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*A viscosity approach to degenerate complex Monge-Ampère equations*

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# A viscosity approach to degenerate complex Monge-Ampère equations

Ahmed Zeriahi[1]

**Abstract.** — This is the content of the lectures given by the author at the winter school KAWA3 held at the University of Barcelona in 2012 from January 30 to February 3. The main goal was to give an account of viscosity techniques and to apply them to degenerate Complex Monge-Ampère equations.

We will survey the main techniques used in the viscosity approach and show how to adapt them to degenerate complex Monge-Ampère equations. The heart of the matter in this approach is the "Comparison Principle" which allows us to prove uniqueness of solutions with prescribed boundary conditions.

We will prove a global viscosity comparison principle for degenerate complex Monge-Ampère equations on compact Kähler manifolds and show how to combine Viscosity methods and Pluripotential methods to get "continuous versions" of the Calabi-Yau and Aubin-Yau Theorems in some degenerate situations. In particular we prove the existence of singular Kähler-Einstein metrics with continuous potentials on compact normal Kähler varieties with mild singularities and ample or trivial canonical divisor.

**Résumé.** — Ce qui suit reproduit les exposés de l'auteur à l'Ecole d'Hiver KAWA 3, qui s'est tenue à l'Université de Barcelone du 30 janvier au 3 février 2012. Le but principal était d'expliquer les techniques de viscosité et de les appliquer aux équations de Monge-Ampère complexes dégénérées.

Nous survolerons les techniques principales de l'approche par la viscosité, et montrerons comment les adapter aux équations de Monge-Ampère complexes dégénérées. Dans cette méthode, le point crucial est le « Principe de Comparaison » qui nous permet de prouver l'unicité des solutions sous des conditions de valeurs au bord.

Nous démontrerons un principe de comparaison de viscosité global pour les équations de Monge-Ampère complexes dégénérées sur les variétés

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compactes kählériennes et montrerons comment combiner les méthodes de viscosité et les méthodes de pluripotentiel pour obtenir des « versions continues » des Théorèmes de Calabi-Yau et Aubin-Yau dans certaines situations dégénérées. En particulier, nous démontrons l'existence de métriques de Kähler-Einstein singulières avec des potentiels continus sur les variétés de Kähler compactes normales avec des singularités modérées et un diviseur canonique ample ou trivial.

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## Introduction

In the late seventies, E. Bedford and B. A. Taylor ([12]) started developing a new method of potential-theoretic nature adapted to the complex structure for solving degenerate complex Monge-Ampère equations in strictly pseudoconvex domains in $\mathbb{C}^n$. They proved a Comparison Principle and, using the Perron method, they were able to solve the Dirichlet problem for degenerate complex Monge-Ampère equation for continuous data. Then, elaborating on this fundamental work, they succeeded in building a complex potential theory, called nowadays "Pluripotential Theory", to study fine properties of plurisubharmonic functions (see [13], [14], [35], [56]).

A quite elaborate theory was developed in the local case, thanks to the contributions of several authors (see among others [15], [16], [26], [57], [27], [28], [9], [18]). There are good surveys on these last developments (see [5, 55, 59]). The Dirichlet problem for non degenerate complex Monge-Ampère equations with smooth data was solved independently by L. Caffarelly, J.J. Kohn, L. Nirenberg and J. Spruck work using methods from the theory of elliptic PDE's ([31])

Pluripotential theory lies at the foundation of the recent approach to degenerate complex Monge-Ampère equations on compact Kähler manifolds, as developed by many authors with applications to Kähler Geometry (see [58], [40], [62], [41], [19], [38], [71], [22], [64], [65]). There is a nice and complete survey on the recent developements in this area (see [65]).

On the other hand, a standard approach to non linear second order degenerate elliptic equations is the method of viscosity solutions introduced first by M.G. Crandall and P.-L. Lions ([32]) at the begining of the eighties in order to prove existence and uniqueness of "solutions" in a generalized sense for first order non linear equations of Hamilton-Jacobi type. But it appeared quickly that this method can be used to prove existence and uniqueness of "generalized weak solutions" to certain fully non linear second order degenerate elliptic PDE's ([33]), especially for those equations for which the notions of "classical" solution (i.e. smooth solution), "generalized" solution (i.e. a solution in the Sobolev space $W^{2,\infty}$) or "weak" solution (i.e. a solution in the sense of distributions) do not make sense. The remarkable fact in this approach is that we can define, as in classical (linear) potential theory for the Laplace operator for example, the notions of subsolution and supersolution in a generalized sense (viscosity sense) for these equations.

The main tool in the viscosity approach is the *Comparison Principle*, which allows comparison of subsolutions and supersolutions with given boundary conditions. This implies uniqueness of viscosity solutions for the associated Dirichlet problem. Then Perron's method can be applied, as in the classical case, to construct the unique solution as the upper envelope of all subsolutions, once we know the existence of a subsolution and a supersolution with the given boundary conditions.

Whereas the viscosity approach has been developed for real Monge-Ampère equations (see [53]), the complex case has not been studied until recently. There have been some recent interest in adapting viscosity methods to solve degenerate elliptic equations on compact or complete Riemannian manifolds (see [3]). This theory can be applied to complex Monge-Ampère equations only in very restrictive cases since it requires the Riemannian curvature tensor to be nonnegative. There is a viscosity approach to the Dirichlet problem for the complex Monge-Ampère equations on smooth domains in Stein manifolds in [48] and [49]. These articles however do not contain any new result for degenerate complex Monge-Ampère equations, since that case is used there as motivation to develop a deep generalization of plurisubharmonic functions to Riemannian manifolds with some special geometric structure. In a recent paper [50], the same authors also develop an interesting application to potential theory in almost complex manifolds and solve the Dirichlet problem in this general context.

The most advanced results about the complex Monge-Ampère equations were obtained quite recently in [42], and we will mostly follow the presentation given there. The main motivation was the problem of continuity of the potentials of the singular Kähler-Einstein metric in a compact Kähler manifold of general type constructed in [41]. Since this paper appeared, there have been recent applications of viscosity methods to the Dirichlet problem for the complex Monge-Ampère equation (see [68]) and more generally for the complex Hessian equation (see [29]).

It is worth observing that there is no general comparison principle which can be applied to a large class of degenerate elliptic fully non linear second order PDE's, including the degenerate complex Monge-Ampère equations we are considering here.

Nevertheless, viscosity methods can be adapted to the complex case and allow us to prove an appropriate Comparison Principle which leads along the same scheme to uniqueness and existence of viscosity solutions.

The first aim of these notes is to present the fundamental ideas behind the viscosity approach. All the material we need can be found in the well known survey [33] (see also [31]). There are also well written papers available in the literature (see [4], [39], [24]) but we will collect here the main ingredients we will need to adapt the viscosity methods to the complex case. The main result which we will use from the viscosity approach is what we call the Jensen-Ishii maximum principle which will be stated whithout proof here, referring to [33].

In order to compare the two approaches, we will start by reviewing the basic tools from Pluripotential theory we will need, namely the comparison principle for the complex Monge-Ampère operator. We will use the pluripotential comparison principle and the Perron method to show how pluripotential theory provides bounded weak solutions to the degenerate complex Monge-Ampère equations we are considering.

The second aim is to show how to adapt the viscosity methods in the context of complex Monge-Ampère equations on domains as well as on compact Kähler manifolds following [42]. We will compare viscosity solutions to pluripotential solutions. The main advantage of the viscosity approach which we will exploit here is, not only that the notion of subsolution makes sense, but that we can also define the notion of supersolution; then a viscosity solution, if it exists, is necessarily continuous. Observe that in the pluripotential theory framework, we can also define the notion of subsolution, but it is not always clear whether a notion of supersolution makes sense and then the continuity of the pluripotential solution, if it exists, is not obtained for free.

Finally the third aim is to show how to combine pluripotential methods and viscosity methods to prove existence and uniqueness of continuous solutions to some degenerate complex Monge-Ampère equations. Moreover using Kolodziej's a priori $C^0-$estimates as extended in [41], we can give a soft proof of the continuous version of Yau's theorem solving the Calabi conjecture which applies for singular compact Kähler varieties with mild singularities (in the sense of the MPP programme [20]) and with ample canonical divisor. In particular we prove that potentials of singular Kähler-Einstein metrics obtained previously in [41] are continuous, whereas they only were known to be bounded. Surprisingly, this allows us to prove continuity of solutions to complex Monge-Ampère equations in a degenerate situation where Pluripotential theory yields only boundedness.

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# 1. Pluripotential solutions to degenerate complex Monge-Ampère equations

## 1.1. Basic facts from Pluripotential Theory

Pluripotential theory deals with plurisubharmonic (psh) functions. These functions appear naturally in many problems of complex analysis where they play the role of soft objects compared to holomorphic functions which are more rigid. This philosophy led P. Lelong to the fundamental notion of positive current ([63]) which play an important role not only in complex analysis but also in Kähler geometry (see [36], [34]).

### 1.1.1. The complex Monge-Ampère operator

Let us recall the construction of Bedford and Taylor and state the main results which will be needed later on. Let $\Omega \subset \mathbb{C}^n$ be a domain and $PSH(\Omega) \subset L^1_{loc}(\Omega)$ be the set of plurisubharmonic functions in $\Omega$.

Here we denote by $d = \partial + \overline{\partial}$ and $d^c := \frac{i}{2\pi}(\overline{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

Following P. Lelong, a current $T$ of bidmension $(p,p)$ on $\Omega$ is by definition a continuous linear form acting on $C^\infty$-smooth differential forms of bidegree $(p,p)$ with compact support in $\Omega$. It is convenient to view a current of bidmension $(p,p)$ on $\Omega$ as a differential form of bidegree $(n-p, n-p)$ with coefficients given by distributions in $\Omega$ (see [63]).

By P. Lelong, if $u \in PSH(\Omega)$ then $dd^c u$ is a closed positive current on $\Omega$, hence a differential form of bidegree $(1,1)$ whose coefficients are complex Borel measure in $\Omega$ (see [63], [34]).

Since plurisubharmonic functions are invariant under holomorphic transformations, the notion of plurisubharmonicity makes sense on complex manifolds. Moreover plurisubharmonic functions appear naturally in complex geometry as local weights for singular metrics on holomorphic line bundles with positive curvature (see [34]).

We will review some basic facts on Pluripotential theory and refer to the original papers of Bedford and Taylor [12, 13, 14] (see also [35], [56]).

Let $u_1, \cdots, u_k$ be $C^2$-smooth psh functions in $\Omega$. Then the following differential $(k,k)$-form $dd^c u_1 \wedge \cdots \wedge dd^c u_k$ has continuous coefficients, hence it can be seen as a closed positive current of bidegree $(k,k)$ in $\Omega$ acting by duality on (smooth) test $(n-k, n-k)$-forms. Moreover we have

$$dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_k) = dd^c u_1 \wedge \cdots \wedge dd^c u_k$$

pointwise and weakly in the sense of currents in $\Omega$.

In particular, for any smooth psh function in $\Omega$, we have

$$(dd^c u)^n = c_n \mathrm{det}\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) \beta_n,$$

pointwise as a smooth form of top degree, where $\beta_n$ is the euclidean volume form on $\mathbb{C}^n$ and $c_n > 0$ is a numerical constant. This formula will be used here to identify $(dd^c u)^n$ to a positive Borel measure on $\Omega$, called the Monge-Ampère measure of $u$ in $\Omega$.

We want to extend this definition to non smooth psh functions. It is natural to use local approximation. Observe first that by localisation and integration by parts, one can easily prove that for any compact sets $K, L$ such that $K \subset L^\circ \Subset \Omega$, there exists a positive constant $C > 0$, depending on $(K, L)$, such that for any smooth psh function $u_1, \cdots, u_n$ in $\Omega$, we have

$$\int_K dd^c u_1 \wedge \cdots \wedge dd^c u_n \leqslant C \Pi_{1 \leqslant k \leqslant n} \|u_k\|_{L^\infty(L)}. \tag{1.1}$$

This inequality, called the *Chern-Levine-Nirenberg inequality*, allows to extend easily the definition of the Monge-Ampère operator to continuous non smooth psh functions by regularisation. Indeed let $u$ be a continuous psh function in $\Omega$ and $u_j := u \star \chi_j$ its regularisation by convolution against a radial approximation of the Dirac unit mass at the origin. Let us prove that the sequence of (smooth) measures $(dd^c u_j)^n$ converges weakly in the sense of Radon measures in $\Omega$. Since $u$ is continuous, by Dini's lemma, the sequence $(u_j)$ decreases to $u$, locally uniformly in $\Omega$. Since the sequence $(u_j)$ is locally uniformly bounded in $\Omega$, it follows from Chern-Levine-Nirenberg inequality (1.2), that the sequence of measures $(dd^c u_j)^n$ has locally uniformly bounded mass. Therefore it is enough to prove the convergence of the sequence of measures $(dd^c u_j)^n$ against any smooth test function. This will be a consequence of the following observation. The problem of convergence being local, it is enough to consider a test function with compact support in a small ball $B \Subset \Omega$. Fix such a smooth test function $h$ with compact support in $B$. Then for any $C^2-$smooth psh functions $\varphi$ and $\psi$ in $\Omega$, we have by Stokes formula,

$$\int_B h((dd^c\varphi)^n - (dd^c\psi)^n) = \int_B (\varphi - \psi) dd^c h \wedge T,$$

where $T := \sum_{i=0}^{n-1} (dd^c\varphi)^i \wedge (dd^c\psi)^{n-1-i})$. Since $h$ is smooth of compact support, it is possible to write it as $h = w_1 - w_2$, where $w_1, w_2$ are smooth psh functions in $\Omega$. Therefore if $D$ is a neighbourhood of $\overline{B}$ such that $B \Subset D \Subset \Omega$, then by Chern-Levine-Nirenberg inequality, there exists a uniform constant $C > 0$, depending only on a bound of the second derivatives of $h$ and on a uniform bound of $\varphi$ and $\psi$, such that

$$\left| \int_B h((dd^c\varphi)^n - (dd^c\psi)^n) \right| \leqslant C \|\varphi - \psi\|_{L^\infty(D)}. \tag{1.2}$$

Now let $u$ be a continuous psh function on $\Omega$ and $(u_j)$ its regularizing sequence by convolution. Then by Dini's lemma, the convergence is uniform in each compact set. It follows from Chern-Levine-Nirenberg and (1.2) that the sequence of measures $(dd^c u_j)^n$ is a Cauchy sequence of Radon measures. Then it converges to a positive Radon measure on $\Omega$. Moreover again by (1.2), the limit does not depend on the approximating sequence $(u_j)$ which converges to $u$ locally uniformly in $\Omega$. This limit is defined to be the Monge-Ampère measure of $u$ and denoted by $(dd^c u)^n$.

It turns out that the hypothesis of continuity on the psh function $u$ is a strong condition. Indeed it is not preserved by standard constructions as upper envelopes, regularized limsup of psh functions which arise naturally when dealing with the Dirichlet problem for the complex Monge-Ampère operator. Therefore it is desirable to define the complex Monge-Ampère operator for non continuous psh functions, say e.g. for bounded psh functions. As one may see from the previous reasoning, it is not clear how to define the complex Monge-Ampère measure of $u$ by approximating $u$ by a decreasing sequence of smooth psh functions, since the convergence in not locally uniform anymore. However one of the main results in pluripotential theory says that plurisubharmonic functions are actually quasi continuous ([13]) and then the convergence is quasi-uniform and the proof above can be extended to the bounded case.

Actually to pass from continuous to bounded psh functions is one of the main problem when dealing with the complex Monge-Ampère operator in contrast to the real Monge-Ampère operator which deals with convex functions which are continuous.

In their first seminal work [12], E.Bedford and B.A.Taylor were able to extend the definition of the complex Monge-Ampère operator to the class of locally bounded psh functions using the notion of closed positive current. Their main observation is the following. Let $T$ be a closed positive current of bidegree $(k,k)$ $(1 \leqslant k \leqslant n-1)$ and $u$ a locally bounded psh function in $\Omega$. It is well known that $T$ can be extended as a differential form with complex Borel measure coefficients on $\Omega$. Then the current $uT$ is well defined by duality, since $u$ is a locally bounded Borel function and hence locally integrable with respect to all the coefficients of $T$. Therefore we can define the current $dd^c(uT)$ in the weak sense. Now the following simple observation is crucial: the current $dd^c(uT)$ is again a closed positive current on $\Omega$. Indeed, since the problem is local we can assume that the regularizing sequence $u_j \searrow u$ in $\Omega$. Then $u_j T \rightharpoonup uT$ in the weak sense of measures in $\Omega$ and by continuity of the operator $dd^c$ for the weak topology, we conclude that $dd^c(u_j T) \rightharpoonup dd^c(uT)$ weakly in the sense of currents in $\Omega$. Now since $u_j$ is smooth, we have by Stokes formula for currents that $dd^c(u_j T) = dd^c u_j \wedge T$ is a positive closed currents. Therefore $dd^c(uT)$ is also a closed positive current in $\Omega$, which will be denoted by $dd^c u \wedge T$ (see [34]).

It is now clear that we can repeat this construction: if $u_1, \cdots, u_k$ are locally bounded psh functions, it is possible to define by induction the current $dd^c u_1 \wedge \cdots \wedge dd^c u_k$ by the formula

$$dd^c u_1 \wedge \cdots \wedge dd^c u_k := dd^c(u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_k),$$

weakly in the sense of currents in $\Omega$, the resulting current being a closed positive current in $\Omega$.

In particular if $u$ is a locally bounded psh function in $\Omega$, then the current of bidegree $(n,n)$ given by $(dd^c u)^n = dd^c u_1 \wedge \cdots \wedge dd^c u_n$, where $u_1 = \cdots = u_n = u$ can be identified to a positive Borel measure denoted by $(dd^c u)^n$ called the Monge-Ampère measure of $u$.

Likely this definition coincides with the previous one when $u$ is a continuous psh function. More generally, using ingenious integration by parts and local approximations, Bedford and Taylor proved the following important convergence theorem ([12]).

Theorem 1.1. — *Let $(u_j)$ and $(v_j)$ be decreasing sequences of locally bounded psh functions in $\Omega$ converging to locally bounded psh functions $u$ and $v$ respectively in $\Omega$. Then the sequence of measures $u_j(dd^c v_j)^n$ converges to the measure $u(dd^c v)^n$ weakly in the sense of measures in $\Omega$. The same weak convergence still holds if $(u_j)$ or $(v_j)$ increases almost everywhere in $\Omega$ to $u$ or $v$ respectively.*

### 1.1.2. The Pluripotential Comparison Principle

From Theorem 1.1, it is possible to derive the following fundamental result, which we will call the (local) maximum principle ([14]).

Theorem 1.2 *(Maximum Principle). — Let $u, v$ be locally bounded psh functions in $\Omega$. Then we have*

$$\mathbf{1}_{\{u<v\}}(dd^c \max\{u,v\})^n = \mathbf{1}_{\{u<v\}}(dd^c v)^n, \tag{1.3}$$

*weakly in the sense of Borel measures in $\Omega$.*

Observe that the identity (1.3) is trivial when $v$ is continuous, since the two psh functions $\max\{u,v\}$ and $v$ coincide on the open set $\{u < v\}$. The main difficulty in the proof of (1.3) is to pass from continuous to bounded psh functions and this is the main feature in Bedford and Taylor work building up a potential theory for plurisubharmonic functions, called Pluripotential Theory (see [12], [13], [14], [35]). It turns out that the set $\{u < v\}$ is actually open for the plurifine topology and it was proved by Bedford and Taylor that the complex Monge-Ampère operator is local in the plurifine topology (see [14]).

From the maximum principle, it is easy to deduce its companion, which will be called the Pluripotential Comparison Principle.

Corollary 1.3 *(Comparison Principle). — Let $u, v$ be locally bounded psh functions in $\Omega \Subset \mathbb{C}^n$ such that $u \geqslant v$ on $\partial\Omega$ i.e. $\{u < v\} = \{z \in \Omega; u(z) < v(z)\} \Subset \Omega$. Then*

$$\int_{\{u<v\}} (dd^c v)^n \leqslant \int_{\{u<v\}} (dd^c u)^n.$$

*If moreover $(dd^c u)^n \leqslant (dd^c v)^n$ in the weak sense in $\Omega$ then $u \geqslant v$ in $\Omega$.*

This result implies uniqueness of the solution to the Dirichlet problem when it exists. Moreover using Perron's method of envelopes of subsolutions, Bedford and Taylor were able to solve the Dirichlet problem for the complex Monge-Ampère operator. Let us state the following consequence of their result which will be used here ([12, 13]).

Theorem 1.4. — *Let $B \Subset \mathbb{C}^n$ be an euclidean ball, $\mu \geqslant 0$ a continuous volume form on $\overline{B}$ and $\gamma$ a continuous function in $\partial B$. Then there exists a unique psh function $U$ in $B$, which extends as a continuous function in $\overline{B}$ solving the following Dirichlet problem:*

$$\begin{cases} (dd^c U)^n = \mu, & \text{weakly in } B, \\ U = \gamma & \text{in } \partial B. \end{cases}$$

We will also need the following fundamental consequence, known as the "balayage method".

Corollary 1.5. — *Let $\Omega \subset \mathbb{C}^n$ be an open set, $B \Subset \Omega$ is a given euclidean open ball and $\mu \geqslant 0$ a continuous volume form on $\overline{B}$.Then for any psh function $u$ in $\Omega$, bounded in a neighbourhood of $\overline{\mathbb{B}}$ and satisfying $(dd^c u)^n \geqslant \mu$ in the weak sense in $\mathbb{B}$, there exists a psh function $U$ in $\Omega$ such that $U = u$ in $\Omega \setminus \overline{B}$, $U \geqslant u$ in $\Omega$ and $(dd^c U)^n = \mu$, weakly in $B$.*

### 1.1.3. The complex Monge-Ampère operator on compact Kähler manifolds

Now let us explain how to extend the previous tools to compact Kähler manifolds following [45]. The notion of psh function makes sense on any complex manifold since it is invariant under holomorphic transformations.

Let $X$ be a (connected) compact Kähler manifold of dimension $n$ and let $\omega$ be a closed smooth $(1,1)-$form on $X$. Then it is well known that locally in each small coordinate chart $U \subset X$, there exist a smooth function $\rho_U$ such that $\omega = dd^c \rho_U$, the function $\rho_U$ is psh in $U$ and called a local potential of $\omega$ (see [34]). Such a local potential is unique up to addition of a pluriharmonic function in $U$

Recall that a function $\varphi : X \longrightarrow [-\infty, +\infty[$ is said to be $\omega$-plurisubharmonic in $X$ ($\omega$-psh for short) if it is upper semicontinuous in $X$ and locally in each small coordinate chart $U$, the function $u = \varphi + \rho_U$ is psh in $U$, where $\rho_U$ is any local potential of $\omega$ in $U$.

Let us denote by $PSH(X, \omega) \subset L^1(X)$ the convex set of $\omega$-psh functions in $X$, where $L^1(X)$ the Lebesgue space with respect to a fixed smooth non degenerate volume form $\mu_0$ on $X$.

Then the $(1,1)$-current $\omega_\varphi := \omega + dd^c \varphi$ is a closed positive current in the sense of Lelong since locally in $U$ it can be written as $\omega_\varphi = dd^c u$, where $u = \varphi + \rho_U$ is psh in $U$ ([63], [34]). It follows from what was said previously that the complex Monge-Ampère operator is well defined for any bounded $\varphi \in PSH(X) \cap L^\infty(X)$ as a positive $(n,n)$-current on $X$ defined locally in each coordinate chart $U$ as

$$\omega_\varphi^n := (dd^c u)^n.$$

This current can be identified to a positive Borel measure on $X$, which will be denoted by $MA(\varphi) = MA_\omega(\varphi)$ (see [45]). Then the maximum principle still holds in this context (see [46]).

Theorem 1.6. — *Let $\varphi, \psi \in PSH(X) \cap L^\infty(X)$. Then*

$$\mathbf{1}_{\{\varphi<\psi\}} MA(\max\{\varphi, \psi\}) = \mathbf{1}_{\{\varphi<\psi\}} MA(\psi),$$

*in the sense of positive Borel measures in $X$.*
*(Maximum Principle).*

*In particular*

$$\int_{\{\varphi<\psi\}} MA(\psi) \leqslant \int_{\{\varphi<\psi\}} MA(\varphi).$$

*(Comparison Principle).*

There is another important result which will be used later.

Theorem 1.7. — *Let $\varphi, \psi \in PSH(X) \cap L^\infty(X)$. Assume that $\psi \leqslant \varphi$ almost everywhere in $X$ with respect to the measure $MA(\varphi)$. Then $\psi \leqslant \varphi$ everywhere in $X$*
*(Domination Principle).*

For more details on these matters we refer to ([44, 45, 57)].

One of the main tools in recent applications of Pluripotential theory to Kähler Geometry is the a priori uniform estimate du to Kolodziej ([58, 60]. We will use here the following version (see [17, 41, 47]).

Theorem 1.8. — *Let $f \in L^p(X, \mu_0)$ with $p > 1$ and $\psi \in PSH(X, \omega) \cap L^\infty(X)$ with $\sup_X \psi = 0$. Then there exists a constant depending on $\|\psi\|_{L^\infty(X)}$) such that for any $\varphi \in PSH(X) \cap L^\infty(X)$ satisfying $MA(\varphi) \leqslant f\mu_0$ with* $\sup_X \varphi = 0$, *we have the following "weak stability" estimates*

$$\sup_X (\psi - \varphi)^+ \leqslant C\|f\|_{L^p(X)}^{1/n} \|(\psi - \varphi)^+\|_{L^1(X)}^\gamma,$$

*where $\gamma = 1/(nq + 2)$ and $q = p/(p - 1)$.*

*In particular we have the following uniform $L^\infty$-estimate*

$$\|\varphi\|_{L^\infty} \leqslant A\|f\|_{L^p(X)}^{1/n},$$

*where $A > 0$ is a uniform constant independent on $\varphi$.*

### 1.2. Solving degenerate complex Monge-Ampère equations

We are mainly interested here in complex Monge-Ampère equations on compact Kähler manifolds related to the Calabi conjecture and the existence of Kähler-Einstein metrics. Let $X$ be a compact Kähler manifold of dimension $n$ and let $\omega$ be a smooth closed semi-positive form on $X$ such that $\int_X \omega^n > 0$.

We will consider the following global complex Monge-Ampère equation.

$$(MA)_{\varepsilon,\mu} \qquad (\omega + dd^c\varphi)^n = e^{\varepsilon\varphi}\mu,$$

where $\varepsilon \geqslant 0$ and $\mu = f\mu_0$ is a degenerate volume form on $X$ with a density $0 \leqslant f \in L^p(X, \mu_0)$ $(p > 1)$ with respect to a fixed smooth non degenerate volume form $\mu_0$ on $X$.

In the case when $\varepsilon = 0$, $\omega > 0$ is a Kähler from on $X$ and $\mu = f\omega^n > 0$ is a smooth non degenerate voolume form on $X$, the corresponding equation is known as the Calabi-Yau equation and there is a necessary condition for a solution to exists : $\int_X \mu = \int_X \omega^n$. Moreover adding a constant to a solution gives a new solution. E. Calabi observed that the smooth solution of the equation $(MA)_{0,\mu}$ (if it exists) is unique up to an additive constant ([23]). These two facts make actually this equation more difficult to handle. It was proved by Yau ([70]), answering the celebrated Calabi's conjecture, that this equation has a smooth solution $\varphi$ on $X$ i.e. there exists $\varphi \in C^\infty(X)$ such that $\omega_\varphi := \omega + dd^c\varphi > 0$ is a Kähler metric on $X$ satisfying the equation $(MA)_{0,\mu}$. In particular he showed that on compact Kähler manifolds for which the first Chern class is zero i.e. $c_1(X) = 0$, any Kähler class contains a (smooth) Ricci-flat Kähler-Einstein metric (see also [66]).

When $\varepsilon > 0$, $\omega > 0$ is a Kähler form and $\mu = f\omega^n > 0$ is a smooth non degenerate volume form on $X$, the equation $(MA)_{\varepsilon,\mu}$ was considered by Aubin and Yau in connection to the problem of existence of Kähler-Einstein metrics on a compact Kähler manifolds of negative first Chern class i.e. $c_1(X) < 0$. As we will see this equation is much more simpler than the Calabi-Yau equation. The uniqueness is an easy consequence of the Comparison Principle. The existence of a smooth solution for the equation $(MA)_{\varepsilon,\mu}$ was proved in 1978 independently by Aubin and Yau ([2, 70]).

The approach used by Aubin and Yau relies on the continuity method and a priori estimates of high order. It turns out that the a priori $C^0$ estimate is the main step in their approach.

In 1998, S. Kolodziej ([58]) gave a new proof of the a priori $C^0$-estimate using methods from Pluripotential Theory. Moreover, using Yau's theorem he was able to extend it to a slightly more degenerate situation in the case when $\varepsilon = 0$ and $0 \leqslant f \in L^p(X)$ $(p > 1)$, $\omega$ being a Kähler form. This allows him to obtain continuous weak solution of the equation $(MA)_{0,\mu}$ in the sense of Bedford and Taylor (a pluripotential solution).

This result was extended in [41] to a more degenerate situation when $\omega \geqslant 0$ is a closed smooth and semi-positive $(1,1)-$form on $X$ such that $\int_X \omega^n > 0$. The weak solution obtained there was shown to be a bounded $\omega$-psh function, but the continuity was proved under an extra assumption which is satisfied when $\omega > 0$ is Kähler.

We first review the main results obtained in [41]. However we will give a direct approach using pluripotential techniques as developed recently in [42], which do not use the continuity method and high order a priori estimates of Yau and Aubin. Namely we will prove the following result.

Theorem 1.9. — *Let $X$ be a compact Kähler manifold of dimension $n$ and $\omega \geqslant 0$ be a smooth closed $(1,1)-$form on $X$ such that $\int_X \omega^n > 0$ and $\mu = f\mu_0$ a volume form on $X$ with density $0 \leqslant f \in L^p(X;\mu_0)$ $(p > 1)$ with respect to a fixed smooth non degenerate volume form $\mu_0 > 0$. Then there is a unique $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ which satisfies the complex Monge-Ampère equation*

$$(\omega + dd^c\varphi)^n = e^\varphi \mu,$$

*in the pluripotential sense in $X$.*

*Moreover the solution $\varphi$ is the upper envelope of the family of pluripotential subsolutions of the equation in $X$ i.e.*

$$\varphi = \sup \mathcal{F}(X,\omega,\mu),$$

*where*

$$\mathcal{F}(X,\omega,\mu) := \{\psi; \psi \in PSH(X,\omega) \cap L^\infty(X), (\omega + dd^c\psi)^n \geqslant e^\psi \mu\}.$$

This theorem implies that for any $\varepsilon > 0$ the complex Monge-Ampère equation $(MA)_{\varepsilon,\mu}$ has a unique bounded $\omega-$plurisubharmonic solution $\varphi_\varepsilon$. It turns out that the family $(\varphi_\varepsilon)$ is uniformly bounded and converges uniformly on $X$ by Theorem 1.8. More precisely we obtain the following result.

Theorem 1.10. — *Let $\omega \geqslant 0$ be a smooth closed semi-positive $(1,1)-$form on $X$ such that $\int_X \omega^n > 0$ and $\mu = f\mu_0$ a volume form on $X$ with density $0 \leqslant f \in L^p(X)$ $(p > 1)$ with respect to a smooth non degenerate volume form $\mu_0 > 0$ such that $\int_X \mu = \int_X \omega^n$. Then there is a unique $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ which satisfies the complex Monge-Ampère equation*

$$(\omega + dd^c\varphi)^n = f\mu_0,$$

*in the pluripotential sense and normalized by $\int_X \varphi\omega^n = 0$.*

These theorems do not say anything about continuity of the solution. It is possible to prove continuity and even Hölder continuity of the solution when $\omega > 0$ is a Kähler form using pluripotential methods (see [58], [41], [61], [37]). But these results do apply in our degenerate situation.

However, in the last section we will show how to combine pluripotential and viscosity techniques to prove continuity in this more general setting.

Altogether this will provide an alternative and independent approach to a weak version of Calabi conjecture [70]: we will only use upper envelope constructions (both in the viscosity and pluripotential sense), a global viscosity and pluripotential comparison principle and Kolodziej's pluripotential techniques providing uniform a priori estimates ([58], [41]).

This method applies to degenerate equations but yields solutions that are merely continuous (Yau's work yields smooth solutions, assuming the cohomology class $\{\omega\}$ is Kähler and the volume form $\mu$ is both positive and smooth).

The pluripotential approach applies equally well to a slightly more degenerate situation (see [42], [37]).

### 1.3. The Perron method of upper envelopes

Before going into the proofs of the results stated in the last section, we will establish a more general result which shows that the Perron method of upper envelopes will provide us with a solution whenever we are able to find a subsolution. This quite general approach might be useful in other situations.

Here we will consider the following degenerate Monge-Ampère equation

$$MA_\omega(\varphi) = e^\varphi \mu, \tag{1.4}$$

where $\omega \geqslant 0$ is a closed $(1,1)$-form in $X$ with continuous psh local potentials, $MA_\omega(\varphi) := (\omega + dd^c\varphi)^n$ is the Monge-Ampère measure of $\varphi$ defined in the weak sense of Bedford and Taylor and $\mu \geqslant 0$ is a degenerate volume form with $L^1$-density with respect to a fixed smooth volume form.

Our aim here is to show that one can solve this equation in the weak sense of Bedford and Taylor in a rather elementary way, at least when $\omega > 0$ is a Kähler form and $\mu$ has a continuous density, by observing that the (unique) solution is the upper envelope of pluripotential subsolutions.

### 1.3.1. Uniqueness of the solution

Here we will give an easy consequence of the Comparison Principle which will show that the upper envelope of subsolutions of the equation (1.4) is the unique candidate to be a solution.

Proposition 1.11. — *Let $\varphi \in PSH(X,\omega) \cap L^\infty(X)$ be a solution to the Monge-Ampère equation (1.4). Then for any $\psi \in PSH(X,\omega) \cap L^\infty(X)$ satisfying the inequality $MA_\omega(\psi) \geqslant e^\psi \mu$ in the weak sense of Borel measures on $X$, we have $\psi \leqslant \varphi$ in $X$. In particular, the solution of the complex Monge-Ampère equation (1.4) is unique (if it exists).*

*Proof.* — We are going to show that the set $\{\varphi < \psi\}$ has zero measure with respect to $\mu$. Indeed by the comparison principle, it follows that

$$\begin{aligned}\int_{\{\varphi<\psi\}} e^\psi \mu &\leqslant \int_{\{\varphi<\psi\}} (\omega + dd^c\psi)^n \\ &\leqslant \int_{\{\varphi<\psi\}} (\omega + dd^c\varphi)^n \\ &= \int_{\{\varphi<\psi\}} e^\varphi \mu \leqslant \int_{\{\varphi<\psi\}} e^\psi \mu.\end{aligned}$$

Therefore we conclude that $\int_{\{\varphi<\psi\}} (e^\varphi - e^\psi)\mu = 0$ and since $e^\varphi - e^\psi \leqslant 0$ on the set $\{\varphi < \psi\}$, it follows that $\mathbf{1}_{\{\varphi<\psi\}} \cdot (e^\varphi - e^\psi) = 0$ $\mu$-almost everywhere on $X$. If we know that $\mu$ has a positive density with respect to a fixed smooth non degenerate volume form on $X$, we will conclude that $\psi \leqslant \varphi$ almost everywhere in $X$ and then everywhere in $X$ by submean-value inequality in any local chart. In the general case, since $e^\varphi \mu = MA(\varphi)$, it follows that the set $\{\varphi < \psi\}$ has measure 0 with respect to the Monge-Ampère measure $MA(\varphi)$ i.e. $\psi \leqslant \varphi$ almost everywhere with respect to $MA(\varphi)$. It follows from the Domination Principle Theorem 1.7 that $\psi \leqslant \varphi$ on $X$. This shows that the equation $(MA)_{1,\mu}$ has at most one solution. □

### 1.3.2. Existence of a solution

The previous subsection suggests a natural candidate to be the solution to the Monge-Ampère equation (1.4): the upper envelope of pluripotential subsolutions, in the spirit of the classical Perron's method used in solving the classical Dirichlet problem.

Therefore it is natural to consider the class $\mathcal{F} = \mathcal{F}(X,\omega,\mu)$ of all pluripotential subsolutions of the equation $(MA)_{1,\mu})$ defined by

$$\mathcal{F} := \left\{\varphi \in PSH(X,\omega) \cap L^\infty(X) \,/\, MA(\psi) \geqslant e^\psi \mu \text{ in } X\right\}.$$

Now the problem of the existence of a solution remains to prove that $\mathcal{F} \neq \emptyset$ and its upper envelope $\varphi := \sup \mathcal{F}$ is again a subsolution.

Lemma 1.12. — *The class $\mathcal{F}(X, \omega, \mu)$ is uniformly bounded from above on $X$ and stable under the regularized supremum. Moreover it is compact in $PSH(X, \omega)$ (for the $L^1(X)$-topology).*

*Proof.* — We can assume that $\mathcal{F} \neq \emptyset$. We show first that $\mathcal{F}$ is uniformly bounded from above. We can assume without loss of generality that $\mu$ is normalized so that $\mu(X) = 1$. Fix $\psi \in \mathcal{F}$. It follows from the convexity of the exponential that

$$\exp\left(\int_X \psi\mu\right) \leqslant \int_X e^{\psi}\mu = \int_X MA(\psi) = \int_X \omega^n.$$

We infer

$$\sup_X \psi \leqslant \int_X \psi\mu + C_\mu \leqslant \log Vol_\omega(X) + C_\mu,$$

where $C_\mu$ is a uniform constant that only depends on the fact that all $\omega$-psh functions are integrable with respect to $\mu$ (see [45]). This shows that $\mathcal{F}$ is uniformly bounded from above by a constant that only depends on $\mu$ and since it is not empty, it is also uniformly bounded from below.

Stability under finite suprema is an easy consequence of the Maximum Principle Thorem 1.6. If $\psi_1, \psi_2 \in PSH(X, \omega) \cap L^\infty(X)$ we have

$$MA(\sup\{\psi_1, \psi_2\}) \geqslant \mathbf{1}_{\{\psi_1 \geqslant \psi_2\}} MA(\psi_1) + \mathbf{1}_{\{\psi_1 < \psi_2\}} MA(\psi_2).$$

For an infinite family $\mathcal{S}$ of subsolutions, the same reasoning can be applied since the regularized supremum of $\mathcal{S}$ can be approximated almost everywhere by a increasing sequence of finite suprema of subsolutions, and the conclusion follows from the continuity of the complex Monge-Ampère operator for increasing sequences of uniformly bounded $\omega-$psh functions (Theorem 1.1).

The compactness can be proved as follows. By the previous considerations, the family $\mathcal{F}$ is relatively compact in $L^1(X)$ (see [45]). It is then enough to show that it is closed. Let $(\psi_j)_{j \in \mathbb{N}}$ be a sequence of $\mathcal{F}$ converging to $\psi \in PSH(X, \omega)$. We know that $\psi$ is bounded. We can assume that $\psi_j$ converges almost everywhere to $\psi$ in $X$. Set $\overline{\psi}_j := (\sup_{k \geqslant j} \psi_k)^*$. Then $(\psi_k)_{k \in \mathbb{N}}$ is a decreasing sequence of $PSH(X, \omega) \cap L^\infty$ wich converges to $\psi$. From the previous facts it follows that $MA(\overline{\psi}_j) \geqslant e^{\overline{\psi}_j}\mu$ for any $j$ and again by the continuity of the complex Monge-Ampère operator for deacreasing sequences, we conclude that $MA(\psi) \geqslant e^{\psi}\mu$ weakly in $X$. □

Now it remains to prove that the upper envelope of $\mathcal{F}$ is a solution. This is the content of the following result.

THEOREM 1.13. — *If the class $\mathcal{F} = \mathcal{F}(X, \omega, \mu) \neq \emptyset$ is not empty, then its upper envelope given by*

$$\varphi := \sup\{\psi \,/\, \psi \in \mathcal{F}\},$$

*is the unique solution to the complex Monge-Ampère equation $(MA)_{1,\mu}$ in the weak sense in $X$ (pluripotential solution).*

*Proof.* — Indeed, from the compactness of the class, it follows that its upper envelope is $\omega-$psh in $X$ (see [51], Proposition 3.4.4), and then it is a subsolution. Moreover by Choquet's lemma, we can find a sequence $\psi_j \in \mathcal{F}$ of bounded $\omega$-psh (pluripotential) subsolutions such that

$$\varphi = (\sup_{j \in \mathbb{N}} \psi_j)^*.$$

Observe that by Lemma 1.12, the family of bounded pluripotential subsolutions is stable under taking maximum so that we can that assume the $\psi_j$'s form a non decreasing sequence of subsolutions. To see that $\varphi$ is a subsolution, we use a local balayage procedure to modify each $\psi_j$ on a given "small ball" $B \subset X$ by constructing a new bounded $\omega-$psh functions $\tilde{\psi}_j$ on $X$ so that they satisfy the local Monge-Ampère equation $(\omega + dd^c \tilde{\psi}_j)^n = e^{\psi_j} \mu$ on $\mathbb{B}$ and $\tilde{\psi}_j \geqslant \psi$ on $X$ and $\tilde{\psi}_j = \psi_j$ on $X \setminus \mathbb{B}$: this is done using Theorem 1.5. By the comparison principle Corollary 1.3, it follows that $(\tilde{\psi}_j)$ is an non increasing sequence of bounded $\omega-$psh functions which increases almost everywhere in $X$ to the function $\varphi$. Since the Monge-Ampère operator is continuous under increasing sequences by Theorem 1.1, it follows that $\varphi$ is a pluripotential solution of $(MA)_{1,\mu}$ in $B$, hence in all of $X$, as $B$ was arbitrary. □

COROLLARY 1.14. — *Assume that $\mu$ is a Borel volume form satisfying the following condition: $\exists u \in PSH(X, \omega) \cap L^\infty(X), \exists A > 0$ such that*

$$(\dagger) \qquad \mu \leqslant A(\omega + dd^c u)^n,$$

*in the weak sense of measures in $X$.*

*Then the class $\mathcal{F}(X, \omega, \mu)$ is not empty, uniformly upper bounded and its upper envelope $\varphi := \sup \mathcal{F}(X, \omega, \mu)$ is the unique bounded pluripotential solution to $(MA)_{1,\mu}$.*

*Proof.* — Set $M := \sup_X u$ and choose $C > 1$ large constant so that $e^{M-C}A \leqslant 1$. Then by the condition (†), the function $\psi_0 := u - C \in \mathcal{F}(X,\omega,\mu)$ is a pluripotential subsolution to $(MA)_{1,\mu}$. Therefore we can apply the previous Theorem. □

### 1.3.3. Proof of Theorem 1.9

We want to apply Corollary 1.14. The fact that the family $\mathcal{F}(X,\omega,\mu)$ is uniformly upper bounded follows from Lemma 1.12.
To prove that it is not empty requires several steps.
1. Assume that $\omega > 0$ is Kähler and $\mu = f\mu_0$ has a bounded density i.e. $f \in L^\infty(X)$. Then for a large constant $A > 0$ we clearly have $\mu \leqslant A\omega^n$ and then the condition (†) is satisfied. Therefore the conclusion of the Theorem follows from Corollary 1.14.
2. Assume that $\omega > 0$ and $\mu = f\mu_0$ has a density $f \in L^p(X)$. Then we approximate $\mu$ by volume forms with bounded densities $\mu_j := \inf\{f, j\}\mu_0$ for $j \in \mathbb{N}$ and apply the previous case to solve the equations

$$(\omega + dd^c\varphi_j)^n = e^{\varphi_j}\mu_j, \tag{1.5}$$

with $\varphi_j \in PSH(X,\omega) \cap L^\infty(X)$. Moreover since $\int_X e^{e_j\varphi_{\varepsilon_j}}\mu = \int_X \mu$, the last statement of the theorem follows from the fact that

$$\int_X \varphi d\mu = \lim_{\varepsilon_j \to 0} \int_X \frac{e^{\varepsilon_j\varphi_{\varepsilon_j}} - 1}{\varepsilon_j} d\mu = 0.$$

Let us prove that $(\varphi_j)$ is bounded in $L^1(X)$. By [45], it is enough to check that the sequence $(\sup_X \varphi_j)$ is bounded. By Lemma 1.12, this sequence is upper bounded. To see that it is lower bounded, observe that

$$e^{\sup_X \varphi_j} \geqslant \frac{\int_X \omega^n}{\mu(X)} = \int_X \omega^n$$

hence the sequence $(\sup_X \varphi_j)$ is bounded from below.

We now assert that $(\varphi_j)$ is decreasing as $j$ increases to $+\infty$. Indeed assume that $1 < j \leqslant k$ and fix $\delta > 0$. It follows from the (pluripotential) comparison principle that

$$\int_{\{\varphi_k \geqslant \varphi_j + \delta\}} (\omega + dd^c\varphi_k)^n \leqslant \int_{\{\varphi_k \geqslant \varphi_j + \delta\}} (\omega + dd^c\varphi_j)^n.$$

Then using the equations (1.5) and the fact that $\mu_k \geqslant \mu_j$, we infer

$$\mathbf{1}_{\{\varphi_k \geqslant \varphi_j + \delta\}}(\omega + dd^c\varphi_k)^n \geqslant e^\delta \mathbf{1}_{\{\varphi_k \geqslant \varphi_j + \delta\}}(\omega + dd^c\varphi_j)^n$$

in the sense of Borel measures on $X$. Therefore it follows that the set $\{\varphi_k \geqslant \varphi_j + \delta\}$ has zero measure with respect to the Monge-Ampère measure $(\omega + dd^c\varphi_j)^n$ i.e. the inequality $\varphi_k - \delta \leqslant \varphi_j$ holds $(\omega + dd^c\varphi_j)^n$-almost everywhere in $X$. From the domination principle it follows that $\varphi_k - \delta \leqslant \varphi_j$ everywhere in $X$. As $\delta > 0$ was arbitrary, we infer $\varphi_k \leqslant \varphi_j$ in $X$.

We let $\varphi = \lim_{j\to+\infty} \varphi_j$ denote the decreasing limit of the functions $\varphi_j$. By construction this is an $\omega$-psh function. It follows from Theorem 1.8 that $\varphi$ is a bounded $\omega$-psh function in $X$. Passing to the limit in (1.5) as $j \to +\infty$, we conclude using Theorem 1.1 $\varphi$ is a (pluripotential) solution to the Monge-Ampère equation $(\omega + dd^c\varphi)^n = e^{\varphi}\,\mu$. This shows that (†) is satisfied hence we can use Corollary 1.14 to conclude.

3. Assume that $\omega \geqslant 0$ and $\mu = f\mu_0$ with $f \in L^p(X)$. Fix a Kähler form $\beta$. By the above there exists, for each $0 < \varepsilon \leqslant 1$, a unique continuous $(\omega + \varepsilon\beta)$-psh function $u_\varepsilon$ such that

$$(\omega + \varepsilon\beta + dd^c u_\varepsilon)^n = e^{u_\varepsilon}\,\mu.$$

As in the previous case we see that $\sup_X u_\varepsilon$ is bounded, as $0 < \varepsilon \leqslant 1$.

We now claim that $(u_\varepsilon)$ is decreasing as $\varepsilon$ decreases to $0^+$. The proof goes in the same lines as in the previous case. Indeed assume that $0 < \varepsilon' \leqslant \varepsilon$ and fix $\delta > 0$. Note that $u_{\varepsilon'}, u_\varepsilon$ are both $(\omega + \varepsilon\beta)$-plurisubharmonic. It follows from the (pluripotential) comparison principle Theorem 1.6 that

$$\int_{\{u_{\varepsilon'} \geqslant u_\varepsilon + \delta\}} (\omega + \varepsilon\beta + dd^c u_{\varepsilon'})^n \leqslant \int_{\{u_{\varepsilon'} \geqslant u_\varepsilon + \delta\}} (\omega + \varepsilon\beta + dd^c u_\varepsilon)^n.$$

Since

$$(\omega + \varepsilon\beta + dd^c u_{\varepsilon'})^n \geqslant (\omega + \varepsilon'\beta + dd^c u_{\varepsilon'})^n \geqslant e^{\delta}(\omega + \varepsilon\beta + dd^c u_\varepsilon)^n$$

on the set $\{u_{\varepsilon'} \geqslant u_\varepsilon + \delta\}$, this shows that the latter set has zero measure with respect to the measure $(\omega + \varepsilon\beta + dd^c u_\varepsilon)^n$ hence by the domination principle Theorem 1.7, it follows that $u_{\varepsilon'} \leqslant u_\varepsilon + \delta$ everywhere in $X$. As $\delta > 0$ was arbitrary, we infer $u_{\varepsilon'} \leqslant u_\varepsilon$ in $X$.

We let $u = \lim_{\varepsilon\searrow 0} u_\varepsilon$ denote the decreasing limit of the functions $u_\varepsilon$. By construction this is an $\omega$-psh function in $X$ and by Theorem 1.8, $u$ is bounded and a (pluripotential) solution of the Monge-Ampère equation $(\omega + dd^c u)^n = e^u\,\mu$. This shows that the condition (†) is satisfied hence the conclusion follow from Corollary 1.14.

#### 1.3.4. Proof of Theorem 1.10

We approximate the equation $(MA)_{0,\mu}$ by the perturbed equations $(MA)_{\varepsilon,\mu}$, where $\varepsilon \searrow 0$. By Theorem 1.9, for each $\varepsilon > 0$ we can find

$\varphi_\varepsilon \in PSH(X,\omega) \cap L^\infty(X)$ such that

$$(\omega + dd^c \varphi_\varepsilon)^n = e^{\varepsilon \varphi_\varepsilon} \mu, \tag{1.6}$$

in the pluripotential sense in $X$. By convexity of the exponential function, we conclude that $\int_X \varphi_\varepsilon \mu \leqslant 0$. Therefore by [45], it follows that there exists a constant $M > 0$ independent of $\varepsilon$ such $\sup_X \varphi_\varepsilon \leqslant M$. On the other hand from (1.6), if follows that $\sup_X \varphi_e \geqslant 0$. Therefore $(\varphi_\varepsilon)$ in bounded in $L^1(X)$. Then there exists a subsequence $(\varphi_{\varepsilon_j})$, with $\varepsilon_j \searrow 0$, which converges in $L^1(X)$ to a $\varphi \in PSH(X,\omega)$ and such that $\varphi_{\varepsilon_j} \to \varphi$ almost everywhere in $X$. We know that $\varphi = (\limsup_{j \to +\infty} \varphi_j)^*$. By Theorem 1.8, if follows that $\varphi_{\varepsilon_j}$ is a bounded sequence in $L^\infty(X)$ and then $\varphi \in PSH(X,\omega) \cap L^\infty(X)$. Let us define $\tilde{\varphi}_j := (\sup_{k \geqslant j} \varphi_{\varepsilon_k})^*$. Then $(\tilde{\varphi}_j)$ is a non increasing sequence of bounded $\omega$-psh functions which converges to $\varphi$ in $X$. Using the comparison principle as in above we see that for any $j \in \mathbb{N}$, we have

$$MA(\tilde{\varphi}_j) \geqslant \inf_{k \geqslant j} e^{\varepsilon_k \varphi_{\varepsilon_k}} \mu.$$

Since $\varepsilon_j \to 0$ and $\varphi_{\varepsilon_j}$ is uniformly bounded, it follows that the right hand side converges weakly to $\mu$ in $X$, while the left hand side converges weakly to $MA(\varphi)$ by Theorem 1.1. Hence $MA(\varphi) \geqslant \mu$ weakly in $X$, which implies $MA(\varphi) = \mu$, since the two volume forms have the same volume in $X$.

## 2. The viscosity approach to degenerate non linear PDE's

Before we introduce the definitions of viscosity sub(super)solutions, let us give as a motivation some examples of degenerate elliptic PDE's to which viscosity methods can be applied. In particular we will give examples where the notion of generalized or weak solution does not make sense.

### 2.1. Classical solutions

Let us start by general considerations. A fully non linear second order PDE can be written in the following general form

$$F(x, u, Du, D^2 u) = 0, \tag{2.1}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \longrightarrow \mathbb{R}$ is a function satisfying some conditions to be made precise in a while, $\Omega \subset \mathbb{R}^N$ is an open set and $\mathcal{S}_N$ is the space of real symmetric matrices of order $N$.

We will say that $u : \Omega \longrightarrow \mathbb{R}$ is a classical solution of the equation (2.1) if $u$ is $C^2$-smooth in $\Omega$ and satisfies the differential identity

$$F(x, u(x), Du(x), D^2 u(x)) = 0, \forall x \in \Omega.$$

It is quite natural to split the equation $F = 0$ into the two different inequalities $F \leqslant 0$ and $F \geqslant 0$. Then if $u$ satisfies the differential inequality $F(x, u(x), Du(x), D^2u(x)) \leqslant 0$ (resp. $F(x, u(x), Du(x), D^2u(x)) \geqslant 0$) pointwise in $\Omega$, we will say that $u$ is a classical subsolution (resp. supersolution) of the equation (2.1). Therefore $u$ is a classical solution of the equation (2.1) iff $u$ is a classical subsolution and a classical supersolution of the equation (2.1)

In order to apply the viscosity approach to the equation (2.1), we need to impose the following FUNDAMENTAL condition on $F$.

*Degenerate ellipticity condition* : for any $x \in \Omega, s \in \mathbb{R}, p \in R^N, Q_1, Q \in \mathcal{S}_N$, we have

$$(DEC) \qquad Q \geqslant 0 \Longrightarrow F(x, s, p, Q_1 + Q) \leqslant F(x, s, p, Q_1).$$

Here $Q \geqslant 0$ means that the symmetric matrix $Q$ is semi-positive i.e. all its eigenvalues are non negative.

The reason why this condition is important for viscosity methods to apply will appear soon.

### 2.2. Examples

Here we are mainly interested in non linear PDE's. However to enlighten the reader about the necessity of this condition to apply viscosity methods, we will recall some basic facts from the theory of linear elliptic second order PDE's.

**Example 1 : Hamilton-Jacobi-Bellman equations**
These are first order equations of the type

$$H(x, u, Du) = 0, \text{in } \Omega, \tag{2.2}$$

associated to a continuous Hamiltonian function $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is an open set.

The simplest example to keep in mind is the Eikonal equation corresponding to the hamiltonian function $H(u) := |Du(x)| - 1$ defined on $]-1, +1[\times\mathbb{R}$. This example will help us to understand the viscosity concepts. Let us consider the Dirichlet problem for the Eikonal equation:

$$|u'(x)| - 1 = 0, \quad u(\pm 1) = 0. \tag{2.3}$$

It is quite clear that this equation has no classical solution. Indeed, a classical solution to this equation should be a $C^1$-smooth function satisfying the equation $|u'(x)| = 1$ pointwise in $]-1,1[$ and the the boundary condition $u(-1) = u(1) = 0$. Such a function do not exist, since by Rolle's theorem it should have at least a critical point in $]-1,1[$.

However the differential equation (2.3) has plenty of generalized solutions i.e. functions $u \in W^{1,\infty}(]-1,1[)$ satisfying the equation $|u'(x)| = 1$ almost everywhere in $]-1,1[$. Indeed the function $u_0(x) := 1-|x|$ is a generalized solution to the Dirichlet problem (2.3). It is easy to cook up piecewise affine functions that satisfies (2.3) on $[-1,+1]$, except a given finite set. Observe that if $u \in W^{1,\infty}(]-1,+1[)$ is a generalized solution to the equation associated to the Hamiltonian function $H(u)$ then $-u$ is a generalized solution of the equation associated to the Hamiltonian $\tilde{H}(u) := -H(-u)$. From the point of view of generalized solutions, the two corresponding equations are the same and $u$ and $-u$ are two different solutions to the same Dirichlet problem. However as we will see, from viscosity point of view they should be considered as different since they correspond to different Hamiltonian functions. Namely, we will see that $u_0$ is the unique viscosity solution of the Dirichlet problem for the Hamiltonian function $H$ with boundary values 0, while $-u_0$ is the unique viscosity solution of the Dirichlet problem for the Hamiltonian function $\tilde{H}$ with boundary values 0.

**Example 2 : Elliptic second order equations:**
An important class of elliptic second order PDE's are the quasi-linear ones, given by

$$-\sum_{j,k} a^{j,k}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} + H(x,u,Du) = 0, \text{ in } \Omega, \tag{2.4}$$

where $a = (a^{j,k})$ is an $N \times N$ symmetric matrix valued function with continuous entries on $\Omega$ satisfying the (uniform) ellipticity condition

$$\sum_{j,k} a^{j,k}(x)\xi_j\xi_k \geqslant \nu|\xi|^2, \ \forall x \in \Omega, \ \forall \xi \in \mathbb{R}^N, \tag{2.5}$$

where $\nu > 0$ is a uniform constant.

These equations are of the type (2.1) associated to the following Hamiltonian:

$$F(x,s,p,Q) := -\mathrm{Tr}(A(x)Q) + H(x,s,p),$$

where $(x,s,p,Q) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$.

Then it is easy to see that the degenerate ellipticity condition for $F$ i.e. the monotonicity property of $F$ with respect to the partial order on symmetric matrices is a generalisation of the ellipticity condition (2.5) as the following exercise shows.

*Exercise 2.1.* — Let $A \in \mathcal{S}_N$ be a real symmetric matrix of order $N$ such that for any $Q \in \mathcal{S}_N$ with $Q \geqslant 0$, we have $Tr(A \cdot Q) \geqslant 0$. Then $A \geqslant 0$.

When $H(x,p) =< b(x), p > +c(x)s + d(x)$, where $b : \Omega \longrightarrow \mathbb{R}^N$ is continuous vector field and $c, d : \Omega \longrightarrow \mathbb{R}$ are a continuous functions, the equation is a linear second order PDE.

The simplest and fundamental example is the Laplace equation equation $-\Delta = f$ or more generally the Helmholtz equation given by $-\Delta u + cu = f$, where $c \in \mathbb{R}$ is a constant.

Denote by $\Delta_c =:= -\Delta + c$ the Helmholtz operator. Then it is well know that the equation $\Delta_c u = f$ has a weak solution $u \in W_0^{1,2}(\Omega)$ for any $f \in L^2(\Omega)$ iff $-c \notin \Lambda$, where $\Lambda \subset \mathbb{R}^+$ is the spectrum of the operator $-\Delta$ (for the Dirichlet problem with zero boundary values) which is known to be a discrete sequence of positive real numbers $\lambda_k \nearrow +\infty$ (this follows from Fredholm's aternative).

In particular when $c \geqslant 0$, the equation $-\Delta u + cu = f$ has a weak solution $u \in W_0^{1,2}(\Omega)$ when $f \in L^2(\Omega)$. Moreover the weak solution is unique since the elliptic operator $\Delta_c$ satisfies the maximum principle precisely when $c \geqslant 0$. Recall also that by Schauder's theory for elliptic operators the solutions are smooth whenever $f$ is smooth (see [44]).

A typical example of non linear but quasi-linear second order elliptic equation is the following one

$$-\varepsilon \Delta u + H(x, u, Du) = 0, \tag{2.6}$$

where $\varepsilon > 0$ is small. This equation can be considered as a small perturbation of the Hamilton-Jacobi equation $H(x, u, Du) = 0$. The small perturbation term $-\varepsilon \Delta u$ is called a *viscosity term* (in Fluid mechanics). In standard cases, the equation is uniformly elliptic and then it's possible to find a unique $C^2$-smooth solution $u_\varepsilon$ of the equation (2.6) with suitable boundary conditions and get uniform $L^\infty-$estimates of $u_\varepsilon$ and $\nabla u_\varepsilon$ independent of $\varepsilon > 0$. This implies by Ascoli's theorem that some subsequence will converge uniformly to a continuous function $u$, but the corresponding subsequence $\nabla u_\varepsilon$ will converge only weakly in $L^\infty$. This is however not sufficient to pass to the limit in (2.6) as $\varepsilon \searrow 0$ to get a generalized or weak solution to the Hamilton-Jacobi equation (2.2). Nevertheless, it is reasonable to consider that the function $u$ should be a solution of the equation (2.6) in some sense. Indeed, we can show by using an easy stability argument for viscosity solutions, that it will be possible to pass to the limit in the sense of viscosity and get a viscosity solution to the Hamilton-Jacobi equation (2.2). This method, known as the "vanishing viscosity method", motivates the introduction of the viscosity concepts and justifies the terminology of viscosity (see [33]).

This method can be applied to the Eikonal equation and explains why we should consider the two hamiltonians $H(u) = |u'| - 1$ and $\tilde{H}(u) = 1 - |u'|$ as different since the corresponding elliptic perturbations approximating them are different.

**Example 3 : Degenerate Real Monge-Ampère equations**
This equation is of the following type

$$-\det(D^2 u) + f(x, u, Du) = 0, \text{ in } \Omega, \tag{2.7}$$

where $\Omega \subset \mathbb{R}^N$ is a convex domain, the solution being a convex function $u : \Omega \longrightarrow \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^+$ is a continuous non negative function on $\Omega$, non decreasing in $u$.
The above equation in degenerate elliptic if restricted to an appropriate convex subset of the space of symmetric matrices. Namely if we define the Hamiltonian function as follows

$$F(x, s, p, Q) := -\det(Q) + f(x, s, p), \text{ if } Q \geqslant 0 \text{ and } F(x, s, p, Q) = +\infty, \text{ if not.}$$

Then $F$ is lower semi-continuous on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_n$, continuous on its domain $\{F < +\infty\}$ and the equation $F(x, u, Du, D^2 u) = 0$ is degenerate elliptic.

**Example 4 : Degenerate Complex Monge-Ampère equations**
We will consider degenerate complex Monge-Ampère equations on open sets $\Omega \subset \mathbb{C}^n$:

$$-\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) + f(z, u, Du) = 0, \text{ in } \Omega, \tag{2.8}$$

the solution should be a bounded plurisubharmonic function $u : \Omega \longrightarrow \mathbb{R}$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}^+$ is a non negative continuous function, monotone increasing in $u$. This equation can be written as $-(dd^c u)^n + f(z, u, Du)\beta^n = 0$. As in the real case, this equation is degenerate elliptic when restricted to an appropriate convex subset of the space $\mathcal{H}_n$ of hermitian matrices. More precisely, identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, let us define the following Hamiltonian function:

$$F(z, s, p, Q) := -\det(Q^{1,1}) + f(z, s, p), \text{ if } Q^{1,1} \geqslant 0,$$

and $F(z, s, p, Q) = +\infty$, if not, where $(z, s, p, Q) \in \Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{S}_{2n}$ and $Q^{1,1} \in \mathcal{H}_n$ is the hermitian $(1,1)$-part of $Q \in \mathcal{S}_{2n}$ considered as a real quadratic form $Q$ on $\mathbb{C}^n$.

Then again $F$ is a lower semi-continuous degnerate elliptic hamiltonian function on $\Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{S}_{2n}$, continuous on its domain $\{F < +\infty\}$ and the equation can be written as $F(x, u, Du, dd^c u) = 0$ , where $dd^c u(x)$ is the complex hessian of $u$ i.e. precisely the hermitian $(1,1)$-part of the quadratic form $D^2 u(x) \in \mathcal{S}_{2n}$.

We are mainly interested in degenerate complex Monge-Ampère equations on a compact Kähler manifold $X$ of the following type

$$-(\omega + dd^c \varphi)^n + e^{\varepsilon \varphi} \mu = 0,$$

where $\omega \geqslant 0$ is a closed real semi-positive $(1,1)-$form on $X$ such that $\int_X \omega^n > 0$ and $\mu \geqslant$ is a continuous volume form on $X$ such that $\int_X \mu = \int_X \omega^n$.

Locally this equation can be written as a complex Monge-Ampère equation of the type considered (2.8), so the degenerate ellipticity condition will be satisfied in an appropriate sense as we will see in the next section.

### 2.3. Definitions of viscosity concepts

We want to consider fully non linear degenerate elliptic equations. As we have seen above, we will mainly consider equations for which we cannot expect in general to find classical solutions (i.e. smooth) or generalized (i.e. in Sobolev spaces) or even weak solutions (i.e. distributions).

On the other hand, it is well know that the classical Maximum Principle is a fundamental tool in the study of (uniformly) elliptic and parabolic equations, when using Schauder theory to get smooth solutions. Indeed the basic idea for solving these equations with prescribed boundary conditions (e.g. in the Dirichlet problem) lies in the construction of ad hoc barriers i.e. subsolutions and supersolutions satisfying the prescribed boundary conditions and the possibility to compare them by using the Maximum Principle.

Therefore we need to define a new notions of “weak” subsolution and supersolution and find a substitute for the classical maximum principle which allows to prove uniqueness of the a solution when subsolutions and supersolutions with appropriate boundary conditions exist. Once the Comparison Principle holds, the existence is usually proved using the Perron method of envelopes of subsolutions.

We will assume in all the rest of this paper that the function $F$ satisfies the following two important conditions which will play a fundamental role in establishing the Viscosity Comparison Principle to get uniqueness of the solution.

**Hypotheses :**

*1. Degenerate ellipticity condition* : $\forall x, \in \Omega,\ s \in \mathbb{R},\ p \in R^N,\ Q_1, Q \in \mathcal{S}_N$,

$(DEC)$ $\qquad Q \geqslant 0 \Longrightarrow F(x, s, p, Q_1 + Q) \leqslant F(x, s, p, Q_1).$

*2. Properness condition* : $\forall x \in \Omega, \forall (s_1, s_2) \in \mathbb{R}^2, \forall p \in R^n, \forall Q \in \mathcal{S}_N$,

$(PRC)$ $\qquad s_1 \leqslant s_2 \Longrightarrow F(x, s_1, p, Q) \leqslant F(x, s_2, p, Q).$

Observe that this last condition is satisfied when $F$ does not depend on $u$, but in this case it is sometimes harder to prove a comparison principle.

A function $F$ satisfying the degenerate ellipticity condition $(DEC)$ and the properness condition$(PRC)$ will be called a *Hamiltonian function* and the equation (2.1) will be called the degenerate elliptic equation associated to the Hamiltonian function $F$.

It is important to understand that as in the linear case, when $F$ is a Hamiltonian function in the above sense, the function $-F$ is not unless it does not depend neither on $u$ nor on $D^2 u$. So this means that the methods of viscosity can be applied to $F$ but not to $-F$. And even when the function $F$ depends only on $Du$ as for the Eikonal example, we should distinguish between the two equations.

The fundamental idea behind the notion of viscosity solution is provided by the following elementary result which emphasizes the role of the Maximum principle and will serve as a motivation for the general definition to be introduced below.

PROPOSITION 2.2 *(Smooth solutions). — Assume that $F$ is degenerate elliptic and let $u \in C^2(\Omega)$. Then we have the following properties:*
*1. The function $u$ is a classical subsolution of the equation (2.1)) iff the following condition holds :*
**(Sub)***: For any $x_0 \in \Omega$ and any $C^2-$smooth function $\varphi$ in a neighbourhood of $x_0$ such $u - \varphi$ takes its local maximum at $x_0$ (we will say that $\varphi$ touches $u$ from above at $x_0$ and write $u \leqslant_{x_0} \varphi$) we have*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant 0.$$

*2. The function $u$ is a classical supersolution of the equation (2.1)) iff the following condition holds :*
**(Super)***: For any $x_0 \in \Omega$ and any $C^2-$smooth function $\psi$ in a neighbourhood of $x_0$ such $u - \psi$ takes its local minimum at $x_0$ (we will say that $\psi$ touches $u$ from below at $x_0$ and write $u \geqslant_{x_0} \psi$) we have*

$$F(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)) \geqslant 0.$$

A $C^2-$smooth function $\varphi$ in a neighbourhood of $x_0$ satisfying the condition $u \leqslant_{x_0} \varphi$ is called an upper test function for $u$ at $x_0$ and a $C^2-$smooth function $\psi$ in a neighbourhood of $x_0$ satisfying the condition $u \geqslant_{x_0} \psi$ is called a lower test function for $u$ at $x_0$.

This result shows that the application of the classical maximum principle and the use of the degenerate ellipticity condition allows to transfer the differentiation from $u$ to upper and lower $C^2$-test functions in a neighbourhood of each point and ask for the differential inequalities $F \leqslant 0$ and $F \geqslant 0$ to hold for the corresponding test function at the given point.

*Proof.* — It is enough to prove the first part. It is clear that he condition **(Sub)** is sufficient for $u$ to be a classical subsolution. Indeed, since $u$ is $C^2$, it can be taken as an upper test function at any point and then it satisfies the corresponding differential inequality.

Let us prove that the condition **(Sub)** is necessary for $u$ to be a classical solution. Indeed assume that $\varphi$ be a $C^2-$smooth function in a neighbourhood of $x_0$ such that $u \leqslant_{x_0} \varphi$. Then $u - \varphi$ is a $C^2-$smooth function in a neighbourhood of $x_0$ which attains its local maximum at $x_0$. By the local maximum principle we have $D(u-\varphi)(x_0) = 0$ and $D^2(u-\varphi)(x_0) \leqslant 0$ in the sense of quadratic forms (or symmetric matrices). Since $D\varphi(x_0) = Du(x_0)$ and $D^2u(x_0) \leqslant D^2\varphi(x_0)$ in the sense of symmetric matrices and $u$ is a classical subsolution, it follows from the degenerate ellipticity condition that

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqslant F(x_0, u(x_0), Du(x_0), D^2u(x_0)) \leqslant 0,$$

which proves the condition **(Sub)**. 2. □

Observe that the main feature of this characterization is to show that the conditions (**Sub**) and (**Super**) use only the values of $u$ but not its first nor second derivatives. Therefore it can be used as a motivation for the following general definitions.

Definition 2.3. — *1. Let $u : \Omega \longrightarrow \mathbb{R}$ be an upper semi-continuous (usc) function in an open set $\Omega \subset \mathbb{R}^N$. We say that $u$ is a viscosity subsolution of the equation $F(x, u, Du, D^2u) = 0$ on $\Omega$ if it satisfies the condition* **(Sub)**. *We will also say that $u$ satisfies the differential inequality $F(x, u, Du, D^2u) \leqslant 0$ in the VSC sense on $\Omega$.*
*2. Let $u : \Omega \longrightarrow \mathbb{R}$ be a lower semi-continuous (lsc) function in an open set $\Omega \subset \mathbb{R}^N$. We say that $u$ is a viscosity supersolution of the equation $F(x, u, Du, D^2u) = 0$ on $\Omega$ if it satisfies the condition* **(Super)**. *We will also say that $u$ satisfies the differential inequality $F(x, u, Du, D^2u) \geqslant 0$ in the VSC sense on $\Omega$.*

To illustrate the importance of the Properness condition ($PRC$), let us give a simple case where it helps to prove uniqueness.

THEOREM 2.4. — *Let $\Omega \Subset \mathbb{R}^N$ be a bounded domain and assume that the Hamiltonian function $F(x, s, p, Q)$ is strictly increasing in the variable $s$. Then the classical comparison principle holds i.e. if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a classical subsolution of the equation (2.1) and $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a classical supersolution of the equation (2.1) such that $u \leqslant v$ on $\partial\Omega$ then $u \leqslant v$ on $\Omega$. In particular the equation (2.1) has at most one classical solution with prescribed continuous boundary values.*

*Proof.* — Since $u - v$ is continuous on the compact set $\overline{\Omega}$, it attains its maximum at some point $x_0 \in \Omega$ i.e. $\max_{\overline{\Omega}}(u-v) = u(x_0) - v(x_0)$. If $x_0 \in \partial\Omega$ then $u(x_0) \leqslant v(x_0)$ and then we are done. Now assume that $x_0 \in \Omega$. Since $u$ and $v$ are $C^2$ at $x_0 \in \Omega$, it follows from the classical maximum principle that $Du(x_0) = Dv(x_0)$ and $D^2u(x_0) \leqslant D^2v(x_0)$. Now since $u$ is a classical subsolution we have

$$F\left(x_0, u(x_0), Du(x_0), D^2u(x_0)\right) \leqslant 0 \leqslant F\left(x_0, v(x_0), Dv(x_0), D^2v(x_0)\right).$$

Therefore by the degenerate ellipticity condition (**DEC**) we have

$$F(x_0, u(x_0), Du_0(x), D^2u(x_0)) \leqslant F(x_0, v(x_0), Du_0(x), D^2u(x_0)).$$

From the Properness condition (**PRC**) it follows that $u(x_0) \leqslant v(x_0)$. □

Observe that the simple reasoning above uses the fact that $F$ is increasing in a crucial way. In the situation where $F$ does not depend on $u$ for example, we cannot conclude so easily. However one can show that the conclusion is still true but the poof requires a more subtle argument based on a more refined Maximum Principle known as the Alexandroff-Backelman-Pucci maximum principle (see [31], [68]).

*Remark 2.5.* — Observe that in the last result it is enough to assume that only one of the functions is a classical subsolution or a classical supersolution. Indeed assume for example that $u$ is a classical subsolution, while $v$ is a viscosity supersolution with $u \leqslant v$ on $\partial\Omega$. Then arguing as above, we get the inequality $u(x) - u(x_0) + v(x_0) \leqslant v(x)$ in a neighbourhood of $x_0$; which means that the $C^2-$function $q(x) := u(x) - u(x_0) + v(x_0)$ is a lower test function for $v$ at $x_0$. Therefore the VSC inequality for $v$ at $x_0$ implies that

$$F(x_0, v(x_0), Dq(x_0), D^2q(x_0) \geqslant 0.$$

On the other hand, since $u$ is a classical subsolution, we have

$$F(x_0, u(x_0), Du(x_0), D^2u(x_0) \leqslant 0.$$

Comparing these two inequalities we get

$$F(x_0, u(x_0), Du(x_0), D^2u(x_0) \leqslant F(x_0, v(x_0), Dq(x_0), D^2q(x_0).$$

Since $F$ is strictly increasing, it follows that $u(x_0) \leqslant v(x_0)$.

The main goal of the first part of this lecture is to prove a general comparison principle for viscosity solutions. We want to do the same reasoning as above, but our functions are not smooth. We therefore need to approximate them keeping the memory of the viscosity differential inequalities they satisfy. This will be done in the next section.

## 2.4. Characterization of viscosity concepts by mean of jets

As we have seen in the previous proofs, the only important thing that matters for the differential inequalities we were proving is the jet of order 2 of the function $u$ at a given point. Therefore to deal with non smooth functions it is useful to develop a sub-differential calculus and define sub(super)-jets of order 2. This will lead to a characterization of viscosity concepts by means of sub(super)-jets of order 2, which is more flexible.

Definition 2.6. — *1. Let $u : \Omega \longrightarrow \mathbb{R}$ be an usc function and $x_0 \in \Omega$. The super-differential jet of order 2 of $u$ at $x_0$ is the set $J^{2,+}u(x_0)$ of all $(p, Q) \in \mathbb{R}^N \times \mathcal{S}_N$ such that for any $\xi \in \mathbb{R}^N$ with $|\xi| << 1$, the following inequality holds*

$$u(x_0 + \xi) \leqslant u(x_0) + p \cdot \xi + \frac{1}{2} < Q \cdot \xi, \xi > +o(|\xi|^2).$$

*2. Let $u : \Omega \longrightarrow \mathbb{R}$ a lsc function and $x_0 \in \Omega$. The sub-differential jet of order 2 of $u$ at $x_0$ is the set $J^{2,-}u(x_0)$ of all $(p, Q) \in \mathbb{R}^N \times \mathcal{S}_N$ such that for any $\xi \in \mathbb{R}^N$ with $|\xi| << 1$, the following inequality holds*

$$u(x_0 + \xi) \geqslant u(x_0) + p \cdot \xi + \frac{1}{2} < Q \cdot \xi, \xi > +o(|\xi|^2).$$

*3. If $u$ is continuous we can define the differential jet of order 2 of $u$ at $x_0$ as the set $J^2u(x_0) := J^{2,+}u(x_0) \cap J^{2,-}u(x_0)$.*

Observe that if $u$ is twice differentiable at $x_0$ then

$$J^{2,+}u(x_0) = \{(Du(x_0), Q); Q \geqslant D^2u(x_0)\},$$

and

$$J^{2,-}u(x_0) = \{(Du(x_0), Q); Q \leqslant D^2u(x_0)\},$$

so that $J^2u(x_0) = \{(Du(x_0), D^2u(x_0)\}$.

For an arbitrary upper semi-continuous function, it may happen that the set $J^{2,+}u(x_0)$ is empty. However there are many points nearby where this set is not empty as the following remark shows.

*Remark 2.7.* — Observe that the function $u(x) := |x|$ is a convex, non negative function which satisfies the condition $J^2 u(0) = \emptyset$.

Actually the set $\{x \in \Omega; J^{2,+}u(x) \neq \emptyset\}$ is dense in $\Omega$. Indeed, fix a point $x_0 \in \Omega$. Since $u$ is upper semi-continuous at $x_0$, for any ball $B = B(x_0, r) \Subset \Omega$ with $r > 0$ small enough there exists $A > 0$ such that $u(x) - A|x - x_0|^2 < u(x_0)$ for $|x - x_0| = r$. Then defining the function $q$ by $q(x) := A|x - x_0|^2$, we see by upper semi-continuity that the function $u - q$ takes its maximum $M$ in $\bar{B}$ at some point $\hat{x} \in \bar{B}$. Now observe that if $|\hat{x} - x_0| = r$ then $M = u(\hat{x}) - q(\hat{x}) < u(x_0) = u(x_0) - q_A(x_0)$, which contradicts the fact that $M$ is the maximum of $u$ in the ball $\bar{B}$. Therefore $\hat{x} \in B$ and then the function $\hat{q} := q - q(\hat{x}) + u(\hat{x})$ is a $C^2$-smooth upper test function for $u$ at the point $\hat{x} \in B$ which means that $(D\hat{q}(\hat{x}), D^2\hat{q}(\hat{x})) \in J^{2,+}u(\hat{x})$.

As we will see the fundamental theorem of Alexandrov says that for a convex function function, the set $\{x \in \Omega; J^{2,+}u(x) \neq \emptyset\}$ is not only dense in $\Omega$ but it is of full Lebesgue measure in the sense that its complement in $\Omega$ is of Lebesgue measure 0 (see Theorem 2.15). The same remarks holds for a lower semi-continuous function which is bounded from below.

Since viscosity sub(super)-solutions are only usc(lsc) functions, it is necessary to extend the previous definitions by introducing the notions of approximate super(sub)-differential jets.

Definition 2.8. — *Let $u : \Omega \longrightarrow \mathbb{R}$ be an usc function and $x_0 \in \Omega$ and $(p, Q) \in \mathbb{R}^N \times \mathcal{S}_N$. We say that $(p, Q) \in \bar{J}^{2,+}u(x_0)$ if there exists a sequence of points $y_j \to x_0$ in $\Omega$ and a sequence $(p_j, Q_j) \in J^{2,+}u(y_j)$ such that $(p_j, Q_j) \to (p, Q)$. In the same way we define $\bar{J}^{2,-}u(x_0)$ for a lower semi-continuous function $u : \Omega \longrightarrow \mathbb{R}$.*

Then we have the following important characterization of viscosity solutions which will be useful.

Theorem 2.9. — *1. Let $u : \Omega \longrightarrow \mathbb{R}$ be an usc function and $x_0 \in \Omega$. Then $u$ is a viscosity subsolution of the equation $F(x, u, Du, D^2u) = 0$ if and only if for any $x_0 \in \Omega$ and any $(p, Q) \in \bar{J}^{2,+}u(x_0)$, we have $F(x_0, u(x_0), p, Q) \leqslant 0$.*
*2. Let $u : \Omega \longrightarrow \mathbb{R}$ be a lsc function and $x_0 \in \Omega$. Then $u$ is a viscosity supersolution of the equation $F(x, u, Du, D^2u) = 0$ if and only if for any $x_0 \in \Omega$ and any $(p, Q) \in \bar{J}^{2,-}u(x_0)$, we have $F(x_0, u(x_0), p, Q) \geqslant 0$.*

*Proof.* — It is enough to prove the first claim. To prove that the condition is sufficient, it is enough to prove that if $\phi$ is an upper test function for $u$ at some point $x_0$ then $(D\phi(x_0), D^2\phi(x_0)) \in J^{2,+}u(x_0)$. Indeed by Taylor's formula for $|\xi| << 1$ and $x = x_0 + \xi \in \Omega$, we have

$$\phi(x) = \phi(x_0) + D\phi(x_0).\xi + \frac{1}{2}D^2\phi(x_0) \cdot (\xi, \xi) + o(|\xi|^2)$$

Since $u \leqslant_{x_0} \phi$ with $u(x_0) = \phi(x_0)$, it follows that for $|\xi| << 1$,

$$u(x) \leqslant u(x_0) + D\phi(x_0).\xi + \frac{1}{2}D^2\phi(x_0) \cdot (\xi, \xi) + o(|\xi|^2)$$

which proves that $(D\phi(x_0), D^2\phi(x_0)) \in J^{2,+}u(x_0)$.

To prove the converse it is enough to assume that $(p, Q) \in J^{2,+}u(x_0)$, since by approximation the results will follow by lower semi-continuity of $F$. This is less trivial and follows from the following elementary but non trivial lemma (see [39], [33]). □

LEMMA 2.10. — *For any $(p, Q) \in J^{2,+}(u)$ there exists a $C^2$ function near $x_0$ such that $D\phi(x_0) = p$, $D^2\phi(x_0) = Q$ and $u \leqslant_{x_0} \phi$ i.e. $J^2\phi(x_0) = \{(p, Q)\}$.*

Let us come back to the following simple example to show that viscosity concepts are the right ones to ensure uniqueness of the solutions.

*Example 2.11.* — We have already observed the advantage of VSC solutions in exhibiting the solution to the equation $F(x, u, Du, D^2u) = H(u') = |u'| - 1 = 0$ on $[-1, +1]$ with the the boundary condition $u(\pm 1) = 0$. The piecewise affine function on $[-1, 1]$ defined by $u_0(x) = 1 - |x|$, which satisfies $|u_0'(x)| = 1$ except at the origin where it is not differentiable, it is a generalized solution. Observe that the equation has infinitely many piecewise affine generalized solutions with the prescribed boundary condition. However it is not difficult to see that among these generalized solutions, $u_0$ is the only one which is a viscosity solution for the equation associated to the Hamiltonian $H(x, u, u') = |u'| - 1$. Indeed observe that the only problem is at the origin. It's easy to see that any upper test function $q$ for u at 0 satisfies the condition $|q'(0)| \leqslant 1$, while there is no lower test function for $u$ at the origin.

On the other hand, it is also clear that $u_0$ is not a subsolution to the equation $1 - |u'| = 0$, since any upper test function $q$ at the origin should satisfy the inequality $1 \leqslant |q'(0)|$, while by the previous observation it has to satisfy the inequality $|q'(0)| \geqslant 1$ hence $|q'(0)| = 1$, which is obviously not the case. The same reasoning as above actually proves that the function $v_0(x) = |x| - 1$ is a VSC solution to the equation $1 - |u'| = 0$ with boundary values 0.

## 2.5. The Jensen-Ishii maximum principle

Let us recall some classical definitions and results used in this approach (see [39], [24], [33]). As we have seen in the case of Monge-Ampère equations it is necessary to assume that our Hamiltonian function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \longrightarrow \mathbb{R} \cup \{+\infty\}$ will be a lower semi-continuous function which is continuous on its domain $\{F < +\infty\}$.

Definition 2.12. — *Let $\varphi : \Omega \longrightarrow \mathbb{R}$ be a function defined in an open set $\Omega \subset \mathbb{R}^N$. The function $\varphi$ is said to be semi-convex on $\Omega$ if there exists a real number $k > 0$ such that the function $x \longmapsto \varphi(x) + \frac{k}{2}|x|^2$ is convex in (each convex subset of) $\Omega$. In this case we also say that $\varphi$ a $k-$convex function in $\Omega$. The function $\varphi$ is said to be $k-$concave in $\Omega$ if $-\varphi$ is $k-$convex in $\Omega$.*

The following notion is quite useful in the context of the viscosity approach.

Definition 2.13. — *We say that a function $\varphi : \Omega \longrightarrow \mathbb{R}$ is twice differentiable at some point $x_0$ (in the sense of Alexandrov) if there exists $p \in \mathbb{R}^N$ and $Q \in \mathcal{S}_N$ such that for $|\xi| << 1$,*

$$(A) \qquad w(x_0 + \xi) = w(x_0) + < p, \xi > + \frac{1}{2} < Q \cdot \xi, \xi > + o(|\xi|^2).$$

Some remarks are in order.

*Remark 2.14.* — 1. The condition $(A)$ means that $J^2 w(x_0) = \{(p, Q)\}$. This implies that $w$ is differentiable at $x_0$ and $Dw(x_0) = p$, but in general it does not mean that $u$ is twice differentiable in the usual sense at $x_0$. Actually $w$ do not need to be differentiable in a neighbourhood of $x_0$. However the quadratic form $Q$ satisfying the asymptotic expansion $(A)$ at $x_0$ is unique and given by

$$Q(\xi) = \lim_{t \to 0} \frac{w(x_0 + t\xi) + w(x_0 - t\xi) - 2w(x_0)}{t^2},$$

for $\xi \in \mathbb{R}^N$. We will denote the quadratic form $Q$ by $Q = D^2 w(x_0)$ and then $J^2 w(x_0) = \{(Dw(x_0), D^2 w(x_0))\}$.
2. It follows from the definitions that if $w$ is a $k$-convex function in $\Omega$ which is twice differentiable at $x_0 \in \Omega$ then $D^2 w(x_0) \geqslant -k I_N$ in the sense of quadratic forms on $\mathbb{R}^N$.

The following fundamental result will be useful in the poof of the generalized maximum principle.

Theorem 2.15 *(Alexandrov [1]).* — *Let $\varphi$ be a $k-$convex function in $\Omega$. Then there exists an exceptional Borel set $E \subset \Omega$ of Lebesgue measure 0 such that $\varphi$ is twice differentiable at any point $x_0 \in \Omega \setminus E$, hence $D^2\varphi(x_0) \geqslant -kI_N$.*

Let us now state Jensen's maximum principle ([54]) which is the main technical result used in the proof of the comparison principle. It is based on the Alexandrov fundamental theorem.

Theorem 2.16. — *Let $w$ be a semi-convex function in an open set $\Omega \subset \mathbb{R}^N$. Assume that the function $w$ reaches its local maximum at some point $x_0 \in \Omega$. Then there exists a sequence $a_j \to x_0$ in $\Omega$ such that $w$ is twice differentiable at each $a_j$ and $(Dw(a_j), D^2w(a_j)) \to (0, Q^+)$ in $\mathbb{R}^N \times \mathcal{S}_N$ and $Q^+ \leqslant \mathbf{0}$, in particular $(0, Q^+) \in \bar{J}^{2,+}w(x_0)$.*

Viscosity sub(super)-solution of our equations need not to be even continuous in general. So to be able to extract some informations from the viscosity differential inequalities they satisfy, it is necessary to approximate them by smooth functions in an appropriate way keeping memory of these differential inequalities. This can be done using sup(inf)-convolution.

Let $u : \Omega \longrightarrow \mathbb{R}$ be a bounded upper semi-continuous function. For $\varepsilon > 0$ small enough and $x \in \Omega_\varepsilon$, we define the sup-convolutions of $u$ as follows:

$$u^\varepsilon(x) := \sup_{y\in\Omega} \left\{ u(y) - \frac{1}{2\varepsilon^2}|y-x|^2 \right\} = \sup_{\{|y-x|\leqslant A\varepsilon\}} \left\{ u(y) - \frac{1}{2\varepsilon^2}|y-x|^2 \right\},$$

where $A > 0$ is large enough so that $A^2 > 2\mathrm{osc}_\Omega u$.

In the same way if $v : \Omega \longrightarrow \mathbb{R}$ is a bounded lower semi-continuous function. For $\varepsilon > 0$ small enough and $x \in \Omega_\varepsilon$, we define the inf-convolutions of $u$ as follows:

$$v_\varepsilon(x) := \inf_{y\in\Omega_\varepsilon} \left\{ v(y) + \frac{1}{2\varepsilon^2}|y-x|^2 \right\} = \inf_{|y-x|\leqslant\varepsilon} \left\{ v(y) + \frac{1}{2\varepsilon^2}|y-x|^2 \right\}.$$

Then it easy to show the following result (see [31]).

Proposition 2.17. — *1. Let $u : \Omega \longrightarrow \mathbb{R}$ be a bounded upper semi-continuous function. Then for $0 < \varepsilon < 1$ small enough, $u_\varepsilon$ is $\varepsilon^{-2}$-convex in $\Omega_\varepsilon$ and decreases to $u$ in $\Omega$ as $\varepsilon \searrow 0$, hence it is twice differentiable* *at almost every point in* $\Omega$*. Moreover* $u_\varepsilon$ *is a subsolution of the equation* $F_\varepsilon(x, w, Dw, D^2 w) = 0$*, where*

$$F_\varepsilon(x, s, p, Q) := \inf\{F(y, s, p, Q); |y - x| \leqslant A\varepsilon\}.$$

*2. Let* $v : \Omega \longrightarrow \mathbb{R}$ *be a bounded lower semi-continuous function. Then for* $0 < \varepsilon < 1$ *small enough,* $v_\varepsilon$ *is* $\varepsilon^{-2}$*-concave in* $\Omega_\varepsilon$ *and increases to* $v$ *in* $\Omega$ *as* $\varepsilon \searrow 0$*, hence twice differentiable at almost every point in* $\Omega$*. Moreover* $v_\varepsilon$ *is a supersolution of the equation* $F^\varepsilon(x, w, Dw, D^2 w) = 0$*, where*

$$F^\varepsilon(x, s, p, Q) := \sup\{F(y, s, p, Q); |y - x| \leqslant A\varepsilon\}.$$

Observe that $F_\varepsilon$ (resp. $F^\varepsilon$) is a continuous Hamiltonian in its domain which increases (resp. decreases) to $F$ in $\Omega$ as $\varepsilon$ decreases to 0.

Using the above result, it is possible to derive a more general maximum principle for upper semi-continuous functions, called Ishii's lemma in the literature. We will refer to it as the Jensen-Ishii's maximum principle, because it is based on a powerful idea of Jensen [54].

THEOREM 2.18. — *Let* $u, v : \Omega \longrightarrow \mathbb{R}$ *be two bounded functions defined in a domain* $\Omega \subset \mathbb{R}^N$ *such that* $u$ *an upper semi-continuous function in* $\Omega$ *and* $v$ *is a lower semi-continuous function in* $\Omega$*. Let* $\phi : \Omega \times \Omega \longrightarrow \mathbb{R}$ *be a* $C^2$*-smooth function. Assume that the function* $w(x, y) := u(x) - v(y) - \phi(x, y)$ *has a local maximum at some point* $(a, b) \in \Omega \times \Omega$*. Then for any* $\alpha > 0$ *there exists* $Q^+, Q^- \in \mathcal{S}_N$ *such that* $(p^+, Q^+) \in \bar{J}^{2,+}u(a)$*,* $(p_-, Q^-) \in \bar{J}^{2,-}v(b)$ *such that* $p^+ = D_x\phi(a, b), p_- = -D_y\phi(a, b)$ *and*

$$-(\frac{1}{\alpha} + \|A\|)I_{2N} \leqslant M(Q^+, -Q^-) \leqslant A + \alpha A^2,$$

*where* $A = D^2\phi(a, b)$*, and* $M(Q^+, -Q^-)$ *is defined as a quadratic form on* $\mathbb{R}^N \times \mathbb{R}^N$ *as follows: if* $Z = (X, Y) \in \mathbb{R}^N \times \mathbb{R}^N$*, then*

$$< M(Q^+, -Q^-) \cdot Z, Z >=< Q^+ \cdot X, X > - < Q^- \cdot Y, Y > .$$

*In particular we have* $Q^+ \leqslant Q^-$ *as quadratic forms on* $\mathbb{R}^N$ *if we choose* $\phi$ *so that* $D_x^2\phi(x, y) = -D_y^2\phi(x, y)$*.*

A complete proof is given in [33]. It uses the regularization by sup/inf convolution and the maximum principle of Jensen. We will see in the next section how it is used to prove the comparison principle.

## 2.6. The viscosity comparison principle in the local case

The main tool for proving uniqueness of solutions with boundary values conditions is the so called (viscosity) Comparison Principle which we will state now.

DEFINITION 2.19. — *We say that the (viscosity) Comparison Principle holds for the equation $F(x,u,Du,D^2u) = 0$ if for any bounded (viscosity) subsolution $u$ in $\Omega$ and any bounded (viscosity) supersolution $v$ a in $\Omega$ such that $u \leqslant v$ on $\partial\Omega$ then $u \leqslant v$ on $\Omega$.*

Under some additional conditions on how the function $F$ depends on the $u$ variable and its gradient , it is possible to prove the comparison principle for the equation $F = 0$ using Jensen-Ishii's Maximum principle (see [33]). Unfortunately there no general satisfactory statement which can be applied in our case. So we will not state any such result here and refer to [53, 33] for various statements.

However we will use the same ideas in the next section and rely on Jensen-Ishii's Maximum principle to prove a comparison principle adapted to the complex Monge-Ampère equations we are considering.

Let us mention that the Comparison Principle implies uniqueness of the viscosity solution with prescribed boundary values. Once the comparison principle is valid, it is quite easy to deduce existence of viscosity solutions using the Perron method (see [33]).

THEOREM 2.20. — *Assume that the family $\mathcal{U}$ of bounded subsolution of the equation $F(x,u,Du,D^2u) = 0$ in $\Omega$ in non empty and locally upper bounded in $\Omega$. Then the function defined by*

$$U := \sup\{u; u \in \mathcal{U}\}$$

*is the maximal subsolution of the equation $F(x,u,Du,D^2u) = 0$ in $\Omega$.*

*Moreover if the viscosity Comparison Principle holds for the equation $F(x,u,Du,D^2u) = 0$ in $\Omega$ and there exists a subsolution $\underline{u}$ and a supersolution $\overline{u}$ such that $\underline{u}_* = \overline{u}^*$ in $\partial\Omega$, then $U$ is the unique viscosity solution de the equation $F(x,u,Du,D^2u) = 0$ with boundary values $U = \underline{u}_* = \overline{u}^*$ in $\partial\Omega$.*

*Proof.* — A complete proof is given in [33]. Let us just give an idea of the proof. The fact that $U$ is a subsolution is a standard fact: on shows that the upper semi-continuous regularization $U^*$ is actually a subsolution of the equation $F = 0$, which implies that $U^* \in \mathcal{U}$ and then $U^* = U$ is a subsolution. Now the powerful idea of Ishii is to consider the lower semi-continuous regularization $\varphi_*$ of $\varphi$ and to show that it is a supersolution of the equation $F = 0$. This is done by contradiction using a bump construction argument (see [52], [33]). We will give it in details in the complex case in the next section (see Theorem 3.15). Then by the comparison principle $\underline{u} \leqslant U \leqslant \overline{u}$ in $\Omega$, hence $\underline{u}_* \leqslant U_* \leqslant \overline{u}$ in $\Omega$. Then at the boundary we will have $\underline{u}_* \leqslant U_*$ and $U^* \leqslant \overline{u}^* = \underline{u}_*$, which implies that $U \leqslant U_*$ at the boundary $\partial\Omega$. Again by the comparison principle we can conclude that $U \leqslant U_*$ in $\Omega$, which finally implies that $U = U_*$ is a viscosity solution of the equation $F(x, u, Du, D^2u) = 0$ with boundary values $U = \underline{u}_* = \overline{u}^*$ in $\partial\Omega$. Uniqueness follows from the comparison principle. □

## 3. The viscosity approach to degenerate complex Monge-Ampère equations

The purpose of this section is to make the connection between the pluripotential theory for the complex Monge-Ampère operators, as founded by Bedford-Taylor [12, 13], and the viscosity approach developed by P.L. Lions and all (see [53, 33]).

### 3.1. Viscosity subsolutions in the complex case

Let $X$ be a (connected) complex manifold of dimension $n$ and $\mu \geqslant 0$ a semi-positive volume form with continuous density with respect to a fixed smooth non degenerate volume form $\mu_0 > 0$ form on $X$. In this section $B$ will denote the unit ball of $\mathbb{C}^n$ or its image under a coordinate chart in $X$.

We will consider the following general complex Monge-Ampère type equations

$$(MAE) \qquad -(dd^c u)^n + e^{g(z,u)+h(Du)}\mu = 0,$$

where $\Omega \Subset \mathbb{C}^n$ is bounded domain, $g$ is a continuous function on $\Omega \times \mathbb{R}$ increasing in the $u$ variable, $h$ is continuous function in $X$ and $\mu$ is a continuous positive volume form on $X$.

To fit in with the viscosity point of view, we identify $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ and define the Hamiltonian function for $(z, s, p, Q) \in \Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{S}_{2n}$ by the formula

$$F(z,s,p,Q) = \left\{ \begin{array}{ll} -(dd^c Q^{1,1})^n + e^{g(z,s)+h(p)}\mu(z) & \text{if } Q^{1,1} \geqslant 0 \\ +\infty & \text{otherwise.} \end{array} \right\},$$

where $Q^{1,1}$ is the hermitian $(1,1)-$part of the (real) quadratic form $Q$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Here we identify a hermitian form with the real $(1,1)$-form associated to it.

With this notation, we see that our Hamiltonian is lower semi-continuous in $(z, s, p, Q) \in \Omega \times \mathbb{R} \times \mathbb{C}^n \times \mathcal{S}_{2n}$, continuous in its domain and satisfy the degenerate ellipticity condition as well as the properness condition stated in the previous section. Therefore we can use the notions of subsolutions and supersolutions as in the previous section and in particular we can apply Jensen-Ishii's maximum principle. Note that if $\mu \geqslant \mu'$ then $(dd^c\varphi)^n \geqslant \mu$ in the viscosity sense implies $(dd^c\varphi)^n \geqslant \mu'$. This holds in particular if $\mu' = 0$.

### 3.1.1. Subsolutions of the equation $(dd^c u)^n = \mu$

Here we restrict ourselves to the special case where $g \equiv 0$ and $h \equiv 0$ and first observe that a function $\varphi$ satisfies $(dd^c\varphi)^n \geqslant 0$ in the viscosity sense if and only if it is plurisubharmonic in $X$.

Proposition 3.1. — *The viscosity subsolutions of the complex Monge-Ampère equation $(dd^c\varphi)^n = 0$ are precisely the plurisubharmonic functions on $X$.*

*Proof.* — Let $\varphi$ be a subsolution of $(dd^c\varphi)^n = 0$. Let $x_0 \in X$ such that $\varphi(x_0) \neq -\infty$. The problem is local so we can assume that $X$ is a domain in $\mathbb{C}^n$. Let $q \in \mathcal{C}^2(V_{x_0})$ such that $\varphi - q$ has a local maximum at $x_0$. Then the hermitian matrix $Q = dd^c q_{x_0}$ satisfies $\det(Q) \geqslant 0$. Moreover for every hermitian semipositive matrix $H$, we also have $\det(Q + H) \geqslant 0$ since, a fortiori for $q_H = q + H(x - x_0)$, $\varphi - q_H$ has a local maximum at $x_0$ too.

It follows from Lemma 3.2 below that $Q = dd^c q_{x_0}$ is actually semi-positive. We infer that for every positive definite hermitian matrix $(h^{i\bar{j}})$ $\Delta_H q(x_0) := h^{i\bar{j}} \frac{\partial^2 q}{\partial z_i \partial \bar{z}_j}(x_0) \geqslant 0$, i.e. $\varphi$ is a viscosity subsolution of the equation $-\Delta_H \varphi = 0$. In appropriate complex coordinates this constant coefficient differential operator is nothing but the Laplace operator. Hence ([51] Proposition 3.2.10' p. 147) applies to the effect that $\varphi$ is $\Delta_H$-subharmonic hence is in $L^1_{loc}(V_{x_0})$ and satisfies $\Delta_H \varphi \geqslant 0$ in the sense of distributions. Let $(w^i)$ be any vector in $\mathbb{C}^n$. Consider a positive hermitian matrix $(h^{i\bar{j}})$ degenerating to the rank one matrix $(w^i \bar{w}^j)$. By continuity, we have $\sum w^i \bar{w}^j \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}$ $\geqslant 0$ in the sense of distributions. Thus $\varphi$ is plurisubharmonic.

Conversely, assume $\varphi$ is plurisubharmonic. Fix $x_0 \in X$, $q \in \mathcal{C}^2(V_{x_0})$ such that $\varphi - q$ has a local maximum at $x_0$. Then, for every small enough ball $B \subset V_{x_0}$ centered at $x_0$, we have

$$\varphi(x_0) - q(x_0) \geqslant \frac{1}{V(B)} \int_B (\varphi - q)\, dV,$$

hence
$$\frac{1}{V(B)}\int_B q\,dV - q(x_0) \geqslant \frac{1}{V(B)}\int_B \varphi\,dV - \varphi(x_0) \geqslant 0.$$
Letting the radius of $B$ tend to 0 it follows, since $q$ is $\mathcal{C}^2$ that $\Delta q_{x_0} \geqslant 0$. Using complex ellipsoids instead of balls[1], we conclude that $\Delta_H q(x_0) \geqslant 0$ for every positive definite hermitian matrix. Thus $dd^c q_{x_0} \geqslant 0$ and $(dd^c\varphi)^n \geqslant 0$ in the viscosity sense. □

The following lemma is easily proven by diagonalizing $Q$:

LEMMA 3.2. — *Let $Q$ be an hermitian matrix such that, for every semipositive hermitian matrix $H$, $\det(Q+H) \geqslant 0$ then $Q$ is semipositive.*

Recall that when $\varphi$ is plurisubharmonic and locally bounded, its Monge-Ampère measure $MA(\varphi) = (dd^c\varphi)^n$ is well defined [12] (as the unique limit of the smooth measures $MA(\varphi_j)$, where $\varphi_j$ is any sequence of smooth psh functions decreasing to $\varphi$). Our next result makes the basic connection between this pluripotential notion and its viscosity counterpart.

PROPOSITION 3.3. — *Let $\varphi$ be a locally bounded upper semi-continuous function in $X$. It satisfies $(dd^c\varphi)^n \geqslant \mu$ in the viscosity sense iff it is plurisubharmonic and its Monge-Ampère measure satisfies $MA(\varphi) \geqslant \mu$ in the pluripotential sense.*

*Proof.* — Assume $\varphi \in PSH \cap L^\infty(B)$ satisfies $MA(\varphi) \geqslant \mu$. Consider $q$ a $\mathcal{C}^2$ function such that $\varphi - q$ achieves a local maximum at $x_0$ and $\varphi(x_0) = q(x_0)$. Since $\varphi$ satisfies $(dd^c\varphi)^n \geqslant 0$ in the viscosity sense, $(dd^c q)^n_{x_0} \geqslant 0$ and $dd^c q_{x_0} \geqslant 0$ by lemma 3.2. Assume $(dd^c q)^n_{x_0} < \mu_{x_0}$. Let $q^\varepsilon := q + \varepsilon\|x - x_0\|^2$. Choosing $\varepsilon > 0$ small enough, we have $0 < (dd^c q^\varepsilon_{x_0})^n < \mu_{x_0}$. Since $\mu$ has continuous density, we can chose a small ball $B'$ containing $x_0$ of radius $r > 0$ such that $\bar{q}^\varepsilon = q^\varepsilon - \varepsilon\frac{r^2}{2} \geqslant \varphi$ near $\partial B'$ and $MA(\bar{q}^\varepsilon) \leqslant MA(\varphi)$. The comparison principle (Theorem 1.6) yields $\bar{q}^\varepsilon \geqslant \varphi$ on $B'$. But this fails at $x_0$. Hence $(dd^c q)^n_{x_0} \geqslant \mu_{x_0}$ and $\varphi$ is a viscosity subsolution.

Conversely assume $\varphi$ is a viscosity subsolution. Fix $x_0 \in M$ such that $\varphi(x_0) \neq -\infty$ and $q \in \mathcal{C}^2$ such that $\varphi - q$ has a local maximum at $x_0$. Then the hermitian matrix $Q = dd^c q_{x_0}$ satisfies $\det(Q) \geqslant \mu_{x_0}$.

Recall that the classical trick (due to Krylov) of considering the complex Monge-Ampère equation as a Bellmann equation relies on the following:

(1) This amounts to a linear change of complex coordinates.

Lemma 3.4 *[43]. — Let $Q$ be a $n \times n$ non negative hermitian matrix, then*

$$\det(Q)^{1/n} = \inf\{\mathrm{tr(HQ)} \,|\, \mathrm{H} \in \mathrm{H}_n^+ \textit{ and } \det(\mathrm{H}) = \mathrm{n}^{-\mathrm{n}}\},$$

*where $H_n^+$ denotes the set of positive hermitian $n \times n$ matrices.*

Applying this to our situation, it follows that for every positive definite hermitian matrix $H = (h_{i\bar{j}})$ with $\det(H) = n^{-n}$,

$$\Delta_H q(x_0) := \sum h_{i\bar{j}} \frac{\partial^2 q}{\partial z_i \partial \bar{z}_j}(x_0) \geqslant \mu^{1/n}(x_0),$$

i.e. $\varphi$ is a viscosity subsolution of the linear equation $\Delta_H \varphi = \mu^{1/n}$.

This is a constant coefficient linear partial differential equation. Assume $\mu^{1/n}$ is $C^\alpha$ with $\alpha > 0$ and choose a $\mathcal{C}^2$ solution of $\Delta_H \varphi = \mu^{1/n}$ in a neighborhood of $x_0$ (see [44]). Then $u = \varphi - f$ satisfies $\Delta_H u \geqslant 0$ in the viscosity sense. Once again, ([51] prop 3.2.10' p. 147) applies to the effect that $u$ is $\Delta_H$-subharmonic hence $\Delta_H \varphi \geqslant \mu^{1/n}$ in the weak sense of positive Radon measures.

Using convolution to regularize $\varphi$ and setting $\varphi_\varepsilon = \varphi * \rho_\varepsilon$ we see that $\Delta_H \varphi_\varepsilon \geqslant (\mu^{1/n})_\varepsilon$. Another application of the above lemma yields

$$(dd^c \varphi_\varepsilon)^n \geqslant ((\mu^{1/n})_\varepsilon)^n.$$

Since $\varphi_\varepsilon$ is decreasing with $\varepsilon$, continuity of $MA(\varphi)$ with respect to such a sequence yields $MA(\varphi) \geqslant \mu$ by Theorem 1.1.

This settles the case when $\mu > 0$ and $\mu$ is Hölder continuous. In case $\mu > 0$ is merely continuous we observe that $\mu = \sup\{\nu | \nu \in \mathcal{C}^\infty,\ \mu \geqslant \nu > 0\}$. Taking into account the fact that any subsolution of $(dd^c \varphi)^n = \mu$ is a subsolution of $(dd^c \varphi)^n = \nu$ provided $\mu \geqslant \nu$ we conclude $MA(\varphi) \geqslant \mu$.

In the general case when $\mu \geqslant 0$, we observe that $\psi_\varepsilon(z) = \varphi(z) + \varepsilon \|z\|^2$ satisfies $(dd^c \psi_\varepsilon)^n \geqslant \mu + \varepsilon^n \lambda$ in the viscosity sense with $\lambda$ the euclidean volume form. Hence $MA(\psi_\varepsilon) \geqslant \mu$, from which we conclude that $MA(\varphi) \geqslant \mu$. □

*Remark 3.5.* — The proof actually works in any class of plurisubharmonic functions in which the Monge-Ampère operator is continuous by decreasing limits of locally bounded functions and the comparison principle holds. When $n \geqslant 2$, these are precisely the finite energy classes studied in [27, 46, 18].

The basic idea of the proof is closely related to the method in [12] and is the topic treated in [69]. An alternative proof by using sup-convolutions will be given in the next section.

We now relax the assumption that $\varphi$ being bounded and connect viscosity subsolutions to pluripotential subsolutions through the following:

THEOREM 3.6. — *Assume that there exists a bounded psh function $\rho$ on $X$ such that $(dd^c\rho)^n \geqslant \mu$ in the weak sense in $X$. Let $\varphi$ be an upper semicontinuous function such that $\varphi \not\equiv -\infty$ on any connected component. The following are equivalent:*

*(i) $\varphi$ satisfies $(dd^c\varphi)^n \geqslant \mu$ in the viscosity sense on $X$;*

*(ii) $\varphi$ is plurisubharmonic and for all $c > 0$, $(dd^c \sup[\varphi, \rho - c])^n \geqslant \mu$ in the pluripotential sense on $X$.*

Observe that these properties are local and that it is possible to find a local strictly psh function such that locally $(dd^c\rho)^n \geqslant \mu$.

*Proof.* — Assume first that $\varphi$ is a viscosity subsolution of $-(dd^c\rho)^n+\mu = 0$. Since $\rho-c$ is also a subsolution, it follows from the maximum principle as in the proof of Lemma 1.12 that $\sup(\varphi, \rho-c)$ is a pluripotential subsolution, hence Proposition 3.3 yields $MA(\sup(\varphi, \rho - c)) \geqslant \mu$ in the viscosity sense.

Conversely, fix $x_0 \in X$ and assume i) holds. If $\varphi$ is locally bounded near $x_0$, Proposition 3.3 implies that $\varphi$ is a pluripotential subsolution near $x_0$.

Assume $\varphi(x_0) \neq -\infty$ but $\varphi$ is not locally bounded near $x_0$. Fix $q \in \mathcal{C}^2$ such that $q \geqslant \varphi$ near $x_0$ and $q(x_0) = \varphi(x_0)$. Then for $c > 0$ big enough we have $q \geqslant \varphi_c = \sup(\varphi, \rho - c)$ and $q(x_0) = \varphi_c(x_0)$, hence $(dd^c q)^n_{x_0} \geqslant \mu_{x_0}$ by Proposition 3.3 again.

Finally if $\varphi(x_0) = -\infty$ there are no $q$ to be tested against the differential inequality, hence it holds for every test function $q$. □

Condition $(ii)$ might seem a bit cumbersome. The point is that the Monge-Ampère operator can not be defined on the whole space of plurisubharmonic functions. When $\varphi$ belongs to its domain of definition, condition $(ii)$ is equivalent to $MA(\varphi) \geqslant \mu$ in the pluripotential sense. To be more precise, we have:

COROLLARY 3.7. — *Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain. Then $\varphi \in \mathcal{E}(\Omega)$, see [28] for the notation, satisfies $(dd^c\varphi)^n \geqslant \mu$ in the viscosity sense iff its Monge-Ampère measure $MA(\varphi)$ satisfies $MA(\varphi) \geqslant \mu$.*

We do not want to recall the definition of the class $\mathcal{E}(\Omega)$ (see [28]). It suffices to say that $\mathcal{E}(\Omega)$ coincides with the domain of definition of the complex Monge-Ampère operator (see [9]) and when $n = 2$, $\mathcal{E}(\Omega) = PSH(\Omega) \cap W^{1,2}_{loc}(\Omega)$ [8].

### 3.1.2. Viscosity subsolutions for $(dd^c u)^n = e^{\varepsilon\varphi}\mu$

We shall consider the complex Monge-Ampère equations

$$-(dd^c\varphi)^n + e^{\varepsilon\varphi}\mu = 0,$$

where $\mu$ is continuous volume form on $X$. Viscosity techniques actually mainly apply to the case $\varepsilon > 0$ and we are going to treat the previous case $\varepsilon = 0$ by a limiting process.

When $\varphi$ is continuous, so is the density of $\tilde{\mu} = e^{\varepsilon\varphi}\mu$: these definitions are then equivalent to the above ones and the first basic properties can be applied. When $\varphi$ is not assumed to be continuous, one needs to carefully check that subsolutions (resp. supersolutions) can still be understood equivalently in the pluripotential or viscosity sense.

Proposition 3.8. — *Let $\varphi : X \to \mathbb{R}$ be a locally bounded u.s.c. function. Then $\varphi$ satisfies $(dd^c\varphi)^n \geqslant e^{\varepsilon\varphi}\mu$ in the viscosity sense in $X$ if and only if it is plurisubharmonic and it does in the pluripotential sense in $X$.*

*Proof.* — We can assume without loss of generality that $\varepsilon = 1$ and $X = \Omega$ is a domain in $\mathbb{C}^n$. When $\varphi$ is continuous, so is the density $\tilde{\mu} := e^{\varphi}v$ and Proposition3.3 above implies that $\varphi$ is a viscosity subsolution of the equation $(dd^c\varphi)^n = e^{\varphi}\mu$ iff it is a pluripotential subsolution of the same equation.

The general case can be handled by approximation. First assume that $\varphi$ is a viscosity subsolution and set $\mu = f\beta_n$, where $f > 0$ is the continuous density of the volume form $\mu$ w.r.t. the euclidean volume form on $\mathbb{C}^n$. We approximate $\varphi$ by sup-convolutions defined for $\delta > 0$ small enough, by

$$\varphi^\delta(x) := \sup_y \left\{ \varphi(y) - \frac{1}{2\delta^2}|x-y|^2 \right\}, \quad x \in \Omega.$$

Observe that if $A > 1$ is a large constant so that $A^2 > 2\mathrm{osc}_\Omega\varphi$, then

$$\varphi^\delta(x) = \sup_{|y|\leqslant A\delta} \left\{ \varphi(x-y) - \frac{1}{2\delta^2}|y|^2 \right\}, \tag{3.1}$$

for $\delta > 0$ small enough, $x \in \Omega_\delta$, where $\Omega_\delta := \{x \in \Omega; dist(x, \partial\Omega) > A\delta\}$.

Thus $(\varphi^\delta)$ is a family of psh (and semi-convex) functions on $\Omega_\delta$, that decrease towards $\varphi$ as $\delta$ decreases to zero. Furthermore, by Proposition 2.17, $\varphi^\delta$ satisfies the following inequality in the sense of viscosity on $\Omega_\delta$

$$(dd^c\varphi^\delta)^n \geqslant e^{\varphi^\delta} f_\delta \beta_n, \text{ with } f_\delta(x) = \inf\{f(y) \,/ |y-x| \leqslant A\delta\}.$$

Since $\varphi^\delta$ is psh and *continuous*, we can invoke Proposition 3.3 and get that

$$(dd^c\varphi^\delta)^n \geqslant e^{\varphi^\delta} f_\delta\, \beta_n \geqslant e^{\varphi} f_\delta\, \beta_n,$$

holds in the pluripotential sense. Since $f_\delta$ increases towards $f$ and the complex Monge-Ampère operator is continuous along decreasing sequences of bounded psh functions (see Theorem 1.1), we finally obtain the inequality $(dd^c\varphi)^n \geqslant e^\varphi \mu$ in the pluripotential sense.

We now treat the other implication. Let $\varphi$ be a psh function satisfying the inequality

$$(dd^c\varphi)^n \geqslant e^\varphi \mu,$$

in the pluripotential sense on $\Omega$. We want to prove that $\varphi$ satisfies the above differential inequality in the sense of viscosity on $\Omega$. If $\varphi$ were continuous then we could use 3.3. But since $\varphi$ is not necessarily continuous we first approximate $\varphi$ using sup-convolution $\varphi^\delta$ as above. Lemma 3.9 below yields the following :

$$(dd^c\varphi^\delta)^n \geqslant e^{\varphi^\delta} f_\delta \beta_n \tag{3.2}$$

in the sense of pluripotential theory in $\Omega_\delta$.

Since $\varphi^\delta$ is continuous we can apply Proposition 3.3 to conclude that $\varphi^\delta$ is a viscosity subsolution of the equation $(dd^c u)^n = e^u f_\delta \beta_n$ on $\Omega_\delta$.

From this we want to deduce that $\varphi$ is a viscosity subsolution of the equation $(dd^c\varphi)^n = e^\varphi f \beta_n$ by passing to the limit as $\delta$ decreases to 0. This is certainly a well know fact in viscosity theory, but let us give a proof here for convenience.

Let $x_0 \in \Omega$, $q$ be a quadratic polynomial such that $\varphi(x_0) = q(x_0)$ and $\varphi \leqslant q$ on a neighbourhood of $x_0$ say on a ball $2B$, where $B := B(x_0, r) \Subset \Omega$. Since $\varphi$ is psh on $\Omega$, it satisfies $(dd^c\varphi)^n \geqslant 0$ in the viscosity sense on $\Omega$ by Proposition 3.1 and then by lemma 3.2, it follows that $dd^c q(x_0) \geqslant 0$. Replacing $q$ by $q(x) + \varepsilon|x - x_0|^2$ and taking $r > 0$ small enough, we can assume that $q$ is psh on the ball $2B$. We want to prove that $(dd^c q(x_0))^n \geqslant e^{\varphi(x_0)} f(x_0)\beta_n$.

Fix $\varepsilon > 0$ small enough. For $x \in B$, set

$$q_\varepsilon(x) := q(x) + 2\varepsilon(|x - x_0|^2 - r^2) + \varepsilon r^2.$$

Observe first that since $\varphi \leqslant q$ on $2B$, we have the following properties :
- if $x \in \partial B$, $\varphi^\delta(x) - q_\varepsilon(x) = \varphi^\delta(x) - q(x) - \varepsilon r^2 < 0$ on $B$, for $\delta >$ small enough.
- If $x = x_0$, we have $\varphi_\delta(x_0) - q^\varepsilon(x_0) = \varphi_\delta(x_0) - q(x_0) + \varepsilon r^2$.
Since $\varphi_\delta(x_0) - q(x_0) \to \varphi(x_0) - q(x_0) + \varepsilon r^2 = \varepsilon r^2$ as $\delta \to 0$, it follows that for $\delta$ small enough, the function $\varphi^\delta(x) - q_\varepsilon(x)$ takes it maximum on $\bar{B}$ at some interior point $x_\delta \in B$ and this maximum satisfies the inequality

$$\lim_{\delta \to 0} \max_{\bar{B}} (\varphi_\delta - q^\varepsilon) = \lim_{\delta \to 0} (\varphi_\delta(x_\delta) - q^\varepsilon(x_\delta)) \geqslant \varepsilon r^2. \tag{3.3}$$

Moreover we claim that $x_\delta \to x_0$ as $\delta \to 0$. Indeed we have

$$\begin{aligned} \varphi^\delta(x_\delta) - q_\varepsilon(x_\delta) &= \varphi^\delta(x_\delta) - q(x_\delta) - 2\varepsilon(|x_\delta - x_0|^2 - r^2) - \varepsilon r^2 \\ &= q^\delta(x_\delta) - q(x_\delta) - 2\varepsilon|x_\delta - x_0|^2 + \varepsilon r^2. \end{aligned}$$

Since $q^\delta(x_\delta) - q(x_\delta)$ converges to 0, it follows that if $x_0'$ is a limit point of the family $(x_\delta)$ in $\bar{B}$, then $\max_{\bar{B}}(\varphi_\delta - q^\varepsilon)$ will converge to a limit which is less or equal to $-2\varepsilon|x_0' - x_0|^2 + \varepsilon r^2$. By the inequality (3.3), this limit is $\geqslant \varepsilon r^2$. Therefore we obtain the inequality $-2\varepsilon|x_0' - x_0|^2 \geqslant 0$ which implies that $x_0' = x_0$ and our claim is proved.

Since $\varphi^\delta - q_\varepsilon$ takes it maximum on $\bar{B}$ at the point $x_\delta \in B$ and $\varphi^\delta$ is a viscosity subsolution of the equation $(dd^c u) \geqslant e^u f_\delta \beta_n$, it follows that

$$(dd^c q_\varepsilon(x_\delta))^n \geqslant e^{\varphi^\delta(x_\delta)} f_\delta(x_\delta)\beta_n = e^{\varphi^\delta(x_\delta) - q_\varepsilon(x_\delta)} e^{q_\varepsilon(x_\delta)} f_\delta(x_\delta)\beta_n.$$

Now observe that

$$\varphi^\delta - q_\varepsilon = (\varphi^\delta - q) + (q - q_\varepsilon)$$

and by Dini's lemma

$$\limsup_{\delta \to 0} \max_{\bar{B}} (\varphi^\delta - q) = \max_{\bar{B}} (\varphi - q) = 0.$$

Therefore

$$\limsup_{\delta \to 0} (\varphi^\delta(x_\delta) - q_\varepsilon(x_\delta)) \geqslant \liminf_{\delta \to 0} \min_{\bar{B}} (q - q_\varepsilon) = \min_{\bar{B}} (-2\varepsilon|x - x_0|^2 + \varepsilon r^2) = -\varepsilon r^2.$$

It follows immediately that

$$(dd^c q_\varepsilon(x_0))^n \geqslant e^{q(x_0) - 2\varepsilon r^2} f(x_0)\beta_n.$$

In the same way, we obtain the required inequality $(dd^c q(x_0))^n \geqslant e^{\varphi(x_0)} f(x_0)\beta_n$, since $q(x_0) = \varphi(x_0)$. $\square$

Lemma 3.9. — *Let $\varphi$ be a bounded plurisubharmonic function in a domain $\Omega \Subset \mathbb{C}^n$ such that*

$$(dd^c\varphi)^n \geqslant e^{\varphi} f\beta_n,$$

*in the pluripotentiel sense in $\Omega$, where $f \geqslant 0$ is a continuous density. Then the sup-convolutions $(\varphi^\delta)$ satisfy*

$$(dd^c\varphi^\delta)^n \geqslant e^{\varphi^\delta} f_\delta\beta_n,$$

*in the pluripotentiel sense in $\Omega_\delta$, where $f_\delta(x) := \inf\{f(y); |y-x| \leqslant A\delta\}$.*

*Proof.* — Fix $\delta > 0$ small enough. For $y \in B(0, A\delta)$, denote by $\psi_y(x) := \varphi(x-y) - \frac{1}{2\delta^2}|y|^2$, $x \in \Omega_\delta$ and observe that $\psi_y$ is a bounded psh function on $\Omega_\delta$ which satisfies the following inequality in the pluripotential sense on $\Omega_\delta$

$$(dd^c\psi_y)^n \geqslant e^{\psi_y} f_\delta\beta_n,$$

thanks to the invariance of the complex Monge-Ampère operator by translation.

Since $\varphi$ is the upper envelope of the family $\{\psi_y; y \in B(0, A\delta)\}$, it follows from a well known topological lemma of Choquet that there is a sequence of points $(y_j)_{j\in\mathbb{N}}$ in the ball $B(0, A\delta)$ such that $\varphi^\delta = (\sup_j \psi_{y_j})^*$ on $\Omega_\delta$. For $j \in \mathbb{N}$, denote by $\theta_j := \sup_{0\leqslant k\leqslant j} \psi_{y_k}$. Then $(\theta_j)$ is an increasing sequence of bounded psh functions on $\Omega_\delta$ which converges a.e. to $\varphi^\delta$ on $\Omega_\delta$. Itfollows from the maximum principle Theorem 1.6 as in the proof of Lemma 1.12that $\theta_j$ is also a pluripotential subsolution of the same equation i.e.

$$(dd^c\theta_j)^n \geqslant e^{\theta_j} f_\delta\beta_n, \tag{3.4}$$

in the pluripotential sense in $\Omega_\delta$.

Now by continuity of the complex Monge-Ampère operator along increasing sequences of bounded psh functions and the fact that $\sup_j \theta_j = \varphi^\delta$ quasi everywhere (see [13]), it follows from (3.4) that $(dd^c\varphi^\delta)^n \geqslant e^{\varphi^\delta} f_\delta\beta_n$ in the pluripotential sense on $\Omega_\delta$. □

### 3.2. Viscosity supersolutions

The definition of supersolutions is more delicate. In the sequel, we use two references on viscosity solutions [33] and [53] since both articles contain some technical points not made in the other one. The outline of the real theory given in [53], sect. V.3, although it suggests a natural definition for supersolutions in the complex case, seems to rely heavily on the continuity of convex functions. Hence, we will introduce a different notion, in the spirit of Definition 3.10.

We will first consider the complex Monge-Ampère equation

$$(dd^c\varphi)^n = \mu,$$

where $\mu \geqslant 0$ is a continuous volume form on some open set $\Omega \subset \mathbb{C}^n$ and $\varepsilon \geqslant 0$. As we have already seen, to fit in the viscosity formalism, we define the Hamiltonian function as

$$F(x, s, Q) := -(dd^c Q^{1,1})^n + \mu(x) = 0 \text{ if } dd^c Q^{1,1} \geqslant 0,$$

and $F(x, s, p, Q) = +\infty$ if not, where $Q^{1,1}$ is the $(1,1)-$form associated to the hermitian $(1,1)-$part of the (real) quadratic form $Q$ on $\mathbb{C}^n$.

Let us denote by $(dd^c Q^{1,1})_+ = dd^c Q^{1,1}$ if $dd^c Q^{1,1} \geqslant 0$ and $(dd^c Q^{1,1})_+ = 0$ if not. Then observe that for a lower test function $q$ for $\varphi$ at $x_0$ i.e. $\varphi \geqslant_{x_0} q$, the condition $(dd^c q(x_0))^n_+ \leqslant \mu(x_0)$ is always satisfied when $dd^c q(x_0)$ is not semi-positive as well as the condition $F(x_0, \varphi(x_0), dd^q(x_0)) \geqslant 0$. Hence the condition $F(x, s, Q) \geqslant 0$ is consistent only when $dd^c Q^{1,1} \geqslant 0$. Therefore the egneral definition of a supersolution can be formulated in the following equivalent way:

Definition 3.10. — *A supersolution of* $(dd^c\varphi)^n = e^{\varepsilon\varphi}\mu$ *is a lower semi-continuous function* $\varphi : \Omega \to \mathbb{R}\cup\{+\infty\}$ *such that* $\varphi \not\equiv +\infty$ *and the following property is satisfied: if for any* $x_0 \in \omega$ *and any* $q \in \mathcal{C}^2$*, defined in a neighborhood of* $x_0$ *such that* $\varphi(x_0) = q(x_0)$ *and* $\varphi - q$ *has a local minimum at* $x_0$*, then*

$$(dd^c q(x_0))^n_+ \leqslant \mu(x_0).$$

As we said before, the viscosity differential inequality (given by the supersolution property) for a lower test function $q$ at $x_0$ do not tell anything about the sign of $dd^c q(x_0)$ and is certainly satisfied whenever $dd^c q(x_0)$ is not semi-positive. However the condition is natural since when $u$ is a smooth function which is a supersolution, then for any lower test function $q$ at a given point $x_0$ we have $dd^c u(x_0) \geqslant dd^c q(x_0)$ by the classical maximum principle. No if we assume that $dd^c q(x_0) \geqslant 0$ we can conclude that $(dd^q)^n_{x_0} \leqslant (dd^c u)^n_{x_0} \leqslant \mu(x_0)$. But if do not assume that $dd^q(x_0)$ is non negative we cannot conclude.

The only way we will use this definition in the sequel is as follows. If $\varphi$ is not a supersolution of the equation then there exists a point $x_0$ and a lower test function $q$ at $x_0$ such that $(dd^c q(x_0))^n_+ > \mu(x_0) \geqslant 0$. Therefore $dd^c q(x_0) \geqslant 0$ and $(dd^c q(x_0))^n > 0$ which implies that $dd^c q(x_0) > 0$.

Supersolutions are less classical objects and are not going to live on the same footing as subsolutions. Whereas subsolutions are automatically plurisubharmonic, this is not necessarily the case of supersolutions.

Given a bounded function $h$, it is natural to consider its plurisubharmonic projection

$$P(h)(x) = P_\Omega(h)(x) := (\sup\{\psi(x) \,/\, \psi \text{ psh on } \Omega \text{ and } \psi \leqslant h\})^*,$$

which is the greatest psh function that lies below $h$ on $\Omega$. Observe that if $h$ is upper semi-continuous on $\Omega$ there is no need of upper regularization and the upper envelope is psh and $\leqslant h$ in $\Omega$.

We will see below that in the previous definition, the lower test function $q$ satisfies $(dd^c q)_+^n \leqslant \mu$ if and only if

$$(dd^c P(q))^n \leqslant \mu.$$

This can be deduced from the fact that if $q$ is $C^2$ in an euclidean ball $B = B(x_0, r)$ then the Monge-Ampère measure $(dd^c P(q))^n$ of its projection $P(q) = P_B(q)$ is concentrated on the set where $P(q) = q$, with

$$(dd^c P(q))^n = \mathbf{1}_{\{P(q)=q\}}(dd^c q)^n.$$

This formula can be easily derived from the (more involved) fact that $P(q)$ is a $\mathcal{C}^{1,1}$-smooth function (see [12], [11]).

Now we can prove the following statement which gives the relationship between the two notions of supersolutions.

Proposition 3.11. — *Let $\Omega \Subset \mathbb{C}^n$ be an open set.*
*1. Let $\psi$ be a bounded plurisubharmonic function in $\Omega$ satisfying $(dd^c\psi)^n \leqslant \mu$ in the pluripotential sense in $\Omega$. Then its lower semi-continuous regularization $\psi_*$ is a viscosity supersolution of the equation $(dd^c\varphi)^n = \mu$ in $\Omega$.*
*2. Let $\varphi$ be a continuous and bounded viscosity supersolution of the equation $-(dd^c u)^n + \mu = 0$ in $\Omega$. Then for any euclidean ball $\mathbb{B} \Subset \Omega$, $\psi := P_{\mathbb{B}}(\varphi)$ is a continuous plurisubharmonic viscosity supersolution of the equation $-(dd^c u)^n + \mu = 0$ in $\mathbb{B}$.*
*3. Let $\varphi$ be a $C^2$-smooth and viscosity supersolution of the equation $-(dd^c u)^n + \mu = 0$ in $\Omega$. Then for any euclidean ball $\mathbb{B} \Subset \Omega$, we have $(dd^c P_{\mathbb{B}}(\varphi))^n \leqslant \mu$ in the pluripotential sense in $\mathbb{B}$.*

*Proof.* — 1. We use the same idea as in the proof of Proposition 3.3. Assume $\psi \in PSH \cap L^\infty(\Omega)$ satisfies $MA(\psi) \leqslant \mu$ in the pluripotential

sense on $\Omega$. Consider $q$ a $\mathcal{C}^2$-smooth function near $x_0$ such that $\psi_*(x_0) = q(x_0)$ and $\psi_* - q$ achieves a local minimum at $x_0$. We want to prove that $(dd^c q(x_0))^n_+ \leqslant \mu(x_0)$. Assume that $(dd^c q(x_0))^n_+ > \mu_{x_0}$. Then $dd^c q(x_0) \geqslant 0$ and $(dd^c q(x_0))^n > \mu_{x_0} > 0$ which implies that $dd^c q(x_0) > 0$. Let $q^\varepsilon := q - 2\varepsilon(\|x - x_0\|^2 - r^2) - \varepsilon r^2$. Since $\mu$ has continuous density, we can choose $\varepsilon > 0$ small enough and a small ball $B(x_0, r)$ containing $x_0$ of radius $r > 0$ such that $dd^c q^\varepsilon > 0$ in $B(x_0, r)$ and $(dd^c q^\varepsilon)^n > \mu$ on the ball $B(x_0, r)$. Thus we have $q^\varepsilon = q - \varepsilon r^2 < \psi_* \leqslant \psi$ near $\partial B(x_0, r)$ while $MA(q^\varepsilon) \geqslant \mu \geqslant MA(\psi)$ in the pluripotential sense on $B(x_0, r)$. The comparison principle Theorem 1.6 yields $q^\varepsilon \leqslant \psi$ on $B(x_0, r)$ hence $q^\varepsilon(x_0) = \liminf_{x \to x_0} q^\varepsilon(x) \leqslant \liminf_{x \to x_0} \psi(x) = \psi_*(x_0)$ i.e. $q(x_0) + \varepsilon r^2 \leqslant \psi_*(x_0) = q(x_0)$, which is a contradiction. Hence $(dd^c q)^n_{x_0} \leqslant \mu_{x_0}$ and $\psi_*$ is a viscosity supersolution.
2. Set $\psi := P(\varphi)$. Then $\psi$ is a continuous psh function by [67]. Fix a point $x_0 \in \Omega$ and consider a super test function $q$ for $\psi$ at $x_0$ i.e. $q$ is a $C^2$ function on a small ball $B(x_0, r) \subset \Omega$ such that $\psi(x_0) = q(x_0)$ and $\psi - q$ attains its minimum at $x_0$. We want to prove that $(dd^c q(x_0))^n_+ \leqslant \mu(x_0)$. Since $\psi \leqslant \varphi$, there are two cases:
- if $\psi(x_0) = \varphi(x_0)$ then $q$ is also a super test function for $\varphi$ at $x_0$ and then $(dd^c q(x_0))^n_+ \leqslant \mu(x_0)$ since $\varphi$ is a supersolution of the same equation,
- if $\psi(x_0) < \varphi(x_0)$, by continuity of $\varphi$ there exists a ball $B(x_0, s)$ $0 < s < r$ such that $\psi = P(\varphi) < \varphi$ on the ball $B(x_0, s)$ and then $(dd^c \psi)^n = 0$ on $B(x_0, s)$ since $(dd^c P(\varphi))^n$ is supported on the contact set $\{P(\varphi) = \varphi\}$. Therefore $\psi$ is a continuous psh function satisfying the inequality $(dd^c \psi)^n = 0 \leqslant \mu$ in the sense of pluripotential theory on the ball $B(x_0, s)$. Assume that $(dd^c q(x_0))^n_+ > \mu(x_0)$. Then by definition, $dd^c q(x_0) > 0$ and $(dd^c q(x_0))^n > \mu(x_0)$. Taking $s > 0$ small enough and $\varepsilon > 0$ small enough we can assume that $q^\varepsilon := q - \varepsilon(|x - x_0|^2 - s^2)$ is psh on $B(x_0, s)$ and $(dd^c q^\varepsilon)^n > \mu \geqslant (dd^c \psi)^n$ on the ball $B(x_0, s)$ while $q^\varepsilon = q \leqslant \psi$ on $\partial B(x_0, s)$. By the pluripotential comparison principle for the complex Monge-Ampère operator, it follows that $q^\varepsilon \leqslant \psi$ on $B(x_0, s)$, thus $q(x_0) + \varepsilon s^2 \leqslant \psi(x_0)$, which is a contradiction.
3. This follows from the observation made before using the argument by Berman and Demailly ([11]). □

### 3.3. The Comparison Principle in the local case

We will consider the following more general complex Monge-Ampère type equations

$$G(u_{j,\bar{k}}) + e^{g(z,u) + h(Du)} = 0, \tag{3.5}$$

where $\Omega \Subset \mathbb{C}^n$ is bounded domain, $G$ is a degenerate elliptic continuous function on the cone $H_n^+$ of semi-positive hermitian forms on $\mathbb{C}^n$, $g$ is a continuous function on $\Omega \times \mathbb{R}$ increasing in the $u$ variable and $h$ is a continuous function on $\mathbb{C}^n$.

Then we will prove the following result.

Theorem 3.12. — *Let $u$ be a subsolution of (3.5) and $v$ a supersolution of (3.5). Assume that $u \leqslant v$ on $\partial\Omega$ then $u \leqslant v$ on $\Omega$.*

*Proof.* — The proof is an adaptation of arguments in [33]. The main idea is to apply the maximum principle to the usc function $u - v$. But since this functions are not smooth, we will apply Jensen-Ishii's maximum principle. Since the function $u - v$ is usc on $\overline{\Omega}$, then its maximum in $\overline{\Omega}$, defined as

$$M := \sup_{\overline{\Omega}}(u - v).$$

is attained at some point in $\overline{\Omega}$. We want to prove that $M \leqslant 0$. Since $u \leqslant v$ on $\partial\Omega$, we can assume that $S := \{x \in \overline{\Omega}; u(x) - v(x) = M\} \subset \Omega$. To apply Jensen-Ishii's maximum principle, we need to double the variable and add a penalty term to make the maximum reached asymptotically on the diagonal. Indeed for $\varepsilon > 0$, define the function

$$\psi_\varepsilon(x, y) := u(x) - v(y) - \frac{1}{2\varepsilon^2}|x - y|^2,$$

which is upper semi-continuous on $\overline{\Omega} \times \overline{\Omega}$. Then it takes its maximum on $\bar{\Omega} \times \bar{\Omega}$ at some point $(x_\varepsilon, y_\varepsilon) \in \bar{\Omega} \times \bar{\Omega}$ i.e.

$$M_\varepsilon := \max_{(x,y)\in\overline{\Omega}^2} \psi_\varepsilon(x, y) = u(x_\varepsilon) - v(y_\varepsilon) - \frac{1}{2\varepsilon^2}|x_\varepsilon - y_\varepsilon|^2.$$

It is quite easy to prove that (see [33])

$$\lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon^2}|x_\varepsilon - y_\varepsilon|^2 = 0$$

and there exists a subsequence $(x_{\varepsilon_j}, y_{\varepsilon_j}) \to (\bar{x}, \bar{x}) \in \overline{\Omega}^2$ such that

$$\lim_{\varepsilon\to 0} M_\varepsilon = M = u(\bar{x}) - v(\bar{x}).$$

Then $\bar{x} \in S$. Since $S \subset \Omega$ from our assumption, it follows that $j >> 1$, $(x_{\varepsilon_j}, y_{\varepsilon_j}) \in \Omega^2$. Therefore we can apply Jensen-Ishii maximum principle. Fix $j >> 1$ and set $p = p(\varepsilon_j) := \frac{1}{\varepsilon_j^2}(x_{\varepsilon_j} - y_{\varepsilon_j})$, there exists $Q^\pm \in \mathcal{S}_{2n}$ such that $(p, Q^+) \in \bar{J}^{2,+}u(x_{\varepsilon_j}), (p, Q^-) \in \bar{J}^{2,-}v(y_{\varepsilon_j})$ and $Q^+ \leqslant Q^-$. It follows from the fact that $(p, Q^+) \in \bar{J}^{2,+}u(x_\varepsilon)$ and the definition of viscosity subsolution that the hermitian $(1,1)-$part $H^+$ of the quadratic form $Q^+$ is semi-positive hence so is the hermitian $(1,1)-$part $H^-$ of $Q^-$ since $0 \leqslant H^+ \leqslant H^-$. Then by the degenerate ellipticity condition on $G$, we get $-G(H^+) \leqslant -G(H^-)$. Therefore applying the viscosity inequalities we obtain

$$e^{g(x_{\varepsilon_j}, u(x_{\varepsilon_j})) + h(p)} \leqslant e^{g(y_{\varepsilon_j}, v(y_{\varepsilon_j})) + h(p)},$$

which implies that for $j >> 1$,

$$g(x_{\varepsilon_j}, u(x_{\varepsilon_j})) \leqslant g(y_{\varepsilon_j}, v(y_{\varepsilon_j})).$$

Now recall that $(x_{\varepsilon_j}, y_{\varepsilon_j}) \to (\bar{x}, \bar{x}) \in \Omega^2$ and $u(x_{\varepsilon_j}) - v(y_{\varepsilon_j}) \to M$. We can always assume that $\lim_j v(x_{\varepsilon_j}) = \ell \in \mathbb{R}$ exists and then $\lim_j u(y_{\varepsilon_j} = \ell + M$.

Passing to the limit, we get $g(\bar{x}, \ell + M) \leqslant g(\bar{x}, \ell)$, which implies that $M \leqslant 0$, since $g$ is increasing in the second variable. □

*Remark 3.13.* — The last result cannot be applied when $g$ does not depend on $s$ i.e; the equation do not involve the function $u$ itself. We do not know if the result is still true in this case. However if the function does not depend on $p$ and is only assumed to be non decreasing, it is possible to prove the comparison principle using instead, the so called Alexandroff-Backelman-Pucci maximum principle (see [31], [68], [29]).

Let us give the following application of the local comparison principle.

Proposition 3.14. — *If $\mu > 0$ is a continuous volume form on a complex manifold $X$ of dimension $n$, then viscosity solutions of the equation $(dd^c\varphi)^n = e^{g(x,\varphi)}\mu$ in $X$ are precisely the continuous psh functions $\varphi$ solutions of the equation $(dd^c\varphi)^n = e^{g(x,\varphi)}\mu$ in the pluripotential sense in $X$.*

*Proof.* — We already know by Proposition 3.3 and Proposition 3.11 that continuous psh (pluripotential) solutions of the equation $(dd^c\varphi)^n = e^{g(x,\varphi}\mu$ on $X$ are viscosity solutions of the equation. To prove the converse, assume that $\varphi$ is a viscosity solution of the equation $(dd^c\varphi)^n = e^{g(x,\varphi)}\mu$. Then by Proposition 3.3, $\varphi$ is a continuous psh function in $X$ which satisfies the inequality $(dd^c\varphi)^n \geqslant e^{g(x,\varphi}\mu$ in the pluripotential sense in $X$. To prove equality assume that $B \Subset X$ is a small coordinate chart in $X$ biholomorphic to an euclidean ball in $\mathbb{C}^n$ and use the balayage construction to find a psh function $\psi$ such that $(dd^c\psi)^n = e^{g(x,\varphi)}\mu$, $\psi = \varphi$ on $X \setminus \overline{B}$ and $\psi \geqslant \varphi$ using Theorem1.5. Then by the pluripotential comparison principle it follows that $\varphi \leqslant \psi$ on $B$. On the other hand, by Proposition 3.3, $\psi$ is a viscosity subsolution of the equation $(dd^c\psi)^n = e^{g(x,\varphi)}\mu$. Since $\varphi$ is a viscosity (super)-solution of the $(dd^c\varphi)^n = e^{g(x,\varphi)}\mu$ on $B$ and $\varphi = \psi$ on $\partial B$, it follows from the viscosity comparison principle that $\psi = f$ in $B$. Hence $\varphi = \psi$ on $B$ and satisfies the equation $(dd^c\varphi)^n = \mu$ on $B$. Since $B$ is arbitrary, it follows that $\varphi$ is a pluripotential solution of the equation $(dd^c\varphi)^n = e^{g(x,\varphi)}\mu$ on $\Omega$. □

## 3.4. Viscosity solution: the Perron's method

Once the global comparison principle holds, one easily constructs continuous solutions by Perron's method as we now explain. Consider the following general equation

$$F(x, u, Du, dd^c u) = 0, \tag{3.6}$$

on a domain $\Omega \subset \mathbb{C}^n$, where $F : \Omega \times \mathbb{R} \times \mathbb{R}^{2n} \times H_n^+ \longrightarrow \mathbb{R}$ and extend it as usual to a real quadratic forms as usual by $F(x, s, p, Q) = F(x, s, p, Q^{1,1})$ if the hermitian $(1, 1)-$part of $Q$ is semi-positive and by $+\infty$ if not.

Theorem 3.15. — *Assume the comparison principle holds for the complex Monge-Ampère type equation (3.6) and the family $\mathcal{U}$ of bounded subsolutions of the equation (3.6) is non empty and locally upper bounded in $\Omega$. Then the following properties:*
*1. The upper envelope*

$$\varphi = \sup\{u \,|\, u \in \mathcal{U}\}$$

*is the maximal subsolution of the equation (3.6).*

*Let $\gamma$ be a continuous function on $\partial\Omega$ and assume that the equation (3.6) has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ such that $\underline{u}_* = \gamma = \overline{u}^*$ on $\partial\Omega$. Then $\varphi$ is the unique viscosity solution of (3.6) such that $u = \gamma$ on $\partial\Omega$.*

*Proof.* — We argue as in [33] p. 22-24. Then lemma 4.2 there implies that the upper envelope $\varphi$ of the subsolutions of (3.6) is a subsolution of (3.6) since $F$ is lsc. Hence $\varphi$ is a subsolution of (3.6).

The Ishii's trick is now to consider the lsc regularisation $\varphi_*$ of $\varphi$. We are going to show that $\varphi_*$ is a supersolution of (3.6). We argue by contradiction using a bump construction. Assume the converse is true. Then we can find $x_0 \in \Omega$ and a lower test function $q$ for $\varphi_*$ at $x_0$ such that $F^+(x_0, \varphi_*(x_0), dq(x_0), dd^c q(x_0)) < 0$. This implies that $Q := dd^c q(x_0) \geqslant 0$ and $F(x_0, \varphi_*(x_0), Dq(x_0), Q) < 0$. Let $(z^1, .., z^n)$ be a coordinate system centered at $x_0$ giving a local isomorphism with the complex unit ball. Define for $\delta > 0$, $r > 0$ small enough and $|z| < 2r$,

$$q_\delta(z) := q(z) - \delta(|z - x_0|^2 - r^2).$$

Then

$$q_\delta(x_0) = \varphi_*(x_0) + \delta r^2,$$
$$Dq_\delta(x_0) = Dq(x_0),$$
$$D^2 q_\delta(x_0) = Q - 2\delta I_n.$$

Then since $F(x_0, \varphi_*(x_0), Dq(x_0), Q) < 0$, it follows by continuity of $F$ in its domain that for $\delta > 0$ small enough we can find $r > 0$ small enough so that for $|z| < 2r$,

$$F(z, u_\delta(z), q_\delta(z)) < 0,$$

which means that $q_\delta$ is a subsolution of our equation in the ball $|z| < 2r$. Now observe that for $|z| = 2r$, $q_\delta(z) = q(z) - \delta r^2 \leqslant \varphi_* - \delta r^2 \leqslant \varphi - \delta r^2$. Therefore the new function $\psi(z) := \max\{\varphi, q_\delta\}$ on the ball $B_r : |z| < 2r$ and $\psi = \varphi$ in $\Omega \setminus B_r$ is a subsolution of the equation in $\Omega$. Since $\varphi$ is the maximal subsolution of the equation on $\Omega$, we conclude that $U \leqslant \varphi$ in $\Omega$, which implies that $q_\delta \leqslant \varphi$ on the ball $B_r$. On the other hand, since $u_\delta(x_0) - \varphi_*(x_0) = \delta r^2$, there is a sequence $(y_j)$ converging to $x_0$ such that $\lim_{j\to+\infty} q_\delta(y_j) - \varphi(y_j) = q_\delta(x_0) - \varphi_*(x_0)$. Then for $j >> 1$ we have $y_j \in B_r$ and $q_\delta(y_j) - \varphi(y_j) > \delta r^2/2 > 0$, which contradicts the inequality $q_\delta \leqslant \varphi$ on the ball $B_r$.

Since $\underline{u} \leqslant \varphi \leqslant \overline{u}$ it follows that $\underline{u}_* \leqslant \varphi_* \leqslant \overline{u}$ in $\Omega$. Then $\varphi \leqslant \overline{u}^* = \gamma$ on $\partial\Omega$, while $\gamma = \underline{u}_* \leqslant \varphi_*$ on $\partial\Omega$ which implies that $\varphi \leqslant \varphi_*$ in $\partial\Omega$. By the comparison principle it implies that $\varphi \leqslant \varphi_*$ in $\Omega$, hence $\varphi = \varphi_*$ is a viscosity solution of the equation (3.6). □

Corollary 3.16. — *Let $\mu > 0$ be is a continuous volume fom $\Omega \Subset \mathbb{C}^n$ and $g$ is continuous and increasing in the second variable. Assume that the family $\mathcal{U}$ of bounded viscosity subsolutions of the following complex Monge-Ampère equation*

$$-(dd^c u)^n + e^{g(x,u)}\mu = 0, \tag{3.7}$$

*is non empty and locally upper bounded in $\Omega$. Then the maximal viscosity subsolution*

$$\varphi = \sup\{u \,|\, u \in \mathcal{U}\},$$

*is a viscosity solution of (3.7). Moreover it is a continuous $\omega$-plurisubharmonic function on $\Omega$ and is also a solution of (3.7) in the pluripotential sense.*

*Proof.* — It remains to see that $\varphi$ is also a solution of (3.7) in the pluripotential sense. Since this is a local property, it is enough to prove it locally. We argue by balayage. Let $B \Subset X$ be a small coordinate neighbourhood which is biholomorphic to an euclidean ball in $\mathbb{C}^n$ such that $\omega$ has a local potential on a neighbourhood of $\overline{B}$. Since $\varphi$ is continuous on $\overline{B}$, we can solve the complex Monge-Ampère equation $(dd^c\psi)^n = e^{g(x,\varphi)}\mu$ on $B$ with boundary values equal to $\varphi$ on $\partial B$ by Theorem 1.5. Then by the pluripotential comparison principle we have $\psi \geqslant \varphi$ on $B$. Therefore the function $u := \psi$ on $B$ and $u = \varphi$ on $\Omega \setminus B$ is a continuous $\omega-$psh function on $\Omega$ and by Proposition 3.1, it is a viscosity subsolution of the equation (3.7). Therefore by the global comparison principle $u \leqslant \varphi$ on $\Omega$, which proves that $\varphi = \psi$ on $B$ and then $\varphi$ satisfies the complex Monge-Ampère equation $(\omega + dd^c\varphi)^n = e^{g(x,\varphi)}\mu$ in the pluripotential sense on $B$ which proves our statement. □

*Remark 3.17.* — As we observed in Remark 3.13, the comparison principle is valid in a more general situation where $\mu \geqslant 0$ and $g(x, s)$ is non decreasing in $s$. Therefore the Theorem above is still valid in this general situation. For a different proof of this last statement see [68].

## 4. The viscosity approach in the Kähler case

We now set the basic frame for the viscosity approach to the following degenerate complex Monge-Ampère equation

$$(DMA)_{\varepsilon,\mu} \qquad (\omega + dd^c\varphi)^n = e^{\varepsilon\varphi}\mu,$$

where $\omega$ is a closed smooth $(1, 1)$-form on a $n$-dimensional connected compact complex manifold $X$, $\mu$ is a volume form with nonnegative continuous density and $\varepsilon \in \mathbb{R}_+$.

As we have seen in the last section, the comparison principle lies at the heart of the viscosity approach. Once it is established, Perron's method can be applied to produce viscosity solutions. Our main goal in this section is to establish the global comparison principle for the equation $(DMA)_{\varepsilon,\mu}$. We only assume $X$ is compact (and $\varepsilon > 0$): the structural feature of $(DMA)_{\varepsilon,\mu}$ allows us to avoid any restrictive curvature assumption on $X$ (unlike e.g. in [3]).

### 4.1. Definitions for the compact case

To fit in with the viscosity point of view, we rewrite the Monge-Ampère equation as

$$-(\omega + dd^c\varphi)^n + e^{\varepsilon\varphi}\mu = 0.$$

Let $x \in X$. If $\kappa \in \Lambda^{1,1}T_xX$ we define $\kappa_+^n$ to be $\kappa^n$ if $\kappa \geqslant 0$ and 0 otherwise.

We let $PSH(X, \omega)$ denote the set of all $\omega$-plurisubharmonic ($\omega$-psh for short) functions on $X$: these are integrable functions $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ such that $dd^c\varphi \geqslant -\omega$ in the sense of currents.

Lemma 4.1. — *Let $\Omega \subset X$ be an open subset and $z : \Omega \to \mathbb{C}^n$ be a holomorphic coordinate chart. Let $h$ be a smooth local potential for $\omega$ defined on $\Omega$. Then $(MA_{\varepsilon,\mu})$ reduces in these $z$-coordinates to the scalar equation*

$$(MA_{\varepsilon,\mu|z}) \qquad e^{\varepsilon u}W - \det(u_{z\bar{z}}) = 0$$

*where* $u = (\varphi + h)|_\Omega \circ z^{-1}$, $z_*\mu = e^{\varepsilon h_{|\Omega} \circ z^{-1}} W d\lambda$ *and* $\lambda$ *is the Lebesgue measure on* $z(\Omega)$.

The proof is straightforward.

In order to deal with degenerate elliptic non linear equations and be able to apply results from [33], we introduce the following Hamiltonian function. If $\varphi_x^{(2)}$ is the 2-jet at $x \in X$ of a $\mathcal{C}^2$ real valued function $\varphi$ we set

$$F(x, \varphi(x), \varphi_x^{(2)}) = \begin{cases} e^{\varepsilon\varphi(x)}\mu_x - (\omega_x + dd^c\varphi_x)^n & \text{if } \omega + dd^c\varphi_x \geqslant 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Then $F$ satisfies the degenerate ellipticity condition as well as the properness condition, but it is only lower semi-continuous. However it is continuous on its domain i.e. where it is finite.

#### 4.1.1. Subsolutions

Recall now the following definition from previous sections:

Definition 4.2. — *A subsolution of* $(DMA)_{\varepsilon,\mu}$ *is an upper semi-continuous function* $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ *such that* $\varphi \not\equiv -\infty$ *and the following property is satisfied: if* $x_0 \in X$ *and* $q \in \mathcal{C}^2$, *defined in a neighborhood of* $x_0$, *is such that* $\varphi(x_0) = q(x_0)$ *and*

$$\varphi - q \quad \text{has a local maximum at } x_0,$$

*then* $F(x_0, \varphi(x_0), q_{x_0}^{(2)}) \leqslant 0$.

#### 4.1.2. (Super)solutions

The definition of supersolutions follows the one given in the local setting:

Definition 4.3. — *A supersolution of* $(DMA)_{\varepsilon,\mu}$ *is a lower semicontinuous function* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *such that* $\varphi \not\equiv +\infty$ *and the following property is satisfied: if* $x_0 \in X$ *and* $q \in \mathcal{C}^2$, *defined in a neighborhood of* $x_0$, *is such that* $\varphi(x_0) = q(x_0)$ *and* $\varphi - q$ *has a local minimum at* $x_0$, *then* $F(x_0, \varphi(x_0), q_{x_0}^{(2)}) \geqslant 0$.

Definition 4.4. — *A viscosity solution of* $(DMA)_{\varepsilon,\mu}$ *is a function that is both a sub-and a supersolution. In particular, viscosity solutions are automatically continuous.*

*A pluripotential solution of* $(DMA)_{\varepsilon,\mu}$ *is an usc function* $\varphi \in L^\infty \cap PSH(X, \omega)$ *such that for every local potential* $\psi$ *of* $\omega$ *we have* $MA(\psi + \varphi) = e^{\varepsilon\varphi}\mu$ *in the weak sense of currents.*

*Classical sub/supersolutions are* $\mathcal{C}^2$ *viscosity sub/supersolutions.*

In this setting, the discussion after Theorem 3.6 yields the following:

Corollary 4.5. — *Let $X$ be a compact Kähler manifold and $\omega$ a smooth closed $(1,1)$ form whose cohomology class $[\omega]$ is big. Let $\varphi$ be any continuous $\omega$-psh function. Then $\varphi$ satisfies $(\omega + dd^c\varphi)^n \geqslant e^{\varepsilon\varphi}\mu$ in the viscosity sense iff $\langle(\omega + dd^c\varphi)^n\rangle \geqslant e^{\varepsilon\varphi}\mu$, where $\langle(\omega + dd^c\varphi)^n\rangle$ is the non-pluripolar Monge-Ampère measure [21].*

## 4.2. The global viscosity comparison principle

Since our conditions ($X$ compact, $\mu \geqslant 0$, $\varepsilon > 0$) are invariant under dilation, we can always reduce to the case $\varepsilon = 1$, a normalisation that we shall often make in the sequel.

We now come to the main result of this section:

Theorem 4.6. — *The global viscosity comparison principle for $(DMA)_{1,\mu}$ holds, provided $\omega$ is a closed $(1,1)$-form on $X$, $\mu > 0$, and $X$ is compact.*

Observe that we do not assume $X$ to be Kähler nor $\omega$ to be semi-positive.

*Proof.* — We choose a constant $C > 0$ such that $\varphi$ and $\psi$ both are $\leqslant C/4$ in $L^\infty$-norm. Since $\varphi - \psi$ is upper semicontinuous on the compact manifold $X$, it follows that its maximum is achieved at some point $x_0 \in X$. Choose complex coordinates $z = (z^1, \dots, z^n)$ near $x_0$ defining a biholomorphism identifying an open neighborhood of $x_0$ to the complex ball $B_4 := B(0,4) \subset \mathbb{C}^n$ of radius 4 sending $x_0$ to the origin in $\mathbb{C}^n$.

We define $h_\omega \in \mathcal{C}^2(\overline{B_4}, \mathbb{R})$ to be a local potential smooth up to the boundary for $\omega$ and extend it smoothly to $X$. We may without lost of generality assume that $\|h_\omega\|_\infty < C/4$. In particular $dd^c h_\omega = \omega$ on $B_4$ and the usc function $u := \varphi \circ z^{-1} + h_\omega \circ z^{-1}$ is a viscosity subsolution of

$$(\star) \qquad (dd^c u)^n = e^u f \cdot \beta_n \text{ in } B_4,$$

with $f := z^*(\mu)/\beta_n > 0$ is a positive and continuous volume form on $B_4$.

On the other hand the lsc function $v := \psi \circ z^{-1} + h_\omega \circ z^{-1}$ is a viscosity supersolution of the same equation.

This is a crucial point: the modified equation still has the same form as the original one.

We want to estimate $\max_X(\varphi - \psi) = \max_{\bar{B}_4}(u - v) = u(0) - v(0) \leqslant 0$ by applying the classical maximum principle as in the local case. Observe that if the functions $u$ and $v$ were twice differentiable at $x_0$ the inequality follows from the maximum principle and the differential sub/super inequalities satisfied by $u$ and $v$ at $x_0$ respectively.

In the general case we proceed as in [33] using the penalty method consisting in doubling the variable and adding a penalty function, but we will be adding two penalty functions. We consider the function $x \longmapsto u(x) - v(x)$ as the restriction to the diagonal in the product $B_3 \times B_3$ of the function $(x, y) \longmapsto u(x) - v(y) - \theta(x, y) - (1/2\delta)|x - y|^2$ where $\theta(x, y)$ is the first penality function which vanishes highly on the diagonal near the origin $(0, 0)$ and is large enough on the boundary of the ball $B_3 \times B_3$ to force the maximum to be attained at an interior point; the second penalty function forces the maximum to be asymptotically attained along the diagonal. The fact that the second derivative of the penalty function is a quadratic form on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ which vanishes on the diagonal, will be crucial.

We now proceed to the construction of the first penalty function $\theta$. We want to construct a smooth function $\theta \in \mathcal{C}^\infty(X^2, \mathbb{R})$ satisfying the following conditions

- $\theta \geqslant 0$,
- $\theta^{-1}(0) = \Delta \cap \{\theta_2 \leqslant -\eta\}$,
- $\theta|_{X^2 \setminus B_2^2} > 3C$,

where $\eta > 0$ is small enough (see below for the definition of $\theta_2$) and $C > 0$.

First we construct a Riemannian metric on $X$ which coincides with the flat Kähler metric $\frac{\sqrt{-1}}{2} dz^k \wedge d\bar{z}^k$ on the ball of center 0 and radius 3. For $(x, y) \in X \times X$ define $d(x, y)$ to be the corresponding Riemannian distance function. The continuous function $d^2$ is of class $\mathcal{C}^2$ near the diagonal and $> 0$ outside the diagonal $\Delta \subset X^2$.

Next we construct a smooth non negative function $\theta_1$ on $X \times X$ by the following formula:

$$\theta_1(x, y) = \chi(x, y). \sum_{i=1}^{n} |z^i(x) - z^i(y)|^{2n+4},$$

where $\chi$ smooth non negative cut off function with $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $B_3^2$ $\chi = 0$ near $\partial B_4^2$.

Then we construct a second smooth function on $X \times X$ with $\theta_2|_{B_2^2} < -1$, $\theta_2|_{M^2 \setminus B_2^2} > 3C$.

Choose $1 \gg \eta > 0$ such that $-\eta$ is a regular value of both $\theta_2$ and $\theta_2|_\Delta$.

We perform convolution of $(\xi, \xi') \mapsto \max(\xi, \xi')$ by a smooth semipositive function $\rho$ such that $B_{\mathbb{R}^2}(0, \eta) = \{\rho > 0\}$ and get a smooth function on $\mathbb{R}^2$ $\max_\eta$ such that:

- $\max_\eta(\xi, \xi') = \max(\xi, \xi')$ if $|\xi - \xi'| \geqslant \eta$,
- $\max_\eta(\xi, \xi') > \max(\xi, \xi')$ if $|\xi - \xi'| < \eta$.

Then the function $\theta$ defined by $\theta := \max_\eta(\theta_1, \theta_2)$ satisfies our requirements.

Fix $\alpha > 0$. We want to apply the Jensen-Ishii's maximum principle to the functions $u, v$ and $\phi = \theta - \frac{1}{2\alpha}|x - y|^2$.

For $\alpha > 0$ small enough, consider $(x_\alpha, y_\alpha) \in \bar{B}_3 \times \bar{B}_3$ such that

$$\begin{aligned} m_\alpha \quad &:= \quad \sup_{(x,y)\in \bar{B}_3^2} \left\{ u(x) - v(y) - \frac{1}{2\alpha}|x - y|^2 - \theta(x, y) \right\} \\ &= \quad u(x_\alpha) - v(y_\alpha) - \theta(x_\alpha, y_\alpha) - \frac{1}{2\alpha}|x_\alpha - y_\alpha|^2. \end{aligned}$$

The supremum is achieved since we are maximizing an usc function on the compact set $\bar{B}_3^2$. We also have

$$m_\alpha \geqslant u(0) - v(0) = \varphi(x_0) - \psi(x_0) \geqslant -C/2, \tag{4.1}$$

for $\alpha > 0$ small enough.

By construction, for $(x, y) \in B_3^2 \setminus B_2^2$, we also have

$$u(x) - v(y) - \theta(x, y) - \frac{1}{2\alpha}|x - y|^2 \leqslant -2C < -C, \tag{4.2}$$

which implies that $(x_\alpha, y_\alpha) \in B_2^2$.

The following result follows easily from the above properties (see [33, Proposition 3.7]):

Lemma 4.7. — *For $\alpha > 0$ small enough we have $|x_\alpha - y_\alpha|^2 = o(\alpha)$. Every limit point $(\hat{x}, \hat{y})$ of $(x_\alpha, y_\alpha)$ satisfies $\hat{x} = \hat{y}$, $(\hat{x}, \hat{x}) \in \Delta \cap \{\theta_2 \leqslant -\eta\}$ and*

$$\begin{aligned} \lim_{\alpha\to 0}(u(x_\alpha) - v(y_\alpha)) \quad &= \quad u(\hat{x}) - v(\hat{x}) \\ &= \quad \varphi(x_0) - \psi(x_0. \end{aligned}$$

Next, we use Jensen-Ishii's maximum principle with $\phi = \frac{1}{2\alpha} d^2 + \theta$. For $0 < \alpha << 1$, everything is localized to $B(0,2)$ hence $d$ reduces to the euclidean distance function. Using the usual formula for the first and second derivatives of its square, we get the following:

Lemma 4.8. — *$\forall \varepsilon > 0$, we can find $(p_*, Q_*), (p^*, Q^*) \in \mathbb{C}^n \times Sym^2_{\mathbb{R}}(\mathbb{C}^n)$ s.t.*

1. *$(p_*, Q_*) \in \overline{J}^{2+} u(x_\alpha)$,*
2. *$(p^*, Q^*) \in \overline{J}^{2-} v(y_\alpha)$, where $p^* = D_x\theta(x_\alpha, y_\alpha) + \frac{1}{2\alpha}(x_\alpha - y_\alpha)$ and $p_* = -D_y\theta(x_\alpha, y_\alpha) + \frac{1}{2\alpha}(x_\alpha - y_\alpha)$*
3. *The block diagonal matrix with entries $(Q_*, Q^*)$ satisfies:*

$$-(\varepsilon^{-1} + \|A\|)I \leqslant \begin{pmatrix} Q_* & 0 \\ 0 & -Q^* \end{pmatrix} \leqslant A + \varepsilon A^2,$$

*where $A = D^2\phi(x_\alpha, y_\alpha)$, i.e.*

$$A = \alpha^{-1} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + D^2\theta(x_\alpha, y_\alpha)$$

*and $\|A\|$ is the spectral radius of $A$ (maximum of the absolute values for the eigenvalues of this symmetric matrix).*

By construction, the Taylor series of $\theta$ at any point in $\Delta \cap \{\theta_2 < -\eta\}$ vanishes up to order $2n$. By transversality, $\Delta \cap \{\theta_2 < -\eta\}$ is dense in $\Delta \cap \{\theta_2 \leqslant -\eta\}$, and this Taylor series vanishes up to order $2n$ on $\Delta \cap \{\theta_2 \leqslant -\eta\}$. In particular,

$$D^2\theta(x_\alpha, y_\alpha) = O(d(x_\alpha, y_\alpha)^{2n}) = o(\alpha^n).$$

This implies $\|A\| \simeq 1/\alpha$. We choose $\alpha = \varepsilon$ and deduce

$$-(2\alpha^{-1})I \leqslant \begin{pmatrix} Q_* & 0 \\ 0 & -Q^* \end{pmatrix} \leqslant \frac{3}{\alpha} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + o(\alpha^n)$$

Looking at the upper and lower diagonal terms we deduce that the eigenvalues of $Q_*, Q^*$ are $O(\alpha^{-1})$. Evaluating the inequality on vectors of the form $(Z, Z)$ we deduce from the $\leqslant$ that the eigenvalues of $Q_* - Q^*$ are $o(\alpha^n)$.

For a fixed $Q \in Sym^2_{\mathbb{R}}(\mathbb{C}^n)$, denote by $H = Q^{1,1}$ its $(1,1)$-part. It is a hermitian matrix. Obviously the eigenvalues of $H_* := Q_*^{1,1}, H^* := Q^{*1,1}$ are

$O(\alpha^{-1})$ but those of $H_* - H^*$ are $o(\alpha^n)$. Since $(p_*, Q_*) \in \overline{J^{2+}} w_*(x_\alpha)$ we deduce from the definition of viscosity solutions that $H_*$ is positive definite and that the product of its $n$ eigenvalues is $\geqslant c > 0$ uniformly in $\alpha$. In particular its smallest eigenvalue is $\geqslant c\alpha^{n-1}$. The relation $H_* + o(\alpha^n) \leqslant H^*$ forces $H^* > 0$ for $\alpha > 0$ small enough. Then we have $\det H_* \leqslant \det H^* + o(\alpha^n)$.

By viscosity inequalities, we get

$$\begin{aligned} e^{u(x_\alpha)} \leqslant \det(H_*) &\leqslant \det(H^*) + o(\alpha^n) \\ &\leqslant e^{v(y_\alpha)} + o(\alpha^n). \end{aligned}$$

Passing to the limit as $\alpha \to 0$, we obtain the inequality $e^{u(\hat{x})} \leqslant e^{v(\hat{x})}$, which implies that $u(\hat{x}) \leqslant v(\hat{x})$. $\square$

*Remark 4.9.* — The miracle with the complex Monge Ampère equation we are studying is that the equation does not depend on the gradient in complex coordinates. In fact, it takes the form $F(Q) - f(x) = 0$. The localisation technique would fail without this structural feature.

### 4.3. Perron's method

Once the global comparison principle holds, one easily constructs continuous (viscosity=pluripotential) solutions by Perron's method as we explained in the last section.

THEOREM 4.10. — *Assume the global comparison principle holds for $(DMA)_{\varepsilon,\mu}$ and that $(DMA)_{\varepsilon,\mu}$ has a bounded subsolution $\underline{u}$ and a bounded supersolution $\overline{u}$. Then the maximal subsolution,*

$$\varphi = \sup\{w \,|\, \underline{u} \leqslant w \leqslant \overline{u} \text{ and } w \text{ is a viscosity subsolution of } (DMA)_{\varepsilon,\mu}\}$$

*is the unique viscosity solution of $(DMA)_{\varepsilon,\mu}$.*

*In particular, it is a continuous $\omega$-plurisubharmonic function in $X$ which is also a solution of $(DMA)_{\varepsilon,\mu}$ in the pluripotential sense.*

*Example 4.11.* — Assume $X$ is a complex projective manifold such that $K_X$ is ample. Let $\omega > 0$ be a Kähler representative of $[K_X]$ and $\mu$ a smooth non degenerate volume form on $X$ with $Ric(\mu) = -\omega$. Then the Monge-Ampère equation $(\omega + dd^c\varphi)^n = e^\varphi \mu$ satisfies all the hypotheses of Theorem 4.10 and has a unique (viscosity=pluripotential) solution $\varphi$. On the other hand, the Aubin-Yau theorem [2],[70] implies that it has a unique smooth solution $\varphi_{KE}$ (and $\omega + dd^c\varphi_{KE}$ is the canonical Kähler-Einstein metric on $X$). Uniqueness of the pluripotential solution insures $\varphi = \varphi_{KE}$ hence the potential of the canonical KE metric on $X$ is the envelope of the subsolutions to $(\omega + dd^c\varphi)^n = e^\varphi \mu$.

## 5. Weak versions of Calabi-Yau and Aubin-Yau theorems

In this section we apply the viscosity approach to show that the canonical singular Kähler-Einstein metrics constructed in [41] have continuous potentials.

### 5.1. Manifolds of general type

Assume $X$ is compact Kähler and $\mu$ is a continuous volume form with semi-positive density. Fix $\beta$ a Kähler form on $X$.

COROLLARY 5.1. — *Assume that $\omega \geqslant 0$ is a closed $(1,1)-$form and $\mu > 0$ is a continuous positive volume form. Then $(DMA)_{\varepsilon,\mu}$ has a unique viscosity solution $\varphi$, which is also the unique solution in the pluripotential sense. Hence it is a continuous $\omega-$psh function.*

*Proof.* — Indeed the global comparison principle holds in this case and Theorem 1.13 and Theorem 4.10 enable us to conclude. □

We are now ready to establish that the (pluripotential) solutions of some Monge-Ampère equations considered in the first section are continuous.

THEOREM 5.2. — *Assume $X$ is a compact Kähler manifold, $\omega$ is a semi-positive $(1,1)$-form with $\int_X \omega^n > 0$ and $\mu \geqslant 0$ is a semi-positive continuous volume form on $X$ normalized by $\mu(X) = 1$. Then there exists a unique continuous $\omega$-plurisubharmonic function $\varphi$ which is the viscosity (equivalently pluripotential) solution to the degenerate complex Monge-Ampère equation*

$$(\omega + dd^c\varphi)^n = e^{\varphi}\mu$$

*Proof.* — Observe that if moreover $\mu$ has positive density, the result is an immediate consequence of Corollary 5.1 together with the unicity statement Proposition 1.11.

It remains to relax the positivity assumption made on $\mu$. From now on $\omega$ is semi-positive and big and $\mu$ is a probability measure with semi-positive continuous density. We can solve

$$(\omega + dd^c\varphi_\varepsilon)^n = e^{\varphi_\varepsilon}[\mu + \varepsilon\beta^n]$$

where $\varphi_\varepsilon$ are continuous $\omega$-psh functions and $0 < \varepsilon \leqslant 1$. As we already observed in the firs section this implies that the family $M_\varepsilon := \sup_X \varphi_\varepsilon, \varepsilon \in ]0,1]$ is bounded.

We infer that $(\varphi_\varepsilon)$ is relatively compact in $L^1(X)$. It follows from Theorem 1.8 that $(\varphi_\varepsilon)$ is actually uniformly bounded, as $\varepsilon$ decreases to zero.

Using again the stability estimates Theorem 1.8, we get

$$||\varphi_\varepsilon - \varphi_{\varepsilon'}||_{L^\infty} \leqslant C\,(||\varphi_\varepsilon - \varphi_{\varepsilon'}||_{L^1})^{\frac{1}{n+2}}.$$

Thus, if $(\epsilon_j)$ is a sequence decreasing to zero as $j$ goes to $+\infty$ such that $(\varphi_{\varepsilon_j})_j$ converges in $L^1$, $(\varphi_{\varepsilon_j})$ is actually a Cauchy sequence of continuous functions, hence it uniformly converges, to the unique continuous pluripotential solution $\varphi$ of $(DMA)_{1,\mu}$. From this, it follows that $(\varphi_\varepsilon)$ has a unique cluster value in $L^1$ when $\epsilon$ decreases to 0 hence converges in $L^1$. The preceding argument yields uniform convergence.

Theorem 4.10 insures that $\varphi$ is also a viscosity subsolution. Remark 6.3 p. 35 in [33] actually enables one to conclude that $\varphi$ is indeed a viscosity solution. □

COROLLARY 5.3. — *If $X^{can}$ is a canonical model of a general type projective manifold then the canonical singular Kähler-Einstein metric on $X^{can}$ constructed in [41] has continuous potentials.*

*Proof.* — This is a straightforward consequence of the above theorem, working in a log resolution of $X^{can}$, where $\omega = c_1(K_X, h)$ is the pull-back of the Fubini-Study form from $X^{can}$ and $v = v(h)$ has continuous semi-positive density, since $X^{can}$ has canonical singularities. □

### 5.2. Continuous Ricci flat metrics

We now turn to the study of the degenerate equations $(DMA)_{0,\mu}$

$$(\omega + dd^c\varphi)^n = \mu$$

on a given compact Kähler manifold $X$. Here $\mu = f\mu_0$ is a degenerate volume form with density $f \in L^p(X)$ $(p > 1)$ and $\omega$ is a smooth semi-positive closed $(1,1)$ form on $X$. We assume that $\mu$ is normalized so that

$$\mu(X) = \int_X \omega^n.$$

This is an obvious necessary condition in order to solve the equation

$$(DMA)_{0,\mu} \qquad\qquad (\omega + dd^c\varphi)^n = \mu,$$

on $X$. Bounded solutions to such equations have been provided in [41] when $\mu$ has $L^p$-density, $p > 1$, by adapting the arguments of [58]. Our aim here is to show that these are actually *continuous*.

Theorem 5.4. — *Let $\mu = f\mu_0$ be a degenerate volume form with density $0 \leqslant f \in L^p(X)$ $(p > 1)$ and $\omega \geqslant 0$ is a smooth semi-positive closed* $(1,1)$ *form on $X$. We assume that $\mu$ is normalized so that*

$$\mu(X) = \int_X \omega^n.$$

*Then the complex Monge-Ampère equation $(DMA)_{0,\mu}$ has a unique continuous pluripotential solution $\varphi$ such that $\int_X \varphi\mu = 0$.*

The plan is to combine the viscosity approach for the family of equations $(\omega + dd^c\varphi)^n = e^{\varepsilon\varphi}\mu$, together with the pluripotential tools developed in [58, 27, 45, 41, 42].

*Proof.* — Assume first that $\mu$ is a continuous positive volume form. For $\varepsilon > 0$ we let $\varphi_\varepsilon$ denote the unique viscosity (or equivalently pluripotential) $\omega$-psh continuous solution of the equation

$$(\omega + dd^c\varphi_\varepsilon)^n = e^{\varepsilon\varphi_\varepsilon}\mu,$$

given by Theorem 5.1. As before we see that $M_\varepsilon := \sup_X \varphi_\varepsilon$ is uniformly bounded. We infer that $(\varphi_\varepsilon)$ is bounded in $L^1$ and the Monge-Ampère measures $(\omega + dd^c\varphi_\varepsilon)^n$ have uniformly bounded densities in $L^\infty$. Once again by Theorem1.8 this family of continuous $\omega$-psh functions is uniformly Cauchy hence converges to a *continuous* pluripotential solution of $(DMA)_{0,\mu}$. This pluripotential solution is also a viscosity solution by ([33], Remark 6.3).

As we already observed in section 1, the solutions of $(DMA)_{0,\mu}$ are unique, up to an additive constant. It is natural to wonder which solution is reached by the the family $\varphi_\varepsilon$. Observe that $\int_X e^{\varepsilon\varphi_\varepsilon}\mu = \int_X \mu = \int_X \omega^n$ thus

$$0 = \int_X \frac{e^{\varepsilon\varphi_\varepsilon} - 1}{\varepsilon}\mu = \int_X \varphi_\varepsilon\mu + o(1)$$

hence the limit $\varphi$ of $\varphi_\varepsilon$ as $\varepsilon$ decreases to zero is the unique solution of $(DMA)_{0,\mu}$ that is normalized by $\int_X \varphi\,\mu = 0$.
Now assume that $\mu = f\mu_0$ has an $L^p-$density with $p > 1$. Let $f_j$ a sequence of smooth positive functions on $X$ such $f_j \to f$ in $L^p(X)$.

By the previous case there for each $j \in \mathbb{N}$, there exists a continuous solution $\varphi_j \in PSH(X,\omega)$ to the equation

$$(\omega + dd^c\varphi_j)^n = f_j\mu_0,$$

with $\int_X \varphi_j\mu = 0$. By [45] the sequence $\varphi_j$ is bounded in $L^1(X)$ and again by Theorem 1.8, the sequence $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence of continuous

$\omega-$psh functions for the uniform norm on $X$, hence it converges to a continuous $\omega-$psh function $\varphi$ which is a solution to the equation $(DMA)_{0,\mu}$. □

Note that the way we have produced solutions (by approximation through the non flat case) is independent of [2, 70].

Now we can prove that the Ricci-flat singular metrics constructed in ([41], Theorem 7.5) have continuous potentials.

Corollary 5.5. — *Let $X$ be a compact $\mathbb{Q}$-Calabi-Yau Kähler space. Then $X$ admits a Ricci-flat singular metric with continuous potentials.*

## 6. Concluding remarks

### 6.1. The continuous Calabi conjecture

The combination of viscosity methods and pluripotential techniques yields a soft approach to solving degenerate complex Monge-Ampère equations of the form

$$(\omega + dd^c\varphi)^n = e^{\varepsilon\varphi}\mu$$

when $\varepsilon \geqslant 0$.

Recall that here $X$ is a compact Kähler n-dimensional manifold, $\mu$ is a semi-positive volume form with $L^p$-density $p > 1$ and $\omega$ is smooth closed $(1,1)$-form whose cohomology class is semi-positive and big (i.e. $\{\omega\}^n > 0$).

Altogether this provides an alternative and independent approach to Yau's solution of the Calabi conjecture [70]: we have only used upper envelope constructions (both in the viscosity and pluripotential sense), a global (viscosity) comparison principle and Kolodziej's pluripotential techniques ([58, 41]).

It applies to degenerate equations but yields solutions that are merely continuous (Yau's work yields smooth solutions, assuming the cohomology class $\{\omega\}$ is Kähler and the measure $\mu$ is both positive and smooth). However it is possible to prove that the solutions are Hölder continuous locally in the ample locus $\Omega_\alpha$ of the class $\{\omega\}$ (see [37]).

Note that a third (variational) approach has been studied recently in [19]. It applies to even more degenerate situations where $\mu$ might be singular, providing solutions with less regularity (that belong to the so called class of finite energy).

### 6.2. The case of a big class

Our approach applies equally well to a slightly more degenerate situation. We still assume here that $(X, \omega_X)$ is a compact Kähler manifold of dimension $n$, but $\mu = f\mu_0$ is merely assumed to have density $f \geqslant 0$ in $L^\infty$ and moreover the smooth real closed $(1,1)$-form $\omega$ is no longer assumed to be semi-positive: we simply assume that its cohomology class $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ is *big*, i.e. contains a Kähler current.

It follows from the work of Demailly [36] that one can find a Kähler current in $\alpha$ with analytic singularities: there exists an $\omega$-psh function $\psi_0$ which is smooth in a Zariski open set $\Omega_\alpha$ and has logarithmic singularities of analytic type along $X \setminus \Omega_\alpha = \{\psi_0 = -\infty\}$, such that $T_0 = \omega + dd^c\psi_0 \geqslant \varepsilon_0\omega_X$ dominates the Kähler form $\varepsilon_0\omega_X$, $\varepsilon_0 > 0$.

We refer the reader to [21] for more preliminary material on this situation. Our aim here is to show that one can solve $(DMA)_{1,\mu}$ in a rather elementary way by observing as in the first section that the (unique) solution is the upper envelope of subsolutions. We let as before

$$\mathcal{F} := \{\varphi \in PSH(X,\omega) \cap L^\infty_{loc}(\Omega_\alpha) \,/\, (\omega + dd^c\varphi)^n \geqslant e^\varphi v \text{ in } \Omega_\alpha\}$$

denote the set of all (pluripotential) subsolutions to $(DMA)_{1,\mu}$ (which only makes sense in $\Omega_\alpha$).

Observe that $\mathcal{F}$ is not empty: since $T_0^n$ dominates a volume form and $\mu$ has density in $L^\infty(X)$, the function $\psi_0 - C$ belongs to $\mathcal{F}$ for $C$ large enough. We assume for simplicity $C = 0$ (so that $\psi_0 \in \mathcal{F}$) and set

$$\mathcal{F}_0 := \{\varphi \in \mathcal{F} \,/\, \varphi \geqslant \psi_0\}.$$

Proposition 6.1. — *The class $\mathcal{F}_0$ is uniformly upper bounded on $X$. It compact (for the $L^1$-topology).*

*Proof.* — The proof is the same as for Lemma 1.12. We first show that $\mathcal{F}_0$ is uniformly bounded from above (by definition it is bounded rom below by $\psi_0$). We can assume without loss of generality that $\mu$ is normalized so that $\mu(X) = 1$. Fix $\psi \in \mathcal{F}_0$. It follows from the convexity of the exponential that

$$\exp\left(\int \psi\mu\right) \leqslant \int e^\psi \mu \leqslant \int (\omega + dd^c\psi)^n \leqslant Vol(\alpha).$$

All integrals here are computed on the Zariski open set $\Omega_\alpha$. We refer the reader to [21] for the definition of the volume of a big class.

We infer

$$\sup_X \psi \leqslant \int \psi\mu + C_\mu \leqslant \log Vol(\alpha) + C_\mu,$$

where $C_\mu$ is a uniform constant that only depends on the fact that all $\omega$-psh functions are integrable with respect to $\mu$ (see [45]). This shows that $\mathcal{F}_0$ is uniformly bounded from above by a constant that only depends on $\mu$ and $Vol(\alpha)$.

We now check that $\mathcal{F}_0$ is compact for the $L^1$-topology. Fix $\psi_j \in \mathcal{F}_0^{\mathbb{N}}$. We can extract a subsequence that converges in $L^1$ and almost everywhere to a function $\psi \in PSH(X,\omega)$. Since $\psi \geqslant \psi_0$, it has a well defined Monge-Ampère measure in $\Omega_\alpha$ and we need to check that $(\omega + dd^c\psi)^n \geqslant e^\psi \mu$. We proceed in the same way as Lemma 1.12. □

It follows that

$$\psi := \sup\{\varphi \,/\, \varphi \in \mathcal{F}_0\},$$

the upper envelope of pluripotential subsolutions to $(DMA)_{1,\mu}$, is a well defined $\omega$-psh function which is locally bounded in $\Omega_\alpha$.

Theorem 6.2. — The function $\psi$ is a pluripotential solution to $(DMA)_{1,\mu}$.

*Proof.* — The proof proceed by balayage locally in $\Omega_\alpha$ as in the proof of Theorem 1.13. □

*Remark 6.3.* — The situation considered above covers in particular the construction of a Kähler-Einstein current on a variety $V$ with ample canonical bundle $K_V$ and canonical singularities, since the canonical volume form becomes, after passing to a desingularisation $X$, a volume form $\mu = f\mu_0$ with density $f \in L^\infty$.

The more general case of log-terminal singularities yields density $f \in L^p$, $p > 1$. One can treat this case by an easy approximation argument: setting $f_j = \min(f, j) \in L^\infty$, one first solves $(\omega + dd^c\varphi_j)^n = e^{\varphi_j} f_j \mu_0$ and observe (by using the comparison principle) that the $\varphi_j's$ form a decreasing sequence which converges to the unique solution of $(\omega + dd^c\varphi)^n = e^\varphi f\mu_0$.

## 6.3. More comparison principles

Let again $B \subset \mathbb{C}^n$ denote the open unit ball and let $B' = (1+\eta)B$ with $\eta > 0$ be a slightly larger open ball. Let $u, u' \in PSH(B')$ be plurisubharmonic functions. By convolution with an adequate non negative kernel of the form $\rho_\epsilon(z) = \epsilon^{-2n}\rho_1(\frac{z}{\epsilon})$ we construct $(u_\epsilon)_{\eta>\epsilon>0}$ a family of smooth plurisubharmonic functions decreasing to $u$ as $\epsilon$ decreases to 0.

Lemma 6.4. —

$$\forall z \in B\ u(z) + u'(z) = \limsup_{n\to\infty} \sup\{u'(x) + u_{1/j}(x) | j \geqslant n,\ |x - z| \leqslant 1/n\}$$

*Proof.* — Indeed, we have, if $2/n < \eta$:

$$\begin{aligned} u(z) + u'(z) &\leqslant \sup\{u'(z) + u_{1/j}(z) | j \geqslant n\} \\ &\leqslant \sup\{u'(x) + u_{1/j}(x) | j \geqslant n,\ |x - z| \leqslant 1/n\} \\ &\leqslant \sup\{u'(x) + u(x) |\quad |x - z| \leqslant 2/n\}. \end{aligned}$$

Since $u + u'$ is upper semicontinuous, we have:

$$u(z) + u'(z) = (u + u')^*(z) = \lim_{n\to\infty} \sup\{u + u'(x) |\quad |x - z| \leqslant 2/n\}.$$

□

Lemma 6.5. — *Let $\phi$ a bounded psh function on $B$ and $\mu$ a continuous non negative volume form such that $e^{-\phi}(dd^c\phi)^n \geqslant \mu$ in the viscosity sense.*

*Let $\psi$ be a bounded psh function and $\nu$ a continuous positive volume form, both defined on $B'$ such that $(dd^c\psi)^n \geqslant \nu$.*

*Then $\exists C, c > 0$ depending only on $\|\psi\|_{L^\infty}, \|\phi\|_{L^\infty}$ such that for every $\epsilon \in [0, 1]$ $\Phi = \phi + \epsilon\psi$ satisfies:*

$$e^{-\Phi}(dd^c\Phi)^n \geqslant (1 - \epsilon)^n e^{-C\epsilon}\mu + c\epsilon^n\nu$$

*in the viscosity sense in $B$.*

*Proof.* — We may assume $\epsilon > 0$ and $\nu$ to be smooth. Let us begin by the case when $\psi$ is of class $C^2$. Let $x_0 \in B$ and $q \in C^2$ such that $q(x_0) = \Phi(x_0)$ and $\Phi - q$ has a local maximum at $x_0$. Then, $\phi - (q - \epsilon\psi)$ has a local maximum at $x_0$.

We deduce:

$$dd^c(q - \epsilon\psi)_{x_0} \geqslant 0$$

$$e^{-q(x_0)+\epsilon\psi(x_0)}(dd^c(q - \epsilon\psi))_{x_0})^n \geqslant \mu_{x_0}.$$

Using the inequality $(dd^c q)^n_{x_0} \geqslant (dd^c(q - \epsilon\psi))_{x_0})^n + \epsilon^n(dd^c\psi)^n$, we conclude.

We now treat the general case. Since $\psi$ is defined on $B'$ we can construct by the above classical mollification a sequence of $C^2$ psh functions $(\psi_{1/k})$ converging to $\psi$ as $k$ goes to $+\infty$.

We know from the proof of Proposition 3.3 that $(dd^c\psi_k)^n \geqslant ((\nu^{1/n})_{1/k})^n = \nu_k$ in both the pluripotential and viscosity sense.

We conclude from the previous case that $\Phi_k = \phi + \epsilon\psi_k$ satisfies

$$c\epsilon^n \nu_k + (1-\epsilon)^n e^{-C\epsilon}\mu \leqslant e^{-\Phi_k}(dd^c\Phi_k)^n$$

in the viscosity sense. Since $\mu_k > 0$, we have:

$$c\epsilon^n \nu_k + (1-\epsilon)^n e^{-C\epsilon}\mu - e^{-\Phi_k}(dd^c\Phi_k)^n_+ \leqslant 0$$

in the viscosity sense.

By Lemma 6.1 p. 34 and Remark 6.3 p. 35 in [33], we conclude that

$$\bar{\Phi} = \limsup_{n\to\infty} \sup\{\Phi_j(x) | j \geqslant n,\ |x-z| \leqslant 1/n\}$$

satisfies the limit inequation

$$e^{-\bar{\Phi}}(dd^c\bar{\Phi})^n_+ \geqslant (1-\epsilon)^n e^{-C\epsilon}\mu + c\epsilon^n\nu$$

in the viscosity sense. Now Lemma 6.4 implies that $\bar{\Phi} = \Phi$. Since $\nu > 0$, the proof is complete. □

THEOREM 6.6. — *Let $X$ be a compact Kähler manifold and $\omega \geqslant 0$ be a semi-kähler smooth form.*

*Then, the global viscosity comparison principle holds for $(DMA)_{1,\mu}$ for any non negative continuous measure $\mu$ with $\mu(X) > 0$.*

*Proof.* — This is a variant of the argument sketched in [53] sect. V.3 p. 56.

Let $\overline{u}$ be a supersolution and $\underline{u}$ be a subsolution. Perturb the supersolution $\overline{u}$ setting $\overline{u}_\delta = \overline{u} + \delta$. This $\overline{u}_\delta$ is a supersolution to $(DMA)_{1,\tilde{w}}$ for every continuous volume form $\tilde{w}$ such that $\tilde{w} \geqslant e^{-\delta}\mu$.

We can always assume that $\mu(X) = \int_X \omega^n$. Choose $\nu > 0$ a continuous positive volume form such that $\nu(X) = \int_X \omega^n$. We can construct $\psi$ a continuous quasiplurisubharmonic functions such that, in the viscosity sense

$$(\omega + dd^c\psi)^n = \nu.$$

Perturb the subsolution $\underline{u}$ setting

$$\underline{u}_\epsilon = (1-\epsilon)\underline{u} + \epsilon\psi.$$

By Lemma 6.5, $\underline{u}_\epsilon$ satisfies, in the viscosity sense

$$e^{-(1+\epsilon)u}(\omega + dd^c u)^n \geqslant \left(\frac{1-\epsilon}{1+\epsilon}\right)^n e^{-C\epsilon}\mu + c\left(\frac{\epsilon}{1+\epsilon}\right)^n \nu$$

This in turn implies that $\underline{u}_\epsilon$ satisfies, in the viscosity sense:

$$e^{-u}(\omega + dd^c u)^n \geqslant e^{-\epsilon\|u\|_\infty}\left[\left(\frac{1-\epsilon}{1+\epsilon}\right)^n e^{-C\epsilon}\mu + c\left(\frac{\epsilon}{1+\epsilon}\right)^n \nu\right].$$

Hence $\underline{u}_\epsilon$ satisfies, in the viscosity sense:

$$e^{-u}(\omega + dd^c u)^n \geqslant \tilde{\nu}$$

whenever $\tilde{\nu} \leqslant e^{-\epsilon\|u\|_\infty}((\frac{1-\epsilon}{1+\epsilon})^n e^{-C\epsilon}v + c(\frac{\epsilon}{1+\epsilon})^n \nu)$.

Choosing $1 \gg \delta \gg \epsilon > 0$, we find a continuous volume form $\tilde{\nu} > 0$ such that $\overline{u}_\delta$ is a supersolution and $\underline{u}_\epsilon$ is a viscosity subsolution of $e^{-u}(\omega + dd^c u)^n = \tilde{\nu}$. Using the viscosity comparison principle for $\tilde{\nu}$, we conclude that $\overline{u}_\delta \geqslant \underline{u}_\epsilon$. Letting $\delta \to 0$, we infer $\overline{u} \geqslant \underline{u}$. □

This comparison principle has been inserted here for completeness. It could have been used instead of the pluripotential-theoretic arguments to establish existence of a viscosity solution in the case $\mu \geqslant 0$ of Theorem 5.2. This could be useful in dealing with similar problems where pluripotential tools are less efficient.

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