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## A holomorphic correspondence at the boundary of the Klein combination locus

SHAUN BULLETT<sup>(1)</sup>, ANDREW CURTIS<sup>(1)</sup>

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**ABSTRACT.** — We investigate an explicit holomorphic correspondence on the Riemann sphere with striking dynamical behaviour: the limit set is a fractal resembling the one-skeleton of a tetrahedron and on each component of the complement of this set the correspondence behaves like a Fuchsian group.

**RÉSUMÉ.** — Nous étudions une correspondance holomorphe explicite sur la sphère de Riemann ayant une dynamique remarquable: l'ensemble limite est un fractal qui ressemble au 1-squelette du tétraèdre et sur chaque composante du complémentaire de cet ensemble, la correspondance est donnée par un groupe Fuchsien.

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### 1. Introduction

A one (complex) parameter family of holomorphic correspondences  $\mathcal{F}_a$  containing matings between the modular group and quadratic polynomials was discovered by the first author and Christopher Penrose nearly twenty years ago [4], and further investigated in [3] and [2]. Two naturally defined subsets of interest in the parameter space are:

1. The *connectivity locus*  $\mathcal{M}$ .
2. The *Klein combination locus*  $\mathcal{K} \supset \mathcal{M}$ .

Both loci will be given precise mathematical definitions below, in Section 2. But we mention now that  $\mathcal{M}$  is conjectured to be the set of values of  $a$  such

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that  $\mathcal{F}_a$  is a mating between  $PSL_2(\mathbb{Z})$  and  $q_c : z \rightarrow z^2 + c$  for some  $c \in M$ , the Mandelbrot set, and that  $\mathcal{K}$  is the interior of the set of values of  $a$  for which the action of  $\mathcal{F}_a$  is ‘faithful and discrete’, in a sense not yet formalised. The set  $\mathcal{M}$  is conjectured to be homeomorphic to the Mandelbrot set and the set  $\mathcal{K}$  is conjectured to be homeomorphic to a disc.

The correspondences  $\mathcal{F}_a$  for  $a \in \mathcal{K} \setminus \mathcal{M}$  may be thought of as ‘matings between  $PSL_2(\mathbb{Z})$  and maps  $q_c$  for which the Julia set is a Cantor set’. A classification of these correspondences will be presented in [5]. Any two such  $\mathcal{F}_a$  are quasiconformally conjugate, provided each has no *critical relations*. The exceptional values of  $a$ , for which  $\mathcal{F}_a$  has some critical relation, form a countable set of isolated points in  $\mathcal{K} \setminus \mathcal{M}$ . Our interest in the current article is in the behaviour of  $\mathcal{F}_a$  as  $a$  approaches the outer boundary of  $\mathcal{K}$  and what happens when  $a$  reaches that boundary. A good analogy is the behaviour of a one (complex) parameter family of Kleinian groups, representations  $G \rightarrow PSL_2(\mathbb{C})$  of a finitely generated abstract group  $G$  indexed by a parameter  $a$ , as  $a$  approaches the boundary of the slice in which the representations are discrete and faithful. In this paper we shall focus on one particular boundary point of  $\mathcal{K}$ , which we call the *Penrose point*, and investigate the dynamical behaviour (Figure 1) of the correspondence which has this as its parameter value, in particular showing that the complement of the limit set has four components and that for each component the subcorrespondence which stabilises it is conjugate to a Fuchsian group. More general results and conjectures concerning correspondences in the family  $\mathcal{F}_a$ , and the structure of  $\mathcal{K}$  and its boundary, will be presented elsewhere.

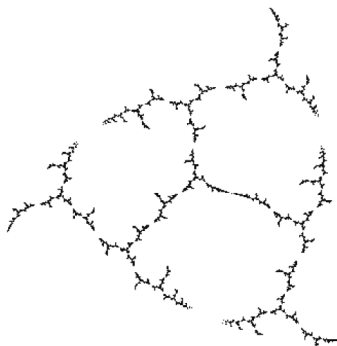


Figure 1. — The limit set of the correspondence  $\mathcal{F}_{a_{1/3}}$ . The ‘gaps’ are an illusion arising from the presence of parabolic points.

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## 2. The family of matings, the connectivity locus $\mathcal{M}$ , and the Klein combination locus $\mathcal{K}$

In this section we briefly introduce the family of holomorphic correspondences  $\mathcal{F}_a$  in which we are interested. For more details the reader should consult the original paper [4].

The first ingredient in the definition of  $\mathcal{F}_a$  is the notion of a covering correspondence. For a given rational map  $q : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  the associated *covering correspondence*  $Cov^q$  is defined by the relation

$$(z, w) \in Cov^q \Leftrightarrow q(z) = q(w).$$

For each  $z \in \hat{\mathbb{C}}$ , by letting  $Cov^q(z) = \{w : (z, w) \in Cov^q\}$  we can consider the correspondence to be a multifunction. Doing so allows us to consider correspondences from a dynamical perspective. Taking this point of view it is often convenient to consider the associated *deleted covering correspondence*  $Cov_0^q$  defined by

$$(z, w) \in Cov_0^q \Leftrightarrow \frac{q(z) - q(w)}{z - w} = 0.$$

Noting that all rational maps are branched covering maps of the sphere to itself we make use of branch cuts to understand how  $Cov^q$  acts as a transformation of the Riemann sphere. For the purposes of the current article we make the following definition of a *fundamental domain* for this action.

**DEFINITION 2.1.** — *A fundamental domain for the action of  $Cov^q$  is any component of  $\hat{\mathbb{C}} \setminus q^{-1}(\ell)$ , where  $\ell$  is any piecewise smooth simple arc which starts and ends at two of the critical values of  $q$  and passes through all the others.*

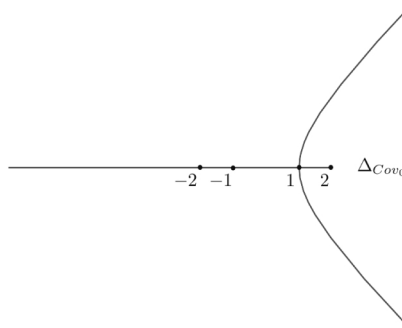


Figure 2. — Fundamental domains for  $Cov_0$ : the curves are the preimage of  $[-\infty, 2]$  under  $Q(z) = z^3 - 3z$ .

We note that if  $\Delta$  is a fundamental domain for  $Cov^q$  then  $\Delta$  is mapped bijectively onto each of the other components of  $\hat{\mathbb{C}} \setminus q^{-1}(\ell)$  by  $Cov_0^q$ .

Up to conjugacy there are only two possibilities for covering correspondences of cubic *polynomials*. This is because in addition to the critical point  $\infty$ , the polynomial either has a double critical point (which we may take to be at 0) or two simple critical points. We restrict attention to the second case and to the particular cubic polynomial  $Q(z) = z^3 - 3z$ . The critical points of  $Q$  are  $-1, 1$  and  $\infty$ , and the corresponding critical values are  $2, -2$  and  $\infty$ . The preimages of  $2$  are  $-1$  and  $2$ , the preimages of  $-2$  are  $1$  and  $-2$ , and the only preimage of  $\infty$  is  $\infty$ . Consider the preimage under  $Q$  of the real interval  $[-\infty, 2]$  (see Figure 2). One possible choice of fundamental domain  $\Delta_{Cov}$  is the right-hand component in Figure 2, with boundary the curves running from  $1$  to  $\infty$ , at asymptotic angles  $\pm\pi/3$ , together with the cut from  $1$  to  $2$ . The correspondence  $Cov_0^Q$  maps the cut  $[1, 2]$  one-to-two onto the interval  $[-2, 1]$ , sending  $1$  to  $-2$  and to  $1$ , and sending  $2$  to the single point,  $-1$ . The point  $-2$  also has a unique image under  $Cov_0^Q$ , namely  $1$ , but all other points of  $\hat{\mathbb{C}}$  (except  $\infty$ ) have two distinct images.

The  $(2 : 2)$  correspondences  $\mathcal{F}_a$  with which we shall be concerned are defined for all  $a \in \hat{\mathbb{C}}$  with  $a \neq 1$ . We set  $\mathcal{F}_a$  to be the composition  $J \circ Cov_0^Q$  where  $J = J_a$  is the (unique) involution of  $\hat{\mathbb{C}}$  which has fixed points  $1$  and  $a$ . Viewing  $J$  as  $Cov_0^q$  where  $q$  is the projection from  $\hat{\mathbb{C}}$  to  $\hat{\mathbb{C}}/J$ , a *fundamental domain*  $\Delta_J$  for  $J$  (in the sense of Definition 1) is a component of the complement of any piecewise smooth simple closed curve which passes through  $1$  and  $a$  and is invariant under  $J$  (for example, any round circle passing through  $1$  and  $a$ ).

For certain values of the parameter  $a$ , there will exist fundamental domains  $\Delta_J$  and  $\Delta_{Cov}$  which satisfy the following *Klein combination condition*:

$$(K) \quad \Delta_{Cov} \cup \Delta_J = \hat{\mathbb{C}} \setminus \{1\}.$$

Recall that we have defined our fundamental domains to be open sets, so the point  $1$ , being on the boundary of both  $\Delta_{Cov}$  and  $\Delta_J$ , cannot lie in their union. Note also that condition  $(K)$  implies that  $\Delta_{Cov} \cap \Delta_J \neq \emptyset$  since  $\hat{\mathbb{C}} \setminus \{1\}$  is connected.

We can satisfy condition  $(K)$  by, for example, taking  $a$  in the half-open real interval  $(1, 7]$ . Choose as  $\Delta_{Cov}$  the right-hand component in Figure 2. For any value of  $a \in (1, 7]$  the circle  $C$  which passes through  $1$  and  $a$  and has centre on the real axis meets the boundary of  $\Delta_{Cov}$  at the single point  $1$ . If we take as  $\Delta_J$  the component of  $\hat{\mathbb{C}} \setminus C$  containing  $\infty$  then the pair  $\Delta_{Cov}, \Delta_J$  satisfy the condition  $(K)$ .

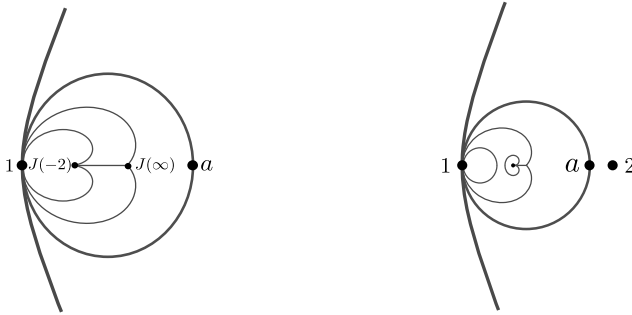


Figure 3. —  $D_J$  and  $D_1 = \mathcal{F}_a(D_J)$ , pictured on the left for  $a \in (2, 7]$  and on the right for  $a \in (1, 2)$ .

When condition  $(K)$  is satisfied, the closed disc  $D_J$  complementary to  $\Delta_J$  is mapped 1 to 2 into itself by the multifunction  $\mathcal{F}_a$ . The image  $D_1 = \mathcal{F}_a(D_J) \subset D_J$  of  $D_J$  will be either one topological disc or two, depending on whether the critical value 2 of the map  $\mathcal{F}_a^{-1}$  lies in  $D_J$  or does not (see Figure 3). In either case, writing  $\Delta$  for  $\Delta_{Cov} \cap \Delta_J$ , we observe that  $D_J \setminus D_1$  consists of the union (modulo boundaries) of three ‘tiles’ which are homeomorphic copies of  $\Delta$ , namely  $J(\Delta)$  and the two images  $J \circ Cov_0(\Delta)$  of  $\Delta$ . These three tiles together make up  $J \circ Cov(\Delta)$ . Writing  $\Lambda_+$  for  $\bigcap_{n=0}^{\infty} \mathcal{F}_a^n(D_J)$ , the ‘ping-pong principle’ underlying the Klein combination theorem for covering correspondences, [1], ensures that  $D_J \setminus \Lambda_+$  is tiled by images of  $\Delta$  under iterates of  $J \circ Cov_0 (= \mathcal{F}_a)$  applied to these three tiles. Similarly, writing  $\Lambda_-$  for  $J(\Lambda_+)$ , the set  $J(D_J) \setminus \Lambda_-$  is tiled by images of  $Cov(\Delta)$  under iterates of  $Cov_0 \circ J (= \mathcal{F}_a^{-1})$ . To state this more precisely, setting  $\Lambda$  to be the union  $\Lambda = \Lambda_+ \cup \Lambda_-$  (which is closed and invariant under  $\mathcal{F}_a$  and  $\mathcal{F}_a^{-1}$ ), and setting  $\Omega$  to be its complement in  $\hat{\mathbb{C}}$ , the Klein combination theorem for covering correspondences tells us that:

1. The action of  $\mathcal{F}_a$  on  $\Omega$  is properly discontinuous and is the free product of the actions of  $Cov$  and  $\{Id, J\}$ , in an appropriate sense (see [1]);
2.  $\Delta = \Delta_{Cov} \cap \Delta_J$  is a fundamental domain for this action, in the sense that  $\Omega = \bigcup_{n \in \mathbb{Z}} \mathcal{F}_a^n(\Delta')$  where  $\Delta' = \Delta \cup (\partial\Delta \setminus \{1\})$  and the ‘tiles’ (the images of  $\Delta'$  under single-valued restrictions of the  $\mathcal{F}_a^n$ ) meet only at their boundaries.

*Remark.* — Theorem 2 of [1] states this result for the simpler case that  $\Delta_{Cov} \cup \Delta_J = \hat{\mathbb{C}}$ , rather than  $\Delta_{Cov} \cup \Delta_J = \hat{\mathbb{C}} \setminus \{1\}$ , and is phrased in terms

of transversals rather than fundamental domains, but the proof in [1] can be adapted to prove the two statements above.

DEFINITION 2.2. — *The Klein combination locus for the family  $\mathcal{F}_a$  is the set  $\mathcal{K}$  of values of  $a \in \hat{\mathbb{C}}$  such that there exist fundamental domains  $\Delta_{Cov}, \Delta_J$  for  $Cov^Q$  and  $J$  satisfying condition (K).*

When condition (K) is satisfied, and  $2 \in D_J$ , the single-valued map  $\mathcal{F}_a^{-1} : D_1 \rightarrow D_J$  is a *pinched quadratic-like map* with pinch point 1, critical point  $J(-1)$  and critical value 2, and for certain values of  $a$  (see [4, 3, 2])  $\mathcal{F}_a$  is a mating in the sense of Definition 2.3 below.

Represent the action of the modular group  $PSL_2(\mathbb{Z})$  on the complex upper half-plane by the *modular correspondence*,

$$(z, w) \in F_{Mod} \Leftrightarrow ((\tau_1(z) - w)(\tau_2(z) - w) = 0.$$

where

$$\tau_1(z) = z + 1 \quad \text{and} \quad \tau_2(z) = \frac{z}{z + 1}.$$

DEFINITION 2.3. — *Let  $q_c : z \rightarrow z^2 + c$  be a quadratic polynomial with connected filled Julia set  $K_c$ . We say that a  $(2 : 2)$  holomorphic correspondence  $F$  is a mating between  $q_c$  and the modular group  $PSL_2(\mathbb{Z})$  if:*

1. *There exist an open subset  $\Omega$  of  $\hat{\mathbb{C}}$  invariant under the action of  $F$ , and a conformal homeomorphism  $h$  from  $\Omega$  to the upper-half plane conjugating the action of  $F$  to  $F_{Mod}$ .*
2. *The complement of  $\Omega$  in  $\hat{\mathbb{C}}$  is the union of two sets  $\Lambda_+$  and  $\Lambda_-$ , with  $\Lambda_+ \cap \Lambda_-$  consisting of a single point, and there exist homeomorphisms  $h_+ : \Lambda_+ \rightarrow K_c$  and  $h_- : \Lambda_- \rightarrow K_c$ , which are conformal on interiors and respectively conjugate the action of  $F^{-1}$  restricted to  $\Lambda_+$ , and that of  $F$  restricted to  $\Lambda_-$ , to the action of  $q_c$  on  $K_c$ .*

For  $a \in \mathcal{K}$ , the set  $\Lambda_+ = \bigcap_{n=1}^{\infty} \mathcal{F}_a^n(D_J)$  is connected if and only if the critical value 2 of the pinched quadratic-like map  $(\mathcal{F}_a)^{-1} : D_1 \rightarrow D_J$  lies in  $(\mathcal{F}_a)^n(D_J)$  for all  $n > 0$ . (For a proof see [4], or the Remark following the proof of Proposition 1 in Section 3 below.)

DEFINITION 2.4. — *The connectivity locus for the family  $\mathcal{F}_a$  is the subset  $\mathcal{M} \subset \mathcal{K}$  of values of  $a$  for which there is a pair of fundamental domains  $\Delta_{Cov}, \Delta_J$  satisfying the condition (K), and with the additional property that  $2 \in \bigcap_{n=1}^{\infty} (\mathcal{F}_a)^n(D_J)$  (where  $D_J = \hat{\mathbb{C}} \setminus \Delta_J$ ).*

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A computer plot of the connectivity locus was presented in the original paper on this family of correspondences [4]. It bears a striking resemblance to the Mandelbrot set  $M$  for quadratic maps. It is conjectured that the family  $\mathcal{F}_a$  contains a mating of  $q_c$  with  $PSL_2(\mathbb{Z})$  for every value of  $c \in M$  and that  $\mathcal{M}$  is homeomorphic to  $M$ . The first conjecture was proved for a large subset of values of  $c \in M$  in [2]. Our interest in the current article is in  $\mathcal{K} \setminus \mathcal{M}$ , and in particular a specific point on the boundary of  $\mathcal{K}$ . The way that we have defined  $\mathcal{K}$  ensures that  $\mathcal{K} \setminus \mathcal{M}$  is an open subset of the parameter space, and thus that the boundary points are outside  $\mathcal{K}$ .

*Remark.* — There is experimental evidence which suggests that for all  $a$  outside the closure  $\overline{\mathcal{K}}$  of  $\mathcal{K}$ , the periodic points of  $\mathcal{F}_a$  are dense in  $\hat{\mathbb{C}}$ . It is tempting to hope that by analogy with the study of deformations of Kleinian groups there might be some definition of ‘faithful and discrete’ for a correspondence action which would allow us to characterise  $\overline{\mathcal{K}}$  as the set of all values of  $a \in \hat{\mathbb{C}}$  for which  $\mathcal{F}_a$  satisfies this yet to be formulated property.

### 3. Dynamics of the correspondence $\mathcal{F}_a$ when $a$ lies in $\mathcal{K} \setminus \mathcal{M}$

When  $a \in \mathcal{K} \setminus \mathcal{M}$  we still have the ‘Klein combination’ set-up for  $\Delta_{Cov}$  and  $\Delta_J$ , and we still have a partition of the Riemann sphere into invariant regions  $\Omega$  and  $\Lambda = \Lambda_+ \cup \Lambda_-$ , but the filled Julia set  $\Lambda_+ = \bigcap_{n=1}^{\infty} (\mathcal{F}_a)^n(D_J)$  of the associated ‘pinched quadratic-like map’ is no longer connected.

PROPOSITION 1. — *For  $a \in \mathcal{K} \setminus \mathcal{M}$ , the set  $\Lambda = \Lambda_+ \cup \Lambda_-$  is a Cantor set.*

*Proof.* — Recall that a subset  $X \subset \hat{\mathbb{C}}$  is a Cantor set if it is non-empty, closed, perfect ( $X$  has no isolated points), and totally disconnected (each component of  $X$  is a single point). We consider the action of  $\mathcal{F}_a = J \circ Cov_0$  as a one-to-two multifunction from  $D_J$  onto  $D_1 = \mathcal{F}_a(D_J)$ . As we have already observed,  $\mathcal{F}_a^{-1} : D_1 \rightarrow D_J$  is a single-valued two-to-one map. Since  $a \notin \mathcal{M}$ , the (unique) critical value 2 of this map does not lie in  $\Lambda_+ = \bigcap_{n=0}^{\infty} \mathcal{F}_a^n(D_J)$ . Hence there exists some least  $n_0$  such that  $2 \notin \mathcal{F}_a^{n_0}(D_J)$ . Now  $\mathcal{F}_a^{n_0-1}(D_J)$  will consist of a single topological disc  $E$  and  $\mathcal{F}_a^{n_0}(D_J)$  will consist of two disjoint topological discs contained in  $E$ , one of which will have the pinch point 1 on its boundary. Label these two discs  $E_0$  and  $E_1$ , where  $E_1$  is the disc which has 1 on its boundary. The two restrictions of  $\mathcal{F}_a$ ,

$$f_0 = (J \circ Cov_0)|_{(E, E_0)}$$

$$f_1 = (J \circ Cov_0)|_{(E, E_1)}$$



are both homeomorphisms which are conformal on interiors. For any finite sequence  $s = s_1, s_2, \dots, s_n$ ,  $s_i \in \{0, 1\}$ , we let

$$E_s = f_{s_1} \circ f_{s_2} \circ \dots \circ f_{s_n}(E),$$

and for an infinite sequence  $s = s_1, s_2, \dots$ , we let

$$E_s = \bigcap_{i=1}^{\infty} f_{s_1} \circ f_{s_2} \circ \dots \circ f_{s_i}(E).$$

Note that  $\Lambda_+$  is precisely the set of all points which are contained within an infinite number of images of  $E$  under mixed iteration of  $f_0$  and  $f_1$ . In other words  $\Lambda = \bigcup_{s \in S} E_s$ , where  $S$  is the set of all possible infinite sequences with elements in  $\{0, 1\}$ .

We now show that each  $E_s$  consists of a single point. We separate the sets  $E_s$  into two distinct types.

Firstly, if  $s$  contains infinitely many 0's then  $E_s$  is contained within infinitely many images of  $E$  under mixed forward iteration of  $f_0$  and  $f_1$ . We let  $A$  be a topological annulus, contained in  $E \setminus (E_0 \cup E_1)$  and such that  $E_0$  is surrounded by  $A$ , i.e.  $E_0$  is contained in the bounded component of  $\hat{\mathbb{C}} \setminus A$ . Then  $E_s$  is surrounded by infinitely many conformal images of  $A$ , all pairwise disjoint. Thus by use of the Grötzsch inequality [6] it follows that each such  $E_s$  is a single point.

Secondly, if  $s$  contains only finitely many 0's then  $E_s$  is an image, under mixed iteration of  $f_0$  and  $f_1$ , of  $E_{111\dots}$ , the intersection of all images of  $E$  under iteration of  $f_1$ . By applying the Denjoy-Wolff Theorem [6] to  $f_1 : E \rightarrow E$ , we know that either  $f_1$  has a fixed point in the interior of  $E$  or there is a fixed point of  $f_1$  on the boundary of  $E$  to which every orbit converges. The fixed points of  $f_1$  are of course fixed points of  $\mathcal{F}_a$ , and there are just four of these, counted with multiplicity, since the equation  $\mathcal{F}_a(z) = z$  can be manipulated into a polynomial equation of degree four. Moreover the fixed point 1 of  $f_1$  has multiplicity two (being parabolic), the branch  $f_0$  of  $\mathcal{F}_a$  has a fixed point  $\zeta \in E_0$  (since  $E_0 = f_0(E)$  has closure contained in  $E$ ), and since  $J(\zeta)$  is also a fixed point of  $\mathcal{F}_a$  (being a fixed point of  $\mathcal{F}_a^{-1}$ ) we have now used up all four fixed points of  $\mathcal{F}_a$ . Therefore  $f_1$  can have no fixed point in the interior of  $E$ . The Denjoy-Wolff Theorem now tells us that all orbits of  $f_1$  on  $E$  must converge to 1. Moreover 1 cannot be isolated, since every point of  $\Lambda_+ \cap E$  has orbit under  $f_1$  accumulating at 1.

That  $\Lambda_+$  is perfect follows from the fact that for every infinite sequence  $s = s_1, s_2, \dots$ , every set in the sequence  $E_{s_1}, E_{s_1, s_2}, E_{s_1, s_2, s_3}, \dots$  contains infinitely many points of  $\Lambda_+$  (in the case of the point 1 this follows from the

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fact that every point of  $\Lambda_+ \setminus \{1\}$  has orbit under  $f_1$  accumulating at 1). That  $\Lambda_+$  is closed follows immediately from the fact that it is the complement of  $\Omega$  in  $D_J$ , together with the point 1 on the boundary of  $D_J$ .

Thus  $\Lambda_+$  is a Cantor set. By symmetry  $J(\Lambda_+)$  is also a Cantor set. The union of two Cantor sets contained in disjoint open discs, except for a single point in common on the boundaries of these discs, is again a Cantor set. So  $\Lambda = \Lambda_+ \cup J(\Lambda_+)$  is a Cantor set.  $\square$

*Remark.* — When  $a \in \mathcal{M}$ , similar reasoning to that in the proof above shows that every  $\mathcal{F}_a^n(D_J)$  is connected, and hence that  $\Lambda_+$  is connected.

It follows from Proposition 1 that for  $a \in \mathcal{K} \setminus \mathcal{M}$  the action of  $\mathcal{F}_a$  on  $\Omega$  can no longer be conjugate to that of the modular group on the complex upper half-plane, since  $\Omega$  is no longer simply-connected. Nevertheless the action of  $\mathcal{F}_a$  on  $\Omega$  remains ‘discontinuous’, in the sense that the space of grand orbits on  $\Omega$  has the structure of an orbifold. As will be shown in [5] and a future article, apart from a countable set of isolated parameter values where the singular points 2 and  $-2$  of  $Cov_0^Q$  lie on the same grand orbit of  $\mathcal{F}_a$ , all the correspondences  $\mathcal{F}_a$  with  $a \in \mathcal{K} \setminus \mathcal{M}$  lie in a single quasi-conformal conjugacy class, and can be obtained from one another by deformations of the complex structure on the orbifold  $\mathcal{O} = \Omega / \langle \mathcal{F}_a \rangle$ . In the final section of this paper we investigate an example where one can follow a ray in deformation space all the way to the boundary.

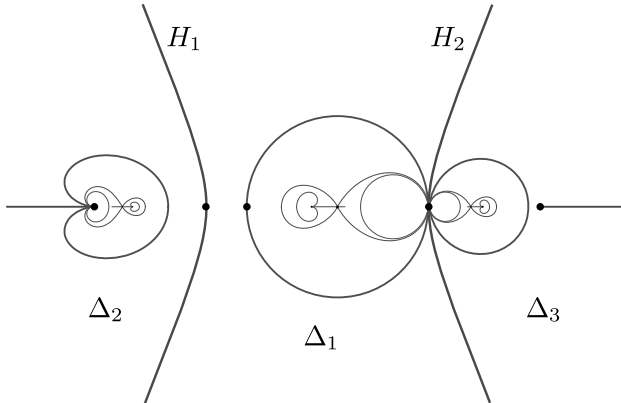


Figure 4. — The fundamental domain  $\Delta_1$  and its first few images under combinations of  $Cov_0$  and  $J$ . (The points marked by black dots on the real axis are:  $-2, -1, a_*, 1$  and  $2$ .)

We begin by describing the dynamics of a *generic* correspondence, that is to say an  $\mathcal{F}_a$  with  $a \in \mathcal{K} \setminus \mathcal{M}$  which does not have any critical coincidences. For example any value of  $a$  in the real interval  $(-1, +1)$  will do. We pick some  $a$  in this interval as the ‘base point’ of our parameter space and denote it by  $a_*$ . We let  $H_1$  denote the image of  $[2, \infty]$  under  $Cov_0$  and let  $H_2$  be the image of  $[-\infty, -2]$ . The region bounded by  $H_1$  and  $H_2$  is a fundamental domain for  $Cov$ , and the Klein Combination Theorem tells us that the region  $\Delta_1$  bounded by  $H_1$ ,  $H_2$ , and the circle through  $a (= a_*)$  and 1 centred on the real axis, is a fundamental domain for the action of  $J \circ Cov_0$  on the union of its images. This fundamental domain and its two images  $\Delta_2$  and  $\Delta_3$  under  $Cov_0$  are shown in Figure 4. Write  $\Delta$  for the union of  $\Delta_1, \Delta_2, \Delta_3$  together with the parts  $[-\infty, -2), [2, \infty], H_1$  and  $H_2 \setminus \{1\}$  of their boundaries which meet only finitely many images of  $\Delta_1$ . The complement of  $\Delta$  consists of closed discs  $D_1, D_2$ , and  $D_3$ , where  $D_1$  is contained in the region between  $H_1$  and  $H_2$  in the figure, and  $D_2$  and  $D_3$  are contained in left-hand and right-hand regions respectively. Now  $J(\Delta)$  is contained in  $D_1$  and  $Cov_0 \circ J(\Delta)$  consists of two components, one contained in  $D_2$  and the other in  $D_3$ . Continued iteration of  $J$  and  $Cov_0$  gives us an infinite collection of images of  $\Delta$  meeting along pairwise common edges. Each of these ‘tiles’ is a simply-connected set with the real axis as a mirror-symmetry line, so the set  $\Omega$ , made up of the union of these images together with their common edges, has as its complement a Cantor set contained in the real line.

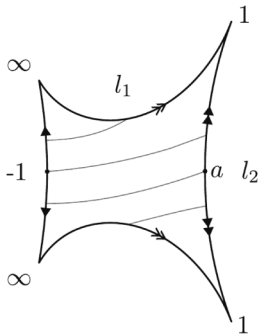


Figure 5. —  $\Delta_1$  marked with its boundary identifications and the geodesic  $\gamma_{1/3}$ .

The orbifold  $\mathcal{O}$ , the grand orbit space of  $\Omega$  under the correspondence, is the quotient of  $\Delta_1$  under the boundary identifications indicated in Figure 5. It is a sphere with four cone points of types  $\pi, \pi, 2\pi/3$  and  $\pi/\infty$  (the last one being a puncture point). One way to deform the complex structure on  $\mathcal{O}$  is to contract a geodesic, for example a geodesic arc from the cone point  $-1$  to the cone point  $a$ . Choices for such an arc correspond to rationals  $p/q$

with  $q$  odd, as follows. If we ‘de-identify’ the point 1 on the boundary of  $\Delta_1$  then we have a (hyperbolic) rectangle. We label two of the sides  $l_1$  and  $l_2$  as shown in the diagram, and we say that a geodesic from  $-1$  to  $a$  has *slope*  $p/q$ , and denote it by  $\gamma_{p/q}$ , if it intersects  $l_1$  and  $l_2$  in  $p$  points and  $q$  points respectively: the example  $\gamma_{1/3}$  is illustrated in Figure 5. The geodesic  $\gamma_{p/q}$  lifts to a *lamination*  $\Gamma_{p/q}$  on  $\Omega$ : we simply mark the same ‘pattern’ on each ‘tile’.

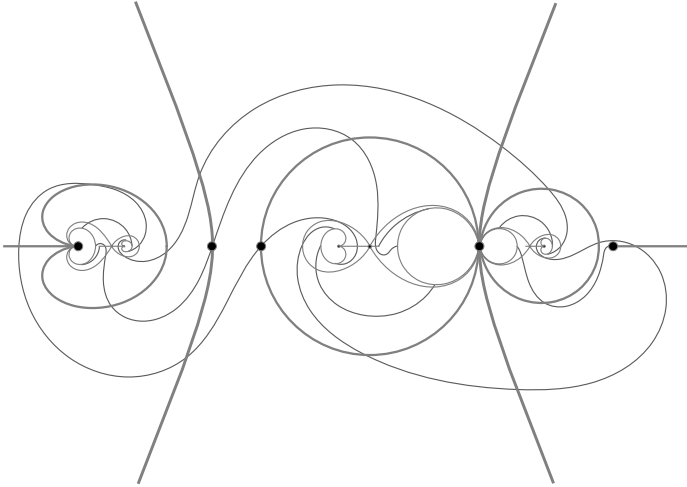


Figure 6. — The global lamination  $\Gamma_{1/3}$ .

The global lamination corresponding to  $\gamma_{1/3}$  is illustrated in Figure 6. It is tempting to try to deform  $\mathcal{F}_{a_*}$  to a correspondence  $\mathcal{F}_a$  with  $a$  on the boundary of  $\mathcal{K}$ , by contracting the leaves of  $\Gamma_{p/q}$  to points, but there are technical difficulties. Even once one has excluded the possibility of a topological obstruction arising from a pair of leaves with the same end points, it is a daunting technical task to construct isotopies shrinking unions of arcs to points in such a way that the resulting correspondence is holomorphic (see [2]). We shall discuss this approach to the structure of the boundary of  $\mathcal{K}$  elsewhere. Here we content ourselves with showing explicitly that the  $\gamma_{1/3}$  pinch point can be reached, and describing the behaviour of the correspondence there. The way that we shall do this is by identifying a unique candidate for the parameter value where the pinched dynamics could occur.

#### 4. An example on the boundary: Penrose point

It is apparent from Figure 6 that the leaf of  $\Gamma_{1/3}$  which passes through  $-1$  also passes through one of the two points  $J \circ Cov_0(a)$ . Thus if we can pinch the geodesic  $\gamma_{1/3}$  on the orbifold  $\mathcal{O}$ , then this will happen at a value of  $a$  where  $-1 \in J \circ Cov_0(a)$ . But then we will also have  $a \in Cov_0(J(-1))$  and so, since  $a$  is fixed by  $J$ , we will now have both  $-1 \in \mathcal{F}_a(a)$  and  $a \in \mathcal{F}_a(J(-1))$ . Since  $J(-1) \in \mathcal{F}_a(-1)$  for any value of  $a$ , we see that the three points  $a, -1, J(-1)$  will become a 3-cycle.

We next note that  $J(-1) \in Cov_0(a)$  if and only if

$$a^2 + aJ(-1) + J(-1)^2 = 3.$$

But

$$J(z) = \frac{(a+1)z - 2a}{2z - (a+1)}$$

so

$$J(-1) = \frac{1+3a}{3+a}.$$

Thus

$$(a^2 - 3)(3+a)^2 + a(1+3a)(3+a) + (1+3a)^2 = 0,$$

which simplifies to

$$a^4 + 9a^3 + 25a^2 - 9a - 26 = 0,$$

and thence to

$$(a^2 + 9a + 26)(a^2 - 1).$$

We deduce that

$$a = -\frac{9}{2} + \frac{\sqrt{23}}{2}i$$

or its complex conjugate, and we denote the value which has positive imaginary part by  $a_{1/3}$ . (The value with negative imaginary part will correspond to pinching  $\gamma_{-1/3}$ .)

Figure 7 illustrates the dynamics of  $\mathcal{F}_a$  for  $a = a_{1/3}$ . The grey curves are the images of the circular arc which runs from  $-1$  through  $+1$  to  $J(-1)$ . Shown in black is the grand orbit of the point  $+1$  under all branches, forward and back, of the iterated correspondence  $\mathcal{F}_a$ . An equivalent description of the black set is that it is the set of all images of the point  $+1$  under finite ‘words’ made up of the symbols  $J$  and  $Cov_0$ .

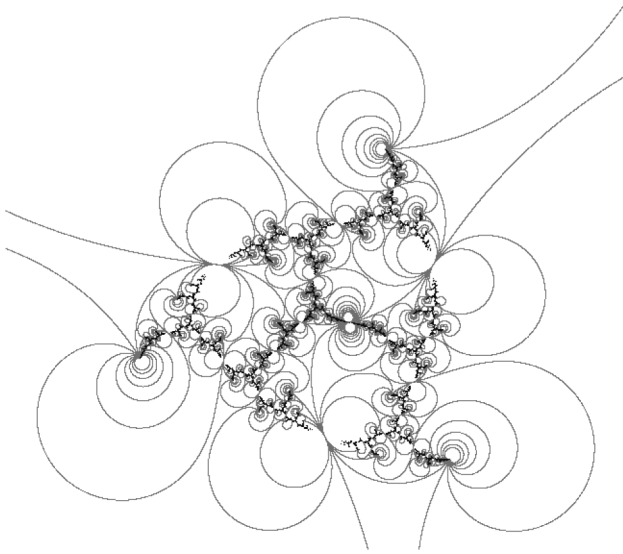


Figure 7. — The correspondence  $\mathcal{F}_{a_{1/3}}$ . The complement  $\Omega$  of the limit set is tiled by ‘ideal hexagons’ (Theorem 1).

*Remark.* — We have assigned  $a_{1/3}$  the name ‘Penrose point’ in honour of Chris Penrose who originally found this example and plotted the limit set in the early 1990’s (unpublished). Our re-discovery of it, and the analysis of its Fatou and limit sets reported on here, provided a key impetus for the formulation of conjectures (which we shall present elsewhere) describing the structure of the boundary of the Klein combination locus  $\mathcal{K}$ . ‘Penrose point’ lies at the tip of a promontory on the ‘outer shoreline’ of  $\mathcal{K}$ .

**THEOREM 4.1.** — *The correspondence  $\mathcal{F}_{a_{1/3}}$  is obtained from  $\mathcal{F}_{a_*}$  by pinching the geodesic  $\gamma_{1/3}$  on the orbifold  $\Omega(\mathcal{F}_{a_*})/\langle \mathcal{F}_{a_*} \rangle$ . Moreover:*

1. *The regular set  $\Omega = \Omega(\mathcal{F}_{a_{1/3}})$  of  $\mathcal{F}_{a_{1/3}}$  is tessellated into ideal hexagons by the images of the circular arc from  $-1$  through  $+1$  to  $J(-1)$ . The action of  $\mathcal{F}_{a_{1/3}}$  on  $\Omega$  is a faithful action of the free product of the cyclic group  $\{Id, J\}$  with the  $(3 : 3)$  correspondence  $Cov$  (where  $Cov_0 = Cov \setminus \{Id\}$  acts on the hexagon containing  $\infty$  as a pair of rotations fixing  $\infty$ ).*
2.  *$\Omega$  has four connected components  $\Omega_k$ ,  $k = 1, 2, 3, 4$ , each conformally homeomorphic to the (open) upper half-plane.*
3. *For each of  $k = 1, 2, 3, 4$  the action of the set of branches of iterates of the correspondence  $\mathcal{F}_{a_{1/3}}$  which stabilise  $\Omega_k$  is conformally*

conjugate to the action of a free product of groups  $C_3 * C_\infty$  on the upper half-plane, where  $C_3$  is generated by an elliptic Möbius transformation of order three and  $C_\infty$  is generated by a parabolic Möbius transformation.

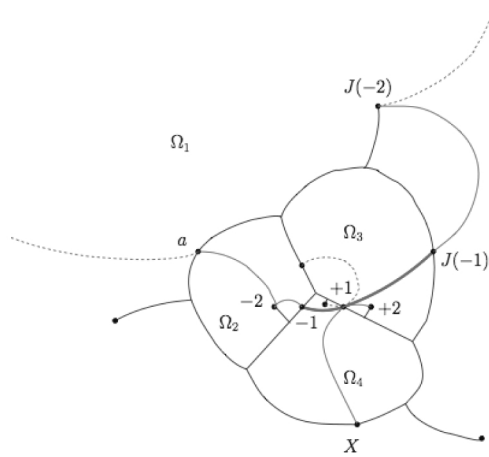


Figure 8. — Schematic picture of the components  $\Omega_i$  of  $\Omega$ , the positions of key points, the arc  $L \cup J(L)$  (thick grey) and its images under  $Cov_0$  (thin grey) and  $J \circ Cov_0$  (dashed). The incomplete dashed arcs at the top and left of the picture are in reality joined by a large loop (see the computer plot on the left in Figure 9). The two images of  $a$  under  $Cov_0$  are  $J(-1)$  and  $X$ .

*Proof.* — The positions of the key points we shall need in the proof are indicated in Figure 8. The circular arc referred to in the statement of the theorem is made up of an arc  $L$  from  $-1$  to  $+1$ , and its image  $J(L)$  from  $+1$  to  $J(-1)$ . Consider the images of  $L \cup J(L)$  under the identity,  $Cov_0$ , and  $J \circ Cov_0$ . These are shown in Figure 8 and the left hand plot of Figure 9. We assert that  $L \cup J(L)$ , together with its image under  $Cov_0$  running from  $-1$  to  $a = a_{1/3}$  (through a cusp at  $-2$ ), and its image under  $J \circ Cov_0$  running from  $a$  to  $J(-1)$  (through a cusp at  $J(-2)$ ), form a piecewise smooth simple closed curve, invariant under  $J$ . Following the same notation as in earlier sections of this paper, denote the component of the complement of this curve containing  $\infty$  by  $\Delta_J$  and the closure of the other component by  $D_J$ . We further assert that  $D_J$  and its images under  $Cov_0$  (plotted on the right in Figure 9) have disjoint interiors, that they are pairwise contiguous along  $L$ ,  $J(L)$  and the branch of  $Cov_0(J(L))$  emanating from  $+1$ , and that the union of  $D_J$  with its two images is therefore again a topological disc bounded by a piecewise smooth curve.

These assertions can be proved as follows (we omit details). Firstly, local analysis around the points of the period three cycle (which is parabolic), and around the (also parabolic) fixed point  $+1$  and its image  $-2$  under  $Cov_0$ , can be used to verify that the intersections of the arcs with neighbourhoods of the end points of  $L$  and their images are arranged as shown. Away from these neighbourhoods the arcs and their images are a definite distance apart and numerical estimation can be used complete the proof of the assertions. The same methods, of local analysis around the period three cycle and numerical estimation away from it, can be used to prove that the cycle can be ‘unpinched’ by a suitable small perturbation of the parameter value  $a_{1/3}$  (see the first remark following this proof), and thus that  $\mathcal{F}_{a_{1/3}}$  is the correspondence obtained from  $\mathcal{F}_{a_*}$  by pinching  $\gamma_{1/3}$ .

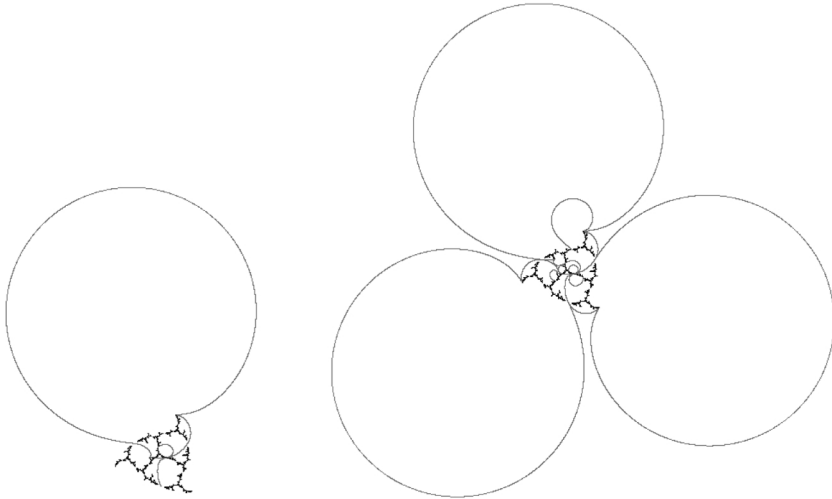


Figure 9. — Computer plots. Left:  $L \cup J(L)$  together with its images under  $Cov_0$  and  $J \circ Cov_0$ . Right: this set together with its image under a further application of  $Cov_0$ .

We deduce from the assertions above that  $D_J$  can be extended as follows, to become a fundamental domain for the action of  $Cov$  on  $\hat{\mathbb{C}}$ . Join  $a$  to  $\infty$  by any smooth curve  $M$  which is disjoint from its images under  $Cov_0$  and from  $\partial\Delta_J$ . Then the simple closed curve made up of  $M$ , the branch of  $Cov_0(M)$  running from  $\infty$  to  $J(-1)$ , and segments of the boundary of  $D_J$  running from  $J(-1)$  to  $-1$  and  $-1$  to  $a$ , will together bound a fundamental domain  $\Delta_{Cov}$  for  $Cov$ . Now observe that the complement  $\Delta_J$  of  $D_J$  is a fundamental domain for  $J$ , and that

$$\overline{\Delta_{Cov}} \cup \overline{\Delta_J} = \hat{\mathbb{C}}.$$



This is precisely the condition we need to apply the ‘ping-pong principle’. The Klein Combination Theorem has a simple statement in the case that the two fundamental domains concerned have disjoint boundaries, or when these boundaries meet at a single point (as in Section 2) but one has to take care when the boundaries meet along arcs, as in our situation here. The intersection  $\Delta = \Delta_J \cap \Delta_{Cov}$  is the interior of a triangle which has two vertices of angle zero and one vertex of angle  $2\pi/3$  (the vertex at  $\infty$ ). Let  $\mathcal{H}$  denote the ideal hexagon formed by the union of  $\Delta$  with its two images under  $Cov_0$ , together with the points which lie on the boundaries of two of these triangles, and  $\infty$  (which lies on the boundary of all three). Thus  $\mathcal{H}$  is the external region (containing  $\infty$ ) in the plot on the right in Figure 9. The ping-pong principle tells us at once that the free product  $\{Id, J\} * Cov$  acts faithfully on the union of images of  $int(\mathcal{H})$ , but we can do better than this and include the edges of  $\mathcal{H}$  (though not its vertices). To see this, consider the following four ideal hexagons, which are disjoint apart from certain of their vertices:  $\mathcal{H}_1 = \mathcal{H}$ ,  $\mathcal{H}_2 = J(\mathcal{H})$ , and the two images of  $\mathcal{H}_2$  under  $Cov_0$ , which we denote  $\mathcal{H}_3$  and  $\mathcal{H}_4$ .

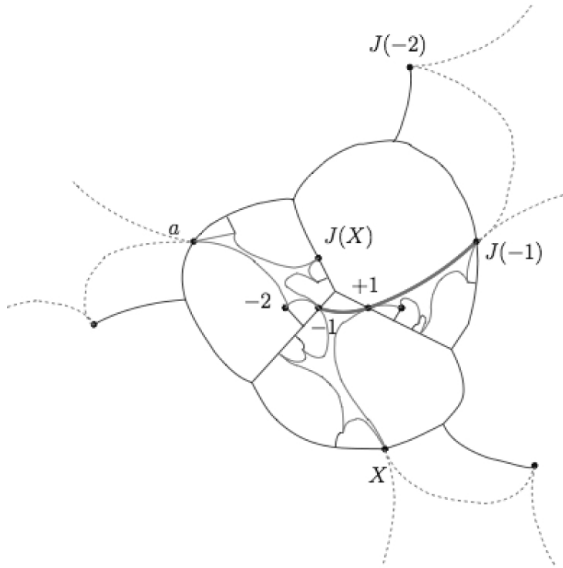


Figure 10. — Sketch of the positions of the four ideal hexagons  $\mathcal{H}_i$ . (The dashed lines are the boundary of  $\mathcal{H}_1$ , which contains  $\infty$ .)

These four hexagons  $\mathcal{H}_i$  are sketched in Figure 10 and can also be identified on the computer plot Figure 7. It is easily checked that any application of  $J$  or a branch of  $Cov_0$  to any of these  $\mathcal{H}_i$  takes it either to another of the

$\mathcal{H}_i$  or to a hexagon that has an edge in common with one of them. Setting  $\Omega$  to be the union of the images of  $\mathcal{H}$  together with its edges (but not its vertices) we deduce that  $F_{a_{1/3}}$  has a proper discontinuous action on  $\Omega$ , that this is a faithful action of the free product of  $\{Id, J\}$  with  $Cov$ , and that the ‘centres’ of the hexagons (the images of the point  $\infty \in \mathcal{H}$ ), each have stabiliser a conjugate of the correspondence  $Cov$ . This gives us Statement 1 of the Theorem. Furthermore, the components of  $\Omega$  are built up inductively from the four  $\mathcal{H}_i$  by adjoining ideal hexagons along edges. Thus  $\Omega$  is the disjoint union of four components, each containing one of the  $\mathcal{H}_i$ , and each homeomorphic to a disc (Statement 2).

The way that the four components of  $\Omega$  are mapped to one another by  $Cov_0$  and  $J$  is determined by where the initial four ideal hexagons  $\mathcal{H}_i$  are mapped. Thus  $Cov_0$  stabilises  $\Omega_1$  and sends each of  $\Omega_2, \Omega_3, \Omega_4$  to the other two, and  $J$  exchanges  $\Omega_1$  with  $\Omega_2$  and also exchanges  $\Omega_3$  with  $\Omega_4$  (see Figure 8 for the labelling of the components). It is easily checked that  $\Omega_2$  is stabilised by  $J \circ Cov_0 \circ J$  (which is a pair of rotations of  $\Omega_2$  of order three, inverse to one another) and by the two branches of  $Cov_0 \circ J \circ Cov_0$  (inverse to one another) which fix the point  $-2$  on the boundary of  $\Omega_2$ . Furthermore, since the images of  $\mathcal{H}_2$  under combinations of these elliptic and parabolic transformations fully tile  $\Omega_2$ , we deduce that they freely generate the set of branches of iterates of  $\mathcal{F}_{a_{1/3}}$  which stabilise  $\Omega_2$ , and that this set of branches forms a group isomorphic to  $C_3 * C_\infty$ .  $\square$

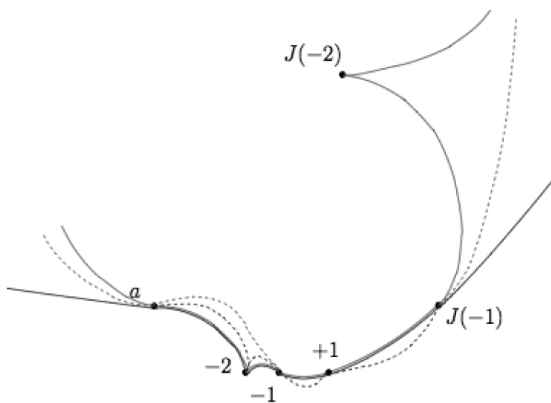


Figure 11. — The boundaries of  $\Delta_J$  (solid grey) and  $\Delta_{Cov}$  (solid black) coincide along the segment between  $a$  and  $J(-1)$ . We obtain new fundamental domains  $\Delta'_J$  and  $\Delta'_{Cov}$  by modifying parts of these boundaries to the positions indicated by the dashed grey and black lines respectively.

*Remarks.* —

1. To show that the parabolic 3-cycle can be ‘unpinched’ to deform the correspondence into one which satisfies the generic set-up of Definition 2.2, we first keep the value of  $a$  fixed at  $a_{1/3}$  and modify  $\Delta_J$  and  $\Delta_{Cov}$  to new fundamental domains  $\Delta'_J$  and  $\Delta'_{Cov}$  for  $J$  and  $Cov$ , in such a way that we still have  $\overline{\Delta'_{Cov}} \cup \overline{\Delta'_J} = \hat{C}$  but now the boundaries of  $\Delta'_J$  and  $\Delta'_{Cov}$  meet at just four points, namely the point  $+1$  and the points of the 3-cycle.

We illustrate how this can be done in Figure 11 (one has to cut thin strips off certain edges of  $\Delta_J$  and  $\Delta_{Cov}$ , and glue them onto other edges). If one now makes a sufficiently small perturbation of the parameter  $a$ , in a direction which splits the (parabolic) 3-cycle into a pair of 3-cycles, one attracting and the other repelling, then  $\Delta'_J$  and  $\Delta'_{Cov}$  will move apart so that their boundaries only meet at  $+1$ .

2. The limit set  $\Lambda$  of  $\mathcal{F}_{a_{1/3}}$  (the complement of  $\Omega$ ) can be described combinatorially as the quotient of the unpinched limit set (a Cantor set) under the identifications induced by contracting the leaves of  $\Gamma_{1/3}$  to points. Alternatively, we may view  $\mathcal{F}_{a_{1/3}}$  on  $\hat{C}$  as a kind of mating of four copies of an action of  $C_3 * C_\infty$  on the unit disc, glued together along their boundaries, and then  $\Lambda$  becomes a quotient space of the union of the four boundary circles of these discs.

3. In this article we have considered just one example of a correspondence at the boundary of the Klein combination locus. At other boundary points the behaviour can be very different. Further examples, and conjectures concerning the overall structure of the boundary of the Klein combination locus, will be presented elsewhere.

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