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ALEXANDER I. KHEYFITS

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Valiron-Titchmarsh Theorem for Subharmonic Functions in $\mathbb{R}^n$ With Masses on a Half-Line

Alexander I. Kheyfits[(1)]

Abstract. — The Valiron-Titchmarsh theorem on asymptotic behavior of entire functions with negative zeros is extended to subharmonic functions in $\mathbb{R}^n$, $n \geqslant 3$, having the Riesz masses on a ray.

Résumé. — Le théorème de Valiron-Titchmarsh sur le comportement asymptotique des fonctions entières avec des zéros négatifs est étendu aux fonctions sous-harmoniques dans $\mathbb{R}^n$, $n \geqslant 3$, ayant les masses de Riesz sur un rayon.

1. Introduction and Statement of Results

In 1913, Georges Valiron published his well-known memoir [24] in *Annales de la faculté des sciences de Toulouse*, which became his dissertation the next year. A century later, this work is still regularly cited. The well-known result of [24] is the sequel Tauberian theorem about entire functions with roots on a half-line. An elegant succinct account of the current research about this class of functions has been recently given by Drasin [11].

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(1) The Graduate Center and Bronx Community College of the City University of New York, BCC, Department of Mathematics and Computer Science, 2155 University Avenue, Bronx, NY 10453, USA
akheyfits@gc.cuny.edu

Article proposé par Xavier Tolsa.

Theorem (Valiron[1] [24, p. 237]).— *Let $f(z), z = re^{\imath\theta}$, be an entire function of non-integer order ρ and finite type with negative zeros, and $n(t)$ be the counting function of its zeros, that is, the number of the zeros of f in the closed disk $\{|z| \leqslant t\}$.*

If there exists the limit over the positive ray

$$\lim_{r\to\infty} r^{-\rho} \log f(r) = \frac{\pi\mathbf{\Delta}}{\sin \pi\rho}, \tag{1.1}$$

where $\mathbf{\Delta}$ is a constant, then there exists the limit

$$\lim_{t\to\infty} t^{-\rho} n(t) = \mathbf{\Delta}. \tag{1.2}$$

The positivity of the counting function is the Tauberian condition of the theorem.

The Valiron theorem was later independently proved by Titchmarsh [23], thus it is often referred to as the Valiron-Titchmarsh theorem on entire functions with negative zeros; its current exposition can be found, for example, in [19, Lect. 12-13]. The result was extended to holomorphic and subharmonic functions in $\mathbb{R}^2$ with associated masses on several rays or on logarithmic spirales, and to functions with many-term asymptotics, see [2, 5, 16, 17] and the references therein. The Abelian converse of the theorem, that is, the implication (1.2) $\Rightarrow$ (1.1) for subharmonic functions with many-term asymptotics and with masses on one ray in $\mathbb{R}^n$, $n \geqslant 3$, was considered by Agranovich [1].

Delange [10] proved that the positive $x-$axis in (1.1) can be replaced by any ray $z = re^{\imath\phi}$ subject to the restriction

$$\phi \neq \frac{2k+1}{2\rho}\pi \text{ with an integer } k,\ -\rho - 1/2 < k < \rho - 1/2. \tag{1.3}$$

This restriction does not appear in the original Valiron theorem, where limit (1.1) is taken over the positive $x-$axis. The phenomenon was later considered in [16] and studied in detail by Azarin [4].

In this paper we extend the Valiron-Titchmarsh theorem to subharmonic functions with masses on a ray in $\mathbb{R}^n$, $n \geqslant 3$. We use the approach based on the General Tauberian Theorem by Wiener [20, Chap. V]. It invokes

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(1) In fact, Valiron proved the theorem for the entire functions of proximate order and so-called directed products.

an integral representation of the indicators of the subharmonic functions under consideration through the associated Legendre functions $P_\nu^\mu(\cos\psi)$ of the first kind. To derive this, likely new representation, we compute in the Appendix the Mellin integral transform of the Weierstrass canonical kernel and of the Riesz kernel $|x-y|^{-\lambda}$. In this section we fix notation, following [14] and [12], and state our results. The proofs are given in section 2.

Introduce in $\mathbb{R}^n = \{x = (x_1, \dots, x_n)\}$ spherical coordinates $x = (r, \theta)$, $r = |x|$, $\theta = (\theta_1, \dots, \theta_{n-1})$, such that $x_1 = r\cos\theta_1$. Here $0 \leqslant \theta_1 \leqslant \pi$ and $0 \leqslant \theta_k \leqslant 2\pi$ for $k = 2, 3, \dots, n-1$. The ball of radius t centered at the origin of $\mathbb{R}^n$ is denoted as B_t, $\overline{B_t}$ being its closure.

Let $u(x)$ be a subharmonic function of finite non-integer order ρ in $\mathbb{R}^n$, $n \geqslant 3$, such that its Riesz associated measure μ is supported by the negative x_1-axis, and let

$$n(t) = t^{2-n}\mu(\overline{B_t})$$

be the counting function of the measure μ. By the Hadamard representation theorem [14, Sect. 4.2.2]

$$u(x) = \int_{\mathbb{R}^n} K_q(x,y)d\mu(y) + Q(x). \tag{1.4}$$

Here K_q is the Weierstrass canonical kernel,

$$K_q(x,y) = -(r^2 + t^2 - 2tr\cos\psi)^{-\frac{n-2}{2}} + t^{2-n}\sum_{j=0}^{q}\left(\frac{r}{t}\right)^j G_j^{\frac{n-2}{2}}(\cos\psi),$$

the integral in (1.4) is convergent uniformly on any compact set and absolutely at the vicinity of infinity, Q is a harmonic polynomial of degree at most q, $q = E(\rho)$ being the integer part of ρ, $0 \leqslant q < \rho < q+1$. Also, let $t = |y|$, $\psi = \widehat{(x,y)}$ be the angle between vectors x and y, and $G_j^{(n-2)/2}(\cos\psi)$ the Gegenbauer polynomials given by the generating function

$$(1 - 2t\cos\psi + t^2)^{-\frac{n-2}{2}} = \sum_{j=0}^{\infty} G_j^{\frac{n-2}{2}}(\cos\psi)\, t^j.$$

Since we are interested in the asymptotic properties of u, which are not affected by the polynomial Q, we assume hereafter that $Q = 0$. Due to the same reason, we suppose, without loss of generality, that the closed unit ball $\overline{B_1}$ is free of the Riesz masses of u, thus $n(t) = 0$ for $0 \leqslant t \leqslant 1$.

For a vector $y = (t, \theta)$ with $\theta_1 = \pi$, all the other angular coordinates are undefined, and we represent such a vector as $y = (t, \pi)$. If a measure

μ is distributed over the negative x_1-axis, the angle between the vectors $x = (r, \theta_1, \ldots, \theta_{n-1})$ with the latitude θ_1, and $y = (t, \pi)$ in representation (1.4) is $\psi = \pi - \theta_1$ for any $\theta_2, \ldots, \theta_{n-1}$, and integral (1.4) becomes

$$u(x) = \int_0^\infty K_q(x, (t, \pi)) d\left(t^{n-2} n(t)\right).$$

Moreover, due to our assumptions, this integral can be transformed (see [12, Eq-ns (9) and (11)]) as

$$u(x) = \int_0^\infty t^{2-n} h_n\left(\frac{r}{t}, \theta_1, q\right) d\left(t^{n-2} n(t)\right), \tag{1.5}$$

where the kernel is

$$h_n(s, \theta_1, q) = -\left(1 + 2s\cos\theta_1 + s^2\right)^{\frac{2-n}{2}} + \sum_{j=0}^{q} (-s)^j G_j^{\frac{n-2}{2}}(\cos\theta_1).$$

Remark 1.1. — It is worth mentioning that in the case under consideration the kernel $h_n(s, \theta_1, q)$ and the integral in formula (1.5) depend only on the angle θ_1, but not upon the other angular coordinates $\theta_2, \ldots, \theta_{n-1}$ of the point x.

Before stating our results, we formulate an Abelian proposition, which is a converse of Valiron's theorem above; its proof is straightforward.

If there exists the limit (1.2), then there exist the limits

$$\lim_{r\to\infty} r^{-\rho} \log f(re^{\imath\theta}) = \frac{\pi\mathbf{\Delta}}{\sin\pi\rho} e^{\imath\rho\theta}$$

for any $\theta \in (-\pi, \pi)$, uniformly in any sector $-\pi + \delta \leqslant \theta \leqslant \pi - \delta$, $0 < \delta < \pi$, therefore, f is a function of the completely regular growth in the plane.

Now we state the Abelian result regarding the subharmonic functions given by (1.5).

PROPOSITION 1.2. — *Let u be a subharmonic function in $\mathbb{R}^n$, $n \geqslant 3$, of non-integer order ρ and finite type, whose Riesz masses are distributed over the negative x_1-axis. If there exists the limit*

$$\lim_{t\to\infty} t^{-\rho} n(t) = \mathbf{\Delta}, \tag{1.6}$$

then for any $x = (r, \theta)$, with $0 \leqslant \theta_1 < \pi$, there exists the limit

$$\lim r^{-\rho} u(x) = H(\theta) = H(\theta_1). \tag{1.7}$$

This limit, that is, the indicator function of the subharmonic function u, is given by

$$H(\theta_1) = (\rho + n - 2)\boldsymbol{\Delta} \int_0^\infty s^{-\rho-1} h_n(s, \theta_1, q) ds. \tag{1.8}$$

Moreover, the indicator can be expressed through the associated Legendre spherical functions of the first kind

$$P_\nu^\mu(\cos\theta_1)$$

on the cut $-1 < \xi = \cos\theta_1 \leqslant 1$ *[6, Chap. 3], as*

$$H(\theta_1) = \frac{\pi 2^{\frac{n-3}{2}} \Gamma(\frac{n-1}{2}) \prod_{k=1}^{n-2} (\rho + k)\ \boldsymbol{\Delta}}{(n-3)! \sin(\pi\rho) \, (\sin\theta_1)^{\frac{n-3}{2}}} \; P_{-\rho-\frac{n-1}{2}}^{\frac{3-n}{2}}(\cos\theta_1).$$

Using the known property, $P_\nu^\mu = P_{-\nu-1}^\mu$*, of the Legendre functions, [6, Sect. 3.3.1(1)], the latter can be rewritten as*

$$H(\theta_1) = \frac{\pi 2^{\frac{n-3}{2}} \Gamma(\frac{n-1}{2}) \prod_{k=1}^{n-2} (\rho + k)\ \boldsymbol{\Delta}}{(n-3)! \sin(\pi\rho) \, (\sin\theta_1)^{\frac{n-3}{2}}} \; P_{\rho+\frac{n-3}{2}}^{\frac{3-n}{2}}(\cos\theta_1). \tag{1.9}$$

Equation (1.8) holds good for $\theta_1 = \pi$ *as well, since in this case both its sides are equal to* $-\infty$*.*

The fact that (1.6) implies (1.7) is not new, it was established by Azarin [3] as early as in 1961, together with an integral representation of the indicator of subharmonic functions in $\mathbb{R}^n$, $n \geqslant 3$, in terms of the fundamental solutions of the Legendre differential equation

$$t(1-t)y''(t) + (n-1)\left(\frac{1}{2} - t\right) y'(t) + \rho(\rho + n - 2)y(t) = 0.$$

The occurrence of the associated Legendre functions in problems like ours was mentioned in [14, p. 160], without explicit formulas though.

A new feature of our result is the explicit representation (1.9) of the indicator in terms of the associated Legendre functions of the first kind, leading to a precise description of the zero sets of the indicators of subharmonic functions at question. We need this description to apply Wiener's tauberian theorem.

The zeros of the spherical functions have been carefully studied. It is known in particular, that if ν is not real, then $P_\nu^\mu(\cos\beta)$ is never zero [15, p. 403], while for any real μ and ν the equation $P_\nu^\mu(\cos\beta) = 0$ has only finitely many roots. In our case $\mu = (3-n)/2$ and $\nu = \rho + (n-3)/2$, hence the function

$$f(\beta) = P_{\rho+(n-3)/2}^{(3-n)/2}(\cos\beta)$$

has finitely many real zeros on $(0, \pi)$.

More precisely [15, p. 386-388], the equation $f(\beta) = 0$ has $E(\rho+1) = q+1$ zeros in the interval $(0, \pi)$, where $E(x)$ is the integer part of x. In particular, if $0 < \rho < 1$, then for any dimension n there is only one root, which was observed in [14, p.161]. For instance [8], if $n = 3$ and $\rho \approx 0.5$, then the only root $\beta_1 \approx 0.7\pi$; if $n = 5$ and $\rho \approx 0.5$, then the unique root $\beta_1 \approx 0.6\pi$.

Let $\Theta_n(\rho) = \{\beta_1^n, \ldots, \beta_{q+1}^n\}$ stand for the set of the roots of the equation $f(\beta) = 0$. Similarly to condition (1.3), our results include the restriction $\phi \notin \Theta_n(\rho)$. Now we state the Tauberian counterpart of Proposition 1.

THEOREM 1.3. — *Let u be a subharmonic function in $\mathbb{R}^n$, $n \geqslant 3$, of non-integer order ρ and finite type, whose Riesz masses are distributed over the negative x_1-axis. Let ϕ, $0 \leqslant \phi < \pi$, be an angle between a vector $x \in \mathbb{R}^n$ and the positive x_1-axis. If $\phi \notin \Theta_n(\rho)$ and the limit*

$$\lim r^{-\rho} u(x) = H(\phi) \tag{1.10}$$

exists, then there exists the limit

$$\lim_{t\to\infty} t^{-\rho} n(t) = M(g; 0) H(\phi)$$

where

$$M(g, 0) = \frac{2^{(n-3)/2}\Gamma((n-2)/2)\sin(\pi\rho)\,(\sin\phi)^{(n-3)/2}}{\pi^{3/2}\prod_{k=1}^{n-3}(\rho+k) P_{\rho+(n-3)/2}^{(3-n)/2}(\cos\phi)}$$

and therefore, by Proposition 1, there exist the limits for any θ_1, $0 \leqslant \theta_1 < \pi$,

$$\lim r^{-\rho} u(x) = H(\theta) \equiv H(\theta_1), \quad x = (r, \theta_1, \ldots, \theta_{n-1}),$$

with

$$H(\theta_1) = \left(\frac{\sin\phi}{\sin\theta_1}\right)^{\frac{n-3}{2}} \frac{P_{\rho+(n-3)/2}^{(3-n)/2}(\cos\theta_1)}{P_{\rho+(n-3)/2}^{(3-n)/2}(\cos\phi)} H(\phi). \tag{1.11}$$

If $\phi \in \Theta_n(\rho)$, the conclusion fails as an example below shows.

Remark 1.4. — If $n = 2$, then there exists an entire function, whose indicator vanishes on finitely many rays, and the function has the completely regular growth on these and only these rays [18, p. 161].

Remark 1.5. — If $n = 2$, the Weierstrass canonical kernel is different from that in (1.4), therefore in this case our proof is invalid. Nonetheless, if $n = 2$, we have [6]

$$P^{1/2}_{-\rho-1/2}(\cos\alpha) = \sqrt{2/(\pi\sin\alpha)}\cos(\rho\alpha),\ 0<\alpha<\pi,$$

thus in this case formula (1.9) becomes the known one, [16, Theor. 1'],

$$H(\theta_1) = \frac{H(\phi)}{\cos\rho\phi}\cos\rho\theta_1.$$

2. Proofs

Proof of Proposition 1.2. — By condition (1.2), $n(t) = \Delta t^\rho + t^\rho\varepsilon(t)$, with $\lim_{t\to\infty}\varepsilon(t) = 0$, thus

$$u(x) = u_1(x) + u_2(x),$$

where

$$u_1(x) = \int_0^\infty K_q(x,(t,\pi))d\left(t^{n-2}\Delta t^\rho\right)$$

$$= (\rho+n-2)\boldsymbol{\Delta}\int_0^\infty t^{\rho+n-3}K_q(x,(t,\pi))dt$$

and $u_2 = u - u_1$.

Due to (1.5),

$$u_1(x) = (\rho+n-2)\boldsymbol{\Delta}\int_0^\infty t^{\rho-1}h_n\left(\frac{r}{t},\theta_1,q\right)dt$$

$$= (\rho+n-2)\boldsymbol{\Delta}\left(\int_0^\infty s^{-\rho-1}h_n\left(s,\theta_1,q\right)ds\right)r^\rho.$$

To get from here (1.7), we have to estimate the second term u_2.

The latter is a subharmonic function of the non-integer order ρ. We need the known bound of the kernel h_n, see for example, [12],

$$\left|h_n(u,\theta,q)\right| \leqslant C\min\left(u^q;u^{q+1}\right) \tag{2.1}$$

for all $u > 0$ with a constant C depending only on n and q.

Since the associated measure of u_2 has the minimal type with respect to the non-integer ρ, the estimates in [21, Chap. 2] or [14, Chap. 4] imply that the function u_2 itself has zero type with respect to ρ, thus proving (1.7)-(1.8).

Equation (1.9) for $\theta_1 \neq \pi$ follows from (1.8) and Corollary 1 in Appendix A.

To verify (1.9) when $\theta_1 = \pi$, we notice that in this case the integral in (1.8) is divergent to $-\infty$. On the other hand, by making use of the known asymptotic formulas for the associated Legendre function [6, Sect. 3.9.1], we straightforwardly find that as $\theta_1 \uparrow \pi$,

$$H(\theta_1) \approx -\frac{(\rho+n-2)\left(\Gamma\left(\frac{n-3}{2}\right)\right)^2 \mathbf{\Delta}}{2(n-4)!} \frac{1}{(\cos(\theta_1/2))^{n-3}} \searrow -\infty$$

for $n \geqslant 4$, and

$$H(\theta_1) \approx (\rho+1)\mathbf{\Delta}\left[2\log(\cos(\theta_1/2)) + \gamma + 2\psi(-\rho) - \pi\cot(\pi\rho)\right] \searrow -\infty$$

if $n = 3$. Here γ is the Euler-Mascheroni constant and ψ the logarithmic derivative of the $\Gamma-$function. $\square$

Before proving Theorem 1, we formulate the following variant of the General Tauberian Theorem of Wiener [13, Sect. 12.8, Theor. 233].

Theorem W. — *Let a function $g \in L(0,\infty)$* be such that

$$\int_0^\infty g(t)t^{-\imath x}dt \neq 0$$

for all real x. Let a bounded real function f be slowly decaying, that is,

$$\underline{\lim}_{x\to\infty}\,(f(y)-f(x)) \geqslant 0$$

as $y > x$ and $\frac{y}{x} \to 1$. If there exists the limit

$$\lim_{x\to\infty}\frac{1}{x}\int_0^\infty g\left(\frac{t}{x}\right)f(t)dt = l\int_0^\infty g(t)dt,$$

then there exists the limit

$$\lim_{x\to\infty} f(x) = l$$

with the same constant l.

Remark 2.1. — A proof of the Valiron-Titchmarsh theorem for analytic functions in $\mathbb{R}^2$ can be based on Montel's theorem [18, p. 464-465], however, Montel's theorem is not valid for subharmonic functions [9].

Proof of Theorem 1.3. — We again represent $u(x)$ by (1.5) and transform it into the equation

$$\frac{u(r,\theta)}{r^\rho} = \frac{1}{r}\int_0^\infty \left(\frac{r}{t}\right)^{1-\rho} h_n\left(\frac{r}{t},\theta_1,q\right) d\alpha(t), \tag{2.2}$$

where

$$d\alpha(t) = t^{3-n-\rho} d\left(t^{n-2}n(t)\right).$$

Set in (2.2) $\theta_1 = \phi$ from (1.10). Denote also $f(t) = \frac{1}{t}\alpha(t)$,

$$g(t) = t^{\rho-1}h_n(1/t,\phi,q),$$

and the integral in (2.2) as $J(r)$. Integrating $J(r)$ by parts, we get

$$J(r) = -\frac{1}{r}\int_0^\infty f(t)\, t\frac{\partial}{\partial t} g\left(\frac{t}{r}\right) dt,$$

since the integrated term vanishes due to (2.1). We have

$$f(t) = \frac{\alpha(t)}{t} = \frac{1}{t}\int_0^t s^{3-n-\rho} d(s^{n-2}n(t)).$$

For a subharmonic function u of order ρ and finite type, its counting function $n(t)$ is non-decreasing and satisfies $n(t) \leqslant Ct^\rho$. Therefore, after integrating $f(t)$ by parts, we get

$$f(t) = \frac{\alpha(t)}{t} = \frac{n(t)}{t^\rho} + \frac{n-3+\rho}{t}\int_0^t \frac{n(t)}{t^\rho} dt. \tag{2.3}$$

Since

$$\frac{n(y)}{y^\rho} - \frac{n(x)}{x^\rho} = \frac{n(y)-n(x)}{x^\rho} + \frac{n(y)}{y^\rho}\left(1-\left(\frac{y}{x}\right)^\rho\right),$$

the integrated term in (2.3) is a slowly decaying function. The same is valid for the integral in (2.3), since

$$\frac{1}{y}\int_0^y \frac{n(s)}{s^\rho} ds - \frac{1}{x}\int_0^x \frac{n(s)}{s^\rho} ds = \frac{1}{y}\int_x^y \frac{n(s)}{s^\rho} ds + \left(\frac{x}{y}-1\right)\frac{1}{x}\int_0^x \frac{n(s)}{s^\rho} ds$$
$$\leqslant C\frac{y-x}{y} + C\frac{|x-y|}{y} \to 0$$

as $x \to \infty$ and $y/x \to 1$.

To apply Theorem W, we write the integral $J(r)$ as

$$J(r) = \frac{1}{r}\int_0^\infty \left(-u\frac{\partial}{\partial u}g(u)\right)\Big|_{u=t/r} f(t)dt,$$

and we have to show that the Mellin transformation of the kernel in (2.2) has no real zeros. Therefore, we are to show that the integral

$$M(g;v) = -\int_0^\infty u\frac{\partial}{\partial u}g(u)u^{-\imath v}du = -\int_0^\infty \frac{\partial}{\partial u}g(u)u^{1-\imath v}du, \qquad (2.4)$$

as a function of v, has no real zeros. Integrating (2.4) by parts, we again notice that the integrated term vanishes, and we have

$$M(g;v) = (1-\imath v)\int_0^\infty g(u)u^{-\imath v}du.$$

We remind that $g(u) = u^{\rho-1}h_n(1/u,\phi,q)$, thus we consider the integral

$$M(h_n;v) = (1-\imath v)\int_0^\infty u^{\rho-1}h_n\left(\frac{1}{u},\phi,q\right)u^{-\imath v}du$$

$$= (1-\imath v)\int_0^\infty h_n(u,\theta_0,q)u^{-1-\rho-\imath v}du,$$

and by virtue of Proposition 2 in the Appendix, equations (A.2)-(A.4) with $s=-\rho-\imath v$, we express $M(h_n;v)$ through the associated Legendre functions of the first kind

$$P^{\frac{3-n}{2}}_{-\rho-\imath v-\frac{n-1}{2}}(\cos\phi).$$

As we have stated above, the latter has no complex roots; while if $v=0$, it has $E(\rho+1)$ zeros, which are excluded by the condition $\phi\notin\Theta_n(\rho)$, thus

$$M(h_n;0) = \frac{\pi 2^{\frac{n-3}{2}}\Gamma(\frac{n-1}{2})\prod_{k=1}^{n-2}(\rho+k)\mathbf{\Delta}}{(n-3)!\sin(\pi\rho)\,(\sin\phi)^{\frac{n-3}{2}}}P^{\frac{3-n}{2}}_{-\rho-\frac{n-1}{2}}(\cos\phi).$$

After some simple algebra we derive from here equation (1.9).

To show that the restriction $\phi\notin\Theta_n(\rho)$ is essential, we consider, in the simplest case $n=3$ and $0<\rho<1$, the function

$$u_0(x) = r^\rho + r^\rho\sin\log\log r\; P_\rho(\cos\theta_1),$$

where as usual, $P_\rho = P^0_\rho$. We straightforwardly verify that the Laplacian

$$\Delta u_0(x)$$

$$= r^{\rho-2}\left\{\rho(\rho+1)+(2\rho+1)\frac{\cos\log\log r}{\log r}P_\rho(\cos\theta_1)+O\left(\frac{1}{\log^2 r}\right)\right\}.$$

It is known [6, Sect.3.9.2 (15)] that $P_\rho(x) \approx \frac{\sin \pi\rho}{\pi} \log \frac{1+x}{2}$ as $x \to -1$, thus $\Delta u_0(x) \geqslant 0$ for $r > r_0$ uniformly in θ_1. Whence, the function u_0 is subharmonic in $\mathbb{R}^3$ outside of a fixed ball $B(r_0)$ if r_0 is large enough. Moreover, if $\phi \notin \Theta_3(\rho)$, then limits (1.8) do not exist since u_0 is oscillating. On the other hand, for $\phi \in \Theta_3(\rho)$, limit (1.10) does exist since $P_\rho(\cos\phi) = 0$. □

Appendix

A. Mellin Transform of the Riesz Kernel

The associated Legendre functions of the first kind $P_\nu^\mu(z)$ are particular solutions of the Legendre differential equation [6, Chap. 3]

$$(1-z^2)y''(z) - 2zy' + \left[\nu(\nu+1) - \mu^2(1-z^2)^{-1}\right] y = 0,$$

where μ and ν are, in general, complex parameters; $P_\nu^\mu(\xi)$ stands for these functions on the cut $-1 < \xi < 1$ [6, Sect. 3.4]. In many instances the Legendre functions are a natural replacement of the trigonometric functions in many-dimensional problems, so that there is a continuous stream of research regarding the spherical functions. In particular, it may be of interest to represent certain analytic objects, like series and integrals as Legendre's functions; among the newest work see, e.g., [22]. The following statement is used in the proof of the main result of this paper.

Instead of the Weierstrass canonical kernel in (1.4), we consider a slightly more general case of the Riesz kernel

$$k_\lambda(t,\xi) = (1 + t^2 + 2t\xi)^{-\lambda}$$

with any λ such that $\Re\lambda > 0$. It is known that its Mellin transform can be represented through the associated Legendre's function,

$$\begin{aligned} &\int_0^\infty t^{\nu-\mu} \left(1 + t^2 + 2t\xi\right)^{\mu-1/2} dt \\ &= \tfrac{\Gamma(1-\mu)\Gamma(\nu-\mu+1)\Gamma(-\mu-\nu)}{2^\mu\Gamma(1-2\mu)} \left(1-\xi^2\right)^{\mu/2} P_\nu^\mu(\xi), \end{aligned} \tag{A.1}$$

where Γ is Euler's $\Gamma-$function, [7, Sect. 6.2, Eq. (22)]. The integral in (A.1) is convergent if

$$\Re\mu - \Re\nu < 1 \text{ and } \Re\mu + \Re\nu < 0.$$

It should be also mentioned that the kernel k_λ is a generating function of the Gegenbauer polynomials $G_j^\lambda(\xi)$, Cf. the case $\lambda = (n-2)/2$ in Section 1,

$$k_\lambda(t,\xi) = (1 + 2t\xi + t^2)^{-\lambda} = \sum_{j=0}^{\infty} (-t)^j G_j^\lambda(\xi).$$

We consider only the case $n \geqslant 3$, since for $n = 2$ the spherical functions are essentially the trigonometric functions, and the integral (A.1) for $n = 2$ is well known, it is equal to $\frac{\pi \cos \rho\theta}{\rho \sin \pi\rho}$. Thus, let

$$h(u) = h(\lambda, q, u, \xi) = -\left(1 + 2u\xi + u^2\right)^{-\lambda} + \sum_{j=0}^{q} (-u)^j G_j^\lambda(\xi)$$

$$= -k_\lambda(u, \xi) + \sum_{j=0}^{q} (-u)^j G_j^\lambda(\xi),$$

which has the same bound as (2.1),

$$|h(\lambda, q, u, \xi)| \leqslant C \min\{u^q; u^{q+1}\},\ 0 < u < \infty,$$

where a positive constant C does not depend on u and ξ.

Proposition A.1. — For any integer $q = 0, 1, 2, \ldots$, real ξ, $-1 < \xi < 1$, and complex λ and s such that

$$0 < \Re\lambda$$

and

$$-q - 1 < \Re s < -q,$$

the Mellin transformation of h is

$$\begin{aligned} M(h, s) &= \textstyle\int_0^\infty \left\{-\left(1 + u^2 + 2u\xi\right)^{-\lambda} + \sum_{j=0}^{q} (-u)^j G_j^\lambda(\xi)\right\} u^{s-1} du \\ &= -\tfrac{\sqrt{\pi}\Gamma(s)\Gamma(2\lambda - s)}{2^{\lambda - 1/2}\Gamma(\lambda)} \left(1 - \xi^2\right)^{\frac{1-2\lambda}{4}} P^{1/2-\lambda}_{s-\lambda-1/2}(\xi). \end{aligned} \tag{A.2}$$

Proof. — The Mellin transform of the kernel h,

$$M(h, s) = \int_0^\infty h(u) u^{s-1} du \tag{A.3}$$

is convergent for $-1 - q < \Re s < -q$. We integrate by parts the integral in (A.3) $q + 1$ times, so that the polynomial part of h vanishes, whence

$$\frac{\partial^{q+1}}{\partial u^{q+1}} k_\lambda(u, \xi) = -\frac{\partial^{q+1}}{\partial u^{q+1}} h(u)$$

and

$$M(h, s) = \frac{(-1)^q}{\prod_{k=0}^{q}(s + k)} \int_0^\infty u^{s+q} \frac{\partial^{q+1}}{\partial u^{q+1}} \left((1 + u^2 + 2u\xi)^{-\lambda}\right) du. \tag{A.4}$$

The latter integral is convergent for non-integer s such that

$$-1-q<\Re s<2\Re\lambda,$$

thus providing the analytic continuation of M(h,s) as a meromorphic function into this broader domain of the $s-$plane.

Now we consider the Mellin transform of the kernel k_λ,

$$M(k_\lambda,s)=\int_0^\infty(1+t^2+2t\xi)^{-\lambda}t^{s-1}dt,$$

which is convergent for

$$-q-1<\Re s<2\Re\lambda. \tag{A.5}$$

Integrating it by parts $q+1$ times, we get

$$M(k_\lambda,s)=\frac{(-1)^{q+1}}{\prod_{k=0}^{q}(s+k)}\int_0^\infty t^{s+q}\frac{\partial^{q+1}}{\partial t^{q+1}}\left((1+t^2+2t\xi)^{-\lambda}\right)dt. \tag{A.6}$$

Due to (A.5), all the integrated terms vanish and the integral is convergent in the wider region $-1-q<\Re s<2\Re\lambda$ excluding the poles at integer points.

By (A.1), we express $M(k_\lambda,s)$ through $P_\nu^\mu(\xi)$ with $\mu=\frac{1}{2}-\lambda$ and $\nu=s-\frac{1}{2}-\lambda$. Finally, combining (A.4) and (A.6) and using Legendre's formula for the $\Gamma-$function with double argument,

$$\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(1/2+z),$$

we get the result.

The Legendre functions $P_\nu^\mu(\xi)$ are entire functions in both μ and ν, thus all the equations are justified due to the principle of analytic continuation. □

Remark A.2. — If $q=0$, a simpler proof can be given. In this case, we can explicitly compute the derivative in (A.4) and split the integral into the two integrals of kind

$$\int_0^\infty u^\alpha(1+u^2+2u\xi)^\beta du$$

with two different α. Then we apply (A.1) to express each of them as $P_\nu^\mu(z)$ and use the recurrence formula [6, Sect. 3.8, Eq-n (18)]

$$(\nu-\mu+1)P_{\nu+1}^\mu(\cos\theta_1)-(\nu+\mu+1)\cos\theta_1 P_\nu^\mu(\cos\theta_1)$$
$$=\sin\theta_1 P_\nu^{\mu+1}(\cos\theta_1)$$

to arrive at (A.2) with $q = 0$. However, the explicit computation of the derivatives for bigger q becomes cumbersome.

In the case, we are interested in, $\lambda = \frac{n-2}{2}$ and $s = -\rho$, and formula (A.2) reads as follows.

Corollary A.3. — *For any integer $q = 0, 1, 2, \ldots$, real ξ, $-1 < \xi < 1$, and complex ρ such that*

$$q < \Re\rho < q + 1,$$

the Mellin transform of the function h_n is

$$M(h_n, \rho) = -\frac{\pi\sqrt{\pi}2^{(3-n)/2}\prod_{k=1}^{n-3}(\rho+k)(1-\xi^2)^{(3-n)/4}}{\sin\pi\rho\,\Gamma((n-2)/2)}P^{(3-n)/2}_{-\rho-(n-1)/2}(\xi),$$

where $\xi = \cos\theta_1$, or using the equation

$$\sqrt{\pi}(n-3)! = 2^{n-3}\Gamma((n-1)/2)\Gamma((n-2)/2),$$

which can be immediately proved by induction,

$$M(h_n, \rho)$$

$$= \frac{\pi 2^{(n-3)/2}\prod_{k=1}^{n-3}(\rho+k)\Gamma((n-1)/2)(1-\xi^2)^{(3-n)/4}}{(n-3)!\sin\pi\rho}P^{(3-n)/2}_{-\rho-(n-1)/2}(\xi).$$

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