ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

TIMOTHY PORTER

Variations on a theme of homotopy

Tome XXII, nº 5 (2013), p. 1045-1089.

<http://afst.cedram.org/item?id=AFST_2013_6_22_5_1045_0>

Annales de la Faculté des Sciences de Toulouse Vol. XXII, n° 5, 2013
pp. 1045–1089

Variations on a theme of homotopy

Timothy Porter[1]

Abstract. — The aim of this article is to bring together various themes from fairly elementary homotopy theory and to examine them, in part, from a historical and philosophical viewpoint.

Résumé. — Le but de cet article est de réunir quelques thèmes de la théorie élémentaire d'homotopie, et de les examiner, au moins partiellement, d'un point de vue historique et philosophique.

Contents

[1] School of Computer Science, University of Bangor, Bangor, Gwynedd, LL57 1UT, U.K.
t.porter@bangor.ac.uk
http://ncatlab.org/timporter/show/HomePage and http://pages.bangor.ac.uk/ mas013/

I would like to thank Sébastien Maronne and Etienne Fieux for the welcome during my stay in Toulouse, and to include, in my thanks, the other organisers and sponsors of this very interesting meeting.

1. Introduction

At several points in the talks at the Workshop, and in the articles here, various constructions and ways of thought were mentioned, for instance, composition of curves, and homotopies, the process of identification modulo homotopy type, and the history of the fundamental groupoid, in particular. In this article, I hope to discuss some ways of thinking of those parts of elementary homotopy theory that seem related to these, to make some comments about them from the perspective of a homotopy theorist, and to introduce various thematic aspects of that 'story' that were not mentioned, yet seem to me to be related or relevant to it.

One theme will be the relationship between homotopy and identification, another will be how elementary homotopy relates to *abstract homotopy theories* via the simple idea of a cylinder functor. (This is also explored in [35] and more briefly in [48].) This will then be linked with homotopy coherence, as this is, I believe, quite central in understanding the question of the behaviour of higher homotopies when thinking about the 'identity' question. Finally we will look at the question of theories in which there are different types of related phenomena which seem to require several types of homotopy in action 'simultaneously' if the determination of the relevant 'homotopy type' is to be feasible.

Throughout I will try to keep in view the various historical and philosophical aspects that are involved in the development of this overall picture.

2. Identity, identification and homotopy

In his talk in the Workshop, Jean-Pierre Marquis examined the notion of 'identity' in philosophy from the point of view of homotopy theory, and stressed the importance of models in the way that homotopy theorists think of spaces and spatial phenomena. The process seems, in part, to involve, to me, one of 'identification', and to start with I want to examine the relationship between the two ideas 'identity' and 'identification' as they are *used* in homotopy theory.

The word 'identification' is important in two related senses (with respect to homotopy theory). Naively one of the aims of classical homotopy theory is to 'identify' spaces, that is, given some space, the theory should come up with 'what it is', e.g. perhaps by listing all possible examples of spaces, which are 'really different' from each other, then the theory should identify which one the given space is. This is the use of 'identify' as meaning 'recognise' or 'classify', to find its 'identity'. If one looks at other similar human activities, one has various common feature. For instance, I try to identify the birds I see. I see a large white bird with longish dark legs, a long straight beak and yellow feet. It is wading in a shallow lagoon, and I identify it by looking in a bird book (list) and comparing known features, location with regards to known places it hangs out, etc. The analogue for a homotopy theorist might be: from the various algebraic and geometric invariants that I have worked out, the given space is of the homotopy type of a wedge (= one point union) of five 2-spheres. The process of recognition establishes the 'identity' of the space within given accepted bounds, namely those made precise by the definition of 'homotopy type' (or 'species' for the case of birds).

Actually in my example, there is another related use of 'identify'. The wedge of 2-spheres that was used is obtained from a union of five 2-spheres by picking a single point in each as a base point and then *identifying* these base points. In this process, 'different' points *become the same*. An extreme case of this is in the formation of the set, $\pi_0(X)$, of (arcwise) connected components of a space. Two points will be *identified* if there is a curve joining them. Of course, the instance of *homotopy type* is an example of the same sort of quotienting or identifying process, this time applied to the class of 'all spaces'. Thinking of the identification of a space, we now have a process that corresponds to it, i.e., *classification* is related to *identification* in both senses.

Where is homotopy in all this? I would like to propose a sort of 'thematic observation' about these situations:

A homotopy is a reason for an identification.

The case of connected components is a good one here. The 'reasons' two points are identified in $\pi_0(X)$ are the paths joining them. Such a curve is a *reason* that they are in the same component. (This is very like the relationship between the intuitions of homotopy theory and the 'proofs as morphisms' idea. To prove two points are in the same component, it is a good idea to give a path joining them.) If there is one path joining the points, there will usually be many different paths.

Of course, curves between points are just a particular case of homotopies between maps, that is, take the two maps from a singleton space to X that pick out the two points. We also have homotopies between curves. Homotopies are themselves morphisms, so we can consider reasons for identifying reasons for identifying two morphisms (and no that is not a slip of the hand while I was doing a copy and paste!) In the formation of the fundamental groupoid, $\Pi_1(X)$, of a space X, one identifies some of the reasons for points to be in the same component, and thus for them to be identified in the formation of $\pi_0(X)$. The fundamental groupoid, thus, takes this type of process one step further. We will examine later some situations where one has to use homotopies between homotopies between homotopies ... Homotopy theory takes ‘reasons’ seriously!

In the next ‘theme’, we turn to a detailed study of homotopy from a nearby viewpoint as this may indicate some of the complexities here.

3. Cylinders

We will give a fairly discursive introduction to parts of abstract homotopy theory exploring, as we go, some of the points that can easily be lost in treatments of the basic notions. The point will thus not be to define and study the notions, but rather to reflect on the definitions and the resulting theory.

One of the basic intuitions behind the notion of homotopy is that of a deformation, of one curve into another, or, more generally, of one continuous map into another. ‘Deformation’ here means that there is a ‘continuously varying family’ of curves, or maps, linking the two given maps. (The idea of ‘deforming’ seems to have a natural ‘time’ aspect, yet that is suppressed in the usual developments. Towards the end of this article, we will briefly examine the reinstatement of this idea of ‘time’ in recent work.) This idea of a ‘continuously varying family’ can be encoded in a simple way using a cylinder. (The dual notion of ‘*cocylinder*’ can also be used, but requires more setting up. Once we have abstracted from the topological cylinder to a cylinder functor, then the notion of cocylinder is easy to give, but this merely ignores the work needed when topologising spaces of curves, that is, in proving that there is a cocylinder for the various topological cases of interest.)

We take $I = [0, 1]$, the closed unit interval in the real line. Suppose we have a space, X, we form a *cylinder* on X by forming the product, $X \times I$. This comes with certain basic structure. There are inclusions of X into $X \times I$ at the bottom and at the top. These, and their abstractions, will be denoted

$$e_0(X), e_1(X) : X \to X \times I$$

with $e_i(X)(x) = (x, i)$ for $i = 0, 1$. There is also a map going the other direction, given by the first projection of the product,

$$\sigma(X) : X \times I \to X,$$

$$\sigma(X)(x, t) = x.$$

(Sometimes this is spoken of as 'crushing' or 'collapsing' the cylinder onto its base.) These maps are natural in both the common sense of the term and in the sense that, if $f : X \to Y$ is a continuous map, we can form

$$f \times I : X \times I \to Y \times I$$

by $(f \times I)(x, t) = (f(x), t)$, and then, for instance, $(f \times I)e_0(X) = e_0(Y)f$. Furthermore these maps clearly satisfy $\sigma e_0 = \sigma e_1 = Id$, i.e., if we are considering assigning $X \times I$ to X as a functor on the category of spaces, e_0, e_1 and σ are natural transformations and the 'natural' level at which to state and consider these equations is at this categorical level.

We can abstract this construction to more general settings and it is useful to pause and examine why this is worth doing. As is briefly discussed right at the start of chapter I of Kamps and Porter, [35]: '*The unification (of ideas) through abstraction leads often to a greater applicability and transparency of proofs giving 'transportability' to other contexts as a bonus.*'

Historically the abstraction of axioms occurred slightly earlier for (co)homology (Eilenberg and Steenrod, 1952) and homological algebra (Cartan and Eilenberg, 1956), than for homotopy theory, (usually placed as relating to Quillen's homotopical algebra lecture notes of 1967). None the less, the abstraction from a topological cylinder as a base for a notion of abstract homotopy is 'early', as it was initiated by Kan in 1955, [36]. Initially he used, as we will, a cylinder to produce a cubically based set-up, since this is perhaps more natural, but in 1957, in [37], he had already started to convert to simplicial sets, as cubical sets had some disadvantages at a technical level. (The history of this transition and the reasons for it have been discussed in several places on the web and in the literature.) Cubical methods, as we will see, are very easy to use in the elementary theory, ..., but we are 'getting ahead of ourselves'! We need the notion of a cylinder functor. (We will use [35] as a reference for further details and developments, including some discussion of the various approaches to abstract homotopy.)

DEFINITION. — Let $\mathcal{C}$ be a category. A *cylinder*, **I**, on $\mathcal{C}$ is a functor, (called a *cylinder functor*),

$$(-) \times I : \mathcal{C} \to \mathcal{C},$$

together with three natural transformations,

$$e_0, e_1 : Id_{\mathcal{C}} \to (-) \times I$$

and

$$\sigma : (-) \times I \to Id_{\mathcal{C}},$$

such that $\sigma e_0 = \sigma e_1 = Id_{\mathcal{C}}$.

It is important to note that writing $X \times I$, for X in $\mathcal{C}$, does not imply that there is some object I in $\mathcal{C}$, nor that this is 'product' with some object. This is just a graphic choice of notation. In many categories 'of interest', a cylinder $(-) \times I$ *is* realisable as a product or tensor product with some object, but this need not be, and, indeed, is not, always the case.

Examples. — There are well known examples of cylinders starting with Top, a category of topological spaces containing $I = [0, 1]$ and closed under products with it. Other examples include

- $Ch(Mod(R))$, the category of chain complexes of modules over some ring, R;
- $Grpd$, the category of groupoids, (i.e., small categories in which all morphisms are isomorphisms) together with functors between them;
- Cat, the category of (small) categories and functors;
- $\mathcal{S}$, the category of simplicial sets;
- $Simp(Ab)$, the category of simplicial Abelian groups,
  and so on.

A cylinder on an opposite category, $\mathcal{C}^{op}$, is called a *cocylinder* on $\mathcal{C}$. Several of the above examples have also a cocylinder defined on them.

An abstract cylinder, $\mathbf{I}$, on $\mathcal{C}$ allows the introduction of a quasi-geometric notion of homotopy between morphisms, by following the model given by classical topological homotopy. We assume, of course, that $\mathcal{C}$ has a cylinder $\mathbf{I}$, as above:

Definition. — Two morphisms, $f_0, f_1 : X \to Y$, in $\mathcal{C}$ are said to be *homotopic*, written $f_0 \simeq f_1$, (or, if there is a risk of confusion due to having two cylinders around, $f_0 \simeq_{\mathbf{I}} f_1$), if there is a morphism,

$$h : X \times I \to Y,$$

in $\mathcal{C}$, such that $f_0 = he_0(X)$ and $f_1 = he_1(X)$. In this case, the morphism, h, is called a *homotopy* between the morphisms defined on the two ends of the cylinder.

We really should say that f_0 and f_1 are *directly homotopic* in the above case and that the homotopy goes *from* f_0 *to* f_1, as, at this level of generality, the relation of 'being homotopic' need neither be transitive nor symmetric. It will be reflexive, since if $f : X \to Y$, the constant homotopy $f\sigma(X) : X \times I \to Y$ shows that $f \simeq f$. Homotopy is also automatically compatible with composition in the sense that, if $f_0 \simeq f_1$ and $g : Y \to Z$, then $gf_0 \simeq gf_1$, and, if $k : W \to X$, then $f_0k \simeq f_1k$.

A researcher used merely to the topological case may find it a bit fastidious to consider such a general case, but there are interesting and useful situations in which transitivity and / or symmetricity fails and that failure is an important piece of information about the context involved. Of course, starting with such a situation, there are well known ways of considering a symmetric transitive closure, but as we will be arguing that the actual homotopy is more important than its mere existence, such closure operations come 'with a price'!

Are transitivity and symmetricity best thought of as *properties* to be *imposed* on an abstract cylinder, just as commutativity is *imposed* on a group in usage such as 'Let A be an Abelian group'? Perhaps not. An alternative approach might be to think of them *structurally* by looking at additional structure that implies those properties. This seems more in tune with the idea of 'homotopy being a reason for identification'. (There seems to be some connection here with the distinctions between 'stuff', 'structure' and 'properties' discussed in the nLab page, 'stuff, structure, property', see [42], but I am not sure what the exact relationship is.)

We will give two different versions of the structural approach. First by imposition of additional structural data as part of the 'signature' of the notion of cylinder, and then, without that extra data, we will consider how, from the basic cylinder structure on **I**, quite natural conditions on an object generated by **I** give useful properties on the relation of homotopy.

4. Adding extra 'structure' to the signature of a cylinder

To work towards a symmetric variant of a cylinder-based notion of homotopy, we can mimic the structure used in elementary introductions to the topological version of the notion. There one frequently uses the involution, $t \mapsto 1 - t$, defined on the unit interval, $[0, 1]$.

DEFINITION. — An *involution* on a cylinder, **I**, is a natural transformation

$$i : (-) \times I \to (-) \times I$$

such that $i \circ i = Id_{(-)\times I}$, and $ie_0 = e_1$ (and hence $ie_1 = e_0$ as well).

Clearly, if I has an involution, then the $\simeq$ defined using it will be symmetric, and homotopies can be 'reversed'. Note that to obtain symmetricity, the 'reason' is inverted in direction. There is not really a new or separate reason why given $f_0 \simeq f_1$, we have $f_1 \simeq f_0$, we merely reverse the direction of the 'reason'. This is so usual that we rarely notice it, but does correspond to what happens in many different contexts. In some proofs of the converse of a result, it is each step of the forward process that is reversible and the result is really obtained by reversing the proof. In other contexts, this may not be the case and the proof of the converse is truly different from the reverse of that of the forward direction.

In our list of examples, for all but two of them, the usual cylinders have involutions. Unfortunately, the important examples of Cat and $\mathcal{S}$ do not.

To handle transitivity, one possible approach is through a *subdivision.* The usual idea is that composition is an *algebraic inverse to subdivision.* (One finds this hinted at quite often, and it is made explicit by Ronnie Brown in many of his articles and books, see, for instance, [9], where, on page 101, he says: *Cubical methods, unlike globular or simplicial methods, allow for a simple algebraic inverse to subdivision, which is crucial for our local-to-global theorems.* We will return to cubical methods slightly later, but note the idea of linking composition and subdivision.)

For subdivision, and thus composition, we need for $\mathcal{C}$ to have pushouts (or, more exactly, for there to be enough pushouts to make sense of the constructions). We take two copies of $X \times I$ and 'glue them end to end', the copy of the first being attached to the bottom of the second using a pushout construction:

$$\begin{array}{ccc} X & \xrightarrow{e_0(X)} & X \times I \\ {\scriptstyle e_1(X)}\downarrow & & \downarrow{\scriptstyle i_2} \\ X \times I & \xrightarrow[i_1]{} & (X \times I)_{e_1(X)} \sqcup_{e_0(X)} (X \times I) \end{array}$$

(This looks like a cylinder made with $[0, 2]$ in the topological case. It is noticeable that the way in which the abstract concept is manipulated is very closely linked to that intuition.) We call this pushout $S(X)$, S for 'subdivided'. The construction makes S into a functor and there are two ends, $e_0^S(X)$, for the bottom of the first cylinder, and $e_1^S(X)$ for the top of the second one. (These can be easily defined diagrammatically and categorically, e.g., $e_0^S(X) = i_1 e_0(X)$.)

Definition. — A *subdivision* on a cylinder, $\mathbf{I}$, is a natural transformation,

$$s : (-) \times I \to S,$$

such that $s(X)e_0(X) = e_0^S(X)$ and $s(X)e_1(X) = e_1^S(X)$.

Sometimes one may demand compatibility with the projections to X as well, but we will not ask for that here.

If now we have $h_1 : f_0 \simeq f_1$ and $h_2 : f_1 \simeq f_2$, then we get a uniquely defined $h : S(X) \to Y$ such that $hi_1 = h_1$ and $hi_2 = h_2$, using the universal property of pushouts. It is then easy to see that $hs(X) : f_0 \simeq f_2$, so $\simeq$ is transitive and, better still, we have a method for composing homotopies. (We will write $hs = h_2 * h_1$ to emphasise the idea of composition.)

Several of the examples have such a subdivision, but the example of $\mathcal{S}$, the case of simplicial sets, does not.

If we want to push on to other deeper results on homotopy, another *structure* that can be used is the abstraction of the monoid structure on $I = [0, 1]$ in the topological case, and a glance at the *explicit* homotopies given in the treatment of the elementary theory of homotopy in many introductory textbooks gives several examples of formulae that use that multiplication, for instance, something of the form $s(1-t)$. Here s and t are variables taking values from $[0, 1]$, so this would give a map from I^2 to I and, as we said, uses the multiplication. This is mirrored by structure on the cylinder.

Definition. — A multiplication, m, on a cylinder is a natural transformation,

$$m : ((-) \times I) \times I \to (-) \times I,$$

such that ...

We will not, in fact, give the conditions that m must satisfy as that would take us somewhat away from the theme we want to explore. The exact conditions would, to some extent, depend on the context and the examples that are being kept in mind. Generally, if one looks at a particular result in elementary homotopy theory, one can usually translate it into the use of these structures with the addition of one or two others. This can be a highly successful exercise, especially when mixed with ideas such as that of the *homotopy extension property* and thus of *cofibration*. For instance, in his book 'Algebraic Homotopy', [3], Hans Baues uses a variable exchange on the square of the cylinder functor, $T : ((-) \times I) \times I \to ((-) \times I) \times I$, which mirrors swapping the two variables, $T(s, t) = (t, s)$, in the unit square. This he combines with cofibrations to get a very powerful and elegant theory.

It should be noticed that whilst the structure is given at one level, however, the 'equations' that have to be satisfied are encoded in the next level up. For instance, if a multiplication, m, exists and is to be useful, it would be usual for it to look as if it were associative, but that is going to need the use of $(((-) \times I) \times I) \times I$, as it is to mirror the equation

$m(m(r,s),t) = m(r,m(s,t))$, which involves three variables. In fact, as we can see even in the case of the reverse / involution, i, the involution does satisfy an equation $i \circ i = Id$, but the reverse homotopy that it defines is a 'reverse' not an inverse. There needs to be a 'reason' for it to be useful, at least a homotopy between $hi * h : f_0 \simeq f_0$ and the identity homotopy on f_0. The equations in ordinary algebraic structures get replaced by homotopies between the structural maps, and that is happening at, at least, one higher 'level of homotopy' than the structure.

This all may look complex, but in fact it can be quite easy to push through in many contexts. On the other hand, it is very rigid and requires lots of choices. For example, there are many different possible choices of a map, i, on $\mathbf{I}$ that would work just as well as the one we selected and described, at least in the topological case, and also many different subdivisions / compositions. Why should we 'privilege' one over the others? To see an alternative, we go back to the beginning and note some structure that is always there even for the simplest cylinder.

5. Cubical enrichment and another type of structure

In the above, we have seen, to a small extent, the use of homotopies between homotopies, i.e., 2-homotopies or double homotopies, which are maps from $(X \times I) \times I$ to Y. There is clearly no limit to the level of homotopy that could be used as we can go on iterating the application of the cylinder functor. We thus may have reasons for identification and 'reasons among such reasons', and so on. In some contexts, these 'higher reasons' initially may seem a bit difficult to interpret so as to see if they may be of use, but that may be because we have not, yet, in those contexts, a sufficient intuition about them.

In the case of spaces, the intuition is clearer and well known. For instance, on a 2-sphere, the equatorial loop is contractible to a point. The contracting homotopies that spring to mind use either the northern or southern hemispheres. There are thus two distinct simply conceived homotopies. There is no homotopy between these two basic homotopies, because of the 2-dimensional hole in the space. Of course, this example can be generalised to higher dimensional spheres. It will use higher level homotopies. These can always be organised into a cubical structure and we will give some definitions so as to be able to discuss this in a bit more detail. (We will use [35], as before, as a reference.)

Definition. — A *cubical set*, Q, consists of a sequence of sets, Q_n, $n \in \mathbb{N}$, and three families of maps,

$$0_n^i, 1_n^i : Q_n \to Q_{n-1},$$

for $i, n \in \mathbb{N}$, $1 \leqslant i \leqslant n$, which are called *face operators*, and

$$\zeta_n^j : Q_n \to Q_{n+1},$$

for $j, n \in \mathbb{N}$, $1 \leqslant j \leqslant n+1$, called *degeneracy operators*, and which are required to satisfy five families of relations, (see [35], p.19).

We will not give these relations in any detail as, for our discussion, it is the intuition behind them that is the only really important feature. The idea is that a cubical set is made up of n-cubes for all n. Any element, $q \in Q_n$, is an n-cube, so it then has 2^n faces which are, themselves, $(n-1)$-cubes. There are n different possible directions and so we have simple intuitions such as $0_n^i(q)$ being the start face or zero face of q in the ith direction, whilst $1_n^i(q)$ will be the target face or one-face in that direction. The start face may also be called the source, or the domain in the ith direction, whilst the one-face is the end face, and so on. Terminology varies.

The faces fit together, of course, so for instance, given $q \in Q_n$, and suppose $i < j$. We have the zero face in the jth direction has a one-face in the ith direction, which will be $1_{n-1}^i 0_n^j(q)$. This is an $(n-2)$ cube and could also be got as $0_{n-1}^{j-1} 1_n^i(q)$. The first of the families of relations mentioned encodes all of these 'face with face' interactions. The others encode relations between different degeneracy operators and between the face and the degeneracy operators.

To understand these relations, it may help to look at the singular cubical set of a space, X. This has $Q_n = Top(I^n, X)$, the set of continuous maps from an n-cube $I^n = [0,1]^n$ to X. If $q : I^n \to X$, then $0_n^i(q) : I^{n-1} \to X$ is given by $0_n^i(q)(t_1, \ldots, t_{n-1}) = q(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$; similarly for $1_n^i(q)$, with 0 replaced by 1, of course. These operators thus restrict the function q to the relevant $(n-1)$-dimensional face of I^n. If we now further restrict to the $j-1$st $(n-2)$-dimensional one face of that face, we will get $q(t_1, \ldots, 0, \ldots, 1, \ldots)$ obtained by inserting 0 and 1 at relevant places in the string of variables. This could also have been done by first inserting the 1 and then the 0. We thus have that $1_{n-1}^{j-1} 0_n^i(q) = 0_{n-1}^i 1_n^j(q)$. (If both faces are zero faces or both are one faces, then this is harder to talk about, but is no more difficult to write down!)

The degeneracy operators correspond, in the singular complex, to an $n+1$-cube that is constant in the jth direction. Of course, we then have doubly degenerate $(n+2)$-cubes and they can be written in two ways, giving another of the families of relations between the operators. The faces of a degenerate n-cube, q, are related to the degeneracies of the faces of q, and so on.

Remark. — Historically, cubical sets were considered by Kan in early papers and then he went over to working with simplicial sets. The question of what basic building blocks, such as simplices, cubes, etc., can be used for homotopy theory was raised by Grothendieck in his notes 'In pursuit of stacks', [26]. Each leads to a 'presheaf category' which can be compared with the category of spaces. (This is discussed in Cisinski, [11], with additional contributions, for instance, in Maltsiniotis, [39].)

Our reason for introducing cubical sets is that any cylinder, **I**, on a category $\mathcal{C}$, generates a cubical set structure on the sets of maps, homotopies, and higher homotopies as follows. We just take, for objects X and Y in $\mathcal{C}$,

$$Q(\mathcal{C})(X,Y)_n = \mathcal{C}(X \times I^n, Y),$$

that is the morphisms from an iterated cylinder $(X) \times I^n$ on X to Y, where, of course, $(-) \times I^n = ((-) \times I^{n-1}) \times I$. The structural maps, $e_0(X)$, $e_1(X)$, $\sigma(X)$, of the cylinder give the basic face operators and degeneracy operators in a fairly simple way. For instance, in $Q(\mathcal{C})(X,Y)_2$, the elements are double homotopies

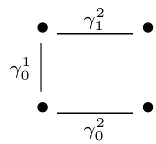
$$q : (X \times I) \times I \to Y,$$

and, for such a q, its faces are the morphisms, $q(e_0(X) \times I)$, $q(e_1(X) \times I)$, $q(e_0(X \times I))$ and $q(e_1(X \times I))$, whilst its degeneracies are $q(\sigma(X) \times I)$ and $q(\sigma(X \times I))$ from $X \times I^3$ to Y.

This defines a functor from $\mathcal{C}^{op} \times \mathcal{C}$ to the category of cubical sets and, actually, provides a cubical set enrichment of $\mathcal{C}$ (in the sense of enriched category theory). It is possible to replace the use of a cylinder based theory by the use of such a cubical enrichment. The resulting theory is more or less identical to that, but does lack a bit of the 'concreteness' that the cylinder based theory seems to have as well as being somewhat harder to manipulate.

As this structure is there whatever cylinder, **I**, that we have, we cannot expect it to behave better than the cylinder, **I**, itself. The change in perspective is to require properties of $Q(\mathcal{C})$ rather than directly of **I**. Of course, properties of $Q(\mathcal{C})$ are really just properties of **I**, but written in a more geometric language. The properties concerned were introduced right at the start of this cubical form of abstract homotopy theory in early work of Daniel Kan, in particular in [36], and they are the cubical form of the well known simplicial Kan conditions.

We will introduce these via a very simple special case. The idea is that we want to be able to fill 'boxes' with 'cubes'. As an example we will take what is called a $(2,1,1)$-box in a cubical set Q. This consists of three 1-cubes (i.e. edges) that fit together like three of the faces of the boundary of a square but without the 1-face in the 1-direction, i.e. a bit like



where the first direction is horizontal, and the second vertical. It, therefore, can be represented as a string, $(\gamma_0^1, -; \gamma_0^2, \gamma_1^2)$ of elements of Q_1 with a gap in the (1,1)-position, and such that $0_1^0\gamma_0^1 = 0_1^1\gamma_0^2$ and $1_1^0\gamma_0^1 = 0_1^0\gamma_1^2$.

The $E(2,1,1)$-*filler condition* on Q states that there is a *filler* for any such $(2,1,1)$-box. More precisely, it states that given any $(\gamma_0^1, -; \gamma_0^2, \gamma_1^2)$, there is a $\gamma \in Q_2$ such that the three equations

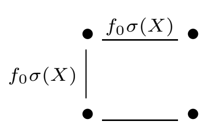
$$\begin{aligned} 0_2^1\gamma &= \gamma_0^1 \\ 0_2^2\gamma &= \gamma_0^2 \\ 1_2^2\gamma &= \gamma_1^2, \end{aligned}$$

are satisfied, so the list of faces of γ gives the $(2,1,1)$-box with the gap filled in. The element γ is a 'filler for the box'.

We will give a proof of the following as it illustrates the use of such conditions as this $E(2,1,1)$.

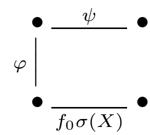
Proposition 5.1. — *If $E(2,1,1)$ holds for $Q(\mathcal{C})(X,Y)$, then $\simeq$ is an equivalence relation on $\mathcal{C}(X,Y)$.*

Proof. — Suppose that we have $f_0, f_1 : X \to Y$ and $\varphi : X \times I \to Y$, a homotopy from f_0 to f_1. We form the $(2,1,1)$-box $(f_0\sigma(X), -; \varphi, f_0\sigma(X))$ in $Q(\mathcal{C})(X,Y)$. (This looks like



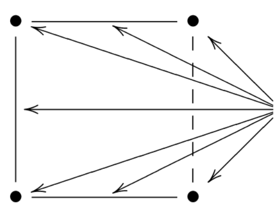
and we note that $f_0\sigma(X)$ is a constant homotopy on f_0.) If $E(2,1,1)$-holds, then there is a γ filling this box, that is, a morphism $\gamma : X \times I^2 \to Y$. We take $\gamma_1^2 := 1_2^1(\gamma) : X \times I \to Y$ and work out the two 'ends' of this homotopy. (They are clear from the picture but can also be calculated from the cubical relations.) We find $\gamma_1^1 : f_1 \simeq f_0$, so $\simeq$ is symmetric.

If we now have $\psi : f_1 \simeq f_2 : X \to Y$, then we form the $(2, 1, 1)$-box, $(\varphi, -; f_0\sigma(X), \psi)$, i.e.,



Now we take a filler, λ, say, and set $\lambda_2^1 := 1_1^1\lambda$, which will be a homotopy from f_0 to f_2. $\square$

In *Top*, fillers can be constructed using 'retractions', or 'collapses' from the square onto the $(2, 1, 1)$-box and then composing with the 'values' assigned to the box.



For the symmetry / 'involution' situation, the end result looks a bit like $f_0\sigma(X) * f_0\sigma(X) * \varphi^{(r)}$, where $\varphi^{(r)}$ is the usual reverse homotopy. For the other situation giving composition of homotopies and thus transitivity, the result looks like $\psi * \varphi * f_0\sigma(X)$, where the triple composite is defined using the subdivision of $[0, 1]$ into three equal parts.

Different retractions yield different fillers. Each filler corresponds to some formula for 'reverse homotopy', i.e., there is no attempt in this approach to come up with some definite preferred reverse, and similarly for the composite. The actual composite will depend on the filler, but if $Q(\mathcal{C})(X, Y)$ satisfies a filler condition for three dimensional boxes, the homotopy class of that composite will be independent of the choices made.

We also note that, in looking for the filler, one is seeking a solution to some equations and in the algebraic examples, this is exactly how one obtains fillers (when they exist). What we said above shows that the solution if it exists need not be unique. The filler in each case can be thought of as a 'reason' for the original homotopy to be reversible, or for the original pair of homotopies to be composable. There is a set of such 'reasons'.

In general, one defines boxes of form (n, ν, k) by writing down $2^n - 1$ elements in Q_{n-1}, that look as if they are all but one of the faces of an n-cube, and in which the missing one is the ν_n^k. A filler for such a (n, ν, k)-

box is then an n-cube whose corresponding (n, ν, k)-box within its boundary matches the given one. (If you want a more formal definition, look at Kamps and Porter, [35].)

A cubical set, Q, satisfies the extension condition $E(n, \nu, k)$ if all (n, ν, k)-boxes have fillers. It satisfies $E(n)$ if it satisfies $E(n, \nu, k)$ for all possible choices of (ν, k).

If we extend these definitions to handling the functorially varying $Q(\mathcal{C})(X, Y)$, then we can ask that the fillers vary naturally with X and Y. This can be set up in various ways, for instance, let the set of (n, ν, k)-boxes in a cubical set, Q, be denoted by (n, ν, k)–$Box(Q)$, then this defines a functor in Q and there is a natural transformation from the functor Q_n, that picks out the n-cubes, to (n, ν, k)–$Box(Q)$ defined in a fairly obvious way by allocating to each n-cube the corresponding element of $(n, \nu, k) - Box(Q)$. If Q satisfies $E(n, \nu, k)$, this natural transformation is a surjection when evaluated at Q. Now apply this to $Q(\mathcal{C})(X, Y)$. If there is a natural transformation from the n-cubes functor to $(n, \nu, k) - Box(Q(\mathcal{C})(X, Y))$, natural in X and Y, and which splits the previous one, then we say that the cylinder, **I**, satisfies $NE(n, \nu, k)$. Finally, if varying this with n, there is compatibility with degeneracy operators, in a fairly obvious sense that we will not go into here, we say **I** satisfies $DNE(n, \nu, k)$. There are also $NE(n)$ and $DNE(n)$ where the condition is satisfied for all (ν, k).

The details of these conditions, NE and DNE, will not be needed as, once again, we are really interested in the intuition behind them and their use. Intuitively they are linked to the situation in the topological case in which the retractions that make things work are in the models for the cubes, so everything is natural in the X and Y. In the more general case, these conditions indicate some sort of almost algorithmic solution to the finding of the fillers. This is the case, for instance, if one works with simplicial groups, where any box can be filled using a systematic filling of the simplicial 'horns' that make it up. The simplicial filling algorithms are very simple to use in low dimensions and guarantee that everything is natural.

As to the use of these conditions, it is clear that the richer and more natural the conditions you have available in an abstract homotopy setting the richer the theory will be. (Sometimes it is possible to compensate for lack of naturalness of the fillers with a filler at one higher dimension.) It is often the case that a cylinder in an application will satisfy all the $DNE(n)$, so there is nothing to worry about when working through generalisations of topologically based results, but it is also interesting to turn around the question and to ask, for a given (abstract homotopy) theorem, what filler conditions will be sufficient to prove it. (This may be useful when looking at situations in which higher level homotopies may not be invertible, for instance.) We give some instances:

- If $\mathbf{I}$ satisfies $NE(2,1,1)$, then there is an $i : X \times I \to X \times I$ such that $ie_0(X) = e_1(X)$, but i may not be natural. If $\mathbf{I}$ satisfies $NE(2)$ and $E(3)$ then i will be natural (or nearly so).

- We can define *cofibrations* by a homotopy extension property with respect to the cylinder for any cylinder. If $\mathbf{I}$ satisfies $E(2)$, then $e_0(X)$ and $e_1(X)$ are cofibrations and, moreover, $\sigma(X)$ is a homotopy equivalence.

- If $\mathcal{C}$ has pushouts and $(-) \times I$ preserves them (and this is the type of categorical condition that is needed for many of the results), then if $\mathbf{I}$ satisfies $E(2)$, there is a functorial factorisation of any morphism as a composite of a cofibration and a homotopy equivalence.

These are fairly 'low key' results, but other deeper examples exist (but would require more setting up so will not be treated in any detail here). For instance, there is a classical theorem of Dold on homotopy equivalences relative to subobjects where the inclusions are cofibrations, and this holds in the abstract case if $\mathbf{I}$ satisfies $DNE(2,1,1)$ and $E(3,1,1)$, see the discussion in Kamps and Porter, [35]. If $DNE(2)$ and $E(3,1,1)$ hold then with a bit of structure relating to pushouts (as above), the category $\mathcal{C}$ has the structure of a (co)fibration category, that is on dualising the axioms given by K. Brown, in [8]. These give a slightly weaker version of the model category structure of Quillen, [55].

In other words with extension conditions in low dimensions, and a limited amount of naturality and compatibility with degeneracies, one gets a rich theory that is very similar to the one that is considered to be the 'industrial standard' for abstract homotopy theories. This makes one wonder if more than these $NE(n)$ for $n = 1, 2, 3$, would be needed to get a completely adequate powerful abstract homotopy theory. Are there 'useful' or 'pretty' results that need higher dimensional filling conditions. The notion of an 'abstract homotopy' begins to approach that of 'identity' a lot more if these low dimensional Kan conditions are satisfied by the cylinder, but that somehow does not 'feel' right. 'Equality' and 'identity' are ideas that, if they are not 'on the nose', feel as if they should require conditions of compatibility in all dimensions.

To examine this a little, and also to introduce the next of the themes of this article, we will briefly look the theory of homotopy coherence.

Before we do that we should make a point about technical convenience: the cubical enrichment, that we introduced above, is often replaced by a simplicial enrichment, i.e., $\mathcal{C}(X, Y)$ will have a structure of a simplicial set rather than a cubical one. The Kan conditions on this will therefore be simplicial ones not cubical ones. This is probably equivalent as a formulation. (I do not know of a published proof of this, explicitly given.) The available fairly simple treatments of homotopy coherence are therefore given simplicially, and although they certainly could be rewritten in cubical terms, this is not a place to do that, so we will place ourselves in a category which is simplicially enriched rather than using the cylinder explicitly. (Such a category will be called a $\mathcal{S}$-category. It will be called *locally Kan* if all the simplicial sets, $\underline{\mathcal{C}}(X, Y)$, are Kan complexes. These locally Kan $\mathcal{S}$-categories seem to be good models for a large class of ∞-categories. They are 'fibrant' in a useful homotopy theory of simplicially enriched categories.) If you want a simple example of such a category, the category of spaces with the simplicial set of maps between X and Y being given by $\underline{\mathbf{Top}}(X, Y)_n = Top(X \times \Delta^n, Y)$, with the natural face and degeneracy maps induced from the usual maps between the simplices. Any such simplicially enriched category can also be cubically enriched, since we can take the cubical set whose set of n-cubes is the set of simplicial maps from the simplicial n-cube $\Delta[1]^n$ to the given mapping simplicial set.

The theory is 'geometrically' nicer to work with if $\mathcal{C}$ is *tensored* or *cotensored*:

If for all $K \in \mathcal{S}$, $X, Y, \in \mathcal{C}$, there is an object $K \bar{\otimes} X$ in $\mathcal{C}$ such that

$$\underline{\mathcal{C}}(K \bar{\otimes} X, Y) \cong \mathcal{S}(K, \underline{\mathcal{C}}(X, Y)$$

naturally in K, X and Y, then $\mathcal{C}$ is said to be *tensored* over $\mathcal{S}$.

Dually, if we require objects $\bar{\mathcal{C}}(K, Y)$ such that

$$\underline{\mathcal{C}}(X, \bar{\mathcal{C}}(K, Y)) \cong \mathcal{S}(K, \underline{\mathcal{C}}(X, Y)$$

then we say $\mathcal{C}$ is *cotensored* over $\mathcal{S}$.

Another version of the result that was mentioned above is the following:

Proposition 5.2 *(cf. Kamps and Porter, [35]). — If $\mathcal{C}$ is a locally Kan $\mathcal{S}$-category tensored over $\mathcal{S}$ then, taking $X \times I = \Delta[1] \bar{\otimes} X$, we get a good cylinder functor such that for the cofibrations relative to* **I** *and weak equivalences taken to be homotopy equivalences, the category $\mathcal{C}$ has a cofibration category structure.*

Of course, we only really need 'locally Kan' in low dimensions for this.

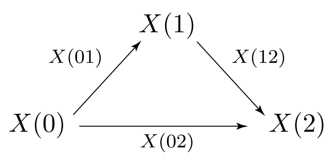
We will look at homotopy coherence primarily in a $\mathcal{S}$-categorical context.

6. Homotopy coherence: an introduction

Useful references for this section, include the book by Kamps and Porter, [35], or, for a short introduction, the *'$\mathcal{S}$-cat notes'*, [49]. Most of this is also in the *Menagerie*, [51], and there are parts developed in the nLab, [42]. Here we will just give enough detail to make the relevance of this to the themes of this article clear, and also to link into later sections. (If the reader needs to follow up further then Lurie's book, [38], gives one way of developing the theory further.)

We start by pointing out the difference between homotopy commutative and homotopy coherent diagrams by looking at a very small example.

Consider a diagram, X, indexed by the small category, [2], and taking values in a category, $\mathcal{C}$, which has an abstract homotopy structure given, say by a cylinder, or a tensored simplicial enrichment.



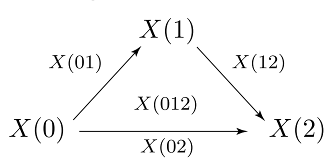
The diagram is, of course, *commutative* if $X(02) = X(12)X(01)$. It is *homotopy commutative* if there *is* a homotopy between $X(02)$ and $X(12)X(01)$.

A diagram indexed by [2] is *homotopy coherent* (and we will often abbreviate this to h.c.) if there is *specified* a homotopy

$$X(012) : X(0) \times I \to X(2),$$

$$X(012) : X(02) \simeq X(12)X(01),$$

so the diagram looks something like:



In terms of our 'thematic observation', the distinction, at this level, and with a very simple type of diagram, between 'homotopy commutative' and 'homotopy coherent' is between the existence of a 'reason' and the 'reason' itself. The specified homotopy is part of the data for the h.c. diagram. A homotopy commutative diagram is a commutative diagram in the homotopy category. The extra data in the h.c. case is not so simply encoded.

If we go to a very slightly more complex case, the distinction becomes more striking. For a diagram indexed by [3], we proceed as follows. Draw a 3-simplex, marking the vertices $X(0), \ldots, X(3)$, the edges, $X(ij)$, for $i < j$, (corresponding to the morphisms between them), and the faces $X(ijk)$, with $i < j < k$, corresponding to homotopies, $X(ijk) : X(i) \times I \to X(k)$. This gives the information 'induced' from the 2-dimensional situation that we saw before. The homotopies $X(ijk)$ fit together to make the sides of a square

$$\begin{array}{ccc} X(13)X(01) & \xrightarrow{X(123)X(01)} & X(23)X(12)X(01) \\ {\scriptstyle X(013)}\uparrow & & \uparrow{\scriptstyle X(23)X(012)} \\ X(03) & \xrightarrow[X(023)]{} & X(23)X(02) \end{array}$$

and the diagram is made h.c. by specifying a second level homotopy

$$X(0123) : X(0) \times I^2 \to X(3)$$

filling this square.

(A point about the notation, really $X(123)X(01)$ stands for $X(123)(X(01) \times I)$, but if we use the longer form the diagrams get very cluttered.)

These can be continued for larger $[n]$. Of course, this is not how the theory is formally specified, but it provides some understanding of the basic idea.

Historical sources: The theory was initially developed by Vogt, [56], following methods introduced with Boardman, [6] (see also the references in that source for other earlier papers on the area). Cordier [12] provided a simple simplicially enriched category theoretic way of working with h.c. diagrams and hence released an 'arsenal' of categorical tools for working with h.c. diagrams. Some of that is worked out in the papers, [14, 15, 16, 17]

We will list some of the results from that theory, illustrating the link with the themes.

(i) If $X : \mathbb{A} \to \mathcal{T}op$ is a commutative diagram, and we replace some of the $X(a)$ by homotopy equivalent $Y(a)$s, with specified homotopy equivalence data:

$$f(a) : X(a) \to Y(a), \quad g(a) : Y(a) \to X(a);$$

$$H(a) : g(a)f(a) \simeq Id, \quad K(a) : f(a)g(a) \simeq Id,$$

then we can combine these data into the construction of a h.c. diagram, Y, based on the objects, $Y(a)$, and homotopy coherent maps,

$$f : X \to Y, \quad g : Y \to X, \text{ etc.},$$

making X and Y 'homotopy equivalent' as h.c. diagrams.

This is 'really' a result about *quasi-categories*, see [32]. The point of it is that if all the $X(a)$s are to be 'identified' with the corresponding $Y(a)$s, because they have the same homotopy type, then we should be able to have a 'diagram' that is based on Y, and it also should end up being 'identifiable' with X. This does seem to be a natural thing to expect. The result can be generalised to input a h.c. diagram, X, instead of a commutative one.

(ii) Vogt, [56]. If $\mathbb{A}$ is a small category, there is a category $Coh(\mathbb{A}, \mathcal{T}op)$ of h. c. diagrams and homotopy classes of h. c. maps between them. Moreover there is an equivalence of categories

$$Coh(\mathbb{A}, \mathcal{T}op) \stackrel{\simeq}{\to} Ho(\mathcal{T}op^{\mathbb{A}})$$

This was extended replacing $\mathcal{T}op$ by a general locally Kan simplicially enriched complete category, $\mathcal{B}$, in [13].

Here there is a general principle in action. The higher dimensional homotopies are kept until the final step. They compose only up to coherent homotopy so one does not get a category of homotopy coherent morphisms between homotopy coherent diagrams, but once one identifies homotopic ones then ..., you guessed, the structure resolves back to a more usual one. (In any case, one gets the homotopy coherent analogue of a category, of course, as all the information has been kept. This would be some form of A_∞-category, or, once again, a quasi-category. This point has been followed up in work by Batanin, [2], Joyal, [33], and notably by Lurie, [38].)

(iii) Cordier (1980), [12]. Given $\mathbb{A}$, a small category, then there is a $\mathcal{S}$-category, $S(\mathbb{A})$, such that a h. c. diagram of type $\mathbb{A}$ in Top is given precisely by an $\mathcal{S}$-functor

$$F : S(\mathbb{A}) \to Top$$

This suggested the extension of homotopy coherent diagrams to other contexts such as that of a general locally Kan $\mathcal{S}$-category, $\mathcal{B}$, and suggests the definition of homotopy coherent diagram in a $\mathcal{S}$-category and thus a h. c. nerve of an $\mathcal{S}$-category.

Definition (Cordier (1980), [12], based on earlier ideas of Vogt, and Boardman-Vogt). — Given a simplicially enriched category, $\mathcal{B}$, the *homotopy coherent nerve* of $\mathcal{B}$, denoted $Ner_{h.c.}(\mathcal{B})$, is the simplicial 'set' with

$$Ner_{h.c.}(\mathcal{B})_n = \mathcal{S}{-}Cat(S[n], \mathcal{B}).$$

To understand simple h. c. diagrams and thus $Ner_{h.c.}(\mathcal{B})$, we will unpack the definition of homotopy coherence. The thing to note is that for any n and $0 \leqslant i < j \leqslant n$, $S[n](i,j) \cong \Delta[1]^{j-i-1}$, the $(j-i-1)$-cube given by the product of $j-i-1$ copies of $\Delta[1]$. We can thus reduce the higher homotopy data to being just that, maps from higher dimensional cubes. This allows one to split the specification of a homotopy coherent diagram into two parts:
(a) specification of certain homotopy coherent *simplices*, i.e., elements in $Ner_{h.c.}(\mathcal{B})$;
and
(b) specification, via a simplicial mapping from $Ner(\mathbb{A})$ to $Ner_{h.c.}(\mathcal{B})$, of how these individual parts (from (a)) of the diagram are glued together.

This idea deserves a bit more dissection. It says essentially that the core of the description of homotopy coherent diagrams is somehow completely 'local'. You have some homotopy theoretic building blocks namely the homotopy coherent simplices and then you specify how to stick them together, but the way in which the information is glued is not dependent on homotopy. It is just the information in the nerve of the indexing category.

The following theorem was proved by Cordier and Porter, [13], but the idea was essentially in Boardman and Vogt's lecture notes, [6], like so much else!

Theorem 6.1 *([13]). — If $\mathcal{B}$ is a locally Kan $\mathcal{S}$-category, then $Ner_{h.c.}(\mathcal{B})$ is a quasi-category.* ■

We have mentioned quasi-categories before, so should now give a few more details. The Kan extension conditions that we looked at in the cubical set context have their analogues in the simplicial context. There we ask for fillers for *horns*, that is, families of $(n-1)$-simplices that fit together as if they were all but one of the faces of an n-simplex. A Kan complex is a simplicial set which has fillers for all horns. The nerve of a small category, $\mathbb{A}$, will only be Kan if $\mathbb{A}$ is a groupoid. A quasi-category is a simplicial set that, like the nerve of a category, has fillers for all horns in which the 'missing face' is neither the zeroth nor the last face. The result says that if we have a locally Kan simplicially enriched category, then its homotopy coherent nerve behave like a category, (up to homotopy!).

There will be some interesting situations in which the simplicial sets of morphisms may have fewer fillers than this and then there is a challenging question as to what sort of homotopy theory will result. We saw that the existence of low dimensional fillers does guarantee a good homotopy theory, and here we can see that when looking at finding specific homotopy coherent diagrams, we are likely to have to *construct* the required homotopies and that will require fillers for only certain boxes or classes of boxes. The interpretation of those situations for their philosophical implications will be very interesting.

We will explore some situations that produce a need for homotopy coherence in the next section and will produce one in which the actual type of homotopy theory is dependent on what fillers are required to exist.

7. Questions and intuitions

The basis for homotopical ideas, in the usual setting, is the intuitions in and around the study of topological spaces, continuous paths and more general maps between spaces, and then deformations of these using continuous maps, typically from cylinders. This influences greatly what we consider 'normal' behaviour of homotopies, and also influences the pictorial representation we make of homotopies. The spaces are often, if not 'usually', thought of as being compact and so are drawn as 'blobs'. What we will explore in this section, after a reflection on some of the limitations of this traditional picture, will be a few examples of quite intuitive topological situations, but where some of our usual pictures are inadequate, as are the usual tools that we use.

To start with it is worth making some general points about invariants in homotopy theory starting with the homotopy groups. In the traditional approach to homotopy theory, (which is, it should be said, a highly successful piece of pure mathematics), having introduced the elementary notion of homotopy of continuous maps between spaces, it is usual to introduce the fundamental group, $\pi_1(X, x_0)$, of a pointed space, that is a space, X, with a chosen base point, x_0. (This has the effect of concentrating attention on arcwise connected spaces.) Sometimes the fundamental groupoid is introduced as well; see Ralf Krömer's talk for a historical perspective on this.

If one is looking at the sort of development of the theory that might be in a first course of lectures on homotopy theory, there is, at this point, often a glance at covering spaces and the actions of $\pi_1(X)$ on sets, but the usual way forward is to concentrate on pointed spaces, (X, x_0), pointed pairs of spaces, (X, A, x_0), and to consider $\pi_1(X, x_0)$ as being $[(S^1, 1), (X, x_0)]$, the set of pointed homotopy classes of maps from the circle, S^1, pointed at 1, to (X, x_0). The derivation of the group structure on $\pi_1(X, x_0)$ is sometimes done via a 'cogroup' structure on $(S^1, 1)$ and which makes it easy to introduce the higher homotopy groups, $\pi_n(X, x_0)$, long exact sequences, etc.

This quite pedagogic treatment follows, at least approximately, the historical development, but it fails to address various points – at least from the point of view of abstract homotopy theory. The first is to ask why do we use S^1 and, more generally, S^n as the spaces that are used for 'probing' the properties of X. In abstract settings (and in some of the examples later) whether involving a cylinder derived theory, or a model category based one, there need not be a single *obvious* analogue of the family of spheres, and so we have to adapt our approach and must avoid the special case of spheres having too much influence on the way we handle, or think of, homotopy invariants in other contexts.

There are intuitions, available already within the topological context, but which generalise to much more general cases and which get us out of this 'sphere-less' context. We can go back to the interval $I = [0, 1]$ and consider any map from a circle into (Y, y_0) as being a map from I to Y which happens to map 0 and 1 to the same point, y_0. This makes one think of S^1 as being $I/\{0, 1\}$, that is, as resulting from 'identifying' the two end points. Provided we can form quotients in our abstract setting, $\mathcal{C}$, we can mimic this for a cylinder $X \times I$ on X and thus get the suspension, ΣX, of X, identifying the two images of X. The classical argument that $[S^1, Y]$ is a group, then, of course, adapts to show that $[\Sigma X, Y]$ is one, provided a replacement for the role of the base point is given. (We will look at some special instances of this slightly later in the examples.) This point about base points makes it clear that a fundamental groupoid analogue would be useful here. We will briefly come back to this later.

The replacement of the spheres by the suspensions works well (setting aside the basepoint problem) as suspensions have a cogroup structure. This gives us $[\Sigma^n X, Y]$ as an analogue of $\pi_n(Y)$, and, with a bit more work, the existence of long cofibration (Puppe) sequences allows long exact sequence of groups, generalising the classical case to be set up in great generality (at least for pointed objects); see Kamps, [34], and treatments in Baues, [3] and Kamps and Porter, [35].

The use of the corresponding fundamental groupoid, $\Pi_1(X, Y)$, is a nice variant in some versions of this theory. It uses $Q(\mathcal{C})(X, Y)$ and takes the fundamental groupoid of that cubical set. This transfers structure on $\mathcal{C}$ from being cubically enriched to being *groupoid enriched.* This means that one more level of the homotopy structure has been used than in the formation of the homotopy category, $Ho(\mathcal{C})$. Rather than just forming a category by identifying homotopic maps, this keeps the homotopies 'up to homotopy'.

Remarks. —

Historical. Such groupoid enriched categories were singled out for study by Fantham and Moore, [21], in 1983, and were quite extensively used as a richer context for abstract homotopy theory until quite recently. (They are still very useful, but their use has been partially superceded by the introduction of simplicially enriched categories, in which the 'hom's are Kan complexes and Kan complexes model the so called weak ∞-groupoids.) Under the name of track categories, groupoid enriched categories are used a lot by Baues and his coworkers, see, for instance, [4]. Their use, and the exact link with the cylinder based / cubical abstract homotopy theory is discussed in [35].

More philosophical ones. The use of groupoid enriched categories, rather than simply the homotopy categories corresponds to accepting there may be multiple *reasons* to *identify* comparison maps between objects of interest, but not probing further to see the *reasons between reasons* as an extra structural level. The use is also related to something that can be clearly observed in *rewriting theory*. In that area the rewrites of terms are explicit reasons why one can replace one expression by another. They thus can be thought of as morphisms. In higher dimensional rewriting, the next level of structure looks at the critical pairs, that is the *ambiguities* as to how to proceed with a 'rewrite'. Typically these occur where there is an overlap between the instructions as to how to proceed. There might be instructions to replace aab by ba, and to replace aaa by the empty word, so given a word with $aaab$ in it, there are two choices. One leads to b directly, the other goes to aba. The track category viewpoint allows calculations to be made as to the ways to complete these rewrites non-ambiguously; see Guiraud and Malbos, [28, 29], for more on this.

We thus have potential analogues of both fundamental groupoids and higher homotopy groups, (but based on other objects than spheres, as there may be none in the category, $\mathcal{C}$, which of course, may not be derived from any topological situation). The suspensions, $\Sigma^n X$, behave like *sphere objects* in $\mathcal{C}$, relative to the object X, in some sense, and give cogroups if viewed in the appropriate category. (Of course, the classical topological case is essentially 'relative to the singleton space'.) As we will see later, looking for analogues of homotopy groups, etc., in other contexts amounts to searching for suitable useful analogues of spheres, but can be also an offshoot of looking for cylinders!

A second point that needs making is that in all this the covering space aspect has got lost. A limited, but important, intuition of generalisations of covering spaces has survived in the theory of fibrations and fibre bundles in general, but it is somewhat strange that $\pi_1 X$ has a beautiful description as the automorphism group of a universal cover of X, but the other higher homotopy groups classify only what they most directly set out to classify, namely, homotopy classes of maps from S^n to the given space, X. This is quite a meagre 'bounty' in reward for all the work needed to set them up!

Of course, Grothendieck in his letters to Larry Breen in about 1975 (see [26]) asked if there was a higher dimensional version of the covering space theory. (I do not know of any earlier mention of this idea.) Grothendieck had, in SGA1, [27], so in 1959-62, used the links between covering spaces and sets with a group action, to define a general categorical setting that not only enabled a fundamental group to be defined for schemes and other algebraic geometric objects, but also made explicit the links between this and Galois theory. His letters to Breen not only suggested that this should generalise to higher 'dimensions', but that the objects thus found might enable a much clearer view of non-abelian versions of cohomology to be given. One key concept in all this is that of an n-type. Some discussion of this is to be found in Baues' article for the Handbook of Algebraic Topology, [5], and it was also mentioned in Jean-Pierre Marquis' talk at the workshop. The use of homotopy n-types seems to be one key to this question of 'higher covering spaces'. Algebraic *models* for n-types are known and their homotopy theory as well.

A final point is that this traditional approach to homotopy theory really only applies well to arcwise connected spaces, and works best if they are compact. If the spaces are not arcwise connected, then, although the definitions of homotopy, homotopy equivalence and thus of homotopy type are still valid, the tools available, such as the homotopy groups are not as useful. There is a well behaved approximation to homotopy type given by (strong) shape theory, (see below), but really one needs a combination of the singular complex and the shape approach. The case of non-compact spaces is similar, in parts, to this, but needs new insights. The traditional methods of homotopy theory use continuous maps, whilst to 'see' the difference between non-compact spaces, it is often necessary to look at the asymptotic behaviour 'out towards infinity'.

We need some simple examples:

- $X_1 = [0,1] \times \mathbb{R}$. This is an infinite strip. It has two 'ends', but is otherwise very boring! It does not change as one goes out towards the ends. It is contractible. The unique map to the singleton space, $\{0\}$, is a homotopy equivalence.

– $X_2 = [0,1] \times [0,\infty)$. This is a half infinite strip. It has just one end and is equally 'boring'! It is contractible.

– Let Y_8 be a figure eight, that is, the one point union of two circles and let X_3 be its universal cover. This looks like an infinite 'thorn bush'! It has infinitely many ends. It is contractible, but is very unlike the previous two examples.

To detect this evidently different behaviour, we need to use *proper maps.* A continuous map, $f : X \to Y$, is continuous because of the behaviour of the inverse image as $f^{-1}O$ is open in X if O is open in Y. On the other hand, if $C \subseteq X$ is compact, then $f(C)$ is compact.

Definition. — A map $f : X \to Y$ is *proper* if for each compact subset, K of Y, $f^{-1}(K)$ is compact.

We will return to 'proper homotopy' and 'proper homotopy type' in more detail, but here we just note that none of the three examples above has the same proper homotopy type as a point, since although the unique map to $\{0\}$ in each case is a homotopy equivalence, none of them is a proper map. We will look at this 'proper' variant of homotopy later.

8. Four case studies and a new direction

We will give several related, but distinct, case studies that illustrate various aspects of the above remarks.

8.1. (Strong) shape theory

First let us review some history, but initially of homology rather than homotopy. The initial development of homology and cohomology was more or less restricted to polyhedra, that is to spaces derived, by a form of geometric realisation, from simplicial complexes. They could thus be triangulated. In the 1920s, homology and cohomology were thus essentially known only for simplicial complexes, and the methods used involved algebraic formulae derived from that simplicial structure. There were attempts to extend the definitions, first to all compact metric spaces and then to all spaces. Leopold Vietoris in 1927 came up with a construction, and then, shortly afterwards Alexandrov, (1929), and then Čech, (1932), gave a different one. The input for both approaches was a space, X, together with an open cover, $\mathcal{U}$, of X.

The *Vietoris complex* of the pair $(X,\mathcal{U})$, which we will denote by $V(X,\mathcal{U})$ is a simplicial complex having the points of X as its vertices and in which a subset, $\langle x_0, \ldots, x_n \rangle$, of vertices is a simplex if there is an open set U in $\mathcal{U}$ containing all of them.

The *Čech complex* or *nerve* of $(X,\mathcal{U})$, denoted $N(X,\mathcal{U})$, is constructed dually, (although this is not immediately obviously the case). The vertices of it are the open sets in $\mathcal{U}$ (or, if you prefer, the elements of a set indexing the open sets in $\mathcal{U}$) and a family, $\langle U_0, \ldots, U_n \rangle$, of such sets forms a simplex if it has non-empty intersection.

Aside: If we think of two points that are in the same open set of $\mathcal{U}$ as being $\mathcal{U}$-near, and try to identify them we get the problems of what happens on overlaps between different open sets of $\mathcal{U}$. This suggests, yet again, the problem of 'identification' and 'classification' of objects. The higher homotopy information is exactly that which is being encoded in $V(X,\mathcal{U})$, or, equivalently, in $N(X,\mathcal{U})$. I leave the reader to chase up this idea and its consequences.

One of the beauties of the Čech construction is that any triangulation of a polyhedron, X, yields an open cover of the space, by the open-stars of vertices of the triangulation. The nerve of this open cover 'is the same' simplicial complex as that used for the triangulation. Conversely, for any open cover of a polyhedral space, we can find a triangulation of the space whose open-star cover is finer than the given one. The intuition behind this is that the nerves of open covers of an arbitrary space *approximate* that space in a way that resembles the way that finer and finer triangulations give a better picture of a polyhedron's 'geometry'. For a polyhedral space, the two pictures, from nerves and triangulations, essentially coincide, but they can not for a more general space as there will there be no analogue of a triangulation.

We need to look at the notion of 'finer' as applied to open covers.

Definition. — Given a space, X and two open covers, $\mathcal{U}$ and $\mathcal{V}$, we say that $\mathcal{U}$ is *finer* than $\mathcal{V}$, written $\mathcal{U} \leqslant \mathcal{V}$, if, for each U in $\mathcal{U}$, there is a V in $\mathcal{V}$ such that $U \subseteq V$. A choice of such a V for each U defines a function, $\varphi : \mathcal{U} \to \mathcal{V}$, such that, for all U, $U \subseteq \varphi(U)$. Such a function is called a *refinement map* for the pair, $(\mathcal{U}, \mathcal{V})$.

The constructions of $V(X,\mathcal{U})$ and $N(X,\mathcal{U})$ depend on $\mathcal{U}$, of course, so we need to compare them for different open covers. Suppose we have $\mathcal{U} \leqslant \mathcal{V}$, then any '$\mathcal{U}$-small' simplex, $\langle x_0, \ldots, x_n \rangle$, i.e., in $V(X,\mathcal{U})$, is also $\mathcal{V}$-small, so we get, without any bother, an inclusion of $V(X,\mathcal{U})$ into $V(X,\mathcal{V})$. That inclusion does not depend on any choice of a refinement map, $\varphi : \mathcal{U} \to \mathcal{V}$, so, if we have $\mathcal{U} \leqslant \mathcal{V} \leqslant \mathcal{W}$, then the inclusions fit together correctly and we can check that this gives us a functor,

$$V(X,-): Cov(X) \to Simp.Comp,$$

where $Cov(X)$ is the category of open covers of X. (This is the category associated to the directed set, $(Cov(X), \leqslant)$.) The functor, $V(X,-)$, is thus an example of an *inverse system* of simplicial complexes.

The situation for $N(X,\mathcal{U})$ is more complicated. Even at the level of vertices, if $\mathcal{U} \leqslant \mathcal{V}$, then any sensible simplicial maps between the nerves must give a corresponding refinement map, and if φ is a refinement map, we can obtain a simplicial map from $N(X,\mathcal{U})$ to $N(X,\mathcal{V})$ simply by mapping a typical simplex, $\langle U_0,\ldots,U_n\rangle$, to the corresponding, $\langle \varphi(U_0),\ldots,\varphi(U_n)\rangle$. This works because $\bigcap U_i \subseteq \bigcap \varphi(U_i)$, so the right hand side is non-empty as least when the left hand side is. This gives $N(X,\varphi)$, but note it does depend on a choice of φ. This causes problems if $\mathcal{U} \leqslant \mathcal{V} \leqslant \mathcal{W}$, as we have, in general, no reason to suppose that it is possible to choose the refinement maps compatibly for all such triples of covers. For instance, there may be many different $\mathcal{V}$s between the other two covers. We thus need not get a functor $N(X,-)$ defined on $Cov(X)$ with values in the category of simplicial complexes. (We do not get a commutative diagram of simplicial complexes, but in fact we do get a homotopy commutative one, which is sufficient for the definition of Čech homology. We will come back to this later.)

To obtain a family of homological invariants, the various constructions then worked with the homology groups of the $V(X,\mathcal{U})$ and $N(X,\mathcal{U})$. In the first case of the Vietoris homology construction, it is clear that on applying a homology group functor to $V(X,-)$, we get an inverse system of Abelian groups, $H_i(V(X,-)) : Cov(X) \to Ab$. These give homological information on the space as viewed through the perspective of finer and finer 'meshes'. (This is, intuitively, very closely related to the idea of the integral as the limit of sums in the integral calculus. I have not examined the original papers of Vietoris to see if this intuition was made explicit there.) The Vietoris homology groups are then defined by taking the (inverse) limit of the system of groups. (Again I do not know to what extent Vietoris built on previous work in obtaining some idea of inverse limit here.)

For the nerve based construction, there is no real problem for compact metric spaces, since one can use open covers by balls of decreasing size to control the possible wildness of the refinement map problem, however there is also no problem in general as, even if in the situation of three covers, $\mathcal{U} \leqslant \mathcal{V} \leqslant \mathcal{W}$, the two induced simplicial maps from $N(X,\mathcal{U})$ to $N(X,\mathcal{W})$, one direct, the other via $N(X,\mathcal{V})$, need not be equal, they will give the same homomorphism from $H_i(N(X,\mathcal{U}))$ to $H_i(N(X,\mathcal{W}))$ as they are *contiguous* maps. ('Contiguity' is a very controlled and constructive form of simplicial homotopy. In fact, the resulting diagrams are homotopy coherent, so there is certainly no problem after applying the homology group constructions.) We thus get the Čech homology groups,

$$\check{H}_i(X) = lim_{\mathcal{U}} H_i(N(X,\mathcal{U})).$$

(There is, however, still a problem. These homology groups do not give long exact sequences as would be expected. The (inverse) limit functor destroys exactness.) The Čech groups were found to be always isomorphic to the Vietoris ones, but the precise reason why was not found until later in a beautiful paper by Dowker, [19]. He showed that for any open cover $\mathcal{U}$, the geometric realisations of $V(X,\mathcal{U})$ and $N(X,\mathcal{U})$ were homotopically equivalent. (This neat result is useful when setting up strong shape theory.)

This was more or less the situation at the end of the 1930s. There was a well developed theory of Čech homology, and it was used in various topological and geometric situations. It coincided with simplicial homology on polyhedral spaces. There was no corresponding homotopy theory known.

Although not directly, or immediately, relevant to homotopy theory, it is worth noting that there was some progress on 'mending' Čech homology's inadequacies in the 1930s and early '40s. Not only did that homology not give long exact sequences in some obvious situations, but various duality results failed to have analogues in the Čech theories. Ordinary (simplicial) homology and cohomology were linked by various duality theorems. Suppose that X was a compact polyhedral subset of $\mathbb{R}^n$ (or more or less equivalently S^n), then the q^{th} homology of $\mathbb{R}^n - X$ and the $(n-q-1)^{th}$-cohomology of X are isomorphic. There was no similar result for Čech homology and cohomology say with X being a compact subset of S^n. First Kolmogoroff, (1936), then Chogoshvili, (1940), and, most importantly, Steenrod, again 1940, modified the definition of a Čech style homology so as to get a homology theory that satisfied the exactness and the duality requirements. (This is discussed in the bibliographic note on pages 403-404 of Sibe Mardešić's book, [41], on *Strong Shape and Homology*.) Steenrod's definition is important as it is possibly the first use of a construction which is very closely related to the homotopy limit. The ideas and intuitions are very geometric and this relates to the later links between strong shape theory and proper homotopy theory, which can, in part, be seen as an extension of the duality theorems mentioned above.

In 1944, the first steps towards a 'Čech homotopy' were given by Christie, [10], who examined what happened if one replaced the homology groups by homotopy groups, but also developed a theory in which the homotopies used in the approximating inverse systems ('nets' in his terminology) were considered as part of the structure. By this means he built into some of his results a certain amount of homotopy coherence.

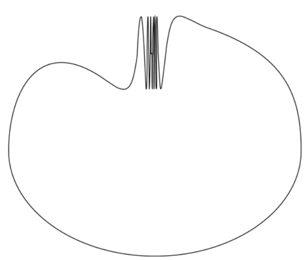
The theory of an adequate homotopy analogue of Čech homology took an enormous step forward in 1968 with Karol Borsuk's development of his shape theory. (This is discussed in Mardešić's article in the *History of Topology*, [40].) For both the historical and philosophical aspects, this theory is important. Homotopy theory can be seen as an attempt to codify the similarities between spatial objects at a very basic level, but the usual methods involve probing spaces with test objects such as simplexes and spheres. In shape theory, the idea is pushed further forward, but in a slightly different direction. The following well known example shows this. (It is usually called the *Warsaw circle* as it was first publicised in this context by the group of mostly Polish topologists work with Borsuk in Warsaw. There is also probably a play on words in comparison with the Vienna circle of philosophers!)

The point of it is that a space may have very little separating it from 'polyhedralness', yet a 'singularity' can cause havoc!

The Warsaw circle, S_W, is the subset of the plane, $\mathbb{R}^2$, specified by

$$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \;\middle|\; -\frac{1}{2\pi} < x \leqslant \frac{1}{2\pi}, x \neq 0\right\} \cup \left\{(0,y) \mid -1 \leqslant y \leqslant 1\right\} \cup C,$$

where C is an arc in $\mathbb{R}^2$ joining $(-\frac{1}{2\pi}, 0)$ and $(\frac{1}{2\pi}, 0)$, disjoint from the other two subsets specified above except at its endpoints.



The space 'looks' like a circle with a bit of 'fuzz' in one part. If one considers an open neighbourhood of S_W in $\mathbb{R}^2$, say all points within $\frac{1}{n}$ of the set, then that neighbourhod looks like an annulus with a thickening at one section. Any open covering of S_W by small open balls will have a nerve that is essentially the same as this, i.e., a circle with a thickened 'bar' at one point, transverse to the circle. (Note that the interval on the y-axis is included to make the space compact.)

Borsuk's shape theory, in some sense, replaced a space by the collection of open neighbourhods. (To make sense of this in more detail, one embeds the compact metric space in the Hilbert cube, and looks at all the open neighbourhoods of that embedded copy, together with the inclusions between them.) The maps between the spaces are expanded to include sequences of maps between the approximations, so that some obvious homotopy relations are satisfied. In the ordinary setting of continuous maps, there are no maps from the circle into S_W that go around the central hole, and the fundamental group of S_W is trivial, as are all its higher homotopy groups, yet it is clearly not contractible. In Borsuk's shape category, there is a morphism from S^1 to S_W, and one in the other direction, which respect more of the intuitive 'shape' of the spaces.

This theory still does not fit well into an abstract homotopy theoretic context. Many of the usual constructs of homotopy theory do not have an analogue in shape theory. That sound negative, but shape theory, as it encompassed many geometric intuitions, has proved very useful. The challenge was to provide a richer theory that would still provide the benefits of shape without the disadvantages. That theory is *Strong Shape Theory*, also called *Steenrod homotopy* as it, in some sense, is the homotopy theory underlying Steenrod's definition of homology, that was mentioned earlier. Strong Shape takes into account the homotopy coherence inherent in the construction of the Čech nerve system. (A starting reference here is the lecture notes of Edwards and Hastings, [20].) The Vietoris construction gives an inverse system of simplicial sets, the Čech construction a homotopy coherent inverse system of such. The first ends up in the category $Pro-\mathcal{S}$, the second seems to land in $Pro-Ho(\mathcal{S})$, i.e., the category of inverse systems 'up to homotopy'. The Vietoris complex is very big and unwealdy, whilst the Čech one is nice and small, and it is easier to see how it reflects the structure of the space being studied. We will go into this in a bit more detail shortly.

There is a more 'philosophical' slant on this idea of shape. In this we propose that we know a space by means of 'finite observations'. Thinking of such observations as being analogous to open sets (think: all points with some measurement above a certain threshold value), then the way the observations fit together is encoded in the nerve of a cover. Alternatively, thinking of observations as never giving arbitrarily fine information on the space, observing a space gives information on what is 'really' there, only up to being in a neighbourhood of the thing being observed. In other words, one only has information on a system of approximating abstract 'spaces', never on the limiting case. One can get additional information by a 'multiscale' process, that is, by looking at the whole *system* of observations and the relationships between the approximating nerves at different scales. Comparison between different objects will then 'logically' involves morphisms between the approximating systems. (Some 'informational' aspects of this idea are explored in [18] and [25], whilst a 'physical' interpretation is attempted in the preprint, [47].)

8.2. Étale homotopy

A similar problem occurs in *étale homotopy theory*, which was the first homotopy theory to make some impact on algebraic geometry. The homotopy constructions relative to this theory are very difficult to do and for much the same reason. Étale homotopy theory was initially developed by Artin and Mazur, [1], based on earlier ideas of Grothendieck and Verdier. It gave a Čech-style homotopy, but was based not on the nerves of coverings, but on a notion of hypercovering. (Any open cover of a space gives not only a simplicial complex by the nerve construction, but by a very simple extension of that construction, it gives a simplicial sheaf. The hypercoverings are generalisations of this construction that yield simplicial sheaves that are more like Kan complexes, i.e., the map to the terminal object is, in a natural sense, a fibration.) The resulting constructions gave pro-objects (i.e., inverse systems) in $Ho(\mathcal{S})$, which were said to give the *étale homotopy type* of the variety or scheme. How does this sort of theory fit into the views sketched out above of what 'homotopy' *is*?

Artin and Mazur realised that their work used $Pro-Ho(\mathcal{S})$, yet that was not the homotopy category of some abstract homotopy theory on $Pro-\mathcal{S}$ as was observed by Quillen early on in [55]. One problem is that the hypercovering approach is not obviously able to be *rigidified* in a really natural way when approached from this direction. By 'rigidified', we mean to replace it by some actual pro-simplicial set whose image in $Pro-Ho(\mathcal{S})$ would be isomorphic to that étale homotopy type. There were approaches put forward using pointed hypercoverings, and these gave some advantages and led to good results, cf. Friedlander's monograph, [23], but the point, in some instances, looks a bit artificial as an additional structure. There seems to be no natural candidate to play the role of the Vietoris complex and it is not clear if such a thing would even make sense.

On the positive side, the Artin-Mazur theory did have notable successes particularly after the further work by Friedlander mentioned above. For any geometric point x of a variety, X, the pro-groups $\pi_i((X;x)_{et})$, $i \geqslant 1$, give a natural definition of homotopy groups in an algebraic geometry context. Artin and Mazur proved that the étale homotopy type, X_{et}, of a smooth complex variety, X, is isomorphic in $Pro-Ho(\mathcal{S})$ to the profinite completion of the topological space $X(\mathbb{C})$. This refined previously known comparison theorems between étale and singular cohomology.

In this setting, i.e., in searching for a useful homotopy theory for algebraic geometric contexts, there was no obvious construction that could play the role of a cylinder or cocylinder. The real advances have had to wait until quite recently and depend, in part as one might expect, on finding a nice homotopy theory on pro-categories that allows some intuitive interpretation in terms of homotopy coherence even if that is not the tool that is used directly.

8.3. Pro-homotopy theory

The problems faced by both shape theory and étale homotopy theory can be summarised by saying that they both needed an adequate homotopy theory in a category of 'pro-spaces', that is, of inverse systems of spaces or simplicial sets. Let us take this apart a bit.

Firstly in these settings of shape and étale homotopy, the objects being studied cannot be usefully 'probed' directly by spheres, simplices, etc., yet we can extract certain homotopy theoretic models encoding some of their 'geometry'. These form *inverse systems* of *approximations* to some 'abstract limiting homotopy type' and morphisms of the original objects induce morphisms of the inverse systems of approximations.

It seems feasible that if these induced morphisms of inverse systems are somehow made up of homotopy equivalences, then really the original objects should be considered to be 'equivalent' in some sense. The problem is not only do the usual constructions of the approximating systems seem to suffer from becoming 'homotopic too early', i.e., ending up in somewhere like $Pro–Ho(\mathcal{S})$, rather than somewhere less vague and 'fluid', but even if we somehow got to $Pro–\mathcal{S}$ or $Pro–Top$, then it is not clear how to interpret the approach, summarised above, as being 'made up of homotopy equivalences'. Trying to use a naive cylinder object also gives somewhat anomalous results, that is, if one wants to find a homotopy theory in $Pro–\mathcal{S}$ or $Pro–Top$ that has homotopy equivalences somewhat like the isomorphisms in the Artin-Mazur's $Pro–Ho(Top)$. Luckily these aspects have closely related solutions. The first steps were taken by Edwards and Hastings in 1976, [20], and other approaches have evolved depending on the applications envisaged, see, for instance, Isaksen, [30, 31], and with Fausk, [22].

Before we examine these briefly, we must first look at the definition of *inverse system*, or rather *pro-object* as they tend to be called nowadays, and also at that of a morphism between two such.

A pro-object in a category $\mathcal{C}$ is a diagram, $X : \mathcal{I} \to \mathcal{C}$, of objects of $\mathcal{C}$ in which the indexing category, $\mathcal{I}$, is cofiltering. This latter condition has two

parts:
(i) for each pair of objects i, j of $\mathcal{I}$, there is an object k and morphisms $k \to i$, $k \to j$,
and
(ii) for each parallel pair of morphisms $i \rightrightarrows j$ between two objects, i and j, of $\mathcal{I}$, there is an object k and a morphism $k \to i$ such that the two composites $k \longrightarrow i \rightrightarrows j$ are equal.

These two properties fit nicely into several of the situations that we have met, for instance, given two triangulations of a polyhedron, there is a common refinement of them; given two open coverings of a space there is a common refinement, etc., which gives 'meaning' to the first condition. The second is sometimes not needed as in many cases that arise in practice, there are never two parallel morphisms between distinct objects, so it is trivially satisfied. On the other hand it is needed in some cases.

The notion of a morphism, $f : X \to Y$, of pro-objects is moderately complex to write down in 'elementary' terms. Clearly it should consist of lots of interrelated morphisms, $f_{ij} : X(i) \to Y(j)$, but the simplest, but not very illuminating description comes via the description of the set of morphisms in the category $Pro{-}\mathcal{C}$:

$$Pro{-}\mathcal{C}(X, Y) := lim_j colim_i \mathcal{C}(X(i), Y(j)).$$

This is admittedly rather opaque, but we will not need the details as we merely use a few facts about these objects and morphisms.

Firstly, if we have a pro-object, $X : \mathcal{I} \to \mathcal{C}$, and we restrict to an initial subcategory of $\mathcal{I}$, then the resulting pro-object will be isomorphic to X. More generally we can compose with an initial functor from some $\mathcal{J}$ to $\mathcal{I}$ and similarly obtain an isomorphic pro-object. We say that the new pro-object is obtained by 'reindexing' the first.

We might propose a cylinder based notion of homotopy for $Pro-\mathcal{C}$, where $\mathcal{C}$ stands for some category with a nice homotopy theory defined on it. The obvious cylinder to try is $X \times I := (X(i) \times I)_{i\in\mathcal{I}}$, in other words, applying the pre-existing cylinder from $\mathcal{C}$ at each index. With that we would have $f, g : X \to Y$ were considered (globally) homotopic if there was a $H : X \times I \to Y$, ..., as expected. This is very much stronger than what the Artin-Mazur notion gives. That latter notion is best illustrated with two *level* morphisms, $\{f_n, g_n : X_n \to Y_n\}$ (and our example will be with $\mathbb{N}$ as indexing category, hence the index n), then they will give equal maps in $Pro{-}Ho(\mathcal{C})$ if, for each n, they are homotopic, $H_n : f_n \simeq g_n$, but without any compatibility, or coherence conditions on these homotopies. For example, the space known as the *dyadic solenoid* is well known as an example both in shape theory and in the theory of dynamical systems. The Čech complex of the dyadic solenoid can be represented as the inverse system, D:

$$\ldots \to S^1 \overset{2}{\to} S^1 \overset{2}{\to} \ldots \to S^1 \overset{2}{\to} S^1$$

where each $S^1 \overset{2}{\to} S^1$ is the degree two map $z \mapsto z^2$. If we calculate $Pro-Ho(Top)(*, D)$, for $*$ a one point space, then each $Ho(Top)(*, D_n)$ is a single point as S^1 is connected, but there are uncountably many global homotopy classes of maps from a point to D, so this global homotopy is very different from the purely discrete, index-by-index version implied by Artin and Mazur's method.

The 'naive' global homotopy idea can also lead to slightly strange effects, but ones that we should not be surprised to see. The following example is due to Edwards and Hastings, [20], p. 54.

Take $X_n = (S^1 \vee [0, \infty)) \times \{0, 1\} \cup_{[n,\infty)\times\{0,1\}} ([n, \infty) \times [0, 1])$. (This looks like two circles with tails, and a ribbon joining the two tails from the point n along the tails out to infinity.) The bonding maps in $X = (X_n)_{n\in\mathbb{N}}$ will be the inclusions of X_n into X_{n-1} at each stage. Now take $Y_n = S^1 \times \{0, 1\} \cup [0, 1]$, i.e., two circles joined by an interval, independent of n and with the bonding maps in the resulting Y, to be just the identity maps at each level. There is an obvious map from X_n to Y_n, crunching the tails down to a point on the respective circles and the ribbon down to the interval. This is a homotopy equivalence. These crunching maps are compatible with the bonding maps so give a level homotopy equivalence from X to Y, but there is no possible level map back from Y to X which could act as a homotopy inverse. That is no surprise, but does return us towards homotopy coherent as there *is* a homotopy coherent map from Y to X.

In this we have tacitly used the fact that if we have a pro-object consisting of morphisms in $\mathcal{C}$, (so in $Pro-(\mathcal{C}^{\to})$, where $\mathcal{C}^{\to}$ is the category of arrows in $\mathcal{C}$), then, automatically, it gives us a morphism in $Pro-\mathcal{C}$. This is just a question of checking it, but we will not do so here. Any morphism of that type, so $(f_i : X_i \to Y_i)_{i\in\mathcal{I}}$, is called a *level morphism*. We thus have that level morphisms are morphisms in $Pro-\mathcal{C}$. Less obviously, given any morphism $f : X \to Y$ in $Pro-\mathcal{C}$, there is a level morphism isomorphic to f. This is obtained by looking at the various representatives of f in the limit-colimit description and building a cofiltering category from it which has initial functors to the indexing categories of X and Y. The end result of this is a level morphism, $\overline{f} : \overline{X} \to \overline{Y}$, together with reindexing isomorphisms between X and $\overline{X}$, and Y and $\overline{Y}$, compatibly, so that

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\ \cong \big| & & \big| \cong \\ X & \xrightarrow{\overline{f}} & Y \end{array}$$

commutes in $Pro-\mathcal{C}$. This is a result of a reindexing lemma to be found in various strengths in various sources, for instance, one can reindex diagrams of certain forms within Pro–$\mathcal{C}$ so as to replace them, up to isomorphism, by pro-diagrams. (But beware the diagrams have to have 'certain forms' and these can be quite constraining.)

This result is the key both to getting a good homotopical structure on pro-categories but also to interpreting it. We will sketch how this goes for pro-simplicial sets. We say that $f : X \to Y$ is a weak equivalence of pro-simplicial sets if it is a retract of a level weak equivalence. (Recall that a weak equivalence of (pointed connected) simplicial sets is a morphism that induces isomorphisms on all homotopy groups. There is an important point here as, technically, these need not be homotopy equivalences unless the simplicial sets concerned are Kan complexes. We will skate over this by assuming that all the ones that we meet are either Kan or have been converted into Kan complexes by one of the usual completion processes for this. (For the purposes of the account here this will suffice.) Because of this, the weak equivalences will be homotopy equivalences for most of our discussion in this section.)

We want, at least, to form a homotopy category, $Ho(Pro-\mathcal{S})$ in which to work for both 'strong' shape and étale homotopy, and for this will invert these weak equivalences in Pro–$\mathcal{S}$. The idea is, in part, that for each indexing category, $\mathcal{I}$, we have the class of (level) homotopy equivalences and, by Vogt's theorem on homotopy coherence, the homotopy category, $Ho(\mathcal{S}^{\mathcal{I}})$, has a faithful interpretation as a category of homotopy coherent diagrams. In constructing $Ho(Pro-\mathcal{S})$, we are taking these categories of homotopy coherent diagrams and are patching them together along the reindexing isomorphisms coming from initial functors on the indexing categories. That is the idea. The link with homotopy coherence was not so evident in [20], but in the more-or-less equivalent approach in [43, 44, 45] this was explicit. The approach of Edwards and Hastings linked their model with proper homotopy theory, (see below) and with Steenrod homology, in both of which coherence plays an evident part. They used the Vietoris complex to develop an approach to a coherent form of shape theory, that was subsequently called *strong shape theory*.

For us it is worth noting that if we have a pointed pro-simplicial set, X, then we can form its homotopy pro-groups, $(\pi_n(X(i)))_{i\in\mathcal{I}}$, We should resist taking the limiting group as the limit functor will destroy information (and exactness of sequences). There are, however, other 'homotopy' groups that are better suited for studying X, and which, in their turn, contain the information on those limit groups. These correspond to two constructions due to Quigley, [52, 53, 54], in his versions of strong shape but which are easier to understand when we pass over to the proper homotopy context, where they were independently developed by E. Brown, [7]. Both here in $Pro-\mathcal{S}$ and there in the 'proper' context, they correspond to different interval-like objects and so to different sphere-like objects, but here they are less 'pretty' to describe.

8.4. Proper homotopy theory

We introduced some of the ideas of proper homotopy theory earlier, but now need to put some detail into the discussion. Proper homotopy theory tries to handle not just deformations of spaces and maps and invariants of such, but also, as it applies to infinite simplicial and CW-complexes and to non-compact manifolds, it handles behaviour 'at infinity'. (Some idea of the theory can be gleaned from the survey article [46], which also gives some indication as to its origins.)

The basic type of space studied by proper homotopy is connected, locally compact and Hausdorff and is also locally connected. (It is often restricted to be σ-compact, so can be written as a union of a nested sequence of compact subsets.)

For such a space, X, we write $\varepsilon(X) = \{c\ell(X\setminus K) \mid K \text{ compact}, K \subseteq X\}$, where $c\ell$ denotes closure. This is a pro-space, and ε is a functor from a suitable category of such 'basic' spaces with proper maps between them, taking values in $Pro-Top$. The simplest invariant of such spaces is the space of (Freudenthal) ends,

$$e(X) = lim\, \pi_0\varepsilon(X).$$

(We note that this is a bit Čech-like in its definition as it is a limit of a standard homotopy invariant applied to a series of 'approximations'.) We used earlier that $\mathbb{R}$ had two ends, and note $\#(e(\mathbb{R})) = 2$ (and so did the infinite strip, X_1), whilst $\#(e(\mathbb{R}_{\geqslant 0}) = 1$ (and so did our half-infinite strip). For our infinite thornbush, $\#(e(X_3)) = 2^{\aleph_0}$. This invariant e is a 'proper' analogue, 'out towards ∞' of the set of connected components, $\pi_0(X)$, for the standard theory. In fact, if M is a compact manifold with boundary ∂M then $M \setminus \partial M$ is a suitable space to which to apply e and $e(M \setminus \partial M) \cong$

$\pi_0(\partial M)$. (This suggests that invariants of the 'ends' should tell one if a given open manifold could be the result of removing the boundary of a compact manifold, and perhaps if such a compact manifold was uniquely determined. The answer is known and is one of the sources of proper homotopy theory, see [46] for a brief discussion and some references. For more details, you would need to dig deeper!)

Note that we called $e(X)$ the *space* of ends. In general we can usefully consider $e(X)$ as a pro-finite space, see [46], p. 133, for a little more on this. In fact we will see that $e(X)$ is not the only way of getting to a 'space of ends'.

Having obtained an analogue of π_0, the 'obvious' next step would be an analogue of π_1. We would have to replace the usual base point needed for the standard version by some (proper) 'base ray', $x : [0, \infty) \to X$, so that we will be putting a 'base point' all the way out to some end, then we could define $lim\, \pi_1(\varepsilon(X), x)$. Oh dear, there are lots of complications here, as two base rays may define the same point in $e(X)$, - and so are in the same 'end' - but the resulting limit groups are far from being isomorphic, (see the example on p. 134 - 135 of [46]).

The main point for us here is to note that this limit group is not recording the information that we might have hoped for. The elements of this group correspond to 'sequences' of homotopy classes of loops in the various $c\ell(X \setminus K)$, where each is homotopic to the next one in the larger of the two sets. (If you are getting the 'message' from earlier then you may have noticed 'homotopic' says a homotopy exists but does not record the homotopy! 'Clearly', we need to bring in some homotopy coherence!) To some extent the limit group was defined simply because it seemed that it could be defined. What might be a better idea would be to ask: what nice geometric test spaces could act as the analogues here of the circle, S^1? There are two fairly obvious ones if we restrict attention to the based situation, and so are looking at spaces with a base ray (as above). One of these is simply the half infinite cylinder, $S^1 \times [0, \infty)$. The other is the half infinite string of circles, $\underline{S}^1 = ([0, \infty) \times \{1\}) \cup (\mathbb{N} \times S^1)$. Both these objects are cogroup objects in our category of spaces and proper maps, so both define some types of 'homotopy group'. They can be thought of as corresponding to two different analogues of an interval, one being $I \times [0, \infty)$, the other the half infinite ladder. Using them we get *two* different 'fundamental groups'. They reflect different aspects of $\varepsilon(X)$, but they are not independent. There are exact sequences joining them that come from the evident inclusion of $\underline{S}^1$ into $S^1 \times [0, \infty)$. There are higher dimensional analogues defined in the obvious way. It is fairly easy to see that the difference between the limit groups that we looked at before and the group based on $S^1 \times [0, \infty)$ is that where the former, as we mentioned, involved sequences of loops each homotopic to the next, in the latter those homotopies are given explicitly by the part of the infinite cylinder between the successive integer 'slices'.

Aside: The homotopy groups related to the $S^n \times [0, \infty)$ can also be calculated from the homotopy limit of $\varepsilon(X)$, and there is a neat construction if $\varepsilon(X)$ is a 'tower' that gives the corresponding $\underline{\pi}_n(X, x))$ from the pro-group. (For more on this see the survey article [46] again.)

The end space functor is the central part of the Edwards-Hastings embedding. There are categories P and P_∞. The first is that of our spaces and proper maps, the second of those same spaces, but with germs at infinity of proper maps as the maps there. The end construction, ε, gives an functor from P_∞ into $Pro-Top$, whilst if we record the homotopy on the whole space as well we get a functor from P into $(Pro-Top, Top)$. Edwards and Hastings proved in [20] that both these were embeddings, and that they induced embeddings at the level of the respective homotopy categories. This allows constructions to travel between the very geometric proper homotopy setting and the very abstract pro-category one.

Edwards and Hastings pushed this one step further and extended the Chapman embedding theorem. Any compact metric space can be embedded in the space $\prod(0, 1/n)$, considered as a subspace of the Hilbert cube, $Q = \prod[0, 1/n]$. Chapman had shown that two compact metric spaces had the same shape if their compliments in Q had the same (weak) proper homotopy, that is, if the resulting $\varepsilon(Q \setminus X)$ and $\varepsilon(Q \setminus Y)$ were isomorphic in $Pro\text{–}Ho(Top)$. Edwards and Hastings essentially showed that strong shape corresponded to proper homotopy of the complements, and hence to working in $Ho(Pro\text{–}Top)$; again the reference for the detailed statements is their lecture notes, [20]. We note the connection to Steenrod 'homotopy' in the title of that source which brings us back to our earlier discussion.

We could say a lot more about proper homotopy that was relevant to our themes, but as some of that is in [46], we will refer the reader to that survey and the papers referred to there.

The point to retain is that homotopy at the 'global' level is studied by the use of information also at the ends, so in some sense locally at infinity. This changes the viewpoint that has to be considered 'normal' for our intuitions. (We will see that even more in the final section below.) There can be local 'reasons for equivalence' yet no global one. There can be non-proper reasons that do not correspond to equivalences out towards infinity, and so on. The compatibility is what is important here.

8.5. Directed homotopy theory

Our final section looks at an emerging form of homotopy, one that has still to get an agreed definitive form, yet it has useful applications and the intuitions are moderately clear to understand. This area is that of directed homotopy, that is a theory of homotopy suited for studying directed spaces, and other similar objects. Partially ordered *sets* are frequently used to model systems in both computer science and physics. The order models 'time', or 'use of resources', and often can not be globally given. For instance, in models for the temporal modal logic S4, the models are preorders, but the time dependency is merely 'before'; there is no clock. Similarly in Physics, in the theory of causal sets, which are 'locally finite' or 'discrete' partial orders, 'causality' is represented by '$\leqslant$' and again no global clock is given. Many physical systems are analysed by models of a space of 'evolving states'. In the study of 'space-time' manifolds, the evolving states are modelled by 'time-like' paths.

The problem is thus to look at some ideas from homotopy theory that suggest approaches on how to model spaces of directed (hence 'time-like') paths in a directed space (and, for us here, to see how that enlghtens our reflections on the notion of homotopy).

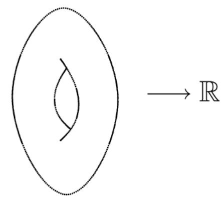
To start off, we will look at a particular type of directed space. These are not the only ones that might be useful, or natural, in a given situation, but we are, here, more interested in building some intuitions

Definition. — A *partially ordered space* or *pospace*, X, is a topological space with a (globally defined) closed partial order, $\leqslant$, so considering $\leqslant$ as a subset of $X \times X$, it is a closed subset.

A directed map (or *dimap* for short), $f : X \to Y$, between two pospaces, X and Y, is a continuous map that respects the partial order, $x \leqslant x' \Rightarrow f(x) \leqslant f(x')$.

Examples. —

1. Give the unit interval $I = [0,1]$, the usual order. This gives it the structure of a pospace that we will denote by $\vec{I}$. A related pospace is the closed interval $[0,r]$ of length $r \geqslant 0$ with its usual order. This will be denoted $\overrightarrow{[0,r]}$.
2. Let M be a compact differentiable manifold and $f : M \to \mathbb{R}$, a Morse function, so that f is smooth with no degenerate critical points. (As a simple example, take a torus "on end" with f a height function,



then f has 4 critical points, one is a minimum, one a maximum and there are two saddle points.) Define a pospace structure on M by $x \leqslant x' \Longleftrightarrow x = x'$ or $f(x) < f(x')$. The idea is to make $t = f(x)$ into a 'time-like variable', in such a way that the 'space-like' slices are the level sets $f^{-1}(t)$.)

What sort of (directed) homotopy should we use with this situation? If we have two 'dipaths', i.e. 'dimaps' from $\vec{I}$ to our directed space, when should they be thought of as being equivalent? Clearly if they are directed homotopic, whatever that should mean, then they should be homotopic, but if there is an ordinary homotopy between them, we would probably want to require that all intermediate paths in the homotopy were themselves dipaths.

As we are to some extent experimenting here, let us just write something down to help us talk about things more exactly:

- a pospace, X, (we will hide mention of the orders);
- two dipaths, $a_0, a_1 : \vec{I} \to X$;
- a homotopy, $h : \vec{I} \times I \to X$, and hence for each t, with $0 \leqslant t \leqslant 1$, an intermediate path, $a_t : \vec{I} \to X$ given by $a_t(s) = h(s, t)$.

Our assumption thus is that we should have that the homotopy, h, should take us through only directed paths, so all the a_t would also be dipaths. Now there is a dichotomy, should 'dihomotopies' be reversible or not?

We could take the above as a definition of homotopy between dipaths, and adapt it to handle homotopies between dimaps between other spaces, $f_0, f_1 : X \to Y$, etc. This will work fine if one is interested in 'equivalent dipaths', for instance in a situation where it is the 'effect' of the passage along the dipath that matters and nearby dipaths give the same 'effect'. On the other hand, there may be more of a irreversible nature to a dihomotopy, for instance it may 'take time' in which case a stronger form of dihomotopy would be to have $h : \vec{I} \times \vec{I} \to X$. These forms would be reflected in the structure of any cylinders considered.

Some of these considerations are discussed in Grandis' book, [24]. Some other questions relating to the possibility of using enriched settings for handling some of the points here are explored in [50], and, of course, the papers referred to in both these sources will allow some idea of the structures that can be used to handle this directed homotopy.

To return, finally, to a point that was mentioned earlier, this is all closely related to the links between homotopy and rewriting. In a rewrite system, not all rewrites need be reversible. Rewrites are specified by the system, so suppose, for instance, that the allowed rewrites include replacing *aa* by the empty word. There may not be a rewrite rule in the system that allows the insertion of an *aa* in place of an empty string, neither directly nor as a consequence of a string of some given rules. In this case, thinking of the rewrites as homotopies, those homotopies will not be reversible. You can add in reverses for such homotopies and will get some ease of handling the rules, but you will also loose information!

In these directed cases, just as in our earlier case studies, the primitive idea of homotopy as deformation, or equivalence, is replaced by something more nuanced, more subtle, but still being able to be handled by many of the methods we have discussed in the various thematic threads we have been exploring.

Bibliography

[1] Artin (M.), Mazur (B.). — Etale homotopy, Lecture Notes in Maths., no. 100, Springer-Verlag, Berlin (1969).

[2] Batanin (M. A.). — "Homotopy coherent category theory and A_∞-structures in monoidal categories", J. Pure. Appl. Alg. 123, p. 67-103 (1998).

[3] Baues (H. J.). — Algebraic homotopy, Cambridge Studies in Advanced Mathematics, vol. 15, Cambridge Univ. Press (1989).

[4] Baues (H. J.). — Combinatorial homotopy and 4-dimensional complexes, Walter de Gruyter (1991).

[5] Baues (H. J.). — "Homotopy types", in Handbook of Algebraic Topology (I.M. James, ed.), Elsevier, p. 1-72 (1995).

[6] Boardman (J. M.), Vogt (R. M.). — Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Maths, no. 347, Springer-Verlag (1973).

[7] Brown (E.). — "Proper homotopy theory in simplicial complexes", in Topology Conference (Virginia Polytech. Inst. and State Univ., Blacksburg, Va., 1973), 1974.

[8] Brown (K. S.). — "Abstract homotopy theory and generalized sheaf cohomology", Trans. Amer. Math. Soc 186, p. 419-458 (1973).

[9] Brown (R.). — "Crossed complexes and higher homotopy groupoids as noncommutative tools for higher dimensional local-to-global problems", in Handbook of Algebra (M. Hazewinkel, ed.), vol. 6, Elsevier B.V. North-Holland, p. 81-124 (2009).

[10] Christie (D. E.). — "Net homotopy for compacta", Trans. Amer. Math. Soc 56, p. 275-308 (1944).

[11] Cisinski (D.-C.). — Les préfaisceaux comme modèles des types d'homotopie, Astérisque, vol. 308, Société Mathématique de France (2006).

[12] Cordier (J.-M.). — "Sur la notion de diagramme homotopiquement cohérent", Cahiers de Top. Géom. Diff. 23, p. 93-112 (1982).

[13] Cordier (J.-M.), Porter (T.). — "Vogt's theorem on categories of homotopy coherent diagrams", Math. Proc. Cambridge Philos. Soc. 100, no. 1, p. 65-90 (1986).

[14] Cordier (J.-M.), Porter (T.). — "Maps between homotopy coherent diagrams", Top. and its Appls. 28, p. 255-275 (1988).

[15] Cordier (J.-M.), Porter (T.). — "Fibrant diagrams, rectifications and a construction of Loday", J. Pure. Applied Alg. 67, p. 111-124 (1990).

[16] Cordier (J.-M.), Porter (T.). — "Categorical aspects of equivariant homotopy", Applied Categorical Structures 4, p. 195-212 (1996).

[17] Cordier (J.-M.), Porter (T.). — "Homotopy coherent category theory", Trans. Amer. Math. Soc. 349, p. 1-54 (1997).

[18] Cordier (J.-M.), Porter (T.). — Shape theory, categorical methods of approximation, Dover (reprint of earlier 1989 edition) (2008).

[19] Dowker (C. H.). — "Homology groups of relations", Annals of Maths 56, p. 8495 (1952).

[20] Edwards (D. A.), Hastings (H. M.). — Čech and Steenrod homotopy theories with applications to geometric topology, Lecture Notes in Maths, vol. 542, Springer-Verlag, (1976).

[21] Fantham (P. H. H.), Moore (E. J.). — "Groupoid enriched categories and homotopy theory", Canad. J. Math. 35, no. 3, p. 385-416 (1983).

[22] Fausk (H.), Isaksen (D.). — "Model structures on pro-categories", Homology, Homotopy and Applications 9, no. 1, p. 367-398 (2007).

[23] Friedlander (E. M.). — Etale homotopy of simplicial schemes, Annals of Mathematics Studies, vol. 104, Princeton University Press, Princeton, N.J. (1982).

[24] Grandis (M.). — Directed algebraic topology, New Mathematical Monographs, vol. 13, Cambridge University Press, Cambridge, (2009), Models of non-reversible worlds.

[25] Gratus (J.), Porter (T.). — "A spatial view of information", Theoretical Computer Science 365, p. 206-215 (2006).

[26] Grothendieck (A.). — "Pursuing stacks", manuscript, 600 + pages, (1983).

[27] Grothendieck (A.). — Revêtements étales et groupe fondamental (SGA 1), Documents Mathématiques (Paris), 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 196061. Directed by A. Grothendieck, With two papers by M. Raynaud, updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; (50 7129)].

[28] Guiraud (Y.), Malbos (P.). — "Coherence in monoidal track categories", Mathematical Structures in Computer Science (2011), p. (to appear).

[29] Guiraud (Y.), Malbos (P.). — "Identities among relations for higher-dimensional rewriting systems", Séminaires et Congrès, SMF 26, p. 145-161 (2011).

[30] Isaksen (D. C.). — "A model structure on the category of pro-simplicial sets", Trans. Amer. Math. Soc. 353, no. 7, p. 28052841 (electronic) (2001).

[31] Isaksen (D. C.). — "Strict model structures for pro-categories", in Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 215, Birkhäuser, Basel, p. 179-198 (2004).

[32] Joyal (A.). — "Quasi-categories and Kan complexes", J. Pure Applied Alg. 175, p. 207-222 (2002).

[33] Joyal (A.) "The theory of quasi-categories and its applications", in Advanced Course on Simplicial Methods in Higher Categories (C. Casacuberta & J. Kock, eds.), i-Math, vol. II, Centre de Recerca Matemàtica, http://www.crm.cat/ HigherCategories/hc2.pdf, 2008.

[34] Kamps (K. H.). — "On exact sequences in homotopy theory", Topol. Appl. Sympos. Bydva (Yugoslavia), p. 136-141 (1972).

[35] Kamps (K. H.), Porter (T.). — Abstract homotopy and simple homotopy theory, World Scientific Publishing Co. Inc., River Edge, NJ (1997).

[36] Kan (D. M.). — "Abstract homotopy I", Proc. Nat. Acad. Sci. USA 41, p. 1092-1096 (1955).

[37] Kan (D. M.). — "On c.s.s complexes", Amer J. Math. 79, p. 449-476 (1958).

[38] Lurie (J.). — Higher topos theory, Annals of mathematics studies, no. 170, Princeton University Press (2009).

[39] Maltsiniotis (G.). — "La catégorie cubique avec connexions est une catégorie test stricte", Homology, Homotopy and Applications 11, no. 2, p. 309-326 (2009).

[40] Mardešić (S.). — "Absolute neighborhood retracts and shape theory", in History of Topology (I. M. James, ed.), Elsevier Science, Amsterdam, p. 247-268 (1999).

[41] Mardešić (S.). — Strong shape and homology, Springer (2000).

[42] nLab. — "nLab", at http://ncatlab.org/nlab/show/HomePage.

[43] Porter (T.). — "Abstract homotopy theory in procategories", Cahiers Top. Géom. Diff. 17, no. 2, p. 113124 (1976).

[44] Porter (T.). — "Coherent prohomotopical algebra", Cahiers Top. Géom. Diff. 18, no. 2, p. 139-179 (1977).

[45] Porter (T.). — "Coherent prohomotopy theory", Cahiers Top. Géom. Diff. 19, no. 1, p. 3-46 (1978).

[46] Porter (T.). — "Proper homotopy theory", in Handbook of Algebraic Topology, North-Holland, Amsterdam, p. 127-167 (1995).

[47] Porter (T.). — "What 'shape' is space-time?", http://arxiv.org/abs/gr-qc/0210075 (2002).

[48] Porter (T.). — "Abstract homotopy theory, the interaction of category theory and homotopy theory", Cubo Matematica Educacional 5, p. 115-165 (2003).

[49] Porter (T.). — "$\mathcal{S}$-categories, $\mathcal{S}$-groupoids, Segal categories and quasicategories", http: //arxiv.org/abs/math/0401274 (2004).

[50] Porter (T.). — "Enriched categories and models for spaces of evolving states", Theoretical Computer Science 405, no. 1-2, p. 88100 (2008).

[51] Porter (T.). — "The Crossed Menagerie: an introduction to crossed gadgetry and co-homology in algebra and topology", 2011, (available from the n-Lab, http: //ncatlab.org/nlab/show/Menagerie).

[52] Quigley (J. B.). — "Shape theory, Approaching and a Hurewicz theorem", Ph.D. Thesis, Indiana University (1970).

[53] Quigley (J. B.). — "An exact sequence from the nth to the $(n-1)$-st fundamental group", Fund. Math. 77, no. 3, p. 195210 (1973).

[54] QUIGLEY (J. B.). — "Equivalence of fundamental and approaching groups of movable pointed compacta", Fund. Math. 91, no. 2, p. 7383 (1976).
[55] QUILLEN (D. G.). — Homotopical algebra, Lecture Notes in Maths., no. 53, Springer-Verlag, (1967).
[56] VOGT (R.). — "Homotopy limits and colimits", Math. Z. 134, p. 11-52 (1973).