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# Numerical characterization of nef arithmetic divisors on arithmetic surfaces

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**Abstract.** — In this paper, we give a numerical characterization of nef arithmetic $\mathbb{R}$-Cartier divisors of $C^0$-type on an arithmetic surface. Namely an arithmetic $\mathbb{R}$-Cartier divisor $\overline{D}$ of $C^0$-type is nef if and only if $\overline{D}$ is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$.

**Résumé.** — Dans le présent article, nous donnons une caractérisation numérique des $\mathbb{R}$-diviseurs arithmétiques nef et de type $C^0$ sur une surface artihmétique. Plus exactement, nous montrons qu'un $\mathbb{R}$-diviseur de Cartier $\overline{D}$ de type $C^0$ est nef si et seulement si $\overline{D}$ est pseudo-effectif et $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$.

## Introduction

Let $X$ be a generically smooth, normal and projective arithmetic surface and let $X \to \mathrm{Spec}(O_K)$ be the Stein factorization of $X \to \mathrm{Spec}(\mathbb{Z})$, where $K$ is a number field and $O_K$ is the ring of integers in $K$. Let $\overline{L}$ be an arithmetic divisor of $C^\infty$-type on $X$ with $\deg(L_K) = 0$ (cf. Conventions and terminology 2). Faltings-Hriljac's Hodge index theorem ([6], [8]) says that

$$\widehat{\deg}(\overline{L}^2) \leqslant 0$$

and the equality holds if and only if $\overline{L} = \widehat{(\phi)} + (0, \eta)$ for some $F_\infty$-invariant locally constant real valued function $\eta$ on $X(\mathbb{C})$ and $\phi \in \mathrm{Rat}(X)^\times_{\mathbb{Q}} := \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. The inequality part of their Hodge index theorem can be generalized as follows: Let $\overline{D}$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor

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of $C^0$-type on $X$, that is, $\overline{D} = \overline{P} - \overline{Q}$ for some nef arithmetic $\mathbb{R}$-Cartier divisors $\overline{P}$ and $\overline{Q}$ of $C^0$-type (cf. Conventions and terminology 2 and 5). If $\deg(D_K) \geqslant 0$, then

$$\widehat{\deg}(\overline{D}^2) \leqslant \widehat{\mathrm{vol}}(\overline{D})$$

(cf. [12, Theorem 6.2], [13, Theorem 6.6.1], Theorem 4.3). This inequality is called the *generalized Hodge index theorem*. It is very interesting to ask the equality condition of the inequality. It is known that if $\overline{D}$ is nef, then $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$ (cf. [12, Corollary 5.5], [13, Proposition-Definition 6.4.1]), so that the problem is the converse. In the case where $\deg(D_K) = 0$ (and hence $\widehat{\mathrm{vol}}(\overline{D}) = 0$), it is nothing more than the equality condition of the Hodge index theorem (cf. Lemma 4.1). Thus the following theorem gives an answer to the above question.

Theorem 0.1 (cf. Theorem 4.3). — *We assume that* $\deg(D_K) > 0$. *Then* $\overline{D}$ *is nef if and only if* $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$.

For the proof of the above theorem, we need the integral formulae of the arithmetic volumes due to Boucksom-Chen [4] and the existence of the Zariski decomposition of big arithmetic divisors [13]. From the point of view of a characterization of nef arithmetic $\mathbb{R}$-Cartier divisors, the following variant of the above theorem is also significant.

Corollary 0.2 (cf. Corollary 4.4). — $\overline{D}$ *is nef if and only if* $\overline{D}$ *is pseudo-effective and* $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$.

Let $\Upsilon(\overline{D})$ be the set of all arithmetic $\mathbb{R}$-Cartier divisors $\overline{M}$ of $C^0$-type on $X$ such that $\overline{M}$ is nef and $\overline{M} \leqslant \overline{D}$. As an application of the above theorem, we have the following numerical characterization of the greatest element of $\Upsilon(\overline{D})$.

Corollary 0.3 (cf. Corollary 5.4). — *We assume that* $X$ *is regular. Let* $\overline{P}$ *be an arithmetic* $\mathbb{R}$*-Cartier divisor of* $C^0$*-type on* $X$. *Then the following are equivalent:*

*(1)* $\overline{P}$ *is the greatest element of* $\Upsilon(\overline{D})$, *that is,* $\overline{P} \in \Upsilon(\overline{D})$ *and* $\overline{M} \leqslant \overline{P}$ *for all* $\overline{M} \in \Upsilon(\overline{D})$.

*(2)* $\overline{P}$ *is an element of* $\Upsilon(\overline{D})$ *with the following property:*

$$\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0 \quad \text{and} \quad \widehat{\deg}(\overline{B}^2) < 0$$

*for all integrable arithmetic* $\mathbb{R}$*-Cartier divisors* $\overline{B}$ *of* $C^0$*-type with* $(0,0) \lneqq \overline{B} \leqslant \overline{D} - \overline{P}$ *(cf. Conventions and terminology 5).*

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## Conventions and terminology

Here we fix several conventions and the terminology of this paper. An *arithmetic variety* means a quasi-projective and flat integral scheme over $\mathbb{Z}$. It is said to be *generically smooth* if the generic fiber over $\mathbb{Z}$ is smooth over $\mathbb{Q}$. Throughout this paper, $X$ is a $(d+1)$-dimensional, generically smooth, normal and projective arithmetic variety. Let $X \to \operatorname{Spec}(O_K)$ be the Stein factorization of $X \to \operatorname{Spec}(\mathbb{Z})$, where $K$ is a number field and $O_K$ is the ring of integers in $K$. For details of the following 2 and 4, see [13] and [15].

**1.** A pair $(M, \|\cdot\|)$ is called a *normed* $\mathbb{Z}$*-module* if $M$ is a finitely generated $\mathbb{Z}$-module and $\|\cdot\|$ is a norm of $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. A quantity

$$\log\left(\frac{\operatorname{vol}\left(\{x \in M_{\mathbb{R}} \mid \|x\| \leqslant 1\}\right)}{\operatorname{vol}(M_{\mathbb{R}}/(M/M_{tor}))}\right) + \log \#(M_{tor})$$

does not depend on the choice of the Haar measure vol on $M_{\mathbb{R}}$, where $M_{tor}$ is the group of torsion elements of $M$. We denote the above quantity by $\hat{\chi}(M, \|\cdot\|)$.

**2.** Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $\operatorname{Div}(X)$ be the group of Cartier divisors on $X$ and let $\operatorname{Div}(X)_{\mathbb{K}} := \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{K}$, whose element is called a $\mathbb{K}$*-Cartier divisor on* $X$. For $D \in \operatorname{Div}(X)_{\mathbb{R}}$, we define $H^0(X, D)$ and $H^0(X_K, D_K)$ to be

$$\begin{cases} H^0(X, D) = \{\phi \in \operatorname{Rat}(X)^{\times} \mid D + (\phi) \geqslant 0\} \cup \{0\}, \\ H^0(X_K, D_K) = \{\phi \in \operatorname{Rat}(X_K)^{\times} \mid D_K + (\phi)_K \geqslant 0 \text{ on } X_K\} \cup \{0\}, \end{cases}$$

where $X_K$ is the generic fiber of $X \to \operatorname{Spec}(O_K)$.

A pair $\overline{D} = (D, g)$ is called an *arithmetic* $\mathbb{K}$*-Cartier divisor of* $C^{\infty}$*-type* (resp. *of* $C^0$*-type*) if the following conditions are satisfied:

(a) $D$ is a $\mathbb{K}$-Cartier divisor on $X$, that is, $D = \sum_{i=1}^{r} a_i D_i$ for some $D_1, \ldots, D_r \in \operatorname{Div}(X)$ and $a_1, \ldots, a_r \in \mathbb{K}$.

(b) $g : X(\mathbb{C}) \to \mathbb{R} \cup \{\pm\infty\}$ is a locally integrable function and $g \circ F_{\infty} = g$ (*a.e.*), where $F_{\infty} : X(\mathbb{C}) \to X(\mathbb{C})$ is the complex conjugation map.

(c) For any point $x \in X(\mathbb{C})$, there exist an open neighborhood $U_x$ of $x$ and a $C^\infty$-function (resp. continuous function) $u_x$ on $U_x$ such that

$$g = u_x + \sum_{i=1}^{r}(-a_i)\log|f_i|^2 \quad (a.e.)$$

on $U_x$, where $f_i$ is a local equation of $D_i$ over $U_x$ for each $i$.

The function $g$ is called a *$D$-Green function of $C^\infty$-type* (resp. *of $C^0$-type*). Note that $dd^c([u_x])$ does not depend on the choice of local equations $f_1, \ldots, f_r$, so that $dd^c([u_x])$ is defined globally on $X(\mathbb{C})$. It is called the *first Chern current of* $\overline{D}$ and is denoted by $c_1(\overline{D})$, that is, $c_1(\overline{D}) = dd^c([g]) + \delta_D$. Note that, if $\overline{D}$ is of $C^\infty$-type, then $c_1(\overline{D})$ is represented by a $C^\infty$-form, which is called the *first Chern form of* $\overline{D}$. Let $\mathcal{C}$ be either $C^\infty$ or $C^0$. The set of all arithmetic $\mathbb{K}$-Cartier divisors of $\mathcal{C}$-type is denoted by $\widehat{\mathrm{Div}}_{\mathcal{C}}(X)_{\mathbb{K}}$. Moreover, the group

$$\left\{(D,g) \in \widehat{\mathrm{Div}}_{\mathcal{C}}(X)_{\mathbb{Q}} \mid D \in \mathrm{Div}(X)\right\}$$

is denoted by $\widehat{\mathrm{Div}}_{\mathcal{C}}(X)$. An element of $\widehat{\mathrm{Div}}_{\mathcal{C}}(X)$ is called an *arithmetic Cartier divisor of $\mathcal{C}$-type*. For $\overline{D} = (D,g), \overline{E} = (E,h) \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{K}}$, we define relations $\overline{D} = \overline{E}$ and $\overline{D} \geqslant \overline{E}$ as follows:

$$\begin{aligned} \overline{D} = \overline{E} \quad &\overset{\text{def}}{\Longleftrightarrow} \quad D = E,\ \ g = h \ (a.e.), \\ \overline{D} \geqslant \overline{E} \quad &\overset{\text{def}}{\Longleftrightarrow} \quad D \geqslant E,\ \ g \geqslant h \ (a.e.). \end{aligned}$$

Let $\mathrm{Rat}(X)^\times_{\mathbb{K}} := \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{K}$, and let

$$(\ )_{\mathbb{K}} : \mathrm{Rat}(X)^\times_{\mathbb{K}} \to \mathrm{Div}(X)_{\mathbb{K}} \quad \text{and} \quad \widehat{(\ )}_{\mathbb{K}} : \mathrm{Rat}(X)^\times_{\mathbb{K}} \to \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{K}}$$

be the natural extensions of the homomorphisms

$$\mathrm{Rat}(X)^\times \to \mathrm{Div}(X) \quad \text{and} \quad \mathrm{Rat}(X)^\times \to \widehat{\mathrm{Div}}_{C^\infty}(X)$$

given by $\phi \mapsto (\phi)$ and $\phi \mapsto \widehat{(\phi)}$, respectively. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type. We define $\hat{\Gamma}^\times(X,\overline{D})$ and $\hat{\Gamma}^\times_{\mathbb{K}}(X,\overline{D})$ to be

$$\begin{cases} \hat{\Gamma}^\times(X,\overline{D}) := \left\{\phi \in \mathrm{Rat}(X)^\times \mid \overline{D} + \widehat{(\phi)} \geqslant (0,0)\right\}, \\ \hat{\Gamma}^\times_{\mathbb{K}}(X,\overline{D}) := \left\{\phi \in \mathrm{Rat}(X)^\times_{\mathbb{K}} \mid \overline{D} + \widehat{(\phi)}_{\mathbb{K}} \geqslant (0,0)\right\}. \end{cases}$$

Note that $\hat{\Gamma}^\times_{\mathbb{Q}}(X,\overline{D}) = \bigcup_{n=1}^\infty \hat{\Gamma}^\times(X,n\overline{D})^{1/n}$. Moreover, we set

$$\hat{H}^0(X,\overline{D}) := \hat{\Gamma}^\times(X,\overline{D}) \cup \{0\} \quad \text{and} \quad \hat{H}^0_{\mathbb{K}}(X,\overline{D}) := \hat{\Gamma}^\times_{\mathbb{K}}(X,\overline{D}) \cup \{0\}.$$

For $\xi \in X$, we define the $\mathbb{K}$-*asymptotic multiplicity of* $\overline{D}$ *at* $\xi$ to be

$$\mu_{\mathbb{K},\xi}(\overline{D}) := \begin{cases} \inf\left\{\operatorname{mult}_\xi(D + (\phi)_{\mathbb{K}}) \mid \phi \in \widehat{\Gamma}^\times_{\mathbb{K}}(X, \overline{D})\right\} & \text{if } \widehat{\Gamma}^\times_{\mathbb{K}}(X, \overline{D}) \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

(for details, see [13, Proposition 6.5.2, Proposition 6.5.3] and [15, Section 2]).

**3.** Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. Let $\phi \in H^0(X(\mathbb{C}), D_{\mathbb{C}})$, that is, $\phi \in \operatorname{Rat}(X(\mathbb{C}))^\times$ and $(\phi) + D_{\mathbb{C}} \geqslant 0$ on $X(\mathbb{C})$. Then $|\phi| \exp(-g/2)$ is represented by a continuous function $|\phi|_g^c$ on $X(\mathbb{C})$ (cf. [13, SubSection 2.5]), so that we may consider $\sup\{|\phi|_g^c(x) \mid x \in X(\mathbb{C})\}$. We denote it by $\|\phi\|_{\overline{D}}$ or $\|\phi\|_g$. Note that, for $\phi \in H^0(X, D)$, $\phi \in \hat{H}^0(X, \overline{D})$ if and only if $\|\phi\|_{\overline{D}} \leqslant 1$. We define $\widehat{\operatorname{vol}}(\overline{D})$ and $\widehat{\operatorname{vol}}_\chi(\overline{D})$ to be

$$\widehat{\operatorname{vol}}(\overline{D}) := \limsup_{m\to\infty} \frac{\log \# \hat{H}^0(X, m\overline{D})}{m^{d+1}/(d+1)!},$$

$$\widehat{\operatorname{vol}}_\chi(\overline{D}) := \limsup_{m\to\infty} \frac{\hat{\chi}(H^0(X, mD), \|\cdot\|_{m\overline{D}})}{m^{d+1}/(d+1)!}.$$

It is well known that $\widehat{\operatorname{vol}}(\overline{D}) \geqslant \widehat{\operatorname{vol}}_\chi(\overline{D})$. More generally, for $\xi_1, \ldots, \xi_l \in X$ and $\mu_1, \ldots, \mu_l \in \mathbb{R}_{\geqslant 0}$, we define $\widehat{\operatorname{vol}}(\overline{D}; \mu_1\xi_1, \ldots, \mu_l\xi_l)$ to be

$$\widehat{\operatorname{vol}}(\overline{D}; \mu_1\xi_1, \ldots, \mu_l\xi_l) := \\ \limsup_{m\to\infty} \frac{\log \# \left(\left\{\phi \in \widehat{\Gamma}^\times(X, m\overline{D}) \mid \operatorname{mult}_{\xi_i}(mD + (\phi)) \geqslant \mu_i \ (\forall i)\right\} \cup \{0\}\right)}{m^{d+1}/(d+1)!}.$$

Note that $\widehat{\operatorname{vol}}(\overline{D}; \mu\xi) = \widehat{\operatorname{vol}}(\overline{D})$ for $0 \leqslant \mu \leqslant \mu_{\mathbb{Q},\xi}(\overline{D})$.

**4.** Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. The effectivity, bigness, pseudo-effectivity and nefness of $\overline{D}$ are defined as follows:

- $\overline{D}$ is effective $\overset{\text{def}}{\iff}$ $\overline{D} \geqslant (0, 0)$.
- $\overline{D}$ is big $\overset{\text{def}}{\iff}$ $\widehat{\operatorname{vol}}(\overline{D}) > 0$.
- $\overline{D}$ is pseudo-effective $\overset{\text{def}}{\iff}$ $\overline{D} + \overline{A}$ is big for any big arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}$ of $C^0$-type.
- $\overline{D} = (D, g)$ is nef $\overset{\text{def}}{\iff}$
  - (a) $\widehat{\deg}(\overline{D}\big|_C) \geqslant 0$ for all reduced and irreducible 1-dimensional closed subschemes $C$ of $X$.
  - (b) $c_1(\overline{D})$ is a positive current.

A decomposition $\overline{D} = \overline{P} + \overline{N}$ is called a *Zariski decomposition of* $\overline{D}$ if the following properties are satisfied:

(1) $\overline{P}$ and $\overline{N}$ are arithmetic $\mathbb{R}$-Cartier divisors of $C^0$-type on $X$.

(2) $\overline{P}$ is nef and $\overline{N}$ is effective.

(3) $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{D})$.

We set

$$\Upsilon(\overline{D}) := \left\{ \overline{M} \;\middle|\; \begin{array}{l} \overline{M} \text{ is an arithmetic } \mathbb{R}\text{-Cartier divisor of } C^0\text{-type} \\ \text{such that } \overline{M} \text{ is nef and } \overline{M} \leqslant \overline{D} \end{array} \right\}.$$

If $\overline{P}$ is the greatest element of $\Upsilon(\overline{D})$ (i.e. $\overline{P} \in \Upsilon(\overline{D})$ and $\overline{M} \leqslant \overline{P}$ for all $\overline{M} \in \Upsilon(\overline{D})$) and $\overline{N} = \overline{D} - \overline{P}$, then $\overline{D} = \overline{P} + \overline{N}$ is a Zariski decomposition of $\overline{D}$ (cf. Proposition B.1).

**5.** Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. According to [18], we say $\overline{D}$ is *integrable* if there are nef arithmetic $\mathbb{R}$-Cartier divisors $\overline{P}$ and $\overline{Q}$ of $C^0$-type such that $\overline{D} = \overline{P} - \overline{Q}$. Note that if either $\overline{D}$ is of $C^\infty$-type, or $c_1(\overline{D})$ is a positive current, then $\overline{D}$ is integrable (cf. [13, Proposition 6.4.2]). Moreover, for integrable arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_0, \ldots, \overline{D}_d$ of $C^0$-type on $X$, the arithmetic intersection number $\widehat{\deg}(\overline{D}_0 \cdots \overline{D}_d)$ is defined in the natural way (cf. [13, SubSection 6.4], [15, SubSection 2.1]). Note that if $\overline{D} = \overline{P} + \overline{N}$ is a Zariski decomposition and $\overline{D}$ is integrable, then $\overline{N}$ is also integrable.

**6.** We assume that $X$ is regular and $d = 1$. Let $D_1, \ldots, D_k$ be $\mathbb{R}$-Cartier divisors on $X$. We set $D_i = \sum_C a_{i,C} C$ for each $i$, where $C$ runs over all reduced and irreducible 1-dimensional closed subschemes on $X$. We define $\max\{D_1, \ldots, D_k\}$ to be

$$\max\{D_1, \ldots, D_k\} := \sum_C \max\{a_{1,C}, \ldots, a_{k,C}\} C.$$

Let $\overline{D}_1 = (D_1, g_1), \ldots, \overline{D}_k = (D_k, g_k)$ be arithmetic $\mathbb{R}$-Cartier divisors of $C^0$-type on $X$. Then $\max\{\overline{D}_1, \ldots, \overline{D}_k\}$ is defined to be

$$\max\{\overline{D}_1, \ldots, \overline{D}_k\} := (\max\{D_1, \ldots, D_k\}, \max\{g_1, \ldots, g_k\}).$$

Note that $\max\{\overline{D}_1, \ldots, \overline{D}_k\}$ is also an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type (cf. [13, Lemma 9.1.2]).

## 1. Relative Zariski decomposition of arithmetic divisors

We assume that $X$ is regular and $d = 1$. The Stein factorization $X \to \operatorname{Spec}(O_K)$ of $X \to \operatorname{Spec}(\mathbb{Z})$ is denoted by $\pi$. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. We say $\overline{D}$ is *relatively nef* if $c_1(\overline{D})$ is a positive current and $\widehat{\deg}(\overline{D}|_C) \geqslant 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$ on $X$. We set

$$\Upsilon_{rel}(\overline{D}) := \left\{ \overline{M} \;\middle|\; \begin{array}{l} \overline{M} \text{ is an arithmetic } \mathbb{R}\text{-Cartier divisor of } C^0\text{-type} \\ \text{such that } \overline{M} \text{ is relatively nef and } \overline{M} \leqslant \overline{D} \end{array} \right\}.$$

Theorem 1.1 (Relative Zariski decomposition). — *If $\deg(D_K) \geqslant 0$, then there is the greatest element $\overline{Q}$ of $\Upsilon_{rel}(\overline{D})$, that is, $\overline{Q} \in \Upsilon_{rel}(\overline{D})$ and $\overline{M} \leqslant \overline{Q}$ for all $\overline{M} \in \Upsilon_{rel}(\overline{D})$. Moreover, if we set $\overline{N} := \overline{D} - \overline{Q}$, then $\overline{Q}$ and $\overline{N}$ satisfy the following properties:*

*(a) $N$ is vertical.*

*(b) $\widehat{\deg}(\overline{Q} \cdot \overline{N}) = 0$.*

*(c) For any $P \in \operatorname{Spec}(O_K)$, $\pi^{-1}(P)_{red} \not\subseteq \operatorname{Supp}(N)$.*

*(d) The natural homomorphism $H^0(X, nQ) \to H^0(X, nD)$ is bijective and $\|\cdot\|_{n\overline{D}} = \|\cdot\|_{n\overline{Q}}$ for each $n \geqslant 0$.*

*(e) $\widehat{\operatorname{vol}}_\chi(\overline{Q}) = \widehat{\operatorname{vol}}_\chi(\overline{D})$.*

Before staring the proof of Theorem 1.1, we need several preparations. Let $D$ be an $\mathbb{R}$-Cartier divisor on $X$. We say $D$ is *$\pi$-nef* if $\deg(D|_C) \geqslant 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$ on $X$. First let us consider the relative Zariski decomposition on finite places.

Lemma 1.2. — *Let $D$ be an $\mathbb{R}$-Cartier divisor on $X$ and let $\Sigma(D)$ be the set of all $\mathbb{R}$-Cartier divisors $M$ on $X$ such that $M$ is $\pi$-nef and $M \leqslant D$. If $\deg(D_K) \geqslant 0$, then there is the greatest element $Q$ of $\Sigma(D)$, that is, $Q \in \Sigma(D)$ and $M \leqslant Q$ for all $M \in \Sigma(D)$. Moreover, if we set $N := D - Q$, then $Q$ and $N$ satisfy the following properties:*

*(a) $N$ is vertical.*

*(b) $\deg(Q|_C) = 0$ for all reduced and irreducible 1-dimensional closed subschemes $C$ in $\operatorname{Supp}(N)$.*

*(c) For any $P \in \operatorname{Spec}(O_K)$, $\pi^{-1}(P)_{red} \not\subseteq \operatorname{Supp}(N)$.*

(d) *The natural homomorphism* $H^0(X, nQ) \to H^0(X, nD)$ *is bijective for each* $n \geqslant 0$.

*Proof.* — Let us begin with following claim:

CLAIM 1.3. — $\Sigma(D) \neq \emptyset$.

*Proof.* — First we assume that $\deg(D_K) = 0$. Then, by using Zariski's lemma (cf. [15, Lemma 1.1.4]), we can find a vertical and effective $\mathbb{R}$-Cartier divisor $E$ such that $\deg(\left.(D-E)\right|_C) = 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$ on $X$, and hence $\Sigma(D) \neq \emptyset$.

Next we assume that $\deg(D_K) > 0$. Let $A$ be an ample Cartier divisor on $X$. As $\deg(D_K) > 0$, $H^0(X_K, mD_K - A_K) \neq \{0\}$ for some positive integer $m$, and hence $H^0(X, mD - A) \neq \{0\}$. Thus, there is $\phi \in \mathrm{Rat}(X)^{\times}$ such that $mD - A + (\phi) \geqslant 0$, that is, $D \geqslant (1/m)(A - (\phi))$, as required. □

CLAIM 1.4. — *If* $L_1, \ldots, L_k$ *are* $\pi$*-nef* $\mathbb{R}$*-Cartier divisors, then* $\max\{L_1, \ldots, L_k\}$ *is also* $\pi$*-nef (cf. Conventions and terminology* 6*).*

*Proof.* — We set $L_i' := \max\{L_1, \ldots, L_k\} - L_i$ for each $i$. Let $C$ be a vertical reduced and irreducible 1-dimensional closed subscheme on $X$. Then there is $i$ such that $C \not\subseteq \mathrm{Supp}(L_i')$. As $L_i'$ is effective, we have $\deg(\left.L_i'\right|_C) \geqslant 0$, so that

$$\deg(\left.\max\{L_1, \ldots, L_k\}\right|_C) = \deg(\left.L_i\right|_C) + \deg(\left.L_i'\right|_C) \geqslant 0.$$

□

For a reduced and irreducible 1-dimensional closed subscheme $C$ on $X$, we set

$$q_C := \sup\{\mathrm{mult}_C(M) \mid M \in \Sigma(D)\},$$

which exists in $\mathbb{R}$ because $\mathrm{mult}_C(M) \leqslant \mathrm{mult}_C(D)$ for all $M \in \Sigma(D)$. We fix $M_0 \in \Sigma(D)$.

CLAIM 1.5. — *There is a sequence* $\{M_n\}_{n=1}^{\infty}$ *of* $\mathbb{R}$*-Cartier divisors in* $\Sigma(D)$ *such that* $M_0 \leqslant M_n$ *for all* $n \geqslant 1$ *and* $\lim_{n\to\infty} \mathrm{mult}_C(M_n) = q_C$ *for all reduced and irreducible* 1*-dimensional closed subschemes* $C$ *in* $\mathrm{Supp}(D) \cup \mathrm{Supp}(M_0)$.

*Proof.* — For each reduced and irreducible 1-dimensional closed subscheme $C$ in $\mathrm{Supp}(D) \cup \mathrm{Supp}(M_0)$, there is a sequence $\{M_{C,n}\}_{n=1}^{\infty}$ in $\Sigma(D)$ such that

$$\lim_{n\to\infty} \mathrm{mult}_C(M_{C,n}) = q_C.$$

If we set

$$M_n = \max\left(\{M_{C,n}\}_{C \subseteq \mathrm{Supp}(D) \cup \mathrm{Supp}(M_0)} \cup \{M_0\}\right),$$

then $M_0 \leqslant M_n$ and $M_n \in \Sigma(D)$ by Claim 1.4. Moreover, as

$$\mathrm{mult}_C(M_{C,n}) \leqslant \mathrm{mult}_C(M_n) \leqslant q_C,$$

$\lim_{n\to\infty} \mathrm{mult}_C(M_n) = q_C$. □

Since $\max\{M_0, M\} \in \Sigma(D)$ for all $M \in \Sigma(D)$ by Claim 1.4, we have

$$\mathrm{mult}_C(M_0) \leqslant q_C \leqslant \mathrm{mult}_C(D).$$

In particular, if $C \not\subseteq \mathrm{Supp}(D) \cup \mathrm{Supp}(M_0)$, then $q_C = 0$, so that we can set $Q := \sum_C q_C C$.

Claim 1.6. — *$Q$ is the greatest element $Q$ in $\Sigma(D)$, that is, $Q \in \Sigma(D)$ and $M \leqslant Q$ for all $M \in \Sigma(D)$.*

*Proof.* — By Claim 1.5, we can see that $Q \in \Sigma(D)$, so that the assertion follows. □

We need to check the properties (a) – (d).

(a) We choose effective $\mathbb{R}$-Cartier divisors $N_1$ and $N_2$ such that $N = N_1+N_2$, $N_1$ is horizontal and $N_2$ is vertical. If $N_1 \neq 0$, then $Q \lneqq Q+N_1 \leqslant D$ and $Q + N_1$ is $\pi$-nef, so that we have $N_1 = 0$, that is, $N$ is vertical.

(b) Let $C$ be a vertical reduced and irreducible 1-dimensional closed subscheme in $\mathrm{Supp}(N)$. If $\deg(Q|_C) > 0$, then $Q+\epsilon C$ is $\pi$-nef and $Q+\epsilon C \leqslant D$ for a sufficiently small $\epsilon > 0$, and hence $\deg(Q|_C) = 0$.

(c) We assume the contrary. Then we can find $\delta > 0$ such that $\delta\pi^{-1}(P) \leqslant N$, so that $Q \lneqq Q + \delta\pi^{-1}(P) \leqslant D$ and $Q + \delta\pi^{-1}(P)$ is $\pi$-nef. This is a contradiction.

(d) It is sufficient to see that if $\phi \in \Gamma^\times(X, nD)$, then $\phi \in \Gamma^\times(X, nQ)$. Since $(-1/n)(\phi) \in \Sigma(D)$, we have $(-1/n)(\phi) \leqslant Q$, that is, $nQ + (\phi) \geqslant 0$. Therefore $\phi \in \Gamma^\times(X, nQ)$. □

Moreover, we need the following lemma.

Lemma 1.7. — *Let $S$ be a connected compact Riemann surface and let $D$ be an $\mathbb{R}$-divisor on $S$ with $\deg(D) \geqslant 0$. Let $g$ be a $D$-Green function of $C^0$-type on $S$ and let $G(D, g)$ be the set of all $D$-Green functions $h$ of*

*$C^0$-type on $S$ such that $c_1(D,h)$ is a positive current and $h \leqslant g$* (a.e.). *Then there is the greatest element $q$ of $G(D,g)$, that is, $q \in G(D,g)$ and $h \leqslant q$* (a.e.) *for all $h \in G(D,g)$. Moreover, $q$ has the following property:*

*(1)* $\|\phi\|_{ng} = \|\phi\|_{nq}$ *for all $\phi \in H^0(S,nD)$ and $n \geqslant 0$.*

*(2)* $\displaystyle\int_S (g-q)c_1(D,q) = 0.$

*Proof.* — The existence of $q$ follows from [3, Theorem 1.4] or [13, Theorem 4.6]. We need to check the properties (1) and (2).

(1) Clearly $\|\phi\|_{nq} \geqslant \|\phi\|_{ng}$ because $q \leqslant g$ (a.e.). Let us consider the converse inequality. We may assume that $\phi \neq 0$. We set

$$q' := \max\left\{q, \frac{1}{n}\log(|\phi|^2/\|\phi\|_{ng}^2)\right\}.$$

Since $D \geqslant (-1/n)(\phi)$ and $(1/n)\log(|\phi|^2/\|\phi\|_{ng}^2)$ is a $(-1/n)(\phi)$-Green function of $C^\infty$-type with the first Chern form zero, by [13, Lemma 9.1.1], $q'$ is a $D$-Green function of $C^0$-type such that $c_1(D,q')$ is a positive current. Note that $\|\phi\|_{ng}^2 \geqslant |\phi|^2 \exp(-ng)$ (a.e.), that is,

$$g \geqslant (1/n)\log(|\phi|^2/\|\phi\|_{ng}^2) \text{ (a.e.)},$$

and hence $q' \in G(D,g)$. Therefore, as $q' \geqslant q$ (a.e.), we have $q = q'$ (a.e.), so that $q \geqslant (1/n)\log(|\phi|^2/\|\phi\|_{ng}^2)$ (a.e.), that is, $\|\phi\|_{ng}^2 \geqslant |\phi|^2\exp(-nq)$ (a.e.), which implies $\|\phi\|_{ng} \geqslant \|\phi\|_{nq}$.

(2) If $\deg(D) = 0$, then the assertion is obvious because $c_1(D,q) = 0$, so that we assume that $\deg(D) > 0$. First we consider the case where $g$ is of $C^\infty$-type. We set $\alpha := c_1(D,g)$ and

$$\varphi := \sup\{\psi \mid \psi \text{ is an } \alpha\text{-plurisubharmonic function on } S \text{ and } \psi \leqslant 0\}$$

(cf. [3]). Then, by [13, Proposition 4.3], $q = g + \varphi$ (a.e.). In particular, $\varphi$ is continuous because $g$ and $q$ are of $C^0$-type. If we set $D = \{x \in S \mid \varphi(x) = 0\}$, then, by [3, Corollary 2.5], $c_1(D,q) = \mathbf{1}_D\alpha$, where $\mathbf{1}_D$ is the indicator function of $D$. Thus

$$\int_S (g-q)c_1(D,q) = 0.$$

Next we consider a general case. Let $g'$ be a $D$-Green function of $C^\infty$-type. We set $g = g' + u$ (a.e.) for some continuous function $u$ on $S$. By using the Stone-Weierstrass theorem, we can find a sequence $\{u_n\}$ of $C^\infty$-functions on $S$ such that $\lim_{n\to\infty} \|u_n - u\|_{\sup} = 0$. We set $g_n := g' + u_n$. Let $q_n$ be the greatest element of $G(D, g_n)$. As

$$g - \|u_n - u\|_{\sup} \leqslant g_n \leqslant g + \|u_n - u\|_{\sup} \text{ (a.e.)},$$

we can see $q - \|u_n - u\|_{\sup} \leqslant q_n \leqslant q + \|u_n - u\|_{\sup}$ (a.e.). Thus, if we set $q_n = g' + v_n$ (a.e.) and $q = g' + v$ (a.e.) for some continuous functions $v_n$ and $v$ on $S$, then $\lim_{n\to\infty} \|v_n - v\|_{\sup} = 0$. Moreover, by using the previous observation,

$$0 \quad = \textstyle\int_S (g_n - q_n) c_1(D, q_n) = \int_S (u_n - v_n) c_1(D, q_n).$$

Since $c_1(D, q_n) = c_1(D, g') + dd^c([v_n]) \geqslant 0$, by using [5, Corollary 3.6] or [15, Lemma 1.2.1], we can see that $c_1(D, q_n)$ converges weakly to $c_1(D, q)$ as functionals on $C^0(S)$. In particular, there is a constant $C$ such that $\int_S c_1(D, q_n) \leqslant C$ for all $n$. Thus

$$\begin{aligned}
&\left| \int_S (u_n - v_n) c_1(D, q_n) - \int_S (u - v) c_1(D, q) \right| \\
&\quad \leqslant \left| \int_S (u_n - v_n) c_1(D, q_n) - \int_S (u - v) c_1(D, q_n) \right| \\
&\qquad\qquad + \left| \int_S (u - v) c_1(D, q_n) - \int_S (u - v) c_1(D, q) \right| \\
&\quad \leqslant \|(u - v) - (u_n - v_n)\|_{\sup} C + \left| \int_S (u - v) c_1(D, q_n) - \int_S (u - v) c_1(D, q) \right|.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \int_S (u_n - v_n) c_1(D, q_n) = \int_S (u - v) c_1(D, q),$$

and hence the assertion follows. □

*Proof of Theorem* 1.1. — Let us start the proof of Theorem 1.1. First we consider the existence of the greatest element of $\Upsilon_{rel}(\overline{D})$. By Lemma 1.2, there is the greatest element $Q$ of $\Sigma(D)$. Note that $D - Q$ is vertical. On the other hand, let $G(\overline{D})$ be the set of all $D$-Green functions $h$ of $C^0$-type such that $c_1(D, h)$ is a positive current and $h \leqslant g$ (a.e.). By Lemma 1.7, there is the greatest element $q$ of $G(\overline{D})$, that is, $q \in G(\overline{D})$ and $h \leqslant q$ (a.e.) for all $h \in G(\overline{D})$. Let us see that $q$ is $F_\infty$-invariant. For this purpose, it is sufficient to see that $F_\infty^*(q) \in G(\overline{D})$ and $h \leqslant F_\infty^*(q)$ (a.e.) for all $h \in G(\overline{D})$. The first assertion follows from [13, Lemma 5.1.2]. Let us see the second assertion. Since $F_\infty^*(h) \in G(\overline{D})$ by [13, Lemma 5.1.2], $F_\infty^*(h) \leqslant$

$q$ (a.e.), and hence $h \leqslant F_\infty^*(q)$ (a.e.). Here we set $\overline{Q} := (Q, q)$. Clearly $\overline{Q} \in \Upsilon_{rel}(\overline{D})$. Moreover, for $\overline{M} \in \Upsilon_{rel}(\overline{D})$, $(M', h') := \max\{\overline{Q}, \overline{M}\} \in \Upsilon_{rel}(\overline{D})$ by Claim 1.4 and [13, Lemma 9.1.1] (for the definition of $\max\{\overline{Q}, \overline{M}\}$, see Conventions and terminology 6). In particular, $M' \in \Sigma(D)$ and $h' \in G(\overline{D})$, and hence $(M', h') = \overline{Q}$, that is, $\overline{M} \leqslant \overline{Q}$, as required.

Finally let us see (a) — (e). As $Q$ is the greatest element of $\Sigma(D)$, (a), (c) and the first assertion of (d) follow from Lemma 1.2. The second assertion of (d) follows from (1) in Lemma 1.7. The property (e) is a consequence of (d). Finally we consider (b). If we set $\overline{N} = (N, k)$, then $\widehat{\deg}(\overline{Q} \cdot (N, 0)) = 0$ by (b) in Lemma 1.2, and $\widehat{\deg}(\overline{Q} \cdot (0, k)) = 0$ by (2) in Lemma 1.7, and hence $\widehat{\deg}(\overline{Q} \cdot \overline{N}) = 0$. □

## 2. Generalized Hodge index theorem for $\widehat{\mathrm{vol}}_\chi$

In this section, we consider a refinement of the generalized Hodge index theorem on an arithmetic surface, that is, the case where $d = 1$. As in Conventions and terminology 5, an arithmetic $\mathbb{R}$-Cartier divisor $\overline{D}$ of $C^0$-type on $X$ is said to be *integrable* if $\overline{D} = \overline{P} - \overline{Q}$ for some nef arithmetic $\mathbb{R}$-Cartier divisors $\overline{P}$ and $\overline{Q}$ of $C^0$-type.

THEOREM 2.1. — *Let $\overline{D}$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$ such that $\deg(D_K) \geqslant 0$. Then $\widehat{\deg}(\overline{D}^2) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{D})$ and the equality holds if and only if $\overline{D}$ is relatively nef. In particular, $\widehat{\deg}(\overline{D}^2) \leqslant \widehat{\mathrm{vol}}(\overline{D})$.*

*Proof.* — Let $\mu : X' \to X$ be a desingularization of $X$ (cf. [11]). Then $\widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\mu^*(\overline{D})^2)$ and $\widehat{\mathrm{vol}}_\chi(\overline{D}) = \widehat{\mathrm{vol}}_\chi(\mu^*(\overline{D}))$. Moreover, $\overline{D}$ is relatively nef if and only if $\mu^*(\overline{D})$ is relatively nef. Therefore we may assume that $X$ is regular.

CLAIM 2.2. — *If $\overline{D}$ is relatively nef, then $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}_\chi(\overline{D})$.*

*Proof.* — We divide the proof into five steps:

**Step 1** (the case where $\overline{D}$ is an arithmetic $\mathbb{Q}$-Cartier divisor of $C^\infty$-type and $c_1(\overline{D})$ is a semi-positive form) : In this case, the assertion follows from Ikoma [9, Theorem 3.5.1].

**Step 2** (the case where $\overline{D}$ is of $C^\infty$-type, $c_1(\overline{D})$ is a positive form and $\widehat{\deg}(\overline{D}\big|_C) > 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$) : We choose arithmetic Cartier divisors $\overline{D}_1, \ldots, \overline{D}_l$ of $C^\infty$-type and real numbers $a_1, \ldots, a_l$ such that $\overline{D} = a_1\overline{D}_1 + \cdots + a_l\overline{D}_l$. Then there is a positive number $\delta_0$ such that $c_1(b_1\overline{D}_1 + \cdots + b_l\overline{D}_l)$ is a positive form for all $b_1, \ldots, b_l \in \mathbb{Q}$ with $|b_i - a_i| \leqslant \delta_0$ ($\forall i = 1, \ldots, l$). Let $C$ be a smooth fiber of $X \to \operatorname{Spec}(O_K)$ over $P$. Then, for $b_1, \ldots, b_l \in \mathbb{Q}$ with $|b_i - a_i| \leqslant \delta_0$ ($\forall i = 1, \ldots, l$),

$$\widehat{\deg}\left( (b_1\overline{D}_1 + \cdots + b_l\overline{D}_l)\big|_C \right) = \deg((b_1D_1+\cdots+b_lD_l)_K) \log \#(O_K/P) > 0.$$

Let $C_1, \ldots, C_r$ be all irreducible components of singular fibers of $X \to \operatorname{Spec}(O_K)$. Then, for each $j = 1, \ldots, r$, there is a positive number $\delta_j$ such that

$$\widehat{\deg}\left( (b_1\overline{D}_1 + \cdots + b_l\overline{D}_l)\big|_{C_j} \right) > 0$$

for all $b_1, \ldots, b_l \in \mathbb{Q}$ with $|b_i - a_i| \leqslant \delta_j$ ($\forall i = 1, \ldots, l$). Therefore, if we set $\delta = \min\{\delta_0, \delta_1, \ldots, \delta_r\}$, then, for $b_1, \ldots, b_l \in \mathbb{Q}$ with $|b_i - a_i| \leqslant \delta$ ($\forall i = 1, \ldots, l$),

$$c_1(b_1\overline{D}_1 + \cdots + b_l\overline{D}_l)$$

is a positive form and $\widehat{\deg}\left( (b_1\overline{D}_1 + \cdots + b_l\overline{D}_l)\big|_C \right) > 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$ on $X$, and hence

$$\widehat{\deg}((b_1\overline{D}_1 + \cdots + b_l\overline{D}_l)^2) = \widehat{\operatorname{vol}}_\chi(b_1\overline{D}_1 + \cdots + b_l\overline{D}_l)$$

by Step 1. Thus the assertion follows by the continuity of $\widehat{\operatorname{vol}}_\chi$ due to Ikoma [9, Corollary 3.4.4].

**Step 3** (the case where $\overline{D}$ is of $C^\infty$-type and $c_1(\overline{D})$ is a semi-positive form) : Let $\overline{A}$ be an ample arithmetic Cartier divisor of $C^\infty$-type on $X$. Then, for any positive $\epsilon$, $c_1(\overline{D}+\epsilon\overline{A})$ is a positive form and $\widehat{\deg}((\overline{D} + \epsilon\overline{A})\big|_C) > 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$ on $X$, so that, by Step 2,

$$\widehat{\deg}((\overline{D} + \epsilon\overline{A})^2) = \widehat{\operatorname{vol}}_\chi(\overline{D} + \epsilon\overline{A}).$$

Therefore the assertion follows by virtue of the continuity of $\widehat{\operatorname{vol}}_\chi$.

**Step 4** (the case where $\deg(D_K) > 0$) : Let $h$ be a $D$-Green function of $C^\infty$-type such that $c_1(D, h)$ is a positive form. Then there is a continuous function $\phi$ on $X(\mathbb{C})$ such that $\overline{D} = (D, h+\phi)$, and hence $c_1(D, h)+dd^c([\phi]) \geqslant 0$. Thus, by [13, Lemma 4.2], there is a sequence $\{\phi_n\}_{n=1}^\infty$ of $F_\infty$-invariant $C^\infty$-functions on $X(\mathbb{C})$ with the following properties:

(a) $\lim_{n\to\infty} \|\phi_n - \phi\|_{\sup} = 0$.

(b) If we set $\overline{D}_n = (D, h + \phi_n)$, then $c_1(\overline{D}_n)$ is a semipositive form.

Then, by Step 3, $\widehat{\deg}(\overline{D}_n^2) = \widehat{\mathrm{vol}}_\chi(\overline{D}_n)$ for all $n$. Note that $\lim_{n\to\infty} \widehat{\mathrm{vol}}_\chi(\overline{D}_n) = \widehat{\mathrm{vol}}_\chi(\overline{D})$ by using the continuity of $\widehat{\mathrm{vol}}_\chi$. Moreover, by [15, Lemma 1.2.1],

$$\lim_{n\to\infty} \widehat{\deg}(\overline{D}_n^2) = \widehat{\deg}(\overline{D}^2),$$

as required.

**Step 5** (general case) : Finally we prove the assertion of the claim. As before, let $\overline{A}$ be an ample arithmetic Cartier divisor of $C^\infty$-type on $X$. Then, for any positive number $\epsilon$, $\deg(D_K + \epsilon A_K) > 0$. Thus, in the same way as Step 3, the assertion follows from Step 4. □

Let us go back to the proof of the theorem. Let $\overline{Q}$ be the greatest element of $\Upsilon_{rel}(\overline{D})$ (cf. Theorem 1.1) and $\overline{N} := \overline{D} - \overline{Q}$. Then, by using Claim 2.2 and the properties (b) and (e) in Theorem 1.1,

$$\widehat{\mathrm{vol}}_\chi(\overline{D}) - \widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}_\chi(\overline{Q}) - \widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\overline{Q}^2) - \widehat{\deg}(\overline{D}^2) = -\widehat{\deg}(\overline{N}^2).$$

On the other hand, if we set $\overline{N} = (N, k)$, then

$$\widehat{\deg}(\overline{N}^2) = \widehat{\deg}((N,0)^2) + \frac{1}{2}\int_{X(\mathbb{C})} k dd^c(k)$$

because $N$ is vertical. By (c) in Theorem 1.1 together with Zariski's lemma, $\widehat{\deg}((N,0)^2) \leqslant 0$ and the equality holds if and only if $N = 0$. Moreover, by [15, Proposition 1.2.3 and Proposition 2.1.1],

$$\int_{X(\mathbb{C})} k dd^c(k) \leqslant 0$$

and the equality holds if and only if $k$ is locally constant. Thus $\widehat{\deg}(\overline{N}^2) \leqslant 0$, that is, $\widehat{\mathrm{vol}}_\chi(\overline{D}) \geqslant \widehat{\deg}(\overline{D}^2)$. Moreover, if $\overline{D}$ is relatively nef, then $\widehat{\mathrm{vol}}_\chi(\overline{D}) = \widehat{\deg}(\overline{D}^2)$ by Claim 2.2. Conversely, if $\widehat{\mathrm{vol}}_\chi(\overline{D}) = \widehat{\deg}(\overline{D}^2)$, that is, $\widehat{\deg}(\overline{N}^2) = 0$, then $N = 0$ and $k$ is locally constant, and hence $\overline{D} = \overline{Q} + (0,k)$ is relatively nef. □

As a corollary of the above theorem, we have the following:

Corollary 2.3. — *We assume that $X$ is regular. The following are equivalent:*

*(1) $\overline{Q}$ is the greatest element of $\Upsilon_{rel}(\overline{D})$.*

*(2) $\overline{Q}$ is an element of $\Upsilon_{rel}(\overline{D})$ with the following properties:*

(i) $D - Q$ *is vertical.*

(ii) $\widehat{\deg}(\overline{Q} \cdot \overline{B}) = 0$ *and* $\widehat{\deg}(\overline{B}^2) < 0$ *for all integrable arithmetic* $\mathbb{R}$*-Cartier divisors* $\overline{B}$ *of* $C^0$*-type with* $(0,0) \lneqq \overline{B} \leqslant \overline{D} - \overline{Q}$.

*Proof.* — First, let us see the following claim:

Claim 2.4. — *Let* $\overline{D}_1$ *and* $\overline{D}_2$ *be arithmetic* $\mathbb{R}$*-Cartier divisors of* $C^0$*-type on* $X$ *such that* $\overline{D}_1 \leqslant \overline{D}_2$*. If the natural map* $H^0(X, nD_1) \to H^0(X, nD_2)$ *is bijective for each* $n \geqslant 0$*, then* $\widehat{\mathrm{vol}}_\chi(\overline{D}_1) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{D}_2)$*,*

*Proof.* — This is obvious because $\|\cdot\|_{n\overline{D}_1} \geqslant \|\cdot\|_{n\overline{D}_2}$. □

(1) $\Longrightarrow$ (2) : By the property (a) in Theorem 1.1, $D - Q$ is vertical. For $0 < \epsilon \leqslant 1$, we set $\overline{D}_\epsilon = \overline{Q} + \epsilon\overline{B}$. Then $\overline{D}_\epsilon$ is integrable and $\widehat{\mathrm{vol}}_\chi(\overline{D}_\epsilon) = \widehat{\mathrm{vol}}_\chi(\overline{Q})$ because

$$\widehat{\mathrm{vol}}_\chi(\overline{Q}) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{D}_\epsilon) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{D}) \quad \text{and} \quad \widehat{\mathrm{vol}}_\chi(\overline{Q}) = \widehat{\mathrm{vol}}_\chi(\overline{D})$$

by Claim 2.4 and the properties (d) and (e) in Theorem 1.1. Thus, by using Theorem 2.1,

$$\widehat{\deg}(\overline{D}_\epsilon^2) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{D}_\epsilon) = \widehat{\mathrm{vol}}_\chi(\overline{Q}) = \widehat{\deg}(\overline{Q}^2),$$

which implies $2\widehat{\deg}(\overline{Q} \cdot \overline{B}) + \epsilon\widehat{\deg}(\overline{B}^2) \leqslant 0$. In particular, $\widehat{\deg}(\overline{Q} \cdot \overline{B}) \leqslant 0$. On the other hand, as $B$ is vertical,

$$\widehat{\deg}(\overline{Q} \cdot \overline{B}) = \widehat{\deg}(\overline{Q} \cdot (B, 0)) + \frac{1}{2}\int_{X(\mathbb{C})} c_1(\overline{Q}) b \geqslant 0$$

where $\overline{B} = (B, b)$. Therefore, $\widehat{\deg}(\overline{Q} \cdot \overline{B}) = 0$ and $\widehat{\deg}(\overline{B}^2) \leqslant 0$. Here we assume that $\widehat{\deg}(\overline{B}^2) = 0$. Note that

$$\widehat{\deg}(\overline{B}^2) = \widehat{\deg}((B, 0)^2) + \frac{1}{2}\int_{X(\mathbb{C})} b dd^c(b).$$

Thus, by using the property (c) in Theorem 1.1, Zariski's lemma and [15, Proposition 1.2.3 and Proposition 2.1.1], $B = 0$ and $b$ is a locally constant function. In particular, $\overline{Q} + \overline{B}$ is relatively nef and $\overline{Q} + \overline{B} \leqslant \overline{D}$, so that $\overline{B} = 0$.

(2) $\Longrightarrow$ (1) : Let $\overline{M}$ be an element of $\Upsilon_{rel}(\overline{D})$. If we set $\overline{A} := \max\{\overline{Q}, \overline{M}\}$ (cf. Conventions and terminology 6) and $\overline{B} = (B, b) := \overline{A} - \overline{Q}$, then $\overline{B}$ is effective, $\overline{A} \leqslant \overline{D}$ and $\overline{A}$ is relatively nef by Claim 1.4 and [13, Lemma 9.1.2]. Moreover,

$$\overline{B} = \overline{A} - \overline{Q} \leqslant \overline{D} - \overline{Q}.$$

If we assume $\overline{B} \gneqq (0,0)$, then, by the property (ii), $\widehat{\deg}(\overline{Q} \cdot \overline{B}) = 0$ and $\widehat{\deg}(\overline{B}^2) < 0$. On the other hand, as $\overline{A}$ is relatively nef, $\overline{B}$ is effective and $B$ is vertical by the property (i),

$$\widehat{\deg}(\overline{B}^2) = \widehat{\deg}(\overline{Q}+\overline{B}\cdot\overline{B}) = \widehat{\deg}(\overline{A}\cdot\overline{B}) = \widehat{\deg}(\overline{A}\cdot(B,0)) + \frac{1}{2}\int_{X(\mathbb{C})} c_1(\overline{A})b \geqslant 0,$$

which is a contradiction, so that $\overline{B} = (0,0)$, that is, $\overline{Q} = \overline{A}$, which means that $\overline{M} \leqslant \overline{Q}$, as required. □

*Remark 2.5.* — Let $\overline{D}$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$ with $\deg(D_K) > 0$. For a positive number $\epsilon$, we set

$$\alpha := \frac{\widehat{\deg}(\overline{D}^2)}{[K:\mathbb{Q}]\deg(D_K)} - 2\epsilon.$$

Then, as $\widehat{\deg}((\overline{D} - (0,\alpha))^2) = 2\epsilon[K:\mathbb{Q}]\deg(D_K) > 0$, by Theorem 2.1, there is

$$\phi \in \hat{H}^0(X, n(D - (0,\alpha))) \setminus \{0\}$$

for some $n > 0$. Note that $\|\phi\|_{n(\overline{D}-(0,\alpha))} = \|\phi\|_{n\overline{D}} \exp((n\alpha)/2)$, so that

$$\phi \in H^0(X, nD) \setminus \{0\} \quad \text{and} \quad \|\phi\|_{n\overline{D}} \leqslant \exp\left(-\frac{n\widehat{\deg}(\overline{D}^2)}{2[K:\mathbb{Q}]\deg(D_K)} + n\epsilon\right),$$

which is nothing more than Autissier's result [2, Proposition 3.3.3].

*Remark 2.6.* — The referee points out that Step 1 of Claim 2.2 can be proved by using Randriambololona's version of the arithmetic Hilbert-Samuel formula [17].

## 3. Necessary condition for the equality $\widehat{\mathrm{vol}} = \widehat{\mathrm{vol}}_\chi$

This section is devoted to consider a necessary condition for the equality $\widehat{\mathrm{vol}} = \widehat{\mathrm{vol}}_\chi$ as an application of the integral formulae due to Boucksom-Chen [4].

First of all, let us review Boucksom-Chen's integral formulae [4] in terms of arithmetic $\mathbb{R}$-Cartier divisors. For details, see [16, Section 1]. We fix a monomial order $\precsim$ on $\mathbb{Z}^d_{\geqslant 0}$, that is, $\precsim$ is a total ordering relation on $\mathbb{Z}^d_{\geqslant 0}$ with the following properties:

(a) $(0,\ldots,0) \precsim A$ for all $A \in \mathbb{Z}^d_{\geqslant 0}$.

(b) If $A \precsim B$ for $A, B \in \mathbb{Z}^d_{\geqslant 0}$, then $A + C \precsim B + C$ for all $C \in \mathbb{Z}^d_{\geqslant 0}$.

The monomial order $\precsim$ on $\mathbb{Z}^d_{\geqslant 0}$ extends uniquely to a totally ordering relation $\precsim$ on $\mathbb{Z}^d$ such that $A+C \precsim B+C$ for all $A, B, C \in \mathbb{Z}^d$ with $A \precsim B$. Indeed, for $A, B \in \mathbb{Z}^d$, we define $A \precsim B$ as follows:

$$A \precsim B \quad \overset{\text{def}}{\iff} \quad \text{there is } C \in \mathbb{Z}^d_{\geqslant 0} \text{ such that } A+C, B+C \in \mathbb{Z}^d_{\geqslant 0} \text{ and } A+C \precsim B+C.$$

It is easy to see that this definition is well-defined and it yields the above extension. Uniqueness is also obvious.

As $X \to \operatorname{Spec}(O_K)$ is the Sten factorization of $X \to \operatorname{Spec}(\mathbb{Z})$, $X_K$ is geometrically integral over $K$. Let $\overline{K}$ be an algebraic closure of $K$ and $X_{\overline{K}} := X \times_{\operatorname{Spec}(K)} \operatorname{Spec}(\overline{K})$. Let $z_P = (z_1, \ldots, z_d)$ be a local system of parameters of $\mathcal{O}_{X_{\overline{K}},P}$ for $P \in X(\overline{K})$. Note that the completion $\widehat{\mathcal{O}}_{X_{\overline{K}},P}$ of $\mathcal{O}_{X_{\overline{K}},P}$ with respect to the maximal ideal of $\mathcal{O}_{X_{\overline{K}},P}$ is naturally isomorphic to $\overline{K}[\![z_1, \ldots, z_d]\!]$. Thus, for $f \in \mathcal{O}_{X_{\overline{K}},P}$, we can put

$$f = \sum_{(a_1,\ldots,a_d)\in\mathbb{Z}^d_{\geqslant 0}} c_{(a_1,\ldots,a_d)} z_1^{a_1} \cdots z_d^{a_d}, \qquad (c_{(a_1,\ldots,a_d)} \in \overline{K}).$$

We define $\operatorname{ord}^{\precsim}_{z_P}(f)$ to be

$$\operatorname{ord}^{\precsim}_{z_P}(f) := \begin{cases} \min_{\precsim} \left\{ (a_1, \ldots, a_d) \mid c_{(a_1,\ldots,a_d)} \neq 0 \right\} & \text{if } f \neq 0, \\ \infty & \text{otherwise,} \end{cases}$$

which gives rise to a rank $d$ valuation, that is, the following properties are satisfied:

(i) $\operatorname{ord}^{\precsim}_{z_P}(fg) = \operatorname{ord}^{\precsim}_{z_P}(f) + \operatorname{ord}^{\precsim}_{z_P}(g)$ for $f, g \in \mathcal{O}_{X_{\overline{K}},P}$.

(ii) $\min\left\{\operatorname{ord}^{\precsim}_{z_P}(f), \operatorname{ord}^{\precsim}_{z_P}(g)\right\} \precsim \operatorname{ord}^{\precsim}_{z_P}(f+g)$ for $f, g \in \mathcal{O}_{X_{\overline{K}},P}$.

By the property (i), $\operatorname{ord}^{\precsim}_{z_P} : \mathcal{O}_{X_{\overline{K}},P} \setminus \{0\} \to \mathbb{Z}^d_{\geqslant 0}$ has the natural extension

$$\operatorname{ord}^{\precsim}_{z_P} : \operatorname{Rat}(X_{\overline{K}})^{\times} \to \mathbb{Z}^d$$

given by $\operatorname{ord}^{\precsim}_{z_P}(f/g) = \operatorname{ord}^{\precsim}_{z_P}(f) - \operatorname{ord}^{\precsim}_{z_P}(g)$. Note that this extension also satisfies the same properties (i) and (ii) as before. Since $\operatorname{ord}^{\precsim}_{z_P}(u) = (0, \ldots, 0)$ for all $u \in \mathcal{O}^{\times}_{X_{\overline{K}},P}$, $\operatorname{ord}^{\precsim}_{z_P}$ induces $\operatorname{Rat}(X_{\overline{K}})^{\times}/\mathcal{O}^{\times}_{X_{\overline{K}},P} \to \mathbb{Z}^d$. The composition of homomorphisms

$$\operatorname{Div}(X_{\overline{K}}) \xrightarrow{\alpha_P} \operatorname{Rat}^{\times}(X_{\overline{K}})/\mathcal{O}^{\times}_{X_{\overline{K}},P} \xrightarrow{\operatorname{ord}^{\precsim}_{z_P}} \mathbb{Z}^d$$

is denoted by $\operatorname{mult}_{z_P}^{\precsim}$, where $\alpha_P : \operatorname{Div}(X_{\overline{K}}) \to \operatorname{Rat}(X_{\overline{K}})^{\times}/\mathcal{O}_{X_{\overline{K}},P}^{\times}$ is the natural homomorphism. Moreover, the homomorphism $\operatorname{mult}_{z_P}^{\precsim} : \operatorname{Div}(X_{\overline{K}}) \to \mathbb{Z}^d$ gives rise to the natural extension $\operatorname{Div}(X_{\overline{K}}) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^d$ over $\mathbb{R}$. By abuse of notation, the above extension is also denoted by $\operatorname{mult}_{z_P}^{\precsim}$.

Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type (cf. Conventions and terminology 2). Let $V_\bullet = \bigoplus_{m \geqslant 0} V_m$ be a graded subalgebra of $R(D_K) := \bigoplus_{m \geqslant 0} H^0(X_K, mD_K)$ over $K$. The Okounkov body $\Delta(V_\bullet)$ of $V_\bullet$ is defined by the closed convex hull of

$$\bigcup_{m>0} \left\{ \operatorname{mult}_{z_P}^{\precsim}(D_{\overline{K}} + (1/m)(\phi)) \in \mathbb{R}^d_{\geqslant 0} \mid \phi \in V_m \otimes_K \overline{K} \setminus \{0\} \right\}.$$

For $t \in \mathbb{R}$, let $V_\bullet^t$ be a graded subalgebra of $V_\bullet$ given by

$$V_\bullet^t := \bigoplus_{m \geqslant 0} \left\langle V_m \cap \hat{H}^0(X, m(\overline{D} + (0, -2t))) \right\rangle_K,$$

where $\left\langle V_m \cap \hat{H}^0(X, m(\overline{D} + (0, -2t))) \right\rangle_K$ means the subspace of $V_m$ generated by $V_m \cap \hat{H}^0(X, m(\overline{D} + (0, -2t)))$ over $K$. Let $G_{(\overline{D};V_\bullet)} : \Delta(V_\bullet) \to \mathbb{R} \cup \{-\infty\}$ be a function given by

$$G_{(\overline{D};V_\bullet)}(x) := \begin{cases} \sup\{t \in \mathbb{R} \mid x \in \Delta(V_\bullet^t)\} & \text{if } x \in \Delta(V_\bullet^t) \text{ for some } t, \\ -\infty & \text{otherwise.} \end{cases}$$

Note that $G_{(\overline{D};V_\bullet)}$ is an upper semicontinuous concave function (cf. [4, SubSection 1.3]). We define $\widehat{\operatorname{vol}}(\overline{D}; V_\bullet)$ and $\widehat{\operatorname{vol}}_\chi(\overline{D}; V_\bullet)$ to be

$$\begin{cases} \widehat{\operatorname{vol}}(\overline{D}; V_\bullet) := \limsup_{m\to\infty} \dfrac{\# \log \left( V_m \cap \hat{H}^0(X, m\overline{D}) \right)}{m^{d+1}/(d+1)!}, \\ \widehat{\operatorname{vol}}_\chi(\overline{D}; V_\bullet) := \limsup_{m\to\infty} \dfrac{\hat{\chi}\left(V_m \cap H^0(X, mD), \|\cdot\|_{m\overline{D}}\right)}{m^{d+1}/(d+1)!}. \end{cases}$$

Moreover, for $\xi \in X_K$, we define $\mu_{\mathbb{Q},\xi}(\overline{D}; V_\bullet)$ as follows:

$$\mu_{\mathbb{Q},\xi}(\overline{D}; V_\bullet) := \begin{cases} \inf\left\{ \operatorname{mult}_\xi \left(D + \frac{1}{m}(\phi)\right) \mid m \in \mathbb{Z}_{>0}, \phi \in V_m \cap \hat{H}^0(X, m\overline{D}) \setminus \{0\} \right\} & \text{if } N(\overline{D}; V_\bullet) \neq \emptyset, \\ \infty & \text{otherwise,} \end{cases}$$

where $N(\overline{D}; V_\bullet) = \{m \in \mathbb{Z}_{>0} \mid V_m \cap \hat{H}^0(X, m\overline{D}) \neq \{0\}\}$. Note that $\widehat{\operatorname{vol}}(\overline{D}; V_\bullet) = \widehat{\operatorname{vol}}(\overline{D})$, $\widehat{\operatorname{vol}}_\chi(\overline{D}; V_\bullet) = \widehat{\operatorname{vol}}_\chi(\overline{D})$ and $\mu_{\mathbb{Q},\xi}(\overline{D}; V_\bullet) = \mu_{\mathbb{Q},\xi}(\overline{D})$

if $V_m = H^0(X_K, mD_K)$ for $m \gg 0$ (cf. Conventions and terminology 2 and 3). Let $\Theta(\overline{D}; V_\bullet)$ be the closure of

$$\left\{x \in \Delta(V_\bullet) \mid G_{(\overline{D};V_\bullet)}(x) > 0\right\}.$$

If $V_\bullet$ contains an ample series (cf. [16, SubSection 1.1]), then, in the similar way as [4, Theorem 2.8] and [4, Theorem 3.1], we have the following integral formulae:

$$\widehat{\mathrm{vol}}(\overline{D}; V_\bullet) = (d+1)![K:\mathbb{Q}] \int_{\Theta(\overline{D};V_\bullet)} G_{(\overline{D};V_\bullet)}(x)dx \tag{3.1}$$

and

$$\widehat{\mathrm{vol}}_\chi(\overline{D}; V_\bullet) = (d+1)![K:\mathbb{Q}] \int_{\Delta(V_\bullet)} G_{(\overline{D};V_\bullet)}(x)dx. \tag{3.2}$$

Note that the arguments in [4] work for an arbitrary monomial order. The boundedness of the Okounkov body is not obvious for an arbitrary monomial order. It can be checked by Theorem C.1. Let $\nu : \mathbb{R}^d \to \mathbb{R}$ be a linear map. If we give the monomial order $\prec_\nu$ on $\mathbb{Z}^d_{\geqslant 0}$ by the following rule:

$$a \prec_\nu b \quad \overset{\text{def}}{\Longleftrightarrow} \quad \text{either } \nu(a) < \nu(b), \text{ or } \nu(a) = \nu(b) \text{ and } a \prec_{\mathrm{lex}} b,$$

then $\nu(a) \leqslant \nu(b)$ for all $a, b \in \mathbb{Z}^d_{\geqslant 0}$ with $a \precsim_\nu b$. Let us begin with the following lemma.

Lemma 3.3. — *If $V_\bullet$ contains an ample series and $\widehat{\mathrm{vol}}(\overline{D}; V_\bullet) > 0$, then we have the following:*

*(1)* $\Theta(\overline{D}; V_\bullet) = \Delta(V^0_\bullet) = \left\{x \in \Delta(V_\bullet) \mid G_{(\overline{D};V_\bullet)}(x) \geqslant 0\right\}$.

*(2) We assume that $\nu$ is given by $\nu(x_1, \ldots, x_d) = x_1 + \cdots + x_r$, where $1 \leqslant r \leqslant d$. We further assume that the monomial order $\precsim$ satisfies the property: $\nu(a) \leqslant \nu(b)$ for all $a, b \in \mathbb{Z}^d_{\geqslant 0}$ with $a \precsim b$. Let $B$ is a reduced and irreducible subvariety of $X_{\overline{K}}$ such that $B$ is given by $z_1 = \cdots = z_r = 0$ around $P$. Then $\mu_{\mathbb{Q},B}(\overline{D}; V_\bullet) = \min\left\{\nu(x) \mid x \in \Theta(\overline{D}; V_\bullet)\right\}$.*

*Proof.* — (1) Note that

$$\left\{x \in \Delta(V_\bullet) \mid G_{(\overline{D};V_\bullet)}(x) > 0\right\} \subseteq \Delta(V^0_\bullet) \subseteq \left\{x \in \Delta(V_\bullet) \mid G_{(\overline{D};V_\bullet)}(x) \geqslant 0\right\}$$

and $\left\{x \in \Delta(V_\bullet) \mid G_{(\overline{D};V_\bullet)}(x) \geqslant 0\right\}$ is closed because $G_{(\overline{D};V_\bullet)}$ is upper semicontinuous. Thus it is sufficient to show that $\left\{x \in \Delta(V_\bullet) \mid G_{(\overline{D};V_\bullet)}(x) \geqslant 0\right\} \subseteq$

$\Theta(\overline{D};V_\bullet)$. Let $x \in \Delta(V_\bullet)$ with $G_{(\overline{D};V_\bullet)}(x) \geqslant 0$. As

$$\widehat{\mathrm{vol}}(\overline{D};V_\bullet) = (d+1)![K:\mathbb{Q}]\int_{\Theta(\overline{D};V_\bullet)} G_{(\overline{D};V_\bullet)}(x)dx > 0$$

by (3.1), we can choose $y \in \Theta(\overline{D};V_\bullet)$ with $G_{(\overline{D};V_\bullet)}(y) > 0$. Then

$$G_{(\overline{D};V_\bullet)}((1-t)x+ty) \geqslant (1-t)G_{(\overline{D};V_\bullet)}(x) + tG_{(\overline{D};V_\bullet)}(y) \geqslant tG_{(\overline{D};V_\bullet)}(y) > 0$$

for all $t \in \mathbb{R}$ with $0 < t \leqslant 1$. Thus $x \in \Theta(\overline{D};V_\bullet)$.

(2) First let us see the following claim:

Claim 3.4. — *For $L \in \mathrm{Div}(X)_{\mathbb{R}}$, $\nu\left(\mathrm{mult}_{z_P}^{\precsim}(L)\right) = \mathrm{mult}_B(L)$.*

*Proof.* — It is sufficient to see that $\nu\left(\mathrm{ord}_{z_P}^{\precsim}(f)\right) = \mathrm{ord}_B(f)$ for $f \in \mathcal{O}_{X_{\overline{K}}}\setminus\{0\}$. We set $f = \sum_{\beta\in\mathbb{Z}^d_{\geqslant 0}} c_\beta z^\beta$ and $\alpha = \mathrm{ord}_{z_P}^{\precsim}(f)$. Note that $\mathrm{ord}_B(f) = \min\{\nu(\beta) \mid c_\beta \neq 0\}$. Thus the assertion follows because $c_\alpha \neq 0$ and $\nu(\alpha) \leqslant \nu(\beta)$ for $\beta \in \mathbb{Z}^d_{\geqslant 0}$ with $c_\beta \neq 0$. □

If we set

$$x_\phi = \mathrm{mult}_{z_P}^{\precsim}(D + (1/m)(\phi))$$

for $\phi \in V_m \cap \hat{H}^0(X, m\overline{D}) \setminus \{0\}$ and $m > 0$, then $G_{(\overline{D};V_\bullet)}(x_\phi) \geqslant 0$ by the definition of $G_{(\overline{D};V_\bullet)}$, and hence, $x_\phi \in \Theta(\overline{D};V_\bullet)$ by (1). Therefore, by Claim 3.4,

$$\min\{\nu(x) \mid x \in \Theta(\overline{D};V_\bullet)\} \leqslant \nu(x_\phi) = \mathrm{mult}_B(D + (1/m)(\phi)),$$

which implies $\min\{\nu(x) \mid x \in \Theta(\overline{D};V_\bullet)\} \leqslant \mu_{\mathbb{Q},B}(\overline{D};V_\bullet)$.

Claim 3.5. —

$$\mu_{\mathbb{Q},B}(\overline{D};V_\bullet) \leqslant \nu\left(\mathrm{mult}_{z_P}^{\precsim}\left(D + (1/m)\left(\sum_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}} c_\phi\phi\right)\right)\right),$$

*where $c_\phi \in \overline{K}$ and $\sum_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}} c_\phi\phi \neq 0$.*

*Proof.* — By the property (ii),

$$\min_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}}\left\{\mathrm{ord}_{z_P}^{\precsim}(\phi)\right\} \precsim \mathrm{ord}_{z_P}^{\precsim}\left(\sum_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}} c_\phi\phi\right)$$

on $\mathbb{Z}^d$, which yields

$$\min_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}}\left\{\nu\left(\operatorname{ord}_{z_P}^{\precsim}(\phi)\right)\right\}\leqslant\nu\left(\operatorname{ord}_{z_P}^{\precsim}\left(\sum_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}}c_\phi\phi\right)\right),$$

and hence

$$\begin{aligned}\min_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}}&\left\{\nu\left(\operatorname{mult}_{z_P}^{\precsim}(D+(1/m)(\phi))\right)\right\}\\ &\leqslant\nu\left(\operatorname{mult}_{z_P}^{\precsim}\left(D+(1/m)\left(\sum_{\phi\in V_m\cap\hat{H}^0(X,m\overline{D})\setminus\{0\}}c_\phi\phi\right)\right)\right).\end{aligned}$$

Thus the claim follows by Claim 3.4. □

By the above claim together with (1),

$$\Theta(\overline{D};V_\bullet)=\Delta(V_\bullet^0)\subseteq\{x\in\Delta(V_\bullet)\mid\mu_{\mathbb{Q},B}(\overline{D};V_\bullet)\leqslant\nu(x)\},$$

which shows that $\min\{\nu(x)\mid x\in\Theta(\overline{D};V_\bullet)\}\geqslant\mu_{\mathbb{Q},B}(\overline{D};V_\bullet)$, as required. □

The following theorem is the main result of this section.

Theorem 3.6. — *If $V_\bullet$ contains an ample series, $\widehat{\operatorname{vol}}(\overline{D};V_\bullet)=\widehat{\operatorname{vol}}_\chi(\overline{D};V_\bullet)>0$ and*

$$\inf\left\{\operatorname{mult}_\xi(D+(1/m)(\phi))\mid m\in\mathbb{Z}_{>0},\ \phi\in V_m\setminus\{0\}\right\}=0$$

*for $\xi\in X_K$, then $\mu_{\mathbb{Q},\xi}(\overline{D};V_\bullet)=0$.*

*Proof.* — First let us consider the following claim:

Claim 3.7. — $\Theta(\overline{D};V_\bullet)=\Delta(V_\bullet)$.

*Proof.* — It is sufficient to see that $\Delta(V_\bullet)^\circ\subseteq\left\{x\in\Delta(V_\bullet)\,|\,G_{(\overline{D};V_\bullet)}(x)\geqslant0\right\}$. We assume the contrary, that is, there is $y\in\Delta(V_\bullet)^\circ$ with $G_{(\overline{D};V_\bullet)}(y)<0$. Then, by using the upper semicontinuity of $G_{(\overline{D};V_\bullet)}$, we can find an open neighborhood $U$ of $y$ such that $U\subseteq\Delta(V_\bullet)^\circ$ and $G_{(\overline{D};V_\bullet)}(x)<0$ for all $x\in U$. Then, as $\Theta(\overline{D};V_\bullet)\subseteq\Delta(V_\bullet)\setminus U$, by the integral formulae of $\widehat{\operatorname{vol}}$ and $\widehat{\operatorname{vol}}_\chi$ (cf. (3.1), (3.2)) and (1) in Lemma 3.3,

$$\frac{\widehat{\operatorname{vol}}_\chi(\overline{D};V_\bullet)}{(d+1)![K:\mathbb{Q}]}=\int_{\Delta(V_\bullet)}G_{(\overline{D};V_\bullet)}(x)dx$$

$$
\begin{aligned}
&= \int_U G_{(\overline{D};V_\bullet)}(x)dx + \int_{\Delta(V_\bullet)\setminus U} G_{(\overline{D};V_\bullet)}(x)dx \\
&\quad < \int_{\Delta(V_\bullet)\setminus U} G_{(\overline{D};V_\bullet)}(x)dx \leqslant \int_{\Theta(\overline{D};V_\bullet)} G_{(\overline{D};V_\bullet)}(x)dx \\
&= \frac{\widehat{\mathrm{vol}}(\overline{D};V_\bullet)}{(d+1)![K:\mathbb{Q}]}.
\end{aligned}
$$

This is a contradiction. □

Let $B$ be the Zariski closure of $\{\xi\}$ in $X$. We choose $P \in X(\overline{K})$ and a local system of parameters $z_P = (z_1, \ldots, z_d)$ at $P$ such that $P$ is a regular point of $B_{\overline{K}}$ and $z_1 = \cdots = z_r = 0$ is a local equation of $B_{\overline{K}}$ at $P$. Let $\nu : \mathbb{R}^d \to \mathbb{R}$ be the linear map given by $\nu(x_1, \ldots, x_d) = x_1 + \cdots + x_r$. We also choose a monomial order $\precsim$ such that $\nu(a) \leqslant \nu(b)$ for all $a, b \in \mathbb{Z}^d_{\geqslant 0}$ with $a \precsim b$. By our assumption,

$$
\inf\{\mathrm{mult}_\xi(D + (1/m)(\phi)) \mid m \in \mathbb{Z}_{>0},\ \phi \in V_m \setminus \{0\}\} = 0.
$$

This means that $\min\{\nu(x) \mid x \in \Delta(V_\bullet)\} = 0$, and hence, by Claim 3.7 and (2) in Lemma 3.3,

$$
\mu_{\mathbb{Q},\xi}(\overline{D}; V_\bullet) = \min\{\nu(x) \mid x \in \Theta(\overline{D}; V_\bullet)\} = 0.
$$

□

Corollary 3.8. — *If $D_K$ is nef and big on the generic fiber $X_K$ and $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}_\chi(\overline{D}) > 0$, then $\mu_{\mathbb{Q},\xi}(\overline{D}) = 0$ for all $\xi \in X_K$.*

*Proof.* — As $D_K$ is nef and big, in the similar way as [13, Proposition 6.5.3], for any $\epsilon > 0$, there is $\phi \in \mathrm{Rat}(X_K)^\times_{\mathbb{Q}}$ such that

$$
D_K + (\phi)_{\mathbb{Q}} \geqslant 0 \quad \text{and} \quad \mathrm{mult}_\xi(D_K + (\phi)_{\mathbb{Q}}) < \epsilon,
$$

which means that

$$
\inf\left\{\mathrm{mult}_\xi(D + (1/m)(\phi)) \mid m \in \mathbb{Z}_{>0},\ \phi \in H^0(X_K, mD_K) \setminus \{0\}\right\} = 0.
$$

Thus the corollary follows from Theorem 3.6. □

## 4. Equality condition for the generalized Hodge index theorem

Here let us give the proof of the main theorem of this paper. We assume that $d = 1$. Let us begin with the following two lemmas.

Lemma 4.1. — *We assume that $X$ is regular. For an integrable arithmetic $\mathbb{R}$-Cartier divisor $\overline{D}$ of $C^0$-type on $X$ (cf. Conventions and terminology 5), we have the following:*

*(1) We assume that* $\deg(D_K) = 0$. *Then* $\widehat{\deg}(\overline{D}^2) = 0$ *if and only if* $\overline{D} = \widehat{(\psi)}_{\mathbb{R}} + (0, \lambda)$ *for some* $\psi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ *and* $\lambda \in \mathbb{R}$. *Moreover, if* $\widehat{\deg}(\overline{D}^2) = 0$ *and* $\overline{D}$ *is pseudo-effective, then* $\overline{D} = \widehat{(\psi)}_{\mathbb{R}} + (0, \lambda)$ *for some* $\psi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ *and* $\lambda \in \mathbb{R}_{\geqslant 0}$.

*(2) The following are equivalent:*

*(a)* $\deg(D_K) = 0$ *and* $\overline{D}$ *is nef.*

*(b)* $\deg(D_K) = 0$, $\overline{D}$ *is pseudo-effective and* $\widehat{\deg}(\overline{D}^2) = 0$.

*Proof.* — (1) First we assume that $\widehat{\deg}(\overline{D}^2) = 0$. By [15, Theorem 2.2.3, Remark 2.2.4], there are $\phi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and an $F_\infty$-invariant locally constant real valued function $\eta$ on $X(\mathbb{C})$ such that $\overline{D} = \widehat{(\phi)}_{\mathbb{R}} + (0, \eta)$. Let $K(\mathbb{C})$ be the set of all embeddings $\sigma : K \hookrightarrow \mathbb{C}$. For each $\sigma \in K(\mathbb{C})$, we set $X_\sigma = X \times^{\sigma}_{\operatorname{Spec}(O_K)} \operatorname{Spec}(\mathbb{C})$, where $\times^{\sigma}_{\operatorname{Spec}(O_K)}$ means the fiber product with respect to $\sigma : K \hookrightarrow \mathbb{C}$. Note that $\{X_\sigma\}_{\sigma \in K(\mathbb{C})}$ gives rise to all connected components of $X(\mathbb{C})$. Let $\eta_\sigma$ be the value of $\eta$ on $X_\sigma$. We set

$$\lambda = \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma \in K(\mathbb{C})} \eta_\sigma \quad \text{and} \quad \xi = \eta - \lambda.$$

Then $\xi_{\bar{\sigma}} = \xi_\sigma$ for all $\sigma \in K(\mathbb{C})$ and $\sum_{\sigma \in K(\mathbb{C})} \xi_\sigma = 0$. Thus, by Dirichlet's unit theorem, there is $u \in O_K^{\times} \otimes \mathbb{R}$ such that $\widehat{(u)}_{\mathbb{R}} = (0, \xi)$. Therefore, we have

$$\overline{D} = \widehat{(\phi u)}_{\mathbb{R}} + (0, \lambda).$$

The converse is obvious. We assume that $\widehat{\deg}(\overline{D}^2) = 0$ and $\overline{D}$ is pseudo-effective. Then $\overline{D} = \widehat{(\psi)}_{\mathbb{R}} + (0, \lambda)$ for some $\psi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$. Let $\overline{A}$ be an ample arithmetic Cartier divisor of $C^\infty$-type. Then,

$$0 \leqslant \widehat{\deg}(\overline{A} \cdot \overline{D}) = \frac{\lambda [K : \mathbb{Q}] \deg(A_K)}{2},$$

and hence $\lambda \geqslant 0$, as required.

(2) (a) $\Longrightarrow$ (b) follows from the non-negativity of $\widehat{\deg}(\overline{D}^2)$ ([13, Proposition 6.4.2], [15, SubSection 2.1]) and the Hodge index theorem ([15, Theorem 2.2.3]). Let us show that (b) $\Longrightarrow$ (a). By (1), $\overline{D} = \widehat{(\psi)}_{\mathbb{R}} + (0, \lambda)$ for some $\psi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}_{\geqslant 0}$. Thus the assertion is obvious. □

Lemma 4.2. — *In this lemma, $X$ is not necessarily an arithmetic surface, that is, $X$ is a $(d+1)$-dimensional, generically smooth, normal and projective arithmetic variety. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. Then,*

$$\widehat{\mathrm{vol}}(\overline{D}) \leqslant \widehat{\mathrm{vol}}(\overline{D}+(0,\epsilon)) \leqslant \widehat{\mathrm{vol}}(\overline{D}) + \frac{\epsilon(d+1)[K:\mathbb{Q}]\mathrm{vol}(D_K)}{2}$$

*for $\epsilon \in \mathbb{R}_{\geqslant 0}$.*

*Proof.* — The first inequality is obvious. Note that $\|\cdot\|_{m(\overline{D}+(0,\epsilon))} = e^{-\frac{m\epsilon}{2}}\|\cdot\|_{m\overline{D}}$ for all $m \geqslant 0$. Thus, by using [12, (3) in Proposition 2.1], there is a constant $C$ such that

$$\frac{\log \#\hat{H}^0(X, m(\overline{D}+(0,\epsilon)))}{m^{d+1}/(d+1)!} \leqslant \frac{\log \#\hat{H}^0(X, m\overline{D})}{m^{d+1}/(d+1)!} + \frac{\epsilon(d+1)[K:\mathbb{Q}]}{2}\frac{\dim_K H^0(X_K, mD_K)}{m^d/d!} + C\frac{\log m}{m}$$

holds for $m \gg 1$. Thus the second inequality follows. □

The following theorem is the main result of this paper.

Theorem 4.3. — *Let $\overline{D}$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$ with* $\deg(D_K) > 0$. *Then $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$ if and only if $\overline{D}$ is nef.*

*Proof.* — Let $\nu : X' \to X$ be a desingularization of $X$ (cf. [11]). Then $\widehat{\deg}(\nu^*(\overline{D})^2) = \widehat{\deg}(\overline{D}^2)$ and $\widehat{\mathrm{vol}}(\nu^*(\overline{D})) = \widehat{\mathrm{vol}}(\overline{D})$. Moreover, $\nu^*(\overline{D})$ is nef if and only if $\overline{D}$ is nef. Therefore, we may assume that $X$ is regular.

By [12, Corollary 5.5] and [13, Proposition-Definition 6.4.1], if $\overline{D}$ is nef, then $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$, so that we need to show that if $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$, then $\overline{D}$ is nef.

First we assume that $\overline{D}$ is big. Note that

$$\widehat{\deg}(\overline{D}^2) \leqslant \widehat{\mathrm{vol}}_\chi(\overline{D}) \leqslant \widehat{\mathrm{vol}}(\overline{D}).$$

Thus, by Theorem 2.1 and Corollary 3.8, $\overline{D}$ is relatively nef and $\mu_{\mathbb{R},\xi}(\overline{D}) = 0$ for $\xi \in X_K$. By [13, Theorem 9.2.1], there is a greatest element $\overline{P}$ of $\Upsilon(\overline{D})$ (cf. Conventions and terminology 4). If we set $\overline{N} := \overline{D} - \overline{P}$, then $\overline{D} = \overline{P} + \overline{N}$ is a Zariski decomposition of $\overline{D}$ (cf. Proposition B.1). Then, by [13, Claim 9.3.5.1] or [16, Theorem 4.1.1],

$$\mathrm{mult}_\xi(N) = \mu_{\mathbb{R},\xi}(\overline{D}) = 0$$

for all $\xi \in X_K$, which implies that $N$ is vertical. In particular, $\widehat{\deg}(\overline{D}\big|_C) \geqslant 0$ for all horizontal reduced and irreducible 1-dimensional closed subschemes $C$ on $X$, and hence $\overline{D}$ is nef because $\overline{D}$ is relatively nef.

Next we assume that $\overline{D}$ is not big. Then $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D}) = 0$. Thus, for $\epsilon \in \mathbb{R}_{>0}$,

$$\epsilon[K:\mathbb{Q}]\deg(D_K) = \widehat{\deg}((\overline{D}+(0,\epsilon))^2) \leqslant \widehat{\mathrm{vol}}(\overline{D}+(0,\epsilon)) \leqslant \epsilon[K:\mathbb{Q}]\deg(D_K)$$

by the generalized Hodge index theorem (cf. Theorem 2.1) and Lemma 4.2, and hence $\overline{D}+(0,\epsilon)$ is big and $\widehat{\deg}((\overline{D}+(0,\epsilon))^2) = \widehat{\mathrm{vol}}(\overline{D}+(0,\epsilon))$. Therefore, by the previous observation, $\overline{D}+(0,\epsilon)$ is nef for all $\epsilon \in \mathbb{R}_{>0}$, which means that $\overline{D}$ is nef. □

As a corollary of the above theorem, we have the following:

COROLLARY 4.4. — *Let $\overline{D}$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. Then $\overline{D}$ is nef if and only if $\overline{D}$ is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$.*

*Proof.* — We need to show that if $\overline{D}$ is pseudo-effective and $\widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{D})$, then $\overline{D}$ is nef. Clearly $\deg(D_K) \geqslant 0$. If $\deg(D_K) > 0$, then the nefness of $\overline{D}$ follows from Theorem 4.3. Moreover, if $\deg(D_K) = 0$, then (2) in Lemma 4.3 implies the assertion. □

## 5. Negative part of Zariski decomposition

We assume that $d = 1$. As an application of Theorem 4.3, let us see that the self-intersection number of the negative part of a Zariski decomposition is negative.

THEOREM 5.1. — *Let $\overline{D}$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$ such that $\deg(D_K) \geqslant 0$. Let $\overline{D} = \overline{P} + \overline{N}$ be a Zariski decomposition of $\overline{D}$ (cf. Conventions and terminology 4). Then $\widehat{\deg}(\overline{N}^2) < 0$ if and only if $\overline{D}$ is not nef.*

*Proof.* — First of all, note that $\overline{D}$ is pseudo-effective. As $\widehat{\deg}(\overline{P}\cdot\overline{N}) = 0$ by the following Lemma 5.2,

$$\widehat{\mathrm{vol}}(\overline{D}) - \widehat{\deg}(\overline{D}^2) = \widehat{\mathrm{vol}}(\overline{P}) - \widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\overline{P}^2) - \widehat{\deg}(\overline{D}^2) = -\widehat{\deg}(\overline{N}^2).$$

In addition, by Corollary 4.4, $\overline{D}$ is not nef if and only if $\widehat{\mathrm{vol}}(\overline{D}) > \widehat{\deg}(\overline{D}^2)$. Thus the assertion follows. □

Lemma 5.2. — *Let $\overline{D}$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. If $\overline{D} = \overline{P} + \overline{N}$ is a Zariski decomposition of $\overline{D}$, then $\widehat{\deg}(\overline{P} \cdot \overline{N}) = 0$ and $\widehat{\deg}(\overline{N}^2) \leqslant 0$.*

*Proof.* — For $0 < \epsilon \leqslant 1$, we set $\overline{D}_\epsilon = \overline{P} + \epsilon\overline{N}$. Then $\overline{D}_\epsilon$ is integrable and $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{D}_\epsilon)$ because

$$\widehat{\mathrm{vol}}(\overline{P}) \leqslant \widehat{\mathrm{vol}}(\overline{D}_\epsilon) \leqslant \widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P}).$$

Thus, by the generalized Hodge index theorem (cf. Theorem 2.1),

$$\widehat{\deg}((\overline{P} + \epsilon\overline{N})^2) = \widehat{\deg}(\overline{D}_\epsilon^2) \leqslant \widehat{\mathrm{vol}}(\overline{D}_\epsilon) = \widehat{\mathrm{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^2),$$

and hence

$$2\widehat{\deg}(\overline{P} \cdot \overline{N}) + \epsilon\widehat{\deg}(\overline{N}^2) \leqslant 0.$$

In particular, $\widehat{\deg}(\overline{P} \cdot \overline{N}) \leqslant 0$. On the other hand, as $\overline{P}$ is nef and $\overline{N}$ is effective, $\widehat{\deg}(\overline{P} \cdot \overline{N}) \geqslant 0$. Thus $\widehat{\deg}(\overline{P} \cdot \overline{N}) = 0$ and $\widehat{\deg}(\overline{N}^2) \leqslant 0$. □

*Remark 5.3.* — If $\overline{D}$ is big, then the Zariski decomposition $\overline{D} = \overline{P} + \overline{N}$ is uniquely determined by [16, Theorem 4.2.1]. Otherwise, it is not necessarily unique.

As a consequence of the above theorem, we have the following numerical characterization of the greatest element of $\Upsilon(\overline{D})$ (cf. Conventions and terminology 4).

Corollary 5.4. — *We assume that $X$ is regular. Let $\overline{D}$ and $\overline{P}$ be arithmetic $\mathbb{R}$-Cartier divisors of $C^0$-type on $X$. Then the following are equivalent:*

*(1) $\overline{P}$ is the greatest element of $\Upsilon(\overline{D})$, that is, $\overline{P} \in \Upsilon(\overline{D})$ and $\overline{M} \leqslant \overline{P}$ for all $\overline{M} \in \Upsilon(\overline{D})$.*

*(2) $\overline{P}$ is an element of $\Upsilon(\overline{D})$ with the following property:*

$$\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0 \quad \text{and} \quad \widehat{\deg}(\overline{B}^2) < 0$$

*for all integrable arithmetic $\mathbb{R}$-Cartier divisors $\overline{B}$ of $C^0$-type with $(0, 0) \lneqq \overline{B} \leqslant \overline{D} - \overline{P}$.*

*Proof.* — (1) $\Longrightarrow$ (2) : By Proposition B.1, $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P})$, so that $\overline{P} + \overline{B}$ is a Zariski decomposition because

$$\widehat{\mathrm{vol}}(\overline{P}) \leqslant \widehat{\mathrm{vol}}(\overline{P} + \overline{B}) \leqslant \widehat{\mathrm{vol}}(\overline{D}).$$

Thus $\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0$ by Lemma 5.2. As $\overline{B} \gneqq (0,0)$ and $\overline{P}$ is the greatest element of $\Upsilon(\overline{D})$, $\overline{P} + \overline{B}$ is not nef, so that $\widehat{\deg}(\overline{B}^2) < 0$ by Theorem 5.1.

(2) $\Longrightarrow$ (1) : Let $\overline{M}$ be an element of $\Upsilon(\overline{D})$. If we set $\overline{A} = \max\{\overline{P}, \overline{M}\}$ (cf. Conventions and terminology 6) and $\overline{B} = \overline{A} - \overline{P}$, then $\overline{B}$ is effective, $\overline{A} \leqslant \overline{D}$ and $\overline{A}$ is nef by [13, Lemma 9.1.2]. Moreover,

$$\overline{B} = \overline{A} - \overline{P} \leqslant \overline{D} - \overline{P}.$$

If we assume $\overline{B} \gneqq (0,0)$, then, by the property, $\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0$ and $\widehat{\deg}(\overline{B}^2) < 0$. On the other hand, as $\overline{A}$ is nef and $\overline{B}$ is effective,

$$0 \leqslant \widehat{\deg}(\overline{A} \cdot \overline{B}) = \widehat{\deg}(\overline{P} + \overline{B} \cdot \overline{B}) = \widehat{\deg}(\overline{B}^2),$$

which is a contradiction, so that $\overline{B} = (0,0)$, that is, $\overline{P} = \overline{A}$, which means that $\overline{M} \leqslant \overline{P}$, as required. □

Corollary 5.5. — *We assume that $X$ is regular. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$ such that $\Upsilon(\overline{D}) \neq \emptyset$. Let $\overline{P}$ be the greatest element of $\Upsilon(\overline{D})$ (cf. [13, Theorem 9.2.1]) and let $\overline{N} := \overline{D} - \overline{P}$. We assume that $N \neq 0$. Let $N = c_1 C_1 + \cdots + c_l C_l$ be the decomposition such that $c_1, \ldots, c_l \in \mathbb{R}_{>0}$ and $C_1, \ldots, C_l$ are distinct reduced and irreducible 1-dimensional closed subschemes on $X$. Let $\overline{C}_1 = (C_1, h_1), \ldots, \overline{C}_l = (C_l, h_l)$ be effective arithmetic Cartier divisors of $C^0$-type such that such that $c_1(\overline{C}_1), \ldots, c_1(\overline{C}_l)$ are positive currents and*

$$c_1 \overline{C}_1 + \cdots + c_l \overline{C}_l \leqslant \overline{N}.$$

*Then*

$$\widehat{\deg}(\overline{P} \cdot \overline{C_1}) = \cdots = \widehat{\deg}(\overline{P} \cdot \overline{C_l}) = 0$$

*and the $(l \times l)$ symmetric matrix given by*

$$\left(\widehat{\deg}(\overline{C}_i \cdot \overline{C}_j)\right)_{\substack{1 \leqslant i \leqslant l \\ 1 \leqslant j \leqslant l}}$$

*is negative definite.*

*Proof.* — For $x = (x_1, \ldots, x_l) \in \mathbb{R}^l$, we set $\overline{B}_x = x_1 \overline{C}_1 + \cdots + x_l \overline{C}_l$ and $\overline{D}_x = \overline{P} + \overline{B}_x$. If $0 \leqslant x_i \leqslant c_i$ for all $i = 1, \ldots, l$, then $\overline{B}_x$ is integrable and $(0,0) \leqslant \overline{B}_x \leqslant \overline{N}$. Thus, by Corollary 5.4,

$$0 = \widehat{\deg}(\overline{P} \cdot \overline{B}_{(c_1, \ldots, c_l)}) = c_1 \widehat{\deg}(\overline{P} \cdot \overline{C}_1) + \cdots + c_l \widehat{\deg}(\overline{P} \cdot \overline{C}_l).$$

Note that $\widehat{\deg}(\overline{P} \cdot \overline{C}_i) \geqslant 0$ for all $i = 1, \ldots, l$. Therefore,

$$\widehat{\deg}(\overline{P} \cdot \overline{C_1}) = \cdots = \widehat{\deg}(\overline{P} \cdot \overline{C_l}) = 0$$

Here we claim the following:

Claim 5.6. — *If* $x \in (\mathbb{R}_{\geqslant 0})^l \setminus \{0\}$, *then* $\widehat{\deg}(\overline{B}_x^2) < 0$.

*Proof.* — Note that $\overline{B}_{tx} = t\overline{B}_x$ and that we can find a positive number $t$ with $tx_i \leqslant c_i$ $(\forall i)$. Thus we may assume that $x_i \leqslant c_i$ $(\forall i)$, and hence the assertion follows by Corollary 5.4. □

We need to see that if $x \in \mathbb{R}^l \setminus \{0\}$, then $\widehat{\deg}(\overline{B}_x^2) < 0$. We can choose

$$y = (y_1, \dots, y_l), z = (z_1, \dots, z_l) \in (\mathbb{R}_{\geqslant 0})^l$$

such that $x = y - z$ and $\{i \mid y_i \neq 0\} \cap \{j \mid z_j \neq 0\} = \emptyset$. Note that either $y \neq 0$ or $z \neq 0$. Moreover, $\widehat{\deg}(\overline{B}_y \cdot \overline{B}_z) \geqslant 0$ because $\overline{B}_y \geqslant (0,0)$, $\overline{B}_z \geqslant (0,0)$, $c_1(\overline{B}_y)$ and $c_1(\overline{B}_z)$ are positive currents, and $B_y$ and $B_z$ have no common reduced and irreducible 1-dimensional closed subschemes. Thus, by using the above claim,

$$\widehat{\deg}(\overline{B}_x^2) = \widehat{\deg}((\overline{B}_y - \overline{B}_z)^2) = \widehat{\deg}(\overline{B}_y^2) + \widehat{\deg}(\overline{B}_z^2) - 2\widehat{\deg}(\overline{B}_y \cdot \overline{B}_z) < 0.$$

□

*Remark 5.7.* — By [13, Theorem 9.3.4, (4.1)], we can find effective arithmetic Cartier divisors $\overline{C}_1, \dots, \overline{C}_l$ of $C^0$-type such that $c_1(\overline{C}_1), \dots, c_1(\overline{C}_l)$ are positive currents and $c_1\overline{C}_1 + \cdots + c_l\overline{C}_l \leqslant \overline{N}$.

*Example 5.8.* — Let $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1])$ and $H_i = \{T_i = 0\}$ for $i = 0, 1$. We fix positive numbers $a_0, a_1$ such that $a_0 < 1$, $a_1 < 1$ and $a_0 + a_1 \geqslant 1$. Let us consider an arithmetic Cartier divisor $\overline{D}$ of $C^\infty$-type given by

$$\overline{D} := (H_0, \log(a_0 + a_1|z|^2)),$$

where $z = T_1/T_0$. Note that $c_1(\overline{D})$ is a positive form. Moreover, $\overline{D}$ is pseudo-effective and not nef (cf. [14, Theorem 2.3]). In [14, Theorem 4.1], we give the greatest element of $\Upsilon(\overline{D})$ as follows: Let $\varphi$ be a continuous function on the interval $[0,1]$ given by

$$\varphi(x) = -(1-x)\log(1-x) - x\log(x) + (1-x)\log(a_0) + x\log(a_1),$$

and let $\vartheta = \min\{x \in [0,1] \mid \varphi(x) \geqslant 0\}$ and $\theta = \max\{x \in [0,1] \mid \varphi(x) \geqslant 0\}$. We set

$$\overline{P} := (\theta H_0 - \vartheta H_1, p(z)), \quad \overline{N}_1 := (\vartheta H_1, n_1(z)) \text{ and } \overline{N}_2 := ((1-\theta)H_0, n_2(z)),$$

where $p(z)$, $n_1(z)$ and $n_2(z)$ are Green functions given by

$$p(z) := \begin{cases} \vartheta \log|z|^2 & \text{if } |z| \leqslant \sqrt{\frac{a_0\vartheta}{a_1(1-\vartheta)}}, \\ \log(a_0 + a_1|z|^2) & \text{if } \sqrt{\frac{a_0\vartheta}{a_1(1-\vartheta)}} \leqslant |z| \leqslant \sqrt{\frac{a_0\theta}{a_1(1-\theta)}}, \\ \theta \log|z|^2 & \text{if } |z| \geqslant \sqrt{\frac{a_0\theta}{a_1(1-\theta)}}. \end{cases}$$

$$n_1(z) := \begin{cases} \log(a_0 + a_1|z|^2) - \vartheta \log |z|^2 & \text{if } |z| \leqslant \sqrt{\frac{a_0\vartheta}{a_1(1-\vartheta)}}, \\ 0 & \text{if } |z| \geqslant \sqrt{\frac{a_0\vartheta}{a_1(1-\vartheta)}}. \end{cases}$$

$$n_2(z) := \begin{cases} 0 & \text{if } |z| \leqslant \sqrt{\frac{a_0\theta}{a_1(1-\theta)}}, \\ \log(a_1 + a_0|z|^{-2}) + (1-\theta)\log |z|^2 & \text{if } |z| \geqslant \sqrt{\frac{a_0\theta}{a_1(1-\theta)}}. \end{cases}$$

Then $\overline{P}$ gives the greatest element of $\Upsilon(\overline{D})$ and $\overline{D} = \overline{P} + (\overline{N}_1 + \overline{N}_2)$. It is easy to see that

$$\widehat{\deg}(\overline{P} \cdot \overline{N}_1) = \widehat{\deg}(\overline{P} \cdot \overline{N}_2) = 0 \quad \text{and} \quad \widehat{\deg}(\overline{N}_1 \cdot \overline{N}_2) = 0.$$

Moreover,

$$\begin{aligned} \widehat{\deg}(\overline{N}_1 \cdot \overline{N}_1) &= \widehat{\deg}(\overline{N}_1 \cdot (\overline{N}_1 - \vartheta\widehat{(z)})) = \widehat{\deg}(\overline{N}_1 \cdot (\vartheta H_0, n_1(z) + \vartheta \log |z|^2)) \\ &= \vartheta \widehat{\deg}(\overline{N}_1\big|_{H_0}) + \frac{1}{2}\int_{\mathbb{P}^1(\mathbb{C})} c_1(\overline{N}_1)(n_1(z) + \vartheta \log |z|^2) \\ &= \frac{1}{2}\int_{|z| \leqslant \sqrt{\frac{a_0\vartheta}{a_1(1-\vartheta)}}} dd^c(\log(a_0 + a_1|z|^2)) \log(a_0 + a_1|z|^2) \\ &= \frac{(1-\vartheta)\log(1-\vartheta) + (\log(a_0)+1)\vartheta}{2}. \end{aligned}$$

In the same way,

$$\widehat{\deg}(\overline{N}_2 \cdot \overline{N}_2) = \frac{\theta\log(\theta) + (\log(a_1)+1)(1-\theta)}{2}.$$

Thus the negative definite symmetric matrix $(\widehat{\deg}(\overline{N}_i \cdot \overline{N}_j))_{i,j=1,2}$ is

$$\begin{pmatrix} \frac{(1-\vartheta)\log(1-\vartheta)+(\log(a_0)+1)\vartheta}{2} & 0 \\ 0 & \frac{\theta\log(\theta)+(\log(a_1)+1)(1-\theta)}{2} \end{pmatrix}.$$

## Appendix A. Relative Zariski decomposition and pseudo-effectivity

We assume that $X$ is regular and $d = 1$. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. In this appendix, we would like to investigate the pseudo-effectivity of the relative Zariski decomposition.

PROPOSITION A.1. — *We assume that* $\deg(D_K) \geqslant 0$. *Let* $\overline{Q}$ *be the greatest element of* $\Upsilon_{rel}(\overline{D})$ *(cf. Section 1). Then* $\overline{D}$ *is pseudo-effective if and only if* $\overline{Q}$ *is pseudo-effective.*

*Proof.* — It is obvious that if $\overline{Q}$ is pseudo-effective, then $\overline{D}$ is also pseudo-effective, so that we assume that $\overline{D}$ is pseudo-effective.

First we consider the case where $\deg(D_K) > 0$. Then, by [13, Proposition 6.3.3], $\overline{D} + (0, \epsilon)$ is big for any $\epsilon \in \mathbb{R}_{>0}$. By the property (d) in Theorem 1.1, the natural inclusion map $H^0(X, nQ) \to H^0(X, nD)$ is bijective and $\| \cdot \|_{n\overline{Q}} = \| \cdot \|_{n\overline{D}}$ for each $n \geqslant 0$. Moreover, as

$$\| \cdot \|_{n(\overline{Q}+(0,\epsilon))} = e^{-n\epsilon/2} \| \cdot \|_{n\overline{Q}} \quad \text{and} \quad \| \cdot \|_{n(\overline{D}+(0,\epsilon))} = e^{-n\epsilon/2} \| \cdot \|_{n\overline{D}},$$

we have $\| \cdot \|_{n(\overline{Q}+(0,\epsilon))} = \| \cdot \|_{n(\overline{D}+(0,\epsilon))}$, and hence $\overline{Q} + (0, \epsilon)$ is big for all $\epsilon \in \mathbb{R}_{>0}$. Thus the assertion follows.

Next we assume that $\deg(D_K) = 0$. By [15, Theorem 2.3.3], there are $\phi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$, a vertical effective $\mathbb{R}$-Cartier divisor $E$ on $X$ and an $F_\infty$-invariant continuous function $\eta$ on $X(\mathbb{C})$ such that $\overline{D} = \widehat{(\phi)}_{\mathbb{R}} + (E, \eta)$ and $\pi^{-1}(P)_{red} \not\subseteq \operatorname{Supp}(E)$ for all $P \in \operatorname{Spec}(O_K)$. For each embedding $\sigma : K \hookrightarrow \mathbb{C}$, let $X_\sigma = X \times^{\sigma}_{\operatorname{Spec}(O_K)} \operatorname{Spec}(\mathbb{C})$ and let $\lambda_\sigma = \min_{x \in X_\sigma} \{\eta(x)\}$. Note that $\lambda_{\bar{\sigma}} = \lambda_\sigma$ for all $\sigma$. Let $\lambda : X(\mathbb{C}) \to \mathbb{R}$ be the local constant function such that the value of $\lambda$ on $X_\sigma$ is $\lambda_\sigma$.

Here let us see that $\overline{Q} = \widehat{(\phi)}_{\mathbb{R}} + (0, \lambda)$ is the greatest element of $\Upsilon_{rel}(\overline{D})$. Otherwise, there is an integrable arithmetic $\mathbb{R}$-Cartier divisor $\overline{B} = (B, b)$ of $C^0$-type such that $(0,0) \lneqq \overline{B} \leqslant \overline{D} - \overline{Q} = (E, \eta - \lambda)$ and $\overline{Q} + \overline{B}$ is relatively nef. Since $b$ is continuous and

$$dd^c([b]) = c_1(\overline{B}) = c_1(\overline{Q} + \overline{B})$$

is a positive current, $b$ is plurisubharmonic on $X(\mathbb{C})$, that is, $b$ is a locally constant function. Let $b_\sigma$ be the value of $b$ on $X_\sigma$. If we choose $x_\sigma \in X_\sigma$ with $\lambda_\sigma = \eta(x_\sigma)$, then

$$0 \leqslant b_\sigma \leqslant \eta(x_\sigma) - \lambda_\sigma = 0,$$

and hence $b = 0$, so that, as $\overline{Q} + \overline{B}$ is relatively nef,

$$0 \leqslant \widehat{\deg}(\overline{Q} + \overline{B} \cdot \overline{B}) = \widehat{\deg}((B, 0)^2).$$

On the other hand, by Zariski's lemma, $\widehat{\deg}((B,0)^2) < 0$. This is a contradiction.

By [15, Lemma 2.3.4 and Lemma 2.3.5], $(E, \lambda)$ is pseudo-effective. On the other hand, by the following Lemma A.2, there is a nef arithmetic $\mathbb{R}$-Cartier divisor $\overline{L}$ of $C^\infty$-type such that $\deg(L_K) > 0$ and $\widehat{\deg}(\overline{L} \cdot (E, 0)) = 0$.

Thus,

$$0 \leqslant \widehat{\deg}(\overline{L} \cdot (E,\lambda)) = \sum_{\sigma} \frac{\deg(L_K)\lambda_\sigma}{2},$$

and hence $\sum_\sigma \lambda_\sigma \geqslant 0$. We set $\lambda' = (1/[K:\mathbb{Q}]) \sum_\sigma \lambda_\sigma$ and $\xi = \lambda - \lambda'$. Then $\lambda' \geqslant 0$, $\sum_\sigma \xi_\sigma = 0$ and $\xi_{\bar\sigma} = \xi_\sigma$ for all $\sigma$, where $\xi_\sigma$ is the value of $\xi$ on $X_\sigma$. Thus, by Dirichlet's unit theorem, $(0,\xi) = \widehat{(u)}_{\mathbb{R}}$ for some $u \in O_K^\times \otimes \mathbb{R}$. Therefore,

$$\overline{Q} = \widehat{(\phi u)}_{\mathbb{R}} + (0,\lambda'),$$

which is pseudo-effective. □

Lemma A.2. — *Let $C_1,\ldots,C_r$ be vertical reduced and irreducible* 1-*dimensional closed subschemes on $X$ such that $\pi^{-1}(P)_{red} \not\subseteq C_1 \cup \cdots \cup C_r$ for all $P \in \operatorname{Spec}(O_K)$. Then there is a nef arithmetic $\mathbb{R}$-Cartier divisor $\overline{L}$ of $C^\infty$-type such that* $\deg(L_K) > 0$ *and* $\widehat{\deg}(\overline{L} \cdot (C_i,0)) = 0$ *for all* $i = 1,\ldots,r$.

*Proof.* — Let $\overline{A}$ be an ample arithmetic Cartier divisor of $C^\infty$-type. By using Zariski's lemma, we can find a vertical effective $\mathbb{R}$-Cartier divisor $E$ such that

$$\widehat{\deg}((E,0)\cdot(C_i,0)) = -\deg(\overline{A}\cdot(C_i,0))$$

for all $i = 1,\ldots,r$ and that $\widehat{\deg}((E,0)\cdot(C,0)) \geqslant 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes $C$ with $C \notin \{C_1,\ldots,C_r\}$. Thus, if we set $\overline{L} := \overline{A} + (E,0)$, then $\overline{L}$ is a nef arithmetic $\mathbb{R}$-Cartier divisor of $C^\infty$-type, $\deg(L_K) > 0$ and $\widehat{\deg}(\overline{L}\cdot(C_i,0)) = 0$ for all $i = 1,\ldots,r$. □

As an corollary, we can give a simpler proof of the main result of [15] in the case where $X$ is a generically smooth, normal projective arithmetic surface.

Corollary A.3. — *Let $X$ be a generically smooth, normal projective arithmetic surface and let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. If* $\deg(D_K) = 0$ *and $\overline{D}$ is pseudo-effective, then there is $\phi \in \operatorname{Rat}(X)^\times_{\mathbb{R}}$ such that $\overline{D} + \widehat{(\phi)}_{\mathbb{R}} \geqslant (0,0)$.*

*Proof.* — Clearly we may assume that $X$ is regular. By Proposition A.1, we may also assume that $\overline{D}$ is relatively nef. By the Hodge index theorem (cf. [15, Theorem 2.2.3]), $\widehat{\deg}(\overline{D}^2) \leqslant 0$. We assume that $\widehat{\deg}(\overline{D}^2) < 0$. Let $\overline{A}$ be an ample arithmetic Cartier divisor of $C^\infty$-type on $X$. As $\widehat{\deg}(\overline{D}^2) < 0$, we can find a sufficiently small positive number $\epsilon$ with $\widehat{\deg}((\overline{D} + \epsilon\overline{A})\cdot\overline{D}) < 0$. Moreover, since $D + \epsilon A$ is ample, there is a positive number $c$ such that $\overline{D} + \epsilon\overline{A} + (0,c)$ is nef. In particular,

$$\widehat{\deg}((\overline{D} + \epsilon\overline{A} + (0,c))\cdot\overline{D}) \geqslant 0.$$

On the other hand,

$$\widehat{\deg}((\overline{D} + \epsilon\overline{A} + (0,c)) \cdot \overline{D}) = \widehat{\deg}((\overline{D} + \epsilon\overline{A}) \cdot \overline{D}) + \frac{c[K:\mathbb{Q}]}{2}\deg(D_K) < 0,$$

which is a contradiction, so that $\widehat{\deg}(\overline{D}^2) = 0$. Therefore, by Lemma 4.1, there is $\psi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}_{\geqslant 0}$ such that $\overline{D} = \widehat{(\psi)}_{\mathbb{R}} + (0,\lambda)$, and hence

$$\overline{D} + \widehat{(\psi^{-1})}_{\mathbb{R}} = (0,\lambda) \geqslant (0,0).$$

□

## Appendix B. Small sections of arithmetic $\mathbb{R}$-divisors

Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of $C^0$-type on $X$. In this appendix, let us consider a generalization of [13, Proposition 9.3.3]. Its proof is much simpler than one of [13, Proposition 9.3.3].

Proposition B.1. — *Let $\overline{P}$ be the greatest element of $\Upsilon(\overline{D})$ (cf. Conventions and terminology 4). Then, for $\phi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$, $\overline{D} + \widehat{(\phi)}_{\mathbb{R}}$ is effective if and only if $\overline{P} + \widehat{(\phi)}_{\mathbb{R}}$ is effective. In particular, the natural inclusion maps*

$$\hat{H}^0(X, n\overline{P}) \hookrightarrow \hat{H}^0(X, n\overline{D}), \quad \hat{H}^0_{\mathbb{Q}}(X, \overline{P}) \hookrightarrow \hat{H}^0_{\mathbb{Q}}(X, \overline{D})$$

$$\text{and} \quad \hat{H}^0_{\mathbb{R}}(X, \overline{P}) \hookrightarrow \hat{H}^0_{\mathbb{R}}(X, \overline{D})$$

*are bijective for each $n \geqslant 0$.*

*Proof.* — We assume that $\overline{D} + \widehat{(\phi)}_{\mathbb{R}}$ is effective. Then $-\widehat{(\phi)}_{\mathbb{R}} \in \Upsilon(\overline{D})$, and hence $-\widehat{(\phi)}_{\mathbb{R}} \leqslant \overline{P}$, that is, $\overline{P} + \widehat{(\phi)}_{\mathbb{R}}$ is effective. The converse is obvious. □

As a corollary of the above proposition, we have the following.

Corollary B.2. — *We assume that $d = 1$. Let $\overline{D} = \overline{P} + \overline{N}$ be a Zariski decomposition of $\overline{D}$ (Conventions and terminology 4). If $\overline{D}$ is big, then the natural inclusion maps*

$$\hat{H}^0(X, n\overline{P}) \hookrightarrow \hat{H}^0(X, n\overline{D}), \quad \hat{H}^0_{\mathbb{Q}}(X, \overline{P}) \hookrightarrow \hat{H}^0_{\mathbb{Q}}(X, \overline{D})$$

$$\text{and} \quad \hat{H}^0_{\mathbb{R}}(X, \overline{P}) \hookrightarrow \hat{H}^0_{\mathbb{R}}(X, \overline{D})$$

*are bijective for each $n \geqslant 0$.*

*Proof.* — Let $\mu : X' \to X$ be a desingularization of $X$ (cf. [11]). Then

$$\mu^*(\overline{D}) = \mu^*(\overline{P}) + \mu^*(\overline{N})$$

is a Zariski decomposition of $\mu^*(\overline{D})$. Thus, by [16, Theorem 4.2.1], $\mu^*(\overline{P})$ gives the greatest element of $\Upsilon(\mu^*(\overline{D}))$. Therefore, by Proposition B.1,

$$\hat{H}^0(X', n\mu^*(\overline{P})) = \hat{H}^0(X', n\mu^*(\overline{D})) \quad \text{and} \quad \hat{H}^0_{\mathbb{K}}(X', \mu^*(\overline{P})) = \hat{H}^0_{\mathbb{K}}(X', \mu^*(\overline{D}))$$

for each $n \geqslant 0$, where $\mathbb{K}$ is either $\mathbb{Q}$ or $\mathbb{R}$. Let us consider the following commutative diagrams:

$$\begin{array}{ccc} \hat{H}^0(X, n\overline{P}) & \longrightarrow & \hat{H}^0(X', n\mu^*(\overline{P})) \\ \downarrow & & \Vert \\ \hat{H}^0(X, n\overline{D}) & \longrightarrow & \hat{H}^0(X', n\mu^*(\overline{D})) \end{array} \qquad \begin{array}{ccc} \hat{H}^0_{\mathbb{K}}(X, \overline{P}) & \longrightarrow & \hat{H}^0_{\mathbb{K}}(X', \mu^*(\overline{P})) \\ \downarrow & & \Vert \\ \hat{H}^0_{\mathbb{K}}(X, \overline{D}) & \longrightarrow & \hat{H}^0_{\mathbb{K}}(X', \mu^*(\overline{D})) \end{array}$$

Note that each horizontal arrow is bijective. Thus the assertions follows. □

## Appendix C. A result on subsemigroups of $\mathbb{R}^d_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$

Let $d$ be a positive integer. Let $v : \mathbb{R}^{d+1} \to \mathbb{R}^d$ and $h : \mathbb{R}^{d+1} \to \mathbb{R}$ be the projections given by

$$v(x_1, \ldots, x_d, x_{d+1}) = (x_1, \ldots, x_d) \quad \text{and} \quad h(x_1, \ldots, x_d, x_{d+1}) = x_{d+1}.$$

Let $\Gamma$ be a sub-semigroup of $\mathbb{R}^d_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$. For a non-negative integer $m$, we set

$$\Gamma_m = v(\Gamma \cap (\mathbb{R}^d \times \{m\})) = v(\{\gamma \in \Gamma \mid h(\gamma) = m\}).$$

More generally, for a subset $X$ of $\mathbb{R}^{d+1}$ and $t \in \mathbb{R}$, $X_t$ is given by

$$X_t = v(X \cap (\mathbb{R}^d \times \{t\})) = v(\{x \in X \mid h(x) = t\}).$$

We define $\Sigma(\Gamma)$ and $\Delta(\Gamma)$ to be

$$\Sigma(\Gamma) = \overline{\text{Cone}}(\Gamma) \quad \text{and} \quad \Delta(\Gamma) = \overline{\text{Conv}}\left(\bigcup_{m>0} \frac{1}{m}\Gamma_m\right),$$

where $\overline{\text{Cone}}(\Gamma)$ and $\overline{\text{Conv}}\left(\bigcup_{m>0} \frac{1}{m}\Gamma_m\right)$ is the topological closures of the cone generated by $\Gamma$ and the convex hull of $\bigcup_{m>0} \frac{1}{m}\Gamma_m$, respectively. For $\theta \in \mathbb{R}^d_{\geqslant 0}$, we define $\Gamma^\theta$ to be

$$\Gamma^\theta := \{(x + \theta m, m) \mid (x, m) \in \Gamma\}.$$

Note that $\Gamma^\theta$ is a sub-semigroup of $\mathbb{R}^d_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0}$. For simplicity, we denote $\Sigma(\Gamma)$, $\Delta(\Gamma)$, $\Sigma(\Gamma^\theta)$ and $\Delta(\Gamma^\theta)$ by $\Sigma$, $\Delta$, $\Sigma^\theta$ and $\Delta^\theta$, respectively.

Theorem C.1. — *We assume that there is $\theta \in \mathbb{R}^d_{\geqslant 0}$ such that $\Gamma^\theta \subseteq \mathbb{Z}^{d+1}_{\geqslant 0}$ and $\Gamma^\theta$ generates $\mathbb{Z}^{d+1}$ as a group, then the following are equivalent:*

*(1) There is a constant $M$ such that $\#(\Gamma_m) \leqslant Mm^d$ for all $m \geqslant 1$.*

*(2) $\Delta$ is bounded.*

*Moreover, under the above equivalent conditions, we have*

$$\lim_{m\to\infty} \frac{\#(\Gamma_m)}{m^d} = \mathrm{vol}(\Delta) > 0.$$

*Proof.* — Note that $\Gamma^\theta_m = \Gamma_m + m\theta$ and $\Delta^\theta = \Delta + \theta$. Therefore, in order to prove the assertion, we may assume that $\theta = 0$, that is, $\Gamma \subseteq \mathbb{Z}^{d+1}_{\geqslant 0}$ and $\Gamma$ generates $\mathbb{Z}^{d+1}$. Let us begin with the following claim:

Claim C.2. —

*(a) $t\Delta \subseteq \Sigma_t$ for all $t > 0$.*

*(b) $\Delta$ has an interior point.*

*(c) $\Gamma_m \subseteq m\Delta \cap \mathbb{Z}^d$ for all $m \geqslant 1$. In particular, if $\Delta$ is bounded, then*

$$\limsup_{m\to\infty} \frac{\#(\Gamma_m)}{m^d} \leqslant \mathrm{vol}_d(\Delta).$$

*(d) If $\#(\Gamma_m) < \infty$ for all $m \geqslant 1$, then*

$$\liminf_{m\to\infty} \frac{\#(\Gamma_m)}{m^d} \geqslant \mathrm{vol}_d(\Delta).$$

*Proof.* — (a) As $(1/m)\Gamma_m \subseteq \Sigma_1$ for $m \geqslant 1$, we have $\Delta \subseteq \Sigma_1$. Thus, for $t > 0$, $t\Delta \subseteq t\Sigma_1 \subseteq \Sigma_t$.

(b) We assume that $\Delta$ has no interior point. Then there is a hyperplane $H$ in $\mathbb{R}^d$ such that $\Delta \subseteq H$. Let $W$ be a subspace of $\mathbb{R}^{d+1}$ generated by $H \times \{1\}$. Note that $\dim_{\mathbb{R}} W = d$.

Here let us see that $\Gamma \subseteq W$. Let $(x, m) \in \Gamma$. If $m > 0$, then $x/m \in \Delta$, so that $(x, m) = m(x/m, 1) \in W$. Otherwise, we choose $(y, n) \in \Gamma$ with $n > 0$. Then, as $(x+y, n) = (x, 0) + (y, n) \in \Gamma$, by the previous observation, $(y, n), (x + y, n) \in W$, and hence $(x, 0) = (x + y, n) - (y, n) \in W$.

By our assumption, $\langle \Gamma \rangle_{\mathbb{R}} = \mathbb{R}^{d+1}$, which contradicts to the observation $\Gamma \subseteq W$.

(c) This is a obvious.

(d) First we assume that $\Gamma$ is finitely generated, that is, there is $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $\Gamma = \mathbb{Z}_{\geqslant 0}\gamma_1 + \cdots + \mathbb{Z}_{\geqslant 0}\gamma_n$. By [10, Proposition 3] (note that the constant $C$ in [10, Proposition 1] can be taken as a positive integer), there is $\gamma \in \Gamma$ such that

$$\Sigma \cap \mathbb{Z}^{d+1} + \gamma \subseteq \Gamma,$$

which implies that $m\Delta \cap \mathbb{Z}^d + v(\gamma) \subseteq \Gamma_{m+h(\gamma)}$. Indeed, for $x \in m\Delta \cap \mathbb{Z}^d$, by (a), $x \in \Sigma_m \cap \mathbb{Z}^d$, and hence

$$x + v(\gamma) \in (\Sigma \cap \mathbb{Z}^{d+1} + \gamma)_{m+h(\gamma)} \subseteq \Gamma_{m+h(\gamma)}.$$

In particular, $\#(m\Delta \cap \mathbb{Z}^d) \leqslant \#(\Gamma_{m+h(\gamma)})$, which yields (d) in the case where $\Gamma$ is finitely generated.

In general, let $\Gamma(1) \subseteq \Gamma(2) \subseteq \cdots \subseteq \Gamma$ be a sequence of sub-semigroups of $\Gamma$ with the following properties:

(i) $\Gamma(i)$ is finitely generated for all $i$.

(ii) $\Gamma(i)$ generates $\mathbb{Z}^{d+1}$ as a group for all $i$.

(iii) $\bigcup_i \Gamma(i) = \Gamma$.

By the previous observation,

$$\liminf_{m\to\infty} \frac{\#(\Gamma_m)}{m^d} \geqslant \liminf_{m\to\infty} \frac{\#(\Gamma(i)_m)}{m^d} \geqslant \mathrm{vol}_d(\Delta(i)),$$

where $\Delta(i) = \Delta(\Gamma(i))$. Note that $\lim_{i\to\infty} \mathrm{vol}(\Delta(i)) = \mathrm{vol}(\Delta)$ because $\Delta$ is the closure of $\bigcup_i \Delta(i)$. Hence we obtain the assertion. □

Let us go back to the proof of the theorem. First we assume (1). Then, by (d), $\mathrm{vol}(\Delta) < \infty$ and $\Delta$ has an interior point by (b). Therefore, $\Delta$ is bounded by Lemma C.3 as described below. Next assume (2). Then (1) follows from (c).

Finally we assume the equivalent conditions (1) and (2). Then, by (c) and (d),

$$\limsup_{m\to\infty} \frac{\#(\Gamma_m)}{m^d} \leqslant \mathrm{vol}_d(\Delta) \leqslant \liminf_{m\to\infty} \frac{\#(\Gamma_m)}{m^d},$$

and hence

$$\lim_{m\to\infty} \frac{\#(\Gamma_m)}{m^d} = \mathrm{vol}_d(\Delta) > 0$$

by (b). □

Lemma C.3. — *Let $K$ be a convex set in $V$ such that $K$ has an interior point. Then the following are equivalent:*

*(1)* *$K$ is bounded.*

*(2)* $\mathrm{vol}(K) < \infty$.

*Proof.* — Clearly (1) implies (2). We assume that $\mathrm{vol}(K) < \infty$ and $K$ is not bounded. Let $a$ be an interior point of $K$. Considering the translation given by $x \mapsto x - a$, we may assume $a = 0$. Then there is a positive number $r$ such that $B \subseteq K$, where $B := \{x \in V \mid \langle x, x\rangle \leqslant r^2\}$. As $K$ is not bounded, for any $M > 0$, there is $x \in K$ such that $\langle x, x\rangle \geqslant M^2$. Let $H_x = \{y \in V \mid \langle x, y\rangle = 0\}$ and let $C$ be the convex hull generated by $B \cap H_x$ and $x$. Clearly $C \subseteq K$. Moreover, as $C$ is a cone over $B \cap H_x$, we can see that

$$\mathrm{vol}(C) = \frac{\mathrm{vol}(B \cap H_x)\sqrt{\langle x, x\rangle}}{d},$$

and hence

$$\mathrm{vol}(K) \geqslant \mathrm{vol}(C) \geqslant \frac{\mathrm{vol}(B \cap H_x)M}{d}.$$

This is a contradiction because $\mathrm{vol}(K) < \infty$. □

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