$. The proof requires a fine touch. First, we observe the following straightforward

Fact 3.3. — *Let X be a spectral space. The evaluation map at a point $x \in X$, $ev_x : \mathbf{Sp}(X) \longrightarrow \mathbf{3}$, given by $ev_x(f) = f(x)$ for $f \in \mathrm{Sp}(X)$, is a character of* $\mathcal{L}_{\mathrm{RS}}$*-structures., i.e., $ev_x \in X_{\mathbf{Sp}(X)}$.*

Let $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ be the map $\mathrm{ev}(x) = ev_x (x \in X)$.

Proposition 3.4. — $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ *is injective and spectral.*

Proof. — (1) ev is injective. This amounts to showing that $\mathrm{Sp}(X)$ separates points in X: for $x \neq y$ in X, there is $g \in \mathrm{Sp}(X)$ so that $g(x) \neq g(y)$, i.e., $ev_x(g) \neq ev_y(g)$, whence $ev_x \neq ev_y$, i.e., $\mathrm{ev}(x) \neq \mathrm{ev}(y)$.

Since X is T_0, if $x \neq y$, there is a quasi-compact open $U \subseteq X$ so that $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$, say the first. Let $g : X \longrightarrow \mathbf{3}$ be defined by: $g\lceil U = 1, g\lceil(X \setminus U) = 0$. Since $g^{-1}[1] = U, g^{-1}[-1] = \emptyset$ are quasi-compact open, $g \in \mathrm{Sp}(X)$ and, clearly, $g(x) = 1, g(y) = 0$. The case $y \in U$ and $x \notin U$ is similar.

(2) ev is spectral. By definition, the family $\{[\![f = 1]\!] \mid f \in \mathrm{Sp}(X)\}$, where $[\![f = 1]\!] = \{\sigma \in X_{\mathbf{Sp}(X)} \mid \sigma(f) = 1\}$ is a subbasis for the spectral topology on $X_{\mathbf{Sp}(X)}$, cf. 1.1.A(3). It suffices, then, to show that $\mathrm{ev}^{-1}[[\![f = 1]\!]]$ is quasi-compact open in X, for $f \in \mathrm{Sp}(X)$. For $x \in X$ we have:

$$x \in \mathrm{ev}^{-1}[[\![f = 1]\!]] \Leftrightarrow \mathrm{ev}(x) = ev_x \in [\![f = 1]\!] \Leftrightarrow ev_x(f) = f(x) = 1,$$

i.e., $\mathrm{ev}^{-1}[[\![f = 1]\!]] = f^{-1}[1]$, a quasi-compact open set, as claimed. $\square$

The surjectivity of ev is the key to establish the anti-equivalence of categories announced above.

Theorem 3.5. — *Let X be a hereditarily normal spectral space. Then,* $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ *is surjective.*

Proof. — Given $\sigma \in X_{\mathbf{Sp}(X)}$ we have to find $x \in X$ so that $\mathrm{ev}(x) = ev_x = \sigma$. To accomplish this we shall use Stone's representation theorem of spectral spaces by bounded distributive lattices (cf. [11], §§1,2). This fundamental result proves the existence of a (functorial) bijective correspondence between the points of a spectral space, X, and the prime filters of the bounded distributive lattice $\overline{\mathcal{K}}(X)$ of **closed constructible** – i.e., complements of quasi-compact open – subsets of X; cf. [11], Thm. 2.1.7 for a more precise statement). We shall construct a prime filter $\mathfrak{p}$ of $\overline{\mathcal{K}}(X)$ such that, if x_0 is the unique point of $\bigcap \mathfrak{p}$, then $\sigma = ev_{x_0}$. Recall that the pointwise order coincides with the representation partial order in $\mathbf{Sp}(X)$ (1.4).

Since $\sigma : \mathbf{Sp}(X) \longrightarrow \mathbf{3}$ is a lattice homomorphism (2.2), the set $\mathfrak{q} = \sigma^{-1}[0,-1]$ is a prime filter of the lattice $\mathrm{Sp}(X)$. For $A \in \overline{\mathcal{K}}(X)$ we define maps $c_A, d_A : X \longrightarrow \mathbf{3}$ as follows: for $x \in X$,

$$c_A(x) = \begin{cases} 0 & \text{if } x \in A \\ -1 & \text{if } x \notin A, \end{cases} \qquad d_A(x) = -c_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A. \end{cases}$$

Since $X \setminus A$ is quasi-compact open, we have $c_A, d_A \in \mathrm{Sp}(X)$. Further, since $d_A \leqslant 0 \leqslant c_A$ and σ is monotone, we get $\sigma(d_A) \in \{0,1\}$ and $\sigma(c_A) \in \{0,-1\}$. Now set:

$$\mathfrak{p} := \{A \in \overline{\mathcal{K}}(X) \mid d_A \in \mathfrak{q}\} = \{A \in \overline{\mathcal{K}}(X) \mid \sigma(d_A) = 0\}.$$

Claim 1. — $\mathfrak{p}$ is a prime filter of $\overline{\mathcal{K}}(X)$.

Proof of Claim 1. — (a) $A \subseteq B$ and $A \in \mathfrak{p}$ imply $B \in \mathfrak{p}$. Clearly, $A \subseteq B \Rightarrow d_A \leqslant d_B$. Since σ is monotone, $0 = \sigma(d_A) \leqslant \sigma(d_B)$, and from $\sigma(d_B) \in \{0,1\}$, it follows $\sigma(d_B) = 0$, i.e., $B \in \mathfrak{p}$.

(b) $\emptyset \notin \mathfrak{p}$. Clear, since $d_\emptyset = 1$.

(c) It is straightforward that $d_{A \cap B} = d_A \wedge d_B$ and $d_{A \cup B} = d_A \vee d_B$. Recalling that σ is a lattice homomorphism, these equalities yield that $\mathfrak{p}$ is closed under meets and is prime.

Claim 2. — For $f \in \mathrm{Sp}(X)$, $f \in \mathfrak{q} \Leftrightarrow f^{-1}[0,-1] \in \mathfrak{p}$.

Proof of Claim 2. — Set $A := f^{-1}[0,-1] \in \overline{\mathcal{K}}(X)$. Note that $d_A \leqslant f$ because $\mathrm{Im}(d_A) = \{0,1\}$ and, for $x \in X$, $d_A(x) = 0 \Rightarrow x \in A \Rightarrow f(x) \in \{0,-1\}$.

$(\Leftarrow)$ If $A \in \mathfrak{p}$, then $d_A \in \mathfrak{q}$ and $d_A \leqslant f$ give $f \in \mathfrak{q}$ ($\mathfrak{q}$ is a filter).

$(\Rightarrow)$ For the converse, first observe that $d_A = c_A \wedge f$. Indeed, for $x \in X$ we have:

– $d_A(x) = 0 \Rightarrow x \in A \Rightarrow c_A(x) = 0$ and $f(x) \in \{0, -1\}$, whence $(c_A \wedge f)(x) = 0$;

– $d_A(x) = 1 \Rightarrow x \notin A \Rightarrow f(x) = 1 \Rightarrow (c_A \wedge f)(x) = 1$.

Now assume $A \notin \mathfrak{p}$; then, $\sigma(d_A) = 1$. Since σ is a lattice homomorphism, $d_A = c_A \wedge f$ yields $1 = \sigma(d_A) = \sigma(c_A) \wedge \sigma(f)$; since $\sigma(c_A) \in \{0, -1\}$, this equality entails $\sigma(f) = 1$, i.e., $f \notin \mathfrak{q}$.

Let x_0 be the unique point in $\bigcap \mathfrak{p}$; then:

Claim 3. — $ev_{x_0} = \sigma$.

Proof of Claim 3. — By Claim 2, for $f \in \mathrm{Sp}(X)$ we have:

$$f \in \mathfrak{q} \Leftrightarrow f^{-1}[0, -1] \in \mathfrak{p} \Leftrightarrow x_0 \in f^{-1}[0, -1] \Leftrightarrow f(x_0) \in \{0, -1\}.$$

Using this equivalence, we argue by cases according to the values of $\sigma(f)$:

– $\sigma(f) = 0 \Rightarrow f \in \mathfrak{q}$ and $-f \in \mathfrak{q} \Rightarrow f(x_0), -f(x_0) \in \{0, -1\} \Rightarrow f(x_0) = 0$.

– $\sigma(f) = -1 \Rightarrow \sigma(-f) = 1 \Rightarrow -f \notin \mathfrak{q} \Rightarrow -f(x_0) = 1 \Rightarrow f(x_0) = -1$.

– $\sigma(f) = 1 \Rightarrow f \notin \mathfrak{q} \Rightarrow f(x_0) = 1$.

This completes the proof of Claim 3 and, hence, of Theorem 3.5. □

Corollary 3.6. — *For a hereditarily normal spectral space X, $\mathrm{ev} : X \longrightarrow X_{\mathbf{Sp}(X)}$ is a homeomorphism of X_{con} onto $(X_{\mathbf{Sp}(X)})_{\mathrm{con}}$.*

Proof. — Immediate from Proposition 3.4 and Theorem 3.5, since ev is a continuous bijection between the compact Hausdorff spaces X_{con} and $(X_{\mathbf{Sp}(X)})_{\mathrm{con}}$. □

To prove that ev is an isomorphism between X and $X_{\mathbf{Sp}(X)}$ in the category of spectral spaces we show:

Proposition 3.7. — *Let X be a hereditarily normal spectral space. Then, $\mathrm{ev}^{-1} : X_{\mathbf{Sp}(X)} \longrightarrow X$ is a spectral map. Hence, ev is a homeomorphism and, therefore, an isomorphism between X and $X_{\mathbf{Sp}(X)}$ in the category of spectral spaces.*

Proof. — By the characterization of spectral maps mentioned in 1.1.B(3.ii) and the preceding Corollary 3.6, to show that ev^{-1} is spectral it only remains to prove that, for $\sigma_1, \sigma_2 \in X_{\mathbf{Sp}(X)}$,

$$\sigma_1 \rightsquigarrow_{X_{\mathbf{Sp}(X)}} \sigma_2 \Rightarrow \mathrm{ev}^{-1}(\sigma_1) \underset{X}{\rightsquigarrow} \mathrm{ev}^{-1}(\sigma_2).$$

Since $\mathrm{ev}^{-1}(\sigma_i)$ is the unique $x_i \in X$ such that $\mathrm{ev}_{x_i} = \sigma_i$, this is equivalent to

$$\mathrm{ev}_{x_1} \rightsquigarrow_{X_{\mathbf{Sp}(X)}} \mathrm{ev}_{x_2} \Rightarrow x_1 \underset{X}{\rightsquigarrow} x_2.$$

Assume $x_1 \underset{X}{\not\rightsquigarrow} x_2$, i.e., $x_2 \notin \overline{\{x_1\}}$. Then, there is a quasi-compact open $U \subseteq X$ such that $x_2 \in U$ and $x_1 \notin U$. Define $f : X \longrightarrow \mathbf{3}$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}$$

Since $f^{-1}[1] = U, f^{-1}[-1] = \emptyset$ are quasi-compact open, $f \in \mathrm{Sp}(X)$ and, clearly, $f(x_2) = 1$, $f(x_1) = 0$, which shows $f \in (\mathrm{ev}_{x_2})^{-1}[1] \setminus (\mathrm{ev}_{x_1})^{-1}[1]$. Hence, $(\mathrm{ev}_{x_2})^{-1}[1] \not\subseteq (\mathrm{ev}_{x_1})^{-1}[1]$. From the characterization of specialization in 1.1.A(3) it follows that $\mathrm{ev}_{x_1} \not\rightsquigarrow_{X_{\mathbf{Sp}(X)}} \mathrm{ev}_{x_2}$, as required. The remaining statement is now immediate. $\square$

4. The spectral hull of a real semigroup. Idempotency

We now take on a reverse tack, consisting in applying the construction of the spectral real semigroup $\mathbf{Sp}(X)$ to the case where X is the character space X_G of a given real semigroup G. The result will be a real semigroup $\mathrm{Sp}(G)$ extending G and having the functorial properties of a hull. This *spectral hull* turns out to be idempotent, i.e., its iteration does not produce a larger RS.

Definition and Notation 4.1. — *Let G be a RS and let X_G be its character space. By [14], Prop. 6.4.1, p. 114, X_G is a hereditarily normal spectral space.*

(*i*) *We define* $\mathrm{Sp}(G)$ *to be the real semigroup* $\mathbf{Sp}(X_G)$ (*see 1.8*).

(*ii*) *We denote by η_G the map of G into $\mathbf{3}^{X_G}$ defined by evaluation at elements $g \in G$:*

$$\eta_G(g) := \mathrm{ev}_g : X_G \longrightarrow \mathbf{3}, \textit{ where, for } \sigma \in X_G, \mathrm{ev}_g(\sigma) := \sigma(g).$$

The next Proposition states some elementary properties of the map η_G.

Proposition 4.2. — *Let G be a RS.*

(i) For all $g \in G, \mathrm{ev}_g$ is a spectral map, i.e., $\eta_G(g) \in \mathrm{Sp}(G)$.

(ii) η_G is a real semigroup homomorphism into $\mathrm{Sp}(G)$.

(iii) η_G is injective.

Proof. — (i) Same argument as for item (2) in Proposition 3.4.

(ii) This is straightforward checking using the fact that the constants, product and representation in $\mathrm{Sp}(G)$ are pointwise defined. Details are left to the reader.

(iii) By the definition of η_G we must show

$$\mathrm{ev}_{g_1} = \mathrm{ev}_{g_2} \Rightarrow g_1 = g_2 \qquad (g_1, g_2 \in G),$$

or, equivalently,

$$g_1 \neq g_2 \Rightarrow \exists \sigma \in X_G(\sigma(g_1) \neq \sigma(g_1)).$$

This is precisely the separation theorem 4.4(3) in [8], pp. 116-117.

Note. 4.2(iii) is also a consequence of the more general Corollary 5.10 below.

Definition and Notation 4.3. — *(a) The map $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ – more precisely, the pair $(\mathrm{Sp}(G), \eta_G)$ – will be called the* **spectral hull** *of G. This name is justified by Theorem 5.3(ii).*

(b) Any RS-homomorphism $f : G \longrightarrow H$ gives rise, by composition, to a dual map, $f^* : X_H \longrightarrow X_G$, given by: for $\gamma \in X_H$,

$$f^*(\gamma) := \gamma \circ f : G \longrightarrow \mathbf{3}.$$

Clearly, $f^*(\gamma) \in X_G$.

Fact 4.4. — *$f^* : X_H \longrightarrow X_G$ is a spectral map.*

Proof. — It suffices to check that, for all $g \in G$, $f^{*-1}[\llbracket g = 1 \rrbracket]$ is quasi-compact open in X_H. For $\gamma \in X_H$ we have,

$$\begin{aligned}\gamma \in f^{*-1}[\llbracket g = 1 \rrbracket] &\Leftrightarrow f^*(\gamma) \in \llbracket g = 1 \rrbracket \Leftrightarrow \gamma \circ f \in \llbracket g = 1 \rrbracket \Leftrightarrow (\gamma \circ f)(g) = 1 \\ &\Leftrightarrow \gamma(f(g)) = 1 \Leftrightarrow \gamma \in \llbracket f(g) = 1 \rrbracket,\end{aligned}$$

i.e., $f^{*-1}[\llbracket g = 1 \rrbracket] = \llbracket f(g) = 1 \rrbracket$, as needed. $\square$

(c) In the preceding setting we define, as in 3.1, a map $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ again by composition: for $g \in \mathrm{Sp}(G)$, $\mathrm{Sp}(f)(g) := g \circ f^*$. By 4.4, $g \circ f^* \in \mathrm{Sp}(H)$.

(d) Fact 4.4 shows that the correspondence $(G \longmapsto X_G, f \longmapsto f^*)$, *for* $G \models RS$ *and RS-morphisms* $f : G \longrightarrow H$ *defines a contravariant functor – written* $*$ *– from the category of real semigroups with RS-morphisms into that of spectral spaces with spectral maps.*

Next we show that the operator **Sp** is idempotent.

THEOREM 4.5. — (Idempotency of **Sp**) *Let* X *be a hereditarily normal spectral space. Then,* $\eta_{\mathbf{Sp}(X)} : \mathbf{Sp}(X) \longrightarrow \mathrm{Sp}(\mathbf{Sp}(X))$ *is an isomorphism of real semigroups.*

Proof. — Applying Proposition 4.2 with $G = \mathbf{Sp}(X)$, it only remains to prove:

(a) $\eta_{\mathbf{Sp}(X)}$ is surjective, and (b) $\eta^{-1}_{\mathbf{Sp}(X)}$ is a RS-homomorphism.

<u>*Proof of* (a)</u>. Since Sp is a contravariant functor (3.1, 3.2), $\mathrm{ev}^{-1} \circ \mathrm{ev} = \mathrm{id}_X$, and $\mathrm{ev} \circ \mathrm{ev}^{-1} = \mathrm{id}_{\mathbf{Sp}(X)}$ imply $\mathrm{Sp}(\mathrm{ev}^{-1} \circ \mathrm{ev}) = \mathrm{Sp}(\mathrm{ev}) \circ \mathrm{Sp}(\mathrm{ev}^{-1}) = \mathrm{id}_{\mathbf{Sp}(X)}$ and $\mathrm{Sp}(\mathrm{ev} \circ \mathrm{ev}^{-1}) = \mathrm{Sp}(\mathrm{ev}^{-1}) \circ \mathrm{Sp}(\mathrm{ev}) = \mathrm{id}_{\mathrm{Sp}(\mathbf{Sp}(X))}$. Hence, for $f \in \mathrm{Sp}(\mathbf{Sp}(X))$, $f = \mathrm{Sp}(\mathrm{ev}^{-1})(\mathrm{Sp}(\mathrm{ev})(f)) = \mathrm{Sp}(\mathrm{ev}^{-1})(f \circ \mathrm{ev})$, cf. 3.1. Then, it suffices to show:

(*) $\eta_{\mathbf{Sp}(X)} = \mathrm{Sp}(\mathrm{ev}^{-1})$.

<u>*Proof of* $(*)$</u>. We must show, for $b \in \mathrm{Sp}(X)$:

$$\mathrm{Sp}(\mathrm{ev}^{-1})(b) = b \circ \mathrm{ev}^{-1} = \eta_{\mathbf{Sp}(X)}(b) = ev_b,$$

i.e., $(b \circ \mathrm{ev}^{-1})(\gamma) = ev_b(\gamma) = \gamma(b)$, for all $\gamma \in X_{\mathbf{Sp}(X)}$. By definition, $\mathrm{ev}^{-1}(\gamma) =$ the unique $x \in X$ such that $\mathrm{ev}(x) = ev_x = \gamma$. Then, $\gamma(b) = ev_x(b) = b(x)$, and $b(x) = b(\mathrm{ev}^{-1}(\gamma))$, i.e., $(b \circ \mathrm{ev}^{-1})(\gamma) = \gamma(b)$, as required.

<u>*Proof of* (b)</u>. In the proof of (a) we noted that

$$\mathrm{Sp}(\mathrm{ev})^{-1} = \mathrm{Sp}(\mathrm{ev}^{-1}) \text{ and } \mathrm{Sp}(\mathrm{ev}^{-1})^{-1} = \mathrm{Sp}(\mathrm{ev}).$$

This, together with (*), gives $\eta^{-1}_{\mathbf{Sp}(X)} = \mathrm{Sp}(\mathrm{ev}^{-1})^{-1} = \mathrm{Sp}(\mathrm{ev})$. Since $\mathrm{Sp}(\varphi)$ is a RS-homomorphism for any spectral map φ (3.2), Proposition 3.4 yields that $\eta^{-1}_{\mathbf{Sp}(X)}$ is a RS-homomorphism, proving (b) and Theorem 4.5. □

Theorem 4.5 can be restated as follows:

COROLLARY 4.6. — *Let* G *be a* spectral *RS* (1.11)*. Then, the map* $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ *is an isomorphism. In other words, every spectral RS is canonically isomorphic to its spectral hull* (*the converse is obviously true*)*.*

5. An anti-equivalence of categories

Our main result in this section is the anti-equivalence of the categories **HNSS** and **SRS** (Theorem 5.4). The commutativity of diagrams required for this result are proven in 5.1 and 5.3. These results also have further important consequences, such as:

(i) The duality of the functors * (4.3(d)) and **Sp** (3.1), and

(ii) Uniqueness of the extension $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ (4.3) of any RS-homomorphism $f : G \longrightarrow H$ $(G, H \models \mathrm{RS})$; in particular, unique extension of any RS-character of G to $\mathrm{Sp}(G)$.

Example 5.6 gives a simple illustration of how a RS sits inside its spectral hull. Finally, Theorem 5.8 is an analog to Theorem 5.2 in [6], pp. 75-77 (and to Theorem III.3.5 in [10]) for spectral real semigroups.

Proposition 5.1. — *Let X, Y be hereditarily normal spectral spaces and let $\varphi : X \longrightarrow Y$ be a spectral map. Let $\mathrm{ev}_X : X \longrightarrow X_{\mathrm{Sp}(X)}$ and $\mathrm{ev}_Y : Y \longrightarrow X_{\mathrm{Sp}(Y)}$ denote the bijections given by 3.4 and 3.5. The following diagram is commutative:*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\mathrm{ev}_X"'] & Y \arrow[d, "\mathrm{ev}_Y"] \\ X_{\mathrm{Sp}(X)} \arrow[r, "\mathrm{Sp}(\varphi)^*"'] & X_{\mathrm{Sp}(Y)} \end{tikzcd}$$

Proof. — Recall that, for $x \in X, \mathrm{ev}_X(x) = ev_x : \mathrm{Sp}(X) \longrightarrow \mathbf{3}$; likewise, $\mathrm{ev}_Y(\varphi(x)) = ev_{\varphi(x)} : \mathrm{Sp}(Y) \longrightarrow \mathbf{3}$. By the definition of * (4.3(b)),

$$(\mathrm{Sp}(\varphi)^* \circ \mathrm{ev}_X)(x) = \mathrm{Sp}(\varphi)^*(ev_x) = ev_x \circ \mathrm{Sp}(\varphi),$$

and

$$(\mathrm{ev}_Y \circ \varphi)(x) = \mathrm{ev}_Y(\varphi(x)) = ev_{\varphi(x)}.$$

Thus, we must check that $ev_{\varphi(x)} = ev_x \circ \mathrm{Sp}(\varphi)$. Let $b \in \mathrm{Sp}(Y)$; then, $\mathrm{Sp}(\varphi)(b) = b \circ \varphi$, and we get:

$$(ev_x \circ \mathrm{Sp}(\varphi))(b) = ev_x(\mathrm{Sp}(\varphi)(b)) = ev_x(b \circ \varphi) = (b \circ \varphi)(x) = b(\varphi(x))$$

$$= ev_{\varphi(x)}(b),$$

as required. □

Notation. — Next we fix a $G \models \mathrm{RS}$ and set $X = X_G$. We rebaptize $\mathrm{ev}_G : X_G \longrightarrow X_{\mathrm{Sp}(G)}$ the map ev_{X_G} considered above, i.e., $\mathrm{ev}_G(\sigma) := ev_\sigma$, for $\sigma \in X_G$. Then,

Fact 5.2. — *With notation as above, the following identities hold:*

(*i*) $\mathrm{ev}_G^{-1} = \eta_G^*$; (*ii*) $\mathrm{Sp}(\eta_G^*) = \eta_{\mathrm{Sp}(G)}$. *Hence,*

(*iii*) η_G^* *is bijective.*

Proof. — (i) Fix $\gamma \in X_{\mathrm{Sp}(G)}$. Since $\mathrm{ev}_G^{-1}(\gamma) =$ the unique $\sigma \in X_G$ so that $\gamma = \mathrm{ev}_G(\sigma) = ev_\sigma$, and $\eta_G^*(\gamma) = \gamma \circ \eta_G$, the identity to be proved boils down to showing that $ev_\sigma \circ \eta_G = \sigma$, for $\sigma \in X_G$. Let $g \in G$; since $\eta_G(g) = ev_g$, we get $ev_\sigma(\eta_G(g)) = \eta_G(g)(\sigma) = ev_g(\sigma) = \sigma(g)$, as desired.

(ii) From (*) in the proof of idempotency (4.5) with $X = X_G$, we have

$$\eta_{\mathrm{Sp}(G)} = \eta_{\mathbf{Sp}(X_G)} = \mathrm{Sp}(\mathrm{ev}_G^{-1}),$$

and the result follows at once from (i).

Item (iii) is clear from 3.4, 3.5 and item (i). □

Our next result is an analog of Thm. 4.17 of [6] (and of Theorem III.3.2 of [10]), a result of crucial importance:

Theorem 5.3. — (*i*) *Let $f : G \longrightarrow H$ be a homomorphism of real semigroups. Then* $\mathrm{Sp}(f)$ (*defined in* 4.3(*c*)) *is the unique RS-homomorphism* $F : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ *making the following diagram commute:*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\eta_G"'] \arrow[dr, phantom, "{[D]}"] & H \arrow[d, "\eta_H"] \\ \mathrm{Sp}(G) \arrow[r, "F"'] & \mathrm{Sp}(H) \end{tikzcd}$$

(*ii*) *Let G be a RS. Then* $\mathrm{Sp}(G)$ *is a* hull *for G in the category* **SRS** *of spectral real semigroups. That is, every RS-morphism $f : G \longrightarrow$* $\mathbf{Sp}(X)$, *X a hereditarily normal spectral space, factors* uniquely *through* $\mathrm{Sp}(G)$, *i.e., there is a unique RS-morphism $h : \mathrm{Sp}(G) \longrightarrow$* $\mathbf{Sp}(X)$ *making the following diagram commute:*

$$\begin{tikzcd} G \arrow[rr, "\eta_G"] \arrow[dr, "f"'] & & \mathrm{Sp}(G) \arrow[dl, "h"] \\ & \mathbf{Sp}(X) & \end{tikzcd}$$

Proof. — (i) We first show that with $F = \mathrm{Sp}(f)$ diagram [D] commutes, i.e., $\eta_H(f(g)) = \mathrm{Sp}(f)(\eta_G(g))$ for all $g \in G$. By the definition of η_G, η_H, and with f^* defined in 4.3(b) , this amounts to $ev_{f(g)} = ev_g \circ f^*$. For $\gamma \in X_H$ we have $ev_{f(g)}(\gamma) = \gamma(f(g))$, and $(ev_g \circ f^*)(\gamma) = ev_g(f^*(\gamma)) = ev_g(\gamma \circ f) = \gamma(f(g))$.

For uniqueness, let $F_1, F_2 : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ be RS-homomorphisms making diagram [D] commute. Applying the functor * to this square we get a commutative diagram

$$\begin{tikzcd} X_G \arrow[rr, "f^*"] & & X_H \\ X_{\mathrm{Sp}(G)} \arrow[u, "\eta^*_G"] & & X_{\mathrm{Sp}(H)} \arrow[u, "\eta^*_H"'] \arrow[ll, shift right, "F^*_1"'] \arrow[ll, shift left, "F^*_2"] \end{tikzcd}$$

whence $\eta^*_G \circ F^*_1 = \eta^*_G \circ F^*_2 (= f^* \circ \eta^*_H)$. Since η^*_G is injective (5.2(iii)), $F^*_1 = F^*_2$, and we show this entails $F_1 = F_2$.

In fact, if $F_1(g) \neq F_2(g)$ for some $g \in \mathrm{Sp}(G)$, since $X_{\mathrm{Sp}(H)}$ separates points in $\mathrm{Sp}(H)$, there is $\gamma \in X_{\mathrm{Sp}(H)}$ so that $(\gamma \circ F_1)(g) \neq (\gamma \circ F_2)(g)$, i.e., $\gamma \circ F_1 \neq \gamma \circ F_2$. By definition $F^*_i = \gamma \circ F_i$, so we get $F^*_1(\gamma) \neq F^*_2(\gamma)$, contradiction.

(ii) Use the commutative square [D] of (i) with $H = \mathbf{Sp}(X)$ and $f : G \longrightarrow \mathbf{Sp}(X)$ the given map. By the idempotency theorem 4.5, $\eta_H : H \longrightarrow \mathrm{Sp}(H)$ is an isomorphism of real semigroups. Setting $h := \eta_H^{-1} \circ \mathrm{Sp}(f)$ proves the commutativity of the triangle in (ii). Uniqueness is clear from (i). $\square$

Putting together some of the preceding results we obtain the anti-equivalence of the categories **HNSS** and **SRS**. This is expressed in rather compact form, using category-theoretic language, by the following:

Theorem 5.4. — (Anti-equivalence theorem) *The functor* $\mathbf{Sp} : \mathbf{HNSS} \longrightarrow \mathbf{SRS}$ *assigning to each hereditarily normal spectral space X the real semigroup* $\mathbf{Sp}(X)$ *is an anti-equivalence of categories. Its quasi-inverse is the functor* $\mathbf{ARS} : \mathbf{SRS} \longrightarrow \mathbf{HNSS}$ *assigning to each $G \in \mathbf{SRS}$ its associated character space X_G.*[10] *The natural transformations establishing this anti-equivalence are as follows:*

(1) *The isomorphism* $\mathrm{Id}_{\mathbf{HNSS}} \longmapsto \mathbf{ARS} \circ \mathbf{Sp}$ *is the natural transformation that sends $X \in \mathbf{HNSS}$ to the homeomorphism* $\mathrm{ev} : X \longrightarrow \mathbf{Sp}(X)$.

(10) I.e., **ARS** is the restriction of the functor * (4.3(d)) to the subcategory **SRS**.

(2) *The isomorphism* $\mathrm{Id}_{\mathbf{SRS}} \longmapsto \mathbf{Sp} \circ \mathbf{ARS}$ *is the natural transformation that sends a spectral RS, G, to the isomorphism* $\eta_G : G \longrightarrow \mathrm{Sp}(G)$.

Proof. — (1) Given a spectral map $\varphi : X \longrightarrow Y$, with $X, Y \in \mathbf{HNSS}$, (1) follows from the commutativity of the diagram in Proposition 5.1 and the fact that the maps $\mathrm{ev}_X : X \longrightarrow X_{\mathrm{Sp}(X)}$ and $\mathrm{ev}_Y : Y \longrightarrow X_{\mathrm{Sp}(Y)}$ are homeomorphisms of spectral spaces (3.7(2)).

(2) Given a RS-morphism $f : G \longrightarrow H$, where $G, H \in \mathbf{SRS}$, (2) follows from the commutativity of the diagram in Theorem 5.3(i) together with the fact that the canonical embeddings η_G, η_H are isomorphisms (4.6). $\square$

Corollary 5.5. — *Let G be a RS. Then every $\sigma \in X_G$ extends* uniquely *to a RS-character of* $\mathrm{Sp}(G)$.

Proof. — Follows from 5.3(ii) by taking $X = \mathbf{1}$ (= the singleton spectral space) and observing that $\mathrm{Sp}(\mathbf{1}) = \mathbf{3}$ (see proof of Corollary 2.2).

Explicitly, the extension $\widehat{\sigma} : \mathrm{Sp}(G) \longrightarrow \mathbf{3}$ of a RS-character $\sigma \in X_G$ is defined by evaluation at σ: for $f \in \mathrm{Sp}(G)$, $\widehat{\sigma}(f) := f(\sigma)$. The reader can readily check that $\widehat{\sigma} \in X_{\mathrm{Sp}(G)}$ and $\widehat{\sigma} \circ \eta_G = \sigma$ (i.e., $\widehat{\sigma}\lceil G = \sigma$ with G canonically embedded into $\mathrm{Sp}(G)$ via η_G). $\square$

Remark. — The uniqueness statements in Theorem 5.3 and Corollary 5.5 indicate that a real semigroup "generates" its spectral hull. In Theorem 6.2 we will show that, in fact, it generates its spectral hull as a lattice.

Example 5.6. — Here is a simple example illustrating the way in which a real semigroup sits inside its spectral hull. We compute the spectral hull of the "free" fan on one generator (Example V.4.2(A) in [10]):

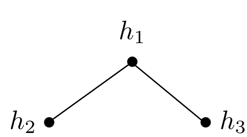
$$F = \{1, 0, -1, x, -x, x^2, -x^2\},$$

with representation given by: for $a, b \in F$,

$$D_F(a, b) := a \cdot \mathrm{Id}(F) \cup b \cdot \mathrm{Id}(F) \cup \{y \in F \mid ya = -yb \text{ and } y = a^2 y\}.$$

– Firstly, F is represented inside $\mathrm{Sp}(F)$ by the seven elements of the form $ev_a, a \in F$.

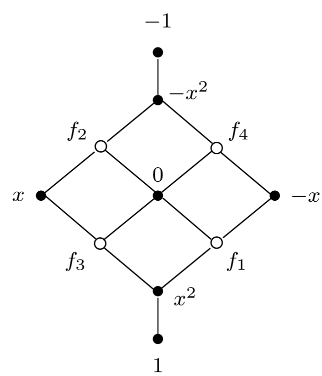
– Besides these, $\mathrm{Sp}(F)$ contains four other elements. Indeed, X_F has the shape:



where $h_1(x) = 0, h_2(x) = 1, h_3(x) = -1$ (the order being specialization). Since the constructible topology is discrete, the spectral characters are the maps of X_F into $\mathbf{3}_{\rm sp}$ that preserve the specialization order. Note, further, that if a spectral character sends h_2 or h_3 to 0, then it must also send h_1 to 0. Direct verification shows that $\mathrm{Sp}(F)(= \mathrm{Sp}(X_F))$ contains the following additional maps:

$$f_1 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto 0 \\ h_3 \mapsto 1, \end{cases} \quad f_2 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto 0 \\ h_3 \mapsto -1, \end{cases} \quad f_3 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto 1 \\ h_3 \mapsto 0, \end{cases} \quad f_4 : \begin{cases} h_1 \mapsto 0 \\ h_2 \mapsto -1 \\ h_3 \mapsto 0, \end{cases}$$

and looks as follows:



Our last result in this section is an analog to Theorem 5.2 in [6], pp. 75–77 (and to Theorem III.3.5 in [10]). It gives, in the context of spectral real semigroups, several characterizations of the surjectivity of the map f^*, dual to a given RS-homomorphism f (4.3(b)), showing, in particular, that this condition is equivalent to the injectivity of $\mathrm{Sp}(f)$.

We shall need the following notion (a "poor man's" version of the Witt equivalence of quadratic forms.)

Definition and Notation 5.7. — *Let G, H, be real semigroups.*

(i) Given forms $\varphi = \langle a_1, \dots, a_n \rangle, \psi = \langle b_1, \dots, b_m \rangle$ with entries in G (of arbitrary*, possibly different, dimension) we set:*

$$\varphi \cong_G \psi : \Leftrightarrow \textit{For all } h \in X_G, \sum_{i=1}^{n} h(a_i) = \sum_{j=1}^{m} h(b_j) \ (\textit{sum in } \mathbb{Z}).$$

(ii) Given a form $\varphi = \langle a_1, \dots, a_n \rangle$ over G and a map $f : G \longrightarrow H, f * \varphi$ denotes the form $\langle f(a_1), \dots, f(a_n) \rangle$.

(iii) A RS-homomorphism $f : G \longrightarrow H$ is a **complete embedding** *if for every pair of forms φ, ψ, over G,*

$$\varphi \cong_G \psi \Leftrightarrow f * \varphi \cong_H f * \psi.$$

Complete embeddings are automatically injective.

THEOREM 5.8. — *Let G, H be real semigroups, and let $f : G \longrightarrow H$ be a RS-morphism. With f^* denoting the dual of f and* $\mathrm{Sp}(f)$ *its spectral extension* $(4.3(b), (c))$*, the following are equivalent:*

(1) *f^* is surjective.*

(2) $\mathrm{Im}(f^*)$ *is dense in $(X_G)_{\mathrm{con}}$ (the constructible topology of X_G).*

(3) $\mathrm{Sp}(f)$ *is injective.*

(4) *For every Pfister form φ over G*[11] *and every $a \in G$, $f(a) \in D_H(f * \varphi) \Rightarrow a \in D_G(\varphi)$.*

(5) *f is a complete embedding.*

Proof. — Recall that f^* is a spectral map (4.4). (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). By Cor. 6.0.2 of [11], $\mathrm{Im}(f^*)$ is a proconstructible subset of X_G, i.e., closed in $(X_G)_{\mathrm{con}}$. This, together with (2), at once implies (1).

(1) $\Rightarrow$ (3). Assume there are $g_1, g_2 \in \mathrm{Sp}(X_G)$ so that $g_1 \neq g_2$ but $\mathrm{Sp}(f)(g_1) = \mathrm{Sp}(f)(g_2)$, i.e., $g_1 \circ f^* = g_2 \circ f^*$ and let $\sigma \in X_G$ be such that $g_1(\sigma) \neq g_2(\sigma)$. Since f^* is assumed surjective, there is $\gamma \in X_H$ such that $f^*(\gamma) = \sigma$. Then,

$$(g_1 \circ f^*)(\gamma) = g_1(f^*(\gamma)) = g_1(\sigma) \neq g_2(\sigma) = (g_2 \circ f^*)(\gamma),$$

contradiction.

(3) $\Rightarrow$ (1). Since "surjective" = "epic" holds in the category of spectral spaces with spectral maps, cf. [11], Cor. 11.3.5(ii), it suffices to prove that f^* is epic, i.e., right-cancellable.

Recall that, if $\rho : X_G \longrightarrow X$ is a spectral map into a spectral space X, then $\mathrm{Sp}(\rho \circ f^*) = \mathrm{Sp}(f^*) \circ \mathrm{Sp}(\rho) = \mathrm{Sp}(f) \circ \mathrm{Sp}(\rho)$, since $\mathrm{Sp}(f) = \mathrm{Sp}(f^*)$, cf. 3.1 and 4.3(c).

Assume $\rho_1, \rho_2 : X_G \longrightarrow X$ are spectral maps into a spectral space X such that $\rho_1 \circ f^* = \rho_2 \circ f^*$. By the preceding paragraph, $\mathrm{Sp}(f) \circ \mathrm{Sp}(\rho_1) = \mathrm{Sp}(f) \circ \mathrm{Sp}(\rho_2)$. Since $\mathrm{Sp}(f)$ is injective (assumption (3)), $\mathrm{Sp}(\rho_1) = \mathrm{Sp}(\rho_2)$, and we show this entails $\rho_1 = \rho_2$. Otherwise, $\rho_1(\sigma) \neq \rho_2(\sigma)$ for some $\sigma \in X_G$.

(11) I.e., a form of the type $\bigotimes_{i=1}^n \langle 1, a_i \rangle$, with $a_1, \ldots, a_n \in G$.

Since X is T_0, there is a quasi-compact open $U \subseteq X$ so that, say, $\rho_1(\sigma) \in U$ and $\rho_2(\sigma) \notin U$ (or the other way around). Let $h : X \longrightarrow \mathbf{3}$ be given by $h\lceil U = 1$ and $h\lceil(X \setminus U) = 0$; h is spectral, i.e., $h \in \mathrm{Sp}(X)$, and $h(\rho_1(\sigma)) = 1 \neq 0 = h(\rho_2(\sigma))$, i.e., $\mathrm{Sp}(\rho_1)(h) \neq \mathrm{Sp}(\rho_2)(h)$, contradiction.

(1) $\Rightarrow$ (4). For this proof we will need:

Proposition. — (Separation Lemma; [8], Thm. 4.4(1), p. 116, and [10], Cor. I.4.7) *Let G be a RS and let φ be either a binary form or a Pfister form, with entries in G. Then, for $a \in G$,*

$$a \in D_G(\varphi) \Leftrightarrow \textit{ For all } \sigma \in X_G, \sigma(a) \in D_{\mathbf{3}}(\sigma * \varphi).$$

Let φ be a Pfister form with entries in G, $a \in G$, and assume $f(a) \in D_H(f * \varphi)$. The Proposition tells us:

(i) The assumption $f(a) \in D_H(f*\varphi)$ is equivalent to $\forall \sigma \in X_H(\sigma(f(a)) \in D_{\mathbf{3}}((\sigma \circ f) * \varphi))$, and

(ii) The conclusion $a \in D_G(\varphi)$ is equivalent to $\forall \gamma \in X_G(\gamma(a) \in D_{\mathbf{3}}(\gamma * \varphi))$.

Since f^* is surjective, for every $\gamma \in X_G$ there is $\sigma \in X_H$ such that $\gamma = f^*(\sigma) = \sigma \circ f$. Then, (ii) follows at once from (i), proving (4).

(1) $\Rightarrow$ (5). Let $\varphi = \langle a_1, \ldots, a_n \rangle, \psi = \langle b_1, \ldots, b_m \rangle$ be arbitrary forms over G, and assume $f * \varphi \cong_H f * \psi$, i.e.,

$$(+) \qquad \textstyle\sum_{i=1}^n \sigma(f(a_i)) = \sum_{j=1}^m \sigma(f(b_j)) \text{ for all } \sigma \in X_H.$$

Given $\gamma \in X_G$, pick $\sigma \in X_H$ so that $\gamma = f^*(\sigma) = \sigma \circ f$. Then, (+) yields $\sum_{i=1}^n \gamma(a_i) = \sum_{j=1}^m \gamma(b_j)$ for all $\gamma \in X_G$, i.e., $\varphi \cong_G \psi$.

The proofs of (4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (2) rest on:

Fact. — *Let G be a RS, and $a_1, \ldots, a_n, b \in G$.*
Let $V = \bigcap_{i=1}^n [\![a_i = 1]\!] \cap [\![b = 0]\!] \subseteq X_G$.
Let φ and ψ respectively denote the Pfister forms $\langle\langle a_1, \ldots, a_n, -b^2 \rangle\rangle$ and $2^{n-1}\langle\langle -1, b^2 \rangle\rangle$, if $n \geqslant 1$, and $\psi = \langle 1, -b^2 \rangle$, $\varphi = 2 \cdot \psi = \psi \oplus \psi$, if $n = 0$. Set $d = \prod_{i=1}^n a_i^2$, if $n \geqslant 1$, and $d = 1$, if $n = 0$. The following hold:

i) If $V \neq \emptyset$, then $-d \notin D_G(\varphi)$ and $d\varphi \ncong_G \psi$.

ii) If $V = \emptyset$, then $-d \in D_G(\varphi)$ and $d\varphi \cong_G \psi$.

Note. In case $V = \bigcap_{i=1}^{n} [\![a_i = 1]\!] (n \geqslant 1)$, just omit the entries $-b^2$ in φ and b^2 in ψ.

Proof of Fact. — By the Proposition above (Separation Lemma), condition $-d \in D_G(\varphi)$ is equivalent to

(†) $\forall \sigma \in X_G \big[\sigma(a_i) \in \{0,1\}$ for $i = 1, \ldots, n$, and $\sigma(-b^2) = 0 \Rightarrow \sigma(-d) = 0\big]$.

Any $\sigma \in V$ verifies $\sigma(a_i) = 1, \sigma(b) = 0$, whence $\sigma(d) = \prod_{i=1}^{n} \sigma(a_i)^2 = 1$, for $n \geqslant 1$ (and, obviously, also for $n = 0$). Thus, (†) fails at every $\sigma \in V$. Hence, $V \neq \emptyset \Rightarrow -d \notin D_G(\varphi)$.

For the second assertion in (i), note that if $n \geqslant 1$, since $\mathrm{sgn}_\sigma(\langle\langle -1, x \rangle\rangle) = \mathrm{sgn}_\sigma(\langle 1, -1, x, -x \rangle) = 0$ holds for all $x \in G$, we have $\mathrm{sgn}_\sigma(\psi) = 0$, for all $\sigma \in X_G$. Next, observe that if $\sigma \in V$, then $\mathrm{sgn}_\sigma(d\varphi) = \sigma(d) \cdot (1 - \sigma(b^2)) \cdot \prod_{i=1}^{n} (1 + \sigma(a_i)) = 2^n$. Hence, $\mathrm{sgn}_\sigma(d\varphi) \neq \mathrm{sgn}_\sigma(\psi)$ whenever $n \geqslant 1$. If $n = 0$, we have $\mathrm{sgn}_\sigma(\psi) = 1$ and $\mathrm{sgn}_\sigma(\varphi) = 2$.

(ii) If $V = \emptyset$, then for any $\sigma \in X_G$ either $\sigma(b) \neq 0$ (i.e., $\sigma(b^2) = 1$) or $\sigma(a_i) \neq 1$ for some $i \in \{1, \ldots, n\}$. If $\sigma(b) \neq 0$ or $\sigma(a_i) = -1$ for some i, (†) holds because its antecedent fails. If $\sigma(b) = 0$ and $\sigma(a_i) = 0$ for some i, then $\sigma(-d) = 0$, and (†) holds. Hence, (†) holds at every $\sigma \in X_G$, which entails $-d \in D_G(\varphi)$.

To prove the last assertion in (ii), it suffices to show that $\mathrm{sgn}_\sigma(d\varphi) = 0$ for all $\sigma \in X_G$. Since $\sigma \notin V$, if $\sigma(b) \neq 0$, the factor $1 - \sigma(b)^2$ is 0, and $\mathrm{sgn}_\sigma(d\varphi) = 0$. Likewise, if $\sigma(a_i) = 0$ for some i, the factor $\sigma(d)$ vanishes, and if $\sigma(a_i) = -1$ for some i, then $1 + \sigma(a_i) = 0$; in either case $\mathrm{sgn}_\sigma(d\varphi) = 0$, as required.

(4) $\Rightarrow$ (2). Assume $\mathrm{Im}(f^*)$ is not dense in $(X_G)_{\mathrm{con}}$. Then, there is a non-empty clopen set U of the form $\bigcap_{i=1}^{n} [\![a_i = 1]\!] \cap [\![b = 0]\!] (a_1, \ldots, a_n, b \in G)$, such that $U \cap \mathrm{Im}(f^*) = \emptyset$, i.e., $f^{*-1}[U] = \emptyset$. Statement (i) of the Fact with $V = U$ yields $-d \notin D_G(\varphi)$, while item (ii) with $V = f^{*-1}[U] = \bigcap_{i=1}^{n} [\![f(a_i) = 1]\!] \cap [\![f(b) = 0]\!] \subseteq X_H$ (and $f * \varphi$) gives $f(-d) \in D_H(f * \varphi)$, contradicting assumption (4).

(5) $\Rightarrow$ (2). Similar to the preceding proof: assuming (2) fails, the last assertion in item (i) of the Fact with $V = U$ yields $d\varphi \not\cong_G \psi$, while that of (ii) applied with $V = f^{*-1}[U]$ gives $f * (d\varphi) \cong_G f * \psi$, contradicting (5). This ends the proof of Theorem 5.8. □

The following are noteworthy consequences of Theorem 5.8.

Corollary 5.9. — *Any injective RS-morphism $f : G \longrightarrow H$ of spectral RSs is a complete embedding.*

Proof. — We use the commutative diagram [D] in Theorem 5.3(i) with $F = \mathrm{Sp}(f)$. Since G, H are spectral, the embeddings η_G and η_H are isomorphisms (4.6); hence $\mathrm{Sp}(f)$ is injective. By 5.8, f is a complete embedding. $\square$

Corollary 5.10. — *For any RS, G, the morphism $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ is a complete embedding.*

Proof. — Follows from the equivalence of (1) and (5) in Theorem 5.8, and Fact 5.2(iii). $\square$

6. The distributive lattice structure of spectral real semigroups

In this section we prove two results concerning the (pure) lattice structure of spectral RSs:

– The spectral real semigroups are exactly the real semigroups whose representation partial order is a *distributive* lattice (Theorem 6.6).

– Any real semigroup generates its spectral hull as a lattice (Theorem 6.2).

We start by checking the following simple, but important property of the lattice structure of spectral real semigroups:

Proposition 6.1. — *Let G, H be RSs, and let $f : G \longrightarrow H$ be a RS-homomorphism. Then, the spectral extension $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$ of f (4.3(c)) is a lattice homomorphism.*

Proof. — Recall that $\mathrm{Sp}(f)(g) = g \circ f^*$ for $g \in \mathrm{Sp}(G)$. Hence, for $\gamma \in X_H$ we have $(g \circ f^*)(\gamma) = g(f^*(\gamma)) = g(\gamma \circ f)$, and:

$$\begin{aligned}((g_1 \vee g_2) \circ f^*)(\gamma) &= (g_1 \vee g_2)(\gamma \circ f) = \max\{g_1(\gamma \circ f), g_2(\gamma \circ f)\} \\ &= (g_1 \circ f^*)(\gamma) \vee (g_2 \circ f^*)(\gamma),\end{aligned}$$

proving preservation of joins. Similarly one establishes that of meets. $\square$

Remarks. — (1) Obviously, the constants $1, 0, -1$ of $\mathrm{Sp}(G)$ and $\mathrm{Sp}(H)$ correspond to each other under $\mathrm{Sp}(f)$; hence, $\mathrm{Sp}(f)$ is a homomorphism of bounded lattices.

(2) F. Miraglia has found an example of an injective RS-morphism $f : G \longrightarrow L$ of a RS, G, with values in a spectral RS, L, whose spectral extension, $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow L (= \mathrm{Sp}(L))$ *is not injective.*

We shall now prove that any real semigroup generates its spectral hull as a lattice.[12]

Theorem 6.2. — *Let G be a RS. Then, for every $f \in \mathrm{Sp}(G)$ there is a finite family $\{F_i \mid i \in I\}$ of finite subsets of G so that $f = \bigvee_{i\in I} \bigwedge_{g \in F_i} \eta_G(g)$; i.e., $\mathrm{Sp}(G)$ is generated as a lattice by $\mathrm{Im}(\eta_G)$.*

Proof. — Recall from 1.1.A(3) that the collection of all finite intersections of sets of the form $[\![x = 1]\!]$ with $x \in G$ is a basis for the spectral topology of X_G. To ease notation we write $\widehat{g}$ for $\eta_G(g) = ev_g (g \in G)$.

Let L denote the sublattice of $\mathrm{Sp}(G)$ generated by $\mathrm{Im}(\eta_G)$, and fix $f \in \mathrm{Sp}(G)$. The construction of a representation of f as in the statement depends on the form of the quasi-compact opens $f^{-1}[1], f^{-1}[-1]$. We split the proof into two cases.

<u>Case I</u>. Both $f^{-1}[1]$ and $f^{-1}[-1]$ are basic quasi-compact opens of X_G, i.e.,

$$(*) \qquad f^{-1}[1] = \bigcap_{i=1}^{k} [\![a_i = 1]\!] \quad \text{and} \quad f^{-1}[-1] = \bigcap_{j=1}^{n} [\![b_j = 1]\!],$$

for some $a_1, \ldots, a_k, b_1, \ldots, b_n \in G$.

Since $D^t_G(\cdot,\cdot) \neq \emptyset$ (axiom [RS3b]) for each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$ we pick an element $t_{ij} \in D^t_G(a_i, -b_j)$ and consider the following element of L:

$$(**) \qquad p := \left[\left(\bigvee_{i=1}^{k} \widehat{a_i} \right) \wedge \widehat{0} \right] \vee \left[\left(\bigwedge_{j=1}^{n} \widehat{-b_j} \right) \wedge \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n} \widehat{t_{ij}} \right].$$

Claim. — $p = f$. Hence, $f \in L$.

Proof of Claim. — To abridge we set $r := \left(\bigvee_{i=1}^{k} \widehat{a_i} \right) \wedge \widehat{0}$ and $s := \left(\bigwedge_{j=1}^{n} \widehat{-b_j} \right) \wedge \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n} \widehat{t_{ij}}$ in (**). The proof proceeds by cases according to the values of f. Let $\sigma \in X_G$.

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[12] For a result in a similar spirit concerning the Boolean hull of a reduced special group, see [6], Prop. 4.10(b).

– $\underline{f(\sigma) = 1}$. By (*), $\sigma(a_i) = \widehat{a}_i(\sigma) = 1$ for all $1 \leqslant i \leqslant k$, whence $r(\sigma) = 1$. On the other hand, since $f^{-1}[1]$ and $f^{-1}[-1]$ are disjoint, by (*) there is a $j_0 \in \{1, \ldots, n\}$ so that $\sigma(b_{j_0}) \in \{0, -1\}$. Fix $i \in \{1, \ldots, k\}$. Since $t_{ij_0} \in D^t_G(a_i, -b_{j_0})$ and $-\sigma(b_{j_0}) \in \{0, 1\}$, we have $\sigma(t_{ij_0}) \in D^t_{\mathbf{3}}(\sigma(a_i), -\sigma(b_{j_0})) = D^t_{\mathbf{3}}(1, -\sigma(b_{j_0})) = \{1\}$ (cf. 1.1.A(2)); that is, $\widehat{t_{ij_0}}(\sigma) = \sigma(t_{ij_0}) = 1$ for all i. It follows that $s(\sigma) = 1$, and hence $p(\sigma) = 1$.

– $\underline{f(\sigma) = -1}$. By (*), $\sigma(b_j) = \widehat{b}_j(\sigma) = 1$ for all $1 \leqslant j \leqslant n$, i.e., $\bigwedge_{j=1}^{n} \widehat{-b_j}(\sigma) = -1$. Since the sets in (*) are disjoint, there is a $i_0 \in \{1, \ldots, k\}$ so that $\sigma(a_{i_0}) \in \{0, -1\}$. Fix $j \in \{1, \ldots, n\}$. Since $t_{i_0 j} \in D^t_G(a_{i_0}, -b_j)$ and $-\sigma(b_j) = -1$, we get $\sigma(t_{i_0 j}) \in D^t_{\mathbf{3}}(\sigma(a_{i_0}), -\sigma(b_j)) = D^t_{\mathbf{3}}(\sigma(a_{i_0}), -1) = \{-1\}$ (1.1.A(2)); that is, $\widehat{t_{i_0 j}}(\sigma) = \sigma(t_{i_0 j}) = -1$ for all $j \in \{1, \ldots, n\}$, which shows that $s(\sigma) = -1$, and hence $p(\sigma) = -1$.

– $\underline{f(\sigma) = 0}$. In this case, $\sigma \notin f^{-1}[1] \cup f^{-1}[-1]$, and (*) implies that there are indices $i_0 \in \{1, \ldots, k\}$ and $j_0 \in \{1, \ldots, n\}$ so that $\sigma(a_{i_0}) \neq 1$ and $\sigma(b_{j_0}) \neq 1$. Then, we have $\sigma(a_{i_0}) \in \{0, -1\}$, whence $(\widehat{a_{i_0}} \wedge \widehat{0})(\sigma) = 0$, and therefore $r(\sigma) = 0$. Likewise, $\sigma(b_{j_0}) \in \{0, -1\}$ yields $\widehat{-b_{j_0}}(\sigma) \in \{0, 1\}$, whence $(\bigwedge_{j=1}^{n} \widehat{-b_j})(\sigma) \leqslant 0$, which in turn gives $s(\sigma) \leqslant 0$. These evaluations together entail $p(\sigma) = 0$, ending the proof of the Claim.

<u>Case II</u>. $f^{-1}[\pm 1]$ are arbitrary quasi-compact opens.

Then, there are basic quasi-compact opens $V_1, \ldots, V_k, U_1, \ldots, U_n$, so that

$$f^{-1}[1] = \bigcup_{i=1}^{k} V_i \quad \text{and} \quad f^{-1}[-1] = \bigcup_{j=1}^{n} U_j.$$

For each pair of indices $i \in \{1, \ldots, k\}, j \in \{1, \ldots, n\}$ we define a map $f_{ij} : X_G \longrightarrow \mathbf{3}$ by: for $\sigma \in X_G$,

$$f_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in V_i \\ -1 & \text{if } \sigma \in U_j \\ 0 & \text{if } \sigma \notin V_i \cup U_j. \end{cases}$$

Since $V_i \cap U_j = \emptyset$, f_{ij} is well defined; clearly, $f_{ij} \in \mathrm{Sp}(X_G)(= \mathrm{Sp}(G))$. Since V_i, U_j are basic quasi-compact opens, Case I implies that each of the functions f_{ij} is in L. On the other hand, straightforward checking according to the values of f shows that $f = \bigvee_{j=1}^{n} \bigwedge_{i=1}^{k} f_{ij}$, yielding $f \in L$. This completes the proof of Theorem 6.2. $\square$

Recalling that the lattice operations of $\mathrm{Sp}(G)$ are definable in the language $\mathcal{L}_{\mathrm{RS}}$ for real semigroups (2.1), we obtain:

Corollary 6.3. — *Let G be a RS. Then,* $\mathrm{Sp}(G) \subseteq \mathrm{dcl}_{\mathrm{RS}}(G, \mathbf{3}^{X_G})$*, the definitional closure of G in $\mathbf{3}^{X_G}$ for the language $\mathcal{L}_{\mathrm{RS}}$ (with $\mathbf{3}^{X_G}$ endowed with the pointwise defined $\mathcal{L}_{\mathrm{RS}}$-structure). In particular,* $\mathrm{Sp}(G)$ *is* rigid *over G: every $\mathcal{L}_{\mathrm{RS}}$-automorphism of $\mathbf{3}^{X_G}$ which pointwise fixes G is the identity on* $\mathrm{Sp}(G)$.

Remark. — For the notion of *definitional closure* of a structure, see [12], pp. 134 ff.

Our last result in this section shows that the spectral real semigroups are exactly the real semigroups for which the representation partial order is a *distributive* lattice.

Warning. — The essential point here is *distributivity*. In fact, there are other classes of real semigroups for which the representation partial order is a lattice (necessarily non-distributive); for example, the RS-fans – a class introduced and investigated in Chapter V of [10] –, have this property.

As a preliminary step we prove:

Lemma 6.4. — *Let G be a RS. Assume that the representation partial order $\leqslant_G$ is a* distributive *lattice. Then, product in G coincides with symmetric difference: for $a, b \in G$,*

$$a \cdot b = (a \wedge -b) \vee (b \wedge -a) (:= a \,\underline{\triangle}\, b).$$

Proof. — To ease notation we write $\leqslant$ for $\leqslant_G$. For $x, y \in G$, Proposition 1.2(d) yields:

$$(*) \qquad x \leqslant y \Leftrightarrow \forall \sigma \in X_G (\sigma(x) \leqslant_{\mathbf{3}} \sigma(y)).$$

We first observe:

(a) $a \wedge -b, b \wedge -a \leqslant a \cdot b$. Hence, $a \,\underline{\triangle}\, b \leqslant a \cdot b$.

(b) $ab \leqslant a \vee b, -a \vee -b$.

Proof of (a). By symmetry, it suffices to prove the first inequality. Using (*) we show $\sigma(a \wedge -b) \leqslant \sigma(ab)$, for all $\sigma \in X_G$.

– Suppose $\sigma(a \wedge -b) = 0$. By monotonicity, $\sigma(a), \sigma(-b) \geqslant 0$, i.e., $\sigma(a) \geqslant 0$ and $\sigma(b) \leqslant 0$. This implies that $\sigma(ab) \neq 1$, for $\sigma(ab) = 1$ implies that $\sigma(a), \sigma(b)$ are both either 1 or -1. Thus, $\sigma(a \wedge -b) = 0 \leqslant \sigma(ab)$.

– If $\sigma(a \wedge -b) = -1$, by monotonicity, $\sigma(a) = \sigma(-b) = -1$, i.e., $\sigma(a) = -1$ and $\sigma(b) = 1$. Hence $\sigma(ab) = -1 = \sigma(a \wedge -b)$.

Proof of (b). Let $\sigma \in X_G$. If $\sigma(ab) = 0$, then at least one of $\sigma(a), \sigma(b)$ is 0, say $\sigma(a) = 0$; by monotonicity, $\sigma(ab) = 0 = \sigma(a) \leqslant \sigma(a \vee b)$ and $\sigma(ab) = 0 = \sigma(-a) \leqslant \sigma(-a \vee -b)$. If $\sigma(ab) = -1$, then, say $\sigma(a) = -1$ and $\sigma(b) = 1$ (or the other way round). By monotonicity, $-1 = \sigma(a) \leqslant \sigma(a \vee b)$ and $-1 = \sigma(-b) \leqslant \sigma(-a \vee -b)$, as required.

The Lemma follows from (b). Indeed, using distributivity we have:

(**) $ab \leqslant (a \vee b) \wedge (-a \vee -b) = (a \wedge -a) \vee (a \wedge -b) \vee (b \wedge -a) \vee (b \wedge -b)$.

The Kleene inequality (1.2(f)) yields $x \wedge -x \leqslant y \vee -y$ for all $x, y \in G$. Using distributivity again obtains:

$$\begin{aligned} a \wedge -a &= (a \wedge -a) \wedge (b \vee -b) = (a \wedge -a \wedge b) \vee (a \wedge -a \wedge -b) \\ &\leqslant (-a \wedge b) \vee (-b \wedge a). \end{aligned}$$

Similarly, $b \wedge -b \leqslant (-a \wedge b) \vee (-b \wedge a)$. The last term in (**) then equals $(-a \wedge b) \vee (-b \wedge a) = a \underline{\triangle} b$, proving $ab \leqslant a \underline{\triangle} b$, as asserted. □

Corollary 6.5. — *Under the assumptions of Lemma 6.4, the following holds for $a, b \in G$:*

(i) $a \cdot (a \vee b), b \cdot (a \vee b) \leqslant ab$.

(ii) For all $x \in G, x \in D_G(a^2, b^2) \Leftrightarrow x = x^2$ *and* $a^2 \wedge b^2 \leqslant x$.

(iii) $(a \vee b)^2 \in D_G(a^2, b^2)$.

Proof. — (i) By Lemma 6.4,

$$\begin{aligned} a \cdot (a \vee b) &= a \underline{\triangle} (a \vee b) = (a \wedge -(a \vee b)) \vee ((a \vee b) \wedge -a) \\ &= (a \wedge -a \wedge -b)) \vee (a \wedge -a) \vee (b \wedge -a) \\ &= (a \wedge -a) \vee (b \wedge -a). \end{aligned}$$

The Kleene inequality $a \wedge -a \leqslant b \vee -b$ implies that $a \wedge -a \leqslant a \underline{\triangle} b = ab$; indeed, by distributivity

$$\begin{aligned} a \wedge -a = (a \wedge -a) \wedge (b \vee -b) &= (a \wedge -a \wedge b)) \vee (a \wedge -a \wedge -b)) \\ &\leqslant (-a \wedge b) \vee (a \wedge -b) = a \underline{\triangle} b. \end{aligned}$$

Since $b \wedge -a \leqslant a \,\underline{\triangle}\, b$, our contention follows. The other inequality in (i) holds by symmetry.

Next, we prove, for $\sigma \in X_G$:

(*) $\sigma(a^2 \wedge\ b^2) = \sigma(a^2) \wedge\ \sigma(b^2)$.

Recall that by Proposition 1.2(g), $\{a^2 \wedge\ b^2\} = D^t_G(a^2, b^2)$ (any G). Since σ preserves D^t,

$$\sigma(a^2 \wedge\ b^2) \in D^t_{\mathbf{3}}(\sigma(a^2), \sigma(b^2)) = D^t_{\mathbf{3}}(\sigma(a)^2, \sigma(b)^2) = \{\sigma(a)^2 \wedge\ \sigma(b)^2\},$$

and (*) follows.

(ii) ($\Rightarrow$) Clearly, $x \in D_G(a^2, b^2)$ implies $x = x^2$. To show $a^2 \wedge\ b^2 \leqslant x$ we check that $\sigma(a^2 \wedge\ b^2) \leqslant \sigma(x)$ for all $\sigma \in X_G$; by (*) it suffices to verify $\sigma(a^2) \wedge\ \sigma(b^2) \leqslant \sigma(x^2)$. But $\sigma(x^2) = 1$ implies $\sigma(x) \neq 0$, whence, from $x = x^2 \in D_G(a^2, b^2)$ we get $1 = \sigma(x^2) = \sigma(a^2)$ or $1 = \sigma(x^2) = \sigma(b^2)$ and hence $\sigma(a^2) \wedge\ \sigma(b^2) = 1$.

($\Leftarrow$) From $x = x^2$ we get $\sigma(x) = \sigma(x^2) \in \{0, 1\}$ for $\sigma \in X_G$. If $\sigma(x) = 1$, then $a^2 \wedge\ b^2 \leqslant x$ and (*) give $\sigma(a^2 \wedge b^2) = \sigma(a^2) \wedge \sigma(b^2) = 1$, and hence either $\sigma(a^2) = 1$ or $\sigma(b^2) = 1$, proving that $x \in D_G(a^2, b^2)$.

(iii) By (ii) it suffices to prove $(a \vee\ b)^2 \geqslant a^2 \wedge\ b^2$. Invoking Lemma 6.4 and using distributivity, we get:

$$\begin{aligned}(a \vee b)^2 &= (a \vee\ b) \,\underline{\triangle}\, (a \vee\ b) = (a \vee\ b) \wedge -(a \vee\ b) = (a \vee\ b) \wedge (-a \wedge -b) = \\ &= (a \wedge -a \wedge -b) \vee (b \wedge -a \wedge -b) \geqslant (a \wedge -a) \wedge (b \wedge -b) = \\ &= (a \,\underline{\triangle}\, a) \wedge (b \,\underline{\triangle}\, b) = a^2 \wedge\ b^2,\end{aligned}$$

as needed. □

Theorem 6.6. — *Let G be a RS and let $\leqslant_G$ denote its representation partial order. Assume that $(G, \leqslant_G)$ is a lattice. The following are equivalent:*

(1) *$(G, \leqslant_G)$ is a distributive lattice.*

(2) *The RS-characters of G are lattice homomorphisms of $(G, \leqslant_G)$ into* $\mathbf{3}$ *(ordered by $1 < 0 < -1$).*

(3) *The canonical embedding $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ is a surjective lattice homomorphism. Hence, $G \simeq \mathrm{Sp}(G)$.*

Each of these conditions is equivalent to:

(4) *G is a spectral RS.*

Proof. — (3) $\Rightarrow$ (4) is obvious, (4) $\Rightarrow$ (3) was proved in 4.6, (4) $\Rightarrow$ (2) in 2.2, and (4) $\Rightarrow$ (1) has been observed in 1.1.C(4).

(1) $\Rightarrow$ (2). We show that every $\sigma \in X_G$ preserves suprema. This suffices, since σ also preserves "$-$" and the De Morgan laws hold in G, i.e., $-(a \wedge b) = -a \vee -b$ and dually. In fact, it suffices to prove $\sigma(a \vee b) \leqslant \sigma(a) \vee \sigma(b)$, as the reverse inequality is immediate from the monotonicity of σ. We argue by cases:

– $\underline{\sigma(a \vee b) = 0}$. From $a \cdot (a \vee b) \leqslant ab$ (6.5(i)) comes $\sigma(a)\sigma(a \vee b) = 0 \leqslant \sigma(a)\sigma(b)$. This shows that $\sigma(a), \sigma(b)$ cannot both be 1, i.e., $\sigma(a) \geqslant 0$ or $\sigma(b) \geqslant 0$, whence $\sigma(a) \vee \sigma(b) \geqslant 0 = \sigma(a \vee b)$.

– $\underline{\sigma(a \vee b) = -1}$. Suppose first that $\sigma(a) = \sigma(b) = 0$. From $(a \vee b)^2 \in D_G(a^2, b^2)$ (6.5(iii)) we get $\sigma(a \vee b)^2 \in D_{\mathbf{3}}(\sigma(a)^2, \sigma(b)^2) = D_{\mathbf{3}}(0, 0) = \{0\}$, whence $\sigma(a \vee b) = 0$, contradiction. So, either $\sigma(a)$ or $\sigma(b)$ is $\neq 0$. If, say, $\sigma(a) = 1$, as above we get $-1 = \sigma(a)\sigma(a \vee b) \leqslant \sigma(a)\sigma(b) = \sigma(b)$, and hence $\sigma(b) = -1$. So, either $\sigma(a)$ or $\sigma(b)$ is -1, and we get $\sigma(a) \vee \sigma(b) = -1 = \sigma(a \vee b)$.

(2) $\Rightarrow$ (3). (i) η_G is a lattice homomorphism. This follows from (2) by direct computation: for $a, b \in G$ and $\sigma \in X_G$,

$$\eta_G(a \wedge b)(\sigma) = ev_{(a \wedge b)}(\sigma) = \sigma(a \wedge b) = \sigma(a) \wedge \sigma(b) = ev_a(\sigma) \wedge ev_b(\sigma)$$

$$= \eta_G(a)(\sigma) \wedge \eta_G(b)(\sigma),$$

and similarly for $\vee$.

(ii) η_G is surjective. This is clear from Theorem 6.2 and (i). $\square$

Corollary 6.7. — *The set of invertible elements of a spectral real semigroup (with induced product operation, representation relation and constants* $1, -1$*) is a Boolean algebra and, hence, a reduced special group.*

Proof. — By Proposition 1.2(e), for $G \models \mathrm{RS}$ and $g \in G$, in the representation partial order $\leqslant_G$ we have $g \wedge -g = g^2$ and $g \vee -g = -g^2$; hence:

g invertible in $G \Leftrightarrow g^2 = 1 \Leftrightarrow g \wedge -g = 1$ and $g \vee -g = -1$.

If G is spectral, by the preceding theorem $\leqslant_G$ is a distributive lattice, and this shows that $-g$ is the Boolean complement of g. $\square$

7. Axiomatization and model-theoretic results

Summarizing the preceding results and those of section 2, we have:

THEOREM 7.1 (Axioms for spectral real semigroups). — *The following statements, together with the axioms for real semigroups, give a first-order axiomatization for the class of spectral real semigroups in the language* $\mathcal{L}_{\mathrm{RS}} = \{\cdot, 1, 0, -1, D\}$*:*

[SRS1] $\forall a \exists c (c = c^2 \ \wedge \ ac = c \ \wedge \ -a \in D(1, -c))$.

Setting:

$a^- :=$ *the unique* c *verifying* [SRS1] (*see* 2.1(*i*)), *and* $a^+ := -((-a)^-)$,

[SRS2] $\forall a \forall b \exists d (d \in D(a, b) \ \wedge \ d^+ = -a^+ \cdot b^+ \ \wedge \ d^- \in D^t(a^-, b^-))$.

Proof. — Theorem 2.1 shows that the spectral RSs verify axioms [SRS1] and [SRS2].

Conversely, any real semigroup, G, verifying axioms [SRS1] and [SRS2] has a lattice structure (defined in the language $\mathcal{L}_{\mathrm{RS}}$), where $a \wedge 0 := a^-$ and $a \wedge b :=$ the unique element d verifying [SRS2]. Since the characters of G preserve the constants, operation and relation in $\mathcal{L}_{\mathrm{RS}}$, these axioms ensure that they are lattice homomorphisms (see Corollary 2.2). Thus, condition (2) of Theorem 6.6 is verified, implying that G is spectral. $\square$

Remark 7.2. — Axioms [SRS1] and [SRS2] are of the form $\forall a\, \psi_1(a)$ and $\forall a \forall b\ \psi_2(a, b)$, where ψ_1, ψ_2 are *positive-primitive* $\mathcal{L}_{\mathrm{RS}}$-formulas, i.e., of the form $\exists x\, \theta_1(a, x), \exists \overline{y}\, \theta_2(a, b, \overline{y})$, with θ_1, θ_2 conjunctions of *atomic* $\mathcal{L}_{\mathrm{RS}}$-formulas and $\overline{y}$ a tuple of variables of suitable length. This is clear for [SRS1] $[\theta_1(z, w) : w = w^2 \ \wedge \ zw = w \ \wedge \ -z \in D(1, -w)]$. For [SRS2] (using uniqueness) replace z^- by $\exists z_1 \theta_1(z, z_1)$ for $z \in \{d, a, b\}$ and, similarly, $z^+ := -((-z)^-)$ by $\exists z_2 \theta_1(-z, z_2)$. Explicitly, in new variables d_i, a_i, b_i $(i = 1, 2)$ (corresponding to d, a, b, respectively), with $\overline{y} = \langle d, d_1, d_2, a_1, a_2, b_1, b_2 \rangle$,

$$\theta_2(a, b, \overline{y}) : d \in D(a, b) \wedge \bigwedge\nolimits_{z \in \{d,a,b\}} \theta_1(z, z_1) \wedge \theta_1(-z, z_2) \wedge \ d_2 = a_2 b_2$$
$$\wedge \ d_1 \in D^t(a_1, b_1).$$

These manipulations, together with Theorem 7.1, yield the following closure properties of the class of spectral real semigroups:

PROPOSITION 7.3. — (1) *The class of spectral real semigroups is closed under inductive limits* (*colimits*) *over a right-directed index set and reduced products*[13] (*in particular, arbitrary products*).

---

(13) Cf. [4], Def. 4.1.6 and §6.2, or [12], §9.4.

(2) *Let* $f : G \longrightarrow H$ *be a* surjective *RS-homomorphism, where* G, H *are RSs. If* G *is spectral, so is* H.

Proof. — (1) Closure under inductive limits and reduced products is known to hold for classes of structures (in an arbitrary language) axiomatized by first-order sentences of the form $\forall \overline{v}(\varphi_1 \rightarrow \varphi_2)$, where φ_1, φ_2 are positive-primitive formulas. (For inductive limits a more general result is proven in [15], Ch. 17, §4; for reduced products, cf. [4], Prop. 6.2.2.). The axioms for RSs and statements [SRS1] and [SRS2] in 7.1 are of this form.

(2) We check that, for arbitrary structures $\mathfrak{A}, \mathfrak{B}$ with language L, say, if $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a *surjective* L-morphism, θ is an L-sentence of the form $\theta : \forall \overline{v} \exists \overline{x} \bigwedge_i \varphi_i(\overline{v}, \overline{x})$, with the φ_i atomic L-formulas, and $\mathfrak{A} \models \theta$, then $\mathfrak{B} \models \theta$.

This is routine model-theoretic verification: Let $\overline{b} \in \mathfrak{B}$, and let $\overline{a}$ be a tuple in $\mathfrak{A}$ such that $f(\overline{a}) = \overline{b}$. Since $\mathfrak{A} \models \theta$ there is $\overline{a'} \in \mathfrak{A}$ so that $\mathfrak{A} \models \bigwedge_i \varphi_i[\overline{a}, \overline{a'}]$. Since the φ_i are atomic and f is a L-morphism, $\mathfrak{B} \models \varphi_i[f(\overline{a}), f(\overline{a'})]$ holds for all i, whence $\mathfrak{B} \models \exists \overline{x} \bigwedge_i \varphi_i[\overline{b}, \overline{x}]$. Since $\overline{b}$ is an arbitrary tuple in $\mathfrak{B}$ we get $\mathfrak{B} \models \theta$. $\square$

Proposition 7.4. — *Let* G *be a RS and* H *be a spectral RS. Then,*

(1) *If* $f : G \longrightarrow H$ *is a pure embedding*[14] *of RSs, then* G *is a spectral RS.*

(2) *The canonical embedding* $\eta_G : G \longrightarrow \mathrm{Sp}(G)$ *of* G *into its spectral hull* is not *pure unless* G *itself is spectral. In the latter case,* η_G *is an isomorphism of* G *onto* $\mathrm{Sp}(G)$.

Proof. — (1) Let $\psi_1(v), \psi_2(v_1, v_2)$ denote the positive-primitive matrices of [SRS1], [SRS2], as in 7.2. Assume $G \not\models$ SRS. Since G is supposed to be a RS, one of [SRS1] or [SRS2] fails in G, say $G \not\models$ [SRS2]. Then, there are $a, b \in G$ so that $G \models \neg\psi_2[a, b]$. On the other hand, since by assumption $H \models$ [SRS2], we have $H \models \psi_2[f(a), f(b)]$. Since ψ_2 is positive-primitive, f is not pure, contradiction.

(2) follows from (1) with $f = \eta_G$. The last assertion is Corollary 4.6. $\square$

8. Quotients of spectral real semigroups

After a brief presentation of a general notion of a quotient in the context of real semigroups and of the relationship between quotients of a RS

(14) A map is *pure* if it reflects positive-existential formulas. For details, see [6], Ch. 5, §3, pp. 91–92.

and proconstructible subsets of its character space, we show that, *in the case of spectral real semigroups* every such proconstructible set determines a congruence of RSs. As a corollary we obtain that the formation of the spectral hull of a RS "commutes" with that of taking quotients under arbitrary RS-congruences.

Definition 8.1. — ([10], Ch. II) *A* **(RS-)congruence** *of a real semigroup G is an equivalence relation $\equiv$ verifying the following requirements:*

(i) $\equiv$ is a congruence of ternary semigroups.

(ii) There is a ternary relation $D_{G/\equiv}$ in the quotient ternary semigroup $(G/\equiv, \cdot, -1, 0, 1)$ so that $(G/\equiv, \cdot, -1, 0, 1, D_{G/\equiv})$ is a real semigroup, and the canonical projection $\pi : G \longrightarrow G/\equiv$ is a RS-morphism.

(iii) (Factoring through π.) For every RS-morphism $f : G \longrightarrow H$ into a real semigroup H such that $a \equiv b$ implies $f(a) = f(b)$ for all $a, b \in G$, there exists a RS-morphism (necessarily unique), $\widehat{f} : G/\equiv \longrightarrow H$, such that $\widehat{f} \circ \pi = f$, i.e. the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/\equiv \arrow[ur, "\widehat{f}"'] & \end{tikzcd}$$

Note that Proposition 7.3(2) yields:

Fact 8.2. — *Any quotient $G/\equiv$ of a spectral RS, G, modulo a RS-congruence $\equiv$ is a spectral RS.*

Proof. — Apply 7.3(2) with $f = \pi$, the canonical quotient map $G \longrightarrow G/\equiv$ given by Definition 8.1(ii). $\square$

Remark. — This Fact applies, in particular, to the various types of quotients of real semigroups studied in [10], §II.3 and in [14], Ch. 6.

The following theorem (proof omitted) describes the relationship between the (RS-)congruences of a real semigroup and the proconstructible subsets of its character space.

Theorem 8.3 (1) ([10], Prop. II.2.5). — *Every RS-congruence $\equiv$ of a real semigroup G determines a* proconstructible *set of characters of G; namely:*

$$\mathcal{H}_{\equiv} = \{p \in X_G \,|\, \textit{There exists } \sigma \in X_{G/\equiv} \textit{ so that } p = \sigma \circ \pi\}.$$

(2) *Conversely,* any *subset* $\mathcal{H}$ *of* X_G *defines an equivalence relation of* G *by setting, for* $a, b \in G$*:*

(†) $a \equiv_{\mathcal{H}} b :\Leftrightarrow$ *For all* $\sigma \in \mathcal{H}$, $\sigma(a) = \sigma(b)$,

having the following properties:

(*i*) $\equiv_{\mathcal{H}}$ *is a congruence of the ternary semigroup* $(G, \cdot, 1, 0, -1)$.

(*ii*) *The quotient ternary semigroup* $G/\equiv_{\mathcal{H}}$ (*denoted* $G/\mathcal{H}$) *carries a ternary relation naturally defined by* (*with* $\pi_{\mathcal{H}} : G \longrightarrow G/\mathcal{H}$ *canonical*)*: for* $a, b, c \in G$,

$$\pi_{\mathcal{H}}(a) \in D_{G/\mathcal{H}}(\pi_{\mathcal{H}}(b), \pi_{\mathcal{H}}(a)) :\Leftrightarrow \forall \sigma \in \mathcal{H}(\sigma(a) \in D_{\mathbf{3}}(\sigma(b), \sigma(c))).$$

(*iii*) *The closure* $\overline{\mathcal{H}}$ *of* $\mathcal{H}$ *in the constructible topology of* X_G *defines the same quotient* $\mathcal{L}_{\mathrm{RS}}$*-structure as* $\mathcal{H}$*, i.e.,* $\equiv_{\mathcal{H}} = \equiv_{\overline{\mathcal{H}}}$ *and* $D_{G/\mathcal{H}} = D_{G/\overline{\mathcal{H}}}$*. So, without loss of generality, we can take* $\mathcal{H}$ *to be proconstructible.*

(3) ([10], Thm. II.2.8) $(G/\mathcal{H}, \cdot, 1, 0, -1, D_{G/\mathcal{H}})$ *verifies all axioms for real semigroups except, possibly, the weak associativity axiom* [RS3a] (*cf.* 1.6).

(4) (*i*) *With the topology induced from the spectral topology in* X_G*, the set* $\mathcal{H}_{\equiv}$ *is homeomorphic to the character space* $X_{G/\equiv}$ (*by the map* $p \in \mathcal{H}_{\equiv} \longmapsto$ *the unique* $\sigma \in X_{G/\equiv}$ *such that* $p = \sigma \circ \pi$).

(*ii*) *Given a RS-congruence* $\equiv$ *of* G*, the equivalence relation on* G *induced by* $\mathcal{H}_{\equiv}$ (*as in* (†) *above*) *coincides with* $\equiv$.

Remarks and Notation 8.4. — (i) Let X be a hereditarily normal spectral space. Since $X = X_{\mathbf{Sp}(X)}$ (via identification by the map ev, cf. 3.7), the equivalence relation $\equiv_Y$ of $\mathbf{Sp}(X)$ defined by a proconstructible set $Y \subseteq X$ (see (†) in 8.3(2)), boils down to: for $a, b \in \mathrm{Sp}(X)$,

$$a \equiv_Y b \Leftrightarrow \forall y \in Y(a(y) = b(y)) \Leftrightarrow a\lceil Y = b\lceil Y,$$

and the corresponding quotient representation relation $D_{\mathbf{Sp}(X)/\equiv_Y}$ becomes: for $a, b, c \in \mathrm{Sp}(X)$ and with $\pi_Y : \mathbf{Sp}(X) \longrightarrow \mathbf{Sp}(X)/\equiv_Y$ canonical,

$$\pi_Y(a) \in D_{\mathbf{Sp}(X)/\equiv_Y}(\pi_Y(b), \pi_Y(c)) :\Leftrightarrow \forall y \in Y(a(y) \in D_{\mathbf{3}}(b(y), c(y))).$$

Routine checking shows that we also have:

$$\pi_Y(a) \in D^t_{\mathbf{Sp}(X)/\equiv_Y}(\pi_Y(b), \pi_Y(c)) \Leftrightarrow \forall y \in Y(a(y) \in D^t_{\mathbf{3}}(b(y), c(y))).$$

To ease notation we shall write D_Y for $D_{\mathbf{Sp}(X)/\equiv_Y}$, and similarly for transversal representation.

(ii) It is a general and well-known fact that the spectral subspaces of a spectral space X are exactly the proconstructible subsets $Y \subseteq X$ with the induced topology. To avoid possible confusion, in the sequel we denote by Y_{sp} the proconstructible subset Y endowed with the (spectral) topology induced from X. The quasi-compact opens of Y_{sp} are exactly the intersections of quasi-compact opens of X with Y, and similarly for the closed constructible subsets of Y_{sp}. These results imply that the specialization order of Y_{sp} is just the restriction of specialization in X. See [11], Thm. 3.3.1. Hence, it is clear that, if in addition X is hereditarily normal, then so is Y_{sp}. In this case, to ease notation, the real semigroup $\mathbf{Sp}(Y_{\mathrm{sp}})$ will be denoted by $\mathbf{Sp}(Y)$. □

Our first result is:

THEOREM 8.5. — *Let X be a hereditarily normal spectral space, and let Y be a proconstructible subset of X. Then,*

(1) *$(\mathbf{Sp}(X)/{\equiv_Y}, D_Y)$ is a real semigroup.*

(2) *$(\mathbf{Sp}(X)/{\equiv_Y}, D_Y)$ is isomorphic to $(\mathbf{Sp}(Y), D_{\mathbf{Sp}(Y)})$.*

(3) *The character space $X_{\mathbf{Sp}(X)/\equiv_Y}$ is homeomorphic to Y_{sp} (for the respective spectral topologies).*

(4) *Every RS-congruence of $\mathbf{Sp}(X)$ is of the form $\equiv_Y$ for a suitable proconstructible set $Y \subseteq X$.*

Proof. — (1) follows from (2) and Theorems 1.7 and 1.8 applied with the spectral space Y_{sp}, see 8.4(ii).

(2) The required isomorphism is the map

$$a/{\equiv_Y} = \pi_Y(a) \stackrel{\varphi}{\longmapsto} a\lceil Y \qquad (a \in \mathrm{Sp}(X)).$$

By 8.4(i) it is clear that φ is a well-defined, injective homomorphism of ternary semigroups. Since representation is pointwise defined in both $\mathbf{Sp}(X)/{\equiv_Y}$ and $\mathbf{Sp}(Y)$, we have

$$\begin{aligned} \pi_Y(a) \in D_{\mathbf{Sp}(X)/\equiv_Y}(\pi_Y(b), \pi_Y(c)) &\Leftrightarrow \forall y \in Y (a(y) \in D_{\mathbf{3}}(b(y), c(y)) \\ &\Leftrightarrow a\lceil Y \in D_{\mathbf{Sp}(Y)}(b\lceil Y, c\lceil Y), \end{aligned}$$

showing that both φ and $\varphi^{-1}\lceil \mathrm{Im}(\varphi)$ preserve representation.

The proof that φ is surjective is more delicate; it boils down to:

CLAIM. — Every map $f \in \mathrm{Sp}(Y)$ extends to a map $g \in \mathrm{Sp}(X)$.

Proof of Claim. — The argument is similar to (a part of) the proof of Theorem 1.8, using Proposition 1.9; we only sketch it.

Recall that $f \in \mathrm{Sp}(Y)$ just means that $f^{-1}[\pm 1]$ are quasi-compact opens in Y_{sp}. To get a spectral map $g : X \longrightarrow \mathbf{3}_{\mathrm{sp}}$ extending f it suffices to construct disjoint quasi-compact opens $U_i (i \in \{\pm 1\})$ of X so that $f^{-1}[i] \subseteq U_i$, and set:

$$g\lceil U_i = i \ (i \in \{\pm 1\}) \ \text{ and } \ g\lceil (X \setminus (U_1 \cup U_{-1})) = 0.$$

For $i \in \{\pm 1\}$ let $\mathrm{Gen}_X(f^{-1}[i]) = \{x \in X \mid \exists y \in f^{-1}[i](x \underset{X}{\rightsquigarrow} y)\}$ be the generization of $f^{-1}[i]$ in X. Arguments similar to those in the proof of 1.8 show:

(i) $\mathrm{Gen}_X(f^{-1}[i])$ is quasi-compact in X $(i \in \{\pm 1\})$;

(ii) $\mathrm{Gen}_X(f^{-1}[1]) \cap \mathrm{Gen}_X(f^{-1}[-1]) = \emptyset$.

By Proposition 1.9 there are disjoint quasi-compact open subsets U_i of X such that $\mathrm{Gen}_X(f^{-1}[i]) \subseteq U_i$ $(i \in \{\pm 1\})$, as required.

(3) follows from (2) using the duality between the categories **RS** of real semigroups and **ARS** of abstract real spectra ([8], Thm. 4.1, p. 115), and the fact that $X_{\mathbf{Sp}(Y)} \simeq Y$ (Proposition 3.7(2)).

(4) is a particular case of Theorem 8.3(4): with notation therein, given a congruence $\equiv$ of $\mathbf{Sp}(X)$, take $Y := \mathcal{H}_{\equiv}$ (a proconstructible subset of $X_{\mathbf{Sp}(X)} = X$; 8.3(1)), and conclude by 8.3(4.ii). $\square$

To establish that the equivalence relation $\equiv_Y$ is a RS-congruence we still have to prove the factorization condition 8.1(iii). This will follow from the next Proposition, which gives a lifting for the quotient representation relation D_Y.

Proposition 8.6. — *Let X be a hereditarily normal spectral space, let $Y \subseteq X$ be proconstructible, and let $a, b, c \in \mathbf{Sp}(X)$. The following are equivalent:*

(1) $\pi_Y(a) \in D_Y(\pi_Y(b), \pi_Y(c))$.

(2) *There exists $a' \in \mathbf{Sp}(X)$ such that $a'\lceil Y = a\lceil Y$ and $a' \in D_{\mathbf{Sp}(X)}(b, c)$.*

Proof. — (2) $\Rightarrow$ (1) is clear from the pointwise definition of both D_Y and $D_{\mathbf{Sp}(X)}$.

(1) $\Rightarrow$ (2). Throughout this proof i stands for ± 1. Let $U_i := a^{-1}[i]$ and $V_i := b^{-1}[i] \cup c^{-1}[i]$, quasi-compact open subsets of X. Assumption (1)

amounts to $U_i \cap Y \subseteq V_i \cap Y$. Set $W_i := U_i \cap V_i$; W_i is quasi-compact open, and $W_1 \cap W_{-1} = \emptyset$. We define a map $a' : X \longrightarrow \mathbf{3}$ by:

$$a' \lceil W_i = i \text{ for } i \in \{\pm 1\}, \text{ and } a' \lceil X \setminus (W_1 \cup W_{-1}) = 0.$$

Clearly, $a' \in \mathbf{Sp}(X)$. We have:

– $\underline{a' \lceil Y = a \lceil Y}$. Let $y \in Y$; for $i \in \{\pm 1\}$ it holds:

$$a(y) = i \Rightarrow y \in U_i \cap Y \subseteq V_i \cap Y \Rightarrow y \in W_i \cap Y \Rightarrow a'(y) = i,$$

and

$$a(y) = 0 \Rightarrow y \in X \setminus (U_1 \cup U_{-1}) \subseteq X \setminus (W_1 \cup W_{-1}) \Rightarrow a'(y) = 0.$$

– $\underline{a' \in D_{\mathbf{Sp}(X)}(b, c)}$. For $x \in X$ and $i \in \{\pm 1\}$ we have:

$$a'(x) = i \Rightarrow x \in W_i \subseteq V_i \Rightarrow b(x) = i \text{ or } c(x) = i,$$

as required. $\square$

Corollary 8.7. — *Let X be a hereditarily normal spectral space, and let $Y \subseteq X$ be a proconstructible subset. The equivalence relation $\equiv_Y$ verifies the factorization condition of Definition* 8.1(*iii*).

Proof. — Given a RS-morphism $f : \mathbf{Sp}(X) \longrightarrow H$ into a RS, H, such that $a \equiv_Y b \Rightarrow f(a) = f(b)$ for $a, b \in \mathbf{Sp}(X)$, it suffices to show that the map $\widehat{f}$: $\mathbf{Sp}(X)/{\equiv_Y} = \mathbf{Sp}(Y) \longrightarrow H$ defined by $\widehat{f} \circ \pi = f$ preserves representation, i.e., for $a, b, c \in \mathbf{Sp}(X)$,

$$\pi_Y(a) \in D_Y(\pi_Y(b), \pi_Y(c)) \Rightarrow f(a) \in D_H(f(b), f(c)).$$

By Proposition 8.6, the antecedent implies that $a' \in D_{\mathbf{Sp}(X)}(b, c)$ for some $a' \in \mathbf{Sp}(X)$ such that $a' \lceil Y = a \lceil Y$, i.e., $a' \equiv_Y a$. By the assumption on f we have $f(a) = f(a')$ and, since f is a RS-morphism, $f(a) = f(a') \in D_H(f(b), f(c))$, as required. $\square$

The spectral hull of a RS-quotient . — As an application of the foregoing results we prove that the spectral hull operation commutes with that of taking quotients under arbitrary RS-congruences:

Theorem 8.8. — *Let $\equiv$ be a RS-congruence of a real semigroup G. Let $Y := \mathcal{H}_{\equiv} \subseteq X_G$ denote the (proconstructible) set of characters defined by $\equiv$ (cf.* 8.3) *and let $\equiv_Y$ denote the RS-congruence of* Sp(G) *induced by Y, as in* 8.4(*i*). *Then we have* $\mathrm{Sp}(G/{\equiv}) \simeq \mathrm{Sp}(G)/{\equiv_Y}$.

Proof. — In Theorem 5.3(i) we proved that – identifying G with $\mathrm{Im}(\eta_G) \subseteq \mathrm{Sp}(X)$ via η_G – any RS-homomorphism $f : G \longrightarrow H$ extends *uniquely* to a RS-morphism $\mathrm{Sp}(f) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(H)$; see 4.3(b) and 4.4. We shall use this with $H = G/\equiv$ and $f = \pi_G =$ the canonical quotient map $G \longrightarrow G/\equiv$.

As in Theorem 8.5, $\pi_Y : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(G)/\equiv_Y$ denotes the corresponding quotient map. We first note:

(a) For $a, b \in \mathrm{Sp}(G)$, $a \equiv_Y b \Rightarrow \mathrm{Sp}(\pi_G)(a) = \mathrm{Sp}(\pi_G)(b)$.

Proof of (a). This implication can be rephrased as

$$a\lceil Y = b\lceil Y \Rightarrow a \circ \pi_G^* = b \circ \pi_G^*.$$

That is, we must show that, for $\sigma \in X_{G/\equiv}$, $(a \circ \pi_G^*)(\sigma) = (b \circ \pi_G^*)(\sigma)$; equivalently, $a(\sigma \circ \pi_G) = b(\sigma \circ \pi_G)$. Since $\sigma \circ \pi_G \in \mathcal{H}_\equiv = Y$ (cf. 8.3(1)) and $a\lceil Y = b\lceil Y$, we conclude $a(\sigma \circ \pi_G) = b(\sigma \circ \pi_G)$, as required.

Since $\equiv_Y$ is a RS-congruence of $\mathrm{Sp}(G)$ (8.5(1) and 8.7), item (a) entails that the map $\mathrm{Sp}(\pi_G) : \mathrm{Sp}(G) \longrightarrow \mathrm{Sp}(G/\equiv)$ induces a RS-morphism $\widehat{\mathrm{Sp}(\pi_G)} : \mathrm{Sp}(G)/\equiv_Y \longrightarrow \mathrm{Sp}(G/\equiv)$ so that $\widehat{\mathrm{Sp}(\pi_G)} \circ \pi_Y = \mathrm{Sp}(\pi_G)$. We show that $\widehat{\mathrm{Sp}(\pi_G)}$ is the required RS-isomorphism.

(b) $\widehat{\mathrm{Sp}(\pi_G)}$ is injective.

Proof of (b). This is just the converse to the implication in (a): for $a, b \in \mathrm{Sp}(G)$,

$$\mathrm{Sp}(\pi_G)(a) = \mathrm{Sp}(\pi_G)(b) \Rightarrow a\lceil Y = b\lceil Y,$$

i.e.,

$$(*) \qquad a \circ \pi_G^* = b \circ \pi_G^* \Rightarrow a\lceil Y = b\lceil Y.$$

Let $p \in Y = \mathcal{H}_\equiv$. By the definition of $\mathcal{H}_\equiv$ (8.3) there is $\sigma \in X_{G/\equiv}$ such that $p = \sigma \circ \pi_G$. From the antecedent of (*) comes

$$a(p) = a(\sigma \circ \pi_G) = (a \circ \pi_G^*)(\sigma) = (b \circ \pi_G^*)(\sigma) = b(\sigma \circ \pi_G) = b(p);$$

since p is an arbitrary element of Y, (*) is proved.

(c) $\widehat{\mathrm{Sp}(\pi_G)}$ is surjective.

Proof of (c). We show $\mathrm{Sp}(\pi_G)$ is surjective. Let $f \in \mathrm{Sp}(G/\equiv)$, i.e., $f : X_{G/\equiv} \longrightarrow \mathbf{3}_{\mathrm{sp}}$ is a spectral map. With $\varphi : Y = \mathcal{H}_\equiv \longrightarrow X_{G/\equiv}$ denoting the (spectral) homeomorphism defined in 8.3(a), we have $f \circ \varphi \in \mathrm{Sp}(Y)$. The Claim in the proof of Theorem 8.5 shows that $f \circ \varphi$ extends to a map $g \in \mathrm{Sp}(X_G) = \mathrm{Sp}(G)$, i.e., $g\lceil Y = f \circ \varphi$. The definition of φ shows that $\varphi = {\pi_G^*}^{-1}$. Hence the last equality yields $f = g \circ \pi_G^* = \mathrm{Sp}(\pi_G)(g)$, proving (c).

(d) $\widehat{\mathrm{Sp}(\pi_G)}$ reflects representation.

Proof of (d). This amounts to proving, for $a, b, c \in \mathrm{Sp}(G)$,

$$\mathrm{Sp}(\pi_G)(a) \in D_{\mathrm{Sp}(G/\equiv)}(\mathrm{Sp}(\pi_G)(b), \mathrm{Sp}(\pi_G)(c))$$

$$\Rightarrow \pi_Y(a) \in D_{\mathrm{Sp}(G)/\equiv_Y}(\pi_Y(b), \pi_Y(c)), \text{ i.e.,}$$

$$(**) \qquad a \circ \pi_G^* \in D_{\mathrm{Sp}(G/\equiv)}(b \circ \pi_G^*, c \circ \pi_G^*) \Rightarrow a\lceil Y \in D_{\mathbf{Sp}(Y)}(b\lceil Y, c\lceil Y).$$

For $z \in \mathrm{Sp}(G)$ and $\sigma \in X_{G/\equiv}$ we have $(z \circ \pi_G^*)(\sigma) = z(\sigma \circ \pi_G)$. So, the antecedent in (**) translates as

$$(***) \qquad \forall \sigma \in X_{G/\equiv} \left[a(\sigma \circ \pi_G) \in D_{\mathbf{3}}(b(\sigma \circ \pi_G), c(\sigma \circ \pi_G)) \right].$$

To establish (d), let $p \in Y = \mathcal{H}_{\equiv}$, i.e., $p = \sigma \circ \pi_G$ for some $\sigma \in X_{G/\equiv}$. Then, (***) yields $a(p) \in D_{\mathbf{3}}(b(p), c(p))$. Since p is an arbitrary element of Y, the conclusion in (**) follows. $\square$

9. Rings whose associated real semigroups are spectral

In this section we prove, first, that the real semigroup associated to any lattice-ordered ring is spectral, a rather direct consequence of the axiomatisation in 7.1(1). This exhibits a very extensive class of examples of spectral RSs arising from rings. An interesting consequence is that the spectral hull of the RS G_A associated to *any* semi-real ring A is canonically isomorphic to the RS $G_{\overline{A}}$ associated to the real closure $\overline{A}$ of A (real closure in the sense of Schwartz [17], see also Prestel-Schwartz, [16]); further, the canonical embedding η_{G_A} of G_A into $\mathrm{Sp}(G_A)$ (4.1(ii)) is induced by the natural map of A into $\overline{A}$ (see also [14], Remark (3), p. 178).

Preliminaries and Notation 9.1. — (A) The RS associated to a ring. Associated to each ring (commutative, unitary, semi-real, i.e., $-1 \notin \sum A^2$) there is a real semigroup $G_A = \{\overline{a} \,|\, a \in A\}$, where $\overline{a} : \mathrm{Sper}\,(A) \longrightarrow \mathbf{3} = \{1, 0, -1\}$ is defined as follows. For $\alpha \in \mathrm{Sper}\,(A)$, let $\pi_\alpha : A \longrightarrow A/\mathrm{supp}(\alpha)$ be the canonical quotient map, and let $\leqslant_\alpha$ denote the total order of $A/\mathrm{supp}(\alpha)$ defined by α. Then,

$$[*] \qquad \overline{a}(\alpha) = \begin{cases} 1 & \Leftrightarrow a \in \alpha \setminus (-\alpha) \quad \Leftrightarrow \pi_\alpha(a) >_\alpha 0 \text{ (in } (A/\mathrm{supp}(\alpha), \leqslant_\alpha)), \\ 0 & \Leftrightarrow a \in \mathrm{supp}(\alpha) \quad \Leftrightarrow \pi_\alpha(a) = 0, \\ -1 & \Leftrightarrow a \in -\alpha \setminus \alpha \quad \Leftrightarrow \pi_\alpha(a) <_\alpha 0. \end{cases}$$

with constants $\overline{1}, \overline{0}, \overline{-1}$, multiplication induced by that of A, and representation (resp. transversal representation) given by: for $a, b, c \in A$,

[R] $\overline{c} \in D_A(\overline{a}, \overline{b}) \Leftrightarrow \forall \alpha \in \mathrm{Sper}\,(A)[\overline{c}(\alpha) = 0 \vee \overline{a}(\alpha)\overline{c}(\alpha) = 1 \vee \overline{b}(\alpha)\overline{c}(\alpha) = 1]$,

[TR] $\overline{c} \in D^t_A(\overline{a}, \overline{b}) \Leftrightarrow \forall \alpha \in \mathrm{Sper}(A)[(\overline{c}(\alpha) = 0 \ \wedge \ \overline{a}(\alpha) = \overline{-b}(\alpha)) \vee$

$$\overline{a}(\alpha)\overline{c}(\alpha) = 1 \ \vee \ \overline{b}(\alpha)\overline{c}(\alpha) = 1].$$

In other words,

$$\overline{a}(\alpha) = \mathrm{sgn}_{\leqslant_\alpha}(\pi_\alpha(a)) = \text{the (strict) sign of } \pi_\alpha(a) \text{ in } (A/\mathrm{supp}(\alpha), \leqslant_\alpha).$$

("Strict" means strictly positive, strictly negative or zero.)

The proof that G_A is, indeed, a real semigroup (couched in the dual language of abstract real spectra), can be found in [14], Thm. 6.1.2, p. 100. Its character space is $X_{G_A} = \mathrm{Sper}\,(A)$. A similar definition can be given for the set $G_{A,T}$ of functions $\overline{a}$ *restricted to* $\mathrm{Sper}\,(A,T) = \{\alpha \in \mathrm{Sper}\,(A) \mid T \subseteq \alpha\}$, where T is a preorder of A, and a similar result holds in this case.

(B) Lattice-ordered rings. We assume known the basics on lattice-ordered rings (abbreviated **ℓ-rings**) for which the reader is referred to [3], Chs. 8, 9.

Throughout this section we assume A is a ℓ-ring. The underlying partial order of A will be denoted by $\leqslant$ (not to be confused with the representation partial order of $G_A = G_{A,\leqslant}$, cf. 1.1). Without risk of confusion, the lattice operations in both A and G_A will be denoted by $\wedge$, $\vee$. In this situation, π_α is a homomorphism of ordered rings of $(A, \leqslant)$ onto $(A/\mathrm{supp}(\alpha), \leqslant_\alpha)$. Since $\leqslant_\alpha$ is a total order, we have:

[†] $\pi_\alpha(a \wedge b) = \min_{\leqslant_\alpha}\{\pi_\alpha(a), \pi_\alpha(b)\}$ and $\pi_\alpha(a \vee b) = \max_{\leqslant_\alpha}\{\pi_\alpha(a), \pi_\alpha(b)\}$.

The lattice operations in A induce binary operations $(\overline{a}, \overline{b}) \mapsto \overline{a \wedge b}$ (resp. $\overline{a \vee b}$) in G_A.

Fact. — *The operations $(\overline{a}, \overline{b}) \mapsto \overline{a \wedge b}$ (resp. $\overline{a \vee b}$) are well-defined: for $a, a', b, b' \in A$,*

$$\overline{a} = \overline{a'} \textit{ and } \overline{b} = \overline{b'} \textit{ imply } \overline{a \wedge b} = \overline{a' \wedge b'} \textit{ and } \overline{a \vee b} = \overline{a' \vee b'}.$$

Sketch of proof. — We just sketch the argument for the case $(\overline{a}, \overline{b}) \mapsto \overline{a \wedge b}$. Obviously it suffices to show: $\overline{a} = \overline{a'} \Rightarrow \overline{a \wedge b} = \overline{a' \wedge b}$.

By [*] in 9.1.A, the assumption $\overline{a}(\alpha) = \overline{a'}(\alpha) (\alpha \in \mathrm{Sper}(A, \leqslant))$ amounts to the fact that $\pi_\alpha(a)$ and $\pi_\alpha(a')$ have the same strict sign in $\leqslant_\alpha$. The conclusion follows from 9.1.B[†] by a case-wise argument according to the values of $(\overline{a \wedge b})(\alpha)$. $\square$

The simple observations that follow will be used in the proof of our main result, together with the characterization of the representation partial order of G_A given in Proposition 1.2(d).

Fact 9.2. — *Let A be a ℓ-ring and let G_A be its associated RS* (9.1.*A*). *For $a, b \in A$ we have:*

(1) *$a \leqslant b \Rightarrow \overline{b} \leqslant \overline{a}$.*

(2) *$\overline{b} \leqslant \overline{a} \Leftrightarrow$ There is $a' \in A$ so that $\overline{a'} = \overline{a}$ and $a' \leqslant b$.*

(3) *$\overline{b} \leqslant \overline{0} \Leftrightarrow$ For all $\alpha \in \mathrm{Sper}(A, \leqslant), \pi_\alpha(b) \geqslant_\alpha 0$.*

(4) *$\overline{a \wedge b} = \overline{a} \vee \overline{b}$ and $\overline{a \vee b} = \overline{a} \wedge \overline{b}$.*

Proof. — (1) By 1.2(d) , we must show, for all $\alpha \in \mathrm{Sper}(A, \leqslant)$:

(i) $\overline{a}(\alpha) = 1 \Rightarrow \overline{b}(\alpha) = 1$, and (ii) $\overline{a}(\alpha) = 0 \Rightarrow \overline{b}(\alpha) \in \{0, 1\}$.

By 9.1.A[*] these conditions are, respectively, equivalent to:

(i′) $\pi_\alpha(a) >_\alpha 0 \Rightarrow \pi_\alpha(b) >_\alpha 0$, and (ii′) $\pi_\alpha(a) = 0 \Rightarrow \pi_\alpha(b) \geqslant_\alpha 0$.

Since $\pi_\alpha : (A, \leqslant) \longrightarrow (A/\mathrm{supp}(\alpha), \leqslant_\alpha)$ is a homomorphism of ordered rings, $a \leqslant b$ implies $\pi_\alpha(a) \leqslant_\alpha \pi_\alpha(b)$, from which (i′) and (ii′) clearly follow.

(2) The implication ($\Leftarrow$) is clear from (1).

($\Rightarrow$) Assume $\overline{b} \leqslant \overline{a}$ (in G_A); then, (i′) and (ii′) above hold. Set $a' := a \wedge b$. It remains to show that $\overline{a}(\alpha) = \overline{a'}(\alpha)$ for $\alpha \in \mathrm{Sper}(A, \leqslant)$; we argue by cases according to the values of $\overline{a}(\alpha)$:

– $\underline{\overline{a}(\alpha) = 1}$. Then, $\pi_\alpha(a) >_\alpha 0$ (see 9.1.A[*]); by (i′), $\pi_\alpha(b) >_\alpha 0$, and we get,

$$0 <_\alpha \min_{\leqslant_\alpha} \{\pi_\alpha(a), \pi_\alpha(b)\} = \pi_\alpha(a \wedge b) = \pi_a(a')$$

i.e., $\overline{a'}(\alpha) = 1$.

– $\underline{\overline{a}(\alpha) = 0}$. By 9.1.A[*], $\pi_\alpha(a) = 0$; (ii′) gives $\pi_\alpha(b) \geqslant_\alpha 0$; thus,

$$\pi_\alpha(a') = \pi_\alpha(a \wedge b) = \min_{\leqslant_\alpha} \{\pi_\alpha(a), \pi_\alpha(b)\} = 0,$$

i.e., $\overline{a'}(\alpha) = 0$.

– $\overline{a}(\alpha) = -1$. An argument similar to the first case yields $\overline{a'}(\alpha) = -1$.

(3) Since $\pi_\alpha(0) = 0$, the characterization of $\overline{b} \leqslant \overline{a}$ set forth in items (i′) and (ii′) of the proof of (1) applied with $a = 0$, yields at once that $\pi_\alpha(b) \geqslant_\alpha 0$ for all $\alpha \in \mathrm{Sper}(A, \leqslant)$.

(4) We only prove the first equality. Since $a \wedge b \leqslant a, b$ (in A), item (1) gives $\overline{a}, \overline{b} \leqslant \overline{a \wedge b}$, whence $\overline{a} \vee \overline{b} \leqslant \overline{a \wedge b}$. To prove the reverse inequality we proceed by cases, according to the values of $(\overline{a \wedge b})(\alpha)$, $\alpha \in \mathrm{Sper}(A, \leqslant)$.

– If $(\overline{a \wedge b})(\alpha) = 1$, there is nothing to prove.

– $(\overline{a \wedge b})(\alpha) = 0$. By 9.1.A[*], $\pi_\alpha(a \wedge b) = 0$, and the first equality in 9.1.B[†] implies that one of $\pi_\alpha(a), \pi_\alpha(b)$ is 0 and the other is $\geqslant_\alpha 0$, i.e., either $\overline{a}(\alpha) = 0$ and $\overline{b}(\alpha) \in \{0, 1\}$ or the other way round. Since the order in **3** is $1 < 0 < -1$, this clearly yields $\overline{a}(\alpha) \vee \overline{b}(\alpha) = 0$.

– $(\overline{a \wedge b})(\alpha) = -1$. From 9.1.A[*] and 9.1.B[†] we get $\min_{\leqslant_\alpha}\{\pi_\alpha(a), \pi_\alpha(b)\} <_\alpha 0$. Then, at least one of $\pi_\alpha(a)$ or $\pi_\alpha(b)$ is $<_\alpha 0$, i.e., $\overline{a}(\alpha) = -1$ or $\overline{b}(\alpha) = -1$, which yields $\overline{a}(\alpha) \vee \overline{b}(\alpha) = -1$. □

Now we turn to the proof of:

THEOREM 9.3. — *Let $(A, \leqslant)$ be a ℓ-ring. The real semigroup G_A associated to A is spectral.*

Proof. — We show that G_A verifies axioms [SRS1] and [SRS2] of 7.1. We repeatedly use the results of 9.2, notably that the order reverses in passing from A to G_A, and the equalities [†] in 9.1.B.

[SRS1]. Fix $a \in A$, and set $c = a \vee 0$. Then, $c \geqslant 0$, whence $\overline{c} \leqslant \overline{0}$, and, by 1.2(c), $\overline{c} \in \mathrm{Id}(G_A)$; in particular, $\overline{c}(\alpha) \in \{0, 1\}$, i.e., $\pi_\alpha(c) \geqslant_\alpha 0$ for $\alpha \in \mathrm{Sper}(A, \leqslant)$. Also, $c \geqslant a$, i.e., $\overline{c} \leqslant \overline{a}$, whence, by 1.1, $-\overline{a} \in D_{G_A}(1, -\overline{c})$.

To prove $\overline{ac} = \overline{c}$, let $\alpha \in \mathrm{Sper}(A, \leqslant)$. The equality obviously holds at α if $\overline{c}(\alpha) = 0$. So, assume $\overline{c}(\alpha) = 1$, i.e., $\pi_\alpha(c) >_\alpha 0$; by 9.1.B[†], $\pi_\alpha(c) = \max_{\leqslant_\alpha}\{\pi_\alpha(a), 0\} >_\alpha 0$ which clearly implies $\pi_\alpha(a) >_\alpha 0$, i.e., $\overline{a}(\alpha) = 1$.

[SRS2]. Given $a, b \in A$, set $d = a \vee b$, i.e., $\overline{d} = \overline{a} \wedge \overline{b}$ (9.2(4)). We show:

(i) $\overline{d} \in D_{G_A}(\overline{a}, \overline{b})$. We must prove: $\overline{d}(\alpha) \neq 0 \Rightarrow \overline{d}(\alpha) = \overline{a}(\alpha)$ or $\overline{d}(\alpha) = \overline{b}(\alpha)$, for $\alpha \in \mathrm{Sper}(A, \leqslant)$. Since the order $\leqslant_\alpha$ is total, the second equality in 9.1.B[†] yields $\pi_\alpha(d) = \pi_\alpha(a)$ or $\pi_\alpha(d) = \pi_\alpha(a)$, which obviously yields the required conclusion.

(ii) $\overline{d}^+ = -\overline{a}^+ \cdot \overline{b}^+$. Since $\overline{z}^+ = \overline{z} \vee \overline{0} \geqslant \overline{0} (z \in A)$, we get $\overline{z}^+(\alpha) \in \{0, -1\}$ for $\alpha \in \mathrm{Sper}(A, \leqslant)$. To prove (ii), we argue by cases according to the values of $\overline{d}^+(\alpha)$.

– $\overline{d}^+(\alpha) = -1$. This means $(\overline{d \wedge 0})(\alpha) = -1$, i.e., $\pi_\alpha(d \wedge 0) <_\alpha 0$; by 9.1.B[†], $\pi_\alpha(d) <_\alpha 0$. Since $d = a \vee b$, the second equality in 9.1.B[†] implies that both $\pi_\alpha(a)$ and $\pi_\alpha(b)$ are $<_\alpha 0$, whence $\overline{a}^+(\alpha) = \overline{b}^+(\alpha) = -1$. It follows that $-\overline{a}^+(\alpha) \cdot \overline{b}^+(\alpha) = -1 = \overline{d}^+(\alpha)$.

– $\overline{d}^+(\alpha) = 0$. Thus, $(\overline{d \wedge 0})(\alpha) = 0$, i.e., $\pi_\alpha(d \wedge 0) = 0$. The first equality in 9.1.B[†] gives $\pi_\alpha(d) \geqslant_\alpha 0$, and (as $d = a \vee b$) the second equality shows that at least one of $\pi_\alpha(a)$ or $\pi_\alpha(b)$ is $\geqslant_\alpha 0$. Then, one of $\pi_\alpha(a \wedge 0)$ or $\pi_\alpha(b \wedge 0)$ equals 0, i.e., either $\overline{a}^+(\alpha) = 0$ or $\overline{b}^+(\alpha) = 0$, proving that (ii) holds at α.

(iii) $(\overline{d})^- \in D^t_{G_A}((\overline{a})^-, (\overline{b})^-)$. For $z \in A$ we have $(\overline{z})^- = \overline{z} \wedge \overline{0} \leqslant \overline{0}$, and hence $(\overline{z})^-(\alpha) \in \{0, 1\}$ for $\alpha \in \mathrm{Sper}(A, \leqslant)$. To establish (iii) we must show:

– $(\overline{d})^-(\alpha) = 0 \Rightarrow (\overline{a})^-(\alpha) = (\overline{b})^-(\alpha) = 0$, and

– $(\overline{d})^-(\alpha) = 1 \Rightarrow (\overline{a})^-(\alpha) = 1$ or $(\overline{b})^-(\alpha) = 1$.

For the first implication, the assumption is $(\overline{d \vee 0})(\alpha) = 0$, i.e., $\pi_\alpha(d \vee 0) = 0$. The second equality in 9.1.B[†] shows that $\pi_\alpha(d) \leqslant_\alpha 0$, which (since $d = a \vee b$) yields $\pi_\alpha(a), \pi_\alpha(b) \leqslant_\alpha 0$. We get $\pi_\alpha(a \vee 0) = \pi_\alpha(b \vee 0) = 0$, i.e., $(\overline{a})^-(\alpha) = (\overline{b})^-(\alpha) = 0$.

For the second implication, the assumption amounts to $\pi_\alpha(d \vee 0) >_\alpha 0$, which implies $\pi_\alpha(d) >_\alpha 0$. The conclusion to be proved amounts to $\pi_\alpha(a) >_\alpha 0$ or $\pi_\alpha(b) >_\alpha 0$, which obviously follows from the second equality in 9.1.B[†] applied with $d = a \vee b$. □

As a consequence of Theorem 9.3 and of previous results, we have:

PROPOSITION 9.4. — *Let A be a semi-real ring, let $\overline{A}$ denote its real closure* (*in the sense of Prestel-Schwartz* [16]), *and let $\iota : A \longrightarrow \overline{A}$ be the natural map. Then,*

(1) *The spectral hull* $\mathrm{Sp}(G_A)$ *of the real semigroup G_A is canonically isomorphic to $G_{\overline{A}}$, the RS associated to $\overline{A}$.*

(2) *The canonical embedding η_{G_A} of G_A into* $\mathrm{Sp}(G_A)$ (*cf.* 4.1(*ii*)) *is induced by the RS-morphism $\overline{\iota} : G_A \longrightarrow G_{\overline{A}}$ given by the ring morphism ι.*

Proof. — The result is a consequence of the following observations:

– The fact that Sper$(A) =$ Sper$(\overline{A})$ (cf. [16], p. 264) entails $\mathrm{Sp}(G_A) = \mathrm{Sp}(G_{\overline{A}})$; indeed, both these RSs consist of the spectral characters of the space $X =$ Sper$(A) =$ Sper$(\overline{A})$ into $\mathbf{3}_{\mathrm{sp}}$ (1.3).

– By Theorem 5.3(i) we have a commutative diagram

$$\begin{tikzcd} G_A \arrow[r, "\bar{\iota}"] \arrow[d, "\eta_{G_A}"'] & G_{\overline{A}} \arrow[d, "\eta_{G_{\overline{A}}}"] \\ \mathrm{Sp}(G_A) \arrow[r, "\mathrm{Sp}(\bar{\iota})"'] & \mathrm{Sp}(G_{\overline{A}}) \end{tikzcd}$$

The previous observation and uniqueness of factorization in 5.3(i) entail that $\mathrm{Sp}(\bar{\iota})$ is the identity of $\mathrm{Sp}(G_A) = \mathrm{Sp}(G_{\overline{A}})$ (the reader can easily check that this identity makes the above diagram commute). Since $\overline{A}$ (ordered by $\overline{A}^2$) is a ℓ-ring (in fact, a reduced f-ring), 9.3 implies that $G_{\overline{A}}$ is a spectral RS. Corollary 4.6 entails, then, that $\eta_{G_{\overline{A}}}$ is an isomorphism of RSs.

Let $\varphi : \mathrm{Sp}(G_A) \longrightarrow G_{\overline{A}}$ be the map $\varphi := \eta^{-1}_{G_{\overline{A}}} \circ \mathrm{Sp}(\bar{\iota})$. By the preceding observation, φ is an isomorphism of RSs, which proves (1). Commutativity of the diagram above then gives $\eta_{G_A} = \varphi^{-1} \circ \bar{\iota}$, which proves (2). $\square$

Remark 9.5. — The well-known Delzell-Madden example of a hereditarily normal spectral space that is not homeomorphic to the real spectrum of any ring, [5], also yields an example of a spectral RS not realizable by a ring: if X denotes this space, the duality RS/ARS ([8], Thm. 4.1) and Proposition 3.7(2) show that $\mathbf{Sp}(X)$ is not isomorphic to G_A for any ring A. Further, in [14], p. 177, Marshall observes, using a dual terminology, that $\mathbf{Sp}(X)$ cannot even be of the form $G_{A,T}$ for a ring A and a preorder T of A.

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