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Topology of arrangements and position of singularities

Enrique Artal Bartolo[1]

Abstract. — This work contains an extended version of a course given in *Arrangements in Pyrénées. School on hyperplane arrangements and related topics* held at Pau (France) in June 2012. In the first part, we recall the computation of the fundamental group of the complement of a line arrangement. In the second part, we deal with characteristic varieties of line arrangements focusing on two aspects: the relationship with the position of the singular points (relative to projective curves of some prescribed degrees) and the notion of essential coordinate components.

Résumé. — Ce travail est un version étendue du cours donné en Juin 2012 à Pau dans le cadre de l'École *Arrangements in Pyrénées. School on hyperplane arrangements and related topics*. Dans la première partie, nous rappelons comment calculer le groupe fondamental du complément d'un arrangement de droites. La deuxième partie est consacrée aux variétés caractéristiques des arrangements de droites. Deux aspects sont étudiés : la relation avec la position des points singuliers (par rapport aux courbes projectives pour certains degrés fixés) et la notion de composantes coordonnées essentielles.

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Dedicado a mi hermana que nos dejó poco después de terminar el curso que da origen a estas notas. Será imposible olvidarte

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Contents

Introduction

The study of hyperplane arrangements has a lot of aspects: combinatorial, topological, geometrical, arithmetical. It depends strongly on the choice of the ground field. In this work, we will focus on the combinatorial aspects of topological properties of hyperplane arrangements; the ground field will be $\mathbb{C}$ though the main source of examples comes from the complexified real arrangements, i.e., arrangements such that all the hyperplanes have real equations. The main result in this relationship between topology and combinatorics is the fact that the combinatorial Orlik-Solomon algebra is isomorphic to the cohomology ring of a hyperplane complement.

The main invariant we are going to study is the fundamental group and this is why we will focus on the case of line arrangements. The fundamental group of the complement of a line arrangement can be computed from the classical Zariski-van Kampen method as it will be explained. It is possible to simplify the way of applying this method, when restricted to the case of line arrangements, as it was shown by Arvola. As a main difference from the general case of algebraic plane curves, the presentation obtained by these methods is minimal, i.e., minimal number of generators and relators (as it can be deduced from several sources [29, 11]). The structure of these fundamental groups is quite complicated and, in general, they are not combinatorial invariants, as shown by Rybnikov.

One main invariant that helps to understand the hidden properties of the fundamental group is the sequence of characteristic varieties. They were introduced by Libgober in the context of hypersurfaces complements; the works of Arapura, Budur, Dimca-Papadima-Suciu and Artal-Cogolludo-Matei (in a more general framework) provide a lot of geometrical properties. It is not yet known if these invariants are of combinatorial nature; it is even not know for some simplified invariants like the Alexander polynomial (also known as the characteristic polynomial of the Milnor fiber).

The core of the work is devoted to the study of characteristic varieties in order to isolate where combinatorics may fail to determine them. These characteristic varieties depend on the cohomology of the complement of the line arrangement twisted by characters of the fundamental group on $\mathbb{C}^*$. For the characters which do ramify along all the lines, a smart approach from Libgober (generalizing previous works originated by Zariski) shows that this homology depend on the position of singularities. These arguments rely on the study of the Pure Hodge Structure of some projective surfaces. This approach has a serious disadvantage: a priori, one must compute the twisted cohomology for an infinite number of characters in order to compute the characteristic varieties. We are again rescued by Libgober which constructs a finite polytope decomposition of a hypercube which completely determines the so-called non-coordinate components of the characteristic varieties. This approach solves theoretically the problem, though with a high computational complexity.

In order to fully understand characteristic varieties one needs to deal with the so-called essential coordinate components where the above approach does not give a complete answer. The deep reason is that for these components one must understand the Mixed Hodge Structure of some quasi-projective surfaces. The goal of our work is to develop a way to compute these coordinate components and to isolate the places where combinatorics may not be enough to compute them.

This last part contains most of the new material, while the previous ones collect in one block properties studied by many authors, though the presentation may differ from the original sources in order to provide a homogeneous structure.

The paper is organized as follows. In Section 1, definitions of hyperplane arrangements and combinatorics are provided. Section 2 deals with the computation of the fundamental group of the complement of a line arrangement, which will be the main invariant to be studied in the rest of the paper. The characteristic varieties, with their first properties, are introduced in Section 3, with the general lines for their computation. This computation is splitted in two parts. The first one (which completely determines the non-coordinate components) is in Section 4. The second part is in Section 5.

1. Definitions and settings

Definition 1.1. — *Let $\Bbbk$ be a field, $n \in \mathbb{N}$. A* hyperplane arrangement *is a finite collection of hyperplanes in one of the following cases:*

1. *Linear hyperplanes of a $\Bbbk$-vector space V of dimension n:* central arrangement*.*
2. *Affine hyperplanes of a $\Bbbk$-affine space E of dimension n:* affine arrangement*.*
3. *Projective hyperplanes of a $\Bbbk$-projective space $\mathbb{P}(V)$ of dimension n:* projective arrangement*.*

These three concepts are closely related. Let us identify V and E with $\Bbbk^n$. Note that central arrangements are a particular case of affine arrangements. If we consider the standard embedding $\Bbbk^n \hookrightarrow \mathbb{P}(\Bbbk^{n+1}) =: \mathbb{P}^n(\Bbbk)$, adding the hyperplane at infinity $H_\infty := \mathbb{P}^n(\Bbbk) \setminus \Bbbk^n$ to the collection, we construct a projective arrangement from an affine arrangement. Finally, a projective arrangement in $\mathbb{P}^n(\Bbbk)$ is essentially the same object as a central arrangement in $\Bbbk^{n+1}$.

Definition 1.2. — *Let $\mathcal{A}$ be a hyperplane arrangement. The* complement $M(\mathcal{A})$ *of the arrangement is the complement of $\bigcup \mathcal{A}$ in the ambient space.*

Remark 1.3. — Let $\mathcal{A}$ be a central arrangement; it is obvious that $M(\mathcal{A})$ coincides as both central and affine arrangement. If $\mathcal{A}$ is an affine arrangement and $\mathcal{A}_\infty$ is the projective arrangement obtained by adding H_∞, then $M(\mathcal{A}) = M(\mathcal{A}_\infty)$. Finally, if $\mathcal{A}$ is a non-empty projective arrangement and $\tilde{\mathcal{A}}$ is the corresponding central arrangement, then there is a natural identification $M(\tilde{\mathcal{A}}) \leftrightarrow M(\mathcal{A}) \times \Bbbk^*$.

The combinatorics of an arrangement $\mathcal{A}$ is the poset $S(\mathcal{A})$ of all the intersections of elements in $\mathcal{A}$, with respect to the reversed inclusion. The combinatorics encodes the properties of $\mathcal{A}$ not depending on the actual equations of the hyperplanes. One central problem in the theory of hyperplane arrangements is to detect which properties of the arrangement depend only on the combinatorics.

In these notes $\Bbbk = \mathbb{C}$ and the properties we are looking for are of topological type. Note that we may chose other fields with topological structure, specially $\mathbb{R}$ or the p-adic number field. We prefer the complex numbers since in this case the arrangement is a codimension-2 topological subspace of the ambient space and we may apply techniques of Geometric Topology.

Example 1.4. — Let $\mathcal{A}$ be a central arrangement in $\mathbb{C}^n$. Orlik and Solomon defined a graded algebra $A(\mathcal{A})$ which depends only on $S(\mathcal{A})$. They proved that $A(\mathcal{A})$ is isomorphic to $H^*(M(\mathcal{A}); \mathbb{Z})$ as a $\mathbb{Z}$-algebra [39, 38].

This result opened an intensive research in order to find which topological invariants are combinatorial. In 1994, G. Rybnikov found two arrangements with the same combinatorics and such that the fundamental groups of their complements are non-isomorphic. This result was finally published in [40], see also [7] for a detailed proof. There are other examples of arrangements with different topology and same combinatorics: in [6], two combinatorially-equivalent arrangements have a different homeomorphism type for $(\mathbb{P}^2(\mathbb{C}), \mathcal{A})$. Though $\pi_1(M(\mathcal{A}))$ is not a combinatorial invariant, it is one of the most important topological invariants of hyperplane arrangements. The Zariski-Lefschetz theory shows that for computing fundamental groups we can restrict our attention to the case of line arrangements.

Theorem 1.5 (Zariski-Lefschetz [45]). — *Let X be a quasi-projective smooth variety in $\mathbb{P}^N(\mathbb{C})$ of dimension n. Let H be a generic hyperplane. Then the morphism $\pi_j(X \cap H) \to \pi_j(X)$ induced by the injection is an isomorphism for $j < n-1$ and epimorphism for $j = n-1$.*

Corollary 1.6. — *The fundamental group of a hyperplane arrangement is also the fundamental group of a line arrangement.*

2. Zariski-van Kampen method and braid monodromy

This method was introduced by Zariski [43] and van Kampen [27] for any plane curve. The relationship with braids was noted by Chisini [14] and exploited by Moishezon [34]. For the case of line arrangements this method was used by Moishezon and Teicher [35]; a precise algorithm depending on the so-called *wiring diagram* was developed by Arvola [11]. For a survey on this method, with advanced examples in the case of line arrangements, see [15]

For technical reasons it is better to consider affine arrangements instead of projective arrangements. There are several ways to construct an affine arrangement from a projective one.

Let $\mathcal{A} := \{\bar{L}_0, \bar{L}_1, \dots, \bar{L}_n\}$ be a line arrangement in $\mathbb{P}^2$. We fix a line $\bar{L}_\infty$ as the line at infinity and consider $\mathbb{C}^2 = \mathbb{P}^2 \setminus \bar{L}_\infty$; we will denote $L_j := \bar{L}_j \cap \mathbb{C}^2$. There are several *natural* choices for $\bar{L}_\infty$.

(L_∞1) We may choose $\bar{L}_\infty = \bar{L}_0$; in this case we will denote $\mathcal{A}^0 := \{L_1, \dots, L_n\}$ the associated affine line arrangement. Note that $M(\mathcal{A}) = M(\mathcal{A}^0)$.

(L_∞2) Choose a generic $\bar{L}_\infty \pitchfork \bigcup \mathcal{A}$. In this case the associated line arrangement is denoted as $\mathcal{A}^\infty := \{L_0, L_1, \dots, L_n\}$. Of course, $M(\mathcal{A}) \neq M(\mathcal{A}^\infty)$ but:

- (a) The topological type of $M(\mathcal{A}^\infty)$ does not depend on the particular choice of $\bar{L}_\infty$ (as far as the genericity condition is preserved).
- (b) As it will be proved later, the group $\pi_1(M(\mathcal{A}^\infty))$ is a central extension of $\pi_1(M(\mathcal{A}))$ by $\mathbb{Z}$.

In this section we fix $\mathcal{A} := \{L_1, \dots, L_n\}$ an affine arrangement in $\mathbb{C}^2$. We consider the set of multiple points of the arrangement:

$$\mathcal{P} := \{P \in \mathbb{C}^2 \mid \exists i < j \text{ s.t. } P \in L_i \cap L_j\}.$$

For $P \in \mathcal{P}$, we denote $m_P := \#\{L \in \mathcal{A} \mid P \in L\}$. In order to apply the Zariski-van Kampen method we need to consider the projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x$. This projection depends on the choice of coordinates x, y.

Definition 2.1. — *The coordinates (and hence the projection) are said to be* generic *if no line L_j is vertical and if $(x, y_1), (x, y_2) \in \mathcal{P}$ then $y_1 = y_2$.*

Remark 2.2. — Later on we will also consider non-generic projections.

2.1. Fibered arrangements

Definition 2.3. — *Let $\mathcal{A} = \{L_1, \dots, L_n\}$ be an affine arrangement in generic coordinates and let $\mathcal{P} = \{(x_1, y_1), \dots, (x_r, y_r)\}$. The* fibered arrangement *associated to $\mathcal{A}$ is*

$$\mathcal{A}^\varphi := \mathcal{A} \cup \{V_{x_i} \mid i = 1, \dots, r\}, \quad V_t := \{x = t\}.$$

Proposition 2.4. — *Let $\mathcal{B} := \{x_1, \dots, x_r\}$. The restriction map $\pi_| : M(\mathcal{A}^\varphi) \to \mathbb{C} \setminus \mathcal{B}$ is a locally trivial fibration, the fiber is homeomorphic to $F := \mathbb{C} \setminus \{n \text{ points}\}$. In particular, the long exact homotopy sequence induces a short exact sequence*

$$1 = \pi_2(\mathbb{C} \setminus \mathcal{B}) \to \pi_1(F) \to \pi_1(M(\mathcal{A}^\varphi)) \to \pi_1(\mathbb{C} \setminus \mathcal{B}) \to 1. \tag{2.1}$$

Sketch of the proof. — From the choice of the vertical lines, all the fibers of $\pi_|$ are homeomorphic to $\mathbb{C} \setminus \{n \text{ points}\}$. In order to deal with proper maps, we embed $\mathbb{C}^k \hookrightarrow \mathbb{P}^k$, $k = 1, 2$, identifying $\mathbb{P}^1$ and $\mathbb{C} \cup \{\infty\}$.

Let $\Sigma_1 \to \mathbb{P}^2$ be the blowing-up of the point at infinity of the vertical lines, E being the exceptional divisor. We also consider $\tilde{\mathcal{B}} = \mathcal{B} \cup \{\infty\}$.

The projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ extends to $\tilde{\pi} : \Sigma_1 \to \mathbb{P}^1$. It is a proper submersion and hence a locally trivial fibration. Let us denote by $\tilde{L}_i$, $\tilde{V}_{x_i}$ and $\tilde{L}_\infty$ the strict transforms of $\bar{L}_i$, $\bar{V}_{x_i}$ and $\bar{L}_\infty$. Note that E and $\bar{L}_i$ are sections of $\tilde{\pi}$. Let us denote $U := \mathbb{P}^1 \setminus \tilde{\mathcal{B}} = \mathbb{C} \setminus \mathcal{B}$, $V := \tilde{\pi}^{-1}(U) \subset \mathbb{C}^2$, $\tilde{L}_{i,U} := \tilde{L}_i \cap V$. The map

$$\tilde{\pi}_| : \left(V, E \cup \bigcup_{i=1}^{n} \tilde{L}_{i,U} \right) \to U$$

is a locally trivial fibration of pairs, since we have taken out the intersection points. Since $M(\mathcal{A}^\varphi) = V \setminus \left(E \cup \bigcup_{i=1}^n \tilde{L}_{i,U} \right)$ the statement for the locally trivial fibration follows. For the exact sequence statement, it follows from $\pi_2(\mathbb{C} \setminus \mathcal{B}) = 0$. ☐

Definition 2.5. — *Let X be a quasi-projective smooth variety, and let $A_1, \dots, A_r \subset X$ be irreducible hypersurfaces. Let $Y := X \setminus \bigcup_{j=1}^{r} A_i$ and $p \in Y$. A* meridian *of A_i in $\pi_1(Y; p)$ is obtained as follows:*

- *Fix a smooth point p_i of $\bigcup_{j=1}^{r} A_j$ such that $p_i \in A_i$.*
- *Fix a* small *closed disk $\mathbb{D}_i$ centered at p_i transversal to A_i such that*

$$\left(\bigcup_{j=1}^{r} A_j \right) \cap \mathbb{D}_i = \{p_i\}$$

and pick a point $q_i \in \partial \mathbb{D}_i$.

- *Let δ_i be the loop based at q_i which runs along $\partial\mathbb{D}_i$ counterclockwise.*
- *Let α_i be a path in Y from p to q_i.*

Then, $\alpha_i \cdot \delta_i \cdot \alpha_i^{-1}$ is such a meridian. The conjugacy class of such a meridian does not depend on the above choices.

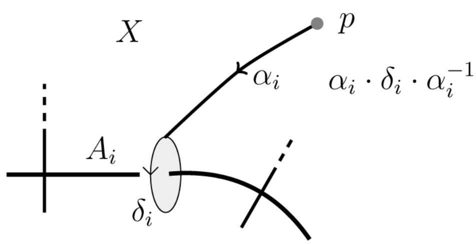

Figure 1. — Meridian

Proposition 2.4 allows to produce a finite presentation of the group $\pi_1(M(\mathcal{A}^\varphi))$; we need to fix bases of the free groups in the ends of the short exact sequence (2.1).

Definition 2.6. — *A* geometric basis *of the free group $\pi_1(\mathbb{C}\setminus\{t_1,\ldots,t_r\}; t_0)$ is a basis of meridians $\mu_1,\ldots,\mu_r$ (μ_i meridian of t_i) such that $(\mu_r \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian of ∞.*

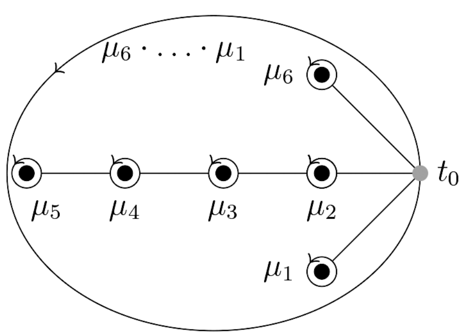

Figure 2. — Geometric basis for $r = 6$

It is useful to have a compact model of $M(\mathcal{A}^\varphi)$. Let $t_x \gg 0$ such that $\mathcal{B} \subset \mathring{\mathbb{D}}_{t_x}$. Consider also $t_y \gg 0$ such that $|y_j| \ll t_y$ and

$$\bigcup \mathcal{A} \cap (\mathbb{D}_{t_x} \times \mathbb{D}_{t_y}) \subset \partial\mathbb{D}_{t_x} \times \mathring{\mathbb{D}}_{t_y}.$$

The inclusion $(\mathbb{D}_{t_x} \times \mathbb{D}_{t_y}) \setminus \bigcup \mathcal{A}^\varphi \hookrightarrow M(\mathcal{A}^\varphi)$ is a homotopy equivalence. Let $p := (t_x, t_y)$ and denote $F := V_{t_x} \setminus \bigcup \mathcal{A}$.

Let us fix a geometric basis $\mu_1, \ldots, \mu_n$ of the free group $\pi_1(F \cap \mathbb{D}_y; p) = \pi_1(F; p)$ and a geometric basis $\alpha_1, \ldots, \alpha_r$ of $\pi_1(\mathbb{D}_x \setminus \mathcal{B}; t_x) = \pi_1(\mathbb{C} \setminus \mathcal{B}; t_x)$. Let us lift $\alpha_1, \ldots, \alpha_r$ to $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ in the line $y = t_y$.

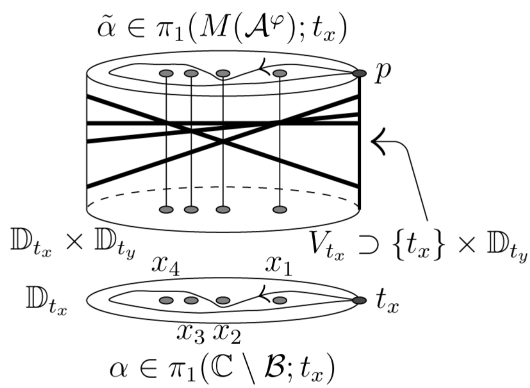

Figure 3. — Polydisk model

Lemma 2.7. — *The elements $\mu_1, \ldots, \mu_n$ and $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ generate $\pi_1(M(\mathcal{A}^\varphi); p)$.*

Proof. — It is a direct consequence of (2.1). □

We state a general result (due to Fujita [22], see also Hamm-Lê [24]) which will be necessary to compute $\pi_1(M(\mathcal{A}))$ from $\pi_1(M(\mathcal{A}^\varphi))$.

Lemma 2.8. — *Let us consider the notation of* Proposition 2.4. *Consider the spaces $Y := X \setminus \bigcup_{j=1}^r A_j$, $Z := X \setminus \bigcup_{j=s+1}^r A_j$, $1 \leqslant s \leqslant r$. The inclusion $Y \hookrightarrow Z$ induces an epimorphism $\pi_1(Y) \to \pi_1(Z)$. Its kernel is generated by the meridians of $A_1, \ldots, A_s$.*

Idea of the Proof. — First, we recall that $\pi_1 = \pi_1^{\mathcal{C}^\infty}$. The surjectivity follows from transversality of the mappings $\mathbb{S}^1 \to Z$ with respect to A_j, $1 \leqslant j \leqslant s$, which implies that we can deform these maps such that the image is disjoint from A_j, $1 \leqslant j \leqslant s$.

In order to describe the kernel the transversality of the mappings $\mathbb{D}^2 \to Z$ with respect to A_j, $1 \leqslant j \leqslant s$ is used. An element α of the kernel is the boundary of a map from the disk to Z. If we move this map to be transversal to A_j, $1 \leqslant j \leqslant s$, the intersection with the image is reduced to a finite number of points. By the definition of meridians, α is a product of meridians and inverse of meridians (depending on the orientation). $\square$

Proposition 2.9. — *Let $\mathcal{A}$ be an affine line arrangement.*

(1) *$\pi_1(M(\mathcal{A}); p)$ is generated by $\mu_1, \dots, \mu_n$.*

(2) *$\mu_\infty := (\mu_n \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian of $\bar{L}_\infty$ in $\pi_1(M(\mathcal{A}); p)$.*

(3) *If $\bar{L}_\infty \pitchfork \bigcup \mathcal{A}$ then μ_∞ is central in $\pi_1(M(\mathcal{A}); p)$.*

Remark 2.10. — Proposition 2.9(1) *is a particular case of the surjectivity statement of Proposition 1.5. Proposition 2.9*(3) *is the centrality statement of* $(L_\infty 2)$(b).

Proof. — The statement (1) is a direct consequence of Lemmas 2.7 and 2.8. The statement (2) comes from the genericity condition and the fact that the projection point $[0:1:0] \notin \bigcup \mathcal{A}$, then the result follows.

Let us prove the statement (3). Let $E := \mathbb{C}^2 \setminus (\mathring{\mathbb{D}}_{t_x} \times \mathring{\mathbb{D}}_{t_y})$ and set $\check{E} := E \setminus \bigcup \mathcal{A}$. By Lemma 2.8 the map $\pi_1(\check{E}; p) \twoheadrightarrow \pi_1(M(\mathcal{A}); p)$. The space $\check{E}$ is homeomorphic to $(\mathbb{C} \setminus \{n-1 \text{ points}\}) \times \mathbb{D}^*$; the meridian μ_∞ corresponds to the boundary of the factor $\mathbb{D}^*$. $\square$

Remark 2.11. — In orden to describe the group $\pi_1(M(\mathcal{A}^\varphi))$, and hence $\pi_1(M(\mathcal{A}))$, we only need the conjugation action of $\tilde{\alpha}_j$ on $\pi_1(F; p) \subset \pi_1(M(\mathcal{A}); p)$.

2.2. Braid action on free groups

Consider $\alpha \in \pi_1(\mathbb{C} \setminus \mathcal{B}; t_x) \equiv \pi_1(\mathbb{D}_{t_x} \setminus \mathcal{B}; t_x)$ represented by a closed loop $\alpha : [0,1] \to \mathbb{D}_{t_x} \setminus \mathcal{B}$ as in Figure 3. We represent $(\alpha([0,1]) \times \mathbb{D}_{t_y}) \setminus \bigcup \mathcal{A}$ in a cylinder where the top and bottom bases are identified, see Figure 4. The lift $\tilde{\alpha}$ is represented as an upward vertical path in Figure 4.

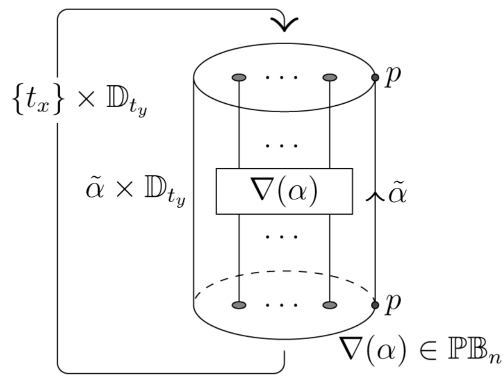

Figure 4. — Action of the braid α

Let $\beta \in \pi_1(F;p)$ (F is represented by the identified bases of the cylinder in Figure 4). Let us consider the loop $\tilde{\alpha}^{-1} \cdot \beta \cdot \tilde{\alpha}$. This loop is homotopic to a loop in the top basis, i.e. it may be represented as an element in $\pi_1(F;p)$. This new loop depends on the trace of $\mathcal{A}$. A more precise description can be done using the homotopy equivalence between the complement of $\mathcal{A}$ in the cylinder and the punctured bases. In order to describe this conjugation we need to introduce the braid groups and their action on free groups.

Let $X_n := \{\mathbf{x} \in \mathbb{C}^n \mid x_i \neq x_j,\ 1 \leqslant i < j \leqslant n\}$ (the complement of the braid arrangement). The symmetric group Σ_n acts freely by entry-permutation on X_n. The quotient X_n/Σ_n is naturally identified with

$$Y_n := \{f(t) \in \mathbb{C}[t] \mid f \text{ monic without multiple roots, } \deg f = n\}.$$

Definition 2.12. — *The fundamental group of Y_n is the* braid group $\mathbb{B}_n$ *in n strands, while the fundamental group of X_n is the* pure braid group $\mathbb{PB}_n$ *in n strands.*

The group $\mathbb{B}_n$ admits the well-known Artin presentation [10]

$$\mathbb{B}_n := \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \underset{1<i+1<j<n}{[\sigma_i, \sigma_j] = 1}, \quad \underset{1\leqslant i<n-1}{\sigma_i \cdot \sigma_{i+1} \cdot \sigma_i = \sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1}} \right\rangle.$$

It is identified with the homotopy classes of n (non-intersecting) paths in $\mathbb{C}$ such that the sets of starting and ending points coincide. The pure braid group consists of classes where all paths are loops.

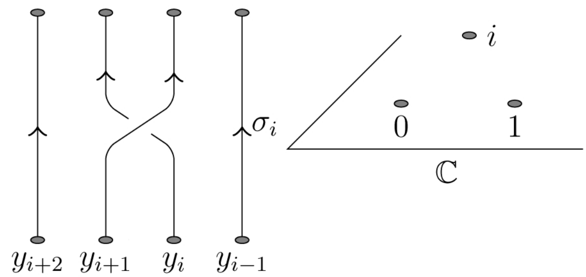

Figure 5

For the sake of simplicity let us fix the point $\mathbf{y} := (y_1, \dots, y_n)$, $y_j := -j$, as base point for the fundamental group. With this identification the generator σ_i is represented as in Figure 5. The group $\mathbb{B}_n$ acts naturally on the free group $\mathbb{F}_n$ generated by $\mu_1, \dots, \mu_n$. It is defined as follows:

$$\begin{array}{rcl} \mathbb{F}_n \times \mathbb{B}_n & \longrightarrow & \mathbb{F}_n \\ (\mu, \tau) & \longmapsto & \mu^\tau; \end{array}$$

where

$$\mu_i^{\sigma_j} := \begin{cases} \mu_{i+1} & \text{if } i = j \\ \mu_{i+1}\mu_i\mu_{i+1}^{-1} =: \mu_{i+1} * \mu_i & \text{if } i = j+1 \\ \mu_i & \text{if } i \neq j, j+1. \end{cases} \tag{2.2}$$

This action can be understood geometrically by identifying $\mathbb{F}_n$ with the fundamental group $\pi_1(\mathbb{C} \setminus \{y_1, \dots, y_n\}; y_0)$, $y_0 := -(n+1)$. This non-canonical identification is determined by the choice of a geometric basis $\mu_1, \dots, \mu_n$ obtained as in Figure 6 using a polygonal path joining $y_0, y_1, \dots, y_n$. Note that the generators of the Artin presentation move along small neighborhoods of the segments joining y_j and y_{j+1}, see Figure 7. If we change the base point, say $\tilde{y}_1, \dots, \tilde{y}_n$, for the braids, we can define an Artin generator system using a polygonal path as in Figure 6; moreover, if we construct a geometric basis of $\pi_1(\mathbb{C} \setminus \{\tilde{y}_1, \dots, \tilde{y}_n\}; \tilde{y}_0)$ using the same polygonal path, then the braid action on the free groups looks again as in (2.2).

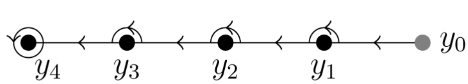

Figure 6

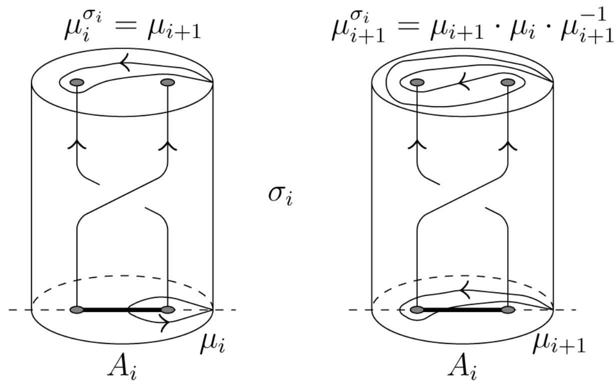

Figure 7

The trace of $\mathcal{A}$ in this cylinder of Figure 4 defines a braid $\nabla(\alpha)$ based at $\mathbf{y}^0 := \{y_1^0, \dots, y_n^0\}$, where $L_j \cap V_{t_x} = \{(t_x, y_j^0)\}$. For a suitable choice of a polygonal path, we can identify $\mathbb{B}_n$ with $\pi_1(Y_n; \mathbf{y}^0)$ and $\pi_1(F; p)$ with $\mathbb{F}_n$.

Under these identifications, a morphism $\nabla : \pi_1(\mathbb{C} \setminus \mathcal{B}; t_x) \to \mathbb{PB}_n \subset \mathbb{B}_n$ is defined where $\mathbb{B}_n$ is the braid group in n strands while $\mathbb{PB}_n$ is the pure braid group. With these arguments we can finally state the Zariski-van Kampen Theorem.

Theorem 2.13. — *The groups $\pi_1(M(\mathcal{A}^\varphi); p)$ and $\pi_1(M(\mathcal{A}); p)$ admit the following finite presentations:*

1. $\pi_1(M(\mathcal{A}^\varphi); p) = \langle \mu_1, \dots, \mu_n, \tilde{\alpha}_1, \dots, \tilde{\alpha}_r \mid \mu_i^{\tilde{\alpha}_j} = \mu_i^{\nabla(\alpha_j)} \rangle$.
2. $\pi_1(M(\mathcal{A}); p) = \langle \mu_1, \dots, \mu_n \mid \mu_i = \mu_i^{\nabla(\alpha_j)} \rangle$.

Example 2.14. — Let $\mathcal{A}$ be an affine arrangement of n lines through $(0,0) \in \mathbb{C}^2$. In this case $\mathcal{B} = \{0\}$; hence $\pi_1(\mathbb{C} \setminus \mathcal{B}) = \langle \alpha \mid - \rangle$ where α is a meridian of 0. It is easily seen that $\nabla(\alpha) = \Delta_n^2$; this element is known as the *full-twist* and is a generator of the center of $\mathbb{B}_n$. A straightforward computation yields

$$\mu_i^{\Delta_n^2} = (\mu_n \cdot \ldots \cdot \mu_1) \cdot \mu_i \cdot (\mu_n \cdot \ldots \cdot \mu_1)^{-1}.$$

Hence $\pi_1(M(\mathcal{A})) = \langle \mu_1, \dots, \mu_n \mid [\mu_1, \dots, \mu_n] = 1 \rangle$, where $[\mu_1, \dots, \mu_n] = 1$ is the set of relations:

$$[\mu_n \cdot \ldots \cdot \mu_1, \mu_i] = 1, \quad 1 \leqslant i < n.$$

Note that we *forget* the relation for $i = n$ since it is a consequence of the previous relations.

Remark 2.15. — For any arrangement $\mathcal{A}$, the behavior of Example 2.14 is the local behavior $\forall P \in \mathcal{P}$.

2.3. Puiseux braid monodromy

The presentation in Theorem 2.13(2) is correct but, in general, it contains more relations than necessary. The purpose of this subsection is to reduce the presentation to a minimal one.

Proposition 2.16. — *For $j \in \{1, \ldots, r\}$ the braid $\nabla(\alpha_j)$ is a product $\tau_j^{-1} \cdot \Delta^2_{a_j,b_j} \cdot \tau_j$, where $1 \leqslant a_j < b_j \leqslant n$ and $\Delta^2_{a_j,b_j}$ is the full-twist involving the $b_j - a_j + 1$ strands from a_j to b_j.*

Proof. — The meridian α_j is a product of $\beta_j \cdot \delta_j \cdot \beta_j^{-1}$ where δ_j is a closed loop based at x'_j, close to x_j. Hence the braid $\nabla(\alpha_j)$ can be decomposed as a product $\tilde{\tau}_j \cdot \Delta_j \cdot \tilde{\tau}_j^{-1}$ where Δ_j is a closed braid based at $(y'_1, \ldots, y'_n)$ which corresponds to the intersections of $L_1, \ldots, L_n$ with $V_{x'_j}$: $L_j \cap V_{x'_j} = \{(x'_j, y'_{\sigma(j)})\}$ for some permutation $\sigma \in \Sigma_n$. Note that the braid $\nabla(\alpha_j)$ is a braid based at $(y^0_1, \ldots, y^0_n)$ where $L_j \cap V_{t_x} = \{(t_x, y^0_j)\}$.

We choose a polygonal path in $V_{x'_j}$ such that $\{y'_{a_j}, \ldots, y'_{b_j}\}$ are points close to the unique multiple point in V_{x_j}, see Figure 8 (where the polygonal path is chosen in $\mathbb{R}$). By means of this polygonal path the open braid $\tilde{\tau}_j$ becomes $\tau_j \in \mathbb{B}_n$ and Δ_j becomes $\Delta^2_{a_j,b_j}$ as in Example 2.14. $\square$

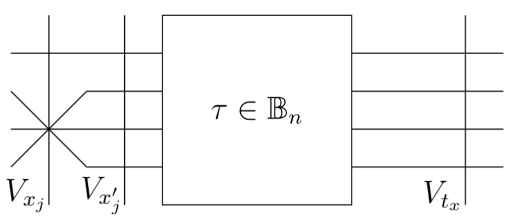

Figure 8

Proposition 2.17. — *Fix $j \in \{1, \ldots, r\}$. Let $\mu_{i,j} := \mu_i^{\tau_j}$. Then, the set of relations $\mu_i = \mu_i^{\nabla(\alpha_j)}$, $1 \leqslant i < n$, can be replaced by $[\mu_{1,j}, \ldots, \mu_{n,j}] = 1$.*

Proof. — Let us fix $j \in \{i, \dots, r\}$. The set of relations $\mu_i = \mu_i^{\nabla(\alpha_j)}$, $i = 1, \dots, n$ is equivalent to the set of relations $\mu = \mu^{\nabla(\alpha_j)}$ for all $\mu \in \pi_1(F)$. This is true since $\mu_1, \dots, \mu_r$ is a basis of $\pi_1(F)$.

Since the action of τ_j defines an automorphism of $\pi_1(F)$, it is also equivalent to the set of relations $\mu^{\tau_j} = (\mu^{\Delta^2_{a_j,b_j}})^{\tau_j}$ for all $\mu \in \pi_1(F)$. Then, it is also equivalent to $\mu_i^{\tau_j} = (\mu_i^{\Delta^2_{a_j,b_j}})^{\tau_j}, a_j \leqslant i < b_j$, since for the other terms the relation is trivial. This set of relations is exactly $[\mu_{1,j}, \dots, \mu_{n,j}] = 1$.

Note that geometrically $\mu_{i,j}$ corresponds to the geometric basis of $\pi_1(F)$ associated with the polygonal path in $V_{x'_j}$ of the proof of Proposition 2.16. □

Example 2.18. — In Figure 9, we see how the basis of the fiber changes before and after a point of multiplicity 5, when we consider geometric bases associated to polygonal paths in $\mathbb{R}$.

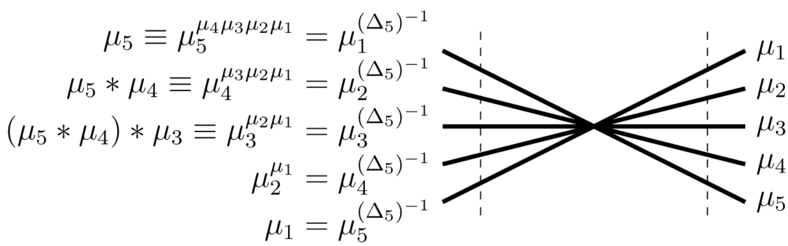

Figure 9

2.4. Complexified real arrangements

An arrangement $\mathcal{A}$ is said to be a *complexified real arrangement* if all its lines are defined by linear equations. For these arrangements there is a natural way to define the polygonal paths: decreasing segments in the real line.

Example 2.19. — For the complexified real arrangement of Figure 10, we can construct the braid monodromy from the braids of the arcs defining the geometric basis. The braids corresponding to the segments between two non-transversal vertical lines are trivial while the braids corresponding to the semi-circles avoiding these lines will be of the form Δ_{a_j,b_j} where a_j, b_j depend on the *height* of the singular point.

Its braid monodromy is defined by

$$\nabla(\alpha_1) = \Delta_{1,3}^2 \quad (2.3)$$
$$\nabla(\alpha_2) = \Delta_{1,3} * \sigma_3^2 \quad (2.4)$$
$$\nabla(\alpha_3) = (\Delta_{1,3} \cdot \sigma_3) * \sigma_2^2 \quad (2.5)$$
$$\nabla(\alpha_4) = (\Delta_{1,3} \cdot \sigma_3 \cdot \sigma_2) * \sigma_1^2 \quad (2.6)$$

The computation is made from the real picture. How this affects the fundamental group is shown in Figure 11.

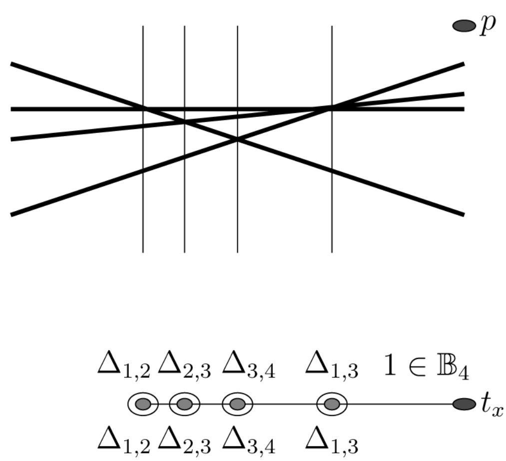

Figure 10

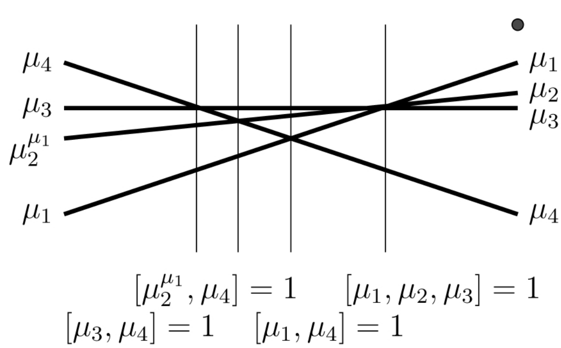

Figure 11

The following result is well known and a proof is derived from the above techniques.

Theorem 2.20. — *Let* $(\mathbb{P}^2, \mathcal{A})$ *be a generic arrangement (all points in* $\mathcal{P}$ *are of multiplicity* 2*). Then,* $\pi_1(M(\mathcal{A}))$ *is abelian.*

Proof. — Choose $\mathcal{A}$ with real equations and $\bar{L}_\infty \in \mathcal{A}$. From the real picture, all commutators of generators appear. Since any generic arrangement is isotopic to a real one, the result follows.

There is an alternative proof which works directly for any generic arrangement $\mathcal{A}$, since it can be obtained as a generic plane section of the coordinate arrangement in $\mathbb{P}^n$ whose complement is homeomorphic to $(\mathbb{C}^*)^n$. This is particularly interesting; the homology of the fundamental group is recovered as the homology of the $K(\pi,1)$-space $(\mathbb{C}^*)^n$. □

2.5. Wiring diagram

Arvola [11] gave the following procedure to compute $\pi_1(M(\mathcal{A}))$ for an affine arrangement which generalizes the way of computing the fundamental group for complexified real arrangements illustrated in Example 2.19.

Choose a normally embedded simple piecewise $\mathcal{C}^\infty$ arc $\Gamma : \mathbb{R} \to \mathbb{C}$ (identified with its image) such that $\{t_x\} \cup \mathcal{B} \subset \Gamma$ (t_x is the image of a big enough real number) and such that no vertex of Γ is in $\mathcal{B}$.

Definition 2.21. — *The* wiring space *is the pair* $(\Gamma \times \mathbb{C}, \bigcup \mathcal{A} \cap (\Gamma \times \mathbb{C}))$.

The wiring space contains essentially all the topological information of the pair $(\mathbb{C}^2, \bigcup \mathcal{A})$. More precisely, if

$$\tilde{\Gamma} := \Gamma \cup \bigcup_{x_i \in \mathcal{B}} \mathbb{D}_\varepsilon(x_i), \quad 0 < \varepsilon \ll 1,$$

then $(\mathbb{C}^2, \bigcup \mathcal{A})$ has the same homotopy type as $(\tilde{\Gamma} \times \mathbb{C}, \bigcup \mathcal{A} \cap (\tilde{\Gamma} \times \mathbb{C}))$.

Note that $\bigcup \mathcal{A} \cap (\Gamma \times \mathbb{C})$ is a union of *real lines*. Under genericity conditions we may choose a projection $\pi_\Gamma : \mathbb{C} \to \mathbb{R}$ such that for $\Pi_\Gamma := (1_\Gamma, \pi_\Gamma) : \Gamma \times \mathbb{C} \to \Gamma \times \mathbb{R} \equiv \mathbb{R}^2$ we have that $\Pi_\Gamma(\bigcup \mathcal{A} \cap (\Gamma \times \mathbb{C}))$ is a union of *n lines* in $\mathbb{R}^2$ with two types of *crossing points*:

- The image of $\mathcal{P}$: *real crossings.*
- Some transversal double points called *virtual crossings*. As in knot theory we draw continuously the upward branch.

Definition 2.22. — *The* wiring diagram *is the pair* $(\mathbb{R}^2, \Pi_\Gamma(\bigcup \mathcal{A} \cap (\Gamma \times \mathbb{C})))$ *with the information on virtual crossings.*

Example 2.23. — The real picture of a complexified real arrangement is a wiring diagram with no virtual crossing.

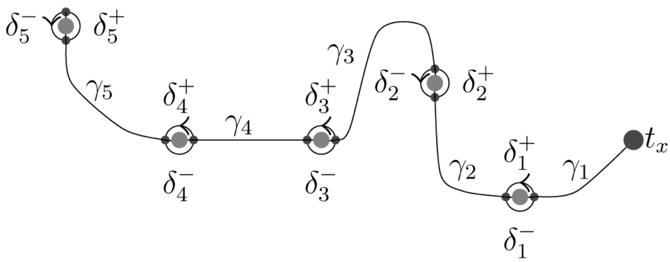

Figure 12. — Γ and associated paths

From $\tilde{\Gamma}$ we recover r arcs $\{\gamma_j\}_{j=1}^r$ (in Γ) and $2r$ cercle arcs $\{\delta_j^\pm\}_{j=1}^r$. From the wiring diagram we associate a braid $\eta_j \in \mathbb{B}_n$ to γ_j (coming from the virtual crossings); again from the diagram we associate the braid Δ_{a_j,b_j} to each $\delta_j^\pm$ (where a_j, b_j depends on the position of the corresponding multiple point). These braids determine the braid monodromy as in Example 2.19.

2.6. Generic and non-generic braid monodromy

We switch now to projective arrangements. There are several ways to associate braid monodromies for such an arrangement.

Definition 2.24. — *Let* $\mathcal{A} := \{\bar{L}_1, \dots, \bar{L}_n\}$ *be a projective arrangement. The* generic braid monodromy *of* $\mathcal{A}$ *is the braid monodromy of* $\mathcal{A}_\infty$*. Such a monodromy is represented by an element* $\nabla(\mathcal{A}) \in (\mathbb{B}_n)^r$, $r := \#\mathcal{P}_\infty$.

Proposition 2.25. — *The fundamental group* $\pi_1(M(\mathcal{A}))$ *is obtained as the quotient of* $\pi_1(M(\mathcal{A}_\infty))$ *by the normal subgroup generated by* $\mu_n \cdot \ldots \cdot \mu_1$.

Proof. — It is a direct consequence of Lemma 2.8 and Proposition 2.9(2). □

More properties can be deduced from a generic braid monodromy.

Proposition 2.26 ([29],[13]). — *The braid monodromy* $\nabla(\mathcal{A})$ *determines the homotopy type of* $M(\mathcal{A}_\infty)$*. Moreover, it determines the topological type of the embedding* $\bigcup \mathcal{A} \subset \mathbb{C}^2$.

Remark 2.27. — In [29], Libgober proves the homotopy part of this proposition using Puiseux braid monodromy.

A braid monodromy $\nabla(\mathcal{A})$ is not uniquely determined. Two operations yield new representations of the braid monodromy of a line arrangement. If simultaneous conjugation by an element $\tau \in \mathbb{B}_n$ is applied, then $\nabla(\mathcal{A})^\tau$ also represents the braid monodromy of the arrangement. The same happens when a Hurwitz action is applied. The Hurwitz action of the i^{th}-Artin generator of $\mathbb{B}_r$ is defined as:

$$(\tau_1, \ldots, \tau_r) \mapsto (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \tau_{i+1} * \tau_i, \tau_{i+2}, \ldots, \tau_r).$$

Moreover, we can obtain the elements of $(\mathbb{PB}_n)^r$ representing the braid monodromy of $\mathcal{A}$ is the orbit of $\nabla(\mathcal{A})$ with respect to the direct product of these commuting actions. See [4] for more details.

Remark 2.28. — A generic braid monodromy $\nabla(\mathcal{A})$ is also useful to compute invariants via braid representations as it was proved by A. Libgober [30].

Sometimes it is either easier or more interesting to compute non-generic braid monodromies. A braid monodromy can be non-generic for various reasons:

(NG1) If $\bar{L}_\infty \not\pitchfork \mathcal{A}$: $(\tau_1, \ldots, \tau_r) \to (\tau_1, \ldots, \tau_r, (\tau_r \cdot \ldots \cdot \tau_1)^{-1}\Delta_n^2)$.

(NG2) If several multiple points are on the same vertical line then decompose the corresponding braids in pairwise commuting braids.

(NG3) If there are some vertical lines in $\mathcal{A}_\infty$, the first step is to compute the braid monodromy of the arrangement without vertical lines without changing the projection. How to obtain a generic braid monodromy is explained in [9]. A generic braid monodromy is easily obtained from a wiring diagram. In the first step we add the vertical lines to the original wiring diagram. In the second step the wiring diagram is turned slightly. Finally a small deformation is performed *near infinity* it order to make non parallel the vertical lines.

Example 2.29. — We illustrate this process with an example in Figure 13.

- *Consider a projective arrangement $\mathcal{A}$, see Figure 13(a).*
- *Take the affine arrangement $\mathcal{A}_0$ and the projection from P (vertical direction), see Figure 13(b).*

- *Change the line at infinity: instead of $\bar{L}_0$, choose a generic vertical line $\bar{L}$ of the pencil of lines through P, see Figure 13(c).*
- *Choose another projection point $Q \in \bar{L}$, see Figure 13(d)*
- *Choose as line at infinity another line through Q.*

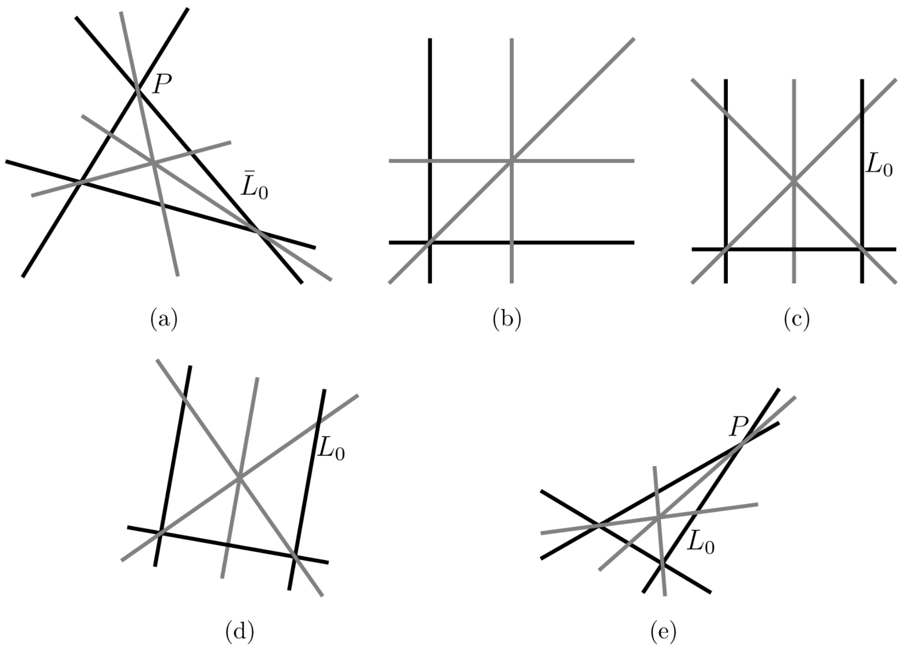

Figure 13

3. Characteristic varieties and twisted cohomology

Let us consider $\mathcal{A} = \{\bar{L}_0, \bar{L}_1, \ldots, \bar{L}_n\}$ a line arrangement in $\mathbb{P}^2$. In this section we will denote $\mathcal{P}_{\mathcal{A}} := \{\bar{L}_i \cap \bar{L}_j \mid 0 \leqslant i < j \leqslant n\}$. We use $\mathcal{A}_0$ in order to compute a minimal presentation of $\pi_1(M(\mathcal{A}))$. Though such a computation is not hard, using either the braid monodromy or the wiring diagram it is, in general, difficult to get properties from a presentation of $\pi_1(M(\mathcal{A}))$.

Using Alexander duality we can see that $H_1(M(\mathcal{A}); \mathbb{Z}) \cong \mathbb{Z}^n$; alternatively, abelianization and Theorem 2.13 can be used.

From now on, we use multiplicative notation for this homology group. A free generator system is given by $t_1, \ldots, t_n$, $t_i \equiv \mu_i \bmod \pi_1(M(\mathcal{A}))'$. To keep the symmetry from the elements of the line arrangement, recall that $(\mu_n \cdot \ldots \cdot \mu_1)^{-1}$ is a meridian of $\bar{L}_0$. Then $t_0 := (t_1 \cdot \ldots \cdot t_n)^{-1}$ is the homology class of a meridian of $\bar{L}_0$.

The abelianization map $ab : \pi_1(M(\mathcal{A}); p) \to H_1(M(\mathcal{A}); \mathbb{Z}) \cong \mathbb{Z}^n$ defines a covering $\rho : \tilde{M}(\mathcal{A}) \to M(\mathcal{A})$ which is called the *universal abelian covering*. Since the deck automorphism group of ρ is canonically identified with $H_1(M(\mathcal{A}); \mathbb{Z})$, we denote $t_1, \ldots, t_n : \tilde{M}(\mathcal{A}) \to \tilde{M}(\mathcal{A})$ the generators of this deck automorphism group (for those elements canonically related with their homonyms).

The homology and cohomology groups $H_1(\tilde{M}(\mathcal{A}); \mathbb{C})$ and $H^1(\tilde{M}(\mathcal{A}); \mathbb{C})$ are canonically $\Lambda_{\mathbb{C}}$-modules, where $\Lambda_{\mathbb{C}} := \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

In order to compute the homology and the cohomology of $\tilde{M}$ we look for a CW-complex having the homotopy type of $\tilde{M}$. In order to find it, let us recall that the space $M(\mathcal{A})$ has the homotopy type of a finite *CW*-complex $K(\mathcal{A})$ of dimension 2. Let us consider $C_*(\mathcal{A}) := C(K(\mathcal{A}); \mathbb{C})$ its chain complex, and consider also $C^*(\mathcal{A})$ the dual cochain complex.

Lifting the cells of $K(\mathcal{A})$, we obtain a *CW*-complex $\tilde{K}(\mathcal{A})$ having the same homotopy type as $\tilde{M}(\mathcal{A})$. We also consider the chain complex $\tilde{C}_*(\mathcal{A}) := C(\tilde{K}(\mathcal{A}); \mathbb{C})$ and its dual cochain complex $\tilde{C}^*(\mathcal{A})$. By construction, $\tilde{C}_*(\mathcal{A})$ and $\tilde{C}^*(\mathcal{A})$ are free $\Lambda_{\mathbb{C}}$-modules. Moreover, $\dim_{\mathbb{C}} C_*(\mathcal{A}) = \operatorname{rank}_{\Lambda_{\mathbb{C}}} \tilde{C}_*(\mathcal{A})$, respecting the graduation.

With these complexes it is possible to compute the groups $H_1(\tilde{M}(\mathcal{A}); \mathbb{C})$ and $H^1(\tilde{M}(\mathcal{A}); \mathbb{C})$ as $\Lambda_{\mathbb{C}}$-modules. These are more tractable invariants of the fundamental group but it is better to look for more effective invariants.

3.1. Twisted cohomology

Following the work of Libgober [29], the CW-complex associated to the minimal presentation can be chosen, see Proposition 2.16 . The cells induce a graded (arbitrarily ordered) basis B of $C^*(\mathcal{A})$. With this basis, the complex

$$C^*(\mathcal{A}) : \ 0 \longrightarrow C^0(\mathcal{A}) \xrightarrow{A_1} C^1(\mathcal{A}) \xrightarrow{A_2} C^2(\mathcal{A}) \longrightarrow 0$$

is determined by matrices A_1, A_2 with $\mathbb{Z}$-coefficients.

In the same way, we can define a graded basis $\tilde{B}$ of $\tilde{C}_*(\mathcal{A})$ as a free $\Lambda_{\mathbb{C}}$-module; each element of $\tilde{B}$ is an arbitrary lift of an element of B. Hence the complex

$$\tilde{C}^*(\mathcal{A}) : \ 0 \longrightarrow \tilde{C}^0(\mathcal{A}) \xrightarrow{\tilde{A}_1} \tilde{C}^1(\mathcal{A}) \xrightarrow{\tilde{A}_2} \tilde{C}^2(\mathcal{A}) \longrightarrow 0$$

is determined by two matrices $\tilde{A}_1, \tilde{A}_2$ with $\Lambda_\mathbb{C}$-coefficients. Recall that $\Lambda_\mathbb{C}$ is the ring of Laurent polynomials in $t_1, \dots, t_n$ with complex coefficients. The evaluation of the matrix $\tilde{A}_i$ at $t_j = 1$, $1 \leqslant j \leqslant n$, yields A_i.

Definition 3.1. — *The* character torus *of $\mathcal{A}$ is defined as:*

$$\mathbb{T}(\mathcal{A}) := H^1(M(\mathcal{A}); \mathbb{C}^*) = \operatorname{Hom}(H_1(M(\mathcal{A}); \mathbb{Z}), \mathbb{C}^*) = \operatorname{Hom}(\pi_1(M(\mathcal{A})), \mathbb{C}^*).$$

A character $\xi \in \mathbb{T}(\mathcal{A})$ induces an evaluation map $\operatorname{ev}_\xi : \Lambda_\mathbb{C} \to \mathbb{C}$ defining a $\Lambda_\mathbb{C}$-module structure on $\mathbb{C}$ denoted by $\mathbb{C}_\xi$. It also induces a local system of coefficients $\underline{\mathbb{C}}_\xi$, which can be associated with a cohomology group $H^1(M(\mathcal{A}); \underline{\mathbb{C}}_\xi)$ which depends only on $\pi_1(M(\mathcal{A}))$.

Definition 3.2. — *The* twisted cohomology $H^1(M(\mathcal{A}); \underline{\mathbb{C}}_\xi)$ *is obtained from the complex $\tilde{C}^*(\mathcal{A}) \otimes_{\Lambda_\mathbb{C}} \mathbb{C}_\xi$ which is obtained by the evaluation of $\tilde{A}_1, \tilde{A}_2$ using* ev_ξ.

3.2. Characteristic varieties

Definition 3.3. — *The* characteristic varieties *of $\mathcal{A}$ are $\mathcal{V}_k(\mathcal{A}) := \{\xi \in \mathbb{T}(\mathcal{A}) \mid \dim_\mathbb{C} H^1(M(\mathcal{A}); \underline{\mathbb{C}}_\xi) \geqslant k\}$.*

Remark 3.4. — Note that the characteristic varieties of $\mathcal{A}$ depend only on $M(\mathcal{A})$; we may define the characteristic varieties of a CW-complex. Moreover, they depend only on the fundamental group, and they may be define for a group. For a finitely generated group, characteristic varieties are subspaces of a finite dimensional torus. It is not hard to see that they are algebraic varieties defined by equations with integer coefficients. Additional restrictions apply for the fundamental groups of quasi-projective varieties.

Theorem 3.5 ([1, 32, 8]). — *The irreducible components of $\mathcal{V}_k(\mathcal{A})$ are subtori translated by torsion elements.*

This theorem is a consequence of a previous result by Arapura, which was refined later by Artal-Cogolludo-Matei.

Theorem 3.6 ([1, 8]). — *If $\Sigma \subset \mathcal{V}_k(\mathcal{A})$ is an irreducible component one of the following (non-exclusive) situations happen:*

1. *There exists a morphism $\Phi : M(\mathcal{A}) \to X$ onto an orbifold X and an irreducible component Σ_X of $\mathcal{V}_k(X)$ such that $\Sigma = \rho^*(\Sigma_X)$.*

2. *Σ is an isolated torsion point.*

Corollary 3.7. — *In order to describe $\mathcal{V}_k(\mathcal{A})$, it is enough to know $H^1(M(\mathcal{A});\underline{\mathbb{C}}_\xi)$ for ξ of torsion type.*

Proof. — Note that a subtorus is the Zariski closure of its torsion points and the same property applies for torsion-translated subtori. □

We are going to combine this with Sakuma's formula [41]. Let $\xi \in \mathbb{T}(\mathcal{A})$ be a character of order h and let $\rho_\xi : M_\xi(\mathcal{A}) \to M(\mathcal{A})$ be the h-fold cyclic covering associated to ξ. We denote by $\tilde{\xi} : M_\xi(\mathcal{A}) \to M_\xi(\mathcal{A})$ the canonical order-h generator of the deck automorphism of ρ_ξ. Let $\tilde{\xi}^* : H^\ell(M_\xi(\mathcal{A});\mathbb{C}) \to H^\ell(M_\xi(\mathcal{A});\mathbb{C})$ be the linear automorphism induced by $\tilde{\xi}$. We denote

$$H^\ell_\xi := \ker\left(\tilde{\xi}^* - \exp\left(\frac{2i\pi}{h}\right)\cdot 1_{H^\ell(M_\xi(\mathcal{A});\mathbb{C})}\right).$$

Proposition 3.8 ([41]). — $H^1_\xi \cong H^1(M(\mathcal{A});\underline{\mathbb{C}}_\xi)$.

As a consequence of Corrolary 3.7 and Proposition 3.8 we obtain the following result which guarantees the algebraic nature of characteristic varieties of arrangements; in fact this result also applies for quasi-projective manifolds.

Corollary 3.9. — *The characteristic varieties $\mathcal{V}_k(\mathcal{A})$ depend on the Betti numbers of the quasi-projective smooth varieties obtained as cyclic covers of $M(\mathcal{A})$.*

Proof. — Corrolary 3.7 implies that the characteristic varieties $\mathcal{V}_k(\mathcal{A})$ depend on the knowledge of $\dim H^1(M(\mathcal{A});\underline{\mathbb{C}}_\xi)$ for all characters ξ of torsion type. Fix a character ξ of order h and keep the above notations. We have the formula

$$\dim H^1(M_\xi(\mathcal{A});\mathbb{C}) = \sum_{j=0}^{h-1} \dim H^1_{\xi^j} = \sum_{j=0}^{h-1} \dim H^1(M(\mathcal{A});\underline{\mathbb{C}}_{\xi^j}).$$

Note also that $\dim H^1_{\xi^j} = \dim H^1_\xi$ if $\gcd(j,h) = 1$ and $0 < j < h$. For example, if $h = p$ is a prime number

$$\dim H^1(M_\xi(\mathcal{A});\mathbb{C}) = \dim H^1(M(\mathcal{A});\mathbb{C}) + (p-1)\dim H^1(M(\mathcal{A});\underline{\mathbb{C}}_\xi)$$

and the first term is n. If $h = p^m$, we have:

$$\dim H^1(M_\xi(\mathcal{A});\mathbb{C}) = \dim H^1(M_{\xi^p}(\mathcal{A});\mathbb{C}) + (p^m - p^{m-1})\dim H^1(M(\mathcal{A});\underline{\mathbb{C}}_\xi).$$

Finally if $h = ap^m$, with $\gcd(a,p) = 1$ and p prime, then

$$\dim H^1(M_\xi(\mathcal{A});\mathbb{C}) = \dim H^1(M_{\xi^p}(\mathcal{A});\mathbb{C}) + \varphi(ap^m)\dim H^1(M(\mathcal{A});\underline{\mathbb{C}}_\xi).$$

□

Characteristic varieties of a hyperplane arrangement depend on the fundamental group and, hence, they are topological invariants. Hyperplane arrangements are also algebraic objects. Let $\mathbb{K}$ be a number field (a finite algebraic extension of $\mathbb{Q}$) and let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2(\mathbb{K})$; any embedding $j : \mathbb{K} \hookrightarrow \mathbb{C}$ induces a line arrangement $\mathcal{A}^j$ in $\mathbb{P}^2(\mathbb{C})$. Note that the arrangements $\mathcal{A}^j$, where j runs along all the embeddings, are combinatorially equivalent. A topological invariant is said to be an *algebraic invariant* if, for line arrangements defined over a number field, it does not depend on the embedding of the number field in $\mathbb{C}$.

Corollary 3.10. — *Characteristic varieties of hyperplane arrangements are algebraic invariants.*

Proof. — It is a consequence of the fact that first Betti numbers of finite covers are algebraic invariants. □

3.3. Cohomology of projective and quasi-projective smooth varieties

The result of last subsection provide some important properties of the characteristic varieties of hyperplane arrangements. In order to be able to compute them, we are goint to apply some facts of the general theory of projective and quasi-projective varieties. We start by fixing a smooth projective completion $X_\xi(\mathcal{A})$ of $M_\xi(\mathcal{A})$ be a smooth projective completion such that

$$D_\xi(\mathcal{A}) := X_\xi(\mathcal{A}) \setminus M_\xi(\mathcal{A}) \tag{3.1}$$

is a normal crossing divisor. We choose this completion such that ρ_ξ extends to a branched covering $\rho_\xi : X_\xi(\mathcal{A}) \to \mathbb{P}^2$. For this covering, $\rho_\xi^{-1}(\bigcup \mathcal{A}) = D_\xi(\mathcal{A})$. We denote by $\mathcal{D}_\xi(\mathcal{A})$ the set of irreducible components of $D_\xi(\mathcal{A})$.

Let us recall what Pure Hodge Theory implies, see for example [23]. There is a decomposition

$$H^1(X_\xi(\mathcal{A});\mathbb{C}) \cong H^1(X_\xi(\mathcal{A});\mathcal{O}_{X_\xi(\mathcal{A})}) \oplus H^0(X_\xi(\mathcal{A});\Omega^1_{X_\xi(\mathcal{A})}) \tag{3.2}$$

such that

$$H^1(X_\xi(\mathcal{A});\mathcal{O}_{X_\xi(\mathcal{A})}) \cong \overline{H^0(X_\xi(\mathcal{A});\Omega^1_{X_\xi(\mathcal{A})})}.$$

Deligne's Mixed Hodge Theory for quasi-projective varieties [17] establishes the following decomposition:

$$H^1(M_\xi(\mathcal{A});\mathbb{C}) \cong H^1(X_\xi(\mathcal{A});\mathcal{O}_{X_\xi(\mathcal{A})}) \oplus H^0(X_\xi(\mathcal{A});\Omega^1_{X_\xi(\mathcal{A})}\log(D_\xi(\mathcal{A}))). \tag{3.3}$$

The first terms of the direct sum decompositions of (3.2) and (3.3) are exactly the same ones. The second terms can be related using Poincaré residues. Let us consider the following short exact sequence of sheaves:

$$0 \longrightarrow \Omega^1_{X_\xi(\mathcal{A})} \longrightarrow \Omega^1_{X_\xi(\mathcal{A})}\log(D_\xi(\mathcal{A})) \longrightarrow \bigoplus_{D\in\mathcal{D}_\xi(\mathcal{A})} i_*\mathcal{O}_D \longrightarrow 0 \tag{3.4}$$

Let us consider two exact sequences associated to the first terms of the associated long exact sequence:

$$0 \longrightarrow H^0(X_\xi;\Omega^1_{X_\xi}) \longrightarrow H^0(X_\xi;\Omega^1_{X_\xi}\log(D_\xi)) \longrightarrow H(\mathcal{A}) \longrightarrow 0 \tag{3.5}$$

$$0 \longrightarrow H(\mathcal{A}) \longrightarrow \bigoplus_{D\in\mathcal{D}_\xi} H^0(D;\mathcal{O}_D) \longrightarrow H^1(X_\xi;\Omega^1_{X_\xi}) \subset H^2(X_\xi;\mathbb{C}) \tag{3.6}$$

where $H(\mathcal{A})$ is the cokernel of the first map (and by exactness, the kernel of the second one).

We deduce the following formulæ:

(F1) $\dim H^1(X_\xi;\mathbb{C}) = 2\dim H^1(X_\xi;\mathcal{O}_{X_\xi})$.

(F2) $\dim H^1(M_\xi;\mathbb{C}) = 2\dim H^1(X_\xi;\mathcal{O}_{X_\xi}) + \dim H(\mathcal{A})$ where

$$H(\mathcal{A}) = \ker\left(\bigoplus_{D\in\mathcal{D}_\xi}\mathbb{C}\langle D\rangle \to H^2(X_\xi;\mathbb{C})\right), \quad \text{see (3.6).}$$

(F3) $\dim H^1(M(\mathcal{A});\mathbb{C}_\xi) = \dim H^1(X_\xi;\mathcal{O}_{X_\xi})^\xi + \dim H^1(X_\xi;\mathcal{O}_{X_\xi})^{\bar\xi} + d_\xi$, where

$$d_\xi := \dim\ker\left(\bigoplus_{D\in\mathcal{D}_\xi}\mathbb{C}\langle D\rangle \to H^2(X_\xi;\mathbb{C})\right)^\xi$$

and the superscript ξ means the eigenspace relative to ξ.

The sum of the two first terms in (F3) depends only on X_ξ; more precisely, this sum coincides with $\dim H^1(X_\xi;\mathbb{C})^\xi$. These two first terms are called the *projective terms*; the remaining term d_ξ depends on the quasi-projective variety $M(\mathcal{A})$ and it is called *quasi-projective term.*

4. Computation of the projective terms

Definition 4.1. — *A character* $\xi \in \mathbb{T}(\mathcal{A})$ *is* fully ramified *if* $\xi(t_j) \neq 1$, $\forall j \in \{0, \ldots, n\}$. *For a general caracter we denote by* $\mathcal{A}^\xi := \{\bar{L}_j \mid \xi(t_j) \neq 1\}$ *the* ramification locus *and by* $\mathcal{A}_0^\xi := \{\bar{L}_j \mid \xi(t_j) = 1\}$ *the* non-ramification locus*.*

Libgober shows that, for fully ramified characters, the quasi-projective terms vanish and everything can be computed inside $X_\xi(\mathcal{A})$.

Theorem 4.2 ([28]). — *If* ξ *is fully ramified then* $H^1(X_\xi; \mathbb{C})^\xi = H^1(M_\xi; \mathbb{C})^\xi$.

Hence, for fully ramified characters ξ, the projective terms determine to which characteristic varieties the character ξ belongs.

In [31], Libgober gives a procedure to compute $\dim H^1(X_\xi; \mathbb{C})^\xi$ which will be stated here in a slightly different way than in the original sources, where the general case of an algebraic plane curve is treated. The case of line arrangements is simpler.

Let us fix from now on a unitary character $\xi \in \mathbb{T}(\mathcal{A})$; in fact, by Corrolary 3.10, we only need to consider torsion characters, but the exposition is essentially the same.

Definition 4.3. — *The* real representative *of* ξ *is given by* $(r_0, r_1, \ldots, r_n) \in [0,1)^{n+1}$ *such that* $\xi(t_j) = \exp(2i\pi r_j)$. *The* level *of* ξ *is* $\ell(\xi) := r_0 + \ldots + r_n \in \mathbb{Z}$ *(recall that* $\prod_{i=0}^n t_i = 1$*).*

Definition 4.4. — *Let* $P \in \mathcal{P}_\mathcal{A}$. *Denote*

$$r_P := \sum_{P \in L_j} r_j \quad \text{and} \quad s_P := \max\{0, \lfloor r_P \rfloor - 1\}.$$

The ideal of quasiadjunction *of* P *with respect to* ξ *is* $\mathcal{J}_{P,\xi} := \mathcal{M}_P^{s_P}$, *where* $\mathcal{M}_P$ *is the maximal ideal of the local ring* $\mathcal{O}_{\mathbb{P}^2,P}$.

Theorem 4.5 ([31]). — *The dimension of* $H^1(X_\xi; \mathcal{O}_{X_\xi})^\xi$ *equals*

$$\dim \operatorname{coker} \left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(\ell(\xi) - 3)) \to \bigoplus_{P \in \mathcal{P}} \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{J}_{P,\xi} \right).$$

The map σ_ξ is defined as follows. Choose a line $\bar{L}_\infty$ disjoint to $\mathcal{P}_\mathcal{A}$. We identify this line as the line at infinity and we choose coordinates x, y for $\mathbb{C}^2$. Then

$$H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(k)) = \{f \in \mathbb{C}[x,y] \mid \deg f \leqslant k\}, \text{ for any } k.$$

The map σ_k results from considering the germ of holomorphic function of a polynomial at any $P \in \mathcal{P}_\mathcal{A}$. Note that in particular, $\mathbb{P}(\ker \sigma_\xi)$ is identified the space of curves of degree $\ell(\xi) - 3$ passing through P with multiplicity at least s_P, $\forall P \in \mathcal{P}_\mathcal{A}$.

Remark 4.6. — If $m_P = 2$ then $\mathcal{J}_{P,\xi} = \mathcal{O}_{\mathbb{P}^2,P}$. We can restrict our attention to $\mathcal{P}_{>2,\mathcal{A}} := \{P \in \mathcal{P}_\mathcal{A} \mid m_P > 2\}$. For $P \in \mathcal{P}_{>2,\mathcal{A}}$, the bound $m_P - 1 > s_P$ holds.

Remark 4.7. — The source and the target of σ_ξ are of combinatorial nature. On the other side, cokerσ_ξ and $\ker \sigma_\xi$ are determined by each other. The space $\mathbb{P}(\ker \sigma_\xi)$ has a geometric interpretation: it consists of the space of curves of degree $\ell(\xi) - 3$ passing through $P \in \mathcal{P}_{>2}$ with multiplicity at least s_P.

Hence, $\dim \ker \sigma_\xi$ is not *a priori* a combinatorial invariant and the same happens for the projective terms. Note also that Theorem 4.5 justifies the relationship between characteristic varieties and the position of singularities, where the term *position* involves not only if the multiple points lie on curves of some given degree but it is also affected by their multiplicity.

Example 4.8. — Let $\mathcal{A} = \{\bar{L}_0, \dots, \bar{L}_n\}$ be an arrangement with only one multiple point $P \in \bar{L}_i$. Consider a character ξ fully ramified of level ℓ. In this case, $s_P = \ell - 1$ and

$$\dim \operatorname{coker} \left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(\ell - 3)) \to \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{M}_P^{\ell-1}\right) = \ell - 1.$$

Note that $\ell(\bar{\xi}) = n + 1 - \ell$.

Example 4.9. — Let us consider the Ceva arrangement consisting of the six lines passing through 4 points $P_1, \dots, P_4$, in general position. Let us consider a character ξ; the images of the meridians of the lines are $t, s, u \in \mathbb{C}^*$, $tsu = 1$, as in Figure 14. We assume that $\ell(\xi) = 4$. Then, $s_{P_i} = 1$ and

$$\dim \operatorname{coker} \left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(1)) \to \mathbb{C}^4\right) = 1.$$

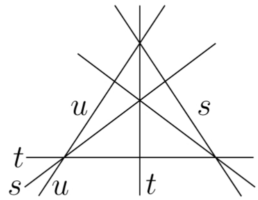

Figure 14. — Ceva arrangement

Example 4.10. — Let us consider the Pappus arrangement $\mathcal{A}$ given by

$$\begin{aligned} xz(4x-y+2z) &= 0 \\ y(x-y-z)(2x+y+2z) &= 0 \\ (x-y)(2x+y+z)(y+2z) &= 0. \end{aligned}$$

The set $\mathcal{P}_{\mathcal{A}}$ consists of nine double points and nine triple points. Let us consider the character ξ defined by $\xi(\mu_i) := \exp(2i\pi\frac{2}{3})$. Its level is $\ell = 6$ and $s_P = 1$ for all triple points. Note that

$$\dim\operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(3)) \to \mathbb{C}^9\right) = 1.$$

This example was studied in [2]. In this paper another arrangement of nine lines is studied, with the same number of double and triple points and where σ_ξ is surjective.

Example 4.11. — Let us consider the Hesse arrangement, i.e. the lines joining the nine inflection points of a smooth cubic, which admits the following equation

$$xyz(x^3y^3z^3 - 27(x^3+y^3+z^3)^3) = 0.$$

It has 9 quadruple points and 12 double points. Let us consider the character ξ defined by $\xi(\mu_i) := -1$. Then $\ell = 6$, $s_P = 1$ and

$$\dim\operatorname{coker}\left(\sigma_\xi : H^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(3)) \to \mathbb{C}^9\right) = 1.$$

4.1. Quasi-adjunction polytopes

This concept is introduced in [31], in order to study all fully-ramified characters using an algorithm involving finitely many computations.

Let us consider the semi-open cube $K := [0,1)^{n+1}$. In this cube we will consider the *level cubes* $K_\ell := K \cap \{r_0 + \ldots + r_n = \ell\}$, $\ell \in \{1,\ldots,n\}$.

Let us fix a point $P \in \mathcal{P}_{>2}$ and associate the following subsets

$$K_{P,k} := \begin{cases} \left\{\sum_{P\in L_j} r_j < 2\right\} & \text{if } k = 0 \\ \left\{k+1 \leqslant \sum_{P\in L_j} r_j < k+2\right\} & \text{if } 1 \leqslant k < m_P - 1 \end{cases}$$

These subsets have the following property. Let ξ be a character and let $(r_0, \ldots, r_n)$ be its real representative. Then:

$$(r_0, \ldots, r_n) \in K_{P,k} \Longleftrightarrow \mathcal{J}_{P,\xi} = \mathcal{M}_P^k,$$

For a fixed ℓ, the sets $\{K_{P,k} \mid P \in \mathcal{P}_\mathcal{A}\}$ induce a finite partition of K_ℓ. The partition subsets are called the *quasi-adjunction polytopes.*

PROPOSITION 4.12. — *Two characters in the same quasi-adjunction polytope share the map σ_ξ.*

Proof. — It is a direct consequence of Theorem 4.5 and the above descriptions. ☐

Remark 4.13. — The structure of characteristic varieties imposes conditions on the polytopes containing a character ξ and its conjugate $\bar{\xi}$, which may impose additional conditions on the position of the points in $\mathcal{P}_{>2}$.

We can summarize the result of this section. First, the information about fully-ramified characters can be obtained studying cyclic coverings of $\mathbb{P}^2$ ramified along $\mathcal{A}$. Second, in order to compute $H^1(M(\mathcal{A}); \underline{\mathbb{C}}_\xi)$ for all fully-ramified characters ξ only a finite number of computations are necessary and they depend on the *position* of the multiple points of $\mathcal{A}$.

For this section we have borrowed ideas of Zariski [44], Esnault-Viehweg [18, 19, 20], Libgober [28], Loeser-Vaquié [33] and the author [3].

5. Coordinate components and quasi-projective term

Let us study now the quasi-projective term of (F3) in page 247. From Theorem 4.2, this term vanishes for fully-ramified characters. In this section we will focus on characters which do not ramify on some lines. Let us introduce some notation in order to properly work with these characters.

Let $\mathcal{A} := \{\bar{L}_0, \bar{L}_1, \ldots, \bar{L}_n\}$ be a line arrangement in $\mathbb{P}^2$. Let us consider the torus $\mathbb{T}(\mathcal{A})$; using the coordinates $(t_0, t_1, \ldots, t_n)$ we consider $\mathbb{T}(\mathcal{A})$ as the subtorus $\prod_{i=0}^n t_i = 1$ in $(\mathbb{C}^*)^{n+1}$.

For any set J such that $\emptyset \neq J \subsetneq \{0,1,\dots,n\}$ we consider the sub-arrangement

$$\mathcal{A}_J := \{\bar{L}_j \mid j \in J\}.$$

Note that the torus $\mathbb{T}(\mathcal{A}_J)$ is in a natural way a subtorus of $\mathbb{T}(\mathcal{A})$, namely

$$\mathbb{T}(\mathcal{A}_J) = \mathbb{T}(\mathcal{A}) \cap \mathbb{T}_J, \quad \mathbb{T}_J := \{(t_0, t_1, \dots, t_n) \in \mathbb{C}^{n+1} \mid t_i = 1 \text{ if } i \notin J\}.$$

The non-trivial characters which are not fully-ramified are exactly those living in

$$\bigcup_{\emptyset \neq J \subsetneq \{0,1,\dots,n\}} \mathbb{T}(\mathcal{A}_J).$$

Under this identification the inclusion $\mathcal{V}_k(\mathcal{A}_J) \subset \mathcal{V}_k(\mathcal{A})$ is obvious. If the equality always holds, then any character (different from $\mathbf{1}$) would be fully-ramified for some sub-arrangement and hence the results of Section 4 would be enough to compute the characteristic varieties of the arrangement $\mathcal{A}$. As we will see in general $\mathcal{V}_k(\mathcal{A}_J) \subsetneq \mathcal{V}_k(\mathcal{A}) \cap \mathbb{T}_J$, i.e., we need more work to describe completely the characteristic varieties of $\mathcal{A}$.

5.1. Coordinate components

As we saw in Section 4, it is better to work with subsets of the characteristic varieties instead of working character by character. Due to their algebraic nature, the most interesting subsets are the irreducible components. We distinguish them in terms of their relative position with the subtori corresponding to sub-arrangements.

DEFINITION 5.1. — *Let Σ be an irreducible component of $\mathcal{V}_k(\mathcal{A})$, $k > 0$. The component is said to be:*

1. Coordinate *if Σ is contained in $\mathbb{T}_J$ for some $\emptyset \neq J \subsetneq \{0,1,\dots,n\}$.*
2. Non-coordinate *if it is not coordinate.*
3. Non-essential coordinate *if Σ is an irreducible component of $\mathcal{V}_k(\mathcal{A}_J)$ for some $\emptyset \neq J \subsetneq \{0,1,\dots,n\}$.*
4. Essential coordinate *if it is coordinate but not non-essential.*

In order to compute the characteristic varieties of $\mathcal{A}$, we may apply induction on the number of lines. For $n = 0$ it is clear that the characteristic varieties are empty. Let us assume that $n > 0$. Non-coordinate components are computed using Section 4 and non-essential coordinate components are assumed to be computed using induction hypothesis. Only essential coordinate components escape from this approach.

Remark 5.2. — The existence of essential coordinate components was proven for line arrangements; in [16], Cohen and Suciu found such components in $\mathcal{V}_2$. Essential coordinate components in $\mathcal{V}_1$ were found in [5] for algebraic curves.

DEFINITION 5.3. — *Fix $\emptyset \neq J \subsetneq \{0, 1, \ldots, n\}$ and denote $\mathcal{V}_{k,J}(\mathcal{A}) := \mathcal{V}_k(\mathcal{A}) \cap \mathbb{T}_J$, $k > 0$. Let Σ_J be an irreducible component of $\mathcal{V}_{k,J}(\mathcal{A})$, $k > 0$. The component is said to be:*

1. J-strict *if Σ_J is an irreducible component of $\mathcal{V}_k(\mathcal{A}_J)$.*
2. J-secant *if there exists Σ, irreducible component of $\mathcal{V}_k(\mathcal{A})$ not contained in $\mathbb{T}_J$ such that $\Sigma_J = \Sigma \cap \mathbb{T}_J$.*
3. J-essential *if Σ_J is neither J-strict nor J-secant.*

Remark 5.4. — Note that J-essential irreducible components of $\mathcal{V}_{k,J}(\mathcal{A})$ are essential coordinate components of $\mathcal{V}_k(\mathcal{A})$. Moreover, they are the only ones which are not covered by the above induction process.

We can translate a result from Libgober in [31] into this language.

THEOREM 5.5 ([31]). — *Essential irreducible components are isolated torsion points.*

5.2. The projective cover

From now on we fix a torsion character $\xi \in \mathbb{T}(\mathcal{A})$ (of order h) and we assume that it is not fully-ramified, i.e., $\mathcal{A}_0^\xi$ is non-empty. For simplicity, we will assume that $\mathcal{A}_0^\xi = \{\bar{L}_m, \ldots, \bar{L}_n\}$, for some $m \in \{1, \ldots, n\}$.

Recall that $M_\xi(\mathcal{A}) = X_\xi \setminus D_\xi$ and $\mathcal{D}_\xi$ is the set of irreducible components of D_ξ, see (3.1). The projective manifold X_ξ is constructed (for any cyclic cover of quasi-projective surface) in [19]. Let us sketch the construction of X_ξ applied to our case.

Let $\pi : Y \to \mathbb{P}^2$ the blowing-up of $\mathcal{P}_{>2,\mathcal{A}}$. Let us denote

$$E_P := \pi^{-1}(P), \quad P \in \mathcal{P}_{>2,\mathcal{A}},$$

the exceptional component of π over P. For $0 \leqslant j \leqslant n$, we denote by $\tilde{L}_j$ the strict transform of $\bar{L}_j$ by π, i.e.,

$$\tilde{L}_j = \overline{\pi^{-1}(\bar{L}_j \setminus \mathcal{P}_{>2,\mathcal{A}})}.$$

Let us denote

$$\tilde{\mathcal{A}} := \{\tilde{L}_0, \ldots, \tilde{L}_n\} \cup \{E_P \mid P \in \mathcal{P}_{>2,\mathcal{A}})\}.$$

The divisor $\bigcup \tilde{\mathcal{A}} = \pi^{-1}(\bigcup \mathcal{A})$ is a normal crossing divisor and π is the minimal composition of blowing-ups with this property. Since the centers of the blowing-ups are in $\bigcup \mathcal{A}$ the map π is an isomorphism from $Y \setminus \bigcup \tilde{\mathcal{A}}$ and $M(\mathcal{A})$, both spaces will be identified.

The character ξ defines an effective divisor C_ξ on Y, with support contained in $\bigcup \tilde{\mathcal{A}}$, such that there exists another divisor B_ξ on Y with the property that C_ξ is linearly equivalent to hB_ξ. Hence, we have an isomorphism $\mathcal{O}(C_\xi) \cong \mathcal{O}(B_\xi)^{\otimes h}$. Let $s : Y \to \mathcal{O}(C_\xi)$ be a holomorphic section such that its divisor is C_ξ. Let

$$X_\xi^{\text{sing}} := \{p \in \mathcal{O}(B_\xi) \mid p^{\otimes n} \in s(Y)\}.$$

The restriction of the projection map of $\mathcal{O}(B_\xi) \to Y$ to X_ξ^{sing} is a compactification of the covering $M_\xi(\mathcal{A}) \to M(\mathcal{A})$. Since $\bigcup \tilde{\mathcal{A}}$ is a normal crossing divisor, the normalization X_ξ^{norm} of X_ξ^{sing} has only a finite number of singular points which are contained in the preimage of $\tilde{\mathcal{P}}$, the set of the double points of $\bigcup \tilde{\mathcal{A}}$. The variety X_ξ is obtained by taking the minimal resolution of X_ξ^{norm}.

Let us describe more properties of $(Y, \tilde{\mathcal{A}})$ in order to describe $(X_\xi, D_\xi(\mathcal{A}))$ later.

Lemma 5.6. — *The intersection of the components of $\tilde{\mathcal{A}}$ is given by:*

1. $E_P^2 = -1$;
2. $(\tilde{L}_j)^2 = 1 - a_j$, $a_j := \#\{P \in \mathcal{P}_{>2} \mid P \in \bar{L}_j\}$;
3. *for* $i \neq j$, $\tilde{L}_i \cap \tilde{L}_j \neq \emptyset \iff \bar{L}_i \cap \bar{L}_j \notin \mathcal{P}_{>2}$;
4. *for* $P \neq Q$, $E_P \cap E_Q = \emptyset$.
5. $E_P \cap \tilde{L}_i \neq \emptyset \iff P \in \bar{L}_i$.

Proof. — It follows from the general properties of blow-ups, see e.g. [12]. ◻

Since $Y \setminus \bigcup \tilde{\mathcal{A}} \equiv M(\mathcal{A})$, its homology in degree one is generated by the homology classes of the meridians of the lines $\bar{L}_j$ in $\mathbb{P}^2$ (denoted by t_j). By the very definition of meridian, t_i is also the homology class of a meridian of

$\tilde{L}_j$ in Y. The divisor $\bigcup \tilde{\mathcal{A}}$ has other irreducible components, the exceptional components E_P. The following Lemma expresses the relations between these meridians.

LEMMA 5.7. — *For $P \in \mathcal{P}_{>2,\mathcal{A}}$, the homology class of a meridian of E_P is given by $t_P := \prod_{P\in \bar{L}_j} t_j \in H_1(M(\mathcal{A});\mathbb{Z})$.*

Proof. — This is a direct consequence of the computation of the fundamental group of a graph manifold, see [36, 25]; it may be seen also as a generalization of [22]. □

We will denote by ρ_ξ a branched covering $X_\xi \to Y$. By construction, the divisor $D_\xi = \rho_\xi^{-1}(\bigcup \tilde{\mathcal{A}})$ is a normal-crossing divisor. In order to describe the set $\mathcal{D}_\xi$ of irreducible components of D_ξ, we need the following definition.

DEFINITION 5.8. — *A component $A \in \mathcal{D}_\xi$ is of* single type *if $\rho_\xi(A) = \{p\}$ is a point (and hence $p \in \tilde{\mathcal{P}}$); we will say that A* belongs to *p. If $\rho_\xi(A)$ is a component $B \in \tilde{\mathcal{A}}$, then A is said to be a B*-component*.*

DEFINITION 5.9. — *Let $B \in \tilde{\mathcal{A}}$. The* neighboring subgroup *G_B of B is the subgroup of $\mathbb{C}^*$ generated by $\xi(t_B)$ and $\xi(t_C)$ for the components $C \in \tilde{\mathcal{A}}$ such that $C \cap B \neq \emptyset$. The* neighboring index *is $n_B := \frac{\#\mathrm{im}\xi}{\# G_B} = \frac{h}{\# G_B}$.*

LEMMA 5.10. — *Let $B \in \tilde{\mathcal{A}}$. The number of B-components of $\mathcal{D}_\xi$ equals the neighboring index n_B.*

Proof. — Let r_B be the order of $\xi(t_B)$ and let $h_B := \frac{h}{r_B}$. This means that for a point $q \in B \setminus \tilde{\mathcal{P}}$ and $q_0 \in \rho^{-1}(q)$ we can find local coordinates (u_0, v_0) centered at q_0, and (u, v) centered at q, such that the map ρ_ξ is expressed as

$$(u_0, v_0) \mapsto (u_0^{r_B}, v_0),$$

where $u_0 = 0$ and $u = 0$ are local equations of B and $\rho_\xi^{-1}(B)$, respectively.

Hence, the covering ρ_ξ defines a branched covering $\hat{\rho}_\xi^{-1}(B) \to B$ of order h_B, where

$$\hat{\rho}_\xi^{-1}(B) := \overline{\rho^{-1}(B \setminus \tilde{\mathcal{P}})},$$

the strict transform of B. The number of connected components of $\hat{\rho}_\xi^{-1}(B)$ is n_B, since the values $\xi(t_C)$, $C \cap B \neq \emptyset$, determine the branching index. □

PROPOSITION 5.11. — *Let $\tilde{\xi} : X_\xi \to X_\xi$ be the extension of the canonical generator of the deck automorphism group of the unbranched covering $M_\xi(\mathcal{A}) \to \mathcal{A}$. Let $A \in \mathcal{D}_\xi$ and let Ω_A be its $\tilde{\xi}$-orbit. Then:*

1. *If A is a single component belonging to $p \in \mathcal{P}$, then $\#\Omega_A < h$.*

2. *If A is a B-component for some $B \in \tilde{\mathcal{A}}$, then $\#\Omega_A = n_B$, in fact, Ω_A is the set B-components.*

Proof. — The statement is obvious for (2.). For (1.), if $p \in \mathcal{P}$, the number of its preimages in X_ξ^{norm} coincides with $\#\Omega_A$. If the preimage of π has h preimages, the branched covering from X_ξ^{norm} is a local automorphism, and then all the preimages are smooth Since the passage from X_ξ^{norm} is the minimal resolution, we deduce that $\#\Omega_A < h$. □

Definition 5.12. — *We say that $B \in \tilde{\mathcal{A}}$ is said* unramified *if $\xi(t_B) = 1$, and* inner unramified *if it is unramified and it is also the case for all its neighbors. We denote by $\mathcal{U}_\xi \subset \tilde{\mathcal{A}}$ the set of the inner unramified components.*

Corollary 5.13. — *Let $A \in \mathcal{D}_\xi$ and let Ω_A be its $\tilde{\xi}$-orbit. Then, $\#\Omega_A = h$ if and only if A is a B-component where B is inner unramified.*

Remark 5.14. — A component $B \in \tilde{\mathcal{A}}$ is unramified in the following cases:

- $B = \tilde{L}_i$ and $\bar{L}_i \in \mathcal{A}_0^\xi$:
- $B = E_P$ and $\prod_{P \in \bar{L}_i} t_i = 1$.

In order to be inner unramified the following conditions must be fulfilled:

- $B = E_P$ and all the lines through P are in $\mathcal{A}_0^\xi$.
- $B = \tilde{L}_i$, all the lines intersecting $\bar{L}_i$ at double points are in $\mathcal{A}_0^\xi$ and all the points $P \in \mathcal{P}_{\geqslant 2,\mathcal{A}} \cap \bar{L}_i$ satisfy $t_P = 1$.

5.3. A twisted intersection form

We are going to compute the quasi-projective term d_ξ in (F3). Recall that this term is the dimension of the ξ-eigenspace of the kernel of the morphism

$$\bigoplus_{D \in \mathcal{D}_\xi} \mathbb{C}\langle D \rangle \longrightarrow H^2(X_\xi; \mathbb{C});$$

where the image of D is its Poincaré dual in $H^2(X_\xi; \mathbb{C})$. This morphism respects the decomposition in eigenspaces defined by ξ on both the source and the target.

Lemma 5.15. — *Let* $U_\xi := \mathbb{C}\langle\mathcal{U}_\xi\rangle$*. For each* $B \in \mathcal{U}_\xi$*, fix (arbitrarily) one* B*-component* B_0*. The* h B*-components are denoted by* $B_0, B_1, \ldots, B_{h-1}$*, where* $B_j := (\rho_\xi)^j(B_0)$*. Then, the map*

$$\begin{array}{rcl} U_\xi & \longrightarrow & \left(\bigoplus_{D\in\mathcal{D}^\xi} \mathbb{C}\langle D\rangle\right)^\xi \\ B & \longmapsto & \frac{1}{\sqrt{h}}\sum_{j=0}^{h-1} \exp\left(\frac{2\sqrt{-1}\pi j}{h}\right) B_j \end{array}$$

is an isomorphism.

Proof. — First note that ξ acts transitively on each orbit, and for an orbit to admit a non-trivial eigenspace for $\exp(\frac{2\sqrt{-1}\pi}{h})$ the necessary and sufficient condition is to have h terms; from Corrolary 5.13, it means that such orbits are the sets of B-components for B an inner ramified component. □

In order to compute the kernel of this morphism we are going to define an intersection form on $\left(\bigoplus_{D\in\mathcal{D}^\xi} \mathbb{C}\langle D\rangle\right)^\xi$ as follows. The intersection form on $H^2(X_\xi;\mathbb{Z})$ induces a non-degenerate hermitian form in $H^2(X_\xi;\mathbb{C})$. For this hermitian form, the decomposition in eigenspaces for $\tilde{\xi}$ is orthogonal. The same applies to $\bigoplus_{D\in\mathcal{D}^\xi} \mathbb{C}\langle D\rangle$ and to its eigenspace.

Definition 5.16. — *The* twisted intersection form $\cdot_\xi$ *for* U_ξ *is the intersection form defined via the isomorphism of Lemma 5.15. We will denote by* A_ξ *the matrix of this form for an order in the basis* $\mathcal{U}_\xi$*.*

Theorem 5.17. — *The twisted intersection form* $\cdot_\xi$ *determines* d_ξ*, more precisely* $d_\xi = \operatorname{corank} A_\xi$

Proof. — This result would be straightforward if the image of $\left(\bigoplus_{D\in\mathcal{D}^\xi} \mathbb{C}\langle D\rangle\right)^\xi$ was $H^2(X_\xi;\mathbb{C})^\xi$. Recall that the twisted intersection form is constructed from the intersection form on $H^2(X_\xi;\mathbb{C})$ which is non-degenerate by Poincaré duality; this one decomposes orthogonally for eigenspaces for ρ_ξ. In particular, the restriction of the intersection to $H^2(X_\xi;\mathbb{C})^\xi$ is non-degenerate and hence the rank of A_ξ measures the dimension of the image (and the corank the dimension of the kernel).

Unfortunately it is not possible to ensure that surjectivity. From the Hodge Index Theorem, the intersection form on $H^2(X_\xi;\mathbb{C})$ has signature $(1, b_2(X_\xi)-1)$, i.e., it can diagonalized with only one $+1$. Let $\bar{L}$ be a generic line in $\mathbb{P}^2$; it is known that $(\bar{L}\cdot\bar{L})_{\mathbb{P}^2}=1$. Let us denote by $\tilde{L}$ its strict transform in Y; since $\bar{L}$ is generic, it does not pass through the centers of the blow-ups, and then $(\tilde{L}\cdot\tilde{L})_Y=1$. Let $\tilde{L}_\xi$ be its preimage in X_ξ; by Poincaré duality, it defines an element in the 1-eigenspace of $H^2(X_\xi;\mathbb{C})$ and $(\tilde{L}_\xi\cdot\tilde{L}_\xi)_{X_\xi}=h>0$. Since the intersection form splits orthogonally for the eigenspaces, we deduce that its restriction to $H^2(X_\xi;\mathbb{C})^\xi$ is not only non-degenerate but it is also negative definite. In particular, it is negative definite (and hence non-degenerate) for any subspace, in particular the image of our morphism. Now, the above argument works. □

The goal now is to compute this matrix A_ξ.

Remark 5.18. — Note that the twisted intersection form for U_ξ is not well defined since the isomorphism of Lemma 5.15 depends on the arbitrary choice of a B-component for B inner unramified. A different choice for a particular B implies that the B-column of A_ξ is multiplied by an h-root of unity, denoted by ζ, while the B-row is multiplied by $\bar{\zeta}$. Anyway, the expected properties for this form will not depend on the particular choices.

Notation 5.19. — The dual graph of $\mathcal{U}_\xi$ is denoted by $\Gamma_{\mathcal{U}_\xi}$. Recall that the vertex of $\Gamma_{\mathcal{U}_\xi}$ corresponds to the inner unramified components, while the edges correspond to the double points.

Among all the possible choices of B-components some are preferred. Choose a *maximal forest* $\mathcal{T}_{\mathcal{U}_\xi}$ of $\Gamma_{\mathcal{U}_\xi}$, i.e. a maximal tree for any connected component of the graph. For each connected component Γ of $\Gamma_{\mathcal{U}_\xi}$ choose a vertex, i.e. an irreducible component B^Γ of $\mathcal{U}_\xi$. For these components, choose arbitrarily a B^Γ-component B^Γ_0. Once these choices are made, we choose the other components by induction on the graph-distance r of a vertex to $\{B^\Gamma\mid\Gamma \text{ connected component of } \Gamma_{\mathcal{U}_\xi}\}$ inside the maximal forest. Fix a vertex related to a component B of $\mathcal{U}_\xi$. If $r=0$ the choice has been made; if $r>0$, there is a unique component B' such that its distance is $r-1$ and it is connected to B. By induction hypothesis the choice of B'_0 has been made. The edge joining B' and B in the maximal forest corresponds to a double point $P\in B\cap B'$. Since the components are unramified, this point has h preimages in X_ξ, and one of them, say P_0 is in B'_0; choose as B_0 the unique B-component such that $P_0\in B_0$.

Definition 5.20. — *A choice of B-components as above is called a* tree choice.

We are going to see how to construct the information about the twisted intersection form for a tree choice. Recall that $b_1(\Gamma)$ coincides with the cardinality of $E(\Gamma_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$, the set of edges of $\Gamma_{\mathcal{U}_\xi}$ which are not in $\mathcal{T}_{\mathcal{U}_\xi}$. For each oriented edge $\vec{e}$, $e \in E(\Gamma_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$, we are going to construct a cycle $\gamma_{\vec{e}} \in H_1(M(\mathcal{A}, \mathbb{Z}))$ as follows. Let B be the origin of $\vec{e}$ and let C be its end. Fix a point $p \in M(\mathcal{A})$ *close* to the double point associated to e. Consider the unique polygonal path in $\mathcal{T}_{\mathcal{U}_\xi}$ joining B and C. Then, lift this path to a cycle $\gamma_{\vec{e}}$ in $M(\mathcal{A})$ very close to $\tilde{\mathcal{A}}$

Remark 5.21. — The following remarks are important:

1. Since each pair of vertices is connected by at most one edge, if B is the origin of $\vec{e}$ and C is its end then we denote $\vec{e} = \overrightarrow{BC}$.
2. We can consider a closed regular neighborhood of $\tilde{\mathcal{A}}$ obtained by plumbing tubular neighborhoods of the irreducible components. Its boundary is a graph 3-manifold (see [42] for its definition and [37] for the plumbing presentation). The cycle $\gamma_{\vec{e}}$ can be constructed in this manifold which is of combinatorial nature. The embedding of this manifold in $M(\mathcal{A})$ is not combinatorial and determines its value in $H_1(M(\mathcal{A}, \mathbb{Z}))$. This embedding has been studied for complexified real arrangements by E. Hironaka [26]; for general arrangements an effective method to compute $\gamma_{\vec{e}}$ as a linear combination of meridians. is given by Florens-Guerville-Marco [21].
3. The cycles $\gamma_{\vec{e}}$ are not well defined. The lift of the polygonal path is not uniquely determined. Essentially, there are two kinds of ambiguities. One can turn around the components A of the polygonal path, which is translated by a multiplication by a power of t_A; since they are components of $\mathcal{U}_\xi$, note that $\xi(t_A) = 1$. For these lifts it is also necessary to *avoid* components A' which are neighbors to components of the polygonal path. This amounts to multiply by $t_{A'}$; by the definition of inner unramified these components also satisfy $\xi(t_{A'}) = 1$.

As a consequence of these commentaries, we have the following result.

Proposition 5.22. — *For $\overrightarrow{BC} \in E(\Gamma_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$, the value of $\xi(\gamma_{\overrightarrow{BC}})$ depends only on $\overrightarrow{BC}$ and will be denoted by $\xi(\overrightarrow{BC})$ (and $\xi(\overleftarrow{BC}) = \xi(\overrightarrow{BC})^{-1}$).*

The vector space has a *non-twisted* intersection form $\cdot$ which corresponds to the one in Y. We are going to compare both intersection forms.

Theorem 5.23. — *Let us consider a tree choice. Let B, C components of $\mathcal{U}_\xi$. Then:*

$$B \cdot_\xi C = \begin{cases} 0 & \text{if } B \cdot C = 0; \\ B \cdot C & \text{if either } B = C \text{ or } \overrightarrow{BC} \subset \mathcal{T}_{\mathcal{U}_\xi}; \\ \xi(\gamma_{\overrightarrow{BC}}) & \text{if } \overrightarrow{BC} \in E(\Gamma_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi}). \end{cases}$$

Proof. — If B and C are disjoint, then $\bigcup_{j=0}^{h-1} B_j$ and $\bigcup_{j=0}^{h-1} C_j$ are also disjoint and then $B \cdot C = B \cdot_\xi C = 0$.

If $\overrightarrow{BC} \subset \mathcal{T}_{\mathcal{U}_\xi}$ then $(B_i \cdot C_j)_{X_\xi} = \delta_{i,j}$. Hence, from the intersection form in X_ξ one obtains:

$$\begin{aligned} B \cdot_\xi C &= \left(\frac{1}{\sqrt{h}} \sum_{j=0}^{h-1} \exp\left(\frac{2\sqrt{-1}\pi j}{h}\right) B_j\right) \cdot \left(\frac{1}{\sqrt{h}} \sum_{j=0}^{h-1} \exp\left(\frac{2\sqrt{-1}\pi j}{h}\right) C_j\right) \\ &= \frac{1}{h} \sum_{j=0}^{h-1} \exp\left(\frac{2\sqrt{-1}\pi j}{h}\right) \exp\left(-\frac{2\sqrt{-1}\pi j}{h}\right) = 1. \end{aligned}$$

If $\overrightarrow{BC} \in E(\Gamma_{\mathcal{U}_\xi}, \mathcal{T}_{\mathcal{U}_\xi})$ let $k \in \{0, 1, \dots, h-1\}$ such that $\xi(\overrightarrow{BC}) = \exp\left(-\frac{2\sqrt{-1}\pi(k)}{h}\right)$. Then, the lifting covering property implies that $B_0 \cdot C_k = 0$. A straightforward computation proves the statement. □

Corollary 5.24. — *If $\Gamma_{\mathcal{U}_\xi}$ is a tree then, for a tree choice, $\cdot_\xi$ coincides with $\cdot$.*

Remark 5.25. — Theorem 5.23 determines precisely which parts of A_ξ are not combinatorial. In particular, if $\Gamma_{\mathcal{U}_\xi}$, the matrix is combinatorial.

5.4. Final comments and examples

Instead of fixing a torsion character we can proceed conversely. Fix a subset $\mathcal{U} \subset \tilde{\mathcal{A}}$, or more generally a subset of lines in $\mathcal{A}$, and consider the characters ξ for which $\mathcal{U} = \mathcal{U}_\xi$. The interesting characters will be those for which $\operatorname{corank} A(\mathcal{U}_\xi) > 0$. Taking into account Theorem 5.5, the non-isolated sets of characters satisfying this property are parts of J-secant components for some J, and hence their existence can be deduced by the study of fully-ramified characters for some sub-arrangement of $\mathcal{A}$ (containing strictly $\mathcal{A}_J$).

We consider now two examples where non-isolated and isolated characters with the above property appear.

Example 5.26. — Let us consider the Ceva arrangement, see Figure 15. Let us consider the lines $\bar{L}_5, \bar{L}_6$ (green lines). For which characters ξ, does $\mathcal{U}_\xi$ contain $\tilde{L}_5, \tilde{L}_6$ and no other strict transform? In Figure 15 the *fat points* are the triple points which induce the other components of $\tilde{\mathcal{A}}$.

Let us denote by ξ such a character. We have imposed $\xi(t_5) = \xi(t_6) = 1$. Let us denote $\xi(t_1) =: x \in \mathbb{C}^*$. Since the triple point $P_{12} := \bar{L}_1 \cap \bar{L}_2$ passes through a green line (for which its strict transform must be inner unramified), we deduce that $\xi(t_{P_{12}}) = 1$, i.e., $\xi(t_2) = x^{-1}$.

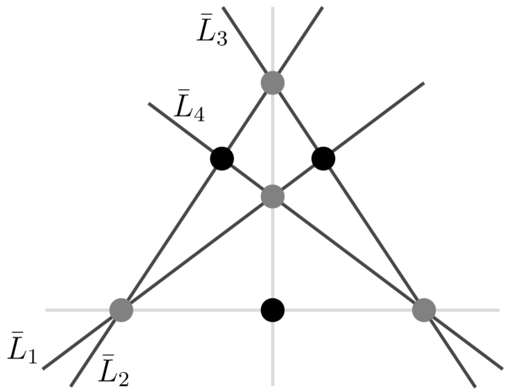

Figure 15. — Ceva arrangement

The same argument applied to $\bar{L}_2 \cap \bar{L}_3$ yields $\xi(t_3) = x$; if applied to $\bar{L}_3 \cap \bar{L}_4$ we obtain $\xi(t_4) = x^{-1}$. It is not hard to see that for any $x \in \mathbb{C}^* \setminus \{1\}$, the character ξ satisfies that $\mathcal{U}_\xi = \{\tilde{L}_5, \tilde{L}_6\}$. The graph $\Gamma_{\mathcal{U}_\xi}$, weighted with the self-intersection in Y is shown in Figure 16. Since it is a tree A_ξ coincides with the usual intersection matrix, i.e.,

$$A_\xi = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Its corank equals one, so for these characters the quasi-projective term is non-zero. Since there is no isolated point, its closure is a J-secant component for $J = \{\bar{L}_j \mid 1 \leqslant j \leqslant 4\}$.

Figure 16. — $\Gamma_{\mathcal{U}_\xi}$

Example 5.27. — Let us study now the extended Ceva arrangement, see Figure 17. It is obtained from Ceva arrangement by adding a line $\bar{L}_7$ joining two double points (which become triple points in this arrangement), namely P_{13} and P_{24}. As for Ceva arrangement, let us consider the lines $\bar{L}_5, \bar{L}_6, \bar{L}_7$ (green lines) and a similar question: For which characters ξ, does $\mathcal{U}_\xi$ contain $\tilde{L}_5, \tilde{L}_6, \bar{L}_7$ and no other strict transform? We have marked in Figure 15 the triple points since they induce the other components of $\tilde{\mathcal{A}}$.

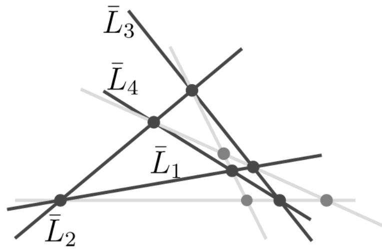

Figure 17. — Extended Ceva arrangement

Let us denote by ξ such a character. We have imposed $\xi(t_5) = \xi(t_6) = \xi(t_7) = 1$. Let us proceed as in Example 5.26. If

$$\xi(t_1) =: x \in \mathbb{C}^* \text{ then } \xi(t_2) = x^{-1}, \xi(t_3) = x \text{ and } \xi(t_4) = x^{-1},$$

using the same triple points. The remaining triple points of the original Ceva arrangement do not impose new conditions. Let us impose now the conditions for the triple points P_{13} and P_{24} which pass through $\bar{L}_7$. They yield the same condition: $x^2 = 1$. Since we do not consider the trivial character, only $x = -1$ is to be considered. Hence the character ξ satisfies that $\mathcal{U}_\xi = \{\tilde{L}_5, \tilde{L}_6, \tilde{L}_7\}$ and it is of order 2. The graph $\Gamma_{\mathcal{U}_\xi}$, weighted with the self-intersection in Y is shown in Figure 18. Let us compute the matrix A_ξ. In order to make a tree choice, we consider the maximal tree which avoids the double point P_{67}. Applying Theorem 5.23:

$$A_\xi = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & \varepsilon \\ 1 & \varepsilon & -1 \end{pmatrix}$$

where $\varepsilon := \xi(\overrightarrow{\tilde{L}_6\tilde{L}_7}) = \xi(\overrightarrow{\tilde{L}_7\tilde{L}_6})$; the last equality holds since ξ is of order 2. In order to compute this value, we may consider the behavior of ρ_ξ in the neighborhood of $\mathcal{U}_\xi$. Two situations may happen: for $\varepsilon = 1$ in Figure 19(a) and for $\varepsilon = -1$ in Figure 19(a). We may compute it using the techniques in [26] (since the arrangement is real) adapted by Florens-Guerville-Marco [21]. In fact, Cohen and Suciu proved in [16] that $\{\xi\}$ is an isolated component of $\mathcal{V}_2(\mathcal{A})$ which is essential coordinate. This means that the right answer is $\varepsilon = -1$ and the corank is 2 as expected.

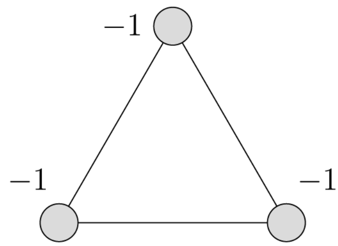

Figure 18. — $\Gamma_{\mathcal{U}_\xi}$

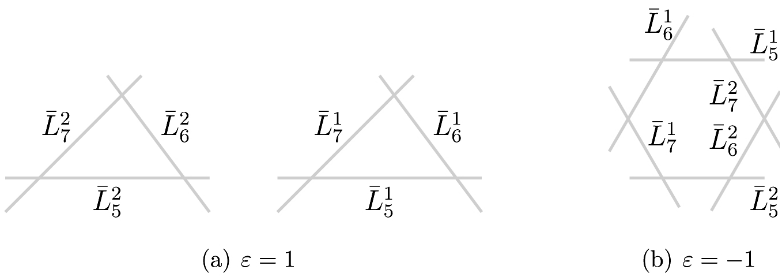

Figure 19. — Options for the preimage of $\mathcal{U}_\xi$ under ρ_ξ

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