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Integrable Osculating Plane Distributions

Gilcione Nonato Costa[(1)]

Abstract. — We give a necessary condition for a holomorphic vector field to induce an integrable osculating plane distribution and, using this condition, we give a characterization of such fields. We also give a generic classification for vector fields which have two invariant coordinate planes.

Résumé. — Nous donnons une condition nécessaire sur les champs de vecteurs holomorphes qui induit une distributuon de plans osculateurs et, en l'utitilisant, nous donnons une caracterisation de ces champs. Cela nous permettra aussi d'obtenir une classification générique des champs de vecteurs ayant deux plans coordonnés invariants.

1. Introduction

In the second half of the 19th century, Arthur Cayley observed that the a linear vector field defined in $\mathbb{R}^3$ or $\mathbb{C}^3$ has an integrable osculating plane distribution (*opd* for short). To be precise, let us consider the orbit $\phi(t)$ of the vector field X such that $\phi'(t) = X(\phi(t))$ with $t \in \mathbb{C}$. The osculating plane associated to X is spanned by the vectors $\phi'(t)$ and

$$\phi''(t) = DX(\phi(t)) \cdot \phi'(t) = DX(\phi(t)) \cdot X(\phi(t)).$$

In other words, the osculating plane is generated by the vector fields X and $Y = DX \cdot X$. Therefore, the osculating plane of a trajectory at a given point is determined by its initial direction and by the direction of the force acting at the given point.

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Article proposé par Vincent Guedj.

Now let $X = Ax + B$ be an affine linear vector field with $A \in M(3,\mathbb{C})$, the set of complex matrices 3×3, $B \in \mathbb{C}^3$. Then $Y(x) = A\cdot X = A^2x + AB$. Taking the Lie bracket, we get

$$[X,Y] = DX\cdot Y - DY\cdot X = A\cdot[A^2x+AB] - A^2\cdot[Ax+B] = 0.$$

Since the Lie bracket vanishes, X and Y span an involutive distribution. This shows that all affine linear vector fields in $\mathbb{C}^3$ have an integrable *opd* . It can be shown that for fields of degree greater than 1 this is not a generic fact. Furthermore, except when X is an affine linear vector field in $\mathbb{C}^3$, it is not easy to find a vector field such that its *opd* is integrable.

In [2], one of the 14 problems proposed by Dominique Cerveau was the description and classification of all the real or complex polynomial vector fields that have this beautiful property. In this article, we will give a partial answer to Cerveau's question, as follows: we will first give a necessary condition for a holomorphic vector field in $\mathbb{C}^3$ to have an integrable *opd*. Then we characterize and classify the polynomial vector fields in $\mathbb{C}^3$ which have two invariant coordinate planes and an integrable *opd*.

Let X be a holomorphic vector field in $\mathbb{C}^3$ and ω_X be the 1-form given by wedge product between X and $Y = DX\cdot X$. If ω_X is integrable then the singular set $\mathrm{Sing}(\omega_X)$ will have at least one component of dimension one, i.e., an analytic curve $\mathcal{C}$. Furthermore, by construction, ω_X is also invariant by X and Y, which shows that $\mathcal{C}$ is invariant by X and Y simultaneously. There are three situations to be considered, depending on the curve $\mathcal{C}$ being contained or not in the singular sets of X or Y. As we will see, if $\mathcal{C}$ is not contained in the singular set $\mathrm{Sing}(X)$ then $\mathcal{C}$ is a straight line. Unless the change of coordinates is linear, the integrability of *opd* is non-invariant by diffeomorphism of $\mathbb{C}^3$. See the example (2.4).

In this article, we will treat in the polynomial vector fields X in $\mathbb{C}^3$ such that the invariant straight line $\mathcal{C}$ is not contained in the singular sets of X and Y. By a linear change of variables, this curve $\mathcal{C}$ may be given as $x_1 = x_2 = 0$ in some coordinate system of $\mathbb{C}^3$. Furthermore, we will impose the two coordinate planes that define the curve invariant $\mathcal{C}$ also be invariant by X and Y. More precisely, we will prove the following theorem:

THEOREM 1.1. — *Consider the space of polynomial vector fields*

$$X(x) = x_1F_0(x)\frac{\partial}{\partial x_1} + x_2G_0(x)\frac{\partial}{\partial x_2} + H_0(x)\frac{\partial}{\partial x_3},$$

such that F_0, G_0, H_0 and $(H_0)_{x_3}$ are non-zero on the x_3-axis. If the opd *induced by X is integrable then one of following condition is generically satisfied*

1. $F_0(x) = bP_0(x)$, $G_0(x) = bQ_0(x)$ *and* $H_0(x) = P_0(x)[a + x_3 b]$ *for some* $a, b \in \mathbb{C}[x_1, x_2]$;

2. $F_0(x) = bP_0(x)$, $G_0(x) = bQ_0(x)$ *and* $H_0(x) = Q_0(x)[a + x_3 b]$ *for some* $a, b \in \mathbb{C}[x_1, x_2]$;

3. $F_0(x) = G_0(x)$ *for all* $x \in \mathbb{C}^3$;

4. $F_0(x) = aP_0(x)$, $G_0(x) = 1 + bP_0(x)$ *and* $H_0(x) = r_0 + \int_0^{x_3} \left[1 + 2bP_0(\xi) + b(b-a)P_0^2(\xi)\right] d\xi_3$ *with* $a, b, r_0 \in \mathbb{C}[x_1, x_2]$ *and* $\xi = (x_1, x_2, \xi_3) \in \mathbb{C}^3$.

Conversely, in the section 4, we will then examine the four conditions given on the theorem 1.1. With the exception of condition (3), in which case the *opd* is always integrable, we must impose additional conditions on F_0, G_0 and H_0 to guarantee the integrability of ω_X.

2. Preliminary

Throughout this paper X will denote a holomorphic vector field and its *opd* will be described by the 1-form ω_X, both defined in a three-dimensional complex manifold M.

DEFINITION 2.1. — *A pair of holomorphic vector fields* $\{X, Y\}$ *will be called an osculating pair if* $Y(x) = DX(x) \cdot X(x)$ *in some coordinate system* $x \in \mathbb{C}^3$.

DEFINITION 2.2. — *The osculating pairs* $\{X_i, Y_i\}$, *for* $i = 1, 2$, *will be called strongly conjugate if there is a unique biholomorphism* f *which conjugates* X_1, X_2 *and* Y_1, Y_2, *simultaneously.*

PROPOSITION 2.3. — *The osculating pairs* $\{X_i, Y_i\}$, *for* $i = 1, 2$, *are strongly conjugate via the biholomorphism* f *if*

$$\left[D^2 f(x) \cdot (Df(x))^{-1} X_1(x)\right] \cdot Df(x) \cdot X_1(x) = 0, \forall x \in \mathbb{C}^3.$$

Proof. — By hypothesis, the osculating pairs $\{X_i, Y_i\}$, for $i = 1, 2$, are strongly conjugate. It follows that

$$X_2(x) = f_* X_1 = Df(f^{-1}(x)) \cdot X_1(f^{-1}(x)),$$

$$Y_2(x) = f_* Y_1 = Df(f^{-1}(x)) \cdot Y_1(f^{-1}(x)),$$

the push-forward of X_1 and Y_1 by f, respectively. Since $Y_1(x) = DX_1(x) \cdot X_1(x)$, we have

$$Y_2(x) = Df(f^{-1}(x)) \cdot DX_1(f^{-1}(x)) \cdot X_1(f^{-1}(x)).$$

On the other hand, $Y_2(x) = DX_2(x) \cdot X_2(x)$ where

$$DX_2(x) = D^2 f(f^{-1}(x)) \cdot Df^{-1}(x) \cdot X_1(f^{-1}(x)) \\ + Df(f^{-1}(x)) \cdot DX_1(f^{-1}(x)) \cdot Df^{-1}(x).$$

Since $Df(f^{-1}(x))Df^{-1}(x) = I_d$, our result follows. □

At this point, we have the first technical difficulty. The property of a vector field to have an integrable *opd* is not in general invariant by a change of coordinates, as we see in this next example.

Example 2.4. — Let $X(x) = \sum_{i=1}^{3} \lambda_i x_i \frac{\partial}{\partial x_i}$ be a linear vector field defined in $\mathbb{C}^3$ with $\lambda_i \neq 0$ for all i. Consider the polynomial diffeomorphism $f(x) = (x_1, x_2 - x_1^2, x_3)$ of $\mathbb{C}^3$ and the vector field $X_2 = f_* X_1$ given by

$$X_2(x) = \lambda_1 x_1 \frac{\partial}{\partial x_1} + (\lambda_2 x_2 + \mu x_1^2) \frac{\partial}{\partial x_2} + \lambda_3 x_3 \frac{\partial}{\partial x_3}$$

where $\mu = \lambda_2 - 2\lambda_1$. After computations, we get

$$\omega_{X_2} = \left[\frac{\lambda_3 - \lambda_2}{\lambda_1 x_1} + \frac{\mu(\lambda_3 - \lambda_2 - 2\lambda_1)x_1}{\lambda_1 \lambda_2 x_2} \right] dx_1 \\ + \frac{\lambda_1 - \lambda_3}{\lambda_2 x_2} dx_2 + \left[\frac{\lambda_2 - \lambda_1}{\lambda_3 x_3} + \frac{\mu(\lambda_1 + \lambda_2)x_1^2}{\lambda_2 \lambda_3 x_2 x_3} \right] dx_3.$$

The integrability condition for ω_{X_2} is

$$\omega_{X_2} \wedge d\omega_{X_2} = \frac{2\mu\lambda_1(\lambda_3 - \lambda_1)x_1}{\lambda_2^2 \lambda_3 x_2^2 x_3} dx_1 \wedge dx_2 \wedge dx_3.$$

Henceforward, ω_{X_2} is integrable if only if $\mu = 0$ or $\lambda_1 = \lambda_3$. Therefore, given that ω_{X_1} is always integrable, this property is not invariant by change of coordinates unless this change to be linear. In fact, if f is linear then $X_2 = f_* X_1$ and $Y_2 = f_* Y_1$ and $[X_2, Y_2] = [f_* X_1, f_* Y_1] = f_*[X_1, Y_1]$.

Corollary 2.5. — *If X_1 and X_2 are holomorphic conjugate via a linear application then the osculating pairs $\{X_i, Y_i\}$, for $i = 1, 2$, are strongly conjugate.*

Proposition 2.6. — *Let X be a holomorphic vector field in $\mathbb{C}^3$. Assume that exists a vector $v \in \mathbb{C}^3$ such that $\langle X, v\rangle = 0$ in some coordinate system $x \in \mathbb{C}^3$. Then the opd induced by X is integrable.*

Proof. — Let

$$X(x) = A_1(x)\frac{\partial}{\partial x_1} + A_2(x)\frac{\partial}{\partial x_2} + A_3(x)\frac{\partial}{\partial x_3}.$$

By a linear change of variables, if necessary, we can assume that $v = (0, 0, 1)$. Since $\langle X, v\rangle = 0$ we have $A_3(x) \equiv 0$. Then $\omega_X = g(x)dx_3$ for some holomorphic function g. It is now easy to show that ω_X is integrable. □

Lemma 2.7. — *Let X_1 be a holomorphic vector field in $\mathbb{C}^3$ such that there exists a holomorphic function f and a field X_2 such that $X_1(x) = f(x) \cdot X_2(x)$. Then ω_{X_1} is integrable if only if ω_{X_2} is integrable.*

Proof. — Since $X_1 = f \cdot X_2$ then $\omega_{X_1} = f^3 \cdot \omega_{X_2}$. Thus, if ω_{X_2} is integrable it follows that $\omega_{X_1} \wedge d\omega_{X_1} = 0$ except on divisor of poles of X_2 of codimension one. Using the Riemann extension theorem, $\omega_{X_1} \wedge d\omega_{X_1}$ extends for all $\mathbb{C}^3$ as a null function. It follows that ω_{X_1} is integrable. On the other hand, it is clear that ω_{X_2} is integrable if ω_{X_1} is integrable. □

From now on, let $\mathcal{F}_i$ be a i-dimensional holomorphic foliation defined on a 3-dimensional manifold M. $\mathcal{F}_2$ will be called invariant by $\mathcal{F}_1$ if $T_p\mathcal{F}_1 \subset T_p\mathcal{F}_2$, for all $p \in M$; in other words, the leaves of $\mathcal{F}_1$ are contained in the leaves of $\mathcal{F}_2$. Locally, $\mathcal{F}_1$ is described by a holomorphic vector field X while $\mathcal{F}_2$ is described by a holomorphic 1-form ω. The condition of invariance of $\mathcal{F}_2$ by $\mathcal{F}_1$ can be written as

$$\omega \cdot X = \sum_{i=1}^{3} A_i(x)B_i(x) = 0$$

where $X = \sum_{i=1}^{3} A_i(x)\frac{\partial}{\partial x_i}$ and $\omega = \sum_{i=1}^{3} B_i(x)dx_i$. See [3] for details.

Proposition 2.8. — *Let $\mathcal{F}_i$, for $i = 1, 2$, be i-dimensional holomorphic foliations defined on a three-dimensional manifold M. Assume the $\mathcal{F}_2$ is invariant by $\mathcal{F}_1$. Then* $\mathrm{Sing}(\mathcal{F}_2)$ *is invariant by $\mathcal{F}_1$.*

Proof. — As before, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be described by the holomorphic vector field X and the 1-form ω, respectively. Since $\mathcal{F}_2$ is a foliation it follows that ω is integrable. Let $p \in \mathrm{Sing}(\mathcal{F}_2) - \mathrm{Sing}(\mathcal{F}_1)$. By a change of variables, we can locally assume that $p = (0,0,0)$ and $X = \dfrac{\partial}{\partial x_3}$. In this situation, the orbits of X are given by $(x_0, y_0, z_0 + t)$ for adequate values of $t \in \mathbb{C}$. In this coordinate system, we have

$$\omega = \sum_{i=1}^{3} B_i(x) dx_i.$$

By hypothesis $\omega \cdot X = B_3(x) \equiv 0$. Since ω is integrable it follows that

$$\omega \wedge d\omega = \left[B_2 \frac{\partial B_1}{\partial x_3} - B_1 \frac{\partial B_2}{\partial x_3} \right] dx_1 \wedge dx_2 \wedge dx_3 = 0.$$

Given that the set $\mathrm{Sing}(\mathcal{F}_2)$ has codimension 2, the germs B_1 and B_2 do not have irreducible factors in common. The Weierstrass preparation theorem shows that there exists an irreducible factor $\phi(x) \in \mathcal{O}_2[x_3]$ of B_1 with multiplicity m. Let $B_1(x) = \phi^m(x) C_1(x)$; then

$$\phi^m(x) C_1(x) \frac{\partial B_2}{\partial x_3} = \left[m\phi^{m-1}(x) \frac{\partial \phi}{\partial x_3} C_1(x) + \phi^m(x) \frac{\partial C_1}{\partial x_3} \right] B_2(x).$$

Since ϕ is not a factor of B_2, it must be a factor of $m\dfrac{\partial \phi}{\partial x_3} C_1(x) + \phi \dfrac{\partial C_1}{\partial x_3}$. But ϕ is not a factor of $C_1(x)$ either and therefore ϕ divides $\dfrac{\partial \phi}{\partial x_3}$. Since $\phi(x) \in \mathcal{O}_2[x_3]$, we have $\dfrac{\partial \phi(x)}{\partial x_3} \equiv 0$. Consequently, all irreducible factors of B_1 are constant with respect to x_3 and the same is true for B_2. Therefore

$$\omega = B_1(x_1, x_2) dx_1 + B_2(x_1, x_2) dx_2.$$

Since $p \in \mathrm{Sing}(\mathcal{F}_2)$ we have $B_1(0,0) = B_2(0,0) = 0$. Thus, for adequate values of $t \in \mathbb{C}$, the line $(0,0,t)$ is contained in $\mathrm{Sing}(\mathcal{F}_2)$ and is a leaf of $\mathcal{F}_1$.
□

THEOREM 2.9. — *Let X be a holomorphic vector field defined on a three-dimensional manifold M with an integrable opd. Then there exists an analytical curve $\mathcal{C} \subset M$ invariant by the osculating pair $\{X, Y\}$.*

Proof. — Let $\mathcal{F}_2$ be the osculating foliation defined by ω_X. Since ω_X is integrable, $\mathrm{Sing}(\mathcal{F}_2)$ has codimension two; in other words, this set has at least a 1-dimensional component $\mathcal{C}$. Furthermore, by construction, $\mathcal{F}_2$ is invariant by the osculating pair $\{X, Y\}$. From proposition (2.8) it follows that $\mathcal{C}$ is invariant by $\{X, Y\}$. □

Theorem 2.9 gives us a necessary condition for the *opd* induced by a holomorphic vector field to be integrable. At this point, it is natural to ask about the type of invariance as well as the properties of this curve. Since $Y = DX \cdot X$, we have three distinct situations for the invariance of $\mathcal{C}$ by $\{X, Y\}$:

1. $\mathcal{C} \not\subset \mathrm{Sing}(X) \subset \mathrm{Sing}(Y)$;
2. $\mathcal{C} \not\subset \mathrm{Sing}(X)$ but $\mathcal{C} \subset \mathrm{Sing}(Y)$;
3. $\mathcal{C} \subset \mathrm{Sing}(X) \subset \mathrm{Sing}(Y)$.

For cases 1 and 2, we have the following proposition

Proposition 2.10. — *Let X be a holomorphic vector field in $\mathbb{C}^3$ such that the $1-$form ω_X is integrable. Consider the analytical curve $\mathcal{C} \subset \mathbb{C}^3$ invariant by $\{X, Y\}$. If $\mathcal{C} \not\subset \mathrm{Sing}(X)$ then $\mathcal{C}$ is a straight line.*

Proof. — Let $t \to \phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t))$ be the parametrization of $\mathcal{C}$ such that $\phi'(t) = X(\phi(t))$ with $t \in \mathbb{C}$. By hypothesis, we have two possible situations: $\mathcal{C} \subset \mathrm{Sing}(Y)$ or $\mathcal{C} \not\subset \mathrm{Sing}(Y)$.

If $\mathcal{C} \not\subset \mathrm{Sing}(Y)$ then there exists a complex function $\lambda(t)$ such that $\phi''(t) = \lambda(t)\phi'(t)$. By integration, we obtain $\phi(t) = \mathbf{a}h(t) + \mathbf{b}$, for a certain complex function $h(t)$ and constants $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$.

If $\mathcal{C} \subset \mathrm{Sing}(Y)$ then $Y((\phi(t)) = DX(\phi(t)) \cdot X(\phi(t)) = 0$. Since $\phi'(t) = X(\phi(t))$, we have that

$$Y((\phi(t)) = DX(\phi(t)) \cdot X(\phi(t)) = DX(\phi(t)) \cdot \phi'(t) = \frac{d}{dt} X(\phi(t)) = 0.$$

Consequently, $X(\phi(t)) = \mathbf{a}$ where $\mathbf{a} \in \mathbb{C}^3$ is a constant. So $\phi'(t) = \mathbf{a}$ and therefore $\phi(t) = \mathbf{a}t + \mathbf{b}$, with $\mathbf{b} \in \mathbb{C}^3$. □

Example 2.11. — Consider $X(x) = \sum_{i=1}^{3} \lambda_i x_i \frac{\partial}{\partial x_i}$ defined in $\mathbb{C}^3$ with $\lambda_i \neq 0$ for $i = 1, 2, 3$. Then $Y(x) = DX(x) \cdot X(x) = \sum_{i=1}^{3} \lambda_i^2 x_i \frac{\partial}{\partial x_i}$ and

$$\omega_X = \frac{\lambda_3 - \lambda_2}{\lambda_1} \frac{dx_1}{x_1} + \frac{\lambda_1 - \lambda_3}{\lambda_2} \frac{dx_2}{x_2} + \frac{\lambda_2 - \lambda_1}{\lambda_3} \frac{dx_3}{x_3}.$$

It is not hard to show the integrability of ω_X. In this situation, the three coordinate axes' are invariant by the osculating pair $\{X, Y\}$.

As we will see, many holomorphic vector fields with an integrable *opd* can be reduced to a vector field of the type given in the next example.

Example 2.12. — Consider

$$X(x) = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2} + f(x)\frac{\partial}{\partial x_3}$$

where f is holomorphic and λ_1, λ_2 are non-zero constants. Then $Y = DX \cdot X$ is given by

$$Y(x) = \lambda_1^2 x_1 \frac{\partial}{\partial x_1} + \lambda_2^2 x_2 \frac{\partial}{\partial x_2} + g(x)\frac{\partial}{\partial x_3}$$

where $g(x) = \lambda_1 x_1 f_{x_1} + \lambda_2 x_2 f_{x_2} + f f_{x_3}$, while the *opd* induced by X is described by the 1-form

$$\omega_X = \frac{g - \lambda_2 f}{\lambda_1 x_1} dx_1 + \frac{\lambda_1 f - g}{\lambda_2 x_2} dx_2 + (\lambda_2 - \lambda_1) dx_3.$$

For ω_X to be integrable we must have

$$\omega_X \wedge d\omega_X = \left[h_2 \frac{\partial h_1}{\partial x_3} - h_1 \frac{\partial h_2}{\partial x_3} + (\lambda_2 - \lambda_1)\left(\frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} \right) \right] dx_1 \wedge dx_2 \wedge dx_3 \equiv 0 \tag{2.1}$$

where $h_1 = \dfrac{g - \lambda_2 f}{\lambda_1 x_1}$ and $h_2 = \dfrac{\lambda_1 f - g}{\lambda_2 x_2}$. Since

$$h_2 \frac{\partial h_1}{\partial x_3} - h_1 \frac{\partial h_2}{\partial x_3} = (\lambda_2 - \lambda_1) \frac{g f_{x_3} - f g_{x_3}}{\lambda_1 \lambda_2 x_1 x_2}$$

it follows from (2.1) that $\omega_X \wedge d\omega_X = (\lambda_2 - \lambda_1) h(x) dx_1 \wedge dx_2 \wedge dx_3$ for some rational function h. In particular, if $\lambda_1 = \lambda_2$ then ω_X is integrable regardless of f, which determines the type of invariance of the x_3-axis by $\{X, Y\}$. More precisely, if $f(0,0,x_3) \equiv 0$ then situation 3 occurs while if $f(0,0,x_3) \not\equiv 0$ and $f_{x_3}(0,0,x_3) \equiv 0$ then situation 2 occurs. Finally, the condition for the situation 1 to occur is $f(0,0,x_3) f_{x_3}(0,0,x_3) \not\equiv 0$.

3. Proof of Theorem 1.1

From now on, we will concentrate on a polynomial vector field X in $\mathbb{C}^3$ that contains a straight line $\mathcal{C}$ invariant by the osculating pair $\{X, Y\}$ and such that $\mathcal{C} \not\subset \mathrm{Sing}(X)$. By a linear change of variables, we can suppose that $\mathcal{C}$ is the x_3-axis in some coordinate system $x \in \mathbb{C}^3$.

Let $f : \mathbb{C}^3 \to \mathbb{C}$ be a holomorphic complex non-null function vanishing along $\mathcal{C}$; f can then be written as:

$$f(x) = x_1 f_1(x_1, x_2, x_3) + x_2 f_2(x_1, x_2, x_3). \tag{3.1}$$

If f_1 and f_2 also vanish on the $x_3 - axis$, they can also be written as in (3.1). Thus f can be rewritten as

$$f(x) = x_1^2 f_{2,0}(x_1, x_2, x_3) + x_1 x_2 f_{1,1}(x_1, x_2, x_3) + x_2^2 f_{0,2}(x_1, x_2, x_3).$$

We repeat this process until we find some function $f_{i,j}$ which does not vanish on the x_3-axis. Then f will be of the form

$$f(x) = \sum_{i+j=m} x_1^i x_2^j f_{i,j}(x). \tag{3.2}$$

where, for some $i, j \in \mathbb{N}$, we have that $f_{i,j}(0, 0, x_3) \not\equiv 0$ and $x_1^i x_2^j f_{i,j}$ are linearly independent. See [4] for details.

Definition 3.1. — *The number m in (3.2) will be called the multiplicity of f along $\mathcal{C}$ and will be denoted by $mult_{\mathcal{C}}(f)$.*

Henceforth, we will denote functions that vanish on the x_3-axis with multiplicity i by capital letters indexed by i, i.e., A_i, B_i and so on.

The conditions for $\mathcal{C}$ to be non-trivially invariant by the vector field

$$X(x) = P(x)\frac{\partial}{\partial x_1} + Q(x)\frac{\partial}{\partial x_2} + R(x)\frac{\partial}{\partial x_3}$$

are $P|_{\mathcal{C}} \equiv 0$, $Q|_{\mathcal{C}} \equiv 0$ and $R|_{\mathcal{C}} \not\equiv 0$, in the other words, $P(x) = \sum_{i=0}^{m} x_1^i x_2^{m-i} p_i(x)$, $Q(x) = \sum_{i=0}^{n} x_1^i x_2^{n-i} q_i(x)$ and $R(x) = \sum_{i=0}^{r} r_i(x_1, x_2) \cdot x_3^i$, with some $r_i(0, 0) \not\equiv 0$ and integers m, n and r. Furthermore, if $R_{x_3}|_{\mathcal{C}} \not\equiv 0$, $\mathcal{C}$ will be also non-trivially invariant by the vector field $Y = DX \cdot X$.

Initially we will consider $\text{mult}_{\mathcal{C}}(P) = \text{mult}_{\mathcal{C}}(Q) = 1$ and the coordinate planes $x_1 = 0$ and $x_2 = 0$ invariant by X. As a consequence, X assumes the following form

$$X(x) = x_1 P_0(x)\frac{\partial}{\partial x_1} + x_2 Q_0(x)\frac{\partial}{\partial x_2} + R_0(x)\frac{\partial}{\partial x_3}$$

with the additional condition $(R_0)_{x_3}|_{\mathcal{C}} \not\equiv 0$. Then, generically, we have

$$Y(x) = A(x)\frac{\partial}{\partial x_1} + B(x)\frac{\partial}{\partial x_2} + C(x)\frac{\partial}{\partial x_3}$$

where

$$\begin{array}{lllll} A & = & A_1 + A_2 & = & x_1\big[P_0^2 + R_0(P_0)_{x_3}\big] + x_1^2 P_0(P_0)_{x_1} + x_1 x_2 Q_0(P_0)_{x_2} \\ B & = & B_1 + B_2 & = & x_2\big[Q_0^2 + R_0(Q_0)_{x_3}\big] + x_1 x_2 P_0(Q_0)_{x_1} + x_2^2 Q_0(Q_0)_{x_2} \\ C & = & C_0 + C_1 & = & R_0(R_0)_{x_3} + x_1 P_0(R_0)_{x_1} + x_2 Q_0(R_0)_{x_2}. \end{array}$$

Taking the wedge product between X and Y we get

$$\omega_X = E(x)dx_1 + F(x)dx_2 + G(x)dx_3$$

where

$$\begin{array}{lllll} E & = & E_1 + E_2 & = & x_2 R_0\big[Q_0(R_0)_{x_3} - R_0(Q_0)_{x_3} - Q_0^2\big] + x_2 Q_0 C_1 - R_0 B_2 \\ F & = & F_1 + F_2 & = & x_1 R_0\big[P_0^2 + R_0(P_0)_{x_3} - P_0(R_0)_{x_3}\big] + R_0 A_2 - x_1 P_0 C_1 \\ G & = & G_2 + G_3 & = & x_1 x_2\big[P_0 Q_0(Q_0 - P_0) + R_0\big(P_0(Q_0)_{x_3} - Q_0(P_0)_{x_3}\big)\big] + \\ & & & & + x_1^2 x_2 P_0 \cdot [P_0(Q_0)_{x_1} - Q_0(P_0)_{x_1}] \\ & & & & + x_1 x_2^2 Q_0 \cdot [P_0(Q_0)_{x_2} - Q_0(P_0)_{x_2}]. \end{array}$$

For ω_X to be integrable we must have

$$\omega_X \wedge d\omega_X = [W_2 + W_3 + \cdots]dx_1 \wedge dx_2 \wedge dx_3 \equiv 0 \tag{3.3}$$

with

$$W_2 = x_1 x_2[M \cdot N_{x_3} - N \cdot M_{x_3}] \tag{3.4}$$

where $M = R_0\bigg(P_0^2 + R_0(P_0)_{x_3} - P_0(R_0)_{x_3}\bigg)$
and $N = R_0\bigg(Q_0(R_0)_{x_3} - R_0(Q_0)_{x_3} - Q_0^2\bigg)$.

Since $W_2 \equiv 0$ is a generic condition, it is enough to discuss (3.3) under this assumption. We will now solve $W_2 \equiv 0$ considering the following cases:

Case 1. $M = \bigg[R_0\bigg(P_0^2 + R_0(P_0)_{x_3} - P_0(R_0)_{x_3}\bigg)\bigg] \equiv 0.$

Since $R_0 \not\equiv 0$, we have

$$\frac{P_0(R_0)_{x_3} - R_0(P_0)_{x_3}}{P_0^2} = \frac{\partial}{\partial x_3}\left(\frac{R_0}{P_0}\right) = 1$$

and consequently

$$R_0(x) = P_0(x)\big[x_3 + k(x_1, x_2)\big]$$

for some rational function $k \in \mathbb{C}(x_1, x_2)$. Let $k(x_1, x_2) = \dfrac{a(x_1, x_2)}{b(x_1, x_2)}$ with $a, b \in \mathbb{C}[x_1, x_2]$ without a factor in common. Multiplying all the components of X by b, we obtain case 1 of theorem 1.1.

Case 2. $N = \left[R_0\left(Q_0(R_0)_{x_3} - R_0(Q_0)_{x_3} - Q_0^2\right)\right] \equiv 0.$

This case is similar to the previous one.

Case 3.

$$\left[R_0\left(P_0^2 + R_0(P_0)_{x_3} - P_0(R_0)_{x_3}\right)\right] \cdot \left[R_0\left(Q_0(R_0)_{x_3} - R_0(Q_0)_{x_3} - Q_0^2\right)\right] \not\equiv 0.$$

Given that $W_2 \equiv 0$, it is not hard to see that

$$\frac{\partial}{\partial x_3}\left[\frac{M}{N}\right] \equiv 0$$

and therefore

$$\frac{M}{N} = \frac{R_0[P_0^2 + R_0(P_0)_{x_3} - P_0(R_0)_{x_3}]}{R_0[Q_0(R_0)_{x_3} - R_0(Q_0)_{x_3} - Q_0^2]} = -\frac{a(x_1, x_2)}{b(x_1, x_2)}$$

for some polynomials a and b without a factor in common.

Therefore, we have

$$\big(aQ_0 - bP_0\big) \cdot (R_0)_{x_3} - R_0 \cdot \big(aQ_0 - bP_0\big)_{x_3} = aQ_0^2 - bP_0^2. \tag{3.5}$$

To solve the partial differential equation (3.5) for R_0, we must consider two distinct situations:

$$(aQ_0 - bP_0) \equiv 0 \quad \text{or} \quad (aQ_0 - bP_0) \not\equiv 0.$$

Suppose first $(aQ_0 - bP_0) \equiv 0$ then we also have $(aQ_0 - bP_0)_{x_3} \equiv 0$. Thus, from (3.5), we get

$$(aQ_0 - bP_0) \equiv (aQ_0^2 - bP_0^2) \equiv 0 \Leftrightarrow \frac{P_0}{Q_0} = \frac{P_0^2}{Q_0^2}.$$

Since P_0 and Q_0 are not identically zero it is follows that $P_0 = Q_0$ for all $x \in \mathbb{C}^3$. We therefore get the third case of theorem 1.1. Furthermore, if this condition is satisfied then ω_X is always integrable.

Now we consider $(aQ_0 - bP_0) \not\equiv 0$. Dividing (3.5) by $(aQ_0 - bP_0)^2$, the integrating factor, we get

$$\frac{\partial}{\partial x_3}\left[\frac{R_0}{aQ_0 - bP_0}\right] = \frac{aQ_0^2 - bP_0^2}{(aQ_0 - bP_0)^2},$$

and therefore

$$R_0(x) = (aQ_0 - bP_0)\left[\int_0^{x_3} \frac{aQ_0^2 - bP_0^2}{(aQ_0 - bP_0)^2} d\xi_3 + r_0(x_1, x_2)\right]$$

where $r_0 \in \mathbb{C}[x_1, x_2]$.

Therefore $(aQ_0 - bP_0) = \Lambda$ is a factor of R_0. Multiplying all components of X by a, we obtain

$$\begin{aligned}\frac{a^2Q_0^2 - abP_0^2}{(aQ_0 - bP_0)^2} &= \frac{\Lambda^2 + 2b\Lambda P_0 + (b^2 - ab)P_0^2}{\Lambda^2} \\ &= 1 + 2b\left(\frac{P_0}{\Lambda}\right) + b(b-a)\left(\frac{P_0}{\Lambda}\right)^2.\end{aligned}$$

Here it is important to emphasize that our aim is to find a polynomial solution for (3.5). For such a solution to exist, Λ must generically, divide P_0. Therefore, $P_0 = \Lambda F_0$ and it follows that $aQ_0 = \Lambda + bP_0 = \Lambda(1 + bF_0)$. Thus X assumes the following form:

$$\begin{aligned} aX(x) &= ax_1P_0\frac{\partial}{\partial x_1} + ax_2Q_0\frac{\partial}{\partial x_2} + aR_0\frac{\partial}{\partial x_3} \\ &= \Lambda\left[ax_1F_0\frac{\partial}{\partial x_1} + x_2(1 + bF_0)\frac{\partial}{\partial x_2} + H_0(x)\frac{\partial}{\partial x_3}\right]\end{aligned}$$

where $H_0(x) = r_0(x_1, x_2) + \int_0^{x_3}\left[1 + 2bF_0(\xi) + b(b-a)F_0^2(\xi)\right]d\xi_3$ with $\xi = (x_1, x_2, \xi_3)$ and r_0 is a polynomial. This finishes the proof of Theorem 1.1. □

4. Classification

In this section, we will analyze the four conditions of theorem 1.1. Except in the case 3, new conditions must be imposed in order to guarantee the integrability of ω_X.

Since cases 1 and 2 are similar, we will consider only the vector fields given in the case 1. Let us begin by considering polynomial vector fields of the following form

$$X = x_1 b(x_1,x_2)P_0(x)\frac{\partial}{\partial x_1} + x_2 b(x_1,x_2)Q_0(x)\frac{\partial}{\partial x_2} + P_0(x)[b(x_1,x_2)x_3 + a(x_1,x_2)]\frac{\partial}{\partial x_3}.$$

In order to simplify our computations, we consider the vector field X_1 given by $X = bP_0X_1$, i.e.,

$$X_1 = x_1\frac{\partial}{\partial x_1} + x_2 Q(x)\frac{\partial}{\partial x_2} + [x_3 + k(x_1,x_2)]\frac{\partial}{\partial x_3}$$

where $Q = Q_0/P_0$ and $k = a/b$. The *opd* induced by X_1 is described by the 1-form

$$\omega_{X_1} = Edx_1 + Fdx_2 + Gdx_3$$

where

$$\left\{\begin{array}{lcl} E & = & E_1 + E_2 = -x_2[x_3+k]H(x) + x_1x_2A(x) + x_2^2B(x) \\ F & = & F_2 = -x_1^2k_{x_1} - x_1x_2Qk_{x_2} \\ G & = & G_2 + G_3 = x_1x_2H(x) + x_1^2x_2Q_{x_1} + x_1x_2^2QQ_{x_2} \end{array}\right. \quad (4.1)$$

with $A(x) = [Qk_{x_1} - (x_3+k)Q_{x_1}]$, $B(x) = Q[Qk_{x_2} - (x_3+k)Q_{x_2}]$ and $H = Q^2 - Q + (x_3+k)Q_{x_3}$. The condition of integrability of ω_{X_1} is written

$$\omega_{X_1}\wedge d\omega_{X_1} = [W_2 + W_3 + \ldots]dx_1\wedge dx_2\wedge dx_3.$$

Since X_1 is obtained from the case 1 of theorem 1.1 we have $W_2 \equiv 0$. After an exhaustive computations, we get

$$W_3 = [x_3+k]H_{x_3}\left[x_1^2x_2k_{x_1} + x_1x_2^2Qk_{x_2}\right]. \quad (4.2)$$

We will impose the condition $W_3 \equiv 0$ for ω_{X_1} to be integrable. Then $W_3 \equiv 0$ if $k_{x_1} \equiv k_{x_2} \equiv 0$ or $H_{x_3} \equiv 0$. For the case where k is a constant, we have the following result:

Proposition 4.1. — *Let X be a polynomial vector field in $\mathbb{C}^3$ such that*

$$X = x_1 b(x_1,x_2)P_0(x)\frac{\partial}{\partial x_1} + x_2 b(x_1,x_2)Q_0(x)\frac{\partial}{\partial x_2} + P_0(x)[b(x_1,x_2)x_3 + a(x_1,x_2)]\frac{\partial}{\partial x_3}$$

with $b(x_1,x_2) = k\cdot a(x_1,x_2)$, where k is a constant. Then the opd induced by X is integrable.

Proof. — As before, let us consider X_1 given that $X = bP_0 \cdot X_1$, i.e.,

$$X_1 = x_1 \frac{\partial}{\partial x_1} + x_2 Q(x) \frac{\partial}{\partial x_2} + (x_3 + k) \frac{\partial}{\partial x_3}$$

where $Q = Q_0/P_0$ and k is a constant. By a linear change of variables, X_1 is of the same type as given in example (2.12) and therefore ω_{X_1} is integrable for all functions Q. □

Now we will examine the condition $H_{x_3} \equiv 0$ given in (4.1). In order to solve this equation we will write $Q(x) = \sum_{i=0}^{\infty} q_i (x_3 + k)^i$ where $q_i \in \mathbb{C}(x_1, x_2)$ are rational functions. It follows that,

$$H = Q^2 + [x_3 + k] \cdot (Q)_{x_3} - Q = \sum_{i=0}^{\infty} [\beta_i + (i-1) q_i](x_3 + k)^i$$

with $\beta_i = \sum_{j=0}^{i} q_{i-j} q_j$. Since $H_{x_3} \equiv 0$ we have $\beta_i + (i-1)q_i = 0$ for $i \in \mathbb{N}$. Therefore, for $i = 1$, we have $\beta_1 = 2q_0 q_1 = 0$. At this point, there are two distinct possibilities: either $q_0 = 0$ or $q_1 = 0$. We observe that we cannot have $q_0 = 0$ and $q_1 = 0$; in fact, $q_0 = q_1 = \cdots = q_{m-1} = 0$, with $m \geqslant 2$ implies $\beta_m = 0$. Since $\beta_m + (m-1)q_m = 0$ we would also have $q_m = 0$. But, given that Q is not identically zero, we obtain a contradiction.

Let us assume that $q_0 \neq 0$. Again there are two distinct possibilities: $q_i = 0$ for all $i \in \mathbb{N}$ or there exists $m \in \mathbb{N}$ such that $q_m \neq 0$. In the former situation we get $Q = Q(x_1, x_2)$ while in the latter we get $\beta_m - (m-1)q_m = (2q_0 + m - 1)q_m = 0$, which gives $q_0 = -(m-1)/2$.

By induction we can show $q_j = 0$ if $j \neq 0 \bmod m$ and $q_{jm} = (-1)^{j+1} \frac{q_m}{m^{j-1}}$. Then,

$$Q = q_0 + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{q_m}{m^{j-1}} (x_3 + k)^{jm} = -\frac{m-1}{2} + \frac{m q_m (z+k)^m}{m + q_m (x_3 + k)^m}.$$

Since $q_m \neq 0$ is a rational function, we can rewrite Q as

$$Q(x) = -\frac{m-1}{2} + \frac{m(x_3 + k)^m}{m\beta + (x_3 + k)^m} \tag{4.3}$$

for some rational function β. Similarly, we obtain the same relation when $q_0 = 0$ and $q_1 \neq 0$. Therefore, equation (4.3) is valid for all $m \in \mathbb{N}$.

For the case where Q is a constant, we obtain the following result:

Theorem 4.2. — *Let X be a holomorphic vector field in $\mathbb{C}^3$ such that*

$$X(x) = x_1 P_0(x)\frac{\partial}{\partial x_1} + x_2 Q_0(x)\frac{\partial}{\partial x_2} + P_0(x)[x_3 + k(x_1, x_2)]\frac{\partial}{\partial x_3}.$$

Assume that $Q_0 = \lambda \cdot P_0$ where λ is a constant. Then the opd induced by X is integrable if

1. *$\lambda \in \{0, 1\}$ or k is an affine linear function;*
2. *$\lambda = \dfrac{-ij \pm \sqrt{ij(i+j-1)}}{j(j-1)}$, $i, j \geqslant 2$, $i, j \in \mathbb{N}$ and $k(x_1, x_2) = \alpha x_1^i x_2^j$ with $\alpha \in \mathbb{C}$.*

Proof. — Let us consider the following vector field

$$X_1 = x_1\frac{\partial}{\partial x_1} + \lambda x_2\frac{\partial}{\partial x_2} + [x_3 + k(x_1, x_2)]\frac{\partial}{\partial x_3}.$$

If $\lambda = 0$ or $\lambda = 1$ the 1-form ω_{X_1} is integrable by proposition (2.6) and the example (2.12), respectively. Furthermore, if k is an affine linear function, ω_{X_1} is integrable by Cayley's observation.

Now, we can suppose that $k(x_1, x_2) = \sum_{n+m=0}^{\infty} \alpha_{m,n} x_1^m x_2^n$ and $\lambda \notin \{0, 1\}$. Thus

$$\omega_{X_1} \wedge d\omega_{X_1} = \lambda(1-\lambda)(x_1 x_2)\left[x_1^2 k_{x_1 x_1} + 2\lambda x_1 x_2 k_{x_1 x_2} + \lambda^2 x_2^2 k_{x_2 x_2}\right]$$

and therefore,

$$x_1^2 k_{x_1 x_1} + 2\lambda x_1 x_2 k_{x_1 x_2} + \lambda^2 x_2^2 k_{x_2 x_2} = \sum_{m+n=2}^{\infty} \alpha_{m,n}\left[m(m-1) + 2mn\lambda + n(n-1)\lambda^2\right] x_1^m x_2^n.$$

Since $\lambda = \dfrac{-ij \pm \sqrt{ij(i+j-1)}}{j(j-1)}$ for some $i, j \geqslant 2$, then it follows that $i(i-1) + 2ij\lambda + j(j-1)\lambda^2 = 0$. Consequently, in order for ω_{X_1} to be integrable, we can take $k(x_1, x_2) = \alpha x_1^i x_2^j$, with $\alpha \in \mathbb{C}$. □

Corollary 4.3. — *Let X be a the polynomial vector field in $\mathbb{C}^3$ given by*

$$X = x_1 b(x_1, x_2) P_0(x)\frac{\partial}{\partial x_1} + x_2 b(x_1, x_2) Q_0(x)\frac{\partial}{\partial x_2} + P_0(x)[b(x_1, x_2)x_3 + a(x_1, x_2)]\frac{\partial}{\partial x_3}$$

with $Q_0 = \lambda \cdot P_0$ *where* λ *is a constant. Then* ω_X *is integrable if one of following conditions is satisfied*

1. $\lambda \in \{0,1\}$ *or* $b(x_1,x_2) = k(x_1,x_2)a(x_1,x_2)$ *with* k *an affine linear function;*
2. $\lambda = \dfrac{-ij \pm \sqrt{ij(i+j-1)}}{j(j-1)}$, $i,j \geqslant 2$, $i,j \in \mathbb{N}$ *and* $b(x_1,x_2) = \alpha x_1^i x_2^j \cdot a(x_1,x_2)$ *with* $\alpha \in \mathbb{C}$*;*

Setting $m = 1$ in the equation (4.3) we have the following result:

Theorem 4.4. — *Let* X *be a holomorphic vector field defined in* $\mathbb{C}^3$ *such that*

$$X(x) = x_1 P_0(x)\frac{\partial}{\partial x_1} + x_2 Q_0(x)\frac{\partial}{\partial x_2} + P_0(x)[x_3 + k(x_1,x_2)]\frac{\partial}{\partial x_3}.$$

Assume that $[x_3 + h(x_1,x_2)]Q_0 = [x_3 + k(x_1,x_2)]P_0$ *for some holomorphic function* h*. Then* ω_X *is integrable if* k *or* h *is a constant function or if* $k \equiv h$*.*

Proof. — As before, we will consider the vector field X_1 such that $X = P_0 \cdot X_1$, i.e.,

$$X_1(x) = x_1\frac{\partial}{\partial x_1} + x_2 Q(x)\frac{\partial}{\partial x_2} + [x_3 + k(x_1,x_2)]\frac{\partial}{\partial x_3}$$

where $Q(x) = \dfrac{x_3 + k(x_1,x_2)}{x_3 + h(x_1,x_2)}$.

If k or h is a constant then ω_{X_1} is integrable by proposition (2.6) while if $h \equiv k$, the integrability of ω_{X_1} follows by example (2.12). In the general case, we must have

$$\omega_X \wedge d\omega_X = [W_2 + W_3 + W_4 + W_5]dx_1 \wedge dx_2 \wedge dx_3 \equiv 0.$$

Given that X_1 has the form given in the case 1 of theorem 1.1 we have $W_2 \equiv 0$ and $W_3 \equiv 0$ because $H_{x_3} \equiv 0$. Therefore, we must have $W_4 \equiv 0$ where

$$\begin{aligned} W_4 \;=\; & \frac{2(k-h)(x_3+k)}{(x_3+h)^5}\Big[x_1^3 x_2 (x_3+h)^2 h_{x_1} k_{x_1} + \\ & + x_1^2 x_2^2 (x_3+k)(x_3+h)(h_{x_1}k_{x_2} + h_{x_2}k_{x_1}) + x_1 x_2^3 (x_3+k)^2 h_{x_2} k_{x_2}\Big]. \end{aligned} \tag{4.4}$$

We now observe that, for ω_{X_1} to be integrable we must have, generically, that

$$x_1^3 x_2 (x_3+h)^2 h_{x_1} k_{x_1} + x_1^2 x_2^2 (x_3+k)(x_3+h)(h_{x_1}k_{x_2} + h_{x_2}k_{x_1}) \\ + x_1 x_2^3 (x_3+k)^2 h_{x_2} k_{x_2} \equiv 0.$$

Dividing the last equation by $x_1 x_2 (x_3+h)^2$, we obtain the quadratic equation

$$x_2^2 Q^2 h_{x_2} k_{x_2} + x_1 x_2 Q (h_{x_1}k_{x_2} + h_{x_2}k_{x_1}) + x_1^2 h_{x_1} k_{x_1} = 0,$$

and therefore $Q = -\dfrac{x_1 k_{x_1}}{x_2 k_{x_2}}$ or $Q = -\dfrac{x_1 h_{x_1}}{x_2 h_{x_2}}$.

In the first case $(x_1 k_{x_1} + x_2 k_{x_2})x_3 + (x_1 h k_{x_1} + x_2 k k_{x_2}) \equiv 0$ for all x_3 and so we must have $(x_1 k_{x_1} + x_2 k_{x_2}) \equiv 0$. This is a contradiction unless k is a constant function. The second case is identical. □

Now we will consider a polynomial vector field of the following form:

$$X(x) = x_1 a(x_1,x_2) P_0(x) \frac{\partial}{\partial x_1} + x_2[1 + b(x_1,x_2)P_0(x)] \frac{\partial}{\partial x_2} + R_0(x) \frac{\partial}{\partial x_3}$$

where $R_0(x) = r_0(x_1,x_2) + \int_0^{x_3} [1 + 2bP_0(\xi) + b(b-a)P_0^2(\xi)] d\xi_3$, with $\xi = (x_1,x_2,\xi_3)$ and a, b and $r_0 \in \mathbb{C}[x_1,x_2]$.

We will restrict our analysis to the case where a and b are nonzero constants and P_0 is a non-constant function. The situations P_0 constant and $b \equiv 0$ were discussed in theorem 4.2.

Theorem 4.5. — *Let X be a polynomial vector field in $\mathbb{C}^3$ given by*

$$X(x) = x_1 P_0(x) \frac{\partial}{\partial x_1} + x_2[1 + \lambda P_0(x)] \frac{\partial}{\partial x_2} + R_0(x) \frac{\partial}{\partial x_3}$$

where P_0 is a non constant function, $R_0 = r_0(x_1,x_2) + \int_0^{x_3} [1 + 2\lambda P_0(\xi) + \lambda(\lambda-1)P_0^2(\xi)] d\xi_3$, with $\xi = (x_1,x_2,\xi_3)$ and $\lambda \neq 0$. Generically, the opd induced by X is integrable if one of the following conditions is satisfied

1. *$\lambda = 1$, $P_0(x) = p_0(x_1,x_2) + p_1(x_1,x_2)\cdot x_3$ with $r_0 = p_0$ and $p_1 = p_0 + 1$ or $r_0 = p_0 + 1$ and $p_0 = p_1$;*
2. *$\lambda \neq 1$, $P_0 = P_0(x_3)$ and r_0 a constant function;*
3. *$\lambda = 1$, $P_0 = P_0(x_3)$ and r_0 are affine linear functions.*

Proof. — As before, the condition of integrability of ω_X is given by

$$\omega_{X_1} \wedge d\omega_{X_1} = [W_2 + W_3 + \cdots]dx_1 \wedge dx_2 \wedge dx_3 \equiv 0$$

where $W_2 \equiv 0$ by theorem 1.1 and

$$W_3 = x_1^2 x_2 P_0 R_0 \Big[- M\big(R_0\big)_{x_1 x_3} + M_{x_3}\big(R_0\big)_{x_1}\Big] \\ + x_1 x_2^2 (1 + \lambda P_0) R_0 \Big[- M\big(R_0\big)_{x_2 x_3} + M_{x_3}\big(R_0\big)_{x_2}\Big]$$

where $M = P_0[1 + \lambda P_0][1 + (\lambda - 1)P_0] - R_0 \cdot (P_0)_{x_3} = P_0 \cdot (R_0)_{x_3} - R_0 \cdot (P_0)_{x_3} - P_0^2$.

In order to obtain the integrability of ω_X we will impose the condition $W_3 \equiv 0$, hence

$$\begin{cases} -M\big(R_0\big)_{x_1 x_3} + M_{x_3}\big(R_0\big)_{x_1} & \equiv \quad 0 \\ -M\big(R_0\big)_{x_2 x_3} + M_{x_3}\big(R_0\big)_{x_2} & \equiv \quad 0 \end{cases} \tag{4.5}$$

Solving (4.5) for M and M_{x_3} we must consider the determinant

$$D = (R_0)_{x_2 x_3} \cdot (R_0)_{x_1} - (R_0)_{x_1 x_3} \cdot (R_0)_{x_2}. \tag{4.6}$$

Let $P_0(x) = \sum_{i=0}^{m} p_i(x_1, x_2) x_3^i$ where $p_i \in \mathbb{C}[x_1, x_2]$, $p_m \not\equiv 0$ and $m = \deg_{x_3}(P_0)$, the degree of P_0 in respect to x_3.

Case 1. $D \not\equiv 0$.

In this case, the homogeneous system (4.5) admits only the trivial solution $M \equiv 0$. But, given that $M = P_0 \cdot (R_0)_{x_3} - R_0 \cdot (P_0)_{x_3} - P_0^2$, we have $M \equiv 0$ if and only if

$$b(x_1, x_2) R_0(x) = P_0(x)[b(x_1, x_2) \cdot x_3 + a(x_1, x_2)] \tag{4.7}$$

for some polynomials a, b. We will consider the situations $\lambda = 1$ and $\lambda \neq 1$ separately. First, let $\lambda \neq 1$; then, since

$$R_0(x) = r_0(x_1, x_2) + \int_0^{x_3} [1 + 2\lambda P_0(\xi) + \lambda(\lambda - 1)P_0^2(\xi)] d\xi_3$$

we obtain $\deg_{x_3}(R_0) = 2m+1$ while from (4.7) $\deg_{x_3}(R_0) = m+1$. Comparing these degrees we get $m = 0$ and it follows that results $P_0 = P_0(x_1, x_2)$. In this situation, we have $M = P_0[1 + \lambda P_0][1 + (\lambda - 1)P_0] \equiv 0$ since P_0 is not a constant, we obtain a contradiction.

Let us now consider $\lambda = 1$. From (4.7) we get $R_0(x) = \sum_{i=0}^{m+1} r_i(x_1, x_2)x_3^i$ with $br_0 = bp_0 + a$ and $r_i = p_{i-1}$ for $i = 1, \dots, m+1$. Given that $R_0 = r_0 + \int_0^{x_3} [1 + 2P_0(\xi)]d\xi_3$ we obtain $r_1 = 1 + 2p_0$ and $r_i = \dfrac{2p_{i-1}}{i}$ for $i = 2, \dots, m+1$. Comparing these two expressions for R_0 we conclude $m = 1$ and $r_0 p_1 = p_0(p_0 + 1)$.

Therefore, if $r_0 = p_0$ then $p_1 = p_0 + 1$. In this situation, X assumes the form:

$$X = x_1 P_0(x)\frac{\partial}{\partial x_1} + x_2[1 + P_0(x)]\frac{\partial}{\partial x_2} + P_0(x) \cdot [1 + x_3]\frac{\partial}{\partial x_3}$$

which induces an integrable ω_X by proposition (4.1). By the same reason, ω_X is integrable if $r_0 = p_0 + 1$ and $p_1 = p_0$.

Furthermore, if $r_0 = \alpha(x_1, x_2)$ is a nontrivial factor of $p_0(x_1, x_2) = \alpha(x_1, x_2)\beta(x_1, x_2)$, then $p_1 = \beta(1+\alpha\beta)$. In this way, X assumes the following form:

$$X = x_1\beta[\alpha+(1+\alpha\beta)x_3]\frac{\partial}{\partial x_1}+x_2(1+\alpha\beta)[1+\beta x_3]\frac{\partial}{\partial x_2}+(1+\beta x_3)[\alpha+(1+\alpha\beta)x_3]\frac{\partial}{\partial x_3}.$$

Dividing X by $\beta[\alpha + (1 + \alpha\beta)x_3]$ we obtain X_1 given by

$$X_1 = x_1\frac{\partial}{\partial x_1} + x_2 Q(x)\frac{\partial}{\partial x_2} + [x_3 + k(x_1, x_2)]\frac{\partial}{\partial x_3}$$

where $k = \dfrac{1}{\beta}$, $Q = \dfrac{x_3 + k(x_1, x_2)}{x_3 + h(x_1, x_2)}$ and $h = \dfrac{\alpha}{1 + \alpha\beta}$. Generically, by proposition (4.4), ω_{X_1} is not integrable because k and h are not constant and $h \not\equiv k$. The same is true if $p_0 + 1 = \alpha\beta$ and $r_0 = \alpha$. This finishes the first case of the theorem.

Case 2. $D \equiv (R_0)_{x_1} \equiv (R_0)_{x_2} \equiv 0$.

Since

$$R_0(x) = r_0(x_1, x_2) + \int_0^{x_3} \left[1 + 2\lambda P_0(\xi) + \lambda(\lambda - 1)P_0^2(\xi)\right]d\xi_3,$$

we get

$$(R_0)_{x_i} = (r_0)_{x_i} + \int_0^{x_3} 2\lambda[1 + (\lambda - 1)P_0] \cdot (P_0)_{x_i} d\xi_3 \equiv 0$$

for $i = 1, 2$. For this to happen we must have $(r_0)_{x_i} = (P_0)_{x_i} = 0$ for $i = 1, 2$ and consequently r_0 is a constant and $P_0 = P_0(x_3)$. In this situation, the *opd* induced by X is described by the 1-form

$$\omega_X = x_1 x_2 M(x) \cdot R_0(x) \left[\lambda \frac{dx_1}{x_1} - \frac{dx_2}{x_2} + \frac{dx_3}{R_0} \right]$$

which is integrable since we also have $R_0 = R_0(x_3)$.

Case 3. $D \equiv (R_0)_{x_2} \equiv 0$ and $(R_0)_{x_1} \not\equiv 0$.

From now on, we will consider only $M \not\equiv 0$ since $M \equiv 0$ has been analyzed in the case 1. As explained in the previous case, we have $r_0 = r_0(x_1)$ and $P_0 = P_0(x_1, x_3) = \sum_{i=0}^{m} p_i(x_1) \cdot x_3^i$ since $(R_0)_{x_2} \equiv 0$.

From the first equation of (4.5) we obtain $b(x_1)M(x) = a(x_1)(R_0)_{x_1}$ for some polynomials $a, b \in \mathbb{C}[x_1]$. So again we must consider the situations $\lambda \neq 1$ and $\lambda = 1$.

We will begin with $\lambda \neq 1$. By hypothesis, $R_0(x) = r_0(x_1) + \int_0^{x_3} [1 + 2\lambda P_0(\xi) + \lambda(\lambda - 1)P_0^2(\xi)] d\xi_3$. It follows that $\deg_{x_3}(R_0) = 2m + 1$ and since $M = P_0(R_0)_{x_3} - R_0(P_0)_{x_3} - P_0^2$ we get $\deg_{x_3}(M) = 3m$. Given that $b(x_1)M(x) = a(x_1)(R_0)_{x_1}$ we have $m \leqslant 1$ because $\deg_{x_3}(R_0)_{x_1} \leqslant 2m + 1$.

If $m = 0$ then $M_{x_3} = P_0 \cdot (R_0)x_3x_3 \equiv 0$ and $(R_0)_{x_1x_3} = 2\lambda[1 + (\lambda - 1)P_0] \cdot (P_0)_{x_1}$. From the first equation of (4.5) we get $M \cdot (R_0)_{x_1x_3} \equiv 0$; it follows that $(R_0)_{x_1x_3} \equiv 0$ since $M \not\equiv 0$. Consequently $(P_0)_{x_1} \equiv 0$, i.e., P_0 is a constant. Therefore, we obtain $m = 1$.

Now, we will compare the coefficients of bM and $a(R_0)_{x_1}$ for $m = 1$:

$$\begin{cases} b\big[(1 + \lambda p_0) \cdot (1 - p_0 + \lambda p_0) \cdot p_0 - r_0 p_1\big] = a r_0' \\ 2b p_0 p_1 (\lambda - 1) \cdot (1 + \lambda p_0) = 2\lambda a[1 + (\lambda - 1)p_0] p_0' \\ b p_1^2 (\lambda - 1) \cdot (1 + 2\lambda p_0) = \lambda a[p_1' + (\lambda - 1) \cdot (p_0' p_1 + p_0 p_1')] \\ \lambda(\lambda - 1) b p_1^3 = \lambda(\lambda - 1) a p_1 p_1' \end{cases} \tag{4.8}$$

From $p_1 \neq 0$ and the third equality of (4.8) we obtain $bp_1^2 = ap_1'$ and $p_1' = \lambda(\lambda-1)[p_0p_1' - p_0'p_1]$. Multiplying the second equality of (4.8) by p_1 we get

$$-p_0p_1' + \lambda(p_0p_1' - p_0'p_1) + \lambda(\lambda-1)(p_0p_1' - p_0'p_1)p_0 = 0.$$

Combining these two equations we obtain $\lambda(p_0p_1' - p_0'p_1) = 0$. Therefore $p_1' = 0$, which gives $p_0' = 0$; it follows that $P_0 = P_0(x_3) = p_0 + p_1 \cdot x_3$. Furthermore, since p_1 is a constant then $b = 0$ and from (4.8) it follows that r_0 is also constant and so we are in the previous case.

Now, let us consider $\lambda = 1$. Given that $R_0(x) = r_0 + \displaystyle\int_0^{x_3} [1 + 2P_0(\xi)]d\xi_3$ and $M = P_0^2 + P_0 - R_0(P_0)_{x_3}$ then $\deg_{x_3}(M) = 2m$ if $m > 1$ and $\deg_{x_3}(M) = 0$, if $m \leqslant 1$. Since $b(x_1)M(x) = a(x_1)(R_0)_{x_1}$ we get $m \leqslant 1$. As in the situation where $\lambda \neq 1$, it is not possible to have $m = 0$ so that $m = 1$, which shows that p_0 and p_1 are both constant. Therefore, X assumes the following form

$$X(x) = x_1P_0(x_3)\frac{\partial}{\partial x_1} + x_2[1 + P_0(x_3)]\frac{\partial}{\partial x_2} + R_0(x)\frac{\partial}{\partial x_3}$$

where $P_0 = P_0(x_3)$ is an affine linear and $R_0(x) = r_0(x_1) + \displaystyle\int_0^{x_3} [1 + 2P_0(\xi)]d\xi$. The *opd* induced by X is described by the 1-form

$$\omega_X = [x_2R_0M + x_1x_2P_0(1+P_0)r_0']dx_1 - [x_1R_0M + x_1^2P_0^2r_0']dx_2 + x_1x_2M\,dx_3$$

where $M = P_0^2 + P_0 - R_0P_0'$. The integrability condition of ω_X is

$$\omega_X \wedge d\omega_X = -x_1^3x_2\big[M(x) \cdot P_0^2(x_3) \cdot r_0''(x_1)\big]dx_1 \wedge dx_2 \wedge dx_3 \equiv 0.$$

Consequently, in this situation ω_X is integrable if r_0 is also an affine linear function.

Case 4. $D \equiv (R_0)_{x_1} \equiv 0$ and $(R_0)_{x_2} \not\equiv 0$.

This case reduces to the previous one by a change of variables.

Case 5. $D \equiv 0$, $(R_0)_{x_1} \not\equiv 0$ and $(R_0)_{x_2} \not\equiv 0$.

Solving the first equation of (4.5) we obtain $f_2(x_1,x_2)M(x) = f_1(x_1,x_2)$ $(R_0)_{x_1}(x)$ for some polynomials $f_1, f_2 \in \mathbb{C}[x_1,x_2]$. Applying the same ideas presented in case 3 of this theorem, we can show that $\deg_{x_3}(P_0) = 1$ and $(p_0)_{x_1} = (p_1)_{x_1} = 0$, therefore $P_0 = P_0(x_2,x_3)$. Repeating this process again but now for the x_2-variable, we obtain $P_0 = P_0(x_3)$ with p_0 and p_1 constants.

As in the case 3, if $\lambda \neq 1$ then r_0 is a constant and consequently, ω_X is integrable because $P_0 = P_0(x_3)$ as proved. For $\lambda = 1$ we will determine the functions $r_0 = r_0(x_1, x_2)$ which ensure the integrability of ω_X. Let

$$X(x) = x_1 P_0(x_3)\frac{\partial}{\partial x_1} + x_2[1 + P_0(x_3)]\frac{\partial}{\partial x_2} + R_0(x)\frac{\partial}{\partial x_3}$$

where $P_0 = P_0(x_3)$ is an affine linear function and $R_0(x) = r_0(x_1, x_2) + \int_0^{x_3}[1 + 2P_0(\xi)]d\xi$. The condition of integrability of ω_X is written as

$$\omega_X \wedge d\omega_X = \\ -M\left[x_1^3 x_2 P_0^2 \cdot (r_0)_{x_1x_1} + 2x_1^2 x_2^2 \cdot P_0(1+P_0)\cdot(r_0)_{x_1x_2} + x_1 x_2^3 (1+P_0)^2 \cdot (r_0)_{x_2x_2}\right]dV$$

where $M = P_0^2 + P_0 - R_0 \cdot P_0'$. Writing $r_0 = \sum_{i+j=0}^{n} a_{i,j} x_1^i x_2^j$ we get

$$\omega_X \wedge d\omega_X = \\ -M\left[\sum_{i+j=2}^{n} a_{i,j} x_1^{i+1} x_2^{j+1}\left[i(i-1)P_0^2 + 2ijP_0(1+P_0) + j(j-1)(1+P_0)^2\right]\right]dV$$

For the existence of nontrivial solutions other than the linear one we must have

$$i(i-1)P_0^2 + 2ijP_0(1 + P_0) + j(j-1)(1 + P_0)^2 \equiv 0$$

for $i + j = l \geqslant 2$. The coefficient of x_3^2 in this last expression is

$$i(i-1)p_1^2 + 2ijp_1^2 + j(j-1)p_1^2$$

since $P_0(x_3) = p_0 + p_1 \cdot x_3$. Given that $p_1 \neq 0$, we have $i(i-1) + 2ij + j(j-1) = (i + j)[(i + j) - 1] = 0$ which is possible only if $(i + j) = 0$ or $(i + j) = 1$. Therefore, in this situation, ω_X is integrable only when r_0 is also an affine linear function.

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