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# Spectral approach for kernel-based interpolation

Bertrand Gauthier[(1)], Xavier Bay[(2)]

**Abstract.** — We describe how the resolution of a kernel-based interpolation problem can be associated with a spectral problem. An integral operator is defined from the embedding of the considered Hilbert subspace into an auxiliary Hilbert space of square-integrable functions. We finally obtain a spectral representation of the interpolating elements which allows their approximation by spectral truncation. As an illustration, we show how this approach can be used to enforce boundary conditions in kernel-based interpolation models and in what it offers an interesting alternative for dimension reduction.

**Résumé.** — Nous décrivons comment la résolution d'un problème d'interpolation à noyaux peut être associée à un problème spectral. Un opérateur intégral est défini à partir d'un plongement du sous-espace hilbertien considéré dans un espace de Hilbert auxiliaire composé de fonctions de carré intégrable. On obtient une représentation spectrale des éléments interpolants permettant leur approximation par troncature du spectre. À titre d'exemple, nous montrons comment cette approche peut être utilisée afin d'intégrer des informations de type conditions aux limites dans un modèle d'interpolation et en quoi elle offre une alternative intéressante pour la réduction de dimension.

## Contents



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## 1. Introduction

This work is devoted to the study of kernel-based interpolation methods (see for instance and among others [26, 23, 5, 20]). In order to cover a relatively wide class of problems, we consider the general framework of interpolation in a *separated topological real vector space* $E$. We denote by $E'$ the topological dual of $E$ and by $\langle \cdot, \cdot \rangle_{E,E'}$ the associated duality bracket. For a linear subspace $M$ of $E'$ and $e \in E$, we say that $f \in E$ is an *interpolator of* $e$ *for* $M$ (or *on* $M$) if

$$\forall e' \in M, \langle f, e' \rangle_{E,E'} = \langle e, e' \rangle_{E,E'} .$$

In this context, we focus on the two linked kernel-based methods that are *optimal interpolation in Hilbert subspaces of* $E$ and *Gaussian process models* based on the conditioning of zero-mean Gaussian processes with sample paths in $E$.

We consider interpolation problems associated with general sets $M$, including more particularly the case where $M$ is infinite dimensional (*infinite data set*). Such a situation for instance occurs when one aims at enforcing *boundary values conditions* in a given interpolation problem. As this overall framework is, in our knowledge, not of the most widespread in the interpolation literature, a significant part of this article is devoted to some recalls.

We propose and analyze an overall process which associates the resolution of kernel-based interpolation problems with the spectral decomposition of particular integral operators. This finally leads to an original spectral representation of the solutions of the considered interpolation problem (Theorem 4.1). By spectral truncations, one then naturally obtains approximations of the interpolating elements which can be proved to be *optimal* in a given sense (Proposition 6.4).

From a theoretical point of view, we want to point out that the spectral properties presented and used in this article are well-known and related to extensions of the Mercer's Theorem. On the applied point of view, the use of spectral methods in approximation and learning problems is not new either. Let us for instance quote the article of F. Cucker and S. Smale [6], where recalls and discussions concerning Mercer kernels and their applications in learning theory can be found. One can also refer to the works of E. Parzen [16] (also mentioned in [5, Section 2.4]), or among others, the articles [27, 14, 18]. The main objective of the present article is to give a theoretical description, in the general context of topological vector spaces, of the processes involved in the association of a kernel-based interpolation problem with a spectral problem. We also aim at showing the potential interests of such an approach.

Let us remark that the construction of the involved integral operator is based on the embedding of the considered Hilbert subspace into an *auxiliary Hilbert space* of squared integrable real-valued functions. The various applications and structures we consider can in this sense be compared with the ones appearing in the work of M. Nashed and G. Wahba [15].

The first part of this article (Section 2) is devoted to the description of optimal interpolation in Hilbert subspaces. In Section 3, we define the notion of *regular embedding adapted to an interpolation problem*. We also show how this embedding defines an integral operator, which is referred to as *problem-adapted*. In Section 4, we use the spectral decomposition of the considered operator in order to study the initial interpolation problem and its approximation by spectral truncation.

In Section 5, we consider the case where the number of data is finite and explicit calculations are carried out to illustrate the use of spectral considerations for the construction of interpolating elements. Section 6 is next devoted to Gaussian process models. The spectral representations considered in the previous sections are extended to the conditioning problem. In particular, we show the *IMSE-optimal* character of the approximation by truncation in this context.

We finally develop (Section 7) a theoretical example of application in which we consider a Hilbert subspace composed of continuously differentiable real-valued functions on $\mathbb{R}^2$. We consider a particular class of kernels and show how to enforce Robin-type constraints on a circle (values and derivatives) in the associated interpolation models. The difference between approximation by truncation and discretization is illustrated.

## 2. Optimal interpolation in Hilbert subspaces

### 2.1. Hilbert subspace and RKHS

The L. Schwartz theory of Hilbert subspaces [21] is an equivalent formalism for the more widespread theory of reproducing kernel Hilbert spaces (RKHS), introduced by N. Aronszajn in [1], this equivalence is for instance discussed in Remark 2.1. The abstract formalism of L. Schwartz is adapted to the framework of topological vector spaces. It also allows to draw interesting parallels with operator theory (see for instance Proposition 3.7).

#### 2.1.1. Hilbert subspace

The general framework of the Hilbert subspaces of $E$ requires the (real) topological vector space $E$ to be also locally convex and quasi-complete (see for instance [19, 21]), what we assume thereafter. Remark that these properties are verified by most of the classical functions spaces, or by Fréchet and Banach spaces. We denote by $E'$ the topological dual space of $E$.

A *Hilbert subspace* $\mathcal{H}$ of $E$ is a linear subspace of $E$ endowed with a Hilbert structure such that the inclusion of the Hilbert space $\mathcal{H}$ into $E$ is *continuous*. We use the notation $\mathcal{H} \in \mathrm{Hilb}(E)$. We then denote by $T_{\mathcal{H}}$ the *Hilbert kernel* naturally associated with $\mathcal{H} \in \mathrm{Hilb}(E)$. We remind that $T_{\mathcal{H}} : E' \to \mathcal{H} \subset E$ is a linear, symmetric and positive application,

*i.e.* $\forall e'$ and $f' \in E', \langle T_{\mathcal{H}} e', f' \rangle_{E,E'} = \langle T_{\mathcal{H}} f', e' \rangle_{E,E'}$ and $\langle T_{\mathcal{H}} e', e' \rangle_{E,E'} \geqslant 0$.

The kernel $T_{\mathcal{H}}$ is in particular characterized by the *representation property*,

$$\forall h \in \mathcal{H}, \forall e' \in E', \langle h, e' \rangle_{E,E'} = (h | T_{\mathcal{H}} e')_{\mathcal{H}}, \tag{2.1}$$

where $(\cdot|\cdot)_{\mathcal{H}}$ is the inner product of $\mathcal{H}$. If $\{h_j, j \in \mathbb{J}\}$ is an orthonormal basis of $\mathcal{H} \in \mathrm{Hilb}(E)$, its associated Hilbert kernel $T_{\mathcal{H}}$ can be written under the form

$$T_{\mathcal{H}} = \sum_{j \in \mathbb{J}} h_j \otimes h_j, \ \ i.e. \ \forall e' \in E', T_{\mathcal{H}} e' = \sum_{j \in \mathbb{J}} \langle h_j, e' \rangle_{E,E'} h_j. \tag{2.2}$$

#### 2.1.2. Reproducing kernel Hilbert space

A RKHS $\mathcal{H}$ of real-valued functions on a set $\mathcal{X}$ is a Hilbert subspace of $\mathbb{R}^{\mathcal{X}}$ (space of real-valued functions on $\mathcal{X}$) endowed with the topology of the pointwise convergence (see [1, 21, 5]).

The *reproducing kernel* $K(\cdot,\cdot)$ of $\mathcal{H}$ is hence linked with the Hilbert kernel $T_{\mathcal{H}}$ by the relation, for all $x$ and $y \in \mathcal{X}$,

$$K(x,y) = \langle T_{\mathcal{H}}\delta_x, \delta_y\rangle_{E,E'}\,, \tag{2.3}$$

where $\delta_x$ is the Dirac measure centered on $x \in \mathcal{X}$ (*i.e.* $\langle f, \delta_x\rangle_{E,E'} = f(x)$ for all $f \in E = \mathbb{R}^{\mathcal{X}}$). This definition is therefore exactly equivalent to the more common definition of a RKHS ; namely that $\mathcal{H}$ is a Hilbert space of real-valued functions on $\mathcal{X}$ such that, for all $x \in \mathcal{X}$, the linear map $L_x : \mathcal{H} \to \mathbb{R}$, $h \mapsto h(x)$, is continuous.

*Remark 2.1.* — The RKHS theory can first appear to be a particular case of the Hilbert subspaces one. In reality, this two theory are equivalent and only differs by their formalism (see [21, 10]). Indeed, one can for instance consider $E$ as a linear subspace of $\mathbb{R}^{E'}$ and then assimilate a Hilbert subspace of $E$ with a RKHS of real-valued functions on $E'$.

## 2.2. Optimal interpolation

Let $\mathcal{H} \in \mathrm{Hilb}(E)$ and $M$ be a linear subspace of $E'$. For a given $\varphi \in \mathcal{H}$, the set of all elements in $\mathcal{H}$ which interpolate $\varphi$ on $M$ can be easily described thanks to the Hilbert subspace structure of $\mathcal{H}$. In what follows, we resume some of the main results concerning the study of such problems. Throughout this article, we will frequently speak about the *interpolation problem associated with* $\mathcal{H} \in \mathrm{Hilb}(E)$ *and* $M$, without necessarily mentioning the element of $\mathcal{H}$ which has to be interpolated.

Let us introduce the set

$$M^0 = \left\{e \in E : \forall e' \in M, \langle e, e'\rangle_{E,E'} = 0\right\}.$$

We define $\mathcal{H}_0 = M^0 \cap \mathcal{H} = T_{\mathcal{H}}(M)^{\perp}$, where $T_{\mathcal{H}}(M)^{\perp}$ denotes the orthogonal, in $\mathcal{H}$, of $T_{\mathcal{H}}(M)$ (and $T_{\mathcal{H}}(M)$ is the set of all $T_{\mathcal{H}}e'$ with $e' \in M$). Then, for a fixed $\varphi \in \mathcal{H}$,

$$\varphi + \left(M^0 \cap \mathcal{H}\right)$$

is the set of all interpolators, in $\mathcal{H}$, of $\varphi$ for $M$.

$\varphi + \left(M^0 \cap \mathcal{H}\right)$ is a non-empty closed affine subspace of $\mathcal{H}$ and is therefore also convex. Thus $\varphi + \left(M^0 \cap \mathcal{H}\right)$ admits a minimal norm element, which we denote $h_{\varphi,M}$ and call *minimal norm interpolator*, or *optimal interpolator*. $h_{\varphi,M}$ is then the orthogonal projection of 0 onto $\varphi + \left(M^0 \cap \mathcal{H}\right)$. Let us remark that this first characterization of the optimal interpolator is essentially *non-constructive*, in the sense that it does not allow the construction of $h_{\varphi,M}$ from the only knowledge of the values of $\varphi$ on $M$.

By definition of the orthogonal projection, $h_{\varphi,M} - 0$ is orthogonal to $\mathcal{H}_0$, *i.e.*

$$h_{\varphi,M} \in \mathcal{H}_0^{\perp} = \left(T_{\mathcal{H}}\left(M\right)^{\perp}\right)^{\perp} = \overline{T_{\mathcal{H}}\left(M\right)}^{\mathcal{H}} = \mathcal{H}_M,$$

with $\mathcal{H}_M$ the closure, in $\mathcal{H}$, of the linear space spanned by $T_{\mathcal{H}}e'$, $e' \in M$. This introduces the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_M$ and implies in particular that $h_{\varphi,M}$ is the only interpolator, in $\mathcal{H}_M$, of $\varphi$ for $M$.

Finally, let $P_{\mathcal{H}_M}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_M$. We know that $\varphi - P_{\mathcal{H}_M}\left[\varphi\right]$ is orthogonal to $\mathcal{H}_M$, thus $\varphi - P_{\mathcal{H}_M}\left[\varphi\right] \in \mathcal{H}_0$, *i.e.* $P_{\mathcal{H}_M}\left[\varphi\right]$ interpolates $\varphi$ for $M$. We finally obtain that

$$h_{\varphi,M} = P_{\mathcal{H}_M}\left[\varphi\right]$$

and this second characterization is suitable for the construction of $h_{\varphi,M}$ from the only knowledge of $\langle\varphi, e'\rangle_{E,E'}$, for $e' \in M$.

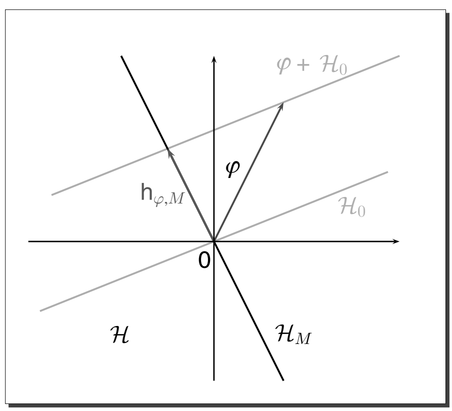


Figure 1. — Schematic representation of optimal interpolation in a Hilbert subspace.

The Hilbert kernel $T_{\mathcal{H}_M}$ of the Hilbert subspace $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$, is linked with $T_{\mathcal{H}}$ by the relation

$$T_{\mathcal{H}_M} = P_{\mathcal{H}_M} T_{\mathcal{H}}.$$

Hence, the knowledge of $T_{\mathcal{H}_M}$ defines the orthogonal projection $P_{\mathcal{H}_M}$ and reciprocally, this result staying true for any closed linear subspace of $\mathcal{H}$. This implies in particular that the Hilbert kernel $T_{\mathcal{H}_0}$ of $\mathcal{H}_0$, $(\cdot|\cdot)_{\mathcal{H}}$, is given by $T_{\mathcal{H}_0} = T_{\mathcal{H}} - T_{\mathcal{H}_M}$.

## 3. Problem adapted integral operators

Let $\mathcal{H} \in \text{Hilb}(E)$ and let $M$ be a linear subspace of $E'$. In all this Section 3, we consider the interpolation problem associated with $\mathcal{H}$ and $M$. We use the same notations and definitions as in Section 2. Let us in particular remind the linked orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_M$.

We introduce the notion of *regular embeddings associated with an interpolation problem* and study the integral operators naturally defined by them. This leads to the construction of specific orthonormal bases of $\mathcal{H}_M$ which are suitable (in the sense of equation (3.14)) for the resolution of the considered interpolation problem.

The results of Sections 3 and 4 hold for any Hilbert subspace $\mathcal{H}$ of $E$, separable or non-separable. However, if $\mathcal{H}$ is non-separable, the existence of a regular embedding requires $\mathcal{H}_M$ to be separable (see Remark 3.4). Let us mention that complementary considerations concerning Section 3 can be found in [10].

### 3.1. Regular embedding and parameterization

Let $(\mathcal{S}, \mathcal{A}, \nu)$ be a general measured set with $\nu$ a $\sigma$-finite measure. We denote by $L^2(\mathcal{S}, \nu)$ the Hilbert space of square-integrable real-valued functions on $\mathcal{S}$ with respect to $\nu$. Let us remind that $L^2(\mathcal{S}, \nu)$ is in fact a quotient space; nevertheless, we make the widespread abuse of notation which consists in assimilating elements of $L^2(\mathcal{S}, \nu)$ with functions on $\mathcal{S}$ (instead of considering *equivalence classes* of $\nu$-almost everywhere equal functions). We will call $L^2(\mathcal{S}, \nu)$ the *auxiliary Hilbert space.*

Let $(\cdot|\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ be respectively the inner product and the norm of $L^2(\mathcal{S}, \nu)$. We recall that

$$\forall f \text{ and } g \in L^2(\mathcal{S}, \nu), (f|g)_{L^2} = \int_{\mathcal{S}} f(s)g(s)d\nu(s).$$

Let us consider an application $\gamma : \mathcal{S} \to E'$. For all $h \in \mathcal{H}$, $\gamma$ allows us to define the function

$$\mathfrak{F}h : \mathcal{S} \to \mathbb{R} \text{ with } \mathfrak{F}h(s) = \langle h, \gamma s\rangle_{E,E'} \text{ for all } s \in \mathcal{S}. \qquad (3.1)$$

We now introduce conditions concerning $L^2(\mathcal{S}, \nu)$, $\gamma$ and $\mathcal{H}$, namely:

C-i. for all $h \in \mathcal{H}$, the function $\mathfrak{F}h : \mathcal{S} \to \mathbb{R}$ is measurable, C-ii. the function $(s,t) \in \mathcal{S} \times \mathcal{S} \mapsto \langle T_{\mathcal{H}}\gamma s, \gamma t\rangle_{E,E'} = (T_{\mathcal{H}}\gamma s | T_{\mathcal{H}}\gamma t)_{\mathcal{H}}$ is measurable,

C-iii. $N = \displaystyle\int_{\mathcal{S}} \|T_{\mathcal{H}}\gamma s\|_{\mathcal{H}}^2 \, d\nu(s) < +\infty.$

Proposition 3.1. — *Under Conditions C-i, C-ii and C-iii, we have $\mathfrak{F}h \in L^2(\mathcal{S},\nu)$ for all $h \in \mathcal{H}$ and*

$$\|\mathfrak{F}h\|_{L^2}^2 \leqslant N \,\|h\|_{\mathcal{H}}^2 . \tag{3.2}$$

*Hence, the linear application $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S},\nu)$, $h \mapsto \mathfrak{F}h$ is continuous.*

*Proof.* — Representation property (2.1), Cauchy-Schwarz inequality applied to $(\cdot|\cdot)_{\mathcal{H}}$ and finally Condition C-iii give

$$\int_{\mathcal{S}} \langle h, \gamma s\rangle_{E,E'}^2 \, d\nu(s) = \int_{\mathcal{S}} (h|T_{\mathcal{H}}\gamma s)_{\mathcal{H}}^2 \, d\nu(s) \leqslant N \,\|h\|_{\mathcal{H}}^2 , \tag{3.3}$$

each integral being well-defined thanks to Conditions C-i and C-ii. □

We now consider the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_M$ and add the following condition on the application $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S},\nu)$,

C-iv. for all $h \in \mathcal{H}$, $\|\mathfrak{F}h\|_{L^2} = 0$ if and only if $h \in \mathcal{H}_0$.

Definition 3.2. — *We call* regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S},\nu)$ adapted to the interpolation problem associated with $\mathcal{H}$ and $M$ *an application $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S},\nu)$ defined from a* parameterization $\gamma : \mathcal{S} \to E'$ *via equation (3.1) and which verifies Conditions C-i, C-ii, C-iii and C-iv.*

Let $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S},\nu)$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S},\nu)$. We consider the linear subspace $\mathfrak{F}(\mathcal{H})$ of $L^2(\mathcal{S},\nu)$ given by $\mathfrak{F}(\mathcal{H}) = \{\mathfrak{F}h, h \in \mathcal{H}\}$ (the image of $\mathcal{H}$ through $\mathfrak{F}$). From C-iv, $\mathfrak{F}(\mathcal{H}_0) = 0$, hence $\mathfrak{F}(\mathcal{H}) = \mathfrak{F}(\mathcal{H}_M)$. We endow this space of the following inner-product:

$$\forall h \text{ and } g \in \mathcal{H}_M, (\mathfrak{F}h|\mathfrak{F}g)_{\mathfrak{F}(\mathcal{H})} = (h|g)_{\mathcal{H}} . \tag{3.4}$$

Proposition 3.3. — *Let $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S},\nu)$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S},\nu)$, then $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ is a Hilbert space. It is isometric to $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$, the isometry being the restriction to $\mathcal{H}_M$ of the regular embedding $\mathfrak{F}$.*

*In addition, the inclusion of the Hilbert space $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ into $L^2(\mathcal{S},\nu)$ is continuous. In other words, $\mathfrak{F}\left(\mathcal{H}_M\right) \in \mathrm{Hilb}\left(L^2(\mathcal{S},\nu)\right)$.*

*Proof.* — The fact that $\mathfrak{F}(\mathcal{H}_M)$ is a Hilbert space isometric to $\mathcal{H}_M$ directly follows from its construction. Further, from Proposition 3.1, we have for all $h \in \mathcal{H}_M$,

$$\|\mathfrak{F}h\|_{L^2}^2 \leqslant N \|h\|_{\mathcal{H}}^2 = N \|\mathfrak{F}h\|_{\mathfrak{F}(\mathcal{H})}^2 ,$$

thus $\mathfrak{F}(\mathcal{H}_M) \in \mathrm{Hilb}\big(L^2(\mathcal{S},\nu)\big)$. $\square$

*Remark 3.4.* — One can for instance consult [9] for a discussion on the conditions appearing in Definition 3.2. The ones we use here are of similar type but specially adapted to the study of the interpolation problem associated with $\mathcal{H}$ and $M$. Indeed, let $\mathfrak{F}$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S},\nu)$ and consider

$$\int_{\mathcal{S}} f(s) \left\langle h, \gamma s\right\rangle_{E,E'} d\nu(s) = (f|\mathfrak{F}h)_{L^2} , \tag{3.5}$$

where $f$ is a fixed element of $L^2(\mathcal{S},\nu)$ and $h \in \mathcal{H}$. From Condition C-iv, the value of expression (3.5) does not vary if one replaces $h$ by $h+h_0$, with $h_0 \in \mathcal{H}_0$. Hence, it only depends of the values of $h$ on $M$ (*i.e.* of $\left\langle h, e'\right\rangle_{E,E'}$ for $e' \in M$), which are the only available informations when considering an interpolation problem associated with $M$ and an element $h$ of $\mathcal{H}$.

When $\mathcal{S}$ is a topological space (endowed with its Borel $\sigma$-algebra) and Conditions C-i, C-ii and C-iii are already verified, C-iv will for instance be realized if for all $h \in \mathcal{H}$, the functions $\mathfrak{F}h$ are continuous and if $M = \mathrm{span}\{\gamma(\mathrm{supp}(\nu))\}$ (*i.e.* $M$ is the linear subspace of $E'$ spanned by $\gamma(\mathrm{supp}(\nu))$ with $\mathrm{supp}(\nu)$ the support of $\nu$).

Finally, note that the existence of a regular embedding $\mathfrak{F}$ associated with the interpolation problem defined by $\mathcal{H}$ and $M$ implies in particular that $\mathcal{H}_M$ *is separable* ; see for instance Proposition 3.8.

*Example 3.5.* — Let us consider a RKHS $\mathcal{H}$ of continuous real-valued functions on a topological space $\mathcal{X}$ and the problem consisting in the interpolation of an element $\varphi$ of $\mathcal{H}$ at given points $x_1, \cdots, x_n$ of $\mathcal{X}$ (*i.e.* $M = \mathrm{span}\{\delta_{x_1}, \cdots, \delta_{x_n}\}$).

One can for instance define a regular embedding for this problem by introducing the measure $\nu = \sum_{i=1}^n w_i \delta_{x_i}$ (with $w_i > 0$) on $\mathcal{S} = \mathcal{X}$ (endowed with its Borel $\sigma$-algebra) and the parameterization $\gamma : x \mapsto \delta_x$ (a different possible parameterization for this problem is given in Section 5).

More generally, if we suppose that the values of $\varphi$ are known on a closed subset $D$ of $\mathcal{X}$ (*i.e.* $M = \mathrm{span}\{\delta_x, x \in D\}$) while keeping the same parameterization $\gamma$, one just has to consider a measure $\nu$ on $\mathcal{X}$ whose support is $D$ and such that Conditions C-iii is also verified.

Note that in this particular example, if one identifies elements of $L^2(\mathcal{S},\nu)$ with functions on $\mathcal{S}=\mathcal{X}$, then the application $\mathfrak{F}$ is in fact the identity operator on $\mathcal{H}$, with $\mathfrak{F}h(x)=h(x)$, for all $h\in\mathcal{H}$ and $x\in\mathcal{X}$.

Proposition 3.6. — *Let $\mathfrak{F}:\mathcal{H}\to L^2(\mathcal{S},\nu)$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S},\nu)$ and consider its adjoint operator ${}^t\mathfrak{F}:L^2(\mathcal{S},\nu)\to\mathcal{H}$ defined by equation (3.7) hereafter. Then for all $f\in L^2(\mathcal{S},\nu)$, ${}^t\mathfrak{F}f\in\mathcal{H}_M$ and we have the following integral representation,*

$$\forall f\in L^2(\mathcal{S},\nu), {}^t\mathfrak{F}f=\int_{\mathcal{S}} f(s)T_{\mathcal{H}}\gamma s\, d\nu(s), \tag{3.6}$$

*this expression having to be understood in the sense of equation (3.8).*

*Proof.* — We remind that ${}^t\mathfrak{F}:L^2(\mathcal{S},\nu)\to\mathcal{H}$ is defined by

$$\forall h\in\mathcal{H},\forall f\in L^2(\mathcal{S},\nu), \left(h|{}^t\mathfrak{F}f\right)_{\mathcal{H}}=\left(\mathfrak{F}h|f\right)_{L^2}. \tag{3.7}$$

From C-iv, we directly deduce that ${}^t\mathfrak{F}f\in\mathcal{H}_M$ for all $f\in L^2(\mathcal{S},\nu)$. Next, by applying the preceding equation to $h=T_{\mathcal{H}}e'$ with $e'\in E'$, we obtain

$$\begin{aligned}\left\langle {}^t\mathfrak{F}f,e'\right\rangle_{E,E'}&=\int_{\mathcal{S}} f(s)\left\langle T_{\mathcal{H}}e',\gamma s\right\rangle_{E,E'} d\nu(s)\\ &=\int_{\mathcal{S}} f(s)\left\langle T_{\mathcal{H}}\gamma s,e'\right\rangle_{E,E'} d\nu(s),\end{aligned} \tag{3.8}$$

which corresponds to equation (3.6) (one can refer to [4] for details about the notion of vectorial integral). □

### 3.2. Integral operator defined by a regular embedding

We still consider the same interpolation problem associated with $\mathcal{H}$ and $M$. Thanks to Proposition 3.3, we know that a regular embedding $\mathfrak{F}$ of $\mathcal{H}_M$ into $L^2(\mathcal{S},\nu)$ defines a Hilbert subspace $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ of $L^2(\mathcal{S},\nu)$. Hence, from the Hilbert subspaces theory, it admits a unique associated Hilbert kernel. If one identifies the continuous dual of $L^2(\mathcal{S},\nu)$ with itself (Riesz-Fréchet representation Theorem), the Hilbert kernel of $\mathfrak{F}(\mathcal{H}_M)$ relatively to $L^2(\mathcal{S},\nu)$ is the unique linear application

$$\mathcal{L}_{\gamma,\nu}:\left(L^2(\mathcal{S},\nu)\right)'=L^2(\mathcal{S},\nu)\to\mathfrak{F}\left(\mathcal{H}_M\right)\subset L^2(\mathcal{S},\nu)$$

which verifies the representation property, for all $h\in\mathcal{H}$ and $f\in L^2(\mathcal{S},\nu)$,

$$\left(\mathfrak{F}h|f\right)_{L^2}=\left(\mathfrak{F}h|\mathcal{L}_{\gamma,\nu}[f]\right)_{\mathfrak{F}(\mathcal{H})}. \tag{3.9}$$

Proposition 3.7. — *Let $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S}, \nu)$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S}, \nu)$ and let $\mathcal{L}_{\gamma,\nu}$ be the Hilbert kernel of $\mathfrak{F}(\mathcal{H}_M) \in \mathrm{Hilb}\left(L^2(\mathcal{S}, \nu)\right)$, then $\mathcal{L}_{\gamma,\nu} = \mathfrak{F}\,{}^t\mathfrak{F}$, i.e. for all $t \in \mathcal{S}$ and $f \in L^2(\mathcal{S}, \nu)$,*

$$\mathcal{L}_{\gamma,\nu}[f](t) = \int_{\mathcal{S}} (T_{\mathcal{H}}\gamma s | T_{\mathcal{H}}\gamma t)_{\mathcal{H}}\, f(s) d\nu(s). \tag{3.10}$$

*Proof.* — By combining equations (3.9), (3.7) and (3.4), we obtain that for $h \in \mathcal{H}$ and $f \in L^2(\mathcal{S}, \nu)$, $(\mathfrak{F}h|f)_{L^2} = (h|{}^t\mathfrak{F}f)_{\mathcal{H}} = (\mathfrak{F}h|\mathfrak{F}\,{}^t\mathfrak{F}f)_{\mathfrak{F}(\mathcal{H})} = (\mathfrak{F}h|\mathcal{L}_{\gamma,\nu}[f])_{\mathfrak{F}(\mathcal{H})}$. We finally deduce equation (3.10) from the integral expression of ${}^t\mathfrak{F}$ given in Proposition 3.6 (equation (3.6)) by applying the preceding relation to $h = T_{\mathcal{H}}\gamma t \in \mathcal{H}$, with $t \in \mathcal{S}$. $\square$

Let us remark that the Hilbert subspace $\mathfrak{F}(\mathcal{H}_M)$ of $L^2(\mathcal{S}, \nu)$ can be assimilated to the RKHS of real-valued functions on $\mathcal{S}$ associated with the reproducing kernel

$$\forall (s,t) \in \mathcal{S} \times \mathcal{S}, \mathcal{K}(s,t) = (T_{\mathcal{H}}\gamma t | T_{\mathcal{H}}\gamma s)_{\mathcal{H}}. \tag{3.11}$$

Hence, $\mathcal{L}_{\gamma,\nu}$ can be seen as a classic integral operator on $L^2(\mathcal{S}, \nu)$ defined by the symmetric and positive kernel $\mathcal{K}(\cdot,\cdot)$ on $\mathcal{S} \times \mathcal{S}$ (see for instance [22, §10] and [9])

We deduce from the theory of integral operators that $\mathcal{L}_{\gamma,\nu}$ is a *Hilbert-Schmidt operator* and therefore a *compact operator*. So $\mathcal{L}_{\gamma,\nu} : L^2(\mathcal{S}, \nu) \to L^2(\mathcal{S}, \nu)$ is diagonalizable and its eigenvalues are positive. We denote by $\lambda_i$ those eigenvalues (repeated according to their algebraic multiplicity) and by $\widetilde{\phi}_i \in L^2(\mathcal{S}, \nu)$ the associated eigenfunctions, with $i \in \mathbb{I}$ (and where $\mathbb{I}$ is a general index set). We remind that $\left\{\widetilde{\phi}_i, i \in \mathbb{I}\right\}$ forms a orthonormal basis of $L^2(\mathcal{S}, \nu)$ and that the set of all *strictly positive* eigenvalues is at most countable.

Proposition 3.8. — *Let $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S}, \nu)$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S}, \nu)$ and consider its associated integral operator*

$$\mathcal{L}_{\gamma,\nu} = \mathfrak{F}\,{}^t\mathfrak{F} : L^2(\mathcal{S}, \nu) \to \mathfrak{F}(\mathcal{H}_M) \subset L^2(\mathcal{S}, \nu).$$

*Denote by $\{\lambda_n, n \in \mathbb{I}_+\}$ the at most countable set (i.e. $\mathbb{I}_+ \subset \mathbb{N}$) of its strictly positive eigenvalues (repeated according to their multiplicity) and let $\widetilde{\phi}_n \in L^2(\mathcal{S}, \nu)$ be their associated eigenfunctions. For all $n \in \mathbb{I}_+$, we define*

$$\phi_n = \frac{1}{\lambda_n}{}^t\mathfrak{F}\widetilde{\phi}_n = \frac{1}{\lambda_n}\int_{\mathcal{S}} \widetilde{\phi}_n(s) T_{\mathcal{H}}\gamma s\, d\nu(s) \in \mathcal{H}_M. \tag{3.12}$$

*Then $\left\{\sqrt{\lambda_n}\phi_n, n \in \mathbb{I}_+\right\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_M$ endowed with the inner product $(\cdot|\cdot)_{\mathcal{H}}$.*

*Proof.* — First remark that from Proposition 3.6, the elements $\phi_n$ of $\mathcal{H}_M$ are well-defined. By definition, we have that, for all $n \in \mathbb{I}_+$ and for all $h \in \mathcal{H}$,

$$\left(\phi_n \middle| h\right)_{\mathcal{H}} = \frac{1}{\lambda_n} \left({}^t\mathfrak{F}\widetilde{\phi}_n \middle| h\right)_{\mathcal{H}} = \frac{1}{\lambda_n} \left(\widetilde{\phi}_n \middle| \mathfrak{F}h\right)_{L^2}. \tag{3.13}$$

As for all $m \in \mathbb{I}_+$, $\mathfrak{F}\phi_m = \widetilde{\phi}_m$, the preceding equation (3.13) applied to $h = \phi_m$, gives that $\left\{\sqrt{\lambda_n}\phi_n n \in \mathbb{I}_+\right\}$ is an orthonormal system of $\mathcal{H}_M$.

From Proposition 3.7 and the properties of Hilbert kernels, we know that the linear subspace spanned by the $\mathcal{L}_{\gamma,\nu}[f]$, $f \in L^2(\mathcal{S}, \nu)$, is dense in $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ (and in particular $\left\{\sqrt{\lambda_n}\widetilde{\phi}_n, n \in \mathbb{I}_+\right\}$ is one of its orthonormal bases). Hence, by the isometry between $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ and $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$, the linear space span$\left\{\sqrt{\lambda_n}\phi_n, n \in \mathbb{I}_+\right\}$ is dense in $\mathfrak{F}(\mathcal{H})$, which concludes the proof. $\square$

In our context of interpolation, the main interest of the elements $\phi_n$ of $\mathcal{H}$, $n \in \mathbb{I}_+$, appearing in Proposition 3.8 is that (see equation (3.13))

$$\forall h \in \mathcal{H}, (\phi_n|h)_{\mathcal{H}} = \frac{1}{\lambda_n} \int_{\mathcal{S}} \widetilde{\phi}_n(s) \left\langle h, \gamma s\right\rangle_{E,E'} d\nu(s). \tag{3.14}$$

Hence, as for equation (3.5) of Remark 3.4, the evaluation of the inner product $(\phi_n|h)_{\mathcal{H}}$ can be directly obtained from the only knowledge of the values of $h$ on $M$.

We are now able to formulate our representation Theorem 4.1, which simply consists in the use of this particular orthonormal basis and of equation (3.14) in order to describe the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_M$.

Before this, we conclude this section by some additional remarks on the structures and applications we have introduced. This will be useful for the rest of our study.

### 3.3. Some important remarks

The definition of a regular embedding $\mathfrak{F}$ of $\mathcal{H}_M$ into $L^2(\mathcal{S}, \nu)$ allows the construction of many applications and structures in addition to the ones studied until now. This section aims at introducing a few of them. Let us mention the article [15], where a similar situation is studied.

#### 3.3.1. Operator on $\mathcal{H}$ defined by a regular embedding

In the same way as an embedding $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S}, \nu)$ defines an integral operator $\mathcal{L}_{\gamma,\nu} = \mathfrak{F}\,{}^t\mathfrak{F}$ on $L^2(\mathcal{S}, \nu)$ (see Proposition 3.7), it also defines an operator on $\mathcal{H}$.

Proposition 3.9. — *Let $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S},\nu)$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S},\nu)$ and consider the framework of Proposition 3.8. We define the following linear operator on $\mathcal{H}$ :*

$$\forall h \in \mathcal{H}, L_{\gamma,\nu}[h] = {}^t\mathfrak{F}\mathfrak{F}h = \int_{\mathcal{S}} \langle h, \gamma s\rangle_{E,E'} T_{\mathcal{H}}\gamma s \, d\nu(s). \tag{3.15}$$

*$L_{\gamma,\nu}$ is a continuous symmetric and positive Hilbert-Schmidt operator on the Hilbert space $\mathcal{H}$, $(\cdot|\cdot)_{\mathcal{H}}$. It is diagonalizable, the eigenspace associated with the null eigenvalues is $\mathcal{H}_0$ and $\sqrt{\lambda_n}\phi_n$, $n \in \mathbb{I}_+$ are the eigenvectors (with $\left\|\sqrt{\lambda_n}\phi_n\right\|_{\mathcal{H}} = 1$) associated with the eigenvalues $\lambda_n$, $n \in \mathbb{I}_+$.*

*Proof.* — The properties of symmetry and positivity of $L_{\gamma,\nu}$ are obvious. Let us give a direct proof of the fact that it is a Hilbert-Schmidt operator. Let $\{h_j, j \in \mathbb{J}\}$ be an orthonormal basis of $\mathcal{H}$. Using equation (2.2) and Fubini's Theorem, we obtain:

$$\sum_{j\in\mathbb{J}} \left\|L_{\gamma,\nu}[h_j]\right\|^2_{\mathcal{H}} = \int_{\mathcal{S}}\int_{\mathcal{S}} (T_{\mathcal{H}}\gamma s|T_{\mathcal{H}}\gamma t)^2_{\mathcal{H}} \, d\nu(s)d\nu(t) \leqslant N^2, \tag{3.16}$$

the last inequality being a consequence of the Cauchy-Schwarz inequality applied to the inner product of $\mathcal{H}$ and of Condition C-iii.

For all $h_0 \in \mathcal{H}_0$, we obviously have ${}^t\mathfrak{F}\mathfrak{F}h_0 = 0$ ; in addition, for $h \in \mathcal{H}$ and $n \in \mathbb{I}_+$,

$$\left({}^t\mathfrak{F}\mathfrak{F}\phi_n\middle|h\right)_{\mathcal{H}} = \lambda_n \left(\frac{1}{\lambda_n}{}^t\mathfrak{F}\widetilde{\phi}_n\middle|h\right)_{\mathcal{H}} = \left(\lambda_n\phi_n|h\right)_{\mathcal{H}}. \tag{3.17}$$

This equation combined with Proposition 3.8 completes the spectral decomposition of $L_{\gamma,\nu}$ and also proves its continuity. □

### 3.3.2. Two additional Hilbert structures

We introduce two Hilbert spaces $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$ and $\overline{\mathcal{H}_M}^{\gamma,\nu}$ which naturally appear when considering a regular embedding $\mathfrak{F}$ of $\mathcal{H}_M$. Note that these two structures will be useful to us for the application of our approach to Gaussian processes conditioning in Section 6.2.

$\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$ is the closure in $L^2(\mathcal{S},\nu)$ of the linear subspace $\mathfrak{F}(\mathcal{H})$. Let us notice that $\left\{\widetilde{\phi}_n, n \in \mathbb{I}_+\right\}$ is obviously one of its orthogonal bases for the inner product $(\cdot|\cdot)_{L^2}$.

Let us now define $\overline{\mathcal{H}_M}^{\gamma,\nu}$. We start by introducing the following symmetric and positive bilinear form on $\mathcal{H}$, for all $h$ and $g \in \mathcal{H}$,

$$(h|g)_{\gamma,\nu} = \big(\mathfrak{F}h\big|\mathfrak{F}g\big)_{L^2} = \int_{\mathcal{S}} \langle h, \gamma s\rangle_{E,E'} \langle g, \gamma s\rangle_{E,E'} \, d\nu(s). \tag{3.18}$$

We also set $\|h\|_{\gamma,\nu}^2 = (h|h)_{\gamma,\nu}$. Condition C-iv implies that the null space of $(\cdot|\cdot)_{\gamma,\nu}$ is $\mathcal{H}_0$ (*i.e.* for $h \in \mathcal{H}$, $\|h\|_{\gamma,\nu} = 0$ if and only if $h \in \mathcal{H}_0$) and $\mathcal{H}_M$ endowed with $(\cdot|\cdot)_{\gamma,\nu}$ is hence a pre-Hilbert space. We then denote by $\overline{\mathcal{H}_M}^{\gamma,\nu}$ the completed of $\mathcal{H}_M$ for $\|\cdot\|_{\gamma,\nu}$.

Remark that the operator $L_{\gamma,\nu}$ (considered as an operator on $\mathcal{H}_M$) can be naturally extended to $\overline{\mathcal{H}_M}^{\gamma,\nu}$ by continuity. $L_{\gamma,\nu}$ is then a Hilbert-Schmidt operator on $\overline{\mathcal{H}_M}^{\gamma,\nu}$, $(\cdot|\cdot)_{\gamma,\nu}$. It is symmetric and positive definite, its eigenvalues are $\lambda_n$, $n \in \mathbb{I}_+$ and each one is associated with the eigenvector $\phi_n$ (and $\|\phi_n\|_{\gamma,\nu} = 1$).

### 3.3.3. Isometries

We are finally in presence of four isometric Hilbert spaces,

$$\mathcal{H}_M, \mathfrak{F}(\mathcal{H}), \overline{\mathfrak{F}(\mathcal{H})}^{L^2} \text{ and } \overline{\mathcal{H}_M}^{\gamma,\nu}.$$

As we have seen in Proposition 3.3, the isometry between $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$ and $\mathfrak{F}(\mathcal{H})$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ is the restriction of $\mathfrak{F}$ to $\mathcal{H}_M$. The continuous extension of this first isometry defines the isometry between $\overline{\mathcal{H}_M}^{\gamma,\nu}$, $(\cdot|\cdot)_{\gamma,\nu}$ and $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$, $(\cdot|\cdot)_{L^2}$.

The isometry between $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$, $(\cdot|\cdot)_{L^2}$ and $\mathfrak{F}(\mathcal{H})$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ is given by

$$\forall n \in \mathbb{I}_+, \widetilde{\phi}_n \leftrightarrow \sqrt{\lambda_n}\widetilde{\phi}_n.$$

It is in fact the restriction of the square-root of $\mathcal{L}_{\gamma,\nu}$ to $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$, with

$$\mathcal{L}_{\gamma,\nu}^{\frac{1}{2}} \left[\sum_{i\in\mathbb{I}} \alpha_i \widetilde{\phi}_i\right] = \sum_{i\in\mathbb{I}} \alpha_i \sqrt{\lambda_i}\widetilde{\phi}_i,$$

where $\sum_{i\in\mathbb{I}} \alpha_i \widetilde{\phi}_i \in L^2(\mathcal{S},\nu)$. We obviously have $\mathcal{L}_{\gamma,\nu} = \mathcal{L}_{\gamma,\nu}^{\frac{1}{2}} \circ \mathcal{L}_{\gamma,\nu}^{\frac{1}{2}}$.

### 3.3.4. Pseudoinverse of a regular embedding

Let us consider the framework of Proposition 3.8. One can define the pseudoinverse (or generalized inverse) $\mathfrak{F}^\dagger$ of $\mathfrak{F}$ by

$$\forall n \in \mathbb{I}_+, \mathfrak{F}^\dagger \widetilde{\phi}_n = \phi_n = \frac{1}{\lambda_n} {}^t\mathfrak{F}\widetilde{\phi}_n \tag{3.19}$$

and for $i \in \mathbb{I}\backslash\mathbb{I}_+$ (*i.e.* $\lambda_i = 0$), $\mathfrak{F}^\dagger \widetilde{\phi}_i = 0$. Then, $\mathfrak{F}^\dagger$ is well-defined from $L^2(\mathcal{S},\nu)$ onto $\overline{\mathcal{H}_M}^{\gamma,\nu}$ and

$$\forall f \in L^2(\mathcal{S},\nu), \mathfrak{F}^\dagger f = \sum_{n\in\mathbb{I}_+} \left(f\middle|\widetilde{\phi}_n\right)_{L^2} \phi_n \in \overline{\mathcal{H}_M}^{\gamma,\nu}. \tag{3.20}$$

The restriction of $\mathfrak{F}^\dagger$ to $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$ defines the inverse of the isometry between $\overline{\mathcal{H}_M}^{\gamma,\nu}$ and $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$. In the same way, its restriction to $\mathcal{H}_M$ gives the inverse of the isometry between $\mathcal{H}_M$ and $\mathfrak{F}(\mathcal{H})$. We have in particular

$$P_{\mathcal{H}_M} = \mathfrak{F}^\dagger\mathfrak{F}, \tag{3.21}$$

which is in fact an equivalent formulation of Theorem 4.1.

## 4. Representation and approximation of the optimal interpolator

For $\mathcal{H} \in \mathrm{Hilb}(E)$, we consider the optimal interpolation problem in $\mathcal{H}$ defined by $\varphi \in \mathcal{H}$ and a linear subspace $M$ of $E'$. In order to apply Section 3 results, we suppose $M$ such that $\mathcal{H}_M$ is separable (thus, $M$ can be an arbitrary linear subspace of $E'$ if $\mathcal{H}$ is itself separable).

### 4.1. Spectral representation for optimal interpolation

Once an orthonormal basis of $\mathcal{H}_M$ known, one can easily express the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_M$. Then, in order to compute the optimal interpolator of $\varphi \in \mathcal{H}$ for $M$ (see Section 2), we need to be able to evaluate the inner-product in $\mathcal{H}$ between $\varphi$ and each elements of the considered basis of $\mathcal{H}_M$ ; and this from the only knowledge of the values of $\varphi$ on $M$ (which are in our context of interpolation the only available informations concerning $\varphi$). This is precisely the property of the orthonormal basis of $\mathcal{H}_M$ associated with a regular embedding $\mathfrak{F}$, its elements indeed verify equation (3.14).

Remark that in order to be applied to a given interpolation problem (associated with $\mathcal{H}$ and $M$), our approach requires the preliminary choice of a measurable space $(\mathcal{S}, \mathcal{A}, \nu)$ and of a parameterization $\gamma : \mathcal{S} \to E'$ allowing the definition of a regular embedding $\mathfrak{F}$ of $\mathcal{H}_M$ into the auxiliary space $L^2(\mathcal{S}, \nu)$. Some considerations concerning this choice are discussed in Section 4.3.

Theorem 4.1. — *Let $\mathfrak{F}$ be a regular embedding of $\mathcal{H}_M$ into $L^2(\mathcal{S}, \nu)$ and consider the orthonormal basis $\left\{\sqrt{\lambda_n}\phi_n n \in \mathbb{I}_+\right\}$ of $\mathcal{H}_M$ associated with $\mathfrak{F}$. Then, for $\varphi \in \mathcal{H}$, we have*

$$P_{\mathcal{H}_M}[\varphi] = \sum_{n \in \mathbb{I}_+} \phi_n \int_{\mathcal{S}} \langle \phi_n, \gamma s \rangle_{E,E'} \langle \varphi, \gamma s \rangle_{E,E'} \, d\nu(s). \tag{4.1}$$

*Proof.* — It is a simple consequence of Proposition 3.8 and equation (3.13),

$$\begin{aligned} P_{\mathcal{H}_M}[\varphi] &= \sum_{n \in \mathbb{I}_+} \sqrt{\lambda_n}\phi_n \left(\sqrt{\lambda_n}\phi_n \middle| \varphi\right)_{\mathcal{H}} = \sum_{n \in \mathbb{I}_+} \lambda_n \phi_n \left(\widetilde{\phi}_n \middle| \mathfrak{F}\varphi\right)_{\mathfrak{F}(\mathcal{H})} \\ &= \sum_{n \in \mathbb{I}_+} \phi_n \left(\widetilde{\phi}_n \middle| \mathfrak{F}\varphi\right)_{L^2}. \end{aligned}$$

See also equation (3.21) for an equivalent formulation of this result. □

Let us remark that the sum appearing in equation (4.1) converges by construction in $\mathcal{H}$. Since $\mathcal{H} \in \mathrm{Hilb}(E)$, it also converges for the initial topology of $E$ and for its weak topology $\sigma(E, E')$. Then, in particular, for all $e' \in E'$,

$$\langle P_{\mathcal{H}_M}[\varphi], e' \rangle_{E,E'} = \sum_{n \in \mathbb{I}_+} \langle \phi_n, e' \rangle_{E,E'} \int_{\mathcal{S}} \langle \phi_n, \gamma s \rangle_{E,E'} \langle \varphi, \gamma s \rangle_{E,E'} \, d\nu(s).$$

Finally, because of the continuous inclusion of $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$ into $\overline{\mathcal{H}_M}^{\gamma,\nu}$, the considered sum also converges for $\|\cdot\|_{\gamma,\nu}$.

## 4.2. Truncated approach and approximation

In practice, even if the spectral decomposition of the operator $\mathcal{L}_{\gamma,\nu} = \mathfrak{F}^t\mathfrak{F}$ is known, it is not always possible, for instance for numerical reasons, to consider all the terms appearing in the Mercer decomposition of $T_{\mathcal{H}_M}$, *i.e.*

$$\forall e' \in E', T_{\mathcal{H}_M} e' = \sum_{n \in \mathbb{I}_+} \lambda_n \langle \phi_n, e' \rangle_{E,E'} \phi_n.$$

A classic alternative simply consists in not considering each of the sum terms, but only a part of them. In this case, one currently speaks about *spectrum truncation* and these are usually the largest eigenvalues which are conserved. The following section aims at giving a first brief study of the use of this alternative in our context. Let us signalize that considerations about the *optimal character* of the approximation by truncation based on the largest eigenvalues are developed in Section 6.4.

Note that we also have to keep in mind that, in the most part of application cases, the true analytical spectral decomposition of $\mathcal{L}_{\gamma,\nu}$ would be unknown. Hence, the study of the behavior of the proposed approach when dealing with approximated spectrum is of great importance in regards of applications.

### 4.2.1. Spectrum truncation

Let us assume that we dispose of an approximated kernel defined from a subset $\mathbb{I}_{trc}$ of $\mathbb{I}_+$, that is, for all $e' \in E'$,

$$T_{\mathcal{H}_M^{trc}} e' = \sum_{n \in \mathbb{I}_{trc}} \lambda_n \left\langle \phi_n, e' \right\rangle_{E,E'} \phi_n.$$

For $\varphi \in \mathcal{H}$, we then obtain an approximation of the optimal interpolator $P_{\mathcal{H}_M}[\varphi]$ that we denote by $P_{\mathcal{H}_M^{trc}}[\varphi]$. We have

$$\begin{aligned} \forall e' \in E', \left\langle P_{\mathcal{H}_M^{trc}}[\varphi], e' \right\rangle_{E,E'} &= \left( \varphi \middle| T_{\mathcal{H}_M^{trc}} e' \right)_{\mathcal{H}} \qquad (4.2) \\ &= \sum_{n \in \mathbb{I}_{trc}} \left\langle \phi_n, e' \right\rangle_{E,E'} \int_{\mathcal{S}} \widetilde{\phi}_n(s) \left\langle \varphi, \gamma s \right\rangle_{E,E'} d\nu(s), \end{aligned}$$

where $\mathcal{H}_M^{trc}$ is the closure in $\mathcal{H}$ of the subspace spanned by the $\phi_n$, for $n \in \mathbb{I}_{trc}$.

Theorem 4.2. — *Let us consider the general framework of Section 4.2. We also introduce the set* $\mathbb{I}_{err}^{trc} = \mathbb{I}_+ \backslash \mathbb{I}_{trc}$, *then*

$$\forall e' \in E', \left\langle \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi], e' \right\rangle_{E,E'}^2 \leqslant \|\varphi\|_{\mathcal{H}}^2 \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n \left\langle \phi_n, e' \right\rangle_{E,E'}^2 \qquad (4.3)$$

$$\textit{and} \quad \left\| \varphi - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi] \right\|_{\gamma,\nu}^2 \leqslant \|\varphi\|_{\mathcal{H}}^2 \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n. \qquad (4.4)$$

*Proof.* — By definition, we have for all $e' \in E'$:

$$\left\langle \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi], e' \right\rangle_{E,E'} = \left( \varphi \middle| \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n \langle \phi_n, e' \rangle_{E,E'} \phi_n \right)_{\mathcal{H}}. \tag{4.5}$$

To obtain expression (4.3), we just have to remark that the Cauchy-Schwarz inequality applied to equation (4.5) gives

$$\left\langle \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi], e' \right\rangle_{E,E'}^2 \leqslant \|\varphi\|_{\mathcal{H}}^2 \left\| \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n \langle \phi_n, e' \rangle_{E,E'} \phi_n \right\|_{\mathcal{H}}^2$$

and that from Proposition 3.8,

$$\left\| \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n \langle \phi_n, e' \rangle_{E,E'} \phi_n \right\|_{\mathcal{H}}^2 = \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n \langle \phi_n, e' \rangle_{E,E'}^2 .$$

Next, from equation (4.3) and the definition of $\|\cdot\|_{\gamma,\nu}^2$ (see section 3.3), we have,

$$\int_{\mathcal{S}} \left\langle \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi], \gamma s \right\rangle_{E,E'}^2 d\nu(s) \leqslant \|\varphi\|_{\mathcal{H}}^2 \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n \int_{\mathcal{S}} \langle \phi_n, \gamma s \rangle_{E,E'}^2 d\nu(s).$$

This inequality gives, combined with the fact that $\|\phi_n\|_{\gamma,\nu}^2 = 1$ and C-iii,

$$\left\| \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi] \right\|_{\gamma,\nu}^2 \leqslant \|\varphi\|_{\mathcal{H}}^2 \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n.$$

To conclude, we remark that by definition of $\|\cdot\|_{\gamma,\nu}$,

$$\left\| \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi] \right\|_{\gamma,\nu}^2 = \left\| \varphi - \mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi] \right\|_{\gamma,\nu}^2 . \tag{4.6}$$

□

We call $\sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n$ the *spectral error term*. It will be in practice evaluated by considering

$$\begin{aligned} \sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n &= \sum_{n \in \mathbb{I}_+} \lambda_n - \sum_{n \in \mathbb{I}_{trc}} \lambda_n \\ &= \int_{\mathcal{S}} \|T_{\mathcal{H}} \gamma s\|_{\mathcal{H}}^2 d\nu(s) - \sum_{n \in \mathbb{I}_{trc}} \lambda_n. \end{aligned} \tag{4.7}$$

A good indicator (see also Section 6.4) of the overall quality of the obtained approximation can classically be found in the ratios

$$\frac{\sum_{n \in \mathbb{I}_{trc}} \lambda_n}{\sum_{n \in \mathbb{I}_+} \lambda_n} = 1 - \frac{\sum_{n \in \mathbb{I}_{err}^{trc}} \lambda_n}{\sum_{n \in \mathbb{I}_+} \lambda_n}.$$

## 4.3. About the choice of the parameterization

In this section, we mention some general considerations concerning the choice of the parameterization. By parameterization, we mean here the overall process leading to the construction of a regular embedding of $\mathcal{H}_M$ into an auxiliary Hilbert space.

### 4.3.1. Computational aspects

In the interpolation context of Theorem 4.1, the parameterization can just appear as a tool allowing to obtain the representation formula (4.1). No matter its choice from the moment it allows the definition of a regular embedding $\mathfrak{F}$ of $\mathcal{H}_M$.

Nevertheless, if one envisages the computation of the elements constituting the orthonormal basis of $\mathcal{H}_M$ associated with the considered embedding (Proposition 3.8), this choice takes importance. Indeed, it in part determines the operator which has to be diagonalized. It then appears reasonable to try to make a choice that defines a simplest as possible spectral problem. One will for instance try to be in a position allowing an analytical resolution, or the use of a particular numerical method.

In such a context, an illustration of what appears to us as relatively judicious choices of parameterizations can be found in [10, Section 3.3]. In this particular example, appropriated choices allow to obtain an analytical expression for many of the involved objects and certain prediction formulas concerning the two parameters Brownian sheet are hence obtained in an original way.

### 4.3.2. The approximation case

In addition of this first consideration, the choice of the parametrization has a direct influence on the behavior of the optimal interpolator approximation obtained by spectral truncation in equation (4.3). Indeed, different choices of parameterization for a same problem lead to different approximations of the optimal interpolator, and this even if the spectral ratios of the considered truncations are equal.

The parameterization directly influences the way $\mathbb{P}_{\mathcal{H}_M^{trc}}[\varphi]$ approximates $\mathbb{P}_{\mathcal{H}_M}[\varphi]$ on $M$. It hence offers a way to modulate the accuracy of the approximation in function of the elements of $M$. This characteristic of *tunable precision* could offer interesting possibilities in regards of applications.

# 5. Finite Case

## 5.1. Context and notations

We suppose that $M$ is of finite dimension, *i.e.* $M = \operatorname{span}\{\mu_1, \cdots, \mu_n\}$, with $n \in \mathbb{N}^*$. Let us define the matrix $\mathbf{T} \in \mathbb{R}^{n\times n}$ by

$$\text{for } 1 \leqslant i, j \leqslant n, \mathbf{T}_{i,j} = (T_{\mathcal{H}}\mu_i | T_{\mathcal{H}}\mu_j)_{\mathcal{H}}. \tag{5.1}$$

For simplicity and without loss of generality, we assume that the $\mu_i \in E'$ are such that the symmetric and positive matrix $\mathbf{T}$ is *invertible*. For convenience, we introduce the following matrix type notations

$$\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)^T \text{ and } \mathbf{T} = \left(T_{\mathcal{H}}\boldsymbol{\mu} | T_{\mathcal{H}}\boldsymbol{\mu}^T\right)_{\mathcal{H}} = \left\langle T_{\mathcal{H}}\boldsymbol{\mu}, \boldsymbol{\mu}^T\right\rangle = \left\langle \boldsymbol{\mu}, T_{\mathcal{H}}\boldsymbol{\mu}^T\right\rangle,$$

where $T_{\mathcal{H}}\boldsymbol{\mu} = (T_{\mathcal{H}}\mu_1, \cdots, T_{\mathcal{H}}\mu_n)^T$ is a column vector. Hence, for $\varphi \in \mathcal{H}$, the optimal interpolator of $\varphi$ for $M$ can be written under the form

$$h_{\varphi,M} = T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{T}^{-1}\langle\boldsymbol{\mu}, \varphi\rangle, \tag{5.2}$$

with $\langle\boldsymbol{\mu}, \varphi\rangle = \left(\langle\varphi, \mu_1\rangle_{E,E'}, \cdots, \langle\varphi, \mu_n\rangle_{E,E'}\right)^T$. Remark for instance that with our notations, $\langle\boldsymbol{\mu}, \varphi\rangle^T = \left\langle\varphi, \boldsymbol{\mu}^T\right\rangle$, and for $e \in E$ and $e' \in E'$, $\langle e, e'\rangle_{E,E'} = \langle e, e'\rangle = \langle e', e\rangle$. So, we will write, for $e' \in E'$, $\langle h_{\varphi,M}, e'\rangle_{E,E'} = \left\langle e', T_{\mathcal{H}}\boldsymbol{\mu}^T\right\rangle\mathbf{T}^{-1}\langle\boldsymbol{\mu}, \varphi\rangle$.

The aim of this section is to prove, by explicit calculations, that the expression of the minimal norm interpolator given in equation (5.2) is equal to the one given in equation (4.1) of Theorem 4.1.

## 5.2. Parameterization

Let us define a trivial parameterization of this problem. Let $\mathcal{S} = \{1, \cdots, n\}$ and consider the measure $\nu$ on $\mathcal{S}$ which assigns a weight $w_i > 0$ to each $i \in \mathcal{S} = \{1, \cdots, n\}$. The auxiliary space $L^2(\mathcal{S}, \nu)$ can then be identified to the space $\mathbb{R}^n$ endowed with the inner-product $(\mathbf{x}|\mathbf{y})_{\mathbf{W}} = \mathbf{x}^T\mathbf{W}\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are two column vectors of $\mathbb{R}^n$ and with $\mathbf{W}$ the matrix

$$\mathbf{W} = \operatorname{diag}(w_1, \cdots, w_n).$$

Let us remark that we identify a vector of $\mathbb{R}^n$ with the column vector of its coefficients in the canonical basis of $\mathbb{R}^n$.

We next consider the application $\gamma : \mathcal{S} \to E'$, given by $\gamma i = \mu_i$ for all $i \in \{1, \cdots, n\}$. The associated application $\mathfrak{F}$ is trivially a regular embedding associated with our problem. It verifies, for all $h \in \mathcal{H}$, $\mathfrak{F}h = \langle \boldsymbol{\mu}, h \rangle \in \mathbb{R}^n$ (and $\mathfrak{F}h(i) = \langle h, \mu_i \rangle_{E,E'}$ for all $i \in \{1, \cdots, n\} = \mathcal{S}$). If we identify $L^2(\mathcal{S}, \nu)$ with $\mathbb{R}^n$, then for $\boldsymbol{\alpha} \in \mathbb{R}^n$

$$ {}^t\mathfrak{F}\boldsymbol{\alpha} = T_{\mathcal{H}} \boldsymbol{\mu}^T \mathbf{W} \boldsymbol{\alpha} \in \mathcal{H}_M. \tag{5.3} $$

We finally obtain that $\mathcal{L}_{\gamma,\nu} = \mathfrak{F}\, {}^t\mathfrak{F}$ in given by

$$ \mathfrak{F}\, {}^t\mathfrak{F}\boldsymbol{\alpha} = \mathbf{T}\mathbf{W}\boldsymbol{\alpha}. \tag{5.4} $$

In the same way, we have (see Proposition 3.9)

$$ \begin{aligned} \forall h \in \mathcal{H}, L_{\gamma,\nu}[h] &= {}^t\mathfrak{F}\mathfrak{F}h = \int_{\mathcal{S}} \langle h, \gamma s \rangle_{E,E'} T_{\mathcal{H}} \gamma s \, d\nu(s) \\ &= \sum_{i=1}^{n} w_i \langle h, \mu_i \rangle_{E,E'} T_{\mathcal{H}} \mu_i = T_{\mathcal{H}} \boldsymbol{\mu}^T \mathbf{W} \langle \boldsymbol{\mu}, h \rangle . \end{aligned} \tag{5.5} $$

The symmetric and positive bilinear form $(\cdot|\cdot)_{\gamma,\nu}$ on $\mathcal{H}$, associated with $\mathfrak{F}$ via equation (3.18), is given by, for $h$ and $g \in \mathcal{H}$,

$$ (h|g)_{\gamma,\nu} = \sum_{i=1}^{n} w_i \langle h, \mu_i \rangle_{E,E'} \langle g, \mu_i \rangle_{E,E'} = \left\langle h, \boldsymbol{\mu}^T \right\rangle \mathbf{W} \langle \boldsymbol{\mu}, g \rangle . $$

### 5.3. Spectral decomposition

Let $\lambda_1 > 0, \cdots, \lambda_n > 0$ be the eigenvalues of $\mathbf{TW}$ and let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ their associated eigenvectors, *i.e.* $\mathbf{TW} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$ with

$$ \boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \cdots, \lambda_n) \text{ and } \mathbf{P} = (\mathbf{v}_1 | \cdots | \mathbf{v}_n) . $$

Note that $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ forms an orthonormal basis of $\mathbb{R}^n$, $(\cdot|\cdot)_{\mathbf{W}}$, *i.e.*

$$ \mathbf{P}^T \mathbf{W} \mathbf{P} = \mathrm{Id}_{n\times n}, \text{ where } \mathrm{Id}_{n\times n} \text{ is the } n \times n \text{ identity matrix.} \tag{5.6} $$

For $k \in \{1, \cdots, n\}$, let

$$ \phi_k = \frac{1}{\lambda_k} {}^t\mathfrak{F}\mathbf{v}_k = \frac{1}{\lambda_k} T_{\mathcal{H}} \boldsymbol{\mu}^T \mathbf{W} \mathbf{v}_k \in \mathcal{H}_M. \tag{5.7} $$

Proposition 5.1 *Under the assumptions of Section 5, we have for all $\varphi \in \mathcal{H}$,*

$$ T_{\mathcal{H}} \boldsymbol{\mu}^T \mathbf{T}^{-1} \langle \boldsymbol{\mu}, \varphi \rangle = \sum_{k=1}^{n} \phi_k \int_{\mathcal{S}} \langle \phi_k, \gamma s \rangle_{E,E'} \langle \varphi, \gamma s \rangle_{E,E'} d\nu(s). $$

*Proof.* — We have

$$\begin{aligned} T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{T}^{-1}\langle\boldsymbol{\mu},\varphi\rangle &= T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{W}\mathbf{W}^{-1}\mathbf{T}^{-1}\langle\boldsymbol{\mu},\varphi\rangle \\ &= T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{W}\mathbf{P}\boldsymbol{\Lambda}^{-1}\mathbf{P}^{-1}\langle\boldsymbol{\mu},\varphi\rangle. \end{aligned}$$

Let us study the terms appearing in this last expression. First, from equation (5.7),

$$T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{W}\mathbf{P}\boldsymbol{\Lambda}^{-1} = (\phi_1, \cdots, \phi_n).$$

Next, using equation (5.6), we find

$$\mathbf{P}^{-1}\langle\boldsymbol{\mu},\varphi\rangle = \left(\mathbf{P}^T\mathbf{W}\mathbf{P}\right)^{-1}\mathbf{P}^T\mathbf{W}\langle\boldsymbol{\mu},\varphi\rangle = \mathbf{P}^T\mathbf{W}\langle\boldsymbol{\mu},\varphi\rangle.$$

To conclude, we remark that, for $k \in \{1, \cdots, n\}$,

$$\int_{\mathcal{S}} \langle\phi_k, \gamma s\rangle_{E,E'} \langle\varphi, \gamma s\rangle_{E,E'} \, d\nu(s) = \sum_{i=1}^{n} w_i \langle\varphi, \mu_i\rangle_{E,E'} \langle\phi_k, \mu_i\rangle_{E,E'}$$

is the $k$-th component of the vector $\mathbf{P}^T\mathbf{W}\langle\boldsymbol{\mu},\varphi\rangle$. □

## 6. Application to Gaussian process models

Optimal interpolation in Hilbert subspaces and Gaussian process models are intrinsically linked. In this section, we recall some of the main properties concerning the conditioning of Gaussian processes in the framework of topological vector spaces. We also apply the spectral approach developed in Sections 3 and 4 to the conditioning problem. The IMSE-optimal character of the approximation by truncation is finally addressed in Section 6.4.

### 6.1. Notations an recalls

Let $\mathcal{H}$ be a *separable* Hilbert subspace of $E$. In all Section 6, we assume that $\mathcal{H}$ is the Cameron-Martin space of a centered Gaussian process $Y$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbf{H}$ the Gaussian Hilbert space associated with $Y$ (see [13]). We remind that $\mathbf{H}$ is a closed linear subspace of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of second order centered real random variables (r.v.) on $(\Omega, \mathcal{F}, \mathbb{P})$.

For the sake of simplicity, we assume that $E$ is a Banach space (see [7, 24] for more general frameworks). We also consider that $Y$ takes its values in $E$ with probability 1 (*i.e.* the triplet $\left(j, \mathcal{H}, \overline{\mathcal{H}}^E\right)$ is an *abstract Wiener space*, where $\overline{\mathcal{H}}^E$ is the completed of $\mathcal{H}$ in $E$ and $j$ the continuous inclusion of $\mathcal{H}$ into $\overline{\mathcal{H}}^E$, see [11]).

We denote by $\mathcal{I} : \mathcal{H} \to \mathbf{H}$ the isometry between $\mathcal{H}$ and $\mathbf{H}$, it verifies

$$\mathbb{E}\left(\mathcal{I}h\mathcal{I}g\right) = \left(h|g\right)_{\mathcal{H}},$$

where $\mathbb{E}\left(\mathcal{I}h\mathcal{I}g\right)$ represents the inner-product in $L^2\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ between the two centered random variables $\mathcal{I}h$ and $\mathcal{I}g \in \mathbf{H}$. Let us also add, for $e' \in E'$,

$$\langle Y, e' \rangle_{E,E'} \overset{\text{(notation)}}{=} Y_{e'} = \mathcal{I}\left(T_{\mathcal{H}}e'\right).$$

One can consult, among others, [3, 25, 10] for more details about the previous notions.

For a linear subspace $M$ of $E'$, $P_{\mathcal{H}_M}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_M$. We then introduce the orthogonal projection $P_{\mathbf{H}_M}$ of $\mathbf{H}$ onto $\mathbf{H}_M = \mathcal{I}\left(\mathcal{H}_M\right)$. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow[\mathcal{I}]{} & \mathbf{H} \\ \big\downarrow P_{\mathcal{H}_M} & & \big\downarrow P_{\mathbf{H}_M} \\ \mathcal{H}_M & \xrightarrow[\mathcal{I}]{} & \mathbf{H}_M \end{array} \tag{6.1}$$

$T_{\mathcal{H}_M} = P_{\mathcal{H}_M} T_{\mathcal{H}}$ is the Hilbert kernel of $\mathcal{H}_M$, hence, by isometry,

$$\forall e' \in E', \mathcal{I}\left(T_{\mathcal{H}_M}e'\right) = P_{\mathbf{H}_M}\left[Y_{e'}\right] \overset{\text{(notation)}}{=} \mathbb{E}\left(Y_{e'}|Y_{f'}, f' \in M\right). \tag{6.2}$$

For all $e' \in E'$, the r.v. $P_{\mathbf{H}_M}\left[Y_{e'}\right]$ is called the *conditional mean of* $Y_{e'}$ *knowing* $Y_{f'}$ *for* $f' \in M$ and $T_{\mathcal{H}_0}$ is the associated *conditional covariance kernel.* The notion of *conditional law* of the process $Y$ is addressed in Section 6.3.

### 6.2. Spectral approach for conditioning

We now consider the general framework of Section 3.

PROPOSITION 6.1. — *Let us consider the centered Gaussian process* $\left(Y_{\gamma s}\right)_{s \in \mathcal{S}}$. *Under the assumptions of Theorem 4.1, the sample paths of* $\left(Y_{\gamma s}\right)_{s \in \mathcal{S}}$ *are in* $L^2(\mathcal{S}, \nu)$ *with probability 1. In addition,*

$$\mathbb{E}\Big[\int_{\mathcal{S}} \left(Y_{\gamma s}\right)^2 d\nu(s)\Big] = \sum_{n \in \mathbb{I}_+} \lambda_n\big(= N\big).$$

*Proof.* — For $s \in \mathcal{S}$, we remind that $Y_{\gamma s} = \mathcal{I}\left(T_{\mathcal{H}}\gamma s\right)$. Let $\{h_j, j \in \mathbb{J}\}$ be an orthonormal basis of $\mathcal{H}$. As in equation (2.2), $\left(Y_{\gamma s}\right)_{s \in \mathcal{S}}$ admits the

Karhunen-Loève expansion

$$\forall s \in \mathcal{S}, Y_{\gamma s} = \sum_{j\in\mathbb{J}} \langle h_j, \gamma s\rangle_{E,E'} \mathcal{I}(h_j), \tag{6.3}$$

where the $\mathcal{I}(h_j) = \zeta_j$, $j \in \mathbb{J}$, form by isometry an orthonormal basis of $\mathbf{H}$ (such independent $\mathcal{N}(0,1)$ r.v. are sometime called *orthogaussian*, see [8]). We then deduce from Condition C-i that the sample paths of $(Y_{\gamma s})_{s\in\mathcal{S}}$ are measurable (as real-valued functions on $\mathcal{S}$) with probability 1.

One can obviously choose the orthonormal basis $\{h_j, j \in \mathbb{J}\}$ of $\mathcal{H}$ such that it coincides on $\mathcal{H}_M$ with its orthonormal basis $\left\{\sqrt{\lambda_n}\phi_n n \in \mathbb{I}_+\right\}$ associated with the considered regular embedding $\mathfrak{F}$ (Proposition 3.8). Then, from C-iii and C-iv,

$$\begin{aligned} \sum_{j\in\mathbb{J}} \int_{\mathcal{S}} \mathbb{E}\left[\left(\mathfrak{F}h_j(s)\zeta_j\right)^2\right] d\nu(s) &= \sum_{j\in\mathbb{J}} \|\mathfrak{F}h_j\|^2_{L^2} \\ &= \sum_{n\in\mathbb{I}_+} \left\|\sqrt{\lambda_n}\widetilde{\phi}_n\right\|^2_{L^2} = \sum_{n\in\mathbb{I}_+} \lambda_n < +\infty. \end{aligned}$$

Thus, the sum $\sum_{j\in\mathbb{J}} \|\mathfrak{F}h_j\zeta_j\|^2_{L^2(\nu\otimes\mathbb{P})}$ is convergent, which, form Tonelli's Theorem, implies the convergence in $L^2(\nu\otimes\mathbb{P})$ of

$$(s,\omega) \mapsto \sum_{j\in\mathbb{J}} \mathfrak{F}h_j(s)\zeta_j(\omega), \text{ with } s \in \mathcal{S} \text{ and } \omega \in \Omega$$

and finally completes the proof. □

Theorem 6.2. — *Let $Y$ be a centered Gaussian process with values in $E$ and covariance kernel $T_{\mathcal{H}}$. Let $M$ be a linear subspace of $E'$. Under the assumptions of Theorem 4.1, we have for all $e' \in E'$,*

$$\mathbb{E}\left(Y_{e'} | Y_{f'}, f' \in M\right) = \sum_{n\in\mathbb{I}_+} \langle \phi_n, e'\rangle_{E,E'} \int_{\mathcal{S}} \widetilde{\phi}_n(s) Y_{\gamma s}\, d\nu(s). \tag{6.4}$$

*In addition, the centered Gaussian process with covariance kernel $T_{\mathcal{H}_M}$ (that is the process corresponding to $\mathbb{E}\left(Y_{e'} | Y_{f'}, f' \in M\right)$, $e' \in E'$) takes its values in $\overline{\mathcal{H}_M}^{\gamma,\nu}$ with probability 1.*

*Proof.* — We have to verify that the right member of equation (6.4) is well-defined. From Proposition 3.8, we know that for all $e' \in E'$,

$$T_{\mathcal{H}_M} e' = \sum_{n\in\mathbb{I}_+} \lambda_n \langle \phi_n, e'\rangle_{E,E'} \phi_n,$$

this series being convergent in $\mathcal{H}$. Hence,

$$\mathcal{I}\left(T_{\mathcal{H}_M} e'\right) = \sum_{n\in\mathbb{I}_+} \lambda_n \left\langle \phi_n, e' \right\rangle_{E,E'} \mathcal{I}\left(\phi_n\right). \tag{6.5}$$

For $n \in \mathbb{I}_+$, we remind that the $\phi_n \in \mathcal{H}_M$ are given by (equation (3.12))

$$\phi_n = \frac{1}{\lambda_n} \int_{\mathcal{S}} \widetilde{\phi}_n(s) T_{\mathcal{H}} \gamma s \, d\nu(s).$$

Now, Proposition 6.1 assures that the expression

$$\mathcal{I}\left(\phi_n\right) = \frac{1}{\lambda_n} \int_{\mathcal{S}} \widetilde{\phi}_n(s) \mathcal{I}\left(T_{\mathcal{H}} \gamma s\right) \, d\nu(s) = \frac{1}{\lambda_n} \int_{\mathcal{S}} \widetilde{\phi}_n(s) Y_{\gamma s} d\nu(s) \tag{6.6}$$

keeps sense under our working hypotheses. The last point of Theorem 6.2 is consequence of expansion (6.4), Proposition 6.1 and isometry between $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$ and $\overline{\mathcal{H}_M}^{\gamma,\nu}$. $\square$

### 6.3. A note on regular conditional probabilities

We now study the conditional laws of the process $Y$ relatively to the knowledge of the values take by its sample paths on $M$. Because, in our study, $\mathcal{H}_M$ can be infinite dimensional, the construction of a regular conditional probability requires some precautions. We consider a case where such a conditional probability exists (see for instance [25], this is for instance always true if $\mathcal{H}_M$ is finite dimensional). We finally introduce an additional assumption which assures the existence of a spectral representation for the mean of the conditional laws and corresponding to a natural extension of equation (4.1).

Let us consider the following condition

$$\overline{\mathcal{H}}^E = \overline{\mathcal{H}_M}^E \oplus \overline{\mathcal{H}_0}^E, \tag{6.7}$$

which means that $\overline{\mathcal{H}}^E = \overline{\mathcal{H}_M}^E + \overline{\mathcal{H}_0}^E$ with $\overline{\mathcal{H}_M}^E \cap \overline{\mathcal{H}_0}^E = \{0\}$.

Condition (6.7) assures us of the existence of the linear continuous projection $\mathfrak{P}$ of $\overline{\mathcal{H}}^E$ onto $\overline{\mathcal{H}_M}^E$ parallel to $\overline{\mathcal{H}_0}^E$ (*i.e.* $\mathfrak{P}h_0 = 0$ for all $h_0 \in \overline{\mathcal{H}_0}^E$).

We then consider the family of Gaussian measures on $\overline{\mathcal{H}}^E$ with mean $\mathfrak{P}[\Psi] \in \overline{\mathcal{H}_M}^E$ for $\Psi \in \overline{\mathcal{H}}^E$ and covariance kernel $T_{\mathcal{H}_0}$. From [25, Theorem 3.11], such a family defines a *regular conditional probability* over $\overline{\mathcal{H}}^E$ relative to the knowledge of $Y$ on $M$. In such case, the following notation is often used

$$\mathfrak{P}[\Psi] = \mathbb{E}\big(Y_{e'} \big| Y_{f'} = \langle \Psi, f' \rangle_{E,E'}, f' \in M\big). \tag{6.8}$$

Note that if we denote by $\mu_Y$ the Gaussian measure on $\overline{\mathcal{H}}^E$ associated with $Y$, the preceding regular conditional probability corresponds to the *disintegration* of $\mu_Y$ relatively to $\mathfrak{P}$.

Now, let us also suppose that $\overline{\mathcal{H}_M}^E$ can be continuously injected into $\overline{\mathcal{H}_M}^{\gamma,\nu}$, what we write

$$\overline{\mathcal{H}_M}^E \hookrightarrow \overline{\mathcal{H}_M}^{\gamma,\nu}. \tag{6.9}$$

We next consider the extension $\check{\mathfrak{F}}$ of $\mathfrak{F}$ to $\overline{\mathcal{H}}^E$ defines by $\check{\mathfrak{F}}h_0 = 0$ for all $h_0 \in \overline{\mathcal{H}_0}^E$ and, on $\overline{\mathcal{H}_M}^E$, by the continuous extension of the restriction of $\mathfrak{F}$ to $\mathcal{H}_M$. From Section 3.3 and condition (6.9), $\check{\mathfrak{F}}$ is well-defined from $\overline{\mathcal{H}}^E$ on $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$.

Then, as for $P_{\mathcal{H}_M} = \mathfrak{F}^\dagger \mathfrak{F}$, we have $\mathfrak{P} = \mathfrak{F}^\dagger \check{\mathfrak{F}}$. Remark that this last expression is well-defined in regards of the definition of $\mathfrak{F}^\dagger$ given in Section 3.3. We finally obtain the spectral representation formula

$$\forall \Psi \in \overline{\mathcal{H}}^E, \mathfrak{P}[\Psi] = \sum_{n \in \mathbb{I}_+} \langle \phi_n, e' \rangle_{E,E'} \int_{\mathcal{S}} \widetilde{\phi}_n(s) \langle \Psi, \gamma s \rangle_{E,E'} \, d\nu(s). \tag{6.10}$$

*Remark 6.3.* — We have the well known equality, for $\varphi \in \mathcal{H}$ and $e' \in E'$,

$$\langle P_{\mathcal{H}_M}[\varphi], e' \rangle_{E,E'} = \mathbb{E}\big(Y_{e'} \big| Y_{f'} = \langle \varphi, f' \rangle_{E,E'}, f' \in M\big). \tag{6.11}$$

## 6.4. Optimal approximation

The results and considerations of Section 4.2 can also be extended to the Gaussian processes case. In this section, we discuss of the optimal character of the approximation by truncation.

Let $Y$ be a centered Gaussian process with values in $E$ and covariance $T_{\mathcal{H}}$. Under the assumptions of Theorem 6.2, we consider a subset $\mathbb{I}_{trc}$ of $\mathbb{I}_+$ composed of the largest eigenvalues of $\mathcal{L}_{\gamma,\nu}$, in the sense that,

$$\text{if } i \in \mathbb{I}_{err}^{trc} = \mathbb{I}_+ \backslash \mathbb{I}_{trc} \text{ and } n \in \mathbb{I}_{trc}, \text{ then } \lambda_i \leqslant \lambda_n. \tag{6.12}$$

Proposition 6.4. — *Let $\mathcal{H}_M^{app}$ be any closed linear subspace of $\mathcal{H}_M$ and denote by $T_{\mathcal{H}_M^{app}}$ its associated Hilbert kernel. For $e' \in E'$, we introduce the two approximations of $Z_{e'} = \mathbb{E}\big(Y_{e'} \big| Y_{f'} = \langle \varphi, f' \rangle_{E,E'}, f' \in M\big)$,*

$$Z_{e'}^{trc} = \mathcal{I}\left(T_{\mathcal{H}_M^{trc}} e'\right) \ \textit{and} \ Z_{e'}^{app} = \mathcal{I}\left(T_{\mathcal{H}_M^{app}} e'\right). \tag{6.13}$$

*If $\mathcal{H}_M^{trc}$ and $\mathcal{H}_M^{app}$ have the same finite dimension $N \in \mathbb{N}^*$, then under equation (6.12),*

$$\mathbb{E}\left( \left\|Y - Z^{trc}\right\|_{\gamma,\nu}^2 \right) \leqslant \mathbb{E}\left( \left\|Y - Z^{app}\right\|_{\gamma,\nu}^2 \right). \tag{6.14}$$

*Proof.* — From Theorem 4.2 and Proposition 6.1, we have $\mathbb{E}\left(\|Z^{trc}\|_{\gamma,\nu}^2\right) = \mathbb{E}\left(\int_{\mathcal{S}} \left(Z_{\gamma s}^{trc}\right)^2 d\nu(s)\right) = \int_{\mathcal{S}} \left\|T_{\mathcal{H}_M^{trc}} \gamma s\right\|_{\mathcal{H}}^2 d\nu(s) = \sum_{n \in \mathbb{I}_{trc}} \lambda_n$

$$\text{and } \mathbb{E}\left( \left\|Y - Z^{trc}\right\|_{\gamma,\nu}^2 \right) = \mathbb{E}\left( \|Y\|_{\gamma,\nu}^2 \right) - \mathbb{E}\left( \left\|Z^{trc}\right\|_{\gamma,\nu}^2 \right). \tag{6.15}$$

Let $f_1, \cdots, f_N$ be an orthonormal basis of $\mathcal{H}_M^{app}$ and consider its decomposition in the orthonormal basis associated with the considered regular embedding, that is

$$\forall i \in \{1, \cdots, N\}, f_i = \sum_{k \in \mathbb{I}_+} \alpha_{i,k} \sqrt{\lambda_k} \phi_k, \text{ with } \alpha_{i,k} \in \mathbb{R}$$

and where, for $i$ and $j \in \{1, \cdots, N\}$, $\sum_{k \in \mathbb{I}_+} \alpha_{i,k} \alpha_{j,k} = \delta_{i,j}$ (Kronecker delta). We hence easily obtain that $\mathbb{E}\left(\|Z^{app}\|_{\gamma,\nu}^2\right) = \sum_{i=1}^N \sum_{k \in \mathbb{I}_+} \alpha_{i,k}^2 \lambda_k$. Next, using for instance convex combinations arguments, we remark that

$$\sum_{n \in \mathbb{I}_{trc}} \lambda_n \geqslant \sum_{i=1}^N \sum_{k \in \mathbb{I}_+} \alpha_{i,k}^2 \lambda_k. \tag{6.16}$$

We finally conclude thanks to equation (6.15). □

In the same way as in equation (4.6), let us remark that we have

$$\mathbb{E}\left( \|Y - Z^{app}\|_{\gamma,\nu}^2 \right) = \mathbb{E}\left( \|Z - Z^{app}\|_{\gamma,\nu}^2 \right). \tag{6.17}$$

Thus, in regards of $\|\cdot\|_{\gamma,\nu}$ (*i.e.* in the sense of equation (6.14)), for $e' \in E'$, $Z_{e'}^{trc}$ is the best approximation of the conditional mean $\mathbb{E}\left(Y_{e'} \middle| Y_{f'} = \langle \varphi, f' \rangle_{E,E'}, f' \in M\right)$ based on $N$ elements of $\mathcal{H}_M$. One can hence speak about a certain *IMSE-optimality* (Integrated Mean Square Error) of the approximation by truncation. This point is for instance illustrated in Section 7.5.

## 7. Example of application

### 7.1. The problem

Let $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{H}$ be the RKHS of real-valued functions on $\mathcal{X}$ (see Section 2.1) associated with the kernel (squared exponential or Gaussian kernel, see for instance [20]), for $x$ and $y \in \mathcal{X}$,

$$K(x,y) = e^{-\frac{\|x-y\|^2}{\sigma^2}}, \text{ with } \sigma > 0 \text{ and } \|\cdot\| \text{ the euclidean norm.}$$

For $m \in \mathbb{N}$, let $\mathcal{E}^m \subset \mathbb{R}^{\mathcal{X}}$ be the subspace of functions of class $C^m$ endowed with the topology of the uniform convergence on the compact subsets of $\mathcal{X}$ for all the derivatives of order $\leqslant m$ (of general order if $m = +\infty$). From [21, Proposition 25], for all $m \in \mathbb{N}$ (and also for $m = +\infty$), $\mathcal{H}$ is a Hilbert subspace of $\mathcal{E}^m$. In what follows, we will consider $\mathcal{H}$ as a Hilbert subspace of $E = \mathcal{E}^1$.

Let $x = (x_1, x_2) \in \mathcal{X}$, we also use polar coordinates, *i.e.* $x = (r_x \cos\alpha_x, r_x \sin\alpha_x)$ with $r_x \in \mathbb{R}_+$ and $\alpha_x \in [0, 2\pi]$. For $x \in \mathcal{X}$, we define $\delta_x \in E'$ and $\eta_x \in E'$ by

$$\forall f \in E, \langle f, \delta_x \rangle_{E,E'} = f(x) \text{ and } \langle f, \eta_x \rangle_{E,E'} = \frac{\partial f}{\partial r_x}(x),$$

$\delta_x$ is the Dirac measure centered on $x$ and $\eta_x$ corresponds to the evaluation of the radial derivative at $x$.

Let $\mathcal{C} \subset \mathbb{R}^2$ be the circle of center 0 and radius $R > 0$. We consider the linear subspaces of $E'$

$$M_D = \operatorname{span}\{\delta_t, t \in \mathcal{C}\} \text{ and } M_N = \operatorname{span}\{\eta_t, t \in \mathcal{C}\}$$

and $M_R = M_D + M_N$ ($D$ and $N$ stand for *Dirichlet* and *Neumann conditions*, $R$ for *Robin*). The aim of this example is to approximate the kernel $K_{0_R}(\cdot,\cdot)$ of the subspace $\mathcal{H}_{0_R}$ of functions $h \in \mathcal{H}$ such that

$$\forall t \in \mathcal{C}, \langle h, \delta_t \rangle_{E,E'} = 0 \text{ and } \langle h, \eta_t \rangle_{E,E'} = 0.$$

Nevertheless, let us remark that what follows contains all the necessary informations to treat the general interpolation problem associated with $\mathcal{H}$, an element $\varphi \in \mathcal{H}$ and $M_R$ (see [10]).

We present a two steps methodology. The first step (Section 7.2) consists in considering independently the interpolation problems in $\mathcal{H}$ associated with $M_N$ and $M_D$. Thanks to the study of a third operator (Section 7.3), we finally combine our results in Section 7.4 and obtain a model in which both values and radial derivatives are controlled on the circle (*Robin condition*). Numerical computations are finally presented in Section 7.5.

### 7.2. The two independent problems

Let us introduce the linear subspaces of $\mathcal{H}$ naturally associated with $M_D$ and $M_N$:

$$\mathcal{H}_{M_D} = \mathcal{H}_{0_D}^{\perp} = \overline{\text{span}\,\{K(t,\cdot), t \in \mathcal{C}\}}^{\mathcal{H}} \text{ and}$$
$$\mathcal{H}_{M_N} = \mathcal{H}_{0_N}^{\perp} = \overline{\text{span}\left\{\frac{\partial K}{\partial r_t}(t,\cdot), t \in \mathcal{C}\right\}}^{\mathcal{H}}.$$

We denote by $T_{\mathcal{H}_{M_D}}$ and $K_{M_D}(\cdot,\cdot)$ the Hilbert kernel and the reproducing kernel of $\mathcal{H}_{M_D}$ respectively. We use similar notations for the kernels associated with $M_N$.

#### 7.2.1. Parameterization

Let $\mathcal{S} = [0, 2\pi]$ endowed with its natural Lebesgue measure (up to the multiplicative constant $R$) and consider the Hilbert space $L^2([0,2\pi])$ of squared integrable real-valued functions (with respect to the Lebesgue measure) on $[0,2\pi]$, endowed with the norm

$$\forall f \in L^2([0,2\pi]), \|f\|_{L^2}^2 = \int_0^{2\pi} f(\theta)^2 R d\theta.$$

$L^2([0,2\pi])$ will play the role of the auxiliary Hilbert space introduced in Section 3.

We pose $s_{R,\theta} = (R\cos\theta, R\sin\theta) \in \mathcal{C}$ and introduce the parameterizations

$$\gamma_D : [0,2\pi] \to M_D, \theta \mapsto \delta_{s_{R,\theta}} \text{ and } \gamma_N : [0,2\pi] \to M_N, \theta \mapsto \eta_{s_{R,\theta}}.$$

One easily verifies that they implicitly define two regular embeddings $\mathfrak{F}_D$ and $\mathfrak{F}_N$ of respectively $\mathcal{H}_{M_D}$ and $\mathcal{H}_{M_N}$ into $L^2([0,2\pi])$ (see also Remark 7.2) which are given by

$$\mathfrak{F}_D h(\theta) = \langle h, \gamma_D\theta\rangle_{E,E'} \text{ and } \mathfrak{F}_N h(\theta) = \langle h, \gamma_N\theta\rangle_{E,E'}, \text{ for all } h \in \mathcal{H} \text{ and } \theta \in [0,2\pi].$$

The associated integral operators $\mathcal{L}_D = \mathfrak{F}_D\,{}^t\mathfrak{F}_D$ and $\mathcal{L}_N = \mathfrak{F}_N\,{}^t\mathfrak{F}_N$ are defined on $L^2([0,2\pi])$ and

$$\mathcal{L}_D[f](\alpha) = \int_0^{2\pi} K\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta, \tag{7.1}$$

$$\mathcal{L}_N[f](\alpha) = \int_0^{2\pi} \frac{\partial^2 K}{\partial r_s \partial r_x} \left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta, \tag{7.2}$$

where $x_{R,\alpha} = (R\cos\alpha, R\sin\alpha) \in \mathcal{C}$, $\alpha \in [0, 2\pi]$ and $f \in L^2\left([0, 2\pi]\right)$.

*Remark 7.1.* — Let $x = (r_x \cos\alpha_x, r_x \sin\alpha_x)$ and $y = (r_y \cos\alpha_y, r_y \sin\alpha_y) \in \mathcal{X}$,

$$K(x, y) = \exp\left\{-\frac{1}{\sigma^2}\left(r_x^2 + r_y^2 - 2r_x r_y \cos(\alpha_x - \alpha_y)\right)\right\},$$

$$\frac{\partial K}{\partial r_x}(x, y) = -\frac{2}{\sigma^2}\big(r_x - r_y\cos(\alpha_x - \alpha_y)\big)K(x, y) \text{ and}$$

$$\frac{\partial^2 K}{\partial r_y \partial r_x}(x, y) = \frac{2}{\sigma^2}\cos(\alpha_x - \alpha_y)K(x, y)$$
$$+\frac{4}{\sigma^4}\big(r_x - r_y\cos(\alpha_x - \alpha_y)\big)\big(r_y - r_x\cos(\alpha_x - \alpha_y)\big)K(x, y).$$

### 7.2.2. Spectral decomposition

Using for instance some arguments of parity, we obtain that the eigenvalues of $\mathcal{L}_D$ are, for $n \geqslant 0$:

$$\lambda_n^D = Re^{-\frac{2R^2}{\sigma^2}} \int_0^{2\pi} e^{\frac{2R^2}{\sigma^2}\cos\theta} \cos(n\theta) d\theta.$$

The ones of $\mathcal{L}_N$ are, for $n \geqslant 0$:

$$\lambda_n^N = \int_0^{2\pi} \left[A\cos\theta + B(1 + \cos^2\theta)\right] e^{-\frac{2R^2}{\sigma^2}(1-\cos\theta)} \cos(n\theta) R d\theta,$$

with $A = \frac{2}{\sigma^2} - \frac{8R^2}{\sigma^4}$ and $B = \frac{4R^2}{\sigma^4}$.

The two operators $\mathcal{L}_D$ and $\mathcal{L}_N$ admit the same eigenfunctions. $\lambda_0^D$ and $\lambda_0^N$ are of *multiplicity 1* and associated with

$$\begin{array}{cccc} \widetilde{\phi}_0 : & [0, 2\pi] & \longrightarrow & \mathbb{R} \\ & \alpha & \longmapsto & \frac{1}{\sqrt{2\pi R}} \end{array} \tag{7.3}$$

For $n \geqslant 1$, $\lambda_n^D$ and $\lambda_n^N$ are of *multiplicity 2* and associated with, for $\alpha \in [0, 2\pi]$,

$$\widetilde{\phi}_n^c(\alpha) = \frac{1}{\sqrt{\pi R}}\cos n\alpha \text{ and } \widetilde{\phi}_n^s(\alpha) = \frac{1}{\sqrt{\pi R}}\sin n\alpha. \tag{7.4}$$

*Remark 7.2.* — The two spaces $\overline{\mathfrak{F}_D(\mathcal{H})}^{L^2}$ and $\overline{\mathfrak{F}_N(\mathcal{H})}^{L^2}$ are the same and correspond to the linear subspace of $2\pi$-periodic functions of $L^2_{loc}(\mathbb{R})$. As the set of all eigenfunctions of each operator $\mathcal{L}_D$ and $\mathcal{L}_N$ coincides with the classical discrete Fourier basis, $\mathcal{L}_D$ and $\mathcal{L}_N$ do not admit other non-null eigenvalue.

Concerning the operator $\mathcal{L}_D$, one can for instance consult [12] where similar spectral problems are studied (see also Remark 7.5).

We are now able to express the orthonormal bases of $\mathcal{H}_{M_D}$ and $\mathcal{H}_{M_N}$, associated with $\mathfrak{F}_D$ and $\mathfrak{F}_N$ respectively (see Proposition 3.8). For $\mathfrak{F}_D$, we introduce the elements $\phi_0^D$, $\phi_n^{cD}$ and $\phi_n^{sD} \in \mathcal{H}_{M_D}$, $n \geqslant 1$, with for instance

$$\forall n \geqslant 1, \forall x \in \mathcal{X}, \phi_n^{sD}(x) = \frac{1}{\lambda_n^D} \int_0^{2\pi} K(s_{R,\theta}, x) \frac{\sin(n\theta)}{\sqrt{\pi R}} R d\theta. \tag{7.5}$$

For $\mathfrak{F}_N$, we introduce $\phi_0^N$, $\phi_n^{cN}$ and $\phi_n^{sN} \in \mathcal{H}_{M_N}$, $n \geqslant 1$, with

$$\forall n \geqslant 1, \forall x \in \mathcal{X}, \phi_n^{cN}(x) = \frac{1}{\lambda_n^N} \int_0^{2\pi} \frac{\partial K}{\partial r_s}(s_{R,\theta}, x) \frac{\cos(n\theta)}{\sqrt{\pi R}} R d\theta. \tag{7.6}$$

Examples of numerical computations are presented in Section 7.5. We are now going to study the behavior on the circle $\mathcal{C}$ of this two families of functions.

## 7.3. An interesting operator

For $x_{R,\alpha} \in \mathcal{C}$, let us consider the values

$$\left\langle \phi_0^N, \delta_{x_{R,\alpha}} \right\rangle_{E,E'}, \left\langle \phi_n^{cN}, \delta_{x_{R,\alpha}} \right\rangle_{E,E'} \text{ and } \left\langle \phi_n^{sN}, \delta_{x_{R,\alpha}} \right\rangle_{E,E'} \text{ for } n \geqslant 1,$$

$$\left\langle \phi_0^D, \eta_{x_{R,\alpha}} \right\rangle_{E,E'}, \left\langle \phi_n^{cD}, \eta_{x_{R,\alpha}} \right\rangle_{E,E'} \text{ and } \left\langle \phi_n^{sD}, \eta_{x_{R,\alpha}} \right\rangle_{E,E'} \text{ for } n \geqslant 1.$$

It appears that those ones are all linked with the integral operator $J_\nu$ given by:

$$\forall f \in L^2\left([0, 2\pi]\right), J_\nu[f](\alpha) = \int_0^{2\pi} \frac{\partial K}{\partial r_s}\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta. \tag{7.7}$$

Indeed, we have for instance, for $n \geqslant 1$,

$$\left\langle \phi_n^{cN}, \delta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{1}{\lambda_n^N} \int_0^{2\pi} \frac{\partial K}{\partial r_s}(s_{R,\theta}, x_{R,\alpha}) \frac{\cos(n\theta)}{\sqrt{\pi R}} R d\theta,$$

$$\left\langle \phi_n^{cD}, \eta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{1}{\lambda_n^D} \int_0^{2\pi} \frac{\partial K}{\partial r_x}(s_{R,\theta}, x_{R,\alpha}) \frac{\cos(n\theta)}{\sqrt{\pi R}} R d\theta$$

and $\dfrac{\partial K}{\partial r_s}(s_{R,\theta}, x_{R,\alpha}) = \dfrac{\partial K}{\partial r_x}(s_{R,\theta}, x_{R,\alpha})$. The operator $J_\nu$ is *self-adjoint but not positive*. For, $n \geqslant 0$, its eigenvalues are

$$\rho_n = \int_0^{2\pi} \frac{-2R}{\sigma^2}(1-\cos\theta)e^{-\frac{2R^2}{\sigma^2}(1-\cos\theta)}\cos(n\theta)Rd\theta.$$

We remark in particular that $\rho_0 < 0$. The eigenvalue $\rho_0$ is of multiplicity 1 and is associated with the same eigenfunction $\widetilde{\phi}_0$ than $\lambda_0^D$ and $\lambda_0^N$. For $n \geqslant 1$, the $\rho_n$ are of multiplicity 2 and are also associated with the same eigenfunctions $\widetilde{\phi}_n^c$ and $\widetilde{\phi}_n^s$ than $\lambda_n^D$ and $\lambda_n^N$. The same argument than the one used in Remark 7.2 assures that $J_\nu$ does not admit other non-null eigenvalue.

The spectrum of the operator $J_{\gamma,\nu}$ has a particular behavior since the number of its negative eigenvalues depends of the ratio between $R$ and $\sigma^2$. The values of $\rho_n$, $0 \leqslant n \leqslant 30$, for $R = 3$ and $\sigma^2 = 2$ are presented in Figure 2.

Concerning the orthonormal basis of $\mathcal{H}_{M_N}$ associated with $\mathfrak{F}_N$, we finally obtain, for $x_{R,\alpha} = (R\cos\alpha, R\sin\alpha) \in \mathcal{C}$,

$$\begin{gathered}\left\langle \phi_0^N, \delta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{\rho_0}{\lambda_0^N}\widetilde{\phi}_0(\alpha) \text{ and for all } n \geqslant 1, \\ \left\langle \phi_n^{cN}, \delta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{\rho_n}{\lambda_n^N}\widetilde{\phi}_n^c(\alpha) \text{ and } \left\langle \phi_n^{sN}, \delta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{\rho_n}{\lambda_n^N}\widetilde{\phi}_n^s(\alpha).\end{gathered} \tag{7.8}$$

As for the one of $\mathcal{H}_{M_N}$ associated with $\mathfrak{F}_N$, we find

$$\begin{gathered}\left\langle \phi_0^D, \eta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{\rho_0}{\lambda_0^D}\widetilde{\phi}_0(\alpha) \text{ and for all } n \geqslant 1, \\ \left\langle \phi_n^{cD}, \eta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{\rho_n}{\lambda_n^D}\widetilde{\phi}_n^c(\alpha) \text{ and } \left\langle \phi_n^{sD}, \eta_{x_{R,\alpha}} \right\rangle_{E,E'} = \frac{\rho_n}{\lambda_n^D}\widetilde{\phi}_n^s(\alpha).\end{gathered} \tag{7.9}$$

### 7.4. Double Constraint

We combine the results of the two preceding sections in order to obtain a model that both takes account of the values of the function and its radial derivative on $\mathcal{C}$. We present an original way to express the kernel $K_{0_R}(\cdot,\cdot)$ of the subspace $\mathcal{H}_{0_R}$ of functions $h \in \mathcal{H}$ such that

$$\forall t \in \mathcal{C}, \langle h, \eta_t \rangle_{E,E'} = 0 \text{ and } \langle h, \delta_t \rangle_{E,E'} = 0.$$

We use Section 7.2 in order to describe the kernels $T_{\mathcal{H}_{0_N}}$ of $\mathcal{H}_{0_N}$ and $T_{\mathcal{H}_{0_D}}$ of $\mathcal{H}_{0_D}$). We now consider the interpolation problem associated with $T_{\mathcal{H}_{0_N}}$ and $M_D$ (arbitrary choice, we could equivalently consider the problem associated with $T_{\mathcal{H}_{0_D}}$ and $M_N$, see Remark 7.4).

*Remark 7.3.* — This two approaches consist, roughly speaking, in considering the two decompositions $P_{\mathcal{H}_{0_R}} = P_{\mathcal{H}_{0_D}} P_{\mathcal{H}_{0_N}}$ or $P_{\mathcal{H}_{0_R}} = P_{\mathcal{H}_{0_N}} P_{\mathcal{H}_{0_D}}$, where $P_{\mathcal{H}_{0_R}}$, $P_{\mathcal{H}_{0_D}}$ and $P_{\mathcal{H}_{0_N}}$ are the orthogonal projections of $\mathcal{H}$ onto $\mathcal{H}_{0_R}$, $\mathcal{H}_{0_D}$ and $\mathcal{H}_{0_N}$ respectively.

So, let us consider the kernel $K_{0_N}(\cdot,\cdot)$ of $\mathcal{H}_{0_N}$, the subspace of functions $h \in \mathcal{H}$ such that $\frac{\partial h}{\partial r_t}(t) = 0$ for all $t \in \mathcal{C}$. We recall that

$$K_{0_N}(x,y) = K(x,y) - K_{M_N}(x,y) \text{ with}$$

$$K_{M_N}(x,y) = \lambda_0^N \phi_0^N(x)\phi_0^N(y) + \sum_{n\geqslant 1} \lambda_n^N \left[\phi_n^{cN}(x)\phi_n^{cN}(y) + \phi_n^{sN}(x)\phi_n^{sN}(y)\right].$$

Thanks to the parameterization $\gamma_D$ and the kernel $K_{0_N}(\cdot,\cdot)$, we define a regular embedding of $\mathcal{H}_{M_N} \cap \mathcal{H}_{0_N}$ into $L^2([0,2\pi])$. We finally obtain the integral operator $\mathcal{L}_{R1}$ defined by, for $f \in L^2([0,2\pi])$ and $\alpha \in [0,2\pi]$:

$$\mathcal{L}_{R1}[f](\alpha) = \int_0^{2\pi} K_{0_N}\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta.$$

From the study of the operator $J_\nu$ (Section 7.3), we obtain that the eigenvalues $\lambda_n^{R1}$, $n \in \mathbb{N}$, of $\mathcal{L}_{R1}$ are given by

$$\lambda_n^{R1} = \lambda_n^D - \frac{\rho_n^2}{\lambda_n^N}. \tag{7.10}$$

The eigenvalue $\lambda_0^{R1}$ is associated with the eigenfunction $\widetilde{\phi}_0$ of equation (7.3), for $n \geqslant 1$, $\lambda_n^{R1}$ is of multiplicity 2 and associated with $\widetilde{\phi}_n^c$ and $\widetilde{\phi}_n^s$ (equation (7.4)).

We finally introduce the elements $\phi_0^{R1}$, $\phi_n^{cR1}$ and $\phi_n^{sR1}$, $n \geqslant 1$, which are associated with $\mathcal{L}_\nu^{R1}$ via Proposition 3.8. Straightforward calculations give

$$\phi_0^{R1} = \frac{1}{\lambda_0^{R1}}\left(\lambda_0^D \phi_0^D - \rho_0 \phi_0^N\right) \text{ and for } n \geqslant 1,$$
$$\phi_n^{cR1} = \frac{1}{\lambda_n^{R1}}\left(\lambda_n^D \phi_n^{cD} - \rho_n \phi_n^{cN}\right) \text{ and } \phi_n^{sR1} = \frac{1}{\lambda_n^{R1}}\left(\lambda_n^D \phi_n^{sD} - \rho_n \phi_n^{sN}\right). \tag{7.11}$$

*Remark 7.4.* — Instead of first considering $M_N$ and next $M_D$, one can operate in an inverse way. This leads to the study of the operator

$$\mathcal{L}_{R2}[f](\alpha) = \int_0^{2\pi} \frac{\partial^2 K_{0_D}}{\partial r_s \partial r_x}\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta,$$

which is associated with $K_{0_D}(\cdot,\cdot)$ and $M_N$ via the parameterization $\gamma_N$. Using the results of Section 7.3, we find that the eigenvalues $\lambda_n^{R2}$, $n \in \mathbb{N}$, of $\mathcal{L}_{R2}$ are

$$\lambda_n^{R2} = \lambda_n^N - \frac{\rho_n^2}{\lambda_n^D}. \tag{7.12}$$

$\lambda_0^{R2}$ is of multiplicity 1 and is associated with $\widetilde{\phi}_0$. $\lambda_n^{R2}$ for $n \geqslant 1$ are of multiplicity 2 and associated with $\widetilde{\phi}_n^c$ and $\widetilde{\phi}_n^s$. We finally obtain

$$\phi_0^{R2} = \frac{1}{\lambda_0^{R2}} \left(\lambda_0^N \phi_0^N - \rho_0 \phi_0^D\right) \text{ and for } n \geqslant 1,$$
$$\phi_n^{cR2} = \frac{1}{\lambda_n^{R2}} \left(\lambda_n^N \phi_n^{cN} - \rho_n \phi_n^{cD}\right) \text{ and } \phi_n^{sR2} = \frac{1}{\lambda_n^{R2}} \left(\lambda_n^N \phi_n^{sN} - \rho_n \phi_n^{sD}\right). \tag{7.13}$$

*Remark 7.5.* — If one conserves the same parameterizations $\gamma_D$ and $\gamma_N$ and the same auxiliary space $L^2([0,2\pi])$, one can in fact obtain similar results for any stationary covariance kernel with required regularity (in the sense that it defines a Hilbert subspace of $E = \mathcal{E}^1$) and of the type

$$K(x,y) = \int_{\mathbb{R}} e^{-i\xi\|x-y\|} d\tau(\xi),$$

with $\|\cdot\|$ the euclidean norm of $\mathbb{R}^2$ (and $i^2 = 1$) and where $\tau$ is a finite symmetric positive measure on $\mathbb{R}$ (see the Bochner's Theorem, for instance in [6]). In particular, from the same arguments of parity than the ones leading to the spectral decomposition of the operators $\mathcal{L}_D$, $\mathcal{L}_N$ and $J_\nu$ of Sections 7.2 and 7.3, we deduce that the eigenfunctions of the integral operators associated with such a kernel $K(\cdot,\cdot)$ are still the same functions $\widetilde{\phi}_0$, $\widetilde{\phi}_n^c$ and $\widetilde{\phi}_n^s$, for $n \geqslant 1$, of equations (7.3) and (7.4) (discrete Fourier basis) ; only the eigenvalues change.

### 7.5. Numerical application

In this last section, we compute some of the involved quantities for $R = 3$ and $\sigma^2 = 2$. All computations have been performed with the free software `R` [17]. In particular, the implied integrals have been evaluated by quadrature (rectangle method).

#### 7.5.1. Spectral computations

We first compute the eigenvalues $\lambda_n^D$, $\lambda_n^N$ and $\rho_n$ of the operators $\mathcal{L}_D$, $\mathcal{L}_N$ and $J_\nu$ for $0 \leqslant n \leqslant 30$. Using equations (7.10) and (7.12), we then directly obtain the values of $\lambda_n^{R1}$ and $\lambda_n^{R2}$. The results are listed in Figure 2 (we do not represent the eigenvalues $\lambda_n^{R2}$).

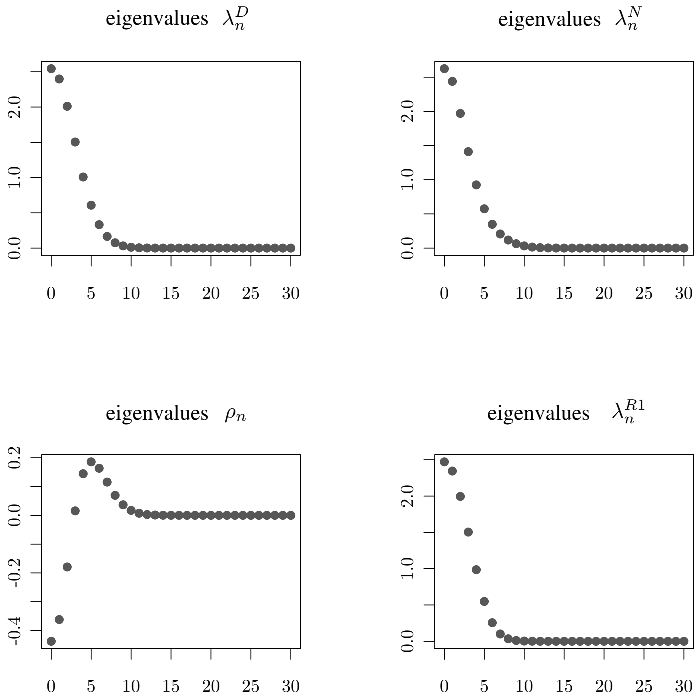


Figure 2. — Eigenvalues $\lambda_n^D$, $\lambda_n^N$, $\rho_n$ and $\lambda_n^{R1}$ for $0 \leqslant n \leqslant 30$, $R = 3$ and $\sigma^2 = 2$.

The second step consists in the computation, thanks to expressions (7.5) and (7.6), of the orthonormal bases of $\mathcal{H}_{M_D}$ and $\mathcal{H}_{M_N}$ associated with $\mathfrak{F}_D$ and $\mathfrak{F}_N$ respectively.

From equation (7.11) and (7.13), we then directly obtain the elements associated with the operators $\mathcal{L}_{R1}$ and $\mathcal{L}_{R2}$. Examples are given in Figure 3. As expected, $\phi_1^{sR2}$ vanishes on the circle $\mathcal{C}$ and the radial derivative of $\phi_8^{cR1}$ also vanishes on $\mathcal{C}$.

### 7.5.2. Approximations by truncation

We now consider approximation by truncation of the different involved kernels. In each case, we conserve the terms which are associated with the most important eigenvalues. We use the same notations and numberings that in Sections 7.2 and 7.4. For $l \in \mathbb{N}^*$, we introduce the following truncated kernels (associated with $M_D$ and $M_N$),

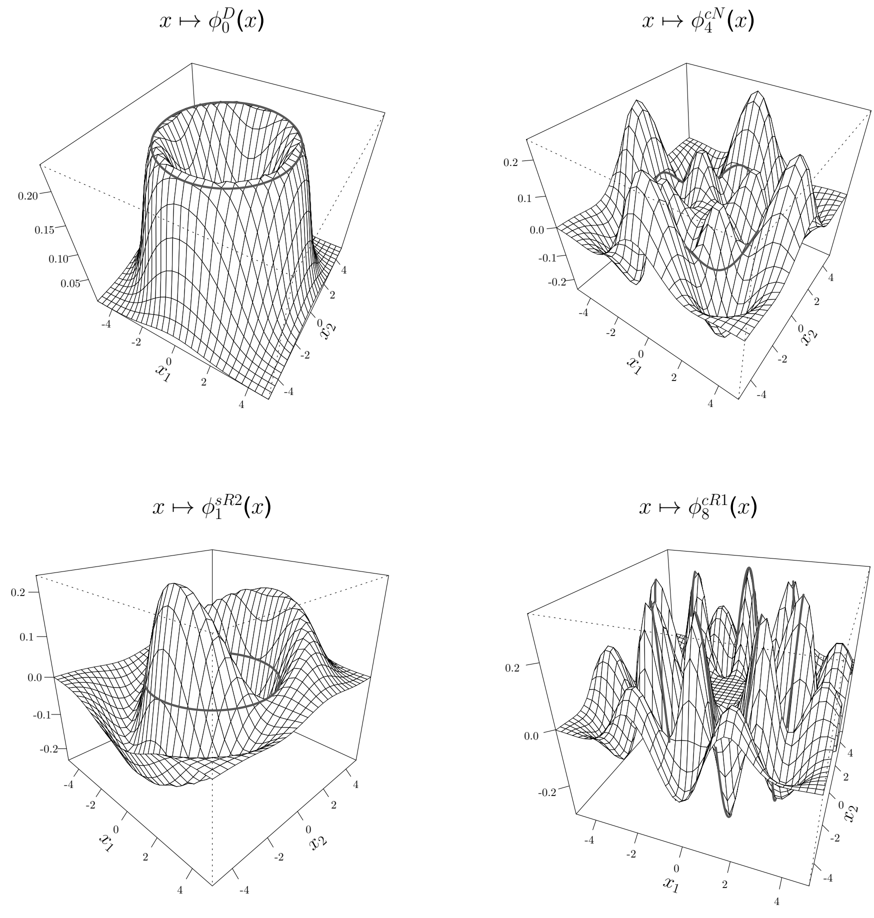


Figure 3. — Graphs of $\phi_0^D$ (*top-left*), $\phi_4^{cN}$ (*top-right*), $\phi_1^{sR2}$ (*bottom-left*) and $\phi_8^{cR1}$ (*bottom-right*) on $[-5,5]^2$, $R=3$ and $\sigma^2=2$.

$$K_{0_D}^{trc}(x,y) = K(x,y) - K_{M_D}^{trc}(x,y) \text{ where}$$

$$K_{M_D}^{trc}(x,y) = \lambda_0^D \phi_0^D(x)\phi_0^D(y) + \sum_{n=1}^{l} \lambda_n^D \left[\phi_n^{cD}(x)\phi_n^{cD}(y) + \phi_n^{sD}(x)\phi_n^{sD}(y)\right] \text{ and}$$

$$K_{0_N}^{trc}(x,y) = K(x,y) - K_{M_N}^{trc}(x,y) \text{ where}$$

$$K_{M_N}^{trc}(x,y) = \lambda_0^N \phi_0^N(x)\phi_0^N(y) + \sum_{n=1}^{l} \lambda_n^N \left[\phi_n^{cN}(x)\phi_n^{cN}(y) + \phi_n^{sN}(x)\phi_n^{sN}(y)\right].$$

Such a kernel $K^{trc}_{M_N}(\cdot,\cdot)$ is hence the approximation by truncation of $K_{M_N}(\cdot,\cdot)$ based one the $N_{trc} = 2l+1$ largest eigenvalues of the operator $\mathcal{L}_N$. Remark that for each eigenvalue (of multiplicity 2) $\lambda_n^N$, $1 \leqslant n \leqslant l$, we have chosen to consider the both associated eigenvectors $\phi_n^{cN}$ and $\phi_n^{sN}$.

We finally define the approximated kernel $K^{trc}_{0_{R1}}(\cdot,\cdot)$ given by,

$$\begin{aligned}\forall x \text{ and } y \in \mathcal{X},\ K^{trc}_{0_{R1}}(x,y) &= K^{trc}_{0_N}(x,y) - \lambda_0^{R1}\phi_0^{R1}(x)\phi_0^{R1}(y) \\ &\quad - \sum_{n=1}^{l} \lambda_n^{R1}\left[\phi_n^{cR1}(x)\phi_n^{R1c}(y) + \phi_n^{sR1}(x)\phi_n^{sR1}(y)\right].\end{aligned}$$

We pose $l = 15$ and compute the kernel $K^{trc}_{0_{R1}}(\cdot,\cdot)$. This approximated kernel is hence based on the $N_{trc} = 31$ largest eigenvalues of the operators $\mathcal{L}_N$ and $\mathcal{L}_{R1}$. Figure 4 shows the sample path of a centered Gaussian process with covariance $K^{trc}_{0_{R1}}(\cdot,\cdot)$ and the graph of the function $x \mapsto K^{trc}_{0_{R1}}(x,x)$.

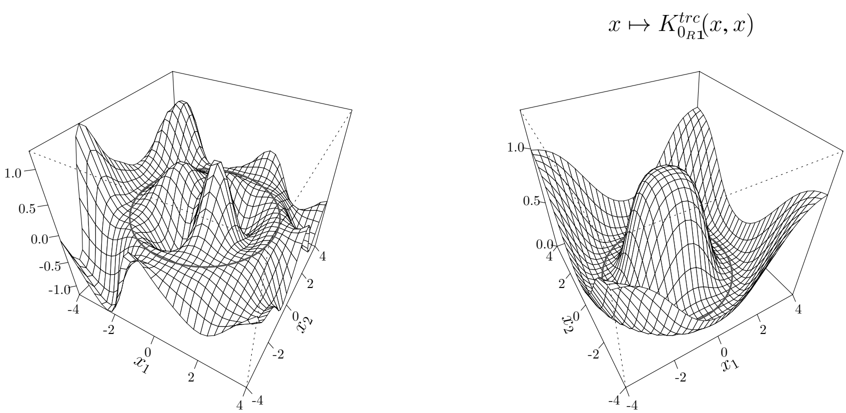


Figure 4. — Sample path of a centered Gaussian process with covariance $K^{trc}_{0_{R1}}(\cdot,\cdot)$ (*left*) and Graphical representation of $x \mapsto K^{trc}_{0_{R1}}(x,x)$ on $[-4,4]^2$ (*right*), $R = 3$, $\sigma^2 = 2$ and $N_{trc} = 31$.

The kernel $K_{0_R}(\cdot,\cdot) = K_{0_{R1}}(\cdot,\cdot)$ vanishes on $\mathcal{C}$ (in particular $K_{0_{R1}}(t,t) = 0$ for all $t \in \mathcal{C}$). Concerning its approximation $K^{trc}_{0_{R1}}(\cdot,\cdot)$, we deduce from Sections 7.3 and 7.4 that for all $t \in \mathcal{C}$, the value of $K^{trc}_{0_{R1}}(t,t)$ is *constant* and

$$\begin{aligned}K^{trc}_{0_{R1}}(t,t) = K^{trc}_{0_D}(t,t) &= K^{trc}_{0_N}(t,t) - \frac{1}{2\pi R}\left(\lambda_0^{R1} + 2\sum_{n=0}^{l}\lambda_n^{R1}\right) \\ &= 1 - \frac{1}{2\pi R}\left(\lambda_0^{D} + 2\sum_{n=1}^{l}\lambda_n^{D}\right).\end{aligned}$$

For $l = 15$, we obtain $K^{trc}_{0_{R1}}(t,t) \approx$ `1.402309e-06` for all $t \in \mathcal{C}$. Additional considerations concerning this approximation are given in Section 7.5).

As for the radial derivative, we also find a constant value, *i.e.* for all $t \in \mathcal{C}$,

$$\begin{aligned} \left.\frac{\partial^2 K^{trc}_{0_{R1}}}{\partial r_x \partial r_y}(x,y)\right|_{x=y=t} &= \left.\frac{\partial^2 K^{trc}_{0_N}}{\partial r_x \partial r_y}(x,y)\right|_{x=y=t} \\ &= \frac{2}{\sigma^2} - \frac{1}{2\pi R}\left(\lambda^N_0 + 2\sum_{n=1}^{l} \lambda^N_n\right). \end{aligned}$$

For $l = 15$, we obtain $\left.\frac{\partial^2 K^{trc}_{0_{R1}}}{\partial r_x \partial r_y}(x,y)\right|_{x=y=t} \approx$ `1.483913e-05`.

### 7.5.3. Comparison with discretization

In this last section, we focus on the interpolation problem associated with $\mathcal{H}$ and $M_D$ and compare the approximations obtained by truncation and discretization.

Hence, we consider a set of $N_{dis} \in \mathbb{N}^*$ points $x_{R,\theta_k} = (R\cos\theta_k, R\sin\theta_k) \in \mathcal{C}$ for $k \in \{0, \cdots, N-1\}$ uniformly distributed on $\mathcal{C}$, that is $\theta_k = \frac{2\pi k}{N_{dis}}$. We then introduce the linear subspace of $E'$

$$M^{dis}_D = \operatorname{span}\left\{\delta_{x_{R,\theta_k}}, k \in \{0, \cdots, N_{dis}-1\}\right\}.$$

We approximate the interpolation problem associated with $M_D$ by replacing it by $M^{dis}_D$ (let us signalize that it is, up to a rotation, the optimal way to discretize this problem in regards of $\|\cdot\|_{\gamma,\nu}$). This finally leads to the orthogonal decomposition $\mathcal{H} = \mathcal{H}^{dis}_{M_D} + \mathcal{H}^{dis}_{0_D}$ and solutions of this new discretized problem can be computed by the classical approach of Section 5.1. We denote by $K^{dis}_{0_D}(\cdot,\cdot)$ the reproducing kernel of the subspace $\mathcal{H}^{dis}_{0_D}$.

In order to compare the overall quality of the approximations obtained by truncation and discretization, we consider the terms

$$\mathrm{Err}^{dis}_D = \int_0^{2\pi} K^{dis}_{0_D}(s_{R,\theta}, s_{R,\theta})\, R d\theta \text{ and } \mathrm{Err}^{trc}_D = \int_0^{2\pi} K^{trc}_{0_D}(s_{R,\theta}, s_{R,\theta})\, R d\theta. \tag{7.14}$$

We remind that this kind of terms naturally arises when one aims at studying, for instance in the discretization case, quantities of the type (see Theorem 4.2 and Proposition 6.4)

$$\left\|\mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}^{dis}_M}[\varphi]\right\|^2_{\gamma,\nu} \text{ with } \varphi \in \mathcal{H}.$$

Because of the optimality of $K_{0_D}^{trc}(\cdot,\cdot)$ in regards of $\|\cdot\|_{\gamma,\nu}$ (see Section 6.4), for $N_{dis} = N_{trc}$, we always have $\mathrm{Err}_D^{trc} \leqslant \mathrm{Err}_D^{dis}$ (note that these two terms tend to zero as $N_{dis}$ and $N_{trc}$ approach $+\infty$, see also Figure 5). For instance, for $R = 3$, $\sigma^2 = 2$ and $N_{dis} = N_{trc} = 31$, we obtain

$$\mathrm{Err}_D^{dis} \approx \texttt{4.438046e-05} \text{ and } \mathrm{Err}_D^{trc} \approx \texttt{2.64329e-05}.$$

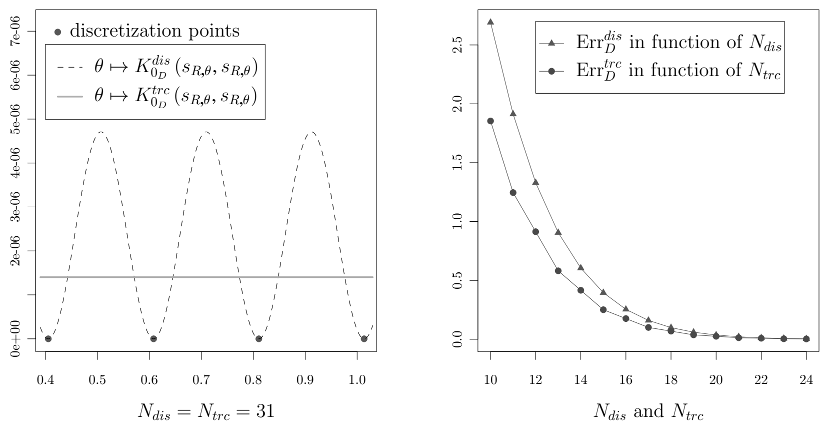


Figure 5. — Graphs of $\theta \mapsto K_{0_D}^{trc}\left(s_{R,\theta}, s_{R,\theta}\right)$ and $\theta \mapsto K_{0_D}^{dis}\left(s_{R,\theta}, s_{R,\theta}\right)$ (with $\theta \in [0, 2\pi]$) for $N_{trc} = N_{dis} = 31$ (*left*) and values of the terms $\mathrm{Err}_D^{dis}$ and $\mathrm{Err}_D^{trc}$ for $10 \leqslant N_{dis}, N_{trc} \leqslant 24$ (*right*) with $R = 3$ and $\sigma^2 = 2$.

Although the gain in considering truncation instead of discretization could in this case appear numerically negligible, one has to remark that the behavior on $\mathcal{C}$ of this two kinds of approximation is totally different (see Figure 5). It can hence be said that truncation leads to a global approximation on $\mathcal{C}$ (with respect to $\|\cdot\|_{\gamma,\nu}$) whereas discretization leads to a local approximation (localized at the discretization points).

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