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On the arithmetic of cross-ratios and generalised Mertens' formulas

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Abstract. — We develop the relation between hyperbolic geometry and arithmetic equidistribution problems that arises from the action of arithmetic groups on real hyperbolic spaces, especially in dimension $\leqslant 5$. We prove generalisations of Mertens' formula for quadratic imaginary number fields and definite quaternion algebras over $\mathbb{Q}$, counting results of quadratic irrationals with respect to two different natural complexities, and counting results of representations of (algebraic) integers by binary quadratic, Hermitian and Hamiltonian forms with error bounds. For each such statement, we prove an equidistribution result of the corresponding arithmetically defined points. Furthermore, we study the asymptotic properties of crossratios of such points, and expand Pollicott's recent results on the Schottky-Klein prime functions.

Résumé. — Nous développons les liens entre la géométrie hyperbolique et les problèmes arithmétiques d'équidistribution, provenant de l'action de groupes arithmétiques sur des espaces hyperboliques réels, surtout en dimension au plus 5. Nous démontrons des généralisations de la formule de Mertens pour les corps de nombres quadratiques imaginaires et les algèbres de quaternion définies sur $\mathbb{Q}$, des résultats de comptage d'irrationnels quadratiques en utilisant deux complexités naturelles, et des résultats de comptage avec termes d'erreur de représentations d'entiers (algébriques) par des formes binaires quadratiques, hermitiennes ou hamiltoniennes. Pour tout tel énoncé, nous démontrons un résultat d'équidistribution des points arithmétiquement définis correspondants. De plus, nous étudions les propriétés asymptotiques des birapports de tels points, et nous étendons les résultats récents de Pollicott sur les fonctions premières de Schottky-Klein.

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1. Introduction

The aim of this paper is to prove various equidistribution results (and their related asymptotic counting results) of arithmetically defined points in low dimensional tori, organised using appropriate complexities. See for instance [26, 15, 43, 3] and their references for other types of results.

We denote by Δ_x the unit Dirac mass at a point x, by $\overset{*}{\rightharpoonup}$ the weak-star convergence of measures, by $\|m\|$ the total mass of a measure m, and by Leb_E the standard Lebesgue measure on a Euclidean space E. Given an imaginary quadratic number field K, we will denote by $\mathcal{O}_K$ its ring of integers, by D_K its discriminant, by ζ_K its zeta function and by $\mathtt{n}$ its norm.

The first two statements are equidistribution results of rational points (satisfying congruence properties) in the complex field and Hamilton's quaternion algebra, analogous to the equidistribution result of Farey fractions in the real field, where the complexity is the norm of the denominator.

Theorem 1.1. — *Let $\mathfrak{m}$ be a (nonzero) fractional ideal of the ring of integers of an imaginary quadratic number field K, with norm $\mathtt{n}(\mathfrak{m})$. As $s \to +\infty$,*

$$\frac{|D_K|\,\zeta_K(2)}{2\,\pi\,s^2} \sum_{\substack{(u,v)\in\mathfrak{m}\times\mathfrak{m}\\ \mathtt{n}(\mathfrak{m})^{-1}\mathtt{n}(v)\leqslant s,\ \mathcal{O}_K u+\mathcal{O}_K v=\mathfrak{m}}} \Delta_{\frac{u}{v}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{C}}\,.$$

Let $\mathbb{H}$ be Hamilton's quaternion algebra over $\mathbb{R}$, with reduced norm $\mathtt{N} : x \mapsto x\,\overline{x}$. Let A be a quaternion algebra over $\mathbb{Q}$, which is definite (that is $A\otimes_{\mathbb{Q}}\mathbb{R} = \mathbb{H}$), with reduced discriminant D_A. Let $\mathcal{O}$ be a maximal order in A, and let $\mathfrak{m}$ be a (nonzero) left ideal of $\mathcal{O}$, with reduced norm $\mathtt{N}(\mathfrak{m})$. See [45] for definitions. We refer to Equation (3.9) for a more general result.

Theorem 1.2. — *As $s \to +\infty$, we have*

$$\frac{\pi^2\,\zeta(3)\,\prod_{p|D_A}(p^3-1)}{360\,D_A\;s^4} \sum_{\substack{(u,v)\in\mathfrak{m}\times\mathfrak{m}\\ \mathtt{N}(\mathfrak{m})^{-1}\mathtt{N}(v)\leqslant s,\ \mathcal{O}u+\mathcal{O}v=\mathfrak{m}}} \Delta_{uv^{-1}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{H}}\,.$$

The next three statements are equidistribution results in $\mathbb{R}$ or $\mathbb{C}$ of points that are constructed arithmetically by using quadratic irrationals.

We first prove that the set of traces $\mathtt{tr}\,\alpha$ of the real quadratic irrationals α over $\mathbb{Q}$, in a given orbit $G\cdot\alpha_0$ by homographies under a finite index subgroup G of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, equidistributes to the Lebesgue measure on $\mathbb{R}$. We use as the complexity of α (the inverse of) the distance to its Galois conjugate α^σ (see [32, 34] for background on this height). Given a real integral quadratic irrational α over $\mathbb{Q}$, we denote by R_α the regulator of $\mathbb{Z}+\alpha\mathbb{Z}$, and by $Q_\alpha(t) = t^2 - (\mathtt{tr}\alpha)\, t + \mathtt{n}(\alpha)$ its associated monic quadratic polynomial (the integrality assumption is only used in this introduction, to simplify the notation). We denote by H_x the stabiliser of $x \in \mathbb{P}_1(\mathbb{R}) = \mathbb{R}\cup\{\infty\}$ in a subgroup H of $\mathrm{PSL}_2(\mathbb{R})$.

Theorem 1.3. — *Let α_0 be a real integral quadratic irrational over $\mathbb{Q}$ and let G be a finite index subgroup of* $\mathrm{PSL}_2(\mathbb{Z})$. *Then, as $\epsilon \to 0$,*

$$\frac{\pi^2\,[\mathrm{PSL}_2(\mathbb{Z}):G]\,\epsilon}{6\,[\mathrm{PSL}_2(\mathbb{Z})_{\alpha_0}:G_{\alpha_0}]\,R_{\alpha_0}} \sum_{\alpha\in G\cdot\alpha_0,\ |\alpha-\alpha^\sigma|\geqslant\epsilon} \Delta_{\mathtt{tr}\,\alpha} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{R}}\,.$$

We refer to Theorems 4.2 and 4.4 for extensions of the above result to quadratic irrationals over an imaginary quadratic number field (using relative traces) or over a rational quaternion algebra, and we only quote in this introduction the following special case of Theorem 4.2.

Corollary 1.4. — *Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio, let K be an imaginary quadratic number field with $D_K \neq -4$, let $\mathfrak{c}$ be a nonzero ideal in $\mathcal{O}_K$, and let $\Gamma_0(\mathfrak{c})$ be the Hecke congruence subgroup*

$$\Big\{\pm\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K)\ :\ c\in\mathfrak{c}\Big\}.$$

Then, as $\epsilon\to 0$,

$$\frac{|D_K|^{\frac{3}{2}}\ \zeta_K(2)\,\mathtt{n}(\mathfrak{c})\prod_{\mathfrak{p}|\mathfrak{c}}\big(1+\frac{1}{\mathtt{n}(\mathfrak{p})}\big)\,\epsilon^2}{4\,\pi^2\,k_{\mathfrak{c}}\,\ln\phi} \sum_{\alpha\in\Gamma_0(\mathfrak{c})\cdot\phi,\,|\alpha-\alpha^\sigma|\geqslant\epsilon} \Delta_{\mathtt{tr}\,\alpha} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{C}}\,.$$

where $k_{\mathfrak{c}}$ is the smallest $k\in\mathbb{N}-\{0\}$ such that the $2k$-th term of Fibonacci's sequence belongs to $\mathfrak{c}$.

Given two real quadratic irrationals α,β over $\mathbb{Q}$, we introduce the *relative height* $h_\alpha(\beta)$ of β with respect to α, measuring how close the (unordered) pair $\{\beta,\beta^\sigma\}$ is to the pair $\{\alpha,\alpha^\sigma\}$, by

$$h_\alpha(\beta) = \frac{\min\{|\beta-\alpha|\,|\beta^\sigma-\alpha^\sigma|,\ |\beta-\alpha^\sigma|\,|\beta^\sigma-\alpha|\}}{|\beta-\beta^\sigma|}\,.$$

See Equation (4.11) for an expression of $h_\alpha(\beta)$ using crossratios of $\alpha,\beta,\alpha^\sigma$ and β^σ. Consider the points

$$x_\alpha^\pm(\beta) = \frac{\mathtt{n}(\beta)-\mathtt{n}(\alpha)\pm\big((\mathtt{n}(\alpha)-\mathtt{n}(\beta))^2+(\mathtt{tr}\,\beta-\mathtt{tr}\,\alpha)(\mathtt{tr}\,\beta\,\mathtt{n}(\alpha)-\mathtt{tr}\,\alpha\,\mathtt{n}(\beta))\big)^{\frac{1}{2}}}{\mathtt{tr}\,\beta-\mathtt{tr}\,\alpha}\,. \tag{1.1}$$

We will prove that when β varies in an orbit of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, the relative height $h_\alpha(\beta)$ is a well defined complexity modulo the stabiliser of α, and, except for finitely many orbits under this stabiliser, the points $x^\pm_\alpha(\beta)$ are well defined and real. The following results says that these points, when β has relative height at most s tending to ∞, equidistribute to the (unique up to scalar) measure on $\mathbb{R}-\{\alpha_0,\alpha_0^\sigma\}$ which is absolutely continuous with respect to the Lebesgue measure and invariant under the stabiliser of $\{\alpha_0,\alpha_0^\sigma\}$ under $\mathrm{PSL}_2(\mathbb{R})$.

Theorem 1.5. — *Let α,β be real integral quadratic irrationals and let G be a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Then, as $s\to+\infty$, for the weak-star convergence on $\mathbb{P}_1(\mathbb{R})-\{\alpha,\alpha^\sigma\}$, we have*

$$\frac{\pi^2\,[\mathrm{PSL}_2(\mathbb{Z}):G]}{24\,[\mathrm{PSL}_2(\mathbb{Z})_\beta:G_\beta]\,R_\beta\, s}\sum_{\beta'\in G\cdot\beta,\ h_\alpha(\beta')\leqslant s}\Delta_{x^-_\alpha(\beta')}+\Delta_{x^+_\alpha(\beta')}\ \overset{*}{\rightharpoonup}\ \frac{dt}{|Q_\alpha(t)|}\,.$$

This result, whose proof uses asymptotic properties of crossratios, implies that

$$\mathrm{Card}\{\beta'\in\mathrm{SO}_{Q_\alpha}(\mathbb{Z})\backslash\mathrm{PSL}_2(\mathbb{Z})\cdot\beta,\ h_\alpha(\beta')\leqslant s\}\sim\frac{48\,R_\alpha\,R_\beta}{\pi^2\,|\alpha-\alpha^\sigma|}\ s\,.$$

We refer to Theorem 4.9 for a version with congruences, and to Theorem 4.10 for an extension of this result to quadratic irrationals over an imaginary quadratic extension of $\mathbb{Q}$.

Towards the same measure, we also have the following equidistribution result of integral representations of quadratic norm forms (see Section 5.3 for generalisations as well as [13] for higher dimensional norm forms).

Theorem 1.6. — *If α is a real quadratic irrational over $\mathbb{Q}$, then, as $s\to+\infty$, for the weak-star convergence on $\mathbb{P}_1(\mathbb{R})-\{\alpha,\alpha^\sigma\}$, we have*

$$\frac{\pi^2}{12\ s}\sum_{(u,v)\in\mathbb{Z}^2,\ (u,v)=1,\ |u^2-\mathtt{tr}\alpha\,uv+\mathtt{n}(\alpha)\,v^2|\leqslant s}\Delta_{\frac{u}{v}}\ \overset{*}{\rightharpoonup}\ \frac{dt}{|Q_\alpha(t)|}.$$

Our final equidistribution result for this introduction is the following equidistribution of coefficients of binary Hermitian forms in an orbit under the Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$, using as complexity their first coefficient (see Subsection 5.1 for extensions, in particular to binary Hamiltonian forms). Given an imaginary quadratic number field K, let

$$f(u,v)=a\,|u|^2+2\,\mathrm{Re}(b\,\overline{u}\,v)+c\,|v|^2$$

be a binary Hermitian form, which is integral over $\mathcal{O}_K$ (that is $a = a(f), c = c(f) \in \mathbb{Z}$ and $b = b(f) \in \mathcal{O}_K$), and indefinite (that is $\operatorname{Disc}(f) = |b|^2 - ac > 0$). We denote by $\cdot$ the action of $\mathrm{SL}_2(\mathbb{C})$ on the set of binary Hermitian forms by precomposition, and by $\mathrm{SU}_f(\mathcal{O}_K)$ the stabiliser of f in $\mathrm{SL}_2(\mathcal{O}_K)$.

Theorem 1.7. — *Let G be a finite index subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$. As $s \to +\infty$,*

$$\frac{[\mathrm{SL}_2(\mathcal{O}_K) : G]\, |D_K|^{\frac{3}{2}}\, \zeta_K(2) \operatorname{Disc}(f)}{\iota_G\, \pi^2 \operatorname{Covol}(\mathrm{SU}_f(\mathcal{O}_K) \cap G)\, s^2} \sum_{f' \in G \cdot f,\ 0 < |a(f')| \leqslant s} \Delta_{\frac{b(f')}{a(f')}} \overset{*}{\rightharpoonup} \operatorname{Leb}_{\mathbb{C}}\,.$$

Identifying the ring of integers $\mathcal{O}_K$ with the upper triangular unipotent subgroup of $\mathrm{SL}_2(\mathcal{O}_K)$, this theorem implies the following counting result of integral binary Hermitian forms (ordered by their first coefficient) in an orbit of the Bianchi group $\mathrm{SL}_2(\mathcal{O}_K)$: as $s \to +\infty$,

$$\operatorname{Card}\{f' \in \mathcal{O}_K \backslash \mathrm{SL}_2(\mathcal{O}_K) \cdot f : 0 < |a(f')| \leqslant s\} \sim \frac{\pi^2\, \iota_G \operatorname{Covol}(\mathrm{SU}_f(\mathcal{O}_K))}{2\, |D_K|^{\frac{1}{2}}\, \zeta_K(2) \operatorname{Disc}(f)}\ s^2.$$

All the above limits and asymptotic formulas come with an error term. The first tool used to prove the above theorems is contained in the geometric equidistribution results of [38]. The particular case of the result of op. cit. that we will use, stated in Section 2, is the following one. Given a lattice Γ in the isometry group of the real n-dimensional hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, and two horoballs or totally geodesic subspaces D^-, D^+ whose stabilisers in Γ have cofinite volume, the initial tangent vectors of the common perpendiculars between D_- and the images under Γ of D^+ equidistribute in the unit normal bundle of D^-. See also [27, 28] for related counting and equidistribution results in real hyperbolic spaces, and [19, 39] in negatively curved symmetric spaces.

The paper is organised according to the arithmetic applications: In Section 3, we apply Section 2 with both D^- and D^+ horoballs to prove generalisations of the classical Mertens' formula, describing the asymptotic behaviour of the average order of Euler's function, for the rings of integers of quadratic imaginary number fields and maximal orders in definite quaternion algebras over $\mathbb{Q}$. In Section 4, we consider counting and equidistribution of quadratic irrationals in terms of two complexities, the height introduced in [32] and the relative height mentioned above. In Subsection 4.1, the geometric result is applied when D^- is a horoball and D^+ is a geodesic line, and in Subsection 4.3 and 4.4 (where we extend Pollicott's result on the asymptotic of crossratios to prove the convergence of generalised Schottky-Klein functions) when both D^- and D^+ are geodesic lines. In the final section, we consider representations of integers by binary quadratic, Hermitian and Hamiltonian forms, applying the geometric result when D^- is a horoball and D^+ is either a geodesic line or a totally geodesic hyperplane or (when considering positive definite forms) a point.

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2. Geometric counting and equidistribution

In this section, we briefly review a simplified version of the geometric counting and equidistribution results proved in [38], whose arithmetic applications will be considered in the other parts of this paper (see also [37] for related references and [39] for the case of locally symmetric spaces).

Let $n \geqslant 2$, let Γ be a discrete nonelementary group of isometries of the n-dimensional real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$, and let $M = \Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ and $T^1M = \Gamma\backslash T^1\mathbb{H}^n_{\mathbb{R}}$ be the quotient orbifolds. Let D^- and D^+ be nonempty proper closed convex subsets in $\mathbb{H}^n_{\mathbb{R}}$, with stabilisers Γ_{D^-} and Γ_{D^+} in Γ, such that the families $(\gamma D^-)_{\gamma\in\Gamma/\Gamma_{D^-}}$ and $(\gamma D^+)_{\gamma\in\Gamma/\Gamma_{D^+}}$ are locally finite in $\mathbb{H}^n_{\mathbb{R}}$ (see [38, §3.3] for more general families in any simply connected complete Riemannian manifold with pinched negative curvature).

We denote by $\partial_\infty\mathbb{H}^n_{\mathbb{R}}$ the boundary at infinity of $\mathbb{H}^n_{\mathbb{R}}$, by $\Lambda\Gamma$ the limit set of Γ and by $(\xi, x, y) \mapsto \beta_\xi(x, y)$ the Busemann cocycle on $\partial_\infty\mathbb{H}^n_{\mathbb{R}} \times \mathbb{H}^n_{\mathbb{R}} \times \mathbb{H}^n_{\mathbb{R}}$ defined by

$$(\xi, x, y) \mapsto \beta_\xi(x, y) = \lim_{t\to+\infty} d(\rho_t, x) - d(\rho_t, y) \ ,$$

where $\rho : t \mapsto \rho_t$ is any geodesic ray in $\mathbb{H}^n_{\mathbb{R}}$ with point at infinity ξ and d is the hyperbolic distance.

For every $v \in T^1\mathbb{H}^n_{\mathbb{R}}$, let $\pi(v) \in \mathbb{H}^n_{\mathbb{R}}$ be its footpoint, and let v_-, v_+ be the points at infinity of the geodesic line $t \mapsto c_v(t)$ in $\mathbb{H}^n_{\mathbb{R}}$ whose tangent vector at time $t = 0$ is v. We denote by $\partial^1_\pm D^\mp$ the *outer/inner unit normal bundle* of $\partial D^\mp$, that is, the set of $v \in T^1\mathbb{H}^n_{\mathbb{R}}$ such that $\pi(v) \in \partial D^\mp$, $v_\pm \in \partial_\infty\mathbb{H}^n_{\mathbb{R}} - \partial_\infty D^\mp$ and the closest point projection on $D^\mp$ of $v_\pm$ is $\pi(v)$. For every γ, γ' in Γ such that γD^- and $\gamma' D^+$ have a common perpendicular (that is, when the closures $\overline{\gamma D^-}$ and $\overline{\gamma' D^+}$ in $\mathbb{H}^n_{\mathbb{R}} \cup \partial_\infty\mathbb{H}^n_{\mathbb{R}}$ are disjoint), we denote by $\alpha_{\gamma,\gamma'}$ this common perpendicular (starting from γD^- at time $t = 0$), by $\ell(\alpha_{\gamma,\gamma'})$ its length, by $v^-_{\gamma,\gamma'} \in \gamma\partial^1_+ D^-$ its initial tangent vector and by $v^+_{\gamma,\,\gamma'} \in \gamma'\partial^1_- D^+$ its terminal tangent vector. The *multiplicity* of $\alpha_{\gamma,\gamma'}$ is

$$m_{\gamma,\gamma'} = \frac{1}{\operatorname{Card}(\gamma\Gamma_{D^-}\gamma^{-1} \cap \gamma'\Gamma_{D^+}\gamma'^{-1})}\,,$$

which equals 1 when Γ acts freely on $T^1\mathbb{H}^n_{\mathbb{R}}$ (for instance when Γ is torsion-free). Let

$$\mathcal{N}_{D^-,\,D^+}(t) = \sum_{\substack{(\gamma,\gamma')\in\Gamma\backslash((\Gamma/\Gamma_{D^-})\times(\Gamma/\Gamma_{D^+}))\\ \overline{\gamma D^-}\,\cap\,\overline{\gamma' D^+}=\emptyset,\ \ell(\alpha_{\gamma,\,\gamma'})\leqslant t}} m_{\gamma,\gamma'} = \sum_{\substack{[\gamma]\in\Gamma_{D^-}\backslash\Gamma/\Gamma_{D^+}\\ \overline{D^-}\,\cap\,\overline{\gamma D^+}=\emptyset,\ \ell(\alpha_{e,\gamma})\leqslant t}} m_{e,\gamma},$$

where Γ acts diagonally on $\Gamma\times\Gamma$. When Γ has no torsion, $\mathcal{N}_{D^-,\,D^+}(t)$ is the number (with multiplicities coming from the fact that $\Gamma_{D^\pm}\backslash D^\pm$ is not assumed to be embedded in M) of the common perpendiculars of length at most t between the images of D^- and D^+ in M. We refer to [38, §4] for the use of Hölder-continuous potentials on $T^1\mathbb{H}^n_{\mathbb{R}}$ to modify this counting function by adding weights, which could be useful for some further arithmetic applications.

Recall the following notions (see for instance [41]). The *critical exponent* of Γ is

$$\delta_\Gamma = \limsup_{N\to+\infty} \frac{1}{N} \ln \operatorname{Card}\{\gamma\in\Gamma : d(x_0,\gamma x_0)\leqslant N\}\,,$$

which is positive, finite and independent of the base point $x_0 \in \mathbb{H}^n_{\mathbb{R}}$. Let $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ be a *Patterson density* for Γ, that is, a family $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{R}}}$ of finite measures on $\partial_\infty\mathbb{H}^n_{\mathbb{R}}$ whose support is $\Lambda\Gamma$, such that $\gamma_*\mu_x = \mu_{\gamma x}$ and

$$\frac{d\mu_x}{d\mu_y}(\xi) = e^{-\delta_\Gamma\beta_\xi(x,\,y)}$$

for all $\gamma\in\Gamma$, $x,y\in\mathbb{H}^n_{\mathbb{R}}$ and $\xi\in\partial_\infty\mathbb{H}^n_{\mathbb{R}}$. For a normalisation, we assume that $\|\mu_{x_0}\| = \operatorname{Vol}(\mathbb{S}^{n-1})$. The *Bowen-Margulis measure* $\widetilde{m}_{\rm BM}$ for Γ on $T^1\mathbb{H}^n_{\mathbb{R}}$ is defined, using Hopf's parametrisation $v\mapsto (v_-, v_+, \beta_{v_+}(x_0,\pi(v))\,)$ of $T^1\mathbb{H}^n_{\mathbb{R}}$, by

$$d\widetilde{m}_{\rm BM}(v) = e^{-\delta_\Gamma(\beta_{v_-}(\pi(v),\,x_0)+\beta_{v_+}(\pi(v),\,x_0))}\,d\mu_{x_0}(v_-)\,d\mu_{x_0}(v_+)\,dt\,.$$

The measure $\widetilde{m}_{\rm BM}$ is nonzero and independent of $x_0\in\mathbb{H}^n_{\mathbb{R}}$. It is invariant under the geodesic flow, the antipodal map $v\mapsto -v$ and the action of Γ, and thus defines a nonzero measure $m_{\rm BM}$ on T^1M, called the *Bowen-Margulis measure* on $M=\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$, which is invariant under the geodesic flow of M and the antipodal map. When $m_{\rm BM}$ is finite, the probability measure $\frac{m_{\rm BM}}{\|m_{\rm BM}\|}$ is then uniquely defined, and it is the unique probability measure of maximal entropy for the geodesic flow (see [29]). This holds for instance when M has finite volume or when Γ is geometrically finite.

Using the endpoint homeomorphism $v \mapsto v_\pm$ from $\partial^1_\pm D^\mp$ to $\partial_\infty \mathbb{H}^n_\mathbb{R} - \partial_\infty D^\mp$, the *skinning measure* $\widetilde{\sigma}_{D^\mp}$ of Γ on $\partial^1_\pm D^\mp$ is defined by

$$d\widetilde{\sigma}_{D^\mp}(v) = e^{-\delta\,\beta_{v_\pm}(\pi(v),\,x_0)}\, d\mu_{x_0}(v_\pm)\,,$$

see [27, §1.2] when $D^\mp$ is a horoball or a totally geodesic subspace in $\mathbb{H}^n_\mathbb{R}$ and [36], [38] for the general case of convex subsets in variable curvature and with a potential.

The measure $\widetilde{\sigma}_{D^\mp}$ is independent of $x_0 \in \mathbb{H}^n_\mathbb{R}$, it is nonzero if $\Lambda\Gamma$ is not contained in $\partial_\infty D^\mp$, and satisfies $\widetilde{\sigma}_{\gamma D^\mp} = \gamma_* \widetilde{\sigma}_{D^\mp}$ for every $\gamma \in \Gamma$. Since the family $(\gamma D^\mp)_{\gamma\in\Gamma}$ is locally finite in $\mathbb{H}^n_\mathbb{R}$, the measure $\sum_{\gamma\in\Gamma/\Gamma_{D^\mp}} \gamma_*\widetilde{\sigma}_{D^\mp}$ is a well defined Γ-invariant locally finite (Borel nonnegative) measure on $T^1\mathbb{H}^n_\mathbb{R}$, hence induces a locally finite measure $\sigma_{D^\mp}$ on $T^1M = \Gamma\backslash T^1\mathbb{H}^n_\mathbb{R}$, called the *skinning measure* of $D^\mp$ in T^1M. We refer to [28, §5] and [36, Theo. 9] for finiteness criteria of the skinning measure $\sigma_{D^\mp}$, in particular satisfied when M has finite volume and if either $D^\mp$ is a horoball centred at a parabolic fixed point of Γ or if $D^\mp$ is a totally geodesic subspace.

The following result on the asymptotic behaviour of the counting function $\mathcal{N}_{D^-,\,D^+}$ in real hyperbolic space is a special case of much more general results [38, Coro. 20, 21, Theo. 28] using [23] for the error term, to check the exponential decay of correlation under the stated assumption on the critical exponent. Furthermore, the equidistribution result of the initial and terminal tangent vectors of the common perpendiculars holds simultaneously in the outer and inner tangent bundles of D^- and D^+. We refer to [37] for a survey of the particular cases known before [38] due to Huber, Margulis, Herrmann, Cosentino, Roblin, Oh-Shah, Martin-McKee-Wambach, Pollicott, and the authors for instance.

For every $t \geqslant 0$, let

$$m_t(x) = \sum_{\gamma\in\Gamma/\Gamma_{D^+},\ \overline{D^-}\cap\overline{\gamma D^+}=\emptyset,\ \alpha_{e,\,\gamma}(0)=x,\ \ell(\alpha_{e,\,\gamma})\leqslant t} m_{e,\gamma}$$

be the multiplicity of a point $x \in \partial D^-$ as the origin of common perpendiculars with length at most t from D^- to the elements of the Γ-orbit of D^+. We denote by Δ_x the unit Dirac mass at a point x.

Theorem 2.1. — *Let Γ, D^-, D^+ be as above. Assume that the measures $m_{\rm BM}, \sigma_{D^-}, \sigma_{D^+}$ are nonzero and finite. Then*

$$\mathcal{N}_{D^-,\,D^+}(t) \;\sim\; \frac{\|\sigma_{D^-}\|\;\|\sigma_{D^+}\|}{\delta_\Gamma\;\|m_{\rm BM}\|}\; e^{\delta_\Gamma\, t}\;,$$

as $t \to +\infty$. If $\delta_\Gamma > \frac{1}{2}$ when $n = 2$ and $\delta_\Gamma > n - 2$ when $n \geqslant 3$, then the error term is $\mathrm{O}(e^{(\delta_\Gamma - \kappa)t})$ for some $\kappa > 0$. Furthermore, the origins of the common perpendiculars equidistribute in the boundary of D^-:

$$\lim_{t \to +\infty} \frac{\delta_\Gamma \, \|m_{\mathrm{BM}}\|}{\|\sigma_{D^-}\| \, \|\sigma_{D^+}\|} \, e^{-\delta_\Gamma \, t} \sum_{x \in \partial D^-} m_t(x) \, \Delta_x \;=\; \frac{\pi_* \widetilde{\sigma}_{D^-}}{\|\sigma_{D^-}\|} \tag{2.1}$$

for the weak-star convergence of measures on the locally compact space $\mathbb{H}^n_{\mathbb{R}}$. □

For smooth functions ψ with compact support on ∂D^-, if $\delta_\Gamma > \frac{1}{2}$ when $n = 2$ and $\delta_\Gamma > n - 2$ when $n \geqslant 3$, there is an error term in the above equidistribution claim evaluated at ψ, of the form $\mathrm{O}(e^{-\kappa\, t}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$, as proved in [38, Theo. 28], using as above [23].

When M has finite volume, we have $\delta_\Gamma = n - 1$, the Bowen-Margulis measure m_{BM} coincides up to a multiplicative constant with the Liouville measure on $T^1 M$, and the skinning measures of points, horoballs and totally geodesic subspaces $D^\pm$ coincide, again up to a multiplicative constant, with the (homogeneous) Riemannian measures on $\partial^1_\pm D^\mp$ induced by the Riemannian metric of $T^1\mathbb{H}^n_{\mathbb{R}}$. Let us denote by $\mathrm{Vol}(N)$ the Riemannian volume of any Riemannian manifold N. These proportionality constants were computed in [37, §7] and [38, Prop. 29]:

$$\|m_{\mathrm{BM}}\| = 2^{n-1}\, \mathrm{Vol}(\mathbb{S}^{n-1})\, \mathrm{Vol}(M)\,,$$

if D^- is a horoball, then

$$\|\sigma_{D^-}\| = 2^{n-1}(n-1)\mathrm{Vol}(\Gamma_{D^-}\backslash D^-),$$

and if D^- is a totally geodesic submanifold in $\mathbb{H}^n_{\mathbb{R}}$ of dimension $k^- \in \{0, \dots, n-1\}$ with pointwise stabiliser of order m^-, then $\widetilde{\sigma}_{D^-} = \mathrm{Vol}_{\partial^1_+ D^-}$, so that

$$\|\sigma_{D^-}\| = \frac{\mathrm{Vol}(\mathbb{S}^{n-k^- -1})}{m^-}\mathrm{Vol}(\Gamma_{D^-}\backslash D^-).$$

See [39, §3] for the computation of the proportionality constants in the complex hyperbolic case.

Using these explicit expressions, we now reformulate Theorem 2.1, considering the following cases.

(1) If D^- and D^+ are totally geodesic submanifolds in $\mathbb{H}^n_{\mathbb{R}}$ of dimensions k^- and k^+ in $\{1, \dots, n-1\}$, respectively, such that $\mathrm{Vol}(\Gamma_{D^-}\backslash D^-)$ and $\mathrm{Vol}(\Gamma_{D^+}\backslash D^+)$ are finite, let

$$c(D^-,\, D^+) = \frac{\operatorname{Vol}(\mathbb{S}^{n-k^- -1})\operatorname{Vol}(\mathbb{S}^{n-k^+ -1})}{2^{n-1}(n-1)\operatorname{Vol}(\mathbb{S}^{n-1})} \, \frac{\operatorname{Vol}(\Gamma_{D^-}\backslash D^-)\operatorname{Vol}(\Gamma_{D^+}\backslash D^+)}{\operatorname{Vol}(M)} \, .$$

(2) If D^- and D^+ are horoballs in $\mathbb{H}^n_{\mathbb{R}}$ centred at parabolic fixed points

$$c(D^-, D^+) = \frac{2^{n-1}(n-1)}{\operatorname{Vol}(\mathbb{S}^{n-1})} \, \frac{\operatorname{Vol}(\Gamma_{D^-}\backslash D^-)\operatorname{Vol}(\Gamma_{D^+}\backslash D^+)}{\operatorname{Vol}(M)} \, .$$

(3) If D^- is a horoball in $\mathbb{H}^n_{\mathbb{R}}$ centred at a parabolic fixed point of Γ and D^+ is a totally geodesic subspace in $\mathbb{H}^n_{\mathbb{R}}$ of dimension $k^+ \in \{1, \dots, n-1\}$ such that $\operatorname{Vol}(\Gamma_{D^+}\backslash D^+)$ is finite, let

$$c(D^-, D^+) = \frac{\operatorname{Vol}(\mathbb{S}^{n-k^+ -1})}{\operatorname{Vol}(\mathbb{S}^{n-1})} \, \frac{\operatorname{Vol}(\Gamma_{D^-}\backslash D^-)\operatorname{Vol}(\Gamma_{D^+}\backslash D^+)}{\operatorname{Vol}(M)} \, .$$

(4) If D^- is a horoball in $\mathbb{H}^n_{\mathbb{R}}$ centred at a parabolic fixed point of Γ and D^+ is a point of $\mathbb{H}^n_{\mathbb{R}}$, let

$$c(D^-, D^+) = \frac{\operatorname{Vol}(\Gamma_{D^-}\backslash D^-)}{\operatorname{Vol}(M)} \, .$$

Corollary 2.2. — *Let Γ be a discrete group of isometries of $\mathbb{H}^n_{\mathbb{R}}$ such that the orbifold $M = \Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ has finite volume. In each of the cases (1) to (4) above, if $m^\pm$ is the cardinality of the pointwise stabiliser of $D^\pm$, then there exists $\kappa > 0$ such that, as $t \to +\infty$,*

$$\mathcal{N}_{D^-,\, D^+}(t) = \frac{c(D^-, D^+)}{m^-\, m^+} \; e^{(n-1)t}\big(1 + \operatorname{O}(e^{-\kappa t})\big) \, .$$

Furthermore, the origins of the common perpendiculars equidistribute in ∂D^- to the induced Riemannian measure: if D^- is a horoball centred at a parabolic fixed point of Γ, then as $t \to +\infty$,

$$\frac{m^+\,(n-1)\operatorname{Vol}(\Gamma_{D^-}\backslash D^-)}{c(D^-, D^+)} \; e^{-(n-1)t} \sum_{x \in \partial D^-} m_t(x)\, \Delta_x \;\overset{*}{\rightharpoonup}\; \operatorname{Vol}_{\partial D^-} \, , \tag{2.2}$$

and if D^- is a totally geodesic subspace in $\mathbb{H}^n_{\mathbb{R}}$ of dimension $k^- \in \{1, \dots, n-1\}$, such that $\operatorname{Vol}(\Gamma_{D^-}\backslash D^-)$ is finite, then as $t \to +\infty$,

$$\frac{m^-\, m^+ \operatorname{Vol}(\Gamma_{D^-}\backslash D^-)}{c(D^-, D^+)} \; e^{-(n-1)t} \sum_{x \in D^-} m_t(x)\, \Delta_x \;\overset{*}{\rightharpoonup}\; \operatorname{Vol}_{D^-} \, . \quad \square \tag{2.3}$$

Again, for smooth functions ψ with compact support on ∂D^-, there is an error term in the above two equidistribution claims evaluated at ψ, of the form $\mathrm{O}(e^{-\kappa\, t}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$.

We end this section by recalling some terminology concerning the isometries of $\mathbb{H}^n_\mathbb{R}$. The *translation length* of an isometry γ of $\mathbb{H}^n_\mathbb{R}$ is

$$\ell(\gamma) = \inf_{x \in \mathbb{H}^n_\mathbb{R}} d(x, \gamma x)\,.$$

If $\ell(\gamma) > 0$, then γ is *loxodromic*. Each loxodromic isometry γ stabilises a unique geodesic line in $\mathbb{H}^n_\mathbb{R}$, called the *translation axis* of γ and denoted by $\operatorname{Axis}\gamma$, on which it acts as a translation by $\ell(\gamma)$. The points at infinity of $\operatorname{Axis}\gamma$ are the two fixed points of γ in $\partial_\infty \mathbb{H}^n_\mathbb{R}$, and we denote by γ^- and γ^+ the attracting and repelling fixed points of γ respectively. Any loxodromic element of Γ is contained in a maximal cyclic subgroup.

An element $\gamma \in \Gamma$ is *primitive* if the cyclic subgroup $\gamma^\mathbb{Z}$ it generates is a maximal cyclic subgroup of Γ. A loxodromic element $\gamma \in \Gamma$ is Γ-*reciprocal* if there is an element in Γ that switches the two fixed points of γ. If γ is Γ-reciprocal, then let $\iota_\Gamma(\gamma) = 2$, otherwise, we set $\iota_\Gamma(\gamma) = 1$. If γ is a primitive loxodromic element of Γ, then the stabiliser of $\operatorname{Axis}\gamma$ is generated by γ, an elliptic element that switches the two points at infinity of the axis of γ if γ is reciprocal, and a (possibly trivial) group of finite order $m_\Gamma(\gamma)$, which is the pointwise stabiliser of $\operatorname{Axis}\gamma$, so that

$$m_\Gamma(\gamma) = \frac{1}{\iota_\Gamma(\gamma)}\,[\operatorname{Stab}_\Gamma(\operatorname{Axis}\gamma) : \gamma^\mathbb{Z}]\,. \tag{2.4}$$

3. Generalised Mertens' formulas

A classical result, known as *Mertens' formula* (see for example [16, Thm. 330], a better error term is due to Walfisz [46]), describes the asymptotic behaviour of the average order $\Phi : \mathbb{N} - \{0\} \to \mathbb{N} - \{0\}$ of Euler's function φ, where $\varphi(n) = \operatorname{Card}((\mathbb{Z}/n\mathbb{Z})^\times)$:

$$\Phi(n) = \sum_{k=1}^{n} \varphi(k) = \frac{3}{\pi^2}\, n^2 + \mathrm{O}(n \ln n)\,.$$

Corollary 2.2 provides a geometric proof of Mertens' formula with a less explicit error term, as follows. The modular group $\Gamma_\mathbb{Q} = \mathrm{PSL}_2(\mathbb{Z})$ isometrically acts on the upper halfplane model of the real hyperbolic plane $\mathbb{H}^2_\mathbb{R}$

via Möbius transformations. Let $\mathcal{H}_\infty$ be the horoball in $\mathbb{H}^2_\mathbb{R}$ that consists of all points with vertical coordinate at least 1, and let Γ_∞ be its stabiliser in $\Gamma_\mathbb{Q}$. The images of $\mathcal{H}_\infty$ under $\Gamma_\mathbb{Q}$ different from $\mathcal{H}_\infty$ are the nonempty intersections with $\mathbb{H}^2_\mathbb{R}$ of the Euclidean disks tangent to the real line at the rational points $\frac{p}{q}$ (where $q > 0$ and $(p,q) = 1$) with Euclidean diameter $\frac{1}{q^2}$. The common perpendicular from $\mathcal{H}_\infty$ to this disc exists if and only if $q > 1$, its length is $\ln q^2$, and its multiplicity is 1. For every $n \in \mathbb{N} - \{0\}$, the cardinality of the set of rational points $\frac{p}{q} \in]0,1]$ with $q \leqslant n$ is $\Phi(n)$. Thus, $\Phi(n) = \mathcal{N}_{\mathcal{H}_\infty,\mathcal{H}_\infty}(\ln n^2) + 1$. Now $\mathrm{Vol}(\Gamma_\infty \backslash \mathcal{H}_\infty) = 1$ and $\mathrm{Vol}(\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R}) = \frac{\pi}{3}$, and we can apply Corollary 2.2 to conclude that for some $\kappa > 0$, as $n \to +\infty$,

$$\Phi(n) = \frac{3}{\pi^2}\, n^2 + \mathrm{O}(n^{2-\kappa})\,. \tag{3.1}$$

Let us reformulate Mertens' formula in a way that is easily extendable in more general contexts (see also [39]). Let the additive group $\mathbb{Z}$ act on $\mathbb{Z}\times\mathbb{Z}$ by horizontal shears (transvections): $k\cdot(u,v) = (u+kv, v)$, and define

$$\psi(s) = \mathrm{Card}\big(\mathbb{Z}\backslash\big\{(u,v) \in \mathbb{Z}\times\mathbb{Z}\ : (u,v) = 1,\ |v| \leqslant s\big\}\big)\,.$$

We easily have $\psi(n) = 2\,\Phi(n)+2$ for every $n \in \mathbb{N}-\{0\}$, so that Mertens' formula (3.1) is equivalent to $\psi(n) = \frac{6}{\pi^2}\, s + \mathrm{O}(s^{1-\kappa})$. Furthermore, a straightforward application of Corollary 2.2 shows that as $s \to +\infty$, we have

$$\frac{\pi^2}{3\,s} \sum_{(u,\,v)=1,\ 1\leqslant v\leqslant s} \Delta_{(\frac{u}{v},\,1)} \overset{*}{\rightharpoonup} \mathrm{Vol}_{\partial\mathcal{H}_\infty}\,.$$

Observing that the pushforward of the measures on $\partial\mathcal{H}_\infty$ by the map $f : (x,1) \mapsto x$ from $\partial\mathcal{H}_\infty$ to $\mathbb{R}$ is linear, is continuous for the weak-star topology, maps the unit Dirac mass at a point p to the unit Dirac mass at $f(p)$ and the volume measure of $\partial\mathcal{H}_\infty$ to the Lebesgue measure, we get the well-known result on the equidistribution of the Farey fractions: as $s \to +\infty$, we have

$$\frac{\pi^2}{6\,s} \sum_{(u,\,v)=1,\ |v|\leqslant s} \Delta_{\frac{u}{v}} \overset{*}{\rightharpoonup} \mathrm{Leb}_\mathbb{R}\,.$$

In subsections 3.1 and 3.2, we generalise the above results to quadratic imaginary number fields and definite quaternion algebras over $\mathbb{Q}$.

3.1. A Mertens' formula for the rings of integers of imaginary quadratic number fields

Let K be an imaginary quadratic number field, with ring of integers $\mathcal{O}_K$, discriminant D_K, zeta function ζ_K and norm $\mathtt{n}$. Let $\omega_K = \mathrm{Card}(\mathcal{O}_K{}^\times)$ (with

$\omega_K = 2$ if $D_K \neq -3, -4$ for future use). Let $\mathfrak{m}$ be a (nonzero) fractional ideal of $\mathcal{O}_K$, with norm $\mathtt{n}(\mathfrak{m})$. Note that the action of the additive group $\mathcal{O}_K$ on $\mathbb{C}\times\mathbb{C}$ by the horizontal shears $k\cdot(u,v) = (u+kv, v)$ preserves $\mathfrak{m}\times\mathfrak{m}$.

We consider the counting function $\psi_{\mathfrak{m}} : [0, +\infty[\, \to \mathbb{N}$ defined by

$$\psi_{\mathfrak{m}}(s) = \operatorname{Card}\big(\mathcal{O}_K \backslash \big\{(u,v) \in \mathfrak{m}\times\mathfrak{m} \;:\; \mathtt{n}(\mathfrak{m})^{-1}\mathtt{n}(v) \leqslant s,\ \mathcal{O}_K u + \mathcal{O}_K v = \mathfrak{m}\big\}\big)\,.$$

Note that $\psi_{\mathfrak{m}}$ depends only on the ideal class of $\mathfrak{m}$ and thus we can assume in the computations that $\mathfrak{m}$ is integral. The *Euler function* φ_K of K is defined on the set of (nonzero) integral ideals $\mathfrak{a}$ of $\mathcal{O}_K$ by $\varphi_K(\mathfrak{a}) = \operatorname{Card}((\mathcal{O}_K/\mathfrak{a})^\times)$, see for example [4, §4A]. Thus,

$$\psi_{\mathcal{O}_K}(s) = \sum_{v\in\mathcal{O}_K,\ 0<\mathtt{n}(v)\leqslant s} \varphi_K(v\mathcal{O}_K)\,,$$

and the first claim of the following result, in the special case $\mathfrak{m} = \mathcal{O}_K$, is an analog of Mertens' formula, due to [14, Satz 2] (with a better error term), see also [6, §4.3]. The equidistribution part of Theorem 3.1 (stated as Theorem 1.1 in the introduction) generalises [6, Th. 4] which covers the case $\mathfrak{m} = \mathcal{O}_K$ without an explicit proportionality constant but with an explicit estimate on the speed of equidistribution. Note that the ideal class group of $\mathcal{O}_K$ is in general nontrivial, hence the extension to general $\mathfrak{m}$ is interesting.

Theorem 3.1. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi_{\mathfrak{m}}(s) = \frac{\pi}{\zeta_K(2)\,\sqrt{|D_K|}}\; s^2 + \mathrm{O}(s^{2-\kappa})\,.$$

Furthermore, as $s \to +\infty$,

$$\frac{|D_K|\,\zeta_K(2)}{2\,\pi\, s^2} \sum_{\substack{(u,v)\in\mathfrak{m}\times\mathfrak{m}\\ \mathtt{n}(\mathfrak{m})^{-1}\mathtt{n}(v)\leqslant s,\ \mathcal{O}_K u+\mathcal{O}_K v=\mathfrak{m}}} \Delta_{\frac{u}{v}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{C}}\,,$$

with error term $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ when evaluated on $\mathcal{C}^\ell$-smooth functions ψ with compact support on $\mathbb{C}$, for ℓ big enough.

Proof. — The Poincaré extension identifies the group $\mathrm{PSL}_2(\mathbb{C})$ acting projectively on the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$ with the group of orientation preserving isometries of the upper halfspace model of the real hyperbolic space $\mathbb{H}^3_{\mathbb{R}}$. We denote the image in $\mathrm{PSL}_2(\mathbb{C})$ of any subgroup G of $\mathrm{SL}_2(\mathbb{C})$ by $\overline{G}$ and of any element g of $\mathrm{SL}_2(\mathbb{C})$ again by g. The *Bianchi group* $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ is a (nonuniform) arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$, whose covolume is given by Humbert's formula (see [10, §8.8 and 9.6])

$$\operatorname{Vol}(\,\overline{\Gamma_K}\,\backslash\mathbb{H}^3_{\mathbb{R}}) = \frac{|D_K|^{\frac{3}{2}}\zeta_K(2)}{4\pi^2}\,. \tag{3.2}$$

Let $\Gamma_{x,\,y}$ be the stabiliser of any $(x,y) \in \mathcal{O}_K \times \mathcal{O}_K$ under the linear action of Γ_K on $\mathcal{O}_K \times \mathcal{O}_K$. In particular, $\Gamma_{1,\,0}$ is the upper triangular unipotent subgroup of Γ_K, whose linear action on $\mathcal{O}_K \times \mathcal{O}_K$ identifies with the action of $\mathcal{O}_K$ via horizontal shears. Furthermore, $x\mathcal{O}_K + y\mathcal{O}_K = u\mathcal{O}_K + v\mathcal{O}_K$ if and only if $(u,v) \in \Gamma_K(x,y)$, and the ideal class group of K corresponds bijectively to the set $\overline{\Gamma_K}\backslash\mathbb{P}^1(K)$ of cusps of the orbifold $\overline{\Gamma_K}\backslash\mathbb{H}^3_\mathbb{R}$ by the map induced by $x\mathcal{O}_K + y\mathcal{O}_K \mapsto \frac{x}{y} \in K \cup \{\infty\}$, see for example Section 7.2 of [10] for details.

Let us now fix $(x,y) \in \mathcal{O}_K \times \mathcal{O}_K$ such that $\mathfrak{m} = x\mathcal{O}_K + y\mathcal{O}_K$. We define $\rho = \frac{x}{y}$. By what we just explained,

$$\psi_\mathfrak{m}(s) = \operatorname{Card}\big(\Gamma_{1,\,0}\backslash\big\{(u,v) \in \Gamma_K(x,y) : |v|^2 \leqslant \mathtt{n}(\mathfrak{m})s\big\}\big)\,.$$

Let us prove the first claim (it follows by integration from the second one, but the essential ingredients of the proof of either statement are the same).

If $y = 0$, let $\gamma_\rho = \mathrm{id}$ and we may assume that $x = 1$ since $\psi_\mathfrak{m}$ depends only on the ideal class of $\mathfrak{m}$. If $y \neq 0$, let

$$\gamma_\rho = \begin{pmatrix} \rho & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})\,.$$

Let $\tau \in \,]0,1]$. Let $\mathcal{H}_\infty$ be the horoball in $\mathbb{H}^3_\mathbb{R}$ that consists of all points with Euclidean height at least $1/\tau$ and let $\mathcal{H}_\rho = \gamma_\rho\mathcal{H}_\infty$. For any $\rho' \in K \cup \{\infty\}$, let $\overline{\Gamma_{\mathcal{H}_{\rho'}}}$ be the stabiliser in $\overline{\Gamma_K}$ of the similarly defined horoball $\mathcal{H}_{\rho'}$. Recall that a subset A of a set endowed with a group action is *precisely invariant* if A meets one of its images by an element of $\overline{\Gamma_K}$ only if A coincides with this image. As in [33] after its Equation (4), we fix τ small enough such that $\mathcal{H}_\infty$ and $\mathcal{H}_\rho$ are precisely invariant under $\overline{\Gamma_K}$.

For any $g \in \mathrm{SL}_2(\mathbb{C})$ such that $g\mathcal{H}_\infty$ and $\mathcal{H}_\infty$ are disjoint, it is easy to check using the explicit expression of the Poincaré extension (see for example [2, Eq. 4.1.4.]) that the length of the common perpendicular of $\mathcal{H}_\infty$ and $g\mathcal{H}_\infty$ is $|\ln(\tau^{-2}|c|^2)|$, where c is the $(2,1)$-entry of g. If $y \neq 0$, then $(u,v) = g(x,y)$ if and only if $(\frac{u}{y}, \frac{v}{y}) = g\gamma_\rho(1,0)$, and the $(2,1)$-entry of $g\gamma_\rho$ is $\frac{v}{y}$. Thus, the length $\ell(\delta_g)$ of the common perpendicular δ_g between $\mathcal{H}_\infty$ and $g\mathcal{H}_\rho = g\gamma_\rho\mathcal{H}_\infty$, if these two horoballs are disjoint, is $\big|\ln(\frac{|v|^2}{\tau^2|y|^2})\big|$ if $y \neq 0$ and $|\ln(\tau^{-2}|v|^2)|$ otherwise.

For the rest of the proof, we concentrate on the case $y \neq 0$. The case $y = 0$ is treated similarly. By discreteness, there are only finitely many double classes $[g]\in\Gamma_{1,0}\backslash\Gamma_K/\Gamma_{x,y}$ such that $\mathcal{H}_\infty$ and $g\mathcal{H}_\rho$ are not disjoint or such that $|v|\leqslant|y|$ or such that the multiplicity of δ_g is different from 1. Since the stabilisers $\Gamma_{1,0}$ and $\Gamma_{x,y}$ do not contain $-\mathrm{id}$, we have

$$\begin{aligned}\psi_{\mathfrak{m}}(s)&=\operatorname{Card}\Big\{[g]\in\Gamma_{1,0}\backslash\Gamma_K/\Gamma_{x,y}\;:\;\ell(\delta_g)\leqslant\ln\frac{\mathtt{n}(\mathfrak{m})s}{\tau^2|y|^2}\;\Big\}+\mathrm{O}(1)\\&=2\operatorname{Card}\Big\{[g]\in\overline{\Gamma_{1,0}}\backslash\overline{\Gamma_K}/\overline{\Gamma_{x,y}}\;:\;\ell(\delta_g)\leqslant\ln\frac{\mathtt{n}(\mathfrak{m})s}{\tau^2|y|^2}\;\Big\}+\mathrm{O}(1)\,.\end{aligned}$$

We use [33, Lem. 7] with $C=\overline{\Gamma_K}$, $A=\overline{\Gamma_{1,0}}$, $A'=\overline{\Gamma_{\mathcal{H}_\infty}}$, $B=\overline{\Gamma_{x,y}}$, $B'=\overline{\Gamma_{\mathcal{H}_\rho}}$, since there are only finitely many $[g]\in A'\backslash C/B'$ such that $g^{-1}A'g\cap B'\neq\{1\}$. Since $[\overline{\Gamma_{\mathcal{H}_\infty}}:\overline{\Gamma_{1,0}}]=[\overline{\Gamma_{\mathcal{H}_\rho}}:\overline{\Gamma_{x,y}}]=\frac{\omega_K}{2}$, we then have

$$\psi_{\mathfrak{m}}(s)=\frac{{\omega_K}^2}{2}\operatorname{Card}\Big\{[g]\in\overline{\Gamma_{\mathcal{H}_\infty}}\backslash\overline{\Gamma_K}/\overline{\Gamma_{\mathcal{H}_\rho}}\;:\;\ell(\delta_g)\leqslant\ln\frac{\mathtt{n}(\mathfrak{m})s}{\tau^2|y|^2}\;\Big\}+\mathrm{O}(1)\,.$$

By [33, Lem. 6] and the last equality of the proof of Theorem 4 on page 1055 of op. cit., we have $\operatorname{Vol}(\overline{\Gamma_{\mathcal{H}_\infty}}\backslash\mathcal{H}_\infty)=\frac{\tau^2\sqrt{|D_K|}}{2\,\omega_K}$ and $\operatorname{Vol}(\overline{\Gamma_{\mathcal{H}_\rho}}\backslash\mathcal{H}_\rho)=\frac{\tau^2\sqrt{|D_K|}}{2\,\omega_K}\,\frac{|y|^4}{\mathtt{n}(\mathfrak{m})^2}$. Thus, Corollary 2.2, applied to $n=3$, $\Gamma=\overline{\Gamma_K}$, $D^-=\mathcal{H}_\infty$ and $D^+=\mathcal{H}_\rho$, gives, since the pointwise stabilisers of $\mathcal{H}_\infty$ and $\mathcal{H}_\rho$ are trivial,

$$\psi_{\mathfrak{m}}(s)=\frac{{\omega_K}^2}{2}\;\frac{2^2\,2\,\tau^4\,|D_K|\,|y|^4\,4\pi^2}{4\,{\omega_K}^2\,\mathtt{n}(\mathfrak{m})^2\,4\pi\,|D_K|^{\frac{3}{2}}\,\zeta_K(2)}\,\frac{\mathtt{n}(\mathfrak{m})^2}{\tau^4|y|^4}\,s^2(1+\mathrm{O}(s^{-\kappa}))\,,$$

which, after simplification, proves the first claim.

To prove the second claim when $y\neq 0$, note that Equation (2.2) in Corollary 2.2 implies that as $t\to+\infty$, with the appropriate error term,

$$\frac{|D_K|\,\zeta_K(2)\,\omega_K}{2\,\pi}\,\frac{\mathtt{n}(\mathfrak{m})^2}{\tau^2\,|y|^4}\,e^{-2t}\sum_{z\in\partial\mathcal{H}_\infty}m_t(z)\,\Delta_z\overset{*}{\rightharpoonup}\operatorname{Vol}_{\partial\mathcal{H}_\infty}\,.$$

Let us use the change of variable $t=\ln\frac{\mathtt{n}(\mathfrak{m})s}{\tau^2|y|^2}$, the fact that the geodesic lines containing the common perpendiculars from $\mathcal{H}_\infty$ to the images of $\mathcal{H}_\rho$ by the elements of $\overline{\Gamma_K}$ are exactly the geodesic lines from ∞ to an element of $\overline{\Gamma_K}\cdot\frac{x}{y}$, and the above facts on disjointness, lengths and multiplicities. This gives, since the map $(u,v)\mapsto u/v$ is ω_K-to-1,

$$\frac{|D_K|\,\zeta_K(2)\,\omega_K\,\tau^2}{2\,\pi\,s^2}\,\frac{1}{\omega_K}\sum_{\substack{(u,v)\in\mathfrak{m}\times\mathfrak{m}\\ \mathtt{n}(\mathfrak{m})^{-1}|v|^2\leqslant s,\;\mathcal{O}_Ku+\mathcal{O}_Kv=\mathfrak{m}}}\Delta_{(\frac{u}{v},\,\frac{1}{\tau})}\overset{*}{\rightharpoonup}\operatorname{Vol}_{\partial\mathcal{H}_\infty}\,,$$

as $s \to +\infty$. The claim now follows from the observation that the pushforward of $\operatorname{Vol}_{\partial \mathcal{H}_\infty}$ by the endpoint map $(x, \frac{1}{\tau}) \mapsto x$ is $\tau^2 \operatorname{Leb}_{\mathbb{C}}$. The proof of the case $y = 0$ is similar. □

3.2. A Mertens' formula for maximal orders of rational quaternion algebras

Let $\mathbb{H}$ be Hamilton's quaternion algebra over $\mathbb{R}$, with $x \mapsto \overline{x}$ its conjugation, $\mathtt{N} : x \mapsto x\overline{x}$ its reduced norm and $\mathtt{Tr} : x \mapsto x + \overline{x}$ its reduced trace. We endow $\mathbb{H}$ with its standard Euclidean structure (making its standard basis orthonormal). Let A be a quaternion algebra over $\mathbb{Q}$, which is definite ($A \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}$), with reduced discriminant D_A. Let $\mathcal{O}$ be a maximal order in A, and let $\mathfrak{m}$ be a (nonzero) left ideal of $\mathcal{O}$, with reduced norm $\mathtt{N}(\mathfrak{m})$ (see [45] for definitions).

The additive group $\mathcal{O}$ acts on the left on $\mathbb{H} \times \mathbb{H}$ by the horizontal shears (transvections) $o \cdot (u, v) = (u + ov, v)$. Let us consider the following counting function of the generating pairs of elements of $\mathfrak{m}$, defined for every $s > 0$, by

$$\psi_{\mathfrak{m}}(s) = \operatorname{Card}\big(\mathcal{O} \backslash \big\{(u, v) \in \mathfrak{m} \times \mathfrak{m} \;:\; \mathtt{N}(\mathfrak{m})^{-1}\mathtt{N}(v) \leqslant s, \quad \mathcal{O}u + \mathcal{O}v = \mathfrak{m}\big\}\big) \,,$$

Theorem 3.2. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi_{\mathfrak{m}}(s) = \frac{90\, D_A^2}{\pi^2\, \zeta(3)\, \prod_{p | D_A} (p^3 - 1)} \;\; s^4 + \mathrm{O}(s^{4-\kappa}) \,,$$

with p ranging over positive rational primes. Furthermore,

$$\frac{\pi^2\, \zeta(3)\, \prod_{p | D_A} (p^3 - 1)}{360\, D_A\; s^4} \qquad \sum_{\substack{(u,v) \in \mathfrak{m} \times \mathfrak{m} \\ \mathtt{N}(\mathfrak{m})^{-1}\mathtt{N}(v) \leqslant s,\ \mathcal{O}u + \mathcal{O}v = \mathfrak{m}}} \qquad \Delta_{uv^{-1}} \;\overset{*}{\rightharpoonup}\; \operatorname{Leb}_{\mathbb{H}}$$

as $s \to +\infty$, with error term $\mathrm{O}(s^{-\kappa} \|\psi\|_\ell)$ when evaluated on $\mathcal{C}^\ell$-smooth functions ψ with compact support on $\mathbb{H}$, for ℓ big enough.

Proof. — We denote by $[\mathfrak{m}]$ the ideal class of a left fractional ideal $\mathfrak{m}$ of $\mathcal{O}$, and by ${}_{\mathcal{O}}\mathcal{I}$ the set of left ideal classes of $\mathcal{O}$, and by

$$\mathcal{O}_r(I) = \{x \in A \;:\; Ix \subset I\}$$

the right order of a $\mathbb{Z}$-lattice I in A (see [45] for definitions). Note that $\psi_{\mathfrak{m}}$ depends only on $[\mathfrak{m}]$. For every (u, v) in $A \times A - \{(0, 0)\}$, consider the two left fractional ideals of $\mathcal{O}$

$$I_{u,\,v} = \mathcal{O}u + \mathcal{O}v \;, \quad K_{u,\,v} = \begin{cases} \mathcal{O}u \cap \mathcal{O}v & \text{if } uv \neq 0 \,, \\ \mathcal{O} & \text{otherwise.} \end{cases}$$

Given $\mathfrak{m}, \mathfrak{m}'$ two left fractional ideals of $\mathcal{O}$ and $s > 0$, let

$$\psi_{\mathfrak{m},\mathfrak{m}'}(s) = \operatorname{Card}\big(\mathcal{O}\backslash\{(u,v) \in \mathfrak{m}\times\mathfrak{m} : \mathtt{N}(v) \leqslant \mathtt{N}(\mathfrak{m})s, I_{u,v} = \mathfrak{m}, [K_{u,v}] = [\mathfrak{m}']\}\big),$$

so that

$$\psi_{\mathfrak{m}} = \sum_{[\mathfrak{m}'] \in \,_{\mathcal{O}}\mathcal{I}} \psi_{\mathfrak{m},\,\mathfrak{m}'}\ . \tag{3.3}$$

We will give in Equation (3.7) an asymptotic to $\psi_{\mathfrak{m},\,\mathfrak{m}'}(s)$ as $s \to +\infty$, and in Equation (3.9) the related equidistribution result, for each $[\mathfrak{m}'] \in \,_{\mathcal{O}}\mathcal{I}$ (interesting in themselves), and then easily infer Theorem 3.2.

We refer for instance to [18], [35, §3] for the following properties of quaternionic homographies. Let $\mathrm{SL}_2(\mathbb{H})$ be the group of 2×2 matrices with coefficients in $\mathbb{H}$ and Dieudonné determinant 1. Recall that the Dieudonné determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\mathtt{N}(ad) + \mathtt{N}(bc) - \mathtt{Tr}(a\,\overline{c}\,d\,\overline{b}\,)\,. \tag{3.4}$$

The group $\mathrm{SL}_2(\mathbb{H})$ acts linearly on the left on the right $\mathbb{H}$-module $\mathbb{H} \times \mathbb{H}$. Therefore it acts projectively on the left on the right projective line $\mathbb{P}^1_r(\mathbb{H}) = (\mathbb{H} \times \mathbb{H} - \{0\})/\mathbb{H}^\times$. The group $\mathrm{PSL}_2(\mathbb{H}) = \mathrm{SL}_2(\mathbb{H})/\{\pm\mathrm{Id}\}$ identifies by the Poincaré extension procedure with the group of orientation preserving isometries of the upper halfspace model $\mathbb{H}\times\,]0, +\infty[$ of the real hyperbolic space $\mathbb{H}^5_{\mathbb{R}}$ of dimension 5, with Riemannian metric

$$ds^2(x) = \frac{ds^2_{\mathbb{H}}(z) + dr^2}{r^2}$$

at the point $x = (z, r)$. For any subgroup G of $\mathrm{SL}_2(\mathbb{H})$, we denote by $\overline{G}$ its image in $\mathrm{PSL}_2(\mathbb{H})$.

Let $\Gamma_{\mathcal{O}} = \mathrm{SL}_2(\mathcal{O}) = \mathrm{SL}_2(\mathbb{H}) \cap \mathcal{M}_2(\mathcal{O})$ be the *Hamilton-Bianchi group* of $\mathcal{O}$, which is a (nonuniform) arithmetic lattice in the connected real Lie group $\mathrm{SL}_2(\mathbb{H})$ (see for instance [31, page 1104]). Given $(x, y) \in \mathcal{O} \times \mathcal{O}$, let $\Gamma_{x,\,y}$ be the stabiliser of (x, y) for the left linear action of $\Gamma_{\mathcal{O}}$. By [35, Rem. 7], the map from the set $\overline{\Gamma_{\mathcal{O}}}\backslash\mathbb{P}^1_r(\mathcal{O})$ of cusps of $\overline{\Gamma_{\mathcal{O}}}\backslash\mathbb{H}^5_{\mathbb{R}}$ into $_{\mathcal{O}}\mathcal{I} \times \,_{\mathcal{O}}\mathcal{I}$ which associates, to the orbit of $[u : v]$ in $\mathbb{P}^1_r(\mathcal{O})$ under $\overline{\Gamma_{\mathcal{O}}}$, the pair of ideal classes $([I_{u,\,v}], [K_{u,\,v}])$ is a bijection. We hence fix a (nonzero) element $(x, y) \in \mathcal{O} \times \mathcal{O}$ such that $[I_{x,\,y}] = [\mathfrak{m}]$ and $[K_{x,\,y}] = [\mathfrak{m}']$, assuming that $x = 1$ if $y = 0$.

Let us prove the first claim. Since the reduced norm of an invertible element of $\mathcal{O}$ is 1, the index in $\Gamma_{1,\,0}$ of its unipotent upper triangular subgroup is $|\mathcal{O}^\times|$, and we have

$$\psi_{\mathfrak{m},\,\mathfrak{m}'}(s) = |\mathcal{O}^\times| \operatorname{Card}\big(\Gamma_{1,\,0}\backslash\{(u,v) \in \Gamma_{\mathcal{O}}(x,y) \ :\ \mathtt{N}(v) \leqslant \mathtt{N}(\mathfrak{m})s\}\big)\,.$$

Let $\tau \in]0,1]$. Let $\mathcal{H}_\infty$ be the horoball in $\mathbb{H}^5_\mathbb{R}$ consisting of the points of Euclidean height at least $1/\tau$. By [31, Lem. 6.7], if c is the $(2,1)$-entry of a matrix $g \in \mathrm{SL}_2(\mathbb{H})$ such that $\mathcal{H}_\infty$ and $g\mathcal{H}_\infty$ are disjoint, then the hyperbolic distance between $\mathcal{H}_\infty$ and $g\mathcal{H}_\infty$ is $|\ln(\tau^{-2}\mathtt{N}(c))|$.

Let $\rho = xy^{-1} \in A \cup \{\infty\}$. If $y = 0$, let $\gamma_\rho = \mathrm{id}$, otherwise let

$$\gamma_\rho = \begin{pmatrix} \rho & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{H}) \,.$$

Let $\mathcal{H}_\rho = \gamma_\rho \mathcal{H}_\infty$, which is a horoball centered at ρ. For every $g \in \Gamma_\mathcal{O}$ such that $\mathcal{H}_\infty$ and $g\mathcal{H}_\rho$ are disjoint, let δ_g be the common perpendicular from $\mathcal{H}_\infty$ to $g\mathcal{H}_\rho = g\gamma_\rho\mathcal{H}_\infty$, with length $\ell(\delta_g) > 0$. Hence if $\mathcal{H}_\infty$ and $g\mathcal{H}_\rho$ are disjoint, since $(u,v) = g(x,y)$ if and only if $(uy^{-1}, vy^{-1}) = g\gamma_\rho(1,0)$ when $y \neq 0$, we have $\ell(\delta_g) = \big|\ln \frac{\mathtt{N}(v)}{\tau^2\mathtt{N}(y)}\big|$ if $y \neq 0$ and $\ell(\delta_g) = |\ln(\tau^{-2}\mathtt{N}(v))|$ otherwise. By discreteness, there are only finitely many double classes $[g] \in \Gamma_{1,0}\backslash\Gamma_\mathcal{O}/\Gamma_{x,y}$ such that $\mathcal{H}_\infty$ and $g\mathcal{H}_\rho$ are not disjoint or such that $\mathtt{N}(v) \leqslant \mathtt{N}(y)$ or such that the multiplicity of δ_g is different from 1. Assume that $y \neq 0$ (the case $y = 0$ is treated similarly). Then, as $s \to +\infty$,

$$\begin{aligned}\psi_{\mathfrak{m},\mathfrak{m}'}(s) &= |\mathcal{O}^\times|\ \mathrm{Card}\big\{[g] \in \Gamma_{1,0}\backslash\Gamma_\mathcal{O}/\Gamma_{x,y}\ :\ \ell(\delta_g) \leqslant \ln \frac{\mathtt{N}(\mathfrak{m})s}{\tau^2\mathtt{N}(y)}\big\} + \mathrm{O}(1)\,. \\ &= 2\,|\mathcal{O}^\times|\ \mathrm{Card}\big\{[g] \in \overline{\Gamma_{1,0}}\,\backslash\overline{\Gamma_\mathcal{O}}/\,\overline{\Gamma_{x,y}}\ :\ \ell(\delta_g) \leqslant \ln \frac{\mathtt{N}(\mathfrak{m})s}{\tau^2\mathtt{N}(y)}\big\} + \mathrm{O}(1)\,.\end{aligned}$$

Let $\overline{\Gamma_{\mathcal{H}_\rho}}$ be the stabiliser in $\overline{\Gamma_\mathcal{O}}$ of the horoball $\mathcal{H}_\rho$. By [33, Lem. 7], the number $\psi_{\mathfrak{m},\mathfrak{m}'}(s)$ is equal, up to a bounded additive constant, to

$$2|\mathcal{O}^\times|[\overline{\Gamma_{\mathcal{H}_\infty}} : \overline{\Gamma_{1,0}}][\overline{\Gamma_{\mathcal{H}_\rho}} : \overline{\Gamma_{x,y}}]\,\mathrm{Card}\big\{[g] \in \overline{\Gamma_{\mathcal{H}_\infty}}\backslash\overline{\Gamma_\mathcal{O}}/\overline{\Gamma_{\mathcal{H}_\rho}} : \ell(\delta_g) \leqslant \ln \frac{\mathtt{N}(\mathfrak{m})s}{\tau^2\mathtt{N}(y)}\big\}.$$

Note that $[\overline{\Gamma_{\mathcal{H}_\infty}} : \overline{\Gamma_{1,0}}] = \frac{|\mathcal{O}^\times|}{2}$. If τ is small enough, then $\mathcal{H}_\infty$ and $\mathcal{H}_\rho$ are precisely invariant under $\overline{\Gamma_\mathcal{O}}$. Hence, using Corollary 2.2, applied to $n = 5$, $\Gamma = \overline{\Gamma_\mathcal{O}}$, $D^- = \mathcal{H}_\infty$ and $D^+ = \mathcal{H}_\rho$, we have, since the pointwise stabilisers of $\mathcal{H}_\infty$ and $\mathcal{H}_\rho$ are trivial,

$$\begin{aligned}\psi_{\mathfrak{m},\mathfrak{m}'}(s) &= |\mathcal{O}^\times|^2[\,\overline{\Gamma_{\mathcal{H}_\rho}} : \overline{\Gamma_{x,y}}\,]\,\mathcal{N}_{D^-,D^+}\big(\ln \frac{\mathtt{N}(\mathfrak{m})s}{\tau^2\mathtt{N}(y)}\big) + \mathrm{O}(1) \\ &= |\mathcal{O}^\times|^2[\,\overline{\Gamma_{\mathcal{H}_\rho}} : \overline{\Gamma_{x,y}}\,]\ c(D^-,D^+)\ \big(\frac{\mathtt{N}(\mathfrak{m})s}{\tau^2\mathtt{N}(y)}\big)^4\big(1 + \mathrm{O}(e^{-\kappa t})\big)\,.\end{aligned}$$

By the Remark before Lemma 15 of [35], we have $\mathrm{Vol}(\,\overline{\Gamma_{\mathcal{H}_\infty}}\,\backslash\mathcal{H}_\infty) = \frac{D_A\,\tau^4}{8\,|\mathcal{O}^\times|^2}$, and by Lemma 15, Equation (33) and the centred equation three lines before

Remark 19 in [35],

$$\mathrm{Vol}(\overline{\Gamma_{\mathcal{H}_\rho}}\backslash\mathcal{H}_\rho)=\frac{D_A\,\mathtt{N}(y)^4\,\tau^4}{16\,|\mathcal{O}_r(K_{x,\,y})^\times|\,[\,\overline{\Gamma_{\mathcal{H}_\rho}}:\overline{\Gamma_{x,\,y}}\,]\,\mathtt{N}(\mathfrak{m})^4}\,. \tag{3.5}$$

By Theorem 3 of [35] (due to Emery, see the appendix of [35]), we have

$$\mathrm{Vol}(M)=\frac{\zeta(3)\prod_{p|D_A}(p^3-1)(p-1)}{11520}\,. \tag{3.6}$$

Recall that $\mathrm{Vol}(\mathbb{S}^4)=\frac{8\pi^2}{3}$. Thus, $\psi_{\mathfrak{m},\,\mathfrak{m}'}(s)$ equals

$$\frac{|\mathcal{O}^\times|^2[\overline{\Gamma_{\mathcal{H}_\rho}}:\overline{\Gamma_{x,y}}]2^4\cdot 4\cdot 3\cdot 11520D_A^2\tau^8\mathtt{N}(y)^4(\mathtt{N}(\mathfrak{m})s)^4(1+\mathrm{O}(s^{-\kappa}))}{8\pi^2\zeta(3)\big(\prod_{p|D_A}(p^3-1)(p-1)\big)8|\mathcal{O}^\times|^2 16|\mathcal{O}_r(K_{x,y})^\times|[\overline{\Gamma_{\mathcal{H}_\rho}}:\overline{\Gamma_{x,y}}]\mathtt{N}(\mathfrak{m})^4\mathtt{N}(y)^4\tau^8}$$

$$=\frac{2160\,D_A^2}{\pi^2\,\zeta(3)\,|\mathcal{O}_r(K_{x,\,y})^\times|\,\prod_{p|D_A}(p^3-1)(p-1)}\;s^4\big(1+\mathrm{O}(s^{-\kappa})\big)\,. \tag{3.7}$$

By page 134 of [9] (see Equation (8) of [35]), we get

$$\sum_{[\mathfrak{m}']\in\,{}_{\mathcal{O}}\mathcal{I}}\frac{1}{|\mathcal{O}_r(\mathfrak{m}')^\times|}=\frac{1}{24}\prod_{p|D_A}(p-1)\,. \tag{3.8}$$

Thus, Equation (3.3) and the above computations give the first claim of Theorem 3.2.

To prove the second claim (when $y\neq 0$, the case $y=0$ being similar), Corollary 2.2 implies that

$$\frac{\pi^2\zeta(3)\big(\prod_{p|D_A}(p^3-1)(p-1)\big)|\mathcal{O}_r(K_{x,y})^\times|[\overline{\Gamma_{\mathcal{H}_\rho}}:\overline{\Gamma_{x,y}}]\mathtt{N}(\mathfrak{m})^4}{4320D_A\mathtt{N}(y)^4\tau^4}e^{-4t}\sum_{z\in\partial\mathcal{H}_\infty}m_t(z)\Delta_z$$

weak-star converges to $\mathrm{Vol}_{\partial\mathcal{H}_\infty}$ as $t\to+\infty$, with the appropriate error term. Let us use the change of variable $t=\ln\frac{\mathtt{N}(\mathfrak{m})s}{\tau^2\mathtt{N}(y)}$, the fact that the geodesic lines containing the common perpendicular from ∞ to an image of $\mathcal{H}_\rho$ by an element of $\overline{\Gamma_{\mathcal{O}}}$ are exactly the geodesic lines from ∞ to an element of $\overline{\Gamma_{\mathcal{O}}}\cdot xy^{-1}$, and the above facts on disjointness, lengths and multiplicities. This gives, since the map $(u,v)\mapsto uv^{-1}$ is $|\mathcal{O}^\times|$-to-1,

$$\frac{\pi^2\zeta(3)\big(\prod_{p|D_A}(p^3-1)(p-1)\big)|\mathcal{O}_r(K_{x,y})^\times|[\overline{\Gamma_{\mathcal{H}_\rho}}:\overline{\Gamma_{x,y}}]}{4320\,D_A\,|\mathcal{O}^\times|s^4\tau^{-4}}\sum_{\substack{(u,v)\in\mathfrak{m}\times\mathfrak{m},\mathtt{N}(\mathfrak{m})^{-1}\mathtt{N}(v)\leqslant s\\ I_{u,v}=\mathfrak{m},[K_{u,v}]=[\mathfrak{m}']}}\Delta_{(uv^{-1},\frac{1}{\tau})}$$

weak-star converges to $\operatorname{Vol}_{\partial\mathcal{H}_\infty}$ as $s \to +\infty$. Since $[\,\overline{\Gamma_{\mathcal{H}_\rho}} : \overline{\Gamma_{x,\,y}}\,] = \frac{|\mathcal{O}^\times|}{2}$ and the pushforward of $\operatorname{Vol}_{\partial\mathcal{H}_\infty}$ by the endpoint map $(x, \frac{1}{\tau}) \mapsto x$ is $\tau^4 \operatorname{Leb}_{\mathbb{H}}$, we have

$$\frac{\pi^2\zeta(3)\big(\prod_{p|D_A}(p^3-1)(p-1)\big)|\mathcal{O}_r(K_{x,y})^\times|}{8640\,D_A s^4} \sum_{\substack{(u,v)\in\mathfrak{m}\times\mathfrak{m},\mathtt{N}(\mathfrak{m})^{-1}\mathtt{N}(v)\leqslant s\\ I_{u,v}=\mathfrak{m},[K_{u,v}]=[\mathfrak{m}']}} \Delta_{uv^{-1}} \overset{*}{\rightharpoonup} \operatorname{Leb}_{\mathbb{H}}, \tag{3.9}$$

The second claim of Theorem 3.2 now follows by dividing both sides by $|\mathcal{O}_r(K_{x,\,y})^\times|$ and summing over $[\mathfrak{m}']$ as above. $\square$

4. Counting and equidistribution of quadratic irrationals and crossratios

Let K be either $\mathbb{Q}$ or an imaginary quadratic number field, and, respectively, $\widehat{K} = \mathbb{R}$ or $\widehat{K} = \mathbb{C}$. Let $\mathcal{O}_K$ be the ring of integers, D_K the discriminant, and ζ_K the zeta function of K. In this section, we denote by $\cdot$ the action by homographies of the group $\Gamma_K = \operatorname{PSL}_2(\mathcal{O}_K)$ on $\mathbb{P}^1(\widehat{K}) = \widehat{K}\cup\{\infty\}$. For every finite index subgroup G of Γ_K, and every $x \in \widehat{K} \cup \{\infty\}$, let G_x be the stabiliser of x in G, with $\Gamma_x = (\Gamma_K)_x$ to simplify the notation.

For every $\alpha \in \widehat{K}$ which is a quadratic irrational over K,

- denote by α^σ its Galois conjugate over K, by $\mathtt{tr}\,\alpha = \alpha + \alpha^\sigma$ and $\mathtt{n}(\alpha) = \alpha\alpha^\sigma$ its relative trace and relative norm, and by

$$Q_\alpha(X) = X^2 - \mathtt{tr}\,\alpha\; X + \mathtt{n}(\alpha)$$

the standard monic quadratic polynomial with roots α and α^σ;

- denote by $h(\alpha) = \frac{2}{|\alpha-\alpha^\sigma|}$ the natural complexity of α in an orbit by homographies of Γ_K, modulo translations by $\mathcal{O}_K$, introduced in [32] and motivated in [34, §4.1] (when α is integral, $\frac{4}{h(\alpha)^2} = \operatorname{Disc} Q_\alpha$ is the discriminant of the minimal polynomial of α);

- if $K = \mathbb{Q}$, let q_α be the least common multiple of the denominators of the rationals $\mathtt{tr}\,\alpha$ and $\mathtt{n}(\alpha)$, let $D_\alpha = {q_\alpha}^2((\mathtt{tr}\,\alpha)^2 - 4\mathtt{n}(\alpha))$ (when α is integral, we have $q_\alpha = 1$ and D_α is the discriminant of the order $\mathbb{Z} + \alpha\mathbb{Z}$), let (t_α, u_α) be the fundamental solution of the Pell-Fermat equation $t^2 - D_\alpha u^2 = 4$, and let $R_\alpha = \operatorname{argcosh}\frac{t_\alpha}{2}$ (which is the regulator of $\mathbb{Z} + \alpha\mathbb{Z}$ when α is integral);

- for every finite index subgroup G of Γ_K, we define the *G-reciprocity index* $\iota_G(\alpha)$ of α as follows: we set $\iota_G(\alpha) = 2$ if α is *G-reciprocal*, that is, if some element of G maps α to α^σ, see [42] and [34, Prop. 4.3] for characterisations (when $K=\mathbb{Q}$, $G=\Gamma_K$ and α is integral, this is equivalent to saying that the order $\mathbb{Z}+\alpha\mathbb{Z}$ contains a unit of norm -1), and we set $\iota_G(\alpha)=1$ otherwise.

To conclude with a geometric remark, for every α as above, there exists a unique primitive loxodromic element $\widehat{\alpha}\in\Gamma_K$ such that the repelling fixed point $\widehat{\alpha}^-$ of $\widehat{\alpha}$ is equal to α (see for instance [32, Lem. 6.2]). Its attractive fixed point $\widehat{\alpha}^+$ is then α^σ. With the notation at the end of Section 2, we have $\iota_G(\alpha)=\iota_G(\widehat{\alpha})$ for every finite index subgroup G of Γ_K. Finally, for all $\gamma\in\Gamma_K$, we have $\widehat{\gamma\cdot\alpha}=\gamma\widehat{\alpha}\gamma^{-1}$.

4.1. Equidistribution and error terms in counting functions of quadratic irrationals

We give in this subsection an error term to the counting asymptotics of [34] of the number of quadratic irrationals $\alpha\in\widehat{K}$ over K with complexity $h(\alpha)$ at most s, in an orbit of a finite index subgroup of Γ_K, and we prove an equidistribution result for the set of traces $\mathtt{tr}\,\alpha$ of these elements, as $s\to+\infty$.

Theorem 4.1. — *Let α_0 be a real quadratic irrational over $\mathbb{Q}$ and let G be a finite index subgroup of $\Gamma_{\mathbb{Q}}$. There exists $\kappa>0$ such that, as $s\to+\infty$,*

$$\begin{aligned}\operatorname{Card}\{\alpha\in G\cdot\alpha_0+[\Gamma_\infty:G_\infty]\mathbb{Z}\ :\ h(\alpha)\leqslant s\}&\\ =\frac{6\,[\Gamma_\infty:G_\infty]\,[\Gamma_{\alpha_0}:G_{\alpha_0}]\,R_{\alpha_0}}{\pi^2\,[\Gamma_{\mathbb{Q}}:G]}&\;s+\mathrm{O}(s^{1-\kappa})\,.\end{aligned}$$

Furthermore, for the weak-star convergence of measures, we have

$$\lim_{s\to+\infty}\ \frac{\pi^2\,[\Gamma_{\mathbb{Q}}:G]}{3\,[\Gamma_{\alpha_0}:G_{\alpha_0}]\,R_{\alpha_0}\,s}\ \sum_{\alpha\in G\cdot\alpha_0,\,h(\alpha)\leqslant s}\Delta_{\mathtt{tr}\,\alpha}\ =\ \mathrm{Leb}_{\mathbb{R}}\,.$$

Note that the subgroup indices (except maybe $[\Gamma_{\alpha_0}:G_{\alpha_0}]$) appearing in the above statement are known, for instance, when G is the principal congruence subgroup modulo a prime p, or the Hecke congruence subgroup modulo p. When $G=\Gamma_{\mathbb{Q}}$, the first claim is known (with a better error term, see for instance [5, page 164]), but it is new in particular when G is not a congruence subgroup of $\Gamma_{\mathbb{Q}}$. For smooth functions ψ with compact support on $\mathbb{R}$, there is an error term in the equidistribution claim evaluated at ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ where $\kappa>0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell\in\mathbb{N}$, see the comment following Theorem 2.1. Theorem 1.3 in the Introduction follows from this result.

Proof. — The proof of the counting result with an error term is similar to that of [34, Théo. 4.4] upon replacing [34, Coro. 3.10] by the above Corollary 2.2 (the factor $\frac{2}{\iota_G(\alpha_0)}$ which appears in [34, Théo. 4.4] comes from the fact that we are counting $\alpha \in G \cdot \{\alpha_0, \alpha_0^\sigma\}$ therein). Noting that $\mathtt{tr}(\alpha + m) = \mathtt{tr}\alpha + 2m$ for all $m \in \mathbb{Z}$, it also follows by applying the equidistribution result with error term on the quotient of $\mathbb{R}$ by $2[\Gamma_\infty : G_\infty]\,\mathbb{Z}$ to the constant function 1.

To prove the second claim, we apply Equation (2.2) with $n = 2$, $\Gamma = G$, D^- the horoball centred at ∞ consisting of the points with vertical coordinates at least 1 in the upper halfplane model of $\mathbb{H}^2_\mathbb{R}$, and D^+ the geodesic line with points at infinity α_0 and α_0^σ, whose stabiliser in G we denote by G_{D^+}. The image of D^+ by an element of G is the geodesic line $]\alpha, \alpha^\sigma[$ with points at infinity α, α^σ for some $\alpha \in G \cdot \alpha_0$. Such an α is uniquely determined if α_0 is not G-reciprocal, otherwise there are exactly two choices, α and α^σ. The origin of the common perpendicular from D^- to $]\alpha, \alpha^\sigma[$ (which exists except for finitely many G_∞-orbits of $\alpha \in G \cdot \alpha_0$) is the point $x_\alpha = (\frac{\mathtt{tr}\,\alpha}{2}, 1) \in \partial D^-$, its length is $\ln h(\alpha)$, and its multiplicity is 1, since $\mathrm{PSL}_2(\mathbb{Z})$ acts freely on $T^1\mathbb{H}^2_\mathbb{R}$. The induced Riemannian measure on ∂D^- is the Lebesgue measure. Therefore, by Equation (2.2) and the value of the constant $c(D^-, D^+)$ given above Corollary 2.2, we have

$$\lim_{s \to +\infty} \frac{\mathrm{Vol}(\mathbb{S}^1)\,\mathrm{Vol}(G\backslash\mathbb{H}^2_\mathbb{R})}{\iota_G(\alpha_0)\,\mathrm{Vol}(\mathbb{S}^0)\,\mathrm{Vol}(G_{D^+}\backslash D^+)\,s} \sum_{\alpha \in G\cdot\alpha_0,\ h(\alpha) \leqslant s} \Delta_{x_\alpha} = \mathrm{Leb}_{\partial D^-}\,. \tag{4.1}$$

We have

$$\mathrm{Vol}(G\backslash\mathbb{H}^2_\mathbb{R}) = [\Gamma_\mathbb{Q} : G]\,\mathrm{Vol}(\Gamma_\mathbb{Q}\backslash\mathbb{H}^2_\mathbb{R}) = [\Gamma_\mathbb{Q} : G]\,\frac{\pi}{3}\,. \tag{4.2}$$

Recall that $\widehat{\alpha_0} \in \Gamma_\mathbb{Q}$ is a primitive loxodromic element with fixed points α_0 and α_0^σ. By for instance [2, p. 173], the translation length $\ell(\widehat{\alpha_0})$ of γ_0 satisfies $\cosh \frac{\ell(\widehat{\alpha_0})}{2} = \frac{|\mathrm{tr}\,\widehat{\alpha_0}\,|}{2}$. By for instance the proof of [34, Prop. 4.1], we have $|\mathrm{tr}\,\widehat{\alpha_0}\,| = t_{\alpha_0}$. Hence

$$\ell(\widehat{\alpha_0}) = 2\,R_{\alpha_0}\,, \tag{4.3}$$

so that

$$\begin{aligned}\mathrm{Vol}(G_{D^+}\backslash D^+) &= \frac{[\Gamma_{\alpha_0} : G_{\alpha_0}]}{\iota_G(\alpha_0)}\mathrm{Vol}(\Gamma_{\alpha_0}\backslash D^+) = \frac{[\Gamma_{\alpha_0} : G_{\alpha_0}]\ell(\widehat{\alpha_0})}{\iota_G(\alpha_0)} \\ &= \frac{2[\Gamma_{\alpha_0} : G_{\alpha_0}]R_{\alpha_0}}{\iota_G(\alpha_0)}.\end{aligned} \tag{4.4}$$

The result now follows from Equations (4.1), (4.2) and (4.4) by applying the pushforwards of measures by the map $f : (x, 1) \mapsto 2x$ from ∂D^-

to $\mathbb{R}^{n-1}$ (which sends the Lebesgue measure to $\frac{1}{2^{n-1}}$ times the Lebesgue measure). $\square$

THEOREM 4.2. — *Let K be an imaginary quadratic number field, let $\alpha_0 \in \mathbb{C}$ be a quadratic irrational over K, let G be a finite index subgroup of Γ_K, and let Λ be the lattice of $\lambda \in \mathcal{O}_K$ such that $\pm\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in G$. There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\begin{aligned}&\operatorname{Card}\{\alpha \in G\cdot\alpha_0 + \Lambda \;:\; h(\alpha) \leqslant s\}\\ &\qquad = \frac{\pi^2\,[\mathcal{O}_K:\Lambda]\,[\Gamma_{\alpha_0}:G_{\alpha_0}]\,\big|\ln\big|\frac{\operatorname{tr}\widehat{\alpha_0}+\sqrt{(\operatorname{tr}\widehat{\alpha_0})^2-4}}{2}\big|\,\big|}{m_{\Gamma_K}(\alpha_0)\,[\Gamma_K:G]\,|D_K|\,\zeta_K(2)}\;s^2 + \mathrm{O}(s^{2-\kappa})\,,\end{aligned}$$

where $\widehat{\alpha_0}$ is a primitive element of Γ_K fixing α_0 with absolute values of eigenvalues different from 1, and $m_G(\alpha_0)$ is the number of elements in G fixing α_0 with absolute values of eigenvalues equal to 1. Furthermore, for the weak-star convergence of measures, we have

$$\lim_{s\to+\infty}\frac{2|D_K|^{\frac{3}{2}}\zeta_K(2)[\Gamma_K:G]\,m_{\Gamma_K}(\alpha_0)}{\pi^2[\Gamma_{\alpha_0}:G_{\alpha_0}]\big|\ln\big|\frac{\operatorname{tr}\widehat{\alpha_0}+\sqrt{(\operatorname{tr}\widehat{\alpha_0})^2-4}}{2}\big|\,\big|\;s^2}\sum_{\alpha\in G\cdot\alpha_0,\; h(\alpha)\leqslant s}\Delta_{\mathtt{tr}\,\alpha} = \mathrm{Leb}_{\mathbb{C}}.$$

For smooth functions ψ with compact support on $\mathbb{C}$, there is an error term in the above equidistribution claim evaluated at ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$.

Proof. — The proof of the counting claim with an error term is similar to that of [34, Théo. 4.6] upon replacing [34, Coro. 3.10] by the above Corollary 2.2, and using the simplification

$$[\Gamma_\infty:\Lambda] = [\Gamma_\infty:\mathcal{O}_K][\mathcal{O}_K:\Lambda] = \frac{|{\mathcal{O}_K}^\times|}{2}\,[\mathcal{O}_K:\Lambda]\,.$$

To prove the second claim, we apply Equation (2.2) with $n = 3$, $\Gamma = G$, D^- the horoball centred at ∞ consisting of the points with vertical coordinates at least 1 in the upper halfspace model of $\mathbb{H}^3_{\mathbb{R}}$, and D^+ the geodesic line with points at infinity α_0 and α_0^σ, whose stabiliser in G we denote by G_{D^+}. The cardinality of the pointwise stabiliser in G of D^+ is $m_G(\alpha_0)$ as defined in the statement of Theorem 4.2 (we have $m_G(\alpha_0) = m_G(\widehat{\alpha_0})$ with the notation at the end of Section 2).

The set of points in ∂D^- fixed by a nontrivial element of the stabiliser of ∞ in Γ_K is discrete since, for instance, the subgroup of translations in a discrete group of isometries of $\mathbb{R}^{n-1}$ has finite index by Bieberbach's theorem. Hence the multiplicity of the common perpendicular from D^- to an image of D^+ by an element of G is different from 1 only when its origin belongs to a discrete subset S of ∂D^-. Furthermore, since $\Gamma_{D^+}\backslash D^+$ is compact, there exists $\epsilon > 0$ (depending only on Γ_K and α_0) such that a geodesic arc, leaving perpendicularly from an image of D^+ and arriving perpendicularly at another image of D^+, has length at least ϵ. Hence the number of common perpendiculars from a given point in S to an image of D^+ grows at most linearly in the length. Therefore these common perpendiculars do not contribute asymptotically to the equidistributing sum, and as in the previous proof, we have

$$\lim_{s\to+\infty} \frac{2\, m_G(\alpha_0)\operatorname{Vol}(\mathbb{S}^2)\operatorname{Vol}(G\backslash\mathbb{H}^3_{\mathbb{R}})}{\iota_G(\alpha_0)\operatorname{Vol}(\mathbb{S}^1)\operatorname{Vol}(G_{D^+}\backslash D^+)s^2} \sum_{\alpha\in G\cdot\alpha_0 : h(\alpha)\leqslant s} \Delta_{(\frac{\operatorname{tr}\alpha}{2},1)} = \operatorname{Leb}_{\partial D^-}. \tag{4.5}$$

Recall that the translation length $\ell(\gamma_0)$ of an element γ_0 of $\mathrm{PSL}_2(\mathbb{C})$ is

$$\ell(\gamma_0) = 2\big|\ln\big|\frac{\operatorname{tr}\gamma_0 + \sqrt{\operatorname{tr}\gamma_0{}^2 - 4}}{2}\big|\big| \tag{4.6}$$

(which is independent of the choice of the square root of the complex number $\operatorname{tr}\gamma_0^2 - 4$ and of the choice of the lift of γ_0 in $\mathrm{SL}_2(\mathbb{C})$ modulo $\{\pm\mathrm{id}\}$). With r the smallest positive integer such that $\widehat{\alpha_0}^{\,r} \in G$, we have $[\Gamma_{\alpha_0} : G_{\alpha_0}] = \frac{m_{\Gamma_K}(\alpha_0)}{m_G(\alpha_0)}\, r$, so that

$$\operatorname{Vol}(G_{D^+}\backslash D^+) = \frac{r}{\iota_G(\alpha_0)}\,\ell(\widehat{\alpha_0}) = \frac{m_G(\alpha_0)\,[\Gamma_{\alpha_0} : G_{\alpha_0}]}{m_{\Gamma_K}(\alpha_0)\,\iota_G(\alpha_0)}\,\ell(\widehat{\alpha_0})\,. \tag{4.7}$$

As in the previous proof, the result now follows from Equations (4.5), (4.6), (4.7) and (3.2) by applying the pushforwards of measures by the map $(x,1)\mapsto 2x$ from ∂D^- to $\mathbb{C}$. □

Corollary 4.3. — *Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio, let K be an imaginary quadratic number field with $D_K \neq -4$, let $\mathfrak{c}$ be a nonzero ideal in $\mathcal{O}_K$, and let $\Gamma_0(\mathfrak{c})$ be the Hecke congruence subgroup $\Big\{\pm\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_K \;:\; c\in\mathfrak{c}\Big\}$. Then there exists $\kappa > 0$ such that, as $s\to+\infty$,*

$$\operatorname{Card}\{\alpha\in\Gamma_0(\mathfrak{c})\cdot\phi + \mathcal{O}_K : h(\alpha)\leqslant s\} = \frac{2\pi^2 k_{\mathfrak{c}}\,\ln\phi}{|D_K|\zeta_K(2)\mathfrak{n}(\mathfrak{c})\prod_{\mathfrak{p}|\mathfrak{c}}\big(1+\frac{1}{\mathfrak{n}(\mathfrak{p})}\big)}s^2 + \mathrm{O}(s^{2-\kappa}),$$

where $k_{\mathfrak{c}}$ is the smallest $k\in\mathbb{N}-\{0\}$ such that the $2k$-th term of the standard Fibonacci sequence belongs to $\mathfrak{c}$, and the product is over the prime ideals in $\mathcal{O}_K$ dividing $\mathfrak{c}$.

Proofs. — The proof of this corollary and of Corollary 1.4 in the introduction, are similar to that of [34, Coro. 4.7]. The element $\widehat{\phi}^{-1}$ is denoted by $\gamma_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ in op. cit., and its translation length is $2 \ln \phi$. It is proven in op. cit. that $m_{\Gamma_0(\mathfrak{c})}(\phi) = 1$, so that

$$\frac{[\Gamma_\phi : (\Gamma_0(\mathfrak{c}))_\phi]}{m_{\Gamma_K}(\phi)} = \frac{[\widehat{\phi}^{\mathbb{Z}} : \widehat{\phi}^{\mathbb{Z}} \cap \Gamma_0(\mathfrak{c})]}{m_{\Gamma_0(\mathfrak{c})}(\phi)} = k_{\mathfrak{c}} . \quad \square$$

Let $\mathbb{H}$ be Hamilton's quaternion algebra over $\mathbb{R}$, let A be a definite quaternion algebra over $\mathbb{Q}$, and let $\mathcal{O}$ be a maximal order in A (we refer to Subsection 3.2 for background, as well as for the notation $\overline{}$, $\mathtt{N}$, $\mathtt{Tr}$, D_A, $\mathrm{PSL}_2(\mathbb{H})$, $\mathrm{PSL}_2(\mathcal{O})$, $\cdot$). For every $x \in \mathbb{H} \cup \{\infty\}$, we denote by G_x the stabiliser of x for the action by homographies of a subgroup G of $\mathrm{PSL}_2(\mathbb{H})$. For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{H})$, we denote by X_γ the largest real root of

$$2x^3 - c_1\, x^2 + 2(c_2 c_3 - 1)x + (c_1 - {c_2}^2 - {c_3}^2)\,,$$

where $c_1 = \mathtt{N}(a+d) + \mathtt{Tr}(ad - bc)$, $c_2 = \frac{1}{2}\mathtt{Tr}(a+d)$, $c_3 = \frac{1}{2}\mathtt{Tr}\big((ad-bc)\overline{a} + (da - cb)\overline{d}\big)$. We will say that an element $\alpha \in \mathbb{H}$ is a *loxodromic quadratic irrational* over $\mathcal{O}$ if there exists $\gamma \in \mathrm{SL}_2(\mathcal{O})$ such that $|X_\gamma| \neq 1$ fixing α, in which case we denote by α^σ its other fixed point (which exists, is unique, and is independent of such a γ, see below for proofs and a justification of the terminology). Note that by the noncommutativity of $\mathbb{H}$, there are several types of quadratic equations over $\mathcal{O}$.

The following result is a counting and equidistribution result in $\mathbb{H}$ of loxodromic quadratic irrationals over $\mathcal{O}$ in a given homographic orbit under $\mathrm{PSL}_2(\mathcal{O})$.

Theorem 4.4. — *Let $\mathcal{O}$ be a maximal order in a definite quaternion algebra over $\mathbb{Q}$, let $\alpha_0 \in \mathbb{H}$ be a loxodromic quadratic irrational over $\mathcal{O}$, let G be a finite index subgroup of* $\mathrm{PSL}_2(\mathcal{O})$*, and let Λ be the lattice of $\lambda \in \mathcal{O}$ such that $\pm\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in G$. There exists $\kappa > 0$ such that, as $\epsilon \to 0$,*

$$\mathrm{Card}\{\alpha \in G \cdot \alpha_0 + \Lambda \ :\ \mathtt{N}(\alpha - \alpha^\sigma) \geqslant \epsilon\} =$$
$$\frac{8640\ D_A\ [\mathcal{O} : \Lambda]\,[\mathrm{PSL}_2(\mathcal{O})_{\alpha_0} : G_{\alpha_0}]\,\big|\ln |X_{\gamma_0} + ({X_{\gamma_0}}^2 - 1)^{1/2}|\,\big|}{\zeta(3)\, m_{\mathrm{PSL}_2(\mathcal{O})}(\alpha_0)\ [\mathrm{PSL}_2(\mathcal{O}) : G]\prod_{p | D_A}(p^3-1)(p-1)}\epsilon^{-2} + \mathrm{O}(\epsilon^{\kappa-2}),$$

where γ_0 is a primitive element of $\mathrm{SL}_2(\mathcal{O})$ *fixing α_0 with $|X_{\gamma_0}| \neq 1$, $m_G(\alpha_0)$ is the number of elements γ in G fixing α_0 with $|X_\gamma| = 1$, and the product* *is over the primes p dividing D_A. Furthermore, the Lebesgue measure $\mathrm{Leb}_{\mathbb{H}}$ on $\mathbb{H}$ is the weak-star limit, as $\epsilon \to 0$, of the measures*

$$\frac{\zeta(3)\, m_{\mathrm{PSL}_2(\mathcal{O})}(\alpha_0)[\mathrm{PSL}_2(\mathcal{O})\!:\!G]\prod_{p|D_A}(p^3-1)(p-1)\ \epsilon^2}{2160\,[\mathrm{PSL}_2(\mathcal{O})_{\alpha_0}\!:\!G_{\alpha_0}]\,\big|\ln|X_{\gamma_0}+({X_{\gamma_0}}^2-1)^{1/2}|\big|}\sum_{\alpha\in G\cdot\alpha_0,\,\mathtt{N}(\alpha-\alpha^\sigma)\geqslant\epsilon}\Delta_{\alpha+\alpha^\sigma}\,.$$

For smooth functions ψ with compact support on $\mathbb{H}$, there is an error term in the equidistribution claim evaluated at ψ, of the form $\mathrm{O}(\epsilon^\kappa\|\psi\|_\ell)$ where $\kappa>0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell\in\mathbb{N}$.

Proof. — First recall (see [30, §3]) that for every $\gamma\in\mathrm{SL}_2(\mathbb{H})$, we have $|X_\gamma|\geqslant 1$, and that γ is loxodromic for its isometric action on the upper halfspace model of $\mathbb{H}^5_{\mathbb{R}}$ by Poincaré's extension (see for instance Equation (14) in [35]) if and only if $|X_\gamma|\neq 1$. If this holds, then (see [30, §3]) γ is a conjugate of $\begin{pmatrix} t_+ & 0\\ 0 & t_-\end{pmatrix}$ with $\mathtt{N}(t_\pm)=X_\gamma\pm(X_\gamma^2-1)^{1/2}$. Since this diagonal matrix acts on the vertical axis $\{0\}\times\,]0,+\infty[$ in $\mathbb{H}^5_{\mathbb{R}}$ by $(0,r)\mapsto(0,\mathtt{N}(t_+)r)$, the translation length $\ell(\gamma)$ of γ is therefore

$$\ell(\gamma)=\big|\ln|X_\gamma+(X_\gamma^2-1)^{1/2}|\,\big|\,. \tag{4.8}$$

As a side remark, if $\gamma\in\mathrm{SL}_2(\mathcal{O})$ satisfies $|X_\gamma|>1$, in particular, it has exactly two fixed points, which are the only two solutions of some quadratic equation $ax+b=xcx+xd$ where $a,b,c,d\in\mathcal{O}$ and $\mathtt{N}(ad)+\mathtt{N}(bc)-\mathtt{Tr}(a\,\overline{c}\,d\,\overline{b}\,)=1$.

To prove the equidistribution claim of Theorem 4.4, we apply Equation (2.2) with $n=5$, $\Gamma=G$, D^- the horoball centred at ∞ consisting of the points with vertical coordinate at least 1 in $\mathbb{H}^5_{\mathbb{R}}$, and D^+ the geodesic line with points at infinity α_0 and α_0^σ, whose stabiliser in G we denote by G_{D^+}. We define $\iota_G(\alpha_0)=2$ if there exists an element in G sending α_0 to ${\alpha_0}^\sigma$ and $\iota_G(\alpha_0)=1$ otherwise. Except for finitely many G_∞-orbits of $\alpha\in G\cdot\alpha_0$, the common perpendicular from D^- to the geodesic line with points at infinity α, α^σ exists and has hyperbolic length $\ln\frac{2}{\mathtt{N}(\alpha-\alpha^\sigma)^{1/2}}$, and its origin is $(\frac{\alpha+\alpha^\sigma}{2},1)\in\partial D^-$. As in the proof of Theorem 4.2, we have

$$\lim_{t\to+\infty}\frac{4m_G(\alpha_0)\mathrm{Vol}(\mathbb{S}^4)\mathrm{Vol}(G\backslash\mathbb{H}^5_{\mathbb{R}})}{\iota_G(\alpha_0)\mathrm{Vol}(\mathbb{S}^3)\mathrm{Vol}(G_{D^+}\backslash D^+)\,e^{4t}}\sum_{\alpha\in G\cdot\alpha_0,\,\mathtt{N}(\alpha-\alpha^\sigma)\geqslant 4e^{-2t}}\Delta_{(\frac{\alpha+\alpha^\sigma}{2},1)}=\mathrm{Leb}_{\partial D^-}. \tag{4.9}$$

We have $\mathrm{Vol}(\mathbb{S}^4)=\frac{8\pi^2}{3}$, $\mathrm{Vol}(\mathbb{S}^3)=2\pi^2$ and, as in the proof of Theorem 4.2,

$$\mathrm{Vol}(G_{D^+}\backslash D^+)=\frac{[\mathrm{PSL}_2(\mathcal{O})_{\alpha_0}:G_{\alpha_0}]\,m_G(\alpha_0)}{\iota_G(\alpha_0)\,m_{\mathrm{PSL}_2(\mathcal{O})}(\alpha_0)}\,\ell(\gamma_0)\,.$$

Taking $\epsilon = 4\,e^{-2\,t}$, the result follows, as in the proof of Theorem 4.2, from Equations (4.9), (4.8) and (3.6) by applying the pushforwards of measures by the map $(x,1) \mapsto 2x$ from ∂D^- to $\mathbb{H}$.

To prove the first claim of Theorem 4.4, note that by [20, Prop. 5.5] for instance, we have

$$\operatorname{Vol}((2\Lambda)\backslash\mathbb{H}) = 2^4\,[\mathcal{O} : \Lambda]\operatorname{Vol}(\mathcal{O}\backslash\mathbb{H}) = 2^4\,[\mathcal{O} : \Lambda]\,\frac{D_A}{4}\,.$$

The measure $\sum_{\alpha\in G\cdot\alpha_0\ \mathtt{N}(\alpha-\alpha^\sigma)\geqslant\epsilon}\Delta_{\alpha+\alpha^\sigma}$ is invariant under translation by 2Λ. The result then follows by considering the measures induced on the compact quotient $(2\Lambda)\backslash\mathbb{H}$, and by applying the equidistribution result with the error term to the constant function 1. $\square$

4.2. Relative complexity of loxodromic elements

This subsection is a geometric one, paving the way to the next one, which is arithmetic. It explains in particular why the crossratios play an important geometric role in this paper.

Let $n \geqslant 2$ and Γ be as in the beginning of Section 2, and let $\gamma_0 \in \Gamma$ be a fixed primitive loxodromic element.

We define the *relative height* $h_{\gamma_0}(\gamma)$ of a primitive loxodromic element $\gamma \in \Gamma$ with respect to γ_0 to be the length of the common perpendicular of the translation axes of γ_0 and γ if they are disjoint and 0 otherwise. Note that

$$h_{\gamma_0^\epsilon}(\gamma^{\epsilon'}) = h_{\gamma_0}(\gamma) = h_{g\gamma_0 g^{-1}}(g\gamma g^{-1}) = h_{\gamma_0}(g'\gamma g'^{-1})$$

for all $\epsilon, \epsilon' \in \{\pm 1\}$, $g \in \Gamma$ and $g' \in \operatorname{Stab}_\Gamma(\operatorname{Axis}\gamma_0)$. Furthermore, it follows from the properness of the action of Γ as in [32, Lemma 3.1] that the set

$$\{[\gamma] \in \gamma_0^{\mathbb{Z}}\backslash\Gamma/\gamma_1^{\mathbb{Z}} \;:\; h_{\gamma_0}(\gamma\gamma_1\gamma^{-1}) \leqslant t\}$$

is finite for all primitive loxodromic elements $\gamma_0, \gamma_1 \in \Gamma$ and $t \geqslant 0$. This shows that the relative height with respect to a given primitive loxodromic element is a reasonable complexity within a given conjugacy class of primitive loxodromic elements of Γ. Theorem 2.1 easily implies the following counting result of primitive loxodromic elements using their relative heights.

Corollary 4.5. — *If the Bowen-Margulis measure $m_{\rm BM}$ of $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ is finite, then for all primitive loxodromic elements $\gamma_0, \gamma_1 \in \Gamma$, as $t \to +\infty$,*

$$\sum_{[\gamma]\,\in\,\gamma_0^{\mathbb{Z}}\backslash\Gamma/\gamma_1^{\mathbb{Z}}\,,\; h_{\gamma_0}(\gamma\gamma_1\gamma^{-1})\leqslant t} m_{e,\gamma} \;\sim\; C_{\gamma_0,\gamma_1}\, e^{\delta_\Gamma t}\,,$$

where

$$C_{\gamma_0,\gamma_1} = \iota_\Gamma(\gamma_0)\, \iota_\Gamma(\gamma_1)\, m_\Gamma(\gamma_0)\, m_\Gamma(\gamma_1)\, \|\sigma_{\mathrm{Axis}(\gamma_0)}\|\, \|\sigma_{\mathrm{Axis}(\gamma_1)}\|\, \delta_\Gamma^{-1}\, \|m_{\mathrm{BM}}\|^{-1},$$

with an error term $\operatorname{O}(e^{(n-1)\,t})$ *if* $\delta_\Gamma > \frac{1}{2}$ *when* $n=2$ *and* $\delta_\Gamma > n-2$ *when* $n \geqslant 3$. *If* $\Gamma\backslash\mathbb{H}^n_\mathbb{R}$ *has finite volume, then*

$$C_{\gamma_0,\gamma_1} = \frac{\operatorname{Vol}(\mathbb{S}^{n-2})^2\, \ell(\gamma_0)\, \ell(\gamma_1)}{2^{n-1}\,(n-1)\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(\Gamma\backslash\mathbb{H}^n_\mathbb{R})}\,.$$

In particular, when Γ is torsion free, as $t \to +\infty$

$$\operatorname{Card}\{[\gamma] \in \gamma_0^\mathbb{Z}\backslash\Gamma/\gamma_1^\mathbb{Z} : h_{\gamma_0}(\gamma\gamma_1\gamma^{-1}) \leqslant t\} \sim C_{\gamma_0,\gamma_1}\, e^{\delta_\Gamma t}.$$

Proof. — By [33, Lem. 7], except above finitely many double classes, the canonical map from $\gamma_0^\mathbb{Z}\backslash\Gamma/\gamma_1^\mathbb{Z}$ to $\operatorname{Stab}_\Gamma(\operatorname{Axis}(\gamma_0))\backslash\Gamma/\operatorname{Stab}_\Gamma(\operatorname{Axis}(\gamma_1))$ is a k-to-1 map, where

$$k = [\operatorname{Stab}_\Gamma(\operatorname{Axis}(\gamma_0)) : \gamma_0^\mathbb{Z}][\operatorname{Stab}_\Gamma(\operatorname{Axis}(\gamma_1)) : \gamma_1^\mathbb{Z}] = \iota_\Gamma(\gamma_0)\iota_\Gamma(\gamma_1) m_\Gamma(\gamma_0) m_\Gamma(\gamma_1)$$

(using Equation (2.4)). When $[\gamma]$ varies in a fiber of this map, the quantities $m_{e,\gamma}$ (defined in Section 2) and $h_{\gamma_0}(\gamma\gamma_1\gamma^{-1})$ are constant. As seen in the proof of Theorem 4.2, if $\Gamma\backslash\mathbb{H}^n_\mathbb{R}$ has finite volume, then

$$\operatorname{Vol}(\operatorname{Stab}_\Gamma(\operatorname{Axis}(\gamma_i))\backslash\operatorname{Axis}(\gamma_i)) = \frac{\ell(\gamma_i)}{\iota_G(\gamma_i)}$$

for $i = 0, 1$. The result then immediately follows from Theorem 2.1 and Corollary 2.2. $\square$

In the end of this subsection, we give an asymptotic formula relating the relative height of two primitive loxodromic elements with the crossratio of its fixed points.

Recall that the *crossratio* of four pairwise distinct points a, b, c, d in $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ or in $\mathbb{P}_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is

$$[a, b, c, d] = \frac{(c-a)\,(d-b)}{(c-b)\,(d-a)}\,,$$

with the standard conventions when one of the points is ∞. Using Ahlfors's terminology, the *absolute crossratio* of four pairwise distinct points a, b, c, d in the one-point compactification $\mathbb{R}^{n-1} \cup \{\infty\}$ of the standard $(n-1)$-dimensional Euclidean space is

$$[[a, b, c, d]] = \frac{\|c-a\|\,\|d-b\|}{\|c-b\|\,\|d-a\|}\,,$$

where $\|\cdot\|$ denotes the standard Euclidean norm and with conventions analogous to the definition of the crossratio when one of the points is ∞.

We will denote by $]a_-, a_+[$ the (oriented) geodesic line in $\mathbb{H}^n_\mathbb{R}$ whose pair of endpoints is a given pair (a_-, a_+) of distinct points in $\mathbb{R}^{n-1} \cup \{\infty\}$. In particular, for every loxodromic element $\gamma \in \Gamma$, we have $\operatorname{Axis}(\gamma) =]\gamma^-, \gamma^+[$.

Lemma 4.6. — *(1) Two geodesic lines $]a_-, a_+[$ and $]b_-, b_+[$ in $\mathbb{H}^2_\mathbb{R}$ or $\mathbb{H}^3_\mathbb{R}$ are orthogonal if and only if $[a_-, a_+, b_-, b_+] = -1$.*

(2) For all primitive loxodromic elements $\gamma_0, \gamma \in \Gamma$, as $h_{\gamma_0}(\gamma) \to +\infty$, we have

$$e^{h_{\gamma_0}(\gamma)} = \frac{4}{[[\gamma_0^-, \gamma^-, \gamma_0^+, \gamma^+]]} + \mathrm{O}(1)\,.$$

Proof. — (1) This is a classical fact, see for example pages 15 and 31 of [11].

(2) This is a corollary of [32, Lem. 2.2]. Note that the crossratio in [32] is the logarithm of the absolute crossratio in this paper. □

4.3. Relative complexity of quadratic irrationals

Let K (and its related terminology) be as in the beginning of Section 4. Let α and β be quadratic irrationals in $\widehat{K}$ over K. We define the *relative height* of β with respect to α as

$$\begin{aligned} h_\alpha(\beta) &= \min\left\{\frac{|\beta-\alpha|\,|\beta^\sigma-\alpha^\sigma|}{|\beta-\beta^\sigma|},\ \frac{|\beta-\alpha^\sigma|\,|\beta^\sigma-\alpha|}{|\beta-\beta^\sigma|}\right\} \\ &= \frac{h(\beta)}{2}\min\{|\mathtt{n}(\beta-\alpha)|,\ \mathtt{n}(\beta-\alpha^\sigma)|\}\,. \end{aligned} \tag{4.10}$$

Another expression of this complexity using the absolute crossratios is, if $\beta \notin \{\alpha, \alpha^\sigma\}$,

$$h_\alpha(\beta) = \frac{|\alpha-\alpha^\sigma|}{\max\{[[\alpha,\beta,\alpha^\sigma,\beta^\sigma]],\ [[\alpha,\beta^\sigma,\alpha^\sigma,\beta]]\}}\,. \tag{4.11}$$

Here are some elementary properties of the relative heights.

Lemma 4.7. — *Let α and β in $\widehat{K}$ be quadratic irrationals over K. Then*

(1) $h_{\alpha^\rho}(\beta^\tau) = h_\alpha(\beta)$ for all $\rho, \tau \in \{\mathrm{id}, \sigma\}$.

(2) $h_\alpha(\beta) = 0$ if and only if $\beta \in \{\alpha, \alpha^\sigma\}$.

(3) $h_\alpha(\gamma\beta) = h_\alpha(\beta)$ for every $\gamma \in \operatorname{Stab}_{\Gamma_K}(\{\alpha, \alpha^\sigma\})$.

Proof. — The first two claims are immediate consequences of the definition, and the last one follows immediately from Equation (4.11) and the invariance of the crossratios under homographies. □

LEMMA 4.8. — *For all α, β in $\widehat{K}$ which are quadratic irrationals over K and $s \geqslant 0$, the set*

$$E_s = \{\beta' \in \Gamma_\alpha \backslash \Gamma_K \cdot \beta \;:\; h_\alpha(\beta') \leqslant s\} \tag{4.12}$$

is finite.

Proof. — Note that this set is well defined by Lemma 4.7 (3). By Equation (4.11), by Lemma 4.6 (2), since $h_{\widehat{\alpha}}(\widehat{\beta}) = h_{\widehat{\alpha}}(\widehat{\beta}^{-1})$ and since the fixed points of $\widehat{\alpha}$ and $\widehat{\beta}$ are α, α^σ and β, β^σ respectively, we have

$$h_\alpha(\beta) = \frac{|\alpha - \alpha^\sigma|}{4}\, e^{h_{\widehat{\alpha}}(\widehat{\beta})} + \mathrm{O}(|\alpha - \alpha^\sigma|)\,. \tag{4.13}$$

The set $\{[\gamma] \in \widehat{\alpha}^{\mathbb{Z}} \backslash \Gamma / \widehat{\beta}^{\mathbb{Z}} \;:\; h_{\widehat{\alpha}}(\gamma\widehat{\beta}\gamma^{-1}) \leqslant t\}$ is finite for all $t \geqslant 0$ (see Subsection 4.2). Note that $\widehat{\alpha}^{\mathbb{Z}}$ and $\widehat{\beta}^{\mathbb{Z}}$ have finite index in Γ_α and Γ_β respectively, and that $\widehat{\gamma\beta} = \gamma\widehat{\beta}\gamma^{-1}$ for all $\gamma \in \Gamma_K$. Therefore the set $E_s = \{\gamma \in \Gamma_\alpha \backslash \Gamma_K / \Gamma_\beta \;:\; h_\alpha(\gamma\beta) \leqslant s\}$ is finite. □

This shows that the relative height with respect to a given quadratic irrational α_0 is, indeed, a reasonable complexity for quadratic irrationals in a given orbit of the modular group Γ_K (modulo the stabiliser of α_0).

Recall that for all quadratic irrationals α and β in $\widehat{K}$ over K, we have given the expression of two points $x^{\pm}_{\alpha}(\beta)$ in $\widehat{K}$ in Equation (1.1) of the introduction. We will see in the proof of Theorems 4.9 and 4.10 (see in particular Lemma 4.11) that these points are well defined when β varies in a given orbit of Γ_K, except for finitely many Γ_α-classes. We are now going to prove the equidistribution of these points, when counted according to their relative heights. We state separately the cases $K = \mathbb{Q}$ and K a quadratic imaginary number field, but it is natural to give a common proof for both statements.

THEOREM 4.9. — *Let $K = \mathbb{Q}$, let G be a subgroup of finite index in $\Gamma_K = \mathrm{PSL}_2(\mathbb{Z})$ and let α_0, β_0 be real quadratic irrationals over $\mathbb{Q}$. Then there exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\begin{aligned}\mathrm{Card}\{\beta \in G_{\alpha_0} \backslash G \cdot \beta_0 \;:\; h_{\alpha_0}(\beta) \leqslant s\} \\ = \frac{24\,[\Gamma_{\alpha_0} : G_{\alpha_0}]\,[\Gamma_{\beta_0} : G_{\beta_0}]\, R_{\alpha_0}\, R_{\beta_0}\, h(\alpha_0)}{\pi^2\,[\Gamma_{\mathbb{Q}} : G]}\; s\,(1 + \mathrm{O}(s^{-\kappa}))\,.\end{aligned}$$

Furthermore, as $s \to +\infty$, we have the following convergence of measures on $\mathbb{R} - \{\alpha_0, \alpha_0^\sigma\}$:

$$\frac{\pi^2\,[\Gamma_{\mathbb{Q}} : G]}{24\,[\Gamma_{\beta_0} : G_{\beta_0}]\,R_{\beta_0}\,s} \sum_{\beta \in G\cdot\beta_0,\ h_{\alpha_0}(\beta) \leqslant s} \Delta_{x^-_{\alpha_0}(\beta)} + \Delta_{x^+_{\alpha_0}(\beta)} \overset{*}{\rightharpoonup} \frac{d\mathrm{Leb}_{\mathbb{R}}(t)}{|Q_{\alpha_0}(t)|}\,. \tag{4.14}$$

THEOREM 4.10. — *Let K be a quadratic imaginary number field, let G be a subgroup of finite index in $\Gamma_K = \mathrm{PSL}_2(\mathcal{O}_K)$ and let $\alpha_0, \beta_0 \in \mathbb{C}$ be quadratic irrationals over K. Then there exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\mathrm{Card}\{\beta \in G_{\alpha_0}\backslash G\cdot\beta_0 \ :\ h_{\alpha_0}(\beta) \leqslant s\} = c\ s^2\,(1 + \mathrm{O}(s^{-\kappa}))\,,$$

where

$$c = \frac{8\pi^3\,[\Gamma_{\alpha_0} : G_{\alpha_0}]\,[\Gamma_{\beta_0} : G_{\beta_0}]\,\big|\ln\big|\frac{\mathrm{tr}\,\widehat{\alpha_0} + \sqrt{\mathrm{tr}\widehat{\alpha_0}^{\,2} - 4}}{2}\big|\big|\ \big|\ln\big|\frac{\mathrm{tr}\,\widehat{\beta_0} + \sqrt{\mathrm{tr}\,\widehat{\beta_0}^{\,2} - 4}}{2}\big|\big|}{m_{\Gamma_K}(\alpha_0)\,m_{\Gamma_K}(\beta_0)\,h(\alpha_0)^{-2}\,[\Gamma_K : G]\,|D_K|^{\frac{3}{2}}\,\zeta_K(2)}\,.$$

Furthermore, as $s \to +\infty$, we have the following convergence of measures on $\mathbb{C} - \{\alpha_0, \alpha_0^\sigma\}$:

$$\frac{m_{\Gamma_K}(\beta_0)[\Gamma_K : G]|D_K|^{\frac{3}{2}}\zeta_K(2)}{16\pi^2[\Gamma_{\beta_0} : G_{\beta_0}]\big|\ln\big|\frac{\mathrm{tr}\,\widehat{\beta_0} + \sqrt{\mathrm{tr}\,\widehat{\beta_0}^{\,2} - 4}}{2}\big|\big|s^2} \sum_{\beta \in G\cdot\beta_0, h_{\alpha_0}(\beta) \leqslant s} \Delta_{x^-_{\alpha_0}(\beta)} + \Delta_{x^+_{\alpha_0}(\beta)} \overset{*}{\rightharpoonup} \frac{d\mathrm{Leb}_{\mathbb{C}}(z)}{|Q_{\alpha_0}(z)|^2}. \tag{4.15}$$

Proof. — Let r and r' be the smallest positive integers such that $\widehat{\alpha_0}^{\,r} \in G$ and $\widehat{\beta_0}^{\,r'} \in G$ respectively. As seen at the end of the proof of Theorem 4.2, for K as in Theorem 4.10, we have

$$[\Gamma_{\alpha_0} : G_{\alpha_0}] = \frac{m_{\Gamma_K}(\alpha_0)}{m_G(\alpha_0)}\,r\,, \tag{4.16}$$

and similarly for β_0. Note that this is clear for $K = \mathbb{Q}$, since then $m_{\Gamma_K}(\alpha_0) = m_G(\alpha_0) = 1$.

To prove the counting claims in the above two theorems, we apply Corollary 4.5 with $n = 2$ or $n = 3$, $\Gamma = G$ (whose critical exponent is $n - 1$), $\gamma_0 = \widehat{\alpha_0}^{\,r}$ and $\gamma_1 = \widehat{\beta_0}^{\,r'}$.

Using in the series of equalities below respectively

- the bijection $\beta \mapsto [\widehat{\beta}]$ from $G_{\alpha_0}\backslash G\cdot\beta_0$ to $G_{\alpha_0}\backslash G/G_{\beta_0}$ (with inverse $[\gamma] \mapsto \gamma\beta_0$), Equation (4.13) and the definition of the complexity $h(\alpha_0)$,

- the fact that $[G_{\alpha_0} : \widehat{\alpha_0}^{\,r\mathbb{Z}}] = m_G(\alpha_0)$ and similarly for β_0, as in the beginning of the proof of Corollary 4.5,

• Corollary 4.5, noting that since $\mathrm{PSL}_2(\mathbb{Z})$ acts freely on $T^1\mathbb{H}^2_{\mathbb{R}}$, all multiplicities $m_{e,\gamma}$ are equal to 1, if $K=\mathbb{Q}$, and otherwise, as seen in the proof of Theorem 4.2, the multiplicities $m_{e,\gamma}$ different from 1 contribute only in a negligible way to the sums,
we have, as $s\to+\infty$,

$$\begin{aligned}
&\operatorname{Card}\{\beta\in G_{\alpha_0}\backslash G\cdot\beta_0\;:\;h_{\alpha_0}(\beta)\leqslant s\}\\
&=\operatorname{Card}\big\{[\gamma]\in G_{\alpha_0}\backslash G/G_{\beta_0}\;:\;h_{\widehat{\alpha_0}}(\gamma\widehat{\beta_0}\gamma^{-1})\leqslant\ln\big(2\,h(\alpha_0)\,s+\mathrm{O}(1)\big)\big\}\\
&=\frac{1+\mathrm{o}(1)}{m_G(\alpha_0)m_G(\beta_0)}\operatorname{Card}\big\{[\gamma]\in\widehat{\alpha_0}^{r\mathbb{Z}}\backslash G/\widehat{\beta_0}^{r'\mathbb{Z}}:h_{\widehat{\alpha_0}}(\gamma\widehat{\beta_0}\gamma^{-1})\leqslant\ln\big(2h(\alpha_0)s+\mathrm{O}(1)\big)\big\}\\
&=\frac{\operatorname{Vol}(\mathbb{S}^{n-2})^2r\ell(\widehat{\alpha_0})r'\ell(\widehat{\beta_0})}{m_G(\alpha_0)m_G(\beta_0)2^{n-1}(n-1)\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(G\backslash\mathbb{H}^n_{\mathbb{R}})}\big(2h(\alpha_0)s\big)^{n-1}(1+\mathrm{O}(s^{-\kappa})).
\end{aligned}$$

When $K=\mathbb{Q}$ and $n=2$, this proves the counting claim in Theorem 4.9, by using Equations (4.2), (4.3) and (4.16), and noting that $m_{\Gamma_K}(\alpha_0)=m_{\Gamma_K}(\beta_0)=1$.

When K is quadratic imaginary and $n=3$, this proves the counting claim in Theorem 4.10, by using Equations (3.2), (4.6) and (4.16).

To prove the equidistribution claims in the above two theorems, we need to use a stronger geometric equidistribution theorem (also proven in [38]) than the one stated in Section 2.

For any $\beta\in G\cdot\beta_0$, let $v_{\alpha_0}(\beta)$ be the initial tangent vector of the common perpendicular from the geodesic line $D^-=\operatorname{Axis}(\widehat{\alpha_0})$ with endpoints α_0 and α_0^σ to that with endpoints β and β^σ, if it exists (and the summations below are on $\beta\in G\cdot\beta_0$ such that it does exist). Let $D^+=\operatorname{Axis}(\widehat{\beta_0})$ and G_{D^+} its stabiliser in G, so that as already seen $\operatorname{Vol}(G_{D^+}\backslash D^+)=\frac{r'\,\ell(\widehat{\beta_0})}{\iota_G(\beta_0)}$. By replacing the equidistribution of initial points by the one of the initial tangent vectors (see [38, Coro. 20]), and by arguments similar to the ones leading to Equations (4.1) and (4.5), we have, by the expression of the skinning measure and Bowen-Margulis measure recalled in Section 2, with $n=2$ if $K=\mathbb{Q}$ and $n=3$ otherwise, as $t\to+\infty$,

$$\frac{2^{n-1}\,(n-1)m_G(\beta_0)\operatorname{Vol}(\mathbb{S}^{n-1})\operatorname{Vol}(G\backslash\mathbb{H}^n_{\mathbb{R}})}{\iota_G(\beta_0)\operatorname{Vol}(\mathbb{S}^{n-2})\operatorname{Vol}(G_{D^+}\backslash D^+)\,e^{(n+1)t}}\sum_{\beta\in G\cdot\beta_0,\,h_{\widehat{\alpha_0}}(\widehat{\beta})\leqslant t}\Delta_{v_{\alpha_0}(\beta)}\overset{*}{\rightharpoonup}\operatorname{Vol}_{\partial^1_+D^-}.$$

Hence, by Equation (4.13), as $s\to+\infty$,

$$\frac{(n-1)\,m_G(\beta_0)\operatorname{Vol}(\mathbb{S}^{n-1})\;\operatorname{Vol}(G\backslash\mathbb{H}^n_{\mathbb{R}})}{\operatorname{Vol}(\mathbb{S}^{n-2})\,r'\,\ell(\widehat{\beta_0})\,h(\alpha_0)^{n-1}\;s^{n-1}}\sum_{\beta\in G\cdot\beta_0,\,h_{\alpha_0}(\beta)\leqslant s}\Delta_{v_{\alpha_0}(\beta)}\overset{*}{\rightharpoonup}\operatorname{Vol}_{\partial^1_+D^-}.\tag{4.17}$$

We give in the following two lemmas the main computations used to deduce from this the equidistribution claims in the above two theorems.

Lemma 4.11. — *If the geodesic lines* $]\alpha_0, \alpha_0^\sigma[$ *and* $]\beta, \beta^\sigma[$ *in* $\mathbb{H}^n_\mathbb{R}$ *are disjoint, then the endpoints of the geodesic line containing their common perpendicular are the points* $x^\pm_{\alpha_0}(\beta)$ *given by Equation* (1.1), *which are hence well defined.*

Proof. — Let $]x^-, x^+[$ be this geodesic line. By Lemma 4.6 (1), and since two nonintersecting nonasymptotic geodesic lines have one and only one geodesic line orthogonal to both of them, the points x^- and x^+ are the two solutions of the pair of equations $[x^-, x^+, \alpha_0, \alpha_0^\sigma] = -1 = [x^-, x^+, \beta, \beta^\sigma]$. An easy computation shows that this system is equivalent to

$$\begin{cases} x^- + x^+ = 2\,\dfrac{\mathtt{n}(\alpha_0) - \mathtt{n}(\beta)}{\mathtt{tr}\,\alpha_0 - \mathtt{tr}\,\beta} \\[3ex] x^- x^+ = \dfrac{\mathtt{tr}\,\beta\,\mathtt{n}(\alpha_0) - \mathtt{tr}\,\alpha_0\,\mathtt{n}(\beta)}{\mathtt{tr}\,\alpha_0 - \mathtt{tr}\,\beta}\,. \end{cases}$$

Solving a quadratic equation gives the result. □

Lemma 4.12. — *For* $n = 2, 3, 5$, *for all distinct* x, y *in* $\partial_\infty \mathbb{H}^n_\mathbb{R} - \{\infty\}$, *the pushforward of the Riemannian measure of the unit normal bundle of the geodesic line* $L =]x, y[$ *in* $\mathbb{H}^n_\mathbb{R}$ *by the positive endpoint map* $v \mapsto z = v_+$ *from* $\partial^1_+ L$ *to* $\partial_\infty \mathbb{H}^n_\mathbb{R} - \{x, y\}$ *is*

$$\frac{\|x-y\|^{n-1}}{\|(z-x)(z-y)\|^{n-1}}\; d\mathrm{Leb}_{\mathbb{R}^{n-1}}(z)\,.$$

By symmetry, the same result holds for the pushforward by the negative endpoint map $v \mapsto z = v_-$. When $n = 2, 3$, $x = \alpha_0$ and $y = \alpha_0^\sigma$, this measure is $\frac{2^{n-1}\, d\mathrm{Leb}_{\widehat{K}}(z)}{h(\alpha_0)^{n-1}\, |Q_{\alpha_0}(z)|^{n-1}}$, with the previous notation.

Proof. — Let $\widehat{K} = \mathbb{R}$ if $n = 2$, $\widehat{K} = \mathbb{C}$ if $n = 3$ and $\widehat{K} = \mathbb{H}$ if $n = 5$ (we write $|z| = \mathtt{N}(z)^{1/2}$ if $z \in \mathbb{H}$), so that $\partial_\infty \mathbb{H}^n_\mathbb{R} - \{\infty\} = \widehat{K}$.

If L_∞ is the geodesic line with points at infinity $0 \in \widehat{K}$ and ∞, then for all $v \in \partial^1_+ L_\infty$, we have $d\mathrm{Vol}_{\partial^1_+ L_\infty}(v) = \frac{dt}{t} d\mathrm{Vol}_{\mathbb{S}^{n-2}}(\sigma)$, where $t > 0$ is the last coordinate of $\pi(v)$ and $\sigma \in \mathbb{S}^{n-2}$ is the parameter of the Euclidean horizontal vector v (note that $d\mathrm{Vol}_{\mathbb{S}^0}$ is the counting measure on $\{-1, 1\}$

if $n=2$). If $z=v_+\in\widehat{K}-\{0\}$ is the positive point at infinity of v, then $z=t\,\sigma$ and

$$d\mathrm{Vol}_{\partial^1_+ L_\infty}(v)=\frac{d\mathrm{Leb}_{\widehat{K}}(z)}{|z|^{n-1}}\;.$$

We may assume that $x>y$ if $n=2$. The homography $A:z\mapsto(z-x)(z-y)^{-1}$ maps x to 0 and y to ∞. Since it may be written $z\mapsto(\lambda z-\lambda x)(\lambda z-\lambda y)^{-1}$ with λ in (the center of) $\widehat{K}$ such that $\lambda^2=(x-y)^{-1}$ if $\widehat{K}=\mathbb{R},\mathbb{C}$ and $\lambda^4=\frac{1}{\mathtt{N}(y-x)}$ if $\widehat{K}=\mathbb{H}$ (see Equation (3.4) to check the Dieudonné determinant is 1), it is the extension at infinity of an isometry of $\mathbb{H}^n_{\mathbb{R}}$ (by Poincaré's extension), sending $\mathrm{Vol}_{\partial^1_+ L}$ to $\mathrm{Vol}_{\partial^1_+ L_\infty}$. Since $|\dot{A}\,(z)|=\frac{|x-y|}{|z-y|^2}$, we have

$$(A^{-1})_*\Big(\frac{d\mathrm{Leb}_{\widehat{K}}(z)}{|z|^{n-1}}\Big)=\frac{|x-y|^{n-1}}{|(z-x)(z-y)|^{n-1}}\;d\mathrm{Leb}_{\widehat{K}}(z)\;.$$

The result follows. $\square$

Using the continuity of the pushforward maps of measures, let us consider the sum of the pushforwards of Equation (4.17) both by the positive and by the negative endpoint map. By the comments following Lemma 4.12, and since $\{x^-_{\alpha_0}(\beta),x^+_{\alpha_0}(\beta)\}=\{v_{\alpha_0}(\beta)_-,v_{\alpha_0}(\beta)_+\}$ by Lemma 4.11, we have, as $s\to+\infty$,

$$\frac{(n-1)m_G(\beta_0)\mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{Vol}(G\backslash\mathbb{H}^n_{\mathbb{R}})}{2^{n-1}\mathrm{Vol}(\mathbb{S}^{n-2})r'\ell(\widehat{\beta_0})s^{n-1}}\sum_{\beta\in G\cdot\beta_0,\,h_{\alpha_0}(\beta)\leqslant s}\Delta_{x^-_{\alpha_0}(\beta)}+\Delta_{x^+_{\alpha_0}(\beta)}\;\overset{*}{\rightharpoonup}\;\frac{2d\mathrm{Leb}_{\widehat{K}}(z)}{|Q_{\alpha_0}(z)|^{n-1}}.$$

The equidistribution claims in Theorems 4.9 and 4.10 follow, as in the end of the proof of their counting claims. $\square$

Remark. — If $\widehat{K}=\mathbb{H}$, the crossratio of a quadruple of pairwise distinct points (a,b,c,d) in $\widehat{K}$ may be defined as

$$[a,b,c,d]=(c-b)^{-1}(c-a)(d-a)^{-1}(d-b)\;.$$

Though this crossratio does not extend continuously to the quadruples of pairwise distinct points in the one-point compactification $\mathbb{H}\cup\{0\}$, and it is only invariant under $\mathrm{SL}_2(\mathbb{H})$ up to conjugation, it also characterises the orthogonality of the geodesic lines in $\mathbb{H}^5_{\mathbb{R}}$ by the same formula (two geodesic lines $]a_-,a_+[$ and $]b_-,b_+[$ in $\mathbb{H}^5_{\mathbb{R}}$ are orthogonal if and only if $[a_-,a_+,b_-,b_+]=-1$). Furthermore, its absolute crossratio $|c-b|^{-1}|c-a||d-a|^{-1}|d-b|$ (we write $|z|=\mathtt{N}(z)^{1/2}$ if $z\in\mathbb{H}$) does extend continuously (as already mentioned) and is invariant under $\mathrm{SL}_2(\mathbb{H})$ (see for instance [8] for all this).

Let A be a definite quaternion algebra over $\mathbb{Q}$, let $\mathcal{O}$ be a maximal order in A, let G be a finite index subgroup of $\mathrm{PSL}_2(\mathcal{O})$, and let $\alpha_0, \beta_0 \in \mathbb{H}$ be two loxodromic quadratic irrationals over $\mathcal{O}$ (see the definitions above Theorem 4.4). We may also define the *relative height* of an element β of $\mathrm{PSL}_2(\mathcal{O}) \cdot \beta_0$ with respect to α_0 by the same formula (4.10). It is still related to absolute crossratios by Equation (4.11), and still satisfies the properties of Lemma 4.7. We do have a counting asymptotic of the elements $\beta \in G_{\alpha_0} \backslash G \cdot \beta_0$ with relative height at most s with respect to α_0, as $s \to +\infty$, analogous to Theorem 4.9 and 4.10, and a related equidistribution result, but now, due to the higher difficulty of solving quadratic equations in the noncommutative $\mathbb{H}$, the points $x^{\pm}_{\alpha_0}(\beta)$ are much less explicit: they can only be defined as the unique two solutions x^-, x^+ of the pair of equations $[x^-, x^+, \alpha_0, \alpha_0^\sigma] = -1 = [x^-, x^+, \beta, \beta^\sigma]$. We leave the precise statement to the interested reader.

4.4. Counting crossratios and application to generalised Schottky-Klein functions

The aim of this subsection is to generalise the work of Pollicott [40], using the asymptotic of crossratios to prove the convergence of Schottky-Klein functions, to a much larger class of Kleinian groups.

Let $\widetilde{M}$ be a complete simply connected Riemannian manifold with (dimension at least 2 and) pinched negative sectional curvature $-b^2 \leqslant K \leqslant -1$. Let Γ be a torsion free (only to simplify the statements) nonelementary (that is, not virtually nilpotent) discrete group of isometries of $\widetilde{M}$, with critical exponent δ_Γ and limit set $\Lambda\Gamma$ (defined as in Section 2). Let D^- and D^+ be two geodesic lines in $\widetilde{M}$ with points at infinity in the *domain of discontinuity* $\Omega\Gamma = \partial_\infty \widetilde{M} - \Lambda\Gamma$ of Γ.

The Bowen-Margulis measure m_{BM} and skinning measures $\sigma_{D^\pm}$ are defined as in Section 2 where $\widetilde{M} = \mathbb{H}^n_{\mathbb{R}}$. We refer to [7] for finiteness criteria of m_{BM}, for instance satisfied if Γ is convex-cocompact, or when Γ is geometrically finite in constant curvature (in which case δ_Γ is the Hausdorff dimension of $\Lambda\Gamma$), but there are many examples. We refer to [1] for mixing criteria of m_{BM} under the geodesic flow, for instance satisfied, when m_{BM} is finite, if $\widetilde{M}$ has constant curvature. Again there are many more examples.

The following result generalises [40, Theo. 1.4] (whose proof used symbolic dynamics and transfer operator techniques), which was only stated for $\widetilde{M} = \mathbb{H}^3_{\mathbb{R}}$, Γ a Schottky group and $D^- = D^+$, with a nonexplicit multiplicative constant.

Theorem 4.13. — *If $m_{\rm BM}$ is finite and mixing, then, as $s \to +\infty$,*

$$\operatorname{Card}\{\gamma \in \Gamma \;:\; d(D^-, \gamma\, D^+) \leqslant s\} \sim \frac{\|\sigma_{D^-}\|\,\|\sigma_{D^+}\|}{\delta_\Gamma\,\|m_{\rm BM}\|}\; e^{\delta_\Gamma\, s}\,.$$

Proof. — Since the endpoints of $D^{\pm}$ are in the domain of discontinuity of Γ, the support of the measure $\widetilde{\sigma}_{D^\pm}$ is compact, hence the skinning measure $\sigma_{D^\pm}$ is finite. The result then follows from [38, Coro. 20], since the stabiliser of $D^{\pm}$ is trivial. $\square$

Until the end of Subsection 4.4, we assume that $\widetilde{M} = \mathbb{H}^3_{\mathbb{R}}$. The next result on asymptotic properties of crossratios is deduced from Theorem 4.13 in the same way that Theorem 1.3 is deduced from Theorem 1.4 in [40].

Corollary 4.14. — *If $m_{\rm BM}$ is finite, then for all z, ξ in $\Omega\Gamma$, as $\epsilon \to 0$,*

$$\operatorname{Card}\{\gamma \in \Gamma \;:\; \big|\,[z, \xi, \gamma z, \gamma\xi] - 1\big| \geqslant \epsilon\} \sim \frac{4^{\delta_\Gamma}\,\|\sigma_{D^-}\|\,\|\sigma_{D^+}\|}{\delta_\Gamma\,\|m_{\rm BM}\|}\; \epsilon^{-\delta_\Gamma}\,.$$

The next result is proven as [40, Coro. 1.2], and defines a much larger class of holomorphic maps in two variables than the Schottky-Klein functions (see op. cit. for motivations and references). We denote by Γ_0 a choice of representatives modulo inverse of the nontrivial elements of Γ.

Corollary 4.15. — *If $m_{\rm BM}$ is finite and $\delta_\Gamma < 1$, then the function*

$$(z, \xi) \mapsto (z - \xi) \prod_{g \in \Gamma_0} \frac{(gz - z)(g\xi - z)}{(g\xi - \xi)(g\xi - z)}\,,$$

where the terms in the product are ordered by the distance of their absolute value to 1, converges uniformly on compact subsets of $\Omega\Gamma \times \Omega\Gamma$.

5. Counting and equidistribution around binary quadratic, Hermitian and Hamiltonian forms

Let $Q(x, y) = a\,x^2 + b\,xy + c\,y^2$ be a binary quadratic form, which is primitive integral (its coefficients a, b, c are relatively prime elements in $\mathbb{Z}$). Let $\operatorname{Disc}_Q = b^2 - 4ac$ be the discriminant of Q. The group $\operatorname{SL}_2(\mathbb{Z})$ acts on the set of integral binary quadratic forms by precomposition, and we denote by $\operatorname{SO}_Q(\mathbb{Z})$ the stabiliser of Q. When Q is positive definite, the following asymptotic is originally due to Gauss with a weaker error bound:

$$\operatorname{Card}\{(u, v) \in \mathbb{Z}\times\mathbb{Z} : (u, v) = 1,\ Q(u, v) \leqslant s\} = \frac{12}{\pi\sqrt{-\operatorname{Disc}_Q}}\,s + \operatorname{O}(s^{131/416})\,,$$

the current error bound is proved in [17]. This asymptotic, with a less explicit error bound, can be given a geometric proof using $\mathrm{PSL}_2(\mathbb{Z})$-orbits of horoballs and points in $\mathbb{H}^2_{\mathbb{R}}$ in the same way as Theorem 5.8 in Subsection 5.2 below.

When Q is indefinite, the group of automorphs $\mathrm{SO}_Q(\mathbb{Z})$ is infinite, and it is appropriate to count $\mathrm{SO}_Q(\mathbb{Z})$-orbits of relatively prime representations of integers by Q. Let

$$R_Q = \ln \frac{t_Q + u_Q\sqrt{\mathrm{Disc}_Q}}{2}$$

be the regulator of Q, where (t_Q, u_Q) is the fundamental solution of the Pell-Fermat equation $t^2 - \mathrm{Disc}_Q\, u^2 = 4$. The asymptotic behaviour of this counting function is (see for example [5, p. 164])

$$\begin{aligned}\mathrm{Card}\big(\mathrm{SO}_Q(\mathbb{Z})\backslash\{(u,v)\in\mathbb{Z}\times\mathbb{Z} : (u,v)=1, |Q(u,v)|\leqslant s\}\big)\\ = \frac{12R_Q}{\pi^2\sqrt{\mathrm{Disc}_Q}}s + \mathrm{O}(\sqrt{s}).\end{aligned} \tag{5.1}$$

A geometric proof of the asymptotic (5.1) with a less explicit error bound can be obtained from the proof of [34, Théo. 4.5], using Corollary 2.2 instead of [34, Cor. 3.9].

The results in Subsection 5.1, generalising the asymptotic formula (5.1) to binary Hermitian and Hamiltonian forms, are the versions with error terms and with related equidistribution results of respectively Theorem 4, Theorem 1, Corollary 3 in [33], and Theorem 13, Corollary 18, Theorem 1, Corollary 2 in [35]. Similarly, the error term in Corollary 8 of [33] is $\mathrm{O}(s^{2-\kappa})$ for some $\kappa > 0$.

Given an arithmetic group Γ of isometries of a totally geodesic subspace X_Γ of $\mathbb{H}^n_{\mathbb{R}}$ or $\mathbb{H}^n_{\mathbb{C}}$, we will denote by $\mathrm{Covol}(\Gamma)$ the volume of the Riemannian orbifold $\Gamma\backslash X_\Gamma$.

5.1. Indefinite binary Hermitian and Hamiltonian forms

Let K and $\mathfrak{m}$ be as in the beginning of Subsection 3.1. Fix a binary Hermitian form $f : \mathbb{C}^2 \to \mathbb{R}$ with

$$f(u,v) = a\,|u|^2 + 2\,\mathrm{Re}(b\,\overline{u}\,v) + c\,|v|^2 \tag{5.2}$$

which is integral over $\mathcal{O}_K$ (its coefficients $a = a(f), b = b(f), c = c(f)$ satisfy $a, c \in \mathbb{Z}$ and $b \in \mathcal{O}_K$). The group $\mathrm{SL}_2(\mathbb{C})$ acts on the right on the set of binary Hermitian forms by precomposition, and the action of the Bianchi subgroup $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ preserves the integral ones. The group $\mathrm{SU}_f(\mathcal{O}_K)$ of automorphs of f consists of the elements $g \in \Gamma_K$ stabilising f, that is, such that $f \circ g = f$.

In this subsection, we assume that f is indefinite (its discriminant $\mathrm{Disc}(f) = |b|^2 - ac$ is positive), a feature which is preserved by the action of $\mathrm{SL}_2(\mathbb{C})$.

Let G be a finite index subgroup of Γ_K. Let $\Gamma_{K,\,x,\,y}$ and $G_{x,\,y}$ be the stabilisers of $(x, y) \in K \times K$ in Γ_K and G respectively. Let $\iota_G = 1$ if $-\mathrm{id} \in G$ and $\iota_G = 2$ otherwise. For every x, y in $\mathcal{O}_K$ not both zero, and for every $s > 0$, let

$$\psi_{f,\,G,\,x,\,y}(s) = \mathrm{Card}\big((\mathrm{SU}_f(\mathcal{O}_K) \cap G)\backslash\big\{(u, v) \in G(x, y)\ :\ \mathtt{n}(\mathcal{O}_K x + \mathcal{O}_K y)^{-1}|f(u, v)| \leqslant s\big\}\big)\ .$$

Theorem 5.1. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi_{f,\,G,\,x,\,y}(s) = \frac{\pi\ \iota_G\ [\Gamma_{K,\,x,\,y} : G_{x,\,y}]\ \mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K) \cap G)}{2\ |D_K|\ \zeta_K(2)\ \mathrm{Disc}(f)\ [\Gamma_K : G]}\quad s^2(1 + \mathrm{O}(s^{-\kappa}))\ .$$

If $y \neq 0$, then, as $s \to +\infty$,

$$\frac{[\Gamma_K : G]\,|D_K|^{\frac{3}{2}}\,\zeta_K(2)\,\mathrm{Disc}(f)\,\mathtt{n}(y)}{\pi^2\,\iota_G\,\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K) \cap G)\,s^2} \sum_{f' \in G\cdot f,\, 0 < |f'(x,y)| \leqslant s} \Delta_{\frac{a(f')x + b(f')y}{f'(x,y)}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{C}}.$$

If Λ is the lattice of $\lambda \in \mathcal{O}_K$ such that $\pm\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in G$, identifying $\mathcal{O}_K$ with the upper triangular unipotent subgroup of Γ_K, we have, as $s \to +\infty$,

$$\mathrm{Card}\{f' \in \Lambda\backslash G \cdot f\ :\ 0 < |a(f')| \leqslant s\}$$
$$= \frac{\pi^2\,\iota_G\,[\mathcal{O}_K : \Lambda]\,\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K) \cap G)}{2\,[\Gamma_K : G]\,|D_K|^{\frac{1}{2}}\,\zeta_K(2)\,\mathrm{Disc}(f)}\quad s^2(1 + \mathrm{O}(s^{-\kappa}))\ .$$

For smooth functions ψ with compact support on $\mathbb{C}$, there is an error term in the above equidistribution claim evaluated at ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$.

Proof. — The first claim is proven in the same way as [33, Thm. 4], using Corollary 2.2 instead of [34, Cor. 4.9].

Let us prove the second claim. Let $n = 3$ and $\widehat{K} = \mathbb{C}$. We denote by $\overline{H}$ the image in $\mathrm{PSL}_2(\widehat{K})$ of any subgroup H of $\mathrm{SL}_2(\widehat{K})$. As in the proof of

Theorem 3.1, we consider the isometric action of $\mathrm{PSL}_2(\widehat{K})$ on (the upper halfspace model of) $\mathbb{H}^n_{\mathbb{R}}$. As explained in [33, §2] (where the convention for binary Hermitian form was to replace b by $\overline{b}$), when $a(f) \neq 0$, the totally geodesic hyperplane

$$\mathcal{C}(f) = \{(z,t) \in \mathbb{H}^n_{\mathbb{R}} \;:\; f(z,1) + |a(f)|\, t^2 = 0\}$$

is preserved by $\mathrm{SU}_f(\mathcal{O}_K)$, its boundary at infinity is the $(n-2)$-sphere of center $-\frac{b(f)}{a(f)}$ and radius $\frac{\sqrt{\mathrm{Disc}(f)}}{|a(f)|}$, the hyperbolic orbifold $\mathrm{SU}_f(\mathcal{O}_K)\backslash\mathcal{C}(f)$ has finite volume (and so does $(\mathrm{SU}_f(\mathcal{O}_K) \cap G)\backslash\mathcal{C}(f)$), and, for every $g \in \mathrm{SL}_2(\widehat{K})$,

$$\mathcal{C}(f \circ g) = g^{-1}\,\mathcal{C}(f)\,.$$

Let $\rho = x\,y^{-1} \in K \cup \{\infty\}$, $\gamma_\rho = \begin{pmatrix} \rho & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(K)$ if $y \neq 0$ and $\gamma_\rho = \mathrm{id}$ otherwise. We define $R_G(f) = 2$ if there exists an element g in G such that $f \circ g = -f$, and $R_G(f) = 1$ otherwise.

To prove the equidistribution claim of Theorem 5.1, we apply Equation (2.2) with $n = 3$, Γ the arithmetic group $\gamma_\rho^{-1}\,\overline{G}\,\gamma_\rho$, D^- the horoball $\mathcal{H}_\infty = \{(z,t) \in \mathbb{H}^n_{\mathbb{R}} \;:\; t \geqslant 1\}$ (noting that ∞ is indeed a parabolic fixed point of Γ, since $\rho = \gamma_\rho\infty$ is a parabolic fixed point of $\overline{G}$) and D^+ the totally geodesic hyperplane $\mathcal{C}(f \circ \gamma_\rho) = \gamma_\rho^{-1}\,\mathcal{C}(f)$. The stabiliser Γ_{D^+} of D^+ in Γ indeed has finite covolume in D^+, since (by conjugation for the first equality and by [33, Eq. (8)] for the second one)

$$\mathrm{Vol}(\Gamma_{D^+}\backslash D^+) = \mathrm{Vol}\big(\mathrm{Stab}_{\overline{G}}(\mathcal{C}(f))\backslash\mathcal{C}(f)\big) = \frac{1}{R_G(f)}\,\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K) \cap G)\,. \tag{5.3}$$

For every $\gamma \in \Gamma$, the origin of the common perpendicular from D^- to $\gamma D^+ = \mathcal{C}(f \circ \gamma_\rho \circ \gamma)$ (when it exists, that is when $a(f \circ \gamma_\rho \circ \gamma) \neq 0$ and the radius of the circle at infinity of $\mathcal{C}(f \circ \gamma_\rho \circ \gamma)$ is strictly less than 1), is $\big(-\frac{b(f\circ\gamma_\rho\circ\gamma)}{a(f\circ\gamma_\rho\circ\gamma)}, 1\big)$, and its hyperbolic length is $\big|\ln\big(\frac{\sqrt{\mathrm{Disc}(f)}}{|a(f\circ\gamma_\rho\circ\gamma)|}\big)\big| = \ln\big(\frac{|a(f\circ\gamma_\rho\circ\gamma)|}{\sqrt{\mathrm{Disc}(f)}}\big)$ (by invariance of the discriminant under $\mathrm{SL}_2(\widehat{K})$). Since D^+ has codimension 1 and $\mathrm{PSL}_2(\widehat{K})$ preserves the orientation of $\mathbb{H}^n_{\mathbb{R}}$, the pointwise stabiliser of D^+ in Γ is trivial. Using a similar comment about the multiplicities as in the proof of Theorem 4.2, we hence have by taking

$t = \ln \frac{s}{\sqrt{\mathrm{Disc}(f)}}$ in Equation (2.2), as $s \to +\infty$,

$$\frac{(n-1)\mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})}{\mathrm{Vol}(\mathbb{S}^{n-2})\mathrm{Vol}(\Gamma_{D^+}\backslash D^+)\big(\frac{s}{\sqrt{\mathrm{Disc}(f)}}\big)^{n-1}} \sum_{\gamma\in\Gamma/\Gamma_{D^+},\, 0<|a(f\circ\gamma_\rho\circ\gamma)|\leqslant s} \Delta_{\big(-\frac{b(f\circ\gamma_\rho\circ\gamma)}{a(f\circ\gamma_\rho\circ\gamma)},1\big)} \overset{*}{\rightharpoonup} \mathrm{Vol}_{\partial D^-}\,.$$

Since the map from $G\cdot f$ to Γ/Γ_{D^+} defined by $f\circ\gamma \mapsto \gamma_\rho^{-1}\gamma\gamma_\rho\Gamma_{D^+}$ is a $R_G(f)$-to-1 map, and since the pushforward of measures by the continuous map $(z,1)\mapsto -z$ from ∂D^- to $\widehat{K}$ sends $\mathrm{Vol}_{\partial D^-}$ to $\mathrm{Leb}_{\widehat{K}}$, we have

$$\frac{(n-1)\mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})\mathrm{Disc}(f)^{\frac{n-1}{2}}}{\mathrm{Vol}(\mathbb{S}^{n-2})\mathrm{Vol}(\Gamma_{D^+}\backslash D^+)R_G(f)s^{n-1}} \sum_{f'\in G\cdot f,\, 0<|a(f'\circ\gamma_\rho)|\leqslant s} \Delta_{\frac{b(f'\circ\gamma_\rho)}{a(f'\circ\gamma_\rho)}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\widehat{K}}. \tag{5.4}$$

Using Humbert's formula (3.2), we have

$$\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3_{\mathbb{R}}) = \mathrm{Vol}(\,\overline{G}\,\backslash\mathbb{H}^3_{\mathbb{R}}) = [\overline{\Gamma_K}:\overline{G}]\mathrm{Vol}(\overline{\Gamma_K}\backslash\mathbb{H}^3_{\mathbb{R}}) = \frac{[\Gamma_K:G]}{\iota_G}\,\frac{|D_K|^{\frac{3}{2}}\zeta_K(2)}{4\pi^2}. \tag{5.5}$$

By Equations (5.4), (5.3) and (5.5), we have

$$\frac{[\Gamma_K:G]\,|D_K|^{\frac{3}{2}}\,\zeta_K(2)\,\mathrm{Disc}(f)}{\iota_G\,\pi^2\,\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O}_K)\cap G)\,s^2} \sum_{f'\in G\cdot f,\ 0<|a(f'\circ\gamma_\rho)|\,\leqslant\, s} \Delta_{\frac{b(f'\circ\gamma_\rho)}{a(f'\circ\gamma_\rho)}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{C}}\,.$$

When $y=0$, this proves Theorem 1.7 in the Introduction. By considering the measures induced on the compact quotient $\Lambda\backslash\mathbb{C}$, whose volume is $\frac{|D_K|}{2}[\mathcal{O}_K:\Lambda]$, and by applying Theorem 1.7 with error term to the constant function 1, we get the last claim of Theorem 5.1.

When $y\neq 0$, replacing s by $\frac{s}{|y|^2}$, using the pushforward of measures on $\mathbb{C}$ by $z\mapsto -\overline{y^{-1}\,z}$, which sends $\mathrm{Leb}_{\mathbb{C}}$ to $|y|^2\mathrm{Leb}_{\mathbb{C}}$, and since $a(f'\circ\gamma_\rho) = f'\circ\gamma_\rho(1,0) = (y\,\overline{y})^{-1}\,f'(x,y)$ and $b(f'\circ\gamma_\rho) = -\big(a(f')\,\overline{\rho}+\overline{b(f')}\,\big)$ by an easy computation, this gives the equidistribution claim in Theorem 5.1. □

For every $s>0$, we consider the integer (depending only on the ideal class of $\mathfrak{m}$)

$$\psi_{f,\mathfrak{m}}(s) = \mathrm{Card}\big(\mathrm{SU}_f(\mathcal{O}_K)\backslash\big\{(u,v)\in\mathfrak{m}\times\mathfrak{m} : \mathtt{n}(\mathfrak{m})^{-1}|f(u,v)|\leqslant s,\, u\mathcal{O}_K+v\mathcal{O}_K=\mathfrak{m}\big\}\big),$$

which is the number of nonequivalent $\mathfrak{m}$-primitive representations by f of rational integers with absolute value at most $s\,\mathtt{n}(\mathfrak{m})$.

By taking $G = \Gamma_K$ and $x, y \in K$ such that $\mathfrak{m} = \mathcal{O}_K x + \mathcal{O}_K y$ (which exist, see for instance [10, §7]) in the first claim of Theorem 5.1, we have the following result.

Theorem 5.2. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi_{f,\,\mathfrak{m}}(s) \sim \frac{\pi \operatorname{Covol}(\mathrm{SU}_f(\mathcal{O}_K))}{2\,|D_K|\;\zeta_K(2)\;\operatorname{Disc}(f)}\;\; s^2(1+\mathrm{O}(s^{-\kappa}))\,. \quad \square$$

Using the results of [22] (or other ways of obtaining formulas for $\operatorname{Covol}(\mathrm{SU}_f(\mathcal{O}_K)))$, one gets very explicit versions of Theorem 5.2 in special cases. A constant $\iota(f) \in \{1,2,3,6\}$ is defined as follows. If $\operatorname{Disc}(f) \equiv 0 \mod 4$, let $\iota(f) = 2$. If the coefficients a and c of the form f as in Equation (5.2) are both even, let $\iota(f) = 3$ if $\operatorname{Disc}(f) \equiv 1 \mod 4$, and let $\iota(f)$ be the remainder modulo 8 of $\operatorname{Disc}(f)$ if $\operatorname{Disc}(f) \equiv 2 \mod 4$. In all other cases, let $\iota(f) = 1$. The following result on integral binary Hermitian forms f over $\mathbb{Q}(i)$ follows from the first claim of Theorem 5.2 using $K = \mathbb{Q}(i)$ as in the proof of [33, Coro. 3].

Corollary 5.3. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\begin{aligned}&\operatorname{Card}\big(\mathrm{SU}_f(\mathbb{Z}[i])\backslash\{(u,v) = \mathbb{Z}[i]^2 \;:\; u\,\mathbb{Z}[i] + v\,\mathbb{Z}[i] = \mathbb{Z}[i],\;\; |f(u,v)| \leqslant s\}\big)\\ &\qquad = \frac{\pi^2}{8\;\iota(f)\;\zeta_{\mathbb{Q}(i)}(2)} \prod_{p|\operatorname{Disc}(f)} \Big(1 + \Big(\frac{-1}{p}\Big)p^{-1}\Big)\;\; s^2(1+\mathrm{O}(s^{-\kappa}))\,,\end{aligned}$$

where p ranges over the odd positive rational primes and $\big(\frac{-1}{p}\big)$ is the Legendre symbol of -1 modulo p. $\square$

Let $\mathbb{H}, A, \mathcal{O}$ and the associated notation be as in the beginning of Section 3.2. Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a binary Hamiltonian form, with

$$f(u,v) = a\,\mathtt{N}(u) + \mathtt{Tr}(\overline{u}\,b\,v) + c\,\mathtt{N}(v)\,, \tag{5.6}$$

which is integral over $\mathcal{O}$ (its coefficients $a = a(f), b = b(f), c = c(f)$ satisfy $a, c \in \mathbb{Z}$ and $b \in \mathcal{O}$). In this subsection, we assume that f is indefinite (its discriminant $\operatorname{Disc}(f) = \mathtt{N}(b) - ac$ is positive). We denote by $\Gamma_{\mathcal{O}} = \mathrm{SL}_2(\mathcal{O})$ the Hamilton-Bianchi group of invertible 2×2 matrices with coefficients in $\mathcal{O}$ (see Subsection 3.2 or [35, §3] for definitions). The group $\mathrm{SL}_2(\mathbb{H})$ acts on the right on the binary Hermitian forms by precomposition and the action of $\Gamma_{\mathcal{O}}$ preserves the integral ones. The group $\mathrm{SU}_f(\mathcal{O})$ of automorphs of f consists of the elements $g \in \Gamma_{\mathcal{O}}$ stabilising f, that is, such that $f \circ g = f$.

Let G be a finite index subgroup of $\Gamma_{\mathcal{O}}$. For all x, y in A not both zero, and for every $s > 0$, let

$$\psi_{f,G,x,y}(s) = \operatorname{Card}\big((\mathrm{SU}_f(\mathcal{O})\cap G)\backslash\big\{(u,v)\in G(x,y) : \mathtt{N}(\mathcal{O}x+\mathcal{O}y)^{-1}|f(u,v)|\leqslant s\big\}\big).$$

Let $\Gamma_{\mathcal{O},\,x,\,y}$ and $G_{x,y}$ be the stabilisers of $(x,y)\in A\times A$ for the left linear actions of $\Gamma_{\mathcal{O}}$ and G respectively. Let $K_{x,\,y}$ be the left fractional ideal $\mathcal{O}$ if $xy = 0$ and $\mathcal{O}x\cap\mathcal{O}y$ otherwise, and let $\mathcal{O}_r(K_{x,\,y})$ be its right order. Let $\iota_G = 1$ if $-\mathrm{id}\in G$, and $\iota_G = 2$ otherwise.

Theorem 5.4. — *There exists $\kappa > 0$ such that, as $s\to+\infty$,*

$$\psi_{f,\,G,\,x,\,y}(s) = \\ \frac{540\,\iota_G\,[\Gamma_{\mathcal{O},\,x,\,y} : G_{x,y}]\ \mathrm{Covol}(\mathrm{SU}_f(\mathcal{O})\cap G)}{\pi^2\zeta(3)|\mathcal{O}_r(K_{x,y})^\times|\mathrm{Disc}(f)^2[\Gamma_{\mathcal{O}} : G]\prod_{p|D_A}(p^3-1)(1-p^{-1})}s^4(1+\mathrm{O}(s^{-\kappa})),$$

with p ranging over the positive rational primes dividing D_A. As $s\to+\infty$, we have

$$\frac{\zeta(3)[\Gamma_{\mathcal{O}} : G]\,\mathrm{Disc}(f)^4\prod_{p|D_A}(p^3-1)(p-1)}{2160\,\iota_G\,\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O})\cap G)\,s^4}\sum_{f'\in G\cdot f\,:\,0<|a(f')|\leqslant s}\Delta_{\frac{b(f')}{a(f')}}\overset{*}{\rightharpoonup}\mathrm{Leb}_{\mathbb{H}}$$

and if $y\neq 0$,

$$\frac{\zeta(3)[\Gamma_{\mathcal{O}} : G]\,\mathrm{Disc}(f)^4\prod_{p|D_A}(p^3-1)(p-1)}{2160\,\iota_G\,\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O})\cap G)\,s^4}\sum_{f'\in G\cdot f,\,0<|f'(x,y)|\leqslant s}\Delta_{\frac{a(f')x+b(f')y}{f'(x,y)}}\overset{*}{\rightharpoonup}\mathrm{Leb}_{\mathbb{H}}.$$

If Λ is the lattice of $\lambda\in\mathcal{O}$ such that $\pm\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}\in G$, identifying $\mathcal{O}$ with the upper triangular unipotent subgroup of $\Gamma_{\mathcal{O}}$, we have, as $s\to+\infty$,

$$\operatorname{Card}\{f'\in\Lambda\backslash G\cdot f\ :\ 0<|a(f')|\leqslant s\} \\ = \frac{540\,\iota_G\,D_A\,\mathrm{Covol}(\mathrm{SU}_f(\mathcal{O})\cap G)}{\zeta(3)\,[\Gamma_{\mathcal{O}} : G]\,\mathrm{Disc}(f)^4\prod_{p|D_A}(p^3-1)(p-1)}\ s^4(1+\mathrm{O}(s^{-\kappa}))\,.$$

For smooth functions ψ with compact support on $\mathbb{H}$, there is an error term in the above equidistribution claim evaluated at ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell\in\mathbb{N}$.

Proof. — The proof is completely analogous to that of Theorem 5.2, mostly replacing $n = 3$ by $n = 5$, $\widehat{K} = \mathbb{C}$ by $\widehat{K} = \mathbb{H}$, K by A, $\mathcal{O}_K$ by $\mathcal{O}$,

$\mathtt{n}$ by $\mathtt{N}$, and references to [33] to references to [35], so that Equation (5.4) is still valid, and then one replaces Humbert's formula (3.2) by Emery's formula (3.6), and the formula $\operatorname{Vol}(\mathcal{O}_K\backslash\mathbb{C}) = \frac{\sqrt{|D_K|}}{4}$ by $\operatorname{Vol}(\mathcal{O}\backslash\mathbb{H}) = \frac{D_A}{4}$. $\square$

Given two left fractional ideals $\mathfrak{m}, \mathfrak{m}'$ of $\mathcal{O}$ and $s \geqslant 0$, let

$$\psi_{f,\mathfrak{m},\mathfrak{m}'}(s) = \operatorname{Card}\Big(\mathrm{SU}_f(\mathcal{O})\backslash\Big\{(u,v)\in\mathfrak{m}\times\mathfrak{m} : \frac{|f(u,v)|}{\mathtt{n}(\mathfrak{m})} \leqslant s, \mathcal{O}u+\mathcal{O}v=\mathfrak{m}, [K_{u,v}]=[\mathfrak{m}']\Big\}\Big).$$

The following result follows from the first claim of Theorem 5.4 as in the proof of [35, Coro. 18].

Corollary 5.5. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi_{f,\mathfrak{m},\mathfrak{m}'}(s) = \frac{540\operatorname{Covol}(\mathrm{SU}_f(\mathcal{O}))}{\pi^2\zeta(3)|\mathcal{O}_r(\mathfrak{m}')^\times|\operatorname{Disc}(f)^2\prod_{p|D_A}(p^3-1)(1-p^{-1})}s^4(1+\mathrm{O}(s^{-\kappa})).$$

For every $s \geqslant 0$, we consider the integer

$$\psi_{f,\mathfrak{m}}(s) = \operatorname{Card}\big(\mathrm{SU}_f(\mathcal{O})\backslash\big\{(u,v)\in\mathfrak{m}\times\mathfrak{m} : \mathtt{N}(\mathfrak{m})^{-1}|f(u,v)| \leqslant s, \mathcal{O}u+\mathcal{O}v=\mathfrak{m}\big\}\big),$$

which is the number of nonequivalent $\mathfrak{m}$-primitive representations by f of rational integers with absolute value at most $s\,\mathtt{N}(\mathfrak{m})$. The next result then follows from Corollary 5.5 as in the proof given in Corollary 19 of Theorem 1 in [35].

Corollary 5.6. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi_{f,\,\mathfrak{m}}(s) = \frac{45\ D_A\ \operatorname{Covol}(\mathrm{SU}_f(\mathcal{O}))}{2\,\pi^2\ \zeta(3)\ \operatorname{Disc}(f)^2\ \prod_{p|D_A}(p^3-1)}\ \ s^4(1+\mathrm{O}(s^{-\kappa}))\ .$$

The group of automorphs of the binary Hamiltonian form $(u,v)\mapsto \mathtt{Tr}(\overline{u}\,v)$ (which is the standard real scalar product on $\mathbb{H}$) is

$$\mathrm{Sp}_1(\mathcal{O}) = \Big\{g\in\mathrm{SL}_2(\mathcal{O})\ :\ {}^t\overline{g}\begin{pmatrix}0&1\\1&0\end{pmatrix}g = \begin{pmatrix}0&1\\1&0\end{pmatrix}\Big\}\,,$$

which is an arithmetic lattice in the symplectic group $\mathrm{Sp}_1(\mathbb{H})$ over Hamilton's quaternion algebra. The following result follows from Corollary 5.6 as in the proof of [35, Coro. 2].

Corollary 5.7. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\mathrm{Card}\big(\mathrm{Sp}_1(\mathcal{O})\backslash\big\{(u,v)\in\mathcal{O}\times\mathcal{O}\ :\ |\mathtt{Tr}(\overline{u}\,v)|\leqslant s,\ \mathcal{O}u+\mathcal{O}v=\mathcal{O}\ \big\}\big) = \frac{D_A}{48\,\zeta(3)}\prod_{p|D_A}\frac{p^2+1}{p^2+p+1}\ \ s^4(1+\mathrm{O}(s^{-\kappa}))\ . \quad \square$$

5.2. Positive definite binary Hermitian and Hamiltonian forms

We first consider the Hermitian case. Let K and $\mathfrak{m}$ be as in the beginning of Subsection 3.1. Let f be an integral binary Hermitian form over $\mathcal{O}_K$ as in the beginning of Subsection 5.1. In this subsection, we assume that f is positive definite (that is $\mathrm{Disc}(f) < 0$ and $a(f) > 0$), a feature which is preserved by the action of the Bianchi group $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$. Note that the group $\mathrm{SU}_f(\mathcal{O}_K)$ of automorphs of f is then finite.

For every $s > 0$, we consider the following integer (its finiteness is part of the following proof), depending only on the ideal class of $\mathfrak{m}$,

$$\psi^+_{f,\mathfrak{m}}(s) = \mathrm{Card}\ \big\{(u,v)\in\mathfrak{m}\times\mathfrak{m}\ :\ \mathtt{n}(\mathfrak{m})^{-1}f(u,v)\leqslant s,\ \mathcal{O}_K u+\mathcal{O}_K v=\mathfrak{m}\big\}\ ,$$

which is the number of nonequivalent $\mathfrak{m}$-primitive representations by f of the rational integers at most $s\,\mathtt{n}(\mathfrak{m})$.

Theorem 5.8. — *There exists $\kappa > 0$ such that, as $s \to +\infty$, we have*

$$\psi^+_{f,\mathfrak{m}}(s) = \frac{2\pi^2}{|D_K|\ \zeta_K(2)\ |\mathrm{Disc}(f)|}\ \ s^2(1+\mathrm{O}(s^{-\kappa}))\ .$$

If G is a finite index subgroup of Γ_K, with $\iota_G = 1$ if $-\mathrm{id}\in G$ and $\iota_G = 2$ otherwise, then, as $s \to +\infty$,

$$\frac{|\mathrm{SU}_f(\mathcal{O}_K)\cap G|\,[\Gamma_K : G]\,|D_K|^{\frac{3}{2}}\zeta_K(2)|\mathrm{Disc}(f)|}{(\pi\ \iota_G\ s)^2}\sum_{f'\in G\cdot f,\,a(f')\leqslant s}\Delta_{\frac{b(f')}{a(f')}}\ \overset{*}{\rightharpoonup}\ \mathrm{Leb}_{\mathbb{C}}.$$

If Λ is the lattice of $\lambda\in\mathcal{O}_K$ such that $\pm\begin{pmatrix}1&\lambda\\0&1\end{pmatrix}\in G$, identifying $\mathcal{O}_K$ with the upper triangular unipotent subgroup of Γ_K, we have, as $s \to +\infty$,

$$\mathrm{Card}\{f'\in\Lambda\backslash G\cdot f\ :\ a(f')\leqslant s\} = \frac{\pi^2\,{\iota_G}^2\,[\mathcal{O}_K:\Lambda]}{2\,|\mathrm{SU}_f(\mathcal{O}_K)\cap G|\,[\Gamma_K:G]\,|D_K|\,\zeta_K(2)\,|\mathrm{Disc}(f)|}\ \ s^2(1+\mathrm{O}(s^{-\kappa}))\ .$$

For smooth functions ψ with compact support on $\mathbb{C}$, there is an error term in the above equidistribution claim evaluated at ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$.

Proof. — Let $n = 3$ and $\widehat{K} = \mathbb{C}$.

We start the proof by describing the space of positive definite binary Hermitian forms in hyperbolic geometry terms. Let $\mathcal{Q}^+$ be the cone of positive definite binary Hermitian forms. The multiplicative group $\mathbb{R}_+ =]0, +\infty[$ acts on $\mathcal{Q}^+$ by multiplication and we define $\overline{\mathcal{Q}}^+ = \mathcal{Q}^+/\mathbb{R}_+$. The map $\Phi : \overline{\mathcal{Q}}^+ \to \mathbb{H}^n_{\mathbb{R}} = \widehat{K} \times \mathbb{R}_+$ induced by $f \mapsto \big(- \frac{b(f)}{a(f)}, \frac{\sqrt{-\mathrm{Disc}(f)}}{a(f)}\big)$ is a homeomorphism, which is (anti-)equivariant in the sense that $\Phi(f \circ g) = g^{-1}\Phi(f)$ for all $g \in \mathrm{SL}_2(\widehat{K})$ and $f \in \mathbb{Q}^+$ (see for example [35, Prop. 22] for a proof for positive definite binary Hamiltonian forms, the above claims being obtained by embedding as usual $\mathbb{C}$ in Hamilton's quaternion algebra $\mathbb{H}$).

Now, let x, y be not both zero in $\mathcal{O}_K$, with $x = 1$ if $y = 0$, and let $I_{x,\,y} = \mathcal{O}_K x + \mathcal{O}_K y$. We will more generally study the following counting function of the representations, in a given orbit of G, of integers by f, defined, for all $s \geqslant 0$, by

$$\psi^+_{f,\,G,\,x,\,y}(s) = \mathrm{Card}\,\big\{(u,v) \in G(x,y)\ :\ \mathtt{n}(I_{x,\,y})^{-1}\, f(u,v) \leqslant s\big\}\,.$$

Let ρ, γ_ρ be as in the proof of Theorem 5.1. As previously, we denote by $\overline{H}$ the image in $\mathrm{PSL}_2(\widehat{K})$ of any subgroup H of $\mathrm{SL}_2(\widehat{K})$. We will apply Corollary 2.2 with Γ the arithmetic group $\gamma_\rho^{-1}\,\overline{G}\,\gamma_\rho$, D^- the horoball $\mathcal{H}_\infty = \{(z,t) \in \mathbb{H}^n_{\mathbb{R}}\ :\ t \geqslant 1\}$ (which is centered at a parabolic fixed point of Γ, and whose pointwise stabiliser is trivial), and D^+ the singleton consisting of $\Phi(f \circ \gamma_\rho)$.

For any $g \in \Gamma_K$, the distance of $\Phi(f \circ g)$ from $\mathcal{H}_\infty$ is $\ln \frac{a(f\circ g)}{\sqrt{-\mathrm{Disc}(f))}}$, when this number is positive, which is the case except for finitely many right classes of g under the stabiliser of $\mathcal{H}_\infty$. Note that $\gamma \in \Gamma$ belongs to the stabiliser Γ_{D^+} of D^+ in Γ if and only if γ^{-1} stabilises the positive homothety class of $f \circ \gamma_\rho$, that is, since $\mathrm{SL}_2(\widehat{K})$ preserves the discriminant, if and only if γ^{-1} fixes $f \circ \gamma_\rho$. Thus $\Gamma_{D^+} = \gamma_\rho^{-1}(\mathrm{PSU}_f(\mathcal{O}_K) \cap \overline{G})\gamma_\rho$, which is also the pointwise stabiliser of D^+, whose order is hence $\frac{\iota_G}{2}|\mathrm{SU}_f(\mathcal{O}_K) \cap G|$.

Let $G_{\mathcal{H}_\rho}$ and $G_{x,\,y}$ be the stabilisers in G of the horoball $\mathcal{H}_\rho = \gamma_\rho \mathcal{H}_\infty$ in $\mathbb{H}^n_{\mathbb{R}}$ and of the pair (x, y) in $\widehat{K} \times \widehat{K}$. We use the (surprising!) convention that $\mathtt{n}(y) = 1$ if $y = 0$ to avoid considering cases. The counting result with error term of Corollary 2.2, and the value of $C(D^-, D^+)$ in case (4) just above it, using cosets and multiplicities arguments already seen, show that

$$\begin{aligned}
&\psi^+_{f,\,G,\,x,\,y}(s)\\
&= \mathrm{Card}\big\{[\gamma]\in G/G_{x,y} : \mathtt{n}(I_{x,y})^{-1} f(\gamma(x,y)) \leqslant s\big\}\\
&= \frac{2}{\iota_G}\ \mathrm{Card}\big\{[\gamma]\in \overline{G}/\overline{G_{x,y}} : f\circ\gamma\circ\gamma_\rho(1,0) \leqslant \frac{s\,\mathtt{n}(I_{x,y})}{\mathtt{n}(y)}\big\}\\
&= \frac{2}{\iota_G}\ [\overline{G_{\mathcal{H}_\rho}} : \overline{G_{x,y}}]\,\mathrm{Card}\big\{[\gamma]\in \overline{G}/\overline{G_{\mathcal{H}_\rho}} : a(f\circ\gamma\circ\gamma_\rho) \leqslant \frac{s\,\mathtt{n}(I_{x,y})}{\mathtt{n}(y)}\big\} + \mathrm{O}(1)\\
&= [\overline{G_{\mathcal{H}_\rho}} : \overline{G_{x,y}}]\,|\mathrm{SU}_f(\mathcal{O}_K)\cap G|\\
&\qquad \mathrm{Card}\big\{[\gamma]\in \Gamma_{D^+}\backslash\Gamma/\Gamma_{D^-} : a(f\circ\gamma_\rho\circ\gamma) \leqslant \frac{s\,\mathtt{n}(I_{x,y})}{\mathtt{n}(y)}\big\} + \mathrm{O}(1)\\
&= [\overline{G_{\mathcal{H}_\rho}} : \overline{G_{x,y}}]\,|\mathrm{SU}_f(\mathcal{O}_K)\cap G|\ \mathcal{N}_{D^+,D^-}\big(\ln \frac{s\,\mathtt{n}(I_{x,y})}{\mathtt{n}(y)\sqrt{-\mathrm{Disc}(f)}}\big) + \mathrm{O}(1)\\
&= \frac{2[\overline{G_{\mathcal{H}_\rho}} : \overline{G_{x,y}}]\mathrm{Vol}(\Gamma_{D^-}\backslash D^-)}{\iota_G \mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})}\Big(\frac{s\mathtt{n}(I_{x,y})}{\mathtt{n}(y)\sqrt{-\mathrm{Disc}(f)}}\Big)^{n-1}(1+\mathrm{O}(s^{-\kappa})). \qquad (5.7)
\end{aligned}$$

As seen in the proof of Theorem 3.1, we have

$$\begin{aligned}
\mathrm{Vol}(\Gamma_{D^-}\backslash D^-) &= \mathrm{Vol}(\,\overline{G_{\mathcal{H}_\rho}}\backslash\mathcal{H}_\rho) = [\,\overline{(\Gamma_K)_{\mathcal{H}_\rho}} : \overline{G_{\mathcal{H}_\rho}}\,]\,\mathrm{Vol}(\,\overline{(\Gamma_K)_{\mathcal{H}_\rho}}\backslash\mathcal{H}_\rho)\\
&= [\,\overline{(\Gamma_K)_{\mathcal{H}_\rho}} : \overline{G_{\mathcal{H}_\rho}}\,]\frac{\sqrt{|D_K|}}{2\,\omega_K}\,\frac{\mathtt{n}(y)^2}{\mathtt{n}(I_{x,\,y})^2}\,. \qquad (5.8)
\end{aligned}$$

We have

$$\begin{aligned}
[\,\overline{(\Gamma_K)_{\mathcal{H}_\rho}} : \overline{G_{\mathcal{H}_\rho}}\,][\,\overline{G_{\mathcal{H}_\rho}} : \overline{G_{x,\,y}}\,] &= [\,\overline{(\Gamma_K)_{\mathcal{H}_\rho}} : \overline{\Gamma_{K,\,x,\,y}}\,][\,\overline{\Gamma_{K,\,x,\,y}} : \overline{G_{x,\,y}}\,]\\
&= \frac{\omega_K}{2}\,[\Gamma_{K,\,x,\,y} : G_{x,\,y}]\,. \qquad (5.9)
\end{aligned}$$

Using Equation (5.5), Equation (5.7) then gives the following formula, interesting in itself,

$$\psi^+_{f,\,G,\,x,\,y}(s) = \frac{2\,\pi^2\,[\Gamma_{K,\,x,\,y} : G_{x,\,y}]}{|D_K|\,\zeta_K(2)\,|\mathrm{Disc}(f)|\,[\Gamma_K : G]}\ s^2(1+\mathrm{O}(s^{-\kappa}))\,.$$

Since any nonzero ideal $\mathfrak{m}$ in $\mathcal{O}_K$ may be written $\mathfrak{m} = \mathcal{O}_K x_{\mathfrak{m}} + \mathcal{O}_k y_{\mathfrak{m}}$ for some $x_{\mathfrak{m}}, y_{\mathfrak{m}}$ not both zero in $\mathcal{O}_K$ and $\psi^+_{f,\,\mathfrak{m}} = \psi^+_{f,\,\Gamma_K,\,x_{\mathfrak{m}},\,y_{\mathfrak{m}}}$, the first claim of Theorem 5.8 follows.

Let us now prove the equidistribution result in Theorem 5.8. The main additional remark is that for all $f' \in G\cdot f$, the initial point of the common perpendicular from $\mathcal{H}_\infty$ to $\Phi(f'\circ\gamma_\rho)$ is $\big(-\frac{b(f'\circ\gamma_\rho)}{a(f'\circ\gamma_\rho)},\,1\big)$. Taking $t =$

$\ln \frac{s}{\sqrt{-\mathrm{Disc}(f)}}$ in Equation (2.2) in Corollary 2.2, gives, as $s \to +\infty$,

$$\frac{2\,|\mathrm{SU}_f(\mathcal{O}_K) \cap G|\,(n-1)\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})}{\iota_G \left(\frac{s}{\sqrt{-\mathrm{Disc}(f)}}\right)^{n-1}} \sum_{\gamma \in \Gamma/\Gamma_{D^+},\, a(f\circ\gamma_\rho\circ\gamma)\leqslant s} \Delta_{\left(-\frac{b(f\circ\gamma_\rho\circ\gamma)}{a(f\circ\gamma_\rho\circ\gamma)},1\right)} \overset{*}{\rightharpoonup} \mathrm{Vol}_{\partial D^-}$$

Hence, since the map from $G \cdot f$ to Γ/Γ_{D^+} defined by $f\circ\gamma \mapsto \gamma_\rho^{-1}\gamma\gamma_\rho\Gamma_{D^+}$ is a bijection, using the pushforward of measures by the map $(z,1)\mapsto -z$ from ∂D^- to $\widehat{K}$, we have

$$\frac{2}{\iota_G}|\mathrm{SU}_f(\mathcal{O}_K)\cap G|(n-1)\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})|\mathrm{Disc}(f)|^{\frac{n-1}{2}} s^{-(n-1)} \sum_{f'\in G\cdot f,\, a(f'\circ\gamma_\rho)\leqslant s} \Delta_{\frac{b(f'\circ\gamma_\rho)}{a(f'\circ\gamma_\rho)}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\widehat{K}}. \tag{5.10}$$

Taking $y = 0$ (then $\gamma_\rho = \mathrm{id}$) and using Equation (5.5), this gives the equidistribution claim in Theorem 5.8. The last claim of Theorem 5.1 follows from it as in the proof of Theorem 5.1.

The case $y \neq 0$, using Equation (5.5), gives the following equidistribution result, interesting in itself, as in the end of the proof of Theorem 5.1:

$$\frac{|\mathrm{SU}_f(\mathcal{O}_K)\cap G|\,[\Gamma_K : G]\,|D_K|^{\frac{3}{2}}\zeta_K(2)|\mathrm{Disc}(f)|\mathtt{n}(y)}{(\pi\ \iota_G\ s)^2} \sum_{f'\in G\cdot f,\, f'(x,y)\leqslant s} \Delta_{\frac{a(f')x+b(f')y}{f'(x,y)}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{C}}\,. \quad \square$$

We now consider the Hamiltonian case. Let $\mathbb{H}, A, \mathcal{O}, \mathfrak{m}$ and the associated notation be as in the beginning of Section 3.2. Let f be an integral binary Hamiltonian form over $\mathcal{O}$, as in Subsection 5.1. In this subsection, we assume that f is positive definite (that is $\mathrm{Disc}(f) < 0$ and $a(f) > 0$), a feature which is preserved by the action (on the right) of the Hamilton-Bianchi group $\Gamma_{\mathcal{O}} = \mathrm{SL}_2(\mathcal{O})$ (by precomposition). Note that the group $\mathrm{SU}_f(\mathcal{O})$ of automorphs of f is then finite.

For every $s > 0$, we consider the integer

$$\psi^+_{f,\,\mathfrak{m}}(s) = \mathrm{Card}\,\big\{(u,v)\in\mathfrak{m}\times\mathfrak{m}\ :\ \mathtt{N}(\mathfrak{m})^{-1}f(u,v)\leqslant s,\quad \mathcal{O}u+\mathcal{O}v=\mathfrak{m}\big\}\,,$$

which is the number of nonequivalent $\mathfrak{m}$-primitive representations by f of the rational integers at most $s\,\mathtt{N}(\mathfrak{m})$.

Theorem 5.9. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\psi^+_{f,\,\mathfrak{m}}(s) \;\sim\; \frac{60\, D_A}{\zeta(3)\, |\mathrm{Disc}(f)|^2\, \prod_{p|D_A}(p^3-1)} \quad s^4(1+\mathrm{O}(s^{-\kappa}))\ .$$

If G is a finite index subgroup of $\Gamma_{\mathcal{O}}$, with $\iota_G = 1$ if $-\mathrm{id} \in G$ and $\iota_G = 2$ otherwise, then, as $s \to +\infty$,

$$\frac{|\mathrm{SU}_f(\mathcal{O}) \cap G|\, [\Gamma_{\mathcal{O}} : G]\, \zeta(3)\, |\mathrm{Disc}(f)|^2 \prod_{p|D_A}(p^3-1)(p-1)}{1440\ \iota_G{}^2\ s^4} \sum_{f' \in G\cdot f,\, a(f') \leqslant s} \Delta_{\frac{b(f')}{a(f')}} \overset{*}{\rightharpoonup} \mathrm{Leb}_{\mathbb{H}}.$$

If Λ is the lattice of $\lambda \in \mathcal{O}$ such that $\pm\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in G$, identifying $\mathcal{O}$ with the upper triangular unipotent subgroup of $\Gamma_{\mathcal{O}}$, we have, as $s \to +\infty$,

$$\begin{aligned}&\mathrm{Card}\{f' \in\ \Lambda\backslash G\cdot f\ :\ a(f') \leqslant s\}\\ &= \frac{360\ \iota_G{}^2[\mathcal{O} : \Lambda] D_A}{|\mathrm{SU}_f(\mathcal{O}) \cap G|\, [\Gamma_{\mathcal{O}} : G]\, \zeta(3)\, |\mathrm{Disc}(f)|^2 \prod_{p|D_A}(p^3-1)(p-1)} s^4(1+\mathrm{O}(s^{-\kappa})).\end{aligned}$$

Proof. — The proof is completely analogous to that of Theorem 5.8, mostly replacing $n = 3$ by $n = 5$, $\widehat{K} = \mathbb{C}$ by $\widehat{K} = \mathbb{H}$, K by A, $\mathcal{O}_K$ by $\mathcal{O}$, $\mathtt{n}$ by $\mathtt{N}$. Given x, y not both zero in $\mathcal{O}$, with $x = 1$ if $y = 0$, let $I_{x,\,y}$ and $K_{x,\,y}$ be as in the beginning of the proof of Theorem 3.2. We also introduce the following counting function of the representations of integers by f, in a given orbit of G:

$$\psi^+_{f,\,G,\,x,\,y}(s) = \mathrm{Card}\ \big\{(u,v) \in G(x,y)\ :\ \mathtt{N}(I_{x,\,y})^{-1}\, f(u,v) \leqslant s\big\}\ .$$

Keeping the relevant notation of the proof Theorem 5.8, Equation (5.7) is still valid. By Equation (3.5) where $\tau = 1$, we have

$$\begin{aligned}\mathrm{Vol}(\Gamma_{D^-}\backslash D^-) &= \mathrm{Vol}(\,\overline{G_{\mathcal{H}_\rho}}\,\backslash\mathcal{H}_\rho) = [\,\overline{(\Gamma_{\mathcal{O}})_{\mathcal{H}_\rho}} : \overline{G_{\mathcal{H}_\rho}}\,]\,\mathrm{Vol}(\,\overline{(\Gamma_{\mathcal{O}})_{\mathcal{H}_\rho}}\,\backslash\mathcal{H}_\rho)\\ &= [\,\overline{(\Gamma_{\mathcal{O}})_{\mathcal{H}_\rho}} : \overline{G_{\mathcal{H}_\rho}}\,]\ \frac{D_A\,\mathtt{N}(y)^4}{16|\mathcal{O}_r(K_{x,y})^\times|[\overline{(\Gamma_{\mathcal{O}})_{\mathcal{H}_\rho}} : \overline{(\Gamma_{\mathcal{O}})_{x,y}}]\mathtt{N}(I_{x,y})^4}.\end{aligned}$$

By Emery's formula (3.6), we have

$$\begin{aligned}\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}) &= \mathrm{Vol}(\,\overline{G}\,\backslash\mathbb{H}^5_{\mathbb{R}}) = [\,\overline{\Gamma_{\mathcal{O}}} : \overline{G}\,]\mathrm{Vol}(\,\overline{\Gamma_{\mathcal{O}}}\,\backslash\mathbb{H}^5_{\mathbb{R}})\\ &= \frac{[\Gamma_{\mathcal{O}} : G]}{\iota_G}\ \frac{\zeta(3)\prod_{p|D_A}(p^3-1)(p-1)}{11520}\ .\end{aligned}$$

Since

$$\frac{[\,\overline{G_{\mathcal{H}_\rho}}:\overline{G_{x,\,y}}\,]\,[\,\overline{(\Gamma_{\mathcal{O}})_{\mathcal{H}_\rho}}:\overline{G_{\mathcal{H}_\rho}}\,]}{[\,\overline{(\Gamma_{\mathcal{O}})_{\mathcal{H}_\rho}}:\overline{(\Gamma_{\mathcal{O}})_{x,\,y}}\,]}=[\,\overline{(\Gamma_{\mathcal{O}})_{x,\,y}}:\overline{G_{x,\,y}}\,]=[(\Gamma_{\mathcal{O}})_{x,\,y}:G_{x,\,y}],$$

we hence have the following counting asymptotic, interesting in itself:

$$\psi^+_{f,G,x,y}(s)=\frac{1440\,D_A[(\Gamma_{\mathcal{O}})_{x,y}:G_{x,y}]}{\zeta(3)[\Gamma_{\mathcal{O}}:G]|\mathcal{O}_r(K_{x,y})^\times||\mathrm{Disc}(f)|^2\prod_{p|D_A}(p^3-1)(p-1)}\,s^4(1+\mathrm{O}(s^{-\kappa})). \tag{5.11}$$

Given two left fractional ideals $\mathfrak{m},\mathfrak{m}'$ of $\mathcal{O}$ and $s\geqslant 0$, let

$$\psi^+_{f,\mathfrak{m},\mathfrak{m}'}(s)=\mathrm{Card}\big\{(u,v)\in\mathfrak{m}\times\mathfrak{m}:\mathtt{n}(\mathfrak{m})^{-1}f(u,v)\leqslant s, I_{u,v}=\mathfrak{m},[K_{u,v}]=[\mathfrak{m}']\big\}.$$

As explained in the proof of Theorem 3.2, taking $(x,y)\in\mathcal{O}\times\mathcal{O}$ such that $[I_{x,\,y}]=[\mathfrak{m}]$ and $[K_{x,\,y}]=[\mathfrak{m}']$, we have $\psi^+_{f,\,\mathfrak{m},\,\mathfrak{m}'}=\psi^+_{f,\,\Gamma_{\mathcal{O}},\,x,\,y}$. Hence

$$\psi^+_{f,\,\mathfrak{m},\,\mathfrak{m}'}(s)=\frac{1440\,D_A}{\zeta(3)|\mathcal{O}_r(\mathfrak{m}')^\times||\mathrm{Disc}(f)|^2\prod_{p|D_A}(p^3-1)(p-1)}\,s^4(1+\mathrm{O}(s^{-\kappa})).$$

Now, the first claim of Theorem 5.9 follows from Equation (3.8), since

$$\psi^+_{f,\,\mathfrak{m}}=\sum_{[\mathfrak{m}']\,\in\,{}_{\mathcal{O}}\mathcal{I}}\psi^+_{f,\,\mathfrak{m},\,\mathfrak{m}'}\;.$$

The last two assertions of Theorem 5.9 are proven in the same way as the last two assertions of Theorem 5.8, since Equation (5.10) is still valid, and if $y\neq 0$, we furthermore have

$$\frac{|\mathrm{SU}_f(\mathcal{O})\cap G|\,[\Gamma_{\mathcal{O}}:G]\,\zeta(3)\,|\mathrm{Disc}(f)|^2\,\mathtt{n}(y)^2\,\prod_{p|D_A}(p^3-1)(p-1)}{1440\,\iota_G{}^2\,s^4}\sum_{f'\in G\cdot f,\;f'(x,\,y)\leqslant s}\Delta_{\frac{a(f')\,x\,+\,b(f')\,y}{f'(x,\,y)}}\overset{*}{\rightharpoonup}\mathrm{Leb}_{\mathbb{H}}\,.\quad\square$$

5.3. Representation of algebraic integers by integral binary quadratic forms

In this final subsection, we study counting and equidistribution problems of the representations of algebraic integers by quadratic norm forms (for related results, see for instance [24], [25, Thm. R], [21, Chap. VI], as well as [12, §3.1] for an ergodic approach).

Let K and its associated notation be as in the beginning of Section 4. Let $\alpha_0 \in \widehat{K}$ be a fixed quadratic irrational over $\mathcal{O}_K$. We also denote by $\mathtt{n}$ the relative norm in $K(\alpha_0)$ over K (which is consistent with the notation of the beginning of Section 4). We consider the (relative) norm form, seen as a map from $K \times K$ to K, defined by

$$N_{\alpha_0} : (u, v) \mapsto \mathtt{n}(u - \alpha_0 v) = u^2 - \mathtt{tr}\, \alpha_0\, uv + \mathtt{n}(\alpha_0)\, v^2 \,,$$

which is the homogeneous form of the minimal polynomial of α_0 over K. Its values on $\mathcal{O}_K \times \mathcal{O}_K$ belong to $\mathcal{O}_K$ if α_0 is an algebraic integer, and to $a\mathcal{O}_K$ for some $a \in K$ in general. We will study the representations of elements of K by this norm form N_{α_0}, in orbits of (finite index subgroups of) the modular groups $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ (a minor change of notation from the beginning of Section 4).

We denote by $\overline{H}$ the image in $\mathrm{PSL}_2(\widehat{K})$ of any subgroup H of $\mathrm{SL}_2(\widehat{K})$. We denote by H_P the stabiliser, for any group action of a group H on a set X, of a point P or a subset P in X. The stabiliser $\mathrm{Stab}_{\Gamma_K} N_{\alpha_0}$ of the norm form N_{α_0}, for the action (on the right) of Γ_K by precomposition by the linear action, is exactly $(\Gamma_K)_{\{\alpha_0,\, \alpha_0^\sigma\}}$.

As a warm-up, when $K = \mathbb{Q}$, we give an equidistribution result of the fractions of the representations, in an aforementioned orbit, of usual rationals by this norm form (which, in this case, is an indefinite quadratic form, hence the classical counting results of the beginning of Section 5 apply).

Theorem 5.10. — *For every finite index subgroup G of $\Gamma_{\mathbb{Q}} = \mathrm{SL}_2(\mathbb{Z})$, with $\iota_G = 1$ if $-\mathrm{id} \in G$ and $\iota_G = 2$ otherwise, we have the following convergence of measures on $\mathbb{R} - \{\alpha_0, \alpha_0^\sigma\}$ as $s \to +\infty$:*

$$\frac{\pi^2\, \iota_G\, [\Gamma_{\mathbb{Q}} : G]}{6\, [(\Gamma_{\mathbb{Q}})_\infty : G_\infty]\; s} \sum_{(u,v) \in G(0,1),\ |\mathtt{n}(u - \alpha_0 v)| \leqslant s} \Delta_{\frac{u}{v}} \overset{*}{\rightharpoonup} \frac{d\mathrm{Leb}_{\mathbb{R}}(t)}{|Q_{\alpha_0}(t)|}\,.$$

For smooth functions ψ with compact support on $\mathbb{R} - \{\alpha_0, \alpha_0, \}$, there is an error term in the above equidistribution claims evaluated at ψ, of the form $\mathrm{O}(s^{-\kappa} \|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$.

For instance, taking $G = \mathrm{SL}_2(\mathbb{Z})$ gives Theorem 1.6 in the introduction, and taking for G the Hecke congruence subgroup modulo $k \in \mathbb{N} - \{0\}$, we have, by the index computation of for instance [44, p. 24],

$$\frac{\pi^2\, k \prod_{\mathfrak{p} | k} \left(1 + \frac{1}{\mathfrak{p}}\right)}{12\, s} \sum_{(u,v) \in \mathbb{Z}^2,\ (u,v)=1,\ v \equiv 0\ [k],\ |\mathtt{n}(u - \alpha_0 v)| \leqslant s} \Delta_{\frac{u}{v}} \overset{*}{\rightharpoonup} \frac{d\mathrm{Leb}_{\mathbb{R}}(t)}{|Q_{\alpha_0}(t)|}\,.$$

Proof. — We take $n = 2$ and $\widehat{K} = \mathbb{R}$ in this proof. We will apply Equation (2.3) with Γ the arithmetic group $\overline{G}$, D^- the geodesic line with points at infinity $\alpha_0, \alpha_0^\sigma$ (on which the group $\widehat{\alpha_0}^{\mathbb{Z}} \cap \overline{G}$ acts with compact quotient) and D^+ the horoball $\mathcal{H}_\infty = \{(z,t) \in \mathbb{H}^n_{\mathbb{R}} = \widehat{K} \times \mathbb{R}_+ \;:\; t \geqslant 1\}$ (which is centered at a parabolic fixed point of $\overline{G}$).

For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathcal{O}_K)$, the distance ℓ_γ between D^- and γD^+, when they are disjoint (which is the case except for finitely many double classes of γ in $\Gamma_{D^-}\backslash\Gamma/\Gamma_{D^+}$), is equal to the distance between $D^+ = \mathcal{H}_\infty$ and the geodesic line $\gamma^{-1}D^-$ with points at infinity $\gamma^{-1}\alpha_0, \gamma^{-1}\alpha_0^\sigma$. Hence by [34, Lem. 4.2]

$$\ell_\gamma = \left| \ln \frac{|\gamma^{-1}\alpha_0 - \gamma^{-1}\alpha_0^\sigma|}{2} \right| = \ln h(\gamma^{-1}\alpha_0) = \ln \big(h(\alpha_0) \,|\, \mathtt{n}(a - c\alpha_0) \,| \big) \,.$$

Let v_γ be the initial tangent vector of the common perpendicular from D^- to γD^+, when they are disjoint, and note that its positive point at infinity $(v_\gamma)_+$ is the point at infinity of the horoball γD^+, which is $\gamma\infty = \frac{a}{c}$.

By coset and multiplicities argument as above, by the version for initial tangent vectors of Equation (2.3) seen in Subsection 4.3, we have

$$\frac{\mathrm{Vol}(\mathbb{S}^{n-1})\,\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}})}{\mathrm{Vol}(\Gamma_{D^+}\backslash D^+)\; e^{(n-1)t}} \sum_{\gamma\in\Gamma/\Gamma_{D^+},\; \ell_\gamma \leqslant t} \overset{*}{\rightharpoonup} \mathrm{Vol}_{\partial^1_+ D^-} \,. \tag{5.12}$$

We have

$$\mathrm{Vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}) = [\,\overline{\Gamma_{\mathbb{Q}}} : \overline{G}\,]\,\mathrm{Vol}(\,\overline{\Gamma_{\mathbb{Q}}}\,\backslash\mathbb{H}^2_{\mathbb{R}}) = \frac{[\Gamma_{\mathbb{Q}} : G]}{\iota_G}\,\frac{\pi}{3}\,,$$

and

$$\mathrm{Vol}(\Gamma_{D^+}\backslash D^+) = [\,\overline{(\Gamma_{\mathbb{Q}})_\infty} : \overline{G_\infty}\,]\,\mathrm{Vol}(\,\overline{(\Gamma_{\mathbb{Q}})_\infty}\,\backslash D^+) = \frac{[(\Gamma_{\mathbb{Q}})_\infty : G_\infty]}{\iota_G}\,.$$

Taking $t = \ln(h(\alpha_0)s)$ in Equation (5.12), since the image of $\mathrm{Vol}_{\partial^1_+ D^-}$ by the pushforward of measures by the positive endpoint map is $\frac{2^{n-1}\, d\mathrm{Leb}_{\widehat{K}}(z)}{h(\alpha_0)^{n-1}\; |Q_{\alpha_0}(z)|^{n-1}}$ by (the comment following) Lemma 4.12, and since the canonical map $G/G_{1,0} \to \overline{G}/\overline{G_\infty} = \Gamma/\Gamma_{D^+}$ is $\frac{2}{\iota_G}$-to-1, the result follows. $\square$

We now assume that K is an imaginary quadratic number field. Let $\mathfrak{m}$ be a (nonzero) fractional ideal of $\mathcal{O}_K$. Let $\alpha_0 \in \widehat{K}$ be a quadratic irrational over K. We consider the counting function $\overline{\psi}_{\mathfrak{m}}$ of nonequivalent $\mathfrak{m}$-primitive representations, by the norm form N_{α_0}, of elements of K (necessary integers in K if α_0 is an algebraic integer), defined by mapping $s \in [0, +\infty[$ to the cardinality of

$$(\mathrm{Stab}_{\Gamma_K} N_{\alpha_0})\backslash\big\{(u,v) \in \mathfrak{m}\times\mathfrak{m} \;:\; \mathtt{n}(\mathfrak{m})^{-1}|\mathtt{n}(u - v\alpha_0)| \leqslant s,\; \mathcal{O}_K u + \mathcal{O}_K v = \mathfrak{m}\big\}\,.$$

Since for any $b \in \mathcal{O}_K$, we have $\mathtt{n}(b\mathfrak{m}) = |b|^2\mathtt{n}(\mathfrak{m})$, the counting function $\overline{\psi}_{\mathfrak{m}}$ depends only on the ideal class of $\mathfrak{m}$.

Theorem 5.11. — *There exists $\kappa > 0$ such that, as $s \to +\infty$,*

$$\overline{\psi}_{\mathfrak{m}}(s) = \frac{2\,\pi^2\, h(\alpha_0)^2\, \Big|\ln\Big|\frac{\mathrm{tr}\,\widehat{\alpha_0} + \sqrt{(\mathrm{tr}\,\widehat{\alpha_0})^2 - 4}}{2}\Big|\,\Big|}{m_{\Gamma_K}(\alpha_0)\,\iota_{\Gamma_K}(\alpha_0)\,|D_K|\,\zeta_K(2)}\; s^2 + \mathrm{O}(s^{2-\kappa})\,.$$

Furthermore, for every finite index subgroup G of $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$, with $\iota_G = 1$ if $-\mathrm{id} \in G$ and $\iota_G = 2$ otherwise, we have the following convergence of measures on $\mathbb{C} - \{\alpha_0, \alpha_0^\sigma\}$ as $s \to +\infty$:

$$\frac{[\Gamma_K : G]\,|D_K|\,\zeta_K(2)\,\iota_G\,\omega_K}{4\,\pi\,[(\Gamma_K)_\infty : G_\infty]\; s^2} \sum_{(u,v)\in G(0,1),\; |\,\mathtt{n}(u-\alpha_0 v)\,| \leqslant s} \Delta_{\frac{u}{v}} \;\overset{*}{\rightharpoonup}\; \frac{d\mathrm{Leb}_{\mathbb{C}}(z)}{|Q_{\alpha_0}(z)|^2}\,.$$

For smooth functions ψ with compact support on $\mathbb{C}$, there is an error term in the above equidistribution claim evaluated at ψ, of the form $\mathrm{O}(s^{-\kappa}\|\psi\|_\ell)$ where $\kappa > 0$ and $\|\psi\|_\ell$ is the Sobolev norm of ψ for some $\ell \in \mathbb{N}$.

We leave to the reader a version of this equidistribution claim where $(0,1)$ is replaced by any fixed pair (x,y) of elements of $\mathcal{O}_K$ which are not both zero (replacing $D^+ = \mathcal{H}_\infty$ by $D^+ = \gamma_\rho\mathcal{H}_\infty$ in its proof, with γ_ρ as in the proof of Theorem 5.1).

For instance, taking $G = \Gamma_K$, we have

$$\frac{|D_K|\,\zeta_K(2)\,\omega_K}{4\,\pi\; s^2} \sum_{(u,v)\in\mathcal{O}_K\times\mathcal{O}_K,\; \mathcal{O}_K u + \mathcal{O}_K v = \mathcal{O}_K,\; |\,\mathtt{n}(u-\alpha_0 v)\,| \leqslant s} \Delta_{\frac{u}{v}} \;\overset{*}{\rightharpoonup}\; \frac{d\mathrm{Leb}_{\mathbb{C}}(z)}{|Q_{\alpha_0}(z)|^2}\,.$$

Proof. — We will prove a stronger counting claim. Let x, y in $\mathcal{O}_K$ be elements which are not both zero, with $x = 1$ if $y = 0$ and the same strange convention that $\mathtt{n}(y) = 1$ if $y = 0$. Let $I_{x,\,y} = x\mathcal{O}_K + y\mathcal{O}_K$, and for every $s > 0$, let

$$\overline{\psi}_{G,x,y}(s) = \mathrm{Card}\big((\mathrm{Stab}_G N_{\alpha_0})\backslash\big\{(u,v) \in G(x,y) : \mathtt{n}(I_{x,y})^{-1}|\mathtt{n}(u - v\alpha_0)| \leqslant s\big\}\big).$$

Let $\rho = x\,y^{-1} \in K \cup \{\infty\}$, $\gamma_\rho = \begin{pmatrix} \rho & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(K)$ if $y \neq 0$ and $\gamma_\rho = \mathrm{id}$ otherwise. Let $\mathcal{H}_\infty = \{(z,t) \in \mathbb{H}^3_{\mathbb{R}} \;:\; t \geqslant 1\}$ and $\mathcal{H}_\rho = \gamma_\rho \mathcal{H}_\infty$. We will apply the counting claim of Corollary 2.2 with $n = 3$, Γ the arithmetic group $\overline{G}$, D^- the geodesic line with points at infinity $\alpha_0, \alpha_0^\sigma$ (whose stabiliser is exactly $\mathrm{Stab}_{\Gamma_K} N_{\alpha_0} \cap G$, and acts with compact quotient on D^-) and D^+ the horoball $\mathcal{H}_\rho$ (which is centered at a parabolic fixed point of $\overline{G}$).

For every $\gamma \in \mathrm{SL}_2(\mathcal{O}_K)$, the distance ℓ_γ between D^- and $\gamma D^+ = \gamma\gamma_\rho\mathcal{H}_\infty$, when they are disjoint, is equal to the distance between $\mathcal{H}_\infty$ and the geodesic line with points at infinity $\gamma_\rho^{-1}\gamma^{-1}\alpha_0, \gamma_\rho^{-1}\gamma^{-1}\alpha_0^\sigma$. Hence again by [34, Lem. 4.2], if $\gamma\gamma_\rho(1,0) = (u',v')$, then $\ell_\gamma = \ln\big(h(\alpha_0)\,|\mathtt{n}(u' - v'\alpha_0)\,|\big)$. If $y \neq 0$, we have $\gamma\gamma_\rho(1,0) = \frac{1}{y}\gamma(x,y)$, hence

$$\ell_\gamma = \ln\Big(\frac{h(\alpha_0)}{\mathtt{n}(y)}\,|\mathtt{n}(u - v\alpha_0)\,|\Big)$$

if $\gamma(x,y) = (u,v)$. By the above convention, this is also true if $y = 0$.

As in the proof of Theorem 3.1, we then have

$$\begin{aligned}
\overline{\psi}_{G,x,y}(s) &= \mathrm{Card}\Big\{[\gamma] \in G_{\{\alpha_0,\alpha_0^\sigma\}}\backslash G/G_{(x,y)} : \ell_\gamma \leqslant \ln\frac{h(\alpha_0)\mathtt{n}(I_{x,y})s}{\mathtt{n}(y)}\Big\} + \mathrm{O}(1)\\
&= \frac{2}{\iota_G}\mathrm{Card}\Big\{[\gamma] \in \overline{G_{\{\alpha_0,\alpha_0^\sigma\}}}\backslash\overline{G}/\overline{G_{(x,y)}} : \ell_\gamma \leqslant \ln\frac{h(\alpha_0)\mathtt{n}(I_{x,y})s}{\mathtt{n}(y)}\Big\} + \mathrm{O}(1)\\
&= \frac{2}{\iota_G}[\overline{G_{\mathcal{H}_\rho}} : \overline{G_{(x,y)}}]\,\mathcal{N}_{D^-,D^+}\Big(\ln\frac{h(\alpha_0)\mathtt{n}(I_{x,y})s}{\mathtt{n}(y)}\Big) + \mathrm{O}(1)\\
&= \frac{2[\overline{G_{\mathcal{H}_\rho}} : \overline{G_{(x,y)}}]}{\iota_G}\,\frac{\mathrm{Vol}(\mathbb{S}^1)\mathrm{Vol}(\Gamma_{D^-}\backslash D^-)\mathrm{Vol}(\Gamma_{D^+}\backslash D^+)}{m_G(\alpha_0)\mathrm{Vol}(\mathbb{S}^2)\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3_{\mathbb{R}})}\\
&\qquad\Big(\frac{h(\alpha_0)\mathtt{n}(I_{x,y})s}{\mathtt{n}(y)}\Big)^2(1 + \mathrm{O}(s^{-\kappa})).
\end{aligned} \tag{5.13}$$

By Equation (4.7), we have $\mathrm{Vol}(\Gamma_{D^-}\backslash D^-) = \frac{m_G(\alpha_0)\,[(\overline{\Gamma_K})_{\alpha_0} : \overline{G}_{\alpha_0}]}{m_{\Gamma_K}(\alpha_0)\,\iota_G(\alpha_0)}\,\ell(\widehat{\alpha_0})$. By Equation (5.8), we have $\mathrm{Vol}(\Gamma_{D^+}\backslash D^+) = [\,\overline{(\Gamma_K)_{\mathcal{H}_\rho}} : \overline{G_{\mathcal{H}_\rho}}\,]\ \frac{\sqrt{|D_K|}\,\mathtt{n}(y)^2}{2\,\omega_K\,\mathtt{n}(I_{x,\,y})^2}$. By Equation (5.5), we have $\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3_{\mathbb{R}}) = \frac{[\Gamma_K:G]\,|D_K|^{\frac{3}{2}}\,\zeta_K(2)}{4\,\pi^2\,\iota_G}$. We also have $[(\overline{\Gamma_K})_{\alpha_0} : \overline{G}_{\alpha_0}] = \frac{[(\Gamma_K)_{\alpha_0}:G_{\alpha_0}]}{\iota_G}$.
By Equation (5.9), we also have $[\,\overline{(\Gamma_K)_{\mathcal{H}_\rho}} : \overline{G_{\mathcal{H}_\rho}}][\overline{G_{\mathcal{H}_\rho}} : \overline{G_{(x,\,y)}}] = \frac{\omega_K}{2}\,[(\Gamma_K)_{(x,\,y)} : G_{(x,\,y)}]$. Therefore Equation (5.13) gives the following re-

sult, interesting in itself,

$$\overline{\psi}_{G,x,y}(s) = \frac{2\pi^2[(\Gamma_K)_{(x,y)} : G_{(x,y)}][(\Gamma_K)_{\alpha_0} : G_{\alpha_0}]h(\alpha_0)^2\big|\ln\big|\frac{\operatorname{tr}\widehat{\alpha_0}+\sqrt{(\operatorname{tr}\widehat{\alpha_0})^2-4}}{2}\big|\big|}{m_{\Gamma_K}(\alpha_0)\,\iota_G(\alpha_0)\,[\Gamma_K : G]|D_K|\,\zeta_K(2)} \\ s^2(1+\mathrm{O}(s^{-\kappa}))\,.$$

Since any nonzero ideal $\mathfrak{m}$ in $\mathcal{O}_K$ may be written $\mathfrak{m} = x_{\mathfrak{m}}\mathcal{O}_K + y_{\mathfrak{m}}\mathcal{O}_K$ for some $x_{\mathfrak{m}}, y_{\mathfrak{m}}$ not both zero in $\mathcal{O}_K$ and $\overline{\psi}_{\mathfrak{m}} = \overline{\psi}_{\Gamma_K,\, x_{\mathfrak{m}},\, y_{\mathfrak{m}}}$, the first claim of Theorem 5.11 follows.

The proof of the equidistribution claim is similar to the one in Theorem 5.10 (which was written with greater care than necessary for this purpose). □

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