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Pentagramma mirificum and elliptic functions (Napier, Gauss, Poncelet, Jacobi, ...)

Vadim Schechtman[(1)]

Abstract. — We give an exposition of unpublished fragments of Gauss where he discovered (using a work of Jacobi) a remarkable connection between Napier pentagons on the sphere and Poncelet pentagons on the plane. As a corollary we find a parametrization in elliptic functions of the classical dilogarithm five-term relation.

Résumé. — On présente des fragments non-publiés de Gauss où il a découvert (en utilisant des résultats de Jacobi) une connexion remarquable entre les pentagones de Napier sur le sphère et les pentagones de Poncelet sur le plan. Comme conséquence on trouve une paramétrisation de la relation à cinq termes du dilogarithme en termes de fonctions elliptiques.

FAUST. *Das Pentagramma macht dir Pein?*
Ei sage mir, du Sohn der Hölle,
Wenn das dich bannt, wie kamst du denn herein?

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Article proposé par Damien Rössler.

Introduction

In this note we give an exposition of forgotten fragments from Gauss [G] where he discovered, with the help of some work of Jacobi, a remarkable connection between Napier pentagons on the sphere and Poncelet pentagons on the plane.

This gives rise to a parametrization of the variety of Napier pentagons using the division by 5 of elliptic functions. As a corollary we will find the classical five-term relation for the dilogarithm in a somewhat exotic disguise, cf. 6.4.

In the last §7 we add some speculations on (probably) related topics.

In the Appendix written by Leopold Flatto one discusses in more details the elliptic parametrization of the variety of Napier pentagons.

The author gave talks on these subjects in Toulouse on January, in Moscow on May and in Bonn on July, 2011.

This note was inspired by a beautiful talk on cluster algebras given by S.Fomin in Caen several years ago. After the talk he told (answering a question) that these algebras have no relation to elliptic functions; the work of Gauss shows that they do, at least in the case of A_2.

Thanks are due to the referee for correcting an orthographic error.

1. Napier's rules

John Napier of Merchiston (1550 - 1617) was a Scottish mathematician and astrologer. His main work is *Mirifici Logarithmorum Canonis Descriptio* (1614). The reader will find in [C] some historical remarks on what follows.

1.1. For a spherical triangle with sides a, b, c and angles α, β, γ,

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$

Consider a right-angled spherical triangle with sides a, b, c and angles $\alpha, \beta, \gamma = \pi/2$. Let us call its *parts* five quantities

$$\tau = (a, b, \pi/2 - \alpha, \pi/2 - c, \pi/2 - \beta)$$

We consider them as positioned on a circle, i.e. ordered in the cyclic order. If we turn them we get the parts of another right-angled triangle:

$$\tau' = n(\tau) = (a', b', \ldots) = (b, \pi/2 - \alpha, \pi/2 - c, \pi/2 - \beta, a)$$

(*Napierian rotation*). Obviously $n^5 = 1$.

Repeat this once more:

$$g(\tau) := n^2(\tau) = (a'', b'', \ldots) = (\pi/2 - \alpha, \pi/2 - c, \pi/2 - \beta, a, b)$$

In other words,

$$(\alpha'', b'', c'') = (\beta, \pi/2 - c, \pi/2 - a),$$

this triangle is obtained by a reflection of the first one in the vertex β; call it *Gaussian reflection.*

1.2. Napier rules

I. sine (middle part) = product of tangents of adjacent parts:

$$\sin b = \tan a \cot \alpha$$

II. sine (middle part) = product of cosines of opposite parts:

$$\sin a = \sin \alpha \sin c$$

2. Pentagramma Gaussianum

2.1. Let us draw a spherical right-angled triangle which we denote $P_3Q_1P_4$ with the angles $\angle Q_1P_3P_4 = p_3, \angle P_3Q_1P_4 = \pi/2, \angle Q_1P_4P_3 = p_4$ and sides $P_3Q_1 = \pi/2 - p_5, Q_1P_4 = \pi/2 - p_2, P_3P_4 = p_1$.

Let us use an abbreviated notation $p' = \pi/2 - p$. So, the Napierian parts of this triangle are

$$\tau_1 = (p'_2, p'_5, p'_3, p'_1, p'_4)$$

Applying the Gaussian reflexion we get the triangles

$$\tau_i := g^{i-1}(\tau_1) = (p'_{i+1}, p'_{i+4}, p'_{i+2}, p'_{i+5}, p'_{i+3})$$

where we treat the indices modulo 5. We denote the i-th triangle $P_{i+2}Q_iP_{i+3}$, where $\angle P_{i+2}Q_iP_{i+3} = \pi/2$, $p_i = P_{i+2}P_{i+3}$.

So we get a spherical pentagon $P_1P_2P_3P_4P_5$; its characteristic property is $P_iP_{i+2} = \pi/2$, cf. the picture on p. 481 of [G].

Set

$$\alpha_i = \tan^2 p_i$$

Gauss' notation:

$$(\alpha, \beta, \gamma, \delta, \epsilon) = (\alpha_1, \ldots, \alpha_5)$$

Napier rules give:

$$\cos p_i = \sin p_{i+1} \cos p_{i-1}$$

$$1 = \cos p_i \tan p_{i+2} \tan p_{i+3}$$

It follows:

$$(\gamma\delta, \delta\epsilon, \epsilon\alpha, \alpha\beta, \beta\gamma) = (\sec^2 p_1, \sec^2 p_2, \sec^2 p_3, \sec^2 p_4, \sec^2 p_5)$$

Relations:

$$1 + \alpha = \gamma\delta,\ 1 + \beta = \delta\epsilon,\ \text{etc.} \tag{2.1}$$

Out of two quantities one can build up the remaining three, for example:

$$\beta = \frac{1+\alpha+\gamma}{\alpha\gamma},\ \delta = \frac{1+\alpha}{\gamma},\ \epsilon = \frac{1+\gamma}{\alpha},$$

etc. and also

$$\gamma = \frac{1+\alpha}{\alpha\beta - 1},\ \delta = \alpha\beta - 1,\ \epsilon = \frac{1+\beta}{\alpha\beta - 1},$$

etc. One has

$$3 + \alpha + \beta + \gamma + \delta + \epsilon = \alpha\beta\gamma\delta\epsilon = \sqrt{(1+\alpha)(1+\beta)(1+\gamma)(1+\delta)(1+\epsilon)}$$

2.2. Gauss' favorite example:

$$(\alpha, \beta, \gamma, \delta, \epsilon) = (9, 2/3, 2, 5, 1/3),\ \alpha\beta\gamma\delta\epsilon = 20$$

2.3. Regular pentagram. If $\alpha = \beta = \ldots$ then

$$\alpha + 1 = \alpha^2,$$

whence

$$\alpha = \frac{1+\sqrt{5}}{2}$$

$$\alpha^5 = \frac{11 + 5\sqrt{5}}{2} = 11.0901699...$$

Also $\alpha^2 = \sec^2 i$, so

$$\cos p_i = \alpha^{-1} = \frac{\alpha}{1+\alpha} = \frac{1+\sqrt{5}}{2}$$

Note that

$$\alpha^{-1} = \frac{\alpha}{1+\alpha} = \frac{-1+\sqrt{5}}{2} = 0.618033988...$$

We have

$$\sin 2\pi/5 = 2s \cdot c$$

where we have denoted $s = \sin \pi/5, c = \cos \pi/5$. On the other hand,

$$\sin 2\pi/5 = \sin 3\pi/5 = s(3 - 4s^2) = s(4c^2 - 1),$$

whence $2c = 4c^2 - 1$, so

$$\cos \pi/5 = \frac{1 + \sqrt{5}}{4} = \frac{\alpha}{2}$$

It follows that

$$2 \cos p_i \cdot \cos \pi/5 = 1$$

We set

$$c' := \cos 2\pi/5 = \frac{-1 + \sqrt{5}}{4}$$

Then

$$cc' = 1/4,\ \ c - c' = 1/2$$

A cone

2.5. Let M be the centrum of the sphere. The vertices P_i all lie on a quadratic cone whith vertex M. Namely, we have

$$\angle P_{k-1} M P_{k+1} = \pi/2$$

So MP_3 is orthogonal to MP_1 and to MP_5.

We define the coordinates in such a way that M lies ar the origin,

$$P_3 = (1, 0, 0),\ \ P_1 = (0, 1, 0)$$

Then

$$P_5 = (0, \cos p_3, \sin p_3)$$

and

$$P_4 = (\cos p_1, 0, \sin p_1)$$

Finally, the ray $MP_2 \perp P_4 M P_5$, so

$$P_2 = (\cos p_5, \cos p_4, -\cos p_3 \sin p_5)$$

The coordinates of P_k satisfy the equation

$$\cos p_2 \cdot (\cos p_1 \cdot z - \sin p_1 \cdot x)(\cos p_3 \cdot z - \sin p_3 \cdot y) = xy,$$

or

$$z^2 - \sqrt{\alpha}xz - \sqrt{\gamma}yz - \frac{1+\alpha+\gamma}{\sqrt{\alpha\gamma}}xy :=$$

$$= z^2 + pxz + qyz + rxy = 0 \qquad (C)$$

Note that

$$r = -\beta\sqrt{\alpha\gamma}$$

Reduction to principal axes.

2.6. In general, given a quadratic form defined by a symmetric matrix A, $v = (x, y, z)^* \mapsto v^t Av$, if we want to find an orthogonal matrix B such that for $v = Bv'$, $v' = (x', y', z')^*$,

$$v^* Av = v'^* B^* ABv' = v'^* A' v'$$

such that $A' = \mathrm{diag}(G, G', G'')$ then $A' = B^* AB = B^{-1}AB$ and the numbers G, G', G'' are the eigenvalues of A.

The columns of B are eigenvectors of A.

2.7. In our case, for the quadrics (C), the matrix is

$$A = \begin{pmatrix} 0 & r/2 & p/2 \\ r/2 & 0 & q/2 \\ p/2 & q/2 & 1 \end{pmatrix}$$

The characteristic polynomial is

$$\det(t \cdot I - A) = t^3 - t^2 - \frac{p^2+q^2+r^2}{4} - \frac{r(pq-r)}{4},$$

so the characteristic equation takes the form

$$t(2t-1)^2 = \omega(t-1) \qquad (E)$$

where

$$\omega := \alpha\beta\gamma\delta\epsilon$$

This is our main equation.

2.8. Let us investigate the real roots of (E). We suppose that $\omega > 0$. The straight line

$$\ell :\ u = \omega(t-1) \qquad (2.2)$$

allways intersects the cubic parabola

$$P :\ u = t(2t-1)^2 \qquad (2.3)$$

at one negative point $t = G < 0$.

On the other hand, ℓ intersects P at two points $t = G', G'' > 1$ iff ω is greater than some critical value ω_0; if $\omega = \omega_0$ then ℓ touches P at $t = G' = G''$, if $\omega < \omega_0$ then there are no points of intersection of ℓ with P other than $t = G$.

The critical value is $\omega_0 = \alpha_0^5$ where

$$\alpha_0 = \frac{1+\sqrt{5}}{2},$$

cf. 2.3. In that case

$$G = -\alpha_0,\ G' = G'' = \alpha_0^2/2 = -c_0 G,$$

$$c_0 = \cos \pi/5$$

Thus, if $\omega \geqslant \alpha_0^5$ then (E) has one negative root, G, and two positive roots, G', G'' which coincide if $\omega = \alpha_0^5$.

One has:

$$GG'G'' = -\omega/4$$

$$(G-1)(G'-1)(G''-1) = -1/4$$

$$(2G-1)(2G'-1)(2G''-1) = -\omega$$

2.9. *Example.*— For $\alpha\beta\gamma\delta\epsilon = 20$, $G = -2.197, G' = 1.069,\ G'' = 2.128$.

3. Gauss' coordinates

3.1. Consider an ellipse

$$E : \frac{x^2}{a^2} + \frac{x^2}{b^2} = 1,$$

$a > b$. Inscribe E into the circle C with the centrum O and radius a. For a point $P = (x, y) \in E$, let $P' = (x, y') \in C$. The *eccentric anomaly (anomalie excentrique)* of P is the angle $\phi = \angle XOP'$ where $X = (a, 0)$. Then

$$x = a\cos\phi,\ y = b\sin\phi$$

3.2. Return to the Napier pentagon $P_1 \ldots P_5$. Let us draw a plane tangent to the sphere at the point of intersection with the axe of the cone and take the central projection of our pentagon on this plane. Let R_i be the projection of P_i to this plane. The points $R_i,\ i = 1, \ldots, 5$, lie on an ellipse with semi-axes $\sqrt{-G/G'}$ and $\sqrt{-G/G''}$.

We put the coordinate axes x, y of the plane along the axes of the ellipse. Let R_i have the coordinates (x_i, y_i). Let ϕ_i be the eccentric anomaly of R_i. We have

$$x_i = g' \cos\phi_i,\ \ y_i = g'' \sin\phi_i \tag{3.1}$$

where

$$g' = \sqrt{-G/G'},\ \ g'' = \sqrt{-G/G''}$$

Let M be the centrum of the sphere, with coordinates $(x, y, z) = (0, 0, 1)$.

Attention: these coordinates differ from 2.5.

Set $\psi_i = \angle P_i M P_{i+1} = \angle R_i M R_{i+1}$; $\mathbf{r}_i = (x_i, y_i, 1)$.

Then

$$\cos a_i = \frac{(\mathbf{r}_i, \mathbf{r}_{i+1})}{|\mathbf{r}_i||\mathbf{r}_{i+1}|} = \frac{x_i x_{i+1} + y_i y_{i+1} + 1}{\sqrt{(x_i^2 + y_i^2 + 1)(x_{i+1}^2 + y_{i+1}^2 + 1)}}$$

and

$$\alpha_i := \tan^2\psi_i = \sec^2\psi_i - 1 =$$

$$= \frac{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 + (x_i y_{i+1} - y_i x_{i+1})^2}{(x_i x_{i+1} + y_i y_{i+1} + 1)^2} = \frac{|\mathbf{r}_i \times \mathbf{r}_{i+1}|^2}{(\mathbf{r}_i, \mathbf{r}_{i+1})^2} \tag{3.2}$$

For the future use, note the quantities

$$\beta_i := \sin^2\psi_i = \frac{\alpha_i}{\alpha_i + 1} = \frac{|\mathbf{r}_i \times \mathbf{r}_{i+1}|^2}{|\mathbf{r}_i|^2 |\mathbf{r}_{i+1}|^2} \tag{3.3}$$

3.3. We have

$$\angle R_{i-1} M R_{i+1} = \pi/2$$

whence

$$x_{i-1} x_{i+1} + y_{i-1} y_{i+1} + 1 = 0 \tag*{$(3.4)_i$}$$

Solving $(3.4)_{i-1}$ and $(3.4)_{i+1}$ for x_i, y_i, we get

$$x_i = \frac{y_{i+2} - y_{i-2}}{x_{i+2} y_{i-2} - y_{i+2} x_{i-2}},\ \ y_i = \frac{x_{i-2} - x_{i+2}}{x_{i+2} y_{i-2} - y_{i+2} x_{i-2}} \tag{3.5}$$

3.4. Next,

$$\frac{x_i x_{i+1}}{2G' - 1} + \frac{y_i y_{i+1}}{2G'' - 1} + \frac{1}{2G - 1} = 0 \tag{3.6}$$

This will be proven in 3.6. It follows:

$$x_i = -\frac{2G' - 1}{2G - 1} \cdot \frac{y_{i+1} - y_{i-1}}{x_{i-1} y_{i+1} - x_{i+1} y_{i-1}},$$

$$y_i = \frac{2G''-1}{2G-1} \cdot \frac{x_{i+1}-x_{i-1}}{x_{i-1}y_{i+1}-x_{i+1}y_{i-1}} \tag{3.7}$$

3.5. Theorem (Gauss, April 20, 1843). — (a)

$$\frac{\sin((\phi_{i-2}+\phi_{i+2})/2)}{\cos((\phi_{i-2}-\phi_{i+2})/2)} = \frac{G}{G''}\sin\phi_i; \ \frac{\cos((\phi_{i-2}+\phi_{i+2})/2)}{\cos((\phi_{i-2}-\phi_{i+2})/2)} = \frac{G}{G'}\cos\phi_i \tag{3.8}$$

(b)

$$\frac{\sin((\phi_{i-1}+\phi_{i+1})/2)}{\cos((\phi_{i-1}-\phi_{i+1})/2)} = \sqrt{\frac{G(G-1)}{G''(G''-1)}}\sin\phi_i = \frac{G(2G-1)}{G''(2G''-1)}\sin\phi_i$$

$$\frac{\cos((\phi_{i-1}+\phi_{i+1})/2)}{\cos((\phi_{i-1}-\phi_{i+1})/2)} = \sqrt{\frac{G(G-1)}{G'(G'-1)}}\cos\phi_i = \frac{G(2G-1)}{G'(2G'-1)}\cos\phi_i \tag{3.9}$$

3.6. *Proof*, cf. [F]. (a) follows from (3.5) and (3.1), taking into account the formulas

$$\sin a - \sin b = 2\cos((a+b)/2)\sin((a-b)/2$$
$$\cos a - \cos b = -2\sin((a+b)/2)\sin((a-b)/2 \tag{3.10}$$

Proof of (3.6). —

It follows from (3.8) (replacing $i+2$ by i) that

$$G'^2\cos^2((\phi_i+\phi_{i+1})/2) + G''^2\sin^2((\phi_i+\phi_{i+1})/2) = G^2\cos^2((\phi_i-\phi_{i+1})/2).$$

Using

$$\cos^2 a = \frac{1+\cos 2a}{2}, \ \sin^2 a = \frac{1-\cos 2a}{2}$$

we get

$$G^2\cos(\phi_i-\phi_{i+1}) + (G''^2-G'^2)\cos(\phi_i+\phi_{i+1}) = G'^2+G''^2-G^2$$

From (3.1) we get

$$(G^2-G'^2+G''^2)G'x_ix_{i+1} + (G^2+G'^2-G''^2)G''y_iy_{i+1} +$$
$$+(-G^2+G'^2+G''^2)G = 0 \tag{3.11}$$

The numbers G, G', G'' are the roots of

$$t(2t-1)^2 = \omega(t-1) \tag{3.12}$$

It follows that

$$G^2 + G'^2 + G''^2 = \frac{1+\omega}{2}$$

Whence

$$(-G^2 + G'^2 + G''^2)G = \left(\frac{1+\omega}{2} - 2G^2\right)G = \frac{\omega}{2} \cdot \frac{1}{2G-1}$$

Similarly we compute the coefficients at $x_i x_{i+1}$ and at $y_i y_{i+1}$ of (3.11) and arrive at (3.6).

Proof of (b). — The second equalities follow from (3.11). The first ones follow from (3.7) in the same manner as one has deduced (a) from (3.1). □

4. Poncelet's problem and division of elliptic functions (Jacobi)

In this Section we describe the work of Jacobi [J2]. For a modern treatment of it see [GH] and references therein; see also the nice discussions in [BKOR] and [Fl].

4.1. One considers a circle with center C of radius R, inside it a smaller circle with center c of radius r, a will be the distance between their centra. The line cC intersects the bigger circle at a point P, so $|CP| = R$, $|cP| = R + a$.

One takes a point A_1 on the bigger circle, draws the tangent to the smaller one till the intersection with the bigger one at the point A_2, and continues similarly. Denote

$$2\phi_i = \angle A_i CP$$

Let us suppose for simplicity that C is inside the smaller circle. Let B be the point where A_1A_2 touches the smaller circle, $B' \in A_1A_2$ the base of the perpendicular from C, and $D \in cB$ the base ofthe perpendicular from C. Then

$$DB = CB' = R\cos(\phi_2 - \phi_1)$$

and

$$cD = a\cos(\phi_2 + \phi_1),$$

so

$$r = cD + DB = R\cos(\phi_2 - \phi_1) + a\cos(\phi_2 + \phi_1)$$

It follows that

$$R\cos(\phi_{i+1} - \phi_i) + a\cos(\phi_{i+1} + \phi_i) = r$$

or

$$(R+a)\cos\phi_{i+1}\cos\phi_i + (R-a)\sin\phi_{i+1}\sin\phi_i = r \qquad (4.1)_i$$

Substracting $(4.1)_i - (4.1)_{i-1}$ and using

$$\frac{\cos x - \cos y}{\sin y - \sin x} = \tan\left(\frac{x+y}{2}\right),$$

we get

$$\tan((\phi_{i+1}+\phi_{i-1})/2)) = \frac{R-a}{R+a}\tan\phi_i \qquad (4.2)$$

4.2. Jacobi elliptic functions. Cf. [J1]. Fix a number $0 \leqslant k < 1$ and define the period

$$K = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} \qquad (4.3)$$

The function "amplitude"

$$\mathrm{am}:\ [0,K] \longrightarrow [0,\pi/2]$$

is defined by $\phi = \mathrm{am}(u)$,

$$u = \int_0^{\phi} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \int_0^{\sin\phi} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$$

Thus

$$\mathrm{am}(K) = \pi/2$$

We extend am to a function $\mathbb{R} \longrightarrow \mathbb{R}$ by $\mathrm{am}(-u) = -\mathrm{am}(u)$, $\mathrm{am}(u+2K) = \mathrm{am}(u) + \pi$.

We set

$$\Delta\mathrm{am}u := \frac{d\mathrm{am}u}{du} = \sqrt{1-k^2\sin^2\mathrm{am}u}$$

One uses the notations:

$$\mathrm{sn}u = \sin\mathrm{am}u,\ \mathrm{cn}u = \cos\mathrm{am}u,\ \mathrm{dn}u = \Delta\mathrm{am}u$$

Thus sn is an odd function and cn and dn are even.

4.3. Addition formulas. The basic formulas are:

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}u\mathrm{cn}v\mathrm{dn}v + \mathrm{cn}u\mathrm{sn}v\mathrm{dn}u}{1-k^2\mathrm{sn}^2u\mathrm{sn}^2v}$$

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}u\mathrm{cn}v - \mathrm{sn}u\mathrm{sn}v\mathrm{dn}u\mathrm{dn}v}{1-k^2\mathrm{sn}^2u\mathrm{sn}^2v}$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}u\mathrm{dn}v - k^2\mathrm{sn}u\mathrm{sn}v\mathrm{cn}u\mathrm{cn}v}{1-k^2\mathrm{sn}^2u\mathrm{sn}^2v}, \tag{4.4}$$

cf. [J1], §18.

The following formulas are consequences of (4.4).

Main formula:

$$\mathrm{cn}(u-v) = \mathrm{cn}u\mathrm{cn}v + \mathrm{sn}u\mathrm{sn}v\mathrm{dn}(u-v) \tag{4.5}$$

Next,

$$\mathrm{sn}(u+v) - \mathrm{sn}(u-v) = \frac{2\mathrm{cn}u\mathrm{sn}v\mathrm{dn}u}{1-k^2\mathrm{sn}^2u\mathrm{sn}^2v}$$

$$\mathrm{cn}(u+v) - \mathrm{cn}(u-v) = -\frac{2\mathrm{sn}u\mathrm{sn}v\mathrm{dn}u\mathrm{dn}v}{1-k^2\mathrm{sn}^2u\mathrm{sn}^2v},$$

whence

$$\tan((\mathrm{am}(x)+\mathrm{am}(y))/2) = \frac{\mathrm{cn}x - \mathrm{cn}y}{\mathrm{sn}y - \mathrm{sn}x} =$$

$$\mathrm{dn}((x-y)/2)\tan\mathrm{am}((x+y)/2) \tag{4.6}$$

4.4. If $\phi_n = \mathrm{am}(\phi + nt)$ then

$$\tan((\phi_0+\phi_2)/2) = \Delta\mathrm{am}t\tan\phi_1$$

So if

$$\Delta\mathrm{am}t = \frac{R-a}{R+a}$$

then (4.2) is satisfied.

We also have

$$\cos\phi_i\cos\phi_{i+1} + \sin\phi_i\sin\phi_{i+1}\cdot\sqrt{1-k^2\sin^2\alpha} = \cos\alpha$$

where $\alpha = \mathrm{am}t$.

So we can find k and α from the equations

$$\sqrt{1-k^2\sin^2\alpha} = \frac{R-a}{R+a}$$

and

$$\cos\alpha = \frac{r}{R+a}$$

wherefrom

$$k^2 = \frac{4Ra}{(R+a)^2 - r^2} = 1 - \frac{(R-a)^2 - r^2}{(R+a)^2 - r^2} \tag{4.5}$$

It follows:

4.5. THEOREM. — *If the process closes up after n steps and m full turns then, defining k through* (4.5), *we have*

$$\int_0^{\alpha} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{m}{n} \int_0^{\pi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

5. Back to pentagramma

5.1. Let us return to (3.4). Denoting $\phi = \phi_i$, $\phi' = \phi_{i+2}$ and using (3.1) we get

$$g'^2 \cos\phi \cos\phi' + g''^2 \sin\phi \sin\phi' = 1$$

or

$$\cos\phi \cos\phi' + \frac{G'}{G''} \sin\phi \sin\phi' = -\frac{G'}{G} \tag{5.1}$$

Now compare this with the Main formula (4.5). We see that (5.1) will be satisfied if $\phi = \mathrm{am}(u), \phi' = \mathrm{am}(u+w)$, $\alpha = \mathrm{am}(w)$,

$$\mathrm{dn} w = \sqrt{1 - k^2 \sin^2 \alpha} = \frac{G'}{G''} \tag{5.2}$$

and

$$\cos\alpha = -\frac{G'}{G}.$$

It follows that

$$\mathrm{cn} w = -\frac{G'}{G} \tag{5.3}$$

and

$$k = \sqrt{\frac{G'^{-2} - G''^{-2}}{G'^{-2} - G^{-2}}} \tag{5.4}$$

We can go backwards.

5.2. THEOREM. — *For $0 \leqslant k < 1$, let K the corresponding complete elliptic integral* (4.3). *Define vectors in $\mathbb{R})^3$:*

$$\mathbf{r}_j(k,u) = \left(\frac{\mathrm{cn}(u + 4jK/5)}{\sqrt{\mathrm{cn}(2K/5)}}, \frac{\sqrt{\mathrm{dn}(2K/5)}\mathrm{sn}(u + 4jK/5)}{\sqrt{\mathrm{cn}(2K/5)}}, 1 \right),$$

$u \in \mathbb{R}, \jmath \in \mathbb{Z}$. Then $\mathbf{r}_j(k,u) = \mathbf{r}_{j+5}(k,u)$. Set

$$\alpha_j(k,u) = \frac{|\mathbf{r}_j(k,u) \times \mathbf{r}_{j+1}(k,u)|^2}{(\mathbf{r}_j(k,u), \mathbf{r}_{j+1}(k,u))^2}$$

Then we have

$$1 + \alpha_j(k,u) = \alpha_{j-2}(k,u)\alpha_{j+2}(k,u) \tag{5.5}$$

In the degenerate case $k = 0$ we have

$$|\mathbf{r}_j(0,u)|^2 = \sqrt{5},\ \alpha_j(0,u) = \frac{1+\sqrt{5}}{2}$$

for all j, u. The ends of vectors $\mathbf{r}_j(0,u)$ form a regular plane pentagon, cf. 2.3.

6. Dilogarithm five-term relation

6.1. *Euler dilogarithm*:

$$Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t} dt,$$

$0 \leqslant x \leqslant 1$.

Rogers dilogarithm:

$$L(x) = Li_2(x) + \frac{1}{2}\log x \log(1-x) = -\frac{1}{2}\int_0^x \left(\frac{\log(1-t)}{t} - \frac{\log t}{1-t}\right) dt,$$

$0 < x < 1$.

6.2. Theorem (W.Spence, 1809).

$$L(x) + L(1-x) = \frac{\pi^2}{6} \tag{6.1}$$

$$L(x)+L(y)-L(xy)-L(x(1-y)/(1-xy))-L(y(1-x))/(1-xy)) = 0, \tag{6.2}$$

$0 < x, y < 1$.

6.3. One can reformulate (6.2) as follows (cf. [GT]):

$$L(x) + L(1-xy) + L(y) + L((1-y)/(1-xy)) + L((1-x)/(1-xy)) = \frac{\pi^2}{2}$$

Define

$$(b_1, \ldots, b_5) = (x, 1-xy, y, (1-y)/(1-xy), (1-x)/(1-xy))$$

and for an arbitrary $n \in \mathbb{Z}$ define b_n by periodicity $b_n = b_{n+5}$.

Then

$$b_{n-1}b_{n+1} = 1 - b_n,\ \ n \in \mathbb{Z}.$$

Define a new sequence a_n by

$$a_n = \frac{b_n}{1-b_n},\ \ b_n = \frac{a_n}{1+a_n}$$

Then

$$a_{n-2}a_{n+2} = 1 + a_n$$

Combining this with 5.2 and (3.3) we get

6.4. Corollary. *Let* $\mathbf{r}_j(k,u)$ *be as in* 5.2. *Set*

$$\beta_j(k,u) = \frac{|\mathbf{r}_j(k,u) \times \mathbf{r}_{j+1}(k,u)|^2}{|\mathbf{r}_j(k,u)|^2|\mathbf{r}_{j+1}(k,u)|^2}$$

Then

$$\sum_{j=1}^{5} L(\beta_j(k,u)) = \frac{\pi^2}{2} \tag{6.2}$$

In the regular case $k=0$ (6.4.1) takes the form

$$L(\alpha^{-1}) = \frac{\pi^2}{10} \tag{6.3}$$

(Landen), cf. [K]. Here as usually $\alpha = (1+\sqrt{5})/2$.

7. Further remarks

7.1. Onsager – Baxter substitution. The idea of uniformising relations in the spherical (resp. hyperbolic) geometry by elliptic functions (cf. [S] for a nice review) plays an important role in Statistical Physics. Onsager uses it in his 1944 paper [O] on the solution of the Ising model. Baxter remarks that the same idea underlies the solution of his star-triangle relations, cf. [B], 7.13.

Baxter cites a book [Gr] in this connection; apparently this is the very same "Greenhill's very odd and individual *Elliptic functions*" which, according to Littlewood, was the Ramanujan's textbook, cf. [L], [H], Ch. XII.

Consider the $N \times N$ square lattice; at each site i there is a spine $\sigma_i = \pm 1$. The Ising partition function is

$$Z_N(K, L) = \sum_\sigma e^{K \sum_{i,j} \sigma_i \sigma_j + L \sum_{i,k} \sigma_i \sigma_k}$$

Here under the exponential the first sum is over all horizontal edges and the second one is over all vertical edges.

Consider the *free energy per site*

$$\psi(K, L) = - \lim_{N \to \infty} \frac{\log Z_N(K, L)}{N},$$

cf. [B], (9.2.10).

Cramers - Wannier duality

Define the dual pair $(K^\vee, L^\vee)$ by

$$\tanh K^\vee = e^{-2L}, \ \tanh L^\vee = e^{-2K} \tag{7.1a}$$

or equivalently

$$\sinh 2K^\vee \sinh 2L = 1, \ \sinh 2L^\vee \sinh 2K = 1 \tag{7.1b}$$

Then

$$\psi(K^\vee, L^\vee) = \psi(K, L) + \frac{1}{2} \log(\sinh 2K \sinh 2L)$$

Critical point: $(K, L) = (L^\vee, K^\vee)$.

Elliptic parameters:

$$k = (\sinh 2K \sinh 2L)^{-1} \tag{7.2}$$

$$\sinh 2K = -i\mathrm{sn}(iu), \ \sinh 2L = i(k\mathrm{sn}(iu))^{-1} \tag{7.3a}$$

or equivalently

$$e^{2K} = \mathrm{cn}iu - i\mathrm{sn}iu,$$

$$e^{2L} = \frac{i\mathrm{dn}iu + 1}{k\mathrm{sn}iu} \tag{7.3b}$$

In these coordinates, the duality transformation looks like

$$k^\vee = 1/k$$

$$\mathrm{sn}iu^\vee = k\mathrm{sn}iu. \tag{7.4}$$

The critical point is $k = 1$ where the elliptic curve degenerates to $\mathbb{C}^*$.

This Onsager – Baxter elliptic curve resembles the Seiberg - Witten curve from gauge theory (which in turn is a particular case of a Hitchin fibration).

7.2. Kirillov – Reshetikhin identities. The following identites also came from the physicists:

$$\sum_{j=1}^{n} L\left(\frac{\sin^2\left(\frac{\pi}{n+3}\right)}{\sin^2\left(\frac{(j+1)\pi}{n+3}\right)}\right) = \frac{2n}{n+3} \cdot \frac{\pi^2}{6}, \tag{7.5}$$

cf. [KR]. Using (6.1) we deduce

$$\sum_{j=1}^{n} L\left(1 - \frac{\sin^2\left(\frac{\pi}{n+3}\right)}{\sin^2\left(\frac{(j+1)\pi}{n+3}\right)}\right) = \frac{n(n+1)}{n+3} \cdot \frac{\pi^2}{6} \tag{7.6}$$

For $n = 2$ this becomes

$$2L\left(1 - \frac{\sin^2(\pi/5)}{\sin^2(2\pi/5)}\right) = \frac{\pi^2}{5} \tag{7.7}$$

which is equivalent to (6.3). One can consider (6.2) as an elliptic deformation of (6.3). So one may wonder if for an arbitrary n the identity (7.5) allows a deformation involving the division by $n + 3$ of elliptic functions.

7.3. McShane's identity. Let us consider, with [Bo], a binary tree Σ dual to the tesselation Ω. Suppose that to each component $X \in \Omega$ a number $x = \phi(X)$ ("colour") is assigned in such a way that

— if a vertex $v \in V(\Sigma)$ is the intersection of $X, Y, Z \in \Omega$ then

$$x^2 + y^2 + z^2 = xyz \tag{7.8}$$

(Markov triple condition);

— if an edge $e \in E(\Sigma)$ meets $X, Y, Z, W \in \Omega$ in such a way that $e = X \cap Y$ and $e \cap Z$ and $e \cap W$ are the ends of e then

$$xy = z + w \tag{7.9}$$

Note that if (7.8) is fulfilled for one vertex and (7.9) for all edges then (7.8) is true for all vertices (Markov). If 3 colours satisfying (7.8) are chosen near one vertex then the condition (7.9) defines uniquely the full colouring of Ω.

Consider a function

$$h(t) = \frac{1 - \sqrt{1 - 4/t^2}}{2}$$

(this function is closely connected with the generating function of Catalan numbers).

A remarkable statement below resembles a two-dimensional analog of 7.2.

7.4. Theorem (G.McShane). Suppose that $\phi(X) > 2$ for all $X \in \Omega$. Then

$$\sum_{X \in \Omega} h(\phi(X)) = \frac{1}{2}.$$

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Appendix

Parametrization of napier pentagons

Leopold Flatto

Theorem 5.2 gives a parametrization of the variety of Napier pentagons. The object of this appendix is to obtain Theorem 2, which is a refinement of Theorem 5.2. We show that when the parameters in question are properly restricted, then the parametrization of Theorem 5.2 gives a one-to-one map from parameter space to the variety of Napier pentagons.

We retain the notation and use the results of the paper. Thus, $R_1, ..., R_5$ are the vertices of the planar pentagon obtained from the spherical Napier pentagon with vertices $P_1, ..., P_5$ via central projection. The vertices, $R_1, ..., R_5$, lie on an ellipse with semi-axes whose lengths are $\sqrt{-\frac{G}{G'}}$ and $\sqrt{-\frac{G}{G''}}$. In terms of the parameter k, $0 \leqslant k < 1$, these lengths are given by,

$$(*) \qquad a(k) = \frac{1}{\sqrt{\operatorname{cn}(\frac{2}{5}K(k), k)}} \qquad b(k) = \sqrt{\operatorname{dn}(\tfrac{2}{5}K(k), k)} \cdot a(k)$$

Let $a(1) = \lim_{k\to 1} a(k)$ and $b(1) = \lim_{k\to 1} b(k)$.

We first prove,

Theorem 1. — *$a(k)$ is a strictly increasing continuous function of k and $b(k)$ is a strictly decreasing continuous function of k, with $a(0) = b(0) = \sqrt{\sqrt{5}-1}$; $a(1) = \infty$ and $b(1) = 1$.*

Thus, distinct values of k give rise to distinct ellipses. As any two of these distinct ellipses intersect in four points, we obtain from Theorem 1,

Theorem 2. — *Let $r_j(k, u)$ be defined as in Theorem 5.2, and let $||r_j||$ be the length of r_j. The Napier pentagons with vertices $P_1, ..., P_5$ are parametrized in a one-to-one manner by,*

$$P_j = \frac{r_j(k, u)}{||r_j(k, u)||}, \ 1 \leqslant j \leqslant 5,$$

where $0 \leqslant k < 1$ and $0 \leqslant u < \frac{4}{5}K(k)$.

The monotonicity of $a(k)$ and $b(k)$ seems difficult to derive from formulas in (*), as the formulas for the derivatives $a'(k)$ and $b'(k)$ are somewhat cumbersome. Hence, instead of working with the parameter k, we use the parameter ω introduced in Section 2.7 of the paper.

The proof of Theorem 1 proceeds as follows:

1) The range of ω is shown to be $\omega_0 \leqslant \omega < \infty$, where $\omega_0 = (\frac{1+\sqrt{5}}{2})^5$.

2) We prove several lemmas establishing the monotonicity of the eigenvalues $G(\omega), G'(\omega), G''(\omega)$ of the 3×3 matrix A defined in Section 2.7, and also the monotonicity of several functions of G, G', G''.

3) We give the proof of Theorem 1, which follows readily from the monotonicity lemmas.

1) *The range of* ω.

Let $\alpha_i = \tan^2 p_i$, $1 \leqslant i \leqslant 5$, where $0 < p_i < \frac{\pi}{2}$. Define α_i for all integers i by letting $\alpha_{i+5} = \alpha_i$. Then $\omega = \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \cdot \alpha_4 \cdot \alpha_5$. We use the following properties of the α_i's which are just a restatement of the Napier rules given in Section 2.1 of the paper.

1a) $0 < \alpha_i < \infty$;

1b) $1 + \alpha_i = \alpha_{i-2}\alpha_{i+2}$.

Lemma 1. — 1a) *and* 1b) *are equivalent to*,

2a) $0 < \alpha_1, \alpha_2 < \infty$, $\alpha_1\alpha_2 > 1$;

2b) $\alpha_3 = \dfrac{\alpha_1 + 1}{\alpha_1\alpha_2 - 1}$, $\alpha_4 = \alpha_1\alpha_2 - 1$, $\alpha_5 = \dfrac{\alpha_2 + 1}{\alpha_1\alpha_2 - 1}$.

Proof. — Suppose 1a) holds. Employing 1b) successively for $i = 4, 1, 2$, we obtain $\alpha_1\alpha_2 > 1$ and formulas 2b). Conversely, we readily check that 2) implies 1). $\square$

Theorem 3. — *The value* $\omega_0 = (\frac{1+\sqrt{5}}{2})^5 = \frac{11+5\sqrt{5}}{2}$ *is attained only when* $\alpha_1, ..., \alpha_5 = \frac{1+\sqrt{5}}{2}$.

Furthermore, ω *assumes all values* $\geqslant \omega_0$.

Proof. — By Lemma 1, we may think of ω as a function of α_1, α_2. Thus,

$$\omega(\alpha_1, \alpha_2) = \frac{\alpha_1\alpha_2(\alpha_1 + 1)(\alpha_2 + 1)}{\alpha_1\alpha_2 - 1},$$

where $0 < \alpha_1, \alpha_2 < \infty$ and $\alpha_1\alpha_2 > 1$.

Let $\theta = \sqrt{\alpha_1 \alpha_2}$. We now think of ω as a function of θ and α_1, so

$$\omega(\theta, \alpha_1) = \frac{\theta^2}{\theta^2 - 1}\left(\theta^2 + 1 + \alpha_1 + \frac{\theta^2}{\alpha_1}\right)$$

where $\alpha_1 > 0, \theta > 1$. For a given $\theta > 1$, the minimum value of $\omega(\theta, \alpha_1)$ is attained uniquely at $\alpha_1 = \theta$. We have,

$$\omega(\theta, \theta) = \frac{\theta^3 + \theta^2}{\theta - 1} \qquad \frac{d}{d\theta}\omega(\theta, \theta) = \frac{2\theta(\theta^2 - \theta - 1)}{(\theta - 1)^2} \text{ for } \theta > 1$$

From the formula for $\frac{d}{d\theta}\omega(\theta, \theta)$, we get that $\omega(\theta, \theta)$ attains a minimum value uniquely at $\theta_0 = \frac{1+\sqrt{5}}{2}$. Hence $\omega(\alpha_1, \alpha_2)$ attains a minimum value uniquely when $\alpha_1, \alpha_2 = \theta_0$. From formulas 2b), we get that when $\alpha_1, \alpha_2 = \theta_0$, so do $\alpha_3, \alpha_4, \alpha_5 = \theta_0$. It follows that the minimum value of ω is $(\frac{1+\sqrt{5}}{2})^5$ and is attained only when $\alpha_1, ..., \alpha_5 = \theta_0$.

Finally, we observe that $\omega(\theta, \theta)$ is continuous and unbounded for $\theta > 1$. Hence $\omega(\theta, \theta)$, and thus also ω, assume all values $\geqslant \omega_0$. $\square$

2) *Monotonicity Lemmas.*

As mentioned in Section 2.7, the eigenvalues G, G', G'', of the matrix A are the roots of the equation $t(2t-1)^2 - \omega(t-1) = 0$. We will presently show that these roots are real when $\omega_0 \leqslant \omega < \infty$. The graphs of $y = t(2t-1)^2$ and $y = \omega(t-1)$, for real t, are sketched in Figure 1.

We show that the line $y = \omega(t-1)$ is tangent to the curve $y = t(2t-1)^2$ when $\omega = \omega_0$. Let tangency occur at $t = t_0$. Then,

$$t_0(2t_0 - 1)^2 = \omega(t_0 - 1) \text{ and } \frac{d}{dt}\left[t(2t-1)^2\right](t_0) = \omega$$

Eliminating ω from the equations yields $t_0 = \frac{3+\sqrt{5}}{2}$. The value of ω is obtained by letting $t = t_0$ in either equation, yielding, $\omega = \omega_0 = \frac{11+5\sqrt{5}}{2} = (\frac{1+\sqrt{5}}{2})^5$. ω is the slope of the line $y = \omega(t-1)$. As ω increases from ω_0 to ∞ the line $y = \omega(t-1)$ pivots about the points $(1, 0)$, going from being tangent to the curve $y = t(2t-1)^2$ to the vertical line passing through $(1, 0)$. It is then geometrically evident from Figure 1 that for $\omega \geqslant \omega_0$ the equation $t(2t-1)^2 = \omega(t-1)$ has real roots $G(\omega)$,$G'(\omega)$, $G''(\omega)$. The roots are labeled so that $G < 0$, $1 < G' \leqslant G''$, equality holding only for $\omega = \omega_0$.

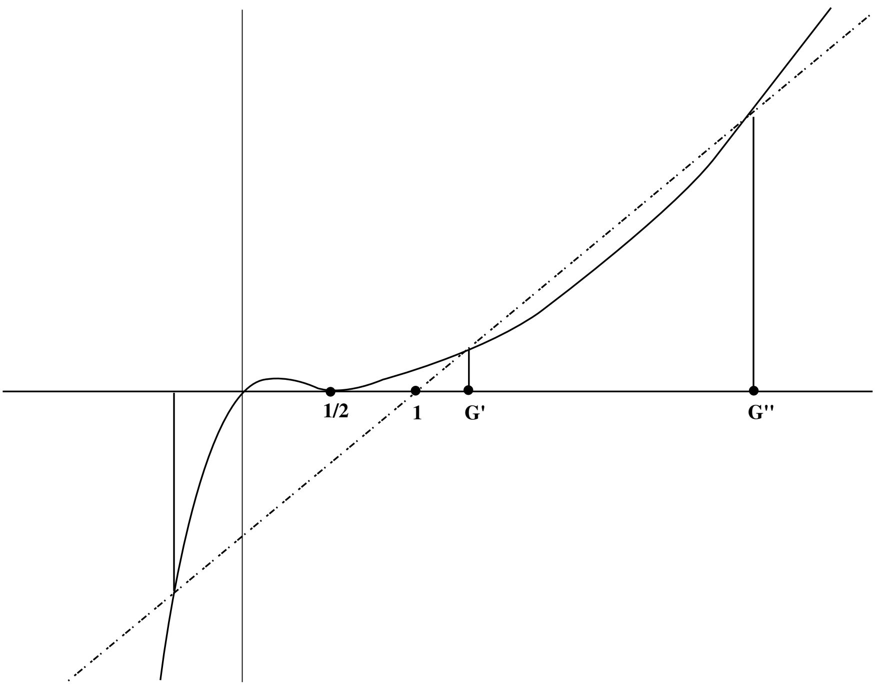

Figure 1

The following lemmas establish the monotonicity of the functions $G(\omega)$, $G'(\omega)$, and $G''(\omega)$ and also the monotonicity of several functions of G,G',G''.

Lemma 2. — *Let $\omega_0 \leqslant \omega < \infty$. G and G' are strictly decreasing continuous functions of ω. G'' is a strictly increasing continuous function of ω. We have, $G'(\omega_0) = G''(\omega_0) = t_0 = \frac{3+\sqrt{5}}{4}$, $G(\omega_0) = 1 - 2t_0 = -\frac{1+\sqrt{5}}{2}$. Let $G'(\infty) = \lim_{\omega\to\infty} G'(\omega)$ with similar meanings for $G''(\infty)$ and $G(\infty)$. Then $G'(\infty) = 1$, $G''(\infty) = \infty$, and $G(\infty) = -\infty$.*

Proof. — As mentioned above, when ω increases from ω_0 to ∞, the line $y = \omega(t-1)$ pivots about the point $(1, 0)$, going from being tangent to the curve $y = t(2t-1)^2$ to the vertical line passing through $(1, 0)$. The monotonicity and continuity properties of the functions G, G', G'' then become geometrically evident from Figure 1 (a rigorous proof of these facts, which we omit, can be given using the implicit function theorem). The only remaining fact requiring proof is the value of $G(\omega_0)$ which follows from $G'(\omega_0) = G''(\omega_0) = t_0$ and $G + G' + G'' = 1$. □

Lemma 3. — 1) *$-G/G'$ is a strictly increasing positive continuous function of ω, with $-G/G'(\omega_0) = (2t_0-1)/t_0 = \sqrt{5}-1$ and $-G/G'(\infty) = \infty$.*

2) *$-G/G''$ is a strictly decreasing positive continuous function of ω, with $-G/G''(\omega_0) = (2t_0-1)/t_0 = \sqrt{5}-1$ and $-G/G''(\infty) = 1$.*

Proof. — 1) Follows from Lemma 2. 2) We have $G + G' + G'' = 1$, so $-G/G'' = 1 + (G'-1)/G''$. Now 2) follows from the above equation and Lemma 2. □

Lemma 4. — a) *Let $k = k(\omega) = \sqrt{\dfrac{(G')^{-2}-(G'')^{-2}}{(G')^{-2}-G^{-2}}}$. $k(\omega)$ is a strictly increasing continuous function with $k(\omega_0) = 0$, $k(\infty) = 1$.*

b) *Hence the inverse function of $k(\omega)$, denoted by $\omega(k)$, is a strictly increasing continuous function for $0 \leqslant k < 1$ with $\omega(0) = \omega_0$ and $\lim_{k\to 1}\omega(k) = \infty$.*

Proof. — We have,

$$1-k^2 = 1-\frac{(G')^{-2}-(G'')^{-2}}{(G')^{-2}-G^{-2}} = 1-\frac{(G/G')^2-(G/G'')^2}{(G/G')^2-1} = \frac{N}{D}$$

where $N = (G/G'')^2-1$ and $D = (G/G')^2-1$. By Lemma 3, N is a strictly decreasing continuous function of ω, with $N(\omega_0) = 5-2\sqrt{5}$, $N(\infty) = 0$, and D is a strictly increasing continuous function of ω with $D(\omega_0) = 5-2\sqrt{5}$, $D(\infty) = \infty$. Hence, $1-k^2 = N/D$ is a strictly decreasing continuous function of ω with $(1-k^2)(\omega_0) = 1$ and $(1-k^2)(\infty) = 0$. The above statement concerning $1-k^2$ is equivalent to Lemma 4a). □

3) *Proof of Theorem 1.* — We have,

$$a(k) = \sqrt{-\frac{G}{G'}(\omega(k))} \text{ and } b(k) = \sqrt{-\frac{G}{G''}(\omega(k))}$$

Theorem 1 follows from the formulas and Lemmas 3,4b. □