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*Traces, lengths, axes and commensurability*

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## Traces, lengths, axes and commensurability

ALAN W. REID<sup>(1)</sup>

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**ABSTRACT.** — The focus of this paper are questions related to how various geometric and analytical properties of hyperbolic 3-manifolds determine the commensurability class of such manifolds. The paper is for the large part a survey of recent work.

**RÉSUMÉ.** — Cet article est bâti autour de la question suivante : comment des propriétés géométriques et analytiques de variétés hyperboliques de dimension 3 déterminent-elles leurs classes de commensurabilité. Cet article est pour la plus grande partie un aperçu de travaux récents.

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### 1. Introduction

This paper is based on three lectures given by the author at the workshop, “*Hyperbolic geometry and arithmetic: a crossview*” held at The Université Paul Sabatier, Toulouse in November 2012. The goal of the lectures was to describe recent work on the extent to which various geometric and analytical properties of hyperbolic 3-manifolds determine the commensurability class of such manifolds. This is the theme of the paper, and is for the most part a survey.

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## 2. The main players

Let  $M = \mathbf{H}^n/\Gamma$  be a closed orientable hyperbolic  $n$ -manifold. The free homotopy classes of closed geodesics on  $M$  correspond to conjugacy classes of hyperbolic elements in  $\Gamma$ . If  $\gamma \in \Gamma$  is a hyperbolic element, then associated to  $\gamma$  is an *axis*  $A_\gamma \subset \mathbf{H}^n$  on which  $\gamma$  translates by an amount,  $\ell_\gamma$  say (the *translation length* of  $\gamma$ ) and also perhaps rotates. The projection of  $A_\gamma$  to  $M$  then determines a closed geodesic in  $M$  whose length is  $\ell_\gamma$ . In what follows, we shall denote the set of axes of all the hyperbolic elements in  $\Gamma$  by  $\mathcal{A}(\Gamma)$ .

The cases of  $n = 2, 3$  will be of most interest to us, and we briefly recall that in this case, if  $\gamma \in \text{PSL}(2, \mathbf{C})$  is a hyperbolic element of trace  $t$ , the eigenvalues of  $\gamma$  are  $\frac{(-t \pm \sqrt{t^2 - 4})}{2}$ . Let  $\lambda_\gamma = \frac{(-t + \sqrt{t^2 - 4})}{2}$  be the eigenvalue satisfying  $|\lambda_\gamma| > 1$ . Then, in the notation above, if  $\gamma$  is now an element in the fundamental group of some hyperbolic manifold, the length  $\ell_\gamma$  of the closed geodesic determined by the projection of  $A_\gamma$  is  $2 \ln |\lambda_\gamma|$ .

With  $M = \mathbf{H}^n/\Gamma$  as above:

The *length spectrum*  $\mathcal{L}(M)$  of  $M$  is the set of all lengths of closed geodesics on  $M$  counted with multiplicities.

The *length set*  $L(M)$  of  $M$  is the set of lengths all closed geodesics counted without multiplicities.

The *rational length set*  $\mathbf{QL}(M)$  of  $M$  is the set of all rational multiples of lengths of closed geodesics of  $M$ .

Recall that if  $M_j = \mathbf{H}^n/\Gamma_j$  ( $j = 1, 2$ ) are closed orientable hyperbolic  $n$ -manifolds, then they are called *commensurable* if  $M_1$  and  $M_2$  have a common finite sheeted cover. Equivalently,  $\Gamma_1$  and some conjugate of  $\Gamma_2$  (in  $\text{Isom}(\mathbf{H}^n)$ ) have a common subgroup of finite index. We will henceforth use this as the definition of commensurability of subgroups. In addition, it will sometimes be convenient to refer to subgroups being *directly commensurable*, and in this case no conjugations are allowed. Given this, the following are elementary observations.

If  $M_1$  and  $M_2$  are commensurable, then:

(i)  $\mathbf{QL}(M_1) = \mathbf{QL}(M_2)$

(ii)  $\mathcal{A}(\Gamma_1) = \mathcal{A}(\Gamma_2)$ ; to be clear, by this we mean that after a choice of conjugates of  $\Gamma_1$  and  $\Gamma_2$ , the set of axes are the same.

Motivated by these observations, in this article we will focus on the following questions.

QUESTION 2.1. — *If  $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ , are  $M_1$  and  $M_2$  commensurable?*

QUESTION 2.2. — *If  $L(M_1) = L(M_2)$ , are  $M_1$  and  $M_2$  commensurable?*

QUESTION 2.3. — *If  $\mathbf{QL}(M_1) = \mathbf{QL}(M_2)$ , are  $M_1$  and  $M_2$  commensurable?*

QUESTION 2.4. — *If  $\mathcal{A}(\Gamma_1) = \mathcal{A}(\Gamma_2)$ , are  $\Gamma_1$  and  $\Gamma_2$  commensurable?*

Note that a positive answer to Question 2.2 provides positive answers to Questions 2.1 and 2.3.

In what follows we use the following terminology. In the case when  $L(M_1) = L(M_2)$ , we say that  $M_1$  and  $M_2$  are *length equivalent*, and when  $\mathcal{L}(M_1) = \mathcal{L}(M_2)$  we say that  $M_1$  and  $M_2$  are *iso-length spectral*.

Since it is known (see [6] pp. 415–417) that for closed hyperbolic manifolds, the spectrum of the Laplace-Beltrami operator acting on  $L^2(M)$ , counting multiplicities, determines  $L(M)$ , a positive answer to Question 2.2 implies a positive answer to the next question. Recall that closed hyperbolic manifolds are called *isospectral* if they have the same spectrum of the Laplace-Beltrami operator counted with multiplicities.

QUESTION 2.5. — *If  $M_1$  and  $M_2$  are isospectral, are they commensurable?*

We note that at present, the known methods of producing isospectral closed hyperbolic  $n$ -manifolds produce commensurable ones. One such method will be discussed further below (see also [22] and [24]).

Addressing these questions for general closed hyperbolic manifolds seems very hard at present, but arithmetic manifolds provide a large and interesting class where considerable progress on these questions can be made.

### 3. Some Number Theoretic Notation

By a number field  $k$  we mean a finite extension of  $\mathbf{Q}$ , the ring of integers of  $k$  will be denoted  $R_k$  and the Galois Closure of  $k$  over  $\mathbf{Q}$  denoted by  $k^{\text{cl}}$ .

### 3.1.

A place  $\nu$  of  $k$  will be one of the canonical absolute values of  $k$ . The finite places of  $k$  correspond bijectively to the prime ideals of  $R_k$ . An infinite place of  $k$  is either real, corresponding to an embedding of  $k$  into  $\mathbf{R}$ , or complex, corresponding to a pair of distinct complex conjugate embeddings of  $k$  into  $\mathbf{C}$ . We denote by  $k_\nu$  the completion of  $k$  at a place  $\nu$ . When  $\nu$  is an infinite place,  $k_\nu$  is isomorphic to  $\mathbf{R}$  or  $\mathbf{C}$  depending on whether  $\nu$  is real or complex.

If  $q$  is a prime power we denote by  $\mathbf{F}_q$  the finite field of cardinality  $q$ .

### 3.2.

Let  $k$  be a field of characteristic different from 2. The standard notation for a quaternion algebra over  $k$  is the following. Let  $a$  and  $b$  be non-zero elements of  $k$ , then  $\left(\frac{a,b}{k}\right)$  (known as the Hilbert Symbol) denotes the quaternion algebra over  $k$  with basis  $\{1, i, j, ij\}$  subject to  $i^2 = a$ ,  $j^2 = b$  and  $ij = -ji$ .

Let  $k$  be a number field, and  $\nu$  a place of  $k$ . If  $B$  is a quaternion algebra defined over  $k$ , the classification of quaternion algebras  $B_\nu = B \otimes_k k_\nu$  over the local fields  $k_\nu$  is quite simple. If  $\nu$  is complex then  $B_\nu$  is isomorphic to  $M(2, k_\nu)$  over  $k_\nu$ . Otherwise there is, up to isomorphism over  $k_\nu$ , a unique quaternion division algebra over  $k_\nu$ , and  $B_\nu$  is isomorphic over  $k_\nu$  to either this division algebra or to  $M(2, k_\nu)$ .

Let  $B$  be a quaternion algebra over the number field  $k$ .  $B$  is *ramified* at a place  $\nu$  of  $k$  if  $B_\nu$  is a division algebra. Otherwise we say  $B$  is *unramified* at  $\nu$ . We shall denote the set of places (resp. finite places) at which  $B$  is ramified by  $\text{Ram } B$  (resp.  $\text{Ram}_f B$ ). The *discriminant* of  $B$  is the  $R_k$ -ideal  $\prod_{\nu \in \text{Ram}_f B} \mathcal{P}_\nu$  where  $\mathcal{P}_\nu$  is the prime ideal associated to the place  $\nu$ .

We summarize for convenience the classification theorem for quaternion algebras over number fields (see [12] Chapter 7).

THEOREM 3.1. —

- *The set  $\text{Ram } B$  is finite, of even cardinality and contains no complex places.*
- *Conversely, suppose  $S$  is a finite set of places of  $k$  which has even cardinality and which contains no complex places. Then there is a quaternion algebra  $B$  over  $k$  with  $\text{Ram } B = S$ , and this  $B$  is unique up to isomorphism over  $k$ .*
- *$B$  is a division algebra of quaternions if and only if  $\text{Ram } B \neq \emptyset$ .*

#### 4. On Question 2.2

It is well-known that geometric and topological constraints are forced on closed hyperbolic manifolds that are isospectral; for example the manifolds have the same volume [16]. On the other hand, for length equivalent manifolds the situation is very different. The following is proved in [9].

**THEOREM 4.1.** — *Let  $M$  be a closed hyperbolic  $n$ -manifold. Then there exist infinitely many pairs of finite covers  $\{M_j, N_j\}$  of  $M$  such that  $L(M_j) = L(N_j)$  and for which  $\text{vol}(M_j)/\text{vol}(N_j) \rightarrow \infty$ .*

We will discuss ideas in the proof of this result. To that end, it is helpful to recall the construction of Sunada [22] that motivates the construction of [9].

##### 4.1.

Let  $G$  be a finite group and  $H_1$  and  $H_2$  subgroups of  $G$ . We say that  $H_1$  and  $H_2$  are *almost conjugate* if they are not conjugate in  $G$  but for every conjugacy class  $C \subset G$  we have:

$$|C \cap H_1| = |C \cap H_2|.$$

For a pair of subgroups  $H_1$  and  $H_2$  as above, we say that  $(H_1, H_2)$  are an almost conjugate pair. Many examples of such finite groups exist (see [2], [7] and [22]). The following is a special case of Sunada's method [22] (see also [2], and [3]).

**THEOREM 4.2.** — *Let  $M$  be a closed hyperbolic  $n$ -manifold,  $G$  a finite group, and  $(H_1, H_2)$  an almost conjugate pair in  $G$ . If  $\pi_1(M)$  admits a homomorphism onto  $G$ , then the finite covers  $M_1$  and  $M_2$  associated to the pullback subgroups of  $H_1$  and  $H_2$  are isospectral and iso-length spectral.*

The isospectral and iso-length spectral manifolds produced by Theorem 4.2 can sometimes be isometric. With more work, one can arrange that they are not, and moreover one can construct non-isometric examples in all dimensions (see [13] for example).

However, for length equivalence far less is required, and indeed in this case the non-isometric condition is much easier to arrange.

Subgroups  $H$  and  $K$  of  $G$  are said to be *elementwise conjugate* if for any conjugacy class  $C \subset G$  the following condition holds:

$$H \cap C \neq \emptyset \quad \text{if and only if} \quad K \cap C \neq \emptyset.$$

With this definition, we have the following from [9] (see Theorem 2.3):

PROPOSITION 4.3. — *Let  $M$  be a closed hyperbolic  $n$ -manifold,  $G$  a group, and  $H$  and  $K$  elementwise conjugate subgroups of  $G$ . Then If  $\pi_1(M)$  admits a homomorphism onto  $G$ , then the covers  $M_H$  and  $M_K$  associated to the pullback subgroups of  $H$  and  $K$  are length equivalent.*

*Proof.* — To prove the proposition, it suffices to show that a closed geodesic  $\gamma$  on  $M$  has a lift to a closed geodesic on  $M_H$  if and only if it has a lift to a closed geodesic on  $M_K$ . Let  $\rho$  denote the homomorphism  $\pi_1(M) \rightarrow G$ . By standard covering space theory,  $\gamma$  has a closed lift to  $M_H$  if and only if  $\rho([\gamma]) \in G$  is conjugate into  $H$ . By assumption this is true for  $H$  if and only if it is true for  $K$ , and the proposition is proved  $\square$

*Example.* — An easy example of a group with elementwise conjugate subgroups is the following.

Let  $G$  be the alternating group  $A_4$ , and set  $a = (12)(34)$  and  $b = (14)(23)$ . Then the subgroups  $H = \{1, a\}$  and the Klein 4-group  $K = \{1, a, b, ab\}$  are elementwise conjugate.

To see this, first note that since  $H < K$ , it only remains to check that if some conjugacy class  $C$  satisfies  $C \cap K \neq \emptyset$  then  $C \cap H \neq \emptyset$ . However, this is clear, since all products of disjoint transpositions are conjugate in  $A_4$ .

#### 4.2.

To prove Theorem 4.1 it suffices to exhibit groups  $G$  that contain elementwise conjugate subgroups and can be surjected by the fundamental groups of closed hyperbolic manifolds. Many such fundamental groups do surject  $A_4$ , but to prove Theorem 4.1 we need other examples. Since the general case is somewhat involved, we will only prove Theorem 4.1 in the case when  $M = \mathbf{H}^n/\Gamma$  and  $\Gamma$  is *large* (i.e.  $\Gamma$  contains a finite index subgroup that surjects a non-abelian free group).

In particular this holds for hyperbolic surfaces, for all closed hyperbolic 3-manifolds (by recent work of Agol [1]), and all arithmetic hyperbolic manifolds in even dimension (and roughly “half” in odd dimensions) by [14].

THEOREM 4.4. — *Let  $M = \mathbf{H}^n/\Gamma$  be a closed hyperbolic  $n$ -manifold and assume that  $\Gamma$  is large. Then for every integer  $n \geq 2$  and every odd prime  $p$ , there exists a finite tower of covers of  $M$*

$$M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{n-1} \rightarrow M_n \rightarrow M,$$

*with each  $M_i \rightarrow M_{i+1}$  of degree  $p$ , such that  $L(M_j) = L(M_k)$  for  $0 \leq j, k \leq n - 1$ .*

*Proof.* — The finite groups in question will be constructed as follows. Recall that the  $n$ -**dimensional special  $\mathbf{F}_p$ -affine group** is the semidirect product  $\mathbf{F}_p^n \rtimes \mathrm{SL}(n, \mathbf{F}_p)$  defined by the standard action of  $\mathrm{SL}(n, \mathbf{F}_p)$  on  $\mathbf{F}_p^n$ . For concreteness, the multiplication in the group is given by:

$$(v, S).(w, T) = (v + Sw, ST) \quad \text{and} \quad (v, S)^{-1} = (-S^{-1}v, S^{-1}).$$

We call any  $\mathbf{F}_p$ -vector subspace  $V$  of  $\mathbf{F}_p^n$  a **translational subgroup** of  $\mathbf{F}_p^n \rtimes \mathrm{SL}(n, \mathbf{F}_p)$ .

Since  $\Gamma$  is large there is a finite index subgroup  $\Delta$  that surjects the group  $G = \mathbf{F}_p^n \rtimes \mathrm{SL}(n, \mathbf{F}_p)$ . Thus it remains to show that we can find appropriate elementwise conjugate subgroups. To that end let  $V$  and  $W$  be translational subgroups of  $G$ . We claim that if  $V$  and  $W$  are both non-trivial then they are elementwise conjugate in  $G$ . To see this, let  $g \in G$ ,  $C$  the  $G$ -conjugacy class of  $g$ , let  $V \neq 0$  be a translational subgroup and assume that  $C \cap V \neq \emptyset$ . By hypothesis there exists  $h \in G$  such that  $hgh^{-1} \in V$ .

Setting  $g = (v_g, \bar{g})$  and  $h = (v_h, \bar{h})$ , we have:

$$hgh^{-1} = (v_h, \bar{h})(v_g, \bar{g})(v_h, \bar{h})^{-1}$$

and a computation shows that the condition  $hgh^{-1} \in V$  forces  $g$  to have the form  $g = (v_g, 1)$ . Since  $\mathrm{SL}(n, \mathbf{F}_p)$  acts transitively on non-trivial elements of  $\mathbf{F}_p^n$  for any  $w \in W$  (non-zero) there exists  $T \in \mathrm{SL}(n, \mathbf{F}_p)$  so that  $Tv = w$ . Then

$$(0, T)(v_g, 1)(0, T)^{-1} = (Tv_g, 1) = (w, 1) \in W.$$

Thus we have shown that some conjugate of  $g$  lies in  $W$  as required.  $\square$

To prove Theorem 4.1 in the general case, other families of groups having elementwise conjugate subgroups are needed and we refer the reader to [9] for details.

## 5. On Questions 2.1 and 2.3

Considerable progress has been made on Questions 2.1 and 2.3 in the setting of arithmetic manifolds (not just hyperbolic ones). In this section we will focus on arithmetic hyperbolic manifolds in dimensions 2 and 3, discussing ideas in the proofs of the following results proved in [19] and [4] respectively. In particular this gives affirmative answers to Questions 2.1, 2.2, 2.3 and 2.5 when  $M_1$  and  $M_2$  are both arithmetic. Indeed, Theorem 5.1 is a mild extension of the result in [19], and was discussed in [20].



**THEOREM 5.1.** — *Let  $M_1 = \mathbf{H}^2/\Gamma_1$  be an orientable arithmetic hyperbolic 2-manifold and assume that  $M_2 = \mathbf{H}^2/\Gamma_2$  is a closed orientable hyperbolic 2-manifold with  $L(\Gamma_1) = L(\Gamma_2)$ . Then  $\Gamma_2$  is arithmetic, and  $\Gamma_1$  and  $\Gamma_2$  are commensurable.*

**THEOREM 5.2.** — *If  $M$  is an arithmetic hyperbolic 3-manifold, then the rational length spectrum and the commensurability class of  $M$  determine one another.*

In dimensions  $\geq 4$ , work of Prasad and Rapinchuk [18] shows that Theorem 5.2 also holds for all even dimensional arithmetic hyperbolic manifolds. However, in odd dimensions the situation is different. For example, they show that for every  $n = 1 \pmod{4}$  there exist arithmetic hyperbolic  $n$ -manifolds for which Question 2.3 has a negative answer.

The motivation for these theorems came from Question 2.5. For closed hyperbolic manifolds, the only known constructions for producing isospectral manifolds are that of Sunada described in §4.1, and constructions building on one due to Vigneras [24]. In particular, all known constructions produce commensurable hyperbolic manifolds. In contrast to Theorems 5.1 and 5.2, there are constructions of lattices in  $\mathrm{SL}(d, \mathbf{R})$  and  $\mathrm{SL}(d, \mathbf{C})$  (for  $d \geq 3$ ) that are isospectral but not commensurable (see [11]).

To discuss the proofs of 5.1 and 5.2 it will be convenient to recall some material on arithmetic Fuchsian and Kleinian groups (we refer the reader to [12] for further details).

### 5.1.

Let  $k$  be a totally real number field, and let  $B$  be a quaternion algebra defined over  $k$  which is ramified at all infinite places except one. Let  $\rho : B \rightarrow M(2, \mathbf{R})$  be an embedding,  $\mathcal{O}$  be an order of  $B$ , and  $\mathcal{O}^1$  the elements of norm one in  $\mathcal{O}$ . Then  $\mathrm{P}\rho(\mathcal{O}^1) < \mathrm{PSL}(2, \mathbf{R})$  is a finite co-area Fuchsian group, which is co-compact if and only if  $B$  is not isomorphic to  $M(2, \mathbf{Q})$ . A Fuchsian group  $\Gamma$  is defined to be arithmetic if and only if  $\Gamma$  is commensurable with some such  $\mathrm{P}\rho(\mathcal{O}^1)$ .

*Notation.* — *Let  $B/k$  be as above and  $\mathcal{O}$  be an order of  $B$ . We will denote the group  $\mathrm{P}\rho(\mathcal{O}^1)$  by  $\Gamma_{\mathcal{O}}^1$ .*

Arithmetic Kleinian groups are obtained in a similar way. In this case we let  $k$  be a number field having exactly one complex place, and  $B$  a quaternion algebra over  $k$  which is ramified at all real places of  $k$ . As above, if  $\mathcal{O}$  is an order of  $B$  and  $\rho : \mathcal{O}^1 \hookrightarrow \mathrm{SL}(2, \mathbf{C})$ , then  $\Gamma_{\mathcal{O}}^1$  is a Kleinian group of

finite co-volume. An *arithmetic Kleinian group*  $\Gamma$  is a subgroup of  $\mathrm{PSL}(2, \mathbf{C})$  commensurable with a group of the type  $\Gamma_{\mathcal{O}}^1$ . An arithmetic Kleinian group is cocompact if and only if the quaternion algebra  $B$  as above is a division algebra.

In both the Fuchsian and Kleinian cases, the isomorphism class of the quaternion algebra  $B/k$  determines a wide commensurability class of groups in  $\mathrm{PSL}(2, \mathbf{R})$  and  $\mathrm{PSL}(2, \mathbf{C})$  respectively (see [12] Chapter 8). By Theorem 3.1 the isomorphism classes of such quaternion algebras will be completely determined by the finite set of places of  $k$  at which  $B$  is ramified.

A hyperbolic orbifold  $\mathbf{H}^2/\Gamma$  or  $\mathbf{H}^3/\Gamma$  will be called arithmetic if  $\Gamma$  is an arithmetic Fuchsian or Kleinian group.

Recall that if  $\Gamma$  is a Fuchsian or Kleinian group, the *invariant trace-field* of  $\Gamma$  is the field

$$k\Gamma = \mathbf{Q}(\{\mathrm{tr}^2\gamma : \gamma \in \Gamma\})$$

and the *invariant quaternion algebra*

$$A\Gamma = \{\Sigma a_j \gamma_j : a_j \in k\Gamma, \gamma_j \in \Gamma^{(2)}\}.$$

As discussed in [12] these are invariants of commensurability. When  $\Gamma$  is arithmetic, the field  $k$  and algebra  $B$  coincide with  $k\Gamma$  and  $A\Gamma$  and are therefore complete commensurability invariants.

We shall call a group  $G$  *derived from a quaternion algebra* if  $G$  is a subgroup of some  $\Gamma_{\mathcal{O}}^1$ , regardless of whether  $G$  has finite index in  $\Gamma_{\mathcal{O}}^1$ .

We remark that independent of arithmeticity, the invariant quaternion algebra of a non-elementary Fuchsian or Kleinian group can be computed efficiently from a pair of non-commuting hyperbolic elements  $\alpha_1, \alpha_2 \in \Gamma^{(2)}$  as

$$\left( \frac{\mathrm{tr}^2(\alpha_1) - 4, \mathrm{tr}[\alpha_1, \alpha_2] - 2}{k} \right)$$

(see [12] Chapter 3 for details).

It will also be convenient to recall the following characterization of arithmeticity (see [23] for the Fuchsian case and [12] for the Kleinian case).

**THEOREM 5.3.** — *Let  $\Gamma$  be a Fuchsian (resp. Kleinian group) of finite co-area (resp. finite co-volume). Then  $\Gamma$  is arithmetic if and only if  $\Gamma^{(2)}$  is derived from a quaternion algebra.*

**THEOREM 5.4.** — *Let  $\Gamma$  be a Fuchsian (resp. Kleinian group) of finite co-area (resp finite co-volume). Then  $\Gamma$  is derived from a quaternion algebra if and only if  $\Gamma$  satisfies the following conditions:*

1.  $\text{tr } \gamma$  is an algebraic integer for all  $\gamma \in \Gamma$ ;
2.  $k\Gamma$  is totally real (resp. a field with one complex place);
3. for all  $\sigma : k\Gamma \rightarrow \mathbf{R}$  (different from the identity embedding when  $k\Gamma$  is totally real) we have  $|\sigma(\text{tr } \gamma)| < 2$  for all non-trivial  $\gamma \in \Gamma$ .

## 5.2.

We now discuss proofs. The basic idea in both proofs (and indeed in the work of [18]) is the following two stage plan. For  $i = 1, 2$ , let  $M_i = \mathbf{H}^n/\Gamma_i$  be arithmetic hyperbolic 2 or 3-manifolds.

**Step 1:** *Show equality of the invariant trace-fields  $k\Gamma_1$  and  $k\Gamma_2$ .*

**Step 2:** *Show that the invariant quaternion algebras  $A\Gamma_1$  and  $A\Gamma_2$  are isomorphic using information about their subfields.*

We begin with the 2-dimensional case. Note that in Theorem 5.1, only  $M_1 = \mathbf{H}^2/\Gamma_1$  is assumed arithmetic, thus we first discuss in this case showing that  $L(M_1) = L(M_2)$  implies  $M_2$  is also arithmetic.

First recall from §2 that if  $\gamma \in \text{PSL}(2, \mathbf{C})$  is a hyperbolic element with eigenvalue  $\lambda_\gamma = \frac{-t + \sqrt{t^2 - 4}}{2}$  being the eigenvalue satisfying  $|\lambda_\gamma| > 1$ , then the length of the closed geodesic determined by the projection of the axis of  $\gamma$  is  $2 \ln |\lambda_\gamma|$ .

Thus, in this case if  $L(M_1) = L(M_2)$ , then up to sign the traces of elements of  $\Gamma_1$  and  $\Gamma_2$  are the same, and so the sets  $\{\text{tr } \gamma^2\}$  are the same.

Let  $A\Gamma_1 = B_1/k$ . From §5.1, since  $\Gamma_1$  is arithmetic,  $\text{tr } \gamma^2$  is an algebraic integer in the totally real field  $k$ . We therefore deduce from the discussion above that the elements of  $\Gamma_2$  have algebraic integer trace and  $k\Gamma_2 = k$ .

To complete the proof that  $\Gamma_2$  is arithmetic we show that  $B_2 = A\Gamma_2$  is ramified at all infinite places except one, for then the characterization theorem (Theorems 5.3 and 5.4) applies.

To prove this we argue as follows. Let  $\alpha_1$  and  $\alpha_2$  be a pair non-commuting hyperbolic elements in  $\Gamma_2^{(2)}$ . From above, a Hilbert Symbol for  $A\Gamma_2$  can be computed as  $\left( \frac{\text{tr}^2(\alpha_1) - 4, \text{tr}[\alpha_1, \alpha_2] - 2}{k} \right)$ . Since  $\Gamma_1$  and  $\Gamma_2$  have the same sets of

traces (up to sign), we can find  $\beta_1, \beta_2 \in \Gamma_1$  such that  $\text{tr}(\beta_1) = \pm \text{tr}(\alpha_1)$  and  $\text{tr}(\beta_2) = \pm \text{tr}([\alpha_1, \alpha_2])$ .

Since  $\text{tr}(\beta_1), \text{tr}(\beta_2) \in \mathbf{k}$ , it follows from [15] Theorem 2.2 that  $\beta_1, \beta_2 \in \Gamma_1 \cap B_1^1$ . Let  $\sigma : k \rightarrow \mathbf{R}$  be a non-identity embedding. Since  $B_1$  is ramified at  $\sigma$  it follows that (recall Theorem 5.4)  $|\sigma(\text{tr } x)| < 2$  for all non-trivial elements  $x \in B^1$ . Thus we can conclude that

$$\sigma(\text{tr}^2(\beta_1) - 4) < 0 \text{ and } \sigma(\text{tr}(\beta_2) - 2) < 0.$$

Hence it follows that

$$\sigma(\text{tr}^2(\alpha_1) - 4) < 0 \text{ and } \sigma(\text{tr}([\alpha_1, \alpha_2]) - 2) < 0,$$

from which it follows that  $B_2$  is ramified at  $\sigma$ . Now  $\sigma$  was an arbitrary non-identity embedding, and so we have shown that  $B_2$  is ramified at all such embeddings as required.

We now describe how to achieve Step 2. This requires the following result. In the statement  $\Gamma$  is either an arithmetic Fuchsian or Kleinian group.

**THEOREM 5.5.** — *Suppose that  $\Gamma$  is derived from a quaternion algebra  $B/k$ .*

(i) *Let  $\gamma$  be a hyperbolic element of  $\Gamma$  with eigenvalue  $\lambda_\gamma$ . The field  $k(\lambda_\gamma)$  is a quadratic extension field of  $k$  which embeds into  $B$ .*

(ii) *Let  $L$  be a quadratic extension of  $k$ . If  $\Gamma$  is Fuchsian assume that  $L$  is not a totally imaginary quadratic extension of  $k$ .*

*Then  $L$  embeds in  $B/k$  if and only if  $L = k(\lambda_\gamma)$  for some hyperbolic  $\gamma \in \Gamma$ . This will be true if and only if no place of  $k$  which splits in  $L$  is ramified in  $B$ .*

To establish commensurability, it suffices to show that  $B_1 \cong B_2$  (see the discussion in §5.2 and [12] Theorem 8.4.6). To that end, we make the following definition.

**DEFINITION.** — *For  $j = 1, 2$  let*

$$\mathcal{N}_j = \{L/k : [L:k] = 2, L \text{ embeds in } B_j \text{ and is not a totally imaginary quadratic extension of } k\}.$$

Clearly if  $B_1$  and  $B_2$  are isomorphic the set of quadratic extensions that embed in one is precisely the set that embeds in the other. The converse is also true (see [12] Chapter 12). However, a slightly different version of this is required to finish the proof of Theorem 5.1. The key point to be proved (see [19] for details) is:

$$B_1 \text{ and } B_2 \text{ are isomorphic if and only if } \mathcal{N}_1 = \mathcal{N}_2 .$$

Given this, the proof is completed as follows. Suppose  $L \in \mathcal{N}_1$ . Theorem 5.5 and commensurability shows that  $L = k(\lambda_\gamma)$  for some hyperbolic  $\gamma \in \Gamma_1$  with eigenvalue  $\lambda_\gamma$ . Since  $L(M_1) = L(M_2)$ , it follows that there exists an element  $\gamma' \in \Gamma_2$  for which  $\lambda_{\gamma'} = \lambda_\gamma$ . Hence  $L = k(\lambda_\gamma) = k(\lambda_{\gamma'})$  embeds in  $B_2$  as required.  $\square$

The proof of Theorem 5.2 proceeds as above, but in this case Step 1 is considerably more difficult to prove. We will discuss some of the ideas in the proof of this part in the special case where the invariant trace-fields  $k_j = k\Gamma_j$  satisfy  $[k_j : k_j \cap \mathbf{R}] > 2$ . This simplifies some of the arguments but gives a flavor of how the proof in the general case proceeds. Step 2 is achieved exactly as in the Fuchsian case.

For  $j = 1, 2$  define:

$$\mathcal{E}_j = \{ \lambda : \lambda \text{ is an e-value of a hyperbolic element of } \Gamma_j \}.$$

Now the claim is the following (see [4] for details).

CLAIM. — For  $j = 1$  or  $2$ ,

$$k_j^{\text{cl}} = \bigcap \{ \mathbf{Q}(\lambda)^{\text{cl}} : \lambda \in \mathcal{E}_j \}.$$

Given this claim, the following result is proved in [4].

**THEOREM 5.6.** — *Let  $k$  and  $k'$  be number fields having exactly one complex place and the same Galois closure  $k^{\text{cl}}$ . Then either  $k$  and  $k'$  are isomorphic, or after replacing  $k'$  by an isomorphic field,  $k$  and  $k'$  are quadratic non-isomorphic extensions of a common totally real subfield  $k^+$ .*

In our case (i.e.  $[k_j : k_j \cap \mathbf{R}] > 2$ ) the former situation applies and so the fields are isomorphic.

*Remark.* — Another number theoretic result proved in [4] is:

**THEOREM 5.7.** — *If  $k$  is a number field with one complex place, then  $k$  is determined up to isomorphism by its zeta function.*

This result contrasts with the fact that there are many examples of number fields which are not determined up to isomorphism by their zeta functions (see [17]). Moreover it is striking that Theorem 5.7 was proved along the way to proving a result about isospectrality. This should be contrasted with the original impetus for Sunada’s construction, namely the existence of non-isomorphic number fields with the same zeta functions. Thus in our setting things appear to have come full circle.

## 6. On Question 2.4

Instead of lengths of closed geodesics (or equivalently translation lengths along axes), we consider the “location” of the axes of hyperbolic elements in arithmetic Fuchsian and Kleinian groups, then again one can get extra information that allows one to prove (see [10]).

**THEOREM 6.1.** — *If  $\Gamma_1$  and  $\Gamma_2$  are arithmetic Fuchsian (resp. Kleinian) groups then Question 2.4 has a positive answer.*

Defining groups  $\Gamma_1$  and  $\Gamma_2$  to be *isoaxial* if  $\mathcal{A}(\Gamma_1) = \mathcal{A}(\Gamma_2)$ , we now proceed to sketch the proof that isoaxial arithmetic Kleinian groups are commensurable. The obvious changes can be made for Fuchsian groups.

Let  $\Gamma$  be a Kleinian group and define

$$\Sigma(\Gamma) = \{\gamma \in \mathrm{PSL}(2, \mathbf{C}) \mid \gamma(\mathcal{A}(\Gamma)) = \mathcal{A}(\Gamma)\}.$$

It is easy to check that  $\Sigma(\Gamma)$  is a subgroup of  $\mathrm{PSL}(2, \mathbf{C})$ . Next recall the *commensurator* of  $\Gamma$  is the subgroup of  $\mathrm{PSL}(2, \mathbf{C})$  defined as:

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{PSL}(2, \mathbf{C}) : g\Gamma g^{-1} \text{ is directly commensurable with } \Gamma\}.$$

We claim that  $\mathrm{Comm}(\Gamma) < \Sigma(\Gamma)$ . To see this, first observe that if  $\Gamma_1$  and  $\Gamma_2$  are commensurable Kleinian groups they contain a finite index subgroup  $\Delta$ . Hence,

$$\mathcal{A}(\Gamma_1) = \mathcal{A}(\Delta) = \mathcal{A}(\Gamma_2).$$

Now if  $x \in \mathrm{Comm}(\Gamma)$ , then  $x\Gamma x^{-1}$  is commensurable with  $\Gamma$ , and so  $x$  preserves the set  $\mathcal{A}(\Gamma)$ . Also notice that if  $\Gamma_1$  and  $\Gamma_2$  are isoaxial Kleinian groups, then for any  $\gamma \in \Gamma_2$ ,  $\mathcal{A}(\Gamma_1) = \mathcal{A}(\gamma\Gamma_1\gamma^{-1})$ , and therefore  $\gamma \in \Sigma(\Gamma_1)$ . Hence  $\Gamma_2 < \Sigma(\Gamma_1)$ .

The key point is now to show that when  $\Gamma$  is arithmetic, we have  $\mathrm{Comm}(\Gamma) = \Sigma(\Gamma)$ . For then, the above discussion implies that  $\Gamma_2 < \mathrm{Comm}(\Gamma_1)$ ,

and it is standard that, if  $\Gamma_2$  is also arithmetic,  $\Gamma_1$  and  $\Gamma_2$  are commensurable.  $\square$

*Remark.* — Unlike Theorem 5.1, for now we are unable to assume that only one of the groups is arithmetic. For example, the question as to whether there could be a finitely generated non-arithmetic Fuchsian group that shares the same set of axes as hyperbolic elements of  $\mathrm{PSL}(2, \mathbf{Z})$  remains open, although we suspect that this is not the case. The above arguments do show that such a group is a subgroup of  $\mathrm{PGL}(2, \mathbf{Q})$ .

## 7. Some Questions

We close with some questions on topics connected to this survey. The main one is the following:

QUESTION 7.1. — *Do Questions 2.1, 2.2, 2.3, 2.4 and 2.5 have positive answers for finitely generated non-arithmetic Fuchsian and Kleinian groups?*

From Theorem 4.4 we have arbitrarily large (but finite) families of length equivalent closed hyperbolic manifolds in all dimensions. Given that we can ask:

QUESTION 7.2. — *Do there exist infinite sets of length equivalent, closed hyperbolic  $n$ -manifolds?*

If we allow finite volume rather than closed then, Schmutz [21] (in dimension 2) and more recently [8] (in dimensions 2 and 3) produced examples of infinitely many finite co-volume but non-cocompact arithmetic groups that are length equivalent. Indeed, in [8], infinitely many examples are constructed that are length equivalent to  $\mathrm{PSL}(2, \mathbf{Z})$ .

Given the examples of Prasad and Rapinchuk mentioned in §5, the following seems particularly interesting.

QUESTION 7.3. — *Can the examples of Prasad and Rapinchuk be isospectral?*

Recall that a closed hyperbolic surface of genus  $g$  is called a *Hurwitz surface* if it has precisely  $84(g - 1)$  orientation preserving isometries. This number is maximal by virtue of Hurwitz's theorem on automorphisms (see [5] for a discussion of such surfaces). Given this high order of symmetry it seems likely that the following question has a positive answer.

QUESTION 7.4. — *Is a Hurwitz surface determined up to isometry by its length spectrum?*

A Hurwitz surface is arithmetic, being a regular cover of the orbifold  $\mathbf{H}^2/\Delta(2, 3, 7)$  where  $\Delta(2, 3, 7)$  is the group of orientation-preserving isometries in the group generated by reflections in the edges of a hyperbolic triangle with interior angles  $(\pi/2, \pi/3, \pi/7)$ . The invariant trace-field is  $k = \mathbf{Q}(\cos \pi/7)$  and the invariant quaternion algebra is the quaternion algebra over  $k$  which is ramified at two real places and unramified at all finite places.

We may deduce from the discussion of the proof of Theorem 5.1 that if  $S$  is a hyperbolic surface that is iso-length spectral with a Hurwitz surface, then  $S$  is arithmetic, and indeed  $S$  will be a cover of  $\mathbf{H}^2/\Delta(2, 3, 7)$  of degree  $84(g-1)$  (although perhaps not regular).

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