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A rate of convergence for the circular law for the complex Ginibre ensemble

ELIZABETH S. MECKES⁽¹⁾, MARK W. MECKES⁽²⁾

ABSTRACT. — We prove rates of convergence for the circular law for the complex Ginibre ensemble. Specifically, we bound the L_p -Wasserstein distances between the empirical spectral measure of the normalized complex Ginibre ensemble and the uniform measure on the unit disc, both in expectation and almost surely. For $1 \leq p \leq 2$, the bounds are of the order $n^{-1/4}$, up to logarithmic factors.

RÉSUMÉ. — Nous établissons des vitesses de convergence pour la loi du cercle de l'ensemble de Ginibre complexe. Plus précisément, nous donnons des bornes supérieures pour les distances de Wasserstein d'ordre p entre la mesure spectrale empirique de l'ensemble de Ginibre complexe normalisé et la mesure uniforme du disque, en espérance et presque sûrement. Si $1 \leq p \leq 2$, les bornes sont de la taille $n^{-1/4}$, à des facteurs logarithmiques près.

1. Introduction

Consider an $n \times n$ random matrix G_n with i.i.d. standard complex Gaussian entries; put slightly differently, G_n is a random element of the set of $n \times n$ matrices over \mathbb{C} , whose distribution has density proportional to $e^{-\operatorname{tr}(GG^*)}$. The random matrix G_n is said to belong to the *complex Ginibre ensemble*.

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Although this ensemble was introduced by Ginibre [7] without any particular application in mind, the eigenvalues of G_n have since been used to model a wide variety of physical phenomena; see in particular [6] and [10] for references.

The central result about the asymptotic behavior of the eigenvalues of G_n is the famous circular law. Let μ_n denote the empirical spectral measure of $\frac{1}{\sqrt{n}}G_n$; that is,

$$\mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k},$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\frac{1}{\sqrt{n}}G_n$. The circular law states that when $n \rightarrow \infty$, μ_n converges in some sense to the uniform measure ν on the unit disc $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$. This was first established by Mehta [12], who showed that the mean empirical spectral measure $\mathbb{E}\mu_n$ converges weakly to ν . A large literature followed, which established the circular law for more general random matrix ensembles, and for stronger forms of convergence, culminating in the recent proof by Tao and Vu [18] of the circular law for random matrices with i.i.d. entries with arbitrary entries with finite variance, in the sense of almost sure weak convergence. The reader is referred to the survey by Bordenave and Chafaï [3] for further history and related results.

The main results of this paper give rates of convergence for the circular law for the complex Ginibre ensemble G_n , both in expectation and almost surely.

THEOREM 1.1. — *There is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and all $p \geq 1$,*

$$\mathbb{E}W_p(\mu_n, \nu) \leq C \max \left\{ \frac{\sqrt{p}}{n^{1/4}}, \left(\frac{\log n}{n} \right)^{\frac{1}{2p}} \right\},$$

where $W_p(\mu, \nu)$ denotes the L_p -Wasserstein distance between probability measures μ and ν .

In particular, in the most widely used Wasserstein metrics, namely $p = 1, 2$, we have

$$\mathbb{E}W_1(\mu_n, \nu) \leq \frac{C}{n^{1/4}} \quad \text{and} \quad \mathbb{E}W_2(\mu_n, \nu) \leq C \left(\frac{\log n}{n} \right)^{\frac{1}{4}}.$$

THEOREM 1.2. — *For each $p \geq 1$ there is a constant $K_p > 0$ such that with probability 1, for sufficiently large n ,*

$$W_p(\mu_n, \nu) \leq K_p \frac{\sqrt{\log n}}{n^{1/4}}$$

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when $1 \leq p \leq 2$, and

$$W_p(\mu_n, \nu) \leq K_p \left(\frac{\log n}{n} \right)^{1/2p}$$

when $p > 2$.

Recall that for any $p \geq 1$, the L_p -Wasserstein distance between two probability measures μ and ν on \mathbb{C} is defined by

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int |w - z|^p d\pi(w, z) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ is the set of all couplings of μ and ν ; i.e., probability measures on $\mathbb{C} \times \mathbb{C}$ with marginals μ and ν (see, e.g., [19]).

A few related results have appeared previously. In [17, Section 14], Tao and Vu sketched an argument giving an almost sure convergence rate for the empirical spectral measure of a random matrix $\frac{1}{\sqrt{n}}M_n$ with i.i.d. entries with a finite moment of order $2 + \varepsilon$. The convergence in this case was in Kolmogorov distance (sup-distance between bivariate cumulative distribution functions), and the rate is of order n^{-c} for some unspecified (but rather small) $c = c(\varepsilon) > 0$. Earlier, Bai [2] established, as an intermediate technical tool, a convergence rate for the empirical spectral measures of the Hermitianized random matrices $(M_n - zI_n)^*(M_n - zI_n)$.

In a different direction, Sandier and Serfaty [15] and Rougerie and Serfaty [14] studied empirical measures of Coulomb gases, which for particular values of certain parameters have the same distribution as μ_n . Among their results are tail bounds for distances between these measures from deterministic equilibrium measures, in terms of metrics which are dual to Sobolev norms on a ball. For a certain choice of parameter, in the 2-dimensional case their metric becomes

$$\sup \left\{ \int_{rD} f d\mu(z) - \int_{rD} f d\nu(z) \mid f \text{ 1-Lipschitz} \right\};$$

without the restriction to rD this would coincide with W_1 , by the Kantorovitch duality theorem.

The basic idea of the proofs of Theorems 1.1 and 1.2 is reasonably simple, but verifying all of the details gets somewhat technical, and so we first give an outline of our approach.

Step 1: We begin by **ordering the eigenvalues** $\{\lambda_k\}_{k=1}^n$ **in a spiral fashion**. Specifically, we define a linear order \prec on \mathbb{C} by making 0 initial, and for nonzero $w, z \in \mathbb{C}$, we declare $w \prec z$ if any of the following holds:

- $\lfloor \sqrt{n} |w| \rfloor < \lfloor \sqrt{n} |z| \rfloor$.
- $\lfloor \sqrt{n} |w| \rfloor = \lfloor \sqrt{n} |z| \rfloor$ and $\arg w < \arg z$.
- $\lfloor \sqrt{n} |w| \rfloor = \lfloor \sqrt{n} |z| \rfloor$, $\arg w = \arg z$, and $|w| \geq |z|$.

Here we are using the convention that $\arg z \in (0, 2\pi]$.

We order the eigenvalues according to \prec : first the eigenvalues in the disc of radius $\frac{1}{\sqrt{n}}$ are listed in order of increasing argument, then the ones in the annulus with inner radius $\frac{1}{\sqrt{n}}$ and outer radius $\frac{2}{\sqrt{n}}$ in order of increasing argument, and so on. (With probability 1, no two eigenvalues of G_n have the same argument; thus the details of the last condition in the definition of \prec are irrelevant and it is included only for completeness.)

Step 2: We define *predicted locations* for (most of) the eigenvalues as follows. Fix some m so that $n - m$ is a perfect square. Then $\tilde{\lambda}_1 = 0$, $\{\tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4\}$ are $\frac{1}{\sqrt{n}}$ times the 3rd roots of unity (in increasing order with respect to \prec), the next five are $\frac{2}{\sqrt{n}}$ times the 5th roots of unity, and so on until $\tilde{\lambda}_{n-m}$.

Formally, given $1 \leq k \leq n - m$, write $\ell = \lceil \sqrt{k} \rceil$ and $q = k - (\ell - 1)^2$, so that

$$k = (\ell - 1)^2 + q \quad \text{and} \quad 1 \leq q \leq 2\ell - 1. \quad (1.1)$$

Now define

$$\tilde{\lambda}_k = \frac{\ell - 1}{\sqrt{n}} e^{2\pi i q / (2\ell - 1)}.$$

Observe that the sequence $(\tilde{\lambda}_k)_{k=1}^{n-m}$ is increasing with respect to \prec .

Step 3: We show that most of the eigenvalues λ_k concentrate around their predicted locations $\tilde{\lambda}_k$. The eigenvalue process of G_n is a determinantal point process, from which concentration inequalities for the number of eigenvalues within subsets of D follow. We apply this concentration property to the number of eigenvalues in an initial segment with respect to the order \prec . Geometric arguments allow one to move from this concentration to concentration of individual eigenvalues around their predicted values.

Step 4: We couple the empirical spectral measure μ_n to the measure ν_n which puts mass $\frac{1}{n}$ at each point $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-m}$, and mass $\frac{m}{n}$ uniformly on the annulus

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$\{z \in \mathbb{C} \mid \sqrt{1 - \frac{m}{n}} \leq |z| \leq 1\}$. The concentration established in the previous step allows us to estimate $W_p(\mu_n, \nu_n)$ via this coupling.

Step 5: The measure ν_n is approximately uniform on D .

This approach adapts those taken by Dallaporta [4] for the Gaussian Unitary Ensemble, and by the authors [11] for random unitary matrices. In those settings, the linear order of the eigenvalues was of critical importance. The lack of a natural order on the complex plane is the major obstacle in adapting the methods of [4, 11] for the Ginibre ensemble, and it is this difficulty which is addressed by the introduction of the spiral order \prec .

The rest of this paper is organized as follows. In Section 2 we dispense with Step 5 of the outline, and collect the main technical tools which will be used in the rest of the paper. In Section 3 we estimate the mean and variance of the number of eigenvalues in an initial segment with respect to the order \prec . In Section 4, we derive estimates for the concentration of individual eigenvalues around their predicted values (Step 3 above), using the results of the previous two sections. Finally, in Section 5, we carry out the coupling argument (Step 4 of the outline) and complete the proofs of Theorems 1.1 and 1.2. We also observe (Theorem 5.1) that our results yield the correct rate of convergence of the mean empirical spectral measure in the total variation metric.

2. Technical tools

We begin by taking care of Step 5 in the outline above. Recall that ν_n is the measure which puts mass $\frac{1}{n}$ at each point $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{n-m}$, and mass $\frac{m}{n}$ uniformly on the annulus $\{z \in \mathbb{C} \mid \sqrt{1 - \frac{m}{n}} \leq |z| \leq 1\}$.

LEMMA 2.1. — For each positive integer n and each $p \geq 1$, $W_p(\nu_n, \nu) < \frac{8}{\sqrt{n}}$.

Proof. — We couple ν_n to ν as follows. The sector

$$S_k := \left\{ z \in \mathbb{C} \mid \frac{\ell - 1}{\sqrt{n}} \leq |z| < \frac{\ell}{\sqrt{n}}, \frac{2\pi(q-1)}{2\ell-1} \leq \arg z \leq \frac{2\pi q}{2\ell-1} \right\},$$

where k , ℓ , and q are related by (1.1), satisfies $\nu(S_k) = 1/n$ for each $1 \leq k \leq n - m$. All of the mass in S_k is coupled to $\tilde{\lambda}_k$, and the identity coupling is used in the annulus $\{z \in \mathbb{C} \mid \sqrt{1 - \frac{m}{n}} \leq |z| \leq 1\}$. If $r \in \left[\frac{\ell}{\sqrt{n}}, \frac{\ell-1}{\sqrt{n}} \right]$ and

$\varphi \in \left[\frac{2\pi(q-1)}{2\ell-1}, \frac{2\pi q}{2\ell-1} \right]$, then

$$\begin{aligned} \left| r e^{i\varphi} - \tilde{\lambda}_k \right| &\leq \left| r e^{i\varphi} - r e^{2\pi i q / (2\ell-1)} \right| + \left| r e^{2\pi i q / (2\ell-1)} - \tilde{\lambda}_k \right| \\ &\leq r \left| \varphi - \frac{2\pi q}{2\ell-1} \right| + \left| r - \frac{\ell-1}{\sqrt{n}} \right| \\ &\leq \frac{2\pi\ell}{(2\ell-1)\sqrt{n}} + \frac{1}{\sqrt{n}} < \frac{8}{\sqrt{n}}. \end{aligned}$$

Therefore

$$W_p(\nu_n, \nu) < \left(\frac{n-m}{n} \left(\frac{8}{\sqrt{n}} \right)^p + \frac{m}{n} (0) \right)^{1/p} \leq \frac{8}{\sqrt{n}}. \quad \square$$

Lemma 2.1 shows that, up to the constant 8, ν_n is an optimal approximation of ν by an empirical measure on n points. Indeed, suppose that x_1, \dots, x_n are any n points in \mathbb{C} , and let $\rho_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$. Then the area of the union of the ε -discs centered at the x_i is at most $n\pi\varepsilon^2$, so $W_1(\rho_n, \nu) \geq (1 - n\varepsilon^2)\varepsilon$, since a fraction at least $(1 - n\varepsilon^2)$ of the mass of ν must move a distance at least ε in transporting ν to ρ_n . Optimizing in ε gives $W_p(\rho_n, \nu) \geq W_1(\rho_n, \nu) \geq \frac{2}{3\sqrt{3n}}$.

PROPOSITION 2.2. — *Let $A \subseteq D$ be measurable, and let $\mathcal{N}(A)$ denote the number of eigenvalues of $\frac{1}{\sqrt{n}}G_n$ lying in A . Then*

$$\mathbb{P}[\mathcal{N}(A) - \mathbb{E}\mathcal{N}(A) \geq t] \leq \exp \left[- \min \left\{ \frac{t^2}{4\sigma^2}, \frac{t}{2} \right\} \right]$$

and

$$\mathbb{P}[\mathbb{E}\mathcal{N}(A) - \mathcal{N}(A) \geq t] \leq \exp \left[- \min \left\{ \frac{t^2}{4\sigma^2}, \frac{t}{2} \right\} \right]$$

for each $t \geq 0$, where $\sigma^2 = \text{Var } \mathcal{N}(A)$.

Proof. — The eigenvalues of G_n form a determinantal point process on \mathbb{C} with the kernel

$$\begin{aligned} K(z, w) &= \frac{1}{\pi} e^{-(|z|^2 + |w|^2)/2} \sum_{k=0}^{n-1} \frac{(z\bar{w})^k}{k!} \\ &= \frac{1}{\pi} e^{-|z-w|^2/2} \left(1 - e^{-z\bar{w}} \sum_{k=n}^{\infty} \frac{(z\bar{w})^k}{k!} \right). \end{aligned} \tag{2.1}$$

The reader is referred to [9] for the definition of a determinantal point process. The fact that the eigenvalues of G_n form such a process follows from the original work of Ginibre [7]; see also [13, Chapter 15].

This fact combines crucially with [9, Theorem 7] (see also [1, Corollary 4.2.24]), which says that $\mathcal{N}(A)$ is distributed exactly as a sum of independent $\{0, 1\}$ -valued random variables, whose parameters are given by the eigenvalues (which necessarily lie in $[0, 1]$) of the operator \mathcal{K} on $L_2(A)$ defined by

$$\mathcal{K}f(z) = \int_A K(z, w)f(w) dw.$$

The statement of the proposition is thus simply Bernstein's classical tail inequality for sums of independent bounded random variables (see, e.g., [16, Lemma 2.7.1]). \square

As discussed in Step 3 of the outline, to bound the deviations of λ_k about its predicted location $\tilde{\lambda}_k$, we first use Proposition 2.2 to bound the deviations of the counting functions for initial segments with respect to the order \prec . Specifically, we will consider $\mathcal{N}(A_{j,\theta})$, where

$$\begin{aligned} A_{j,\theta} &:= \left\{ z \in \mathbb{C} \mid z \prec \frac{j}{\sqrt{n}} e^{i\theta} \right\} \\ &= \left\{ z \in \mathbb{C} \mid |z| < \frac{j}{\sqrt{n}} \right\} \cup \left\{ z \in \mathbb{C} \mid \frac{j}{\sqrt{n}} \leq |z| < \frac{j+1}{\sqrt{n}}, 0 < \arg z \leq \theta \right\}, \end{aligned}$$

for $1 \leq j \leq \sqrt{n} - 1$ and $0 < \theta \leq 2\pi$ (see Figure 1).

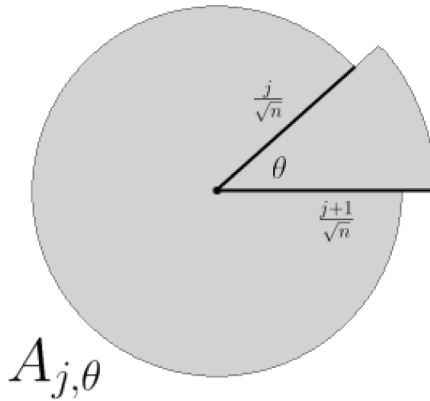


Figure 1

Finally, we conclude this section by collecting a few known formulas and estimates which will be used repeatedly below. The following integral formula can be proved by repeated integration by parts; we omit the proof.

LEMMA 2.3. — *If k is a nonnegative integer and $a > 0$, then*

$$\frac{1}{k!} \int_a^\infty s^k e^{-s} ds = e^{-a} \sum_{\ell=0}^k \frac{a^\ell}{\ell!},$$

and consequently

$$\frac{1}{k!} \int_0^a s^k e^{-s} ds = e^{-a} \sum_{\ell=k+1}^\infty \frac{a^\ell}{\ell!}.$$

The following inequality follows from a standard Chernoff bound argument for Poisson random variables.

LEMMA 2.4. — *If $0 < \lambda \leq n$, then $\sum_{k=n}^\infty \frac{\lambda^k}{k!} \leq \left(\frac{e\lambda}{n}\right)^n$.*

Proof. — Let X have a Poisson distribution with parameter λ . Assuming for simplicity that $\lambda < n$, let $t = \log(n/\lambda) > 0$. Then

$$\sum_{k=n}^\infty \frac{\lambda^k}{k!} = e^\lambda \mathbb{P}[X \geq n] \leq e^{\lambda - tn} \mathbb{E} e^{tX} = e^{\lambda e^t - tn} = \left(\frac{e\lambda}{n}\right)^n. \quad \square$$

Finally, we will use the following uniform version of Stirling's approximation.

LEMMA 2.5. — *For each positive integer n , $\sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n} \leq n! \leq en^{n+\frac{1}{2}} e^{-n}$.*

Proof. — The following version of Stirling's approximation appears as [5, (9.15)]:

$$\sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}.$$

The lemma is trivially true when $n = 1$, and for $n \geq 2$, the lemma follows since $\sqrt{2\pi} e^{1/12n} \leq \sqrt{2\pi} e^{1/24} < e$. \square

3. Means and variances

Concentration inequalities for the random variables $\mathcal{N}(A_{j,\theta})$ about their means follow from Proposition 2.2, but in order to make use of them, fairly sharp estimates on the means and variances of the $\mathcal{N}(A_{j,\theta})$ are needed. These estimates, like the proof of Proposition 2.2, make use of the determinantal point process structure of the eigenvalues of G_n .

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PROPOSITION 3.1. — *If $A \subseteq D$,*

$$\frac{n|A|}{\pi} - e\sqrt{n} \leq \mathbb{E}\mathcal{N}(A) \leq \frac{n|A|}{\pi},$$

where $|A|$ denotes the area of A . Moreover, if $A \subseteq \left(1 - \sqrt{\frac{\log n}{n}}\right) D$, then

$$\frac{n|A|}{\pi} - e^2 \leq \mathbb{E}\mathcal{N}(A) \leq \frac{n|A|}{\pi}.$$

Proof. — The determinantal point process structure of the eigenvalues of G_n implies that that $\mathbb{E}\mathcal{N}(A) = \int_{\sqrt{n}A} K(z, z) dz$ (where dz denotes integration with respect to Lebesgue measure on \mathbb{C}), so that

$$\begin{aligned} \mathbb{E}\mathcal{N}(A) &= \frac{1}{\pi} \int_{\sqrt{n}A} \left(1 - \sum_{k=n}^{\infty} e^{-|z|^2} \frac{|z|^{2k}}{k!}\right) dz \\ &= \frac{n|A|}{\pi} - \frac{1}{\pi} \int_{\sqrt{n}A} \sum_{k=n}^{\infty} e^{-|z|^2} \frac{|z|^{2k}}{k!} dz. \end{aligned}$$

Using Lemma 2.4 and then integrating in polar coordinates,

$$\begin{aligned} \frac{1}{\pi} \int_{\sqrt{n}A} \sum_{k=n}^{\infty} e^{-|z|^2} \frac{|z|^{2k}}{k!} dz &\leq \left(\frac{e}{n}\right)^n \int_{\sqrt{n}D} e^{-|z|^2} |z|^{2n} dz \\ &= 2 \left(\frac{e}{n}\right)^n \int_0^{\sqrt{n}} e^{-r^2} r^{2n+1} dr < \left(\frac{e}{n}\right)^n n! \leq \pi e\sqrt{n}, \end{aligned}$$

by Stirling's approximation (Lemma 2.5).

If $A \subseteq rD$ for $r \leq 1$ then, using Lemmas 2.4, 2.3, and 2.4 again,

$$\begin{aligned} \frac{1}{\pi} \int_{\sqrt{nr}D} \sum_{k=n}^{\infty} e^{-|z|^2} \frac{|z|^{2k}}{k!} dz &\leq \left(\frac{e}{n}\right)^n \int_0^{r^2n} e^{-s} s^n ds \\ &= e^{-r^2n} \left(\frac{e}{n}\right)^n n! \sum_{\ell=n+1}^{\infty} \frac{(r^2n)^\ell}{\ell!} \\ &\leq e^{-r^2n} e\sqrt{n}(er^2)^n \leq e^2\sqrt{n}e^{-n(1-r^2)^2/2}, \end{aligned}$$

since $\log(1 - \varepsilon) \leq -\varepsilon - \varepsilon^2/2$ for $0 < \varepsilon < 1$. Finally, let $r = 1 - \sqrt{\frac{\log n}{n}}$. Then $\sqrt{n}e^{-n(1-r^2)^2/2} \leq \sqrt{n}e^{-n(1-r)^2/2} = 1$. \square

We will also need estimates for the expected number of eigenvalues outside of discs of radius $R \geq 1$.

PROPOSITION 3.2. — *For each $R \geq 1$,*

$$\mathbb{E}\mathcal{N}(\mathbb{C} \setminus RD) \leq \frac{1}{\sqrt{2\pi}} \sqrt{n} e^n R^{2(n-1)} e^{-nR^2}.$$

Proof. — Again using the determinantal point process kernel in (2.1),

$$\begin{aligned} \mathbb{E}\mathcal{N}(\mathbb{C} \setminus RD) &= \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{1}{k!} \int_{\mathbb{C} \setminus \sqrt{n}RD} e^{-|z|^2} |z|^{2k} dz \\ &= \sum_{k=0}^{n-1} \frac{1}{k!} \int_{nR^2}^{\infty} r^k e^{-r} dr \\ &= e^{-nR^2} \sum_{k=0}^{n-1} \sum_{\ell=0}^k \frac{(nR^2)^\ell}{\ell!} \\ &= e^{-nR^2} \sum_{\ell=0}^{n-1} \frac{(nR^2)^\ell}{\ell!} (n - \ell) \\ &= ne^{-nR^2} \left(\sum_{\ell=0}^{n-1} \frac{(nR^2)^\ell}{\ell!} - R^2 \sum_{\ell=0}^{n-2} \frac{(nR^2)^\ell}{\ell!} \right) \\ &= ne^{-nR^2} \left(\frac{(nR^2)^{n-1}}{(n-1)!} - (R^2 - 1) \sum_{\ell=0}^{n-2} \frac{(nR^2)^\ell}{\ell!} \right) \\ &\leq ne^{-nR^2} \frac{(nR^2)^{n-1}}{(n-1)!} \\ &\leq \frac{1}{\sqrt{2\pi}} e^{-nR^2} \sqrt{n} e^n R^{2(n-1)} \end{aligned}$$

by Stirling's approximation. □

PROPOSITION 3.3. — *For each $1 \leq j \leq \sqrt{n} - 1$ and $0 \leq \theta \leq 2\pi$,*

$$\text{Var} \mathcal{N}(A_{j,\theta}) \leq 16j.$$

The constant 16 in the statement of Proposition 3.3 is not optimal and is included only for the sake of concreteness.

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Proof. — By an argument in [8, Appendix B],

$$\begin{aligned}
\text{Var}(\mathcal{N}(A_{j,\theta})) &= \int_{\{|z|\leq j\}} \int_{\{|w|\geq j+1\}} |K(z,w)|^2 dw dz \\
&+ \int_{\{|z|\leq j\}} \int_{\{j\leq |w|\leq j+1, \arg w \geq \theta\}} |K(z,w)|^2 dw dz \\
&+ \int_{\{j\leq |z|\leq j+1, \arg z \leq \theta\}} \int_{\{|w|\geq j+1\}} |K(z,w)|^2 dw dz \\
&+ \int_{\{j\leq |z|\leq j+1, \arg z \leq \theta\}} \int_{\{j\leq |w|\leq j+1, \arg w \geq \theta\}} |K(z,w)|^2 dw dz
\end{aligned} \tag{3.1}$$

Observe that

$$|K(r_1 e^{i\varphi_1}, r_2 e^{i\varphi_2})|^2 = \frac{1}{\pi^2} \sum_{k,\ell=0}^{n-1} \frac{1}{k!\ell!} e^{-(r_1^2+r_2^2)} (r_1 r_2)^{k+\ell} e^{i(k-\ell)(\varphi_1-\varphi_2)}.$$

Integrating in polar coordinates, the first integral in (3.1) is

$$\begin{aligned}
&\frac{1}{\pi^2} \sum_{k,\ell=0}^{n-1} \frac{1}{k!\ell!} \int_0^j r^{k+\ell+1} e^{-r^2} dr \int_{j+1}^\infty r^{k+\ell+1} e^{-r^2} dr \int_0^{2\pi} e^{i\varphi(k-\ell)} d\varphi \\
&\hspace{20em} \int_0^{2\pi} e^{i\varphi(\ell-k)} d\varphi \\
&= \sum_{k=0}^{n-1} \frac{1}{k!^2} \int_0^{j^2} s^k e^{-s} ds \int_{(j+1)^2}^\infty s^k e^{-s} ds \\
&\leq \sum_{k=0}^{j^2} \frac{1}{k!} \int_{j^2}^\infty s^k e^{-s} ds + \sum_{k=(j+1)^2}^{n-1} \frac{1}{k!} \int_0^{(j+1)^2} s^k e^{-s} ds + (2j+1).
\end{aligned}$$

Here we have used that the angular integrals are nonzero only if $k = \ell$, and that the integrals in the second line are bounded by $k!$. Note also that if $j^2 < n-1 < (j+1)^2$, the second term is not needed, and if $j^2 \geq n-1$, then the second and third terms are not needed. By Lemma 2.3 and Stirling's approximation,

$$\begin{aligned}
 \sum_{k=0}^{j^2} \frac{1}{k!} \int_{j^2}^{\infty} s^k e^{-s} ds &= \sum_{k=0}^{j^2} e^{-j^2} \sum_{\ell=0}^k \frac{j^{2\ell}}{\ell!} \\
 &= e^{-j^2} \sum_{\ell=0}^{j^2} \sum_{k=\ell}^{j^2} \frac{j^{2\ell}}{\ell!} = e^{-j^2} \sum_{\ell=0}^{j^2} \frac{j^{2\ell}}{\ell!} (j^2 - \ell + 1) \\
 &\leq 1 + e^{-j^2} \left(\sum_{\ell=0}^{j^2} \frac{j^{2(\ell+1)}}{\ell!} - \sum_{\ell=1}^{j^2} \frac{j^{2\ell}}{(\ell-1)!} \right) \\
 &= 1 + e^{-j^2} \frac{j^{2(j^2+1)}}{(j^2)!} \leq 1 + \frac{j}{\sqrt{2\pi}},
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{k=(j+1)^2}^{n-1} \frac{1}{k!} \int_0^{(j+1)^2} s^k e^{-s} ds &= e^{-(j+1)^2} \sum_{k=(j+1)^2}^{n-1} \sum_{\ell=k+1}^{\infty} \frac{(j+1)^{2\ell}}{\ell!} \\
 &= e^{-(j+1)^2} \sum_{\ell=(j+1)^2+1}^{\infty} \frac{(j+1)^{2\ell}}{\ell!} (\ell - (j+1)^2) \\
 &= e^{-(j+1)^2} \left(\sum_{\ell=(j+1)^2+1}^{\infty} \frac{(j+1)^{2\ell}}{(\ell-1)!} - \sum_{\ell=(j+1)^2+1}^{\infty} \frac{(j+1)^{2(\ell+1)}}{\ell!} \right) \\
 &= e^{-(j+1)^2} \frac{(j+1)^{2((j+1)^2+1)}}{(j+1)^{2!}} \leq \frac{j+1}{\sqrt{2\pi}}.
 \end{aligned}$$

The second integral in (3.1) is equal to

$$\begin{aligned}
 \frac{1}{\pi^2} \sum_{k,\ell=0}^{n-1} \frac{1}{k!\ell!} \int_0^j r^{k+\ell+1} e^{-r^2} dr \int_j^{j+1} r^{k+\ell+1} e^{-r^2} dr \int_0^{2\pi} e^{i\varphi(k-\ell)} \\
 d\varphi \int_{\theta}^{2\pi} e^{i\varphi(\ell-k)} d\varphi \\
 = \left(1 - \frac{\theta}{2\pi}\right) \sum_{k=0}^{n-1} \left(\frac{1}{k!} \int_0^{j^2} s^k e^{-s} ds \right) \left(\frac{1}{k!} \int_{j^2}^{(j+1)^2} s^k e^{-s} ds \right) \\
 \leq \left(1 - \frac{\theta}{2\pi}\right) \sum_{k=0}^{n-1} \frac{1}{k!} \int_{j^2}^{(j+1)^2} s^k e^{-s} ds
 \end{aligned}$$

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since the first angular integral is nonzero only for $k = \ell$, and the third integral in (3.1) is similarly bounded by

$$\frac{\theta}{2\pi} \sum_{k=0}^{n-1} \frac{1}{k!} \int_{j^2}^{(j+1)^2} s^k e^{-s} ds.$$

The function $s \mapsto s^k e^{-s}$ is unimodal for $s > 0$ and takes on its maximum value at $s = k$, so

$$\int_{j^2}^{(j+1)^2} s^k e^{-s} ds \leq \begin{cases} (2j+1)j^{2k} e^{-j^2} & \text{when } k \leq j^2, \\ (2j+1)(j+1)^{2k} e^{-(j+1)^2} & \text{when } k \geq (j+1)^2, \text{ and} \\ k! & \text{always.} \end{cases}$$

From this it follows that the sum of the second and third integrals in (3.1) is bounded by

$$\sum_{k=0}^{n-1} \frac{1}{k!} \int_{j^2}^{(j+1)^2} s^k e^{-s} ds \leq 3(2j+1). \quad (3.2)$$

The final integral in (3.1) is equal to

$$\frac{1}{\pi^2} \sum_{k,\ell=0}^{n-1} \frac{1}{k!\ell!} \left(\int_j^{j+1} r^{k+\ell+1} e^{-r^2} dr \right)^2 \int_0^\theta e^{i\varphi(k-\ell)} d\varphi \int_\theta^{2\pi} e^{i\varphi(\ell-k)} d\varphi. \quad (3.3)$$

For $k \neq \ell$,

$$\int_\theta^{2\pi} e^{i\varphi(\ell-k)} d\varphi = - \int_0^\theta e^{i\varphi(\ell-k)} d\varphi = - \overline{\int_0^\theta e^{i\varphi(k-\ell)} d\varphi}$$

so each summand in (3.3) with $k \neq \ell$ is negative. Thus (3.3) is bounded by

$$\begin{aligned} & \frac{\theta(2\pi-\theta)}{\pi^2} \sum_{k=0}^{n-1} \left(\frac{1}{k!} \int_j^{j+1} r^{2k+1} e^{-r^2} dr \right)^2 \\ &= \frac{\theta}{2\pi} \left(1 - \frac{\theta}{2\pi} \right) \sum_{k=0}^{n-1} \left(\frac{1}{k!} \int_{j^2}^{(j+1)^2} s^k e^{-s} ds \right)^2, \end{aligned}$$

which by (3.2) is less than $\frac{3}{4}(2j+1)$. \square

4. Deviations

The goal of this section is to obtain sharp concentration results for the eigenvalues λ_k about their predicted locations $\tilde{\lambda}_k$. Recall that we only defined $\tilde{\lambda}_k$ for a restricted range of k ; for the outermost eigenvalues, for which we did not define $\tilde{\lambda}_k$, we will make use of the following sloppy estimate.

LEMMA 4.1. — *For each k and any random variable $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$,*

$$\mathbb{E} |\lambda_k - \alpha|^p \leq 4^p + \left(\frac{4}{3}\right)^{p-1} \left(\frac{2}{n}\right)^{\frac{p}{2}} \Gamma\left(1 + \frac{p}{2}\right).$$

and

$$\mathbb{P}[|\lambda_k - \alpha| \geq t] \leq e^{-nt^2/4}$$

for $t > 4$.

Proof. — For any $t \geq 1$,

$$\mathbb{P}[|\lambda_k - \alpha| \geq t] \leq \mathbb{P}[|\lambda_k| \geq t - 1] \leq \mathbb{P}[\mathcal{N}(\mathbb{C} \setminus (t - 1)D) \geq 1] \leq \mathbb{E}\mathcal{N}(\mathbb{C} \setminus (t - 1)D), \quad (4.1)$$

by Markov's inequality. Proposition 3.2 implies that for any $R > 1$,

$$\mathbb{E}\mathcal{N}(\mathbb{C} \setminus RD) \leq \exp\left[-\left(R^2 - \frac{3}{2} - 2\log R\right)n\right],$$

since $\log n \leq n$. For $R > 3$,

$$R^2 - \frac{3}{2} - 2\log R > \frac{R^2}{2},$$

and so

$$\mathbb{E}\mathcal{N}(\mathbb{C} \setminus RD) \leq e^{-nR^2/2}. \quad (4.2)$$

Combining (4.1) and (4.2), we obtain

$$\begin{aligned} \mathbb{E} |\lambda_k - \alpha|^p &= \int_0^\infty pt^{p-1} \mathbb{P}[|\lambda_k - \alpha| \geq t] dt \\ &\leq \int_0^4 pt^{p-1} dt + \int_4^\infty pt^{p-1} e^{-n(t-1)^2/2} dt \\ &\leq 4^p + \left(\frac{4}{3}\right)^{p-1} p \int_3^\infty s^{p-1} e^{-ns^2/2} ds \\ &\leq 4^p + \left(\frac{4}{3}\right)^{p-1} \left(\frac{2}{n}\right)^{\frac{p}{2}} \Gamma\left(1 + \frac{p}{2}\right) \quad \square \end{aligned}$$

and

$$\mathbb{P}[|\lambda_k - \alpha| \geq t] \leq e^{-n(t-1)^2/2} \leq e^{-nt^2/4}$$

for $t > 4$.

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We will need stronger concentration for most of the eigenvalues, which we get as a consequence of the following.

PROPOSITION 4.2. — *For each $1 \leq j \leq \sqrt{n-1} - 1$, $0 \leq \theta \leq 2\pi$, and $t > 0$,*

$$\mathbb{P} \left[\mathcal{N}(A_{j,\theta}) - \frac{n|A_{j,\theta}|}{\pi} \geq t \right] \leq \exp \left[-\min \left\{ \frac{t^2}{64j}, \frac{t}{2} \right\} \right].$$

If $j \leq \sqrt{n} - \sqrt{\log n} - 1$, then

$$\mathbb{P} \left[\frac{n|A_{j,\theta}|}{\pi} - \mathcal{N}(A_{j,\theta}) \geq t \right] \leq 3 \exp \left[-\min \left\{ \frac{t^2}{256j}, \frac{t}{4} \right\} \right].$$

Proof. — The first claim follows immediately from Propositions 2.2, 3.1, and 3.3. For the second, the assumption on j implies that $A_{j,\theta} \subseteq \left(1 - \sqrt{\frac{\log n}{n}}\right) D$, and so by Propositions 2.2, 3.1, and 3.3,

$$\begin{aligned} \mathbb{P} \left[\frac{n|A_{j,\theta}|}{\pi} - \mathcal{N}(A_{j,\theta}) \geq t \right] &\leq \mathbb{P} \left[\mathbb{E}\mathcal{N}(A_{j,\theta}) - \mathcal{N}(A_{j,\theta}) \geq t - e^2 \right] \\ &\leq \exp \left[-\min \left\{ \frac{(t - e^2)^2}{64j}, \frac{t - e^2}{2} \right\} \right] \end{aligned}$$

for $t > e^2$. If $t \geq 2e^2$, then $t - e^2 \geq t/2$, so

$$\mathbb{P} \left[\mathbb{E}\mathcal{N}(A_{j,\theta}) - \mathcal{N}(A_{j,\theta}) \geq t \right] \leq \exp \left[-\min \left\{ \frac{t^2}{256j}, \frac{t}{4} \right\} \right].$$

On the other hand, if $t < 2e^2$, then

$$\exp \left[-\min \left\{ \frac{t^2}{256j}, \frac{t}{4} \right\} \right] > e^{-e^4/64} > 1/3,$$

which implies the second claim. \square

The concentration inequalities for the $\mathcal{N}(A_{j,\theta})$ together with geometric arguments yield the following concentration for individual eigenvalues.

THEOREM 4.3. — *There are constants $C, c > 0$ such that for those k with $\ell = \lceil \sqrt{k} \rceil \leq \sqrt{n} - \sqrt{\log n}$,*

- *when $9 \leq s \leq \pi(\ell - 1) + 2$,*

$$\mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| > \frac{s}{\sqrt{n}} \right] \leq C \exp \left[-\min \left\{ \frac{(s-9)^2}{256\pi^2(\ell-1)}, \frac{s-9}{4\pi} \right\} \right];$$

- when $s > \pi(\ell - 1) + 2$,

$$\mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| > \frac{s}{\sqrt{n}} \right] \leq C e^{-cs^2}.$$

Proof. — Trivially,

$$\mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| \geq t \right] = \mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| \geq t \text{ and } \lambda_k \prec \tilde{\lambda}_k \right] + \mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| \geq t \text{ and } \lambda_k \succ \tilde{\lambda}_k \right].$$

Case 1: $\lambda_k \prec \tilde{\lambda}_k$.

With probability 1, $\lambda_k \prec \tilde{\lambda}_k$ implies that either

$$\frac{\ell - 1}{\sqrt{n}} \leq |\lambda_k| < \frac{\ell}{\sqrt{n}} \quad \text{and} \quad \arg \lambda_k < \arg \tilde{\lambda}_k = \frac{2\pi q}{2\ell - 1}.$$

or

$$|\lambda_k| < \left| \tilde{\lambda}_k \right| = \frac{\ell - 1}{\sqrt{n}}.$$

Observe that, in either case, $\left| \lambda_k - \tilde{\lambda}_k \right| < \frac{2\ell - 1}{\sqrt{n}}$.

If $\left| \lambda_k - \tilde{\lambda}_k \right| \geq s/\sqrt{n}$ and $a(\theta, \varphi)$ denotes the length of the shorter arc on the unit circle between $e^{i\theta}$ and $e^{i\varphi}$, then the elementary estimate

$$\left| R e^{i\theta} - r e^{i\varphi} \right| \leq r \cdot a(\theta, \varphi) + |R - r|,$$

implies that when $|\lambda_k| \in \left[\frac{\ell - 2}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right)$ and $s \geq 1$,

$$a \left(\arg \lambda_k, \frac{2\pi q}{2\ell - 1} \right) \geq \frac{s - 1}{\ell - 1}. \quad (4.3)$$

Suppose that $\frac{s-1}{\ell-1} < \frac{2\pi q}{2\ell-1}$. Since either $|\lambda_k| < \frac{\ell-1}{\sqrt{n}}$ or $\arg \lambda_k < \frac{2\pi q}{2\ell-1}$, Inequality (4.3) implies that

$$\lambda_k \prec \frac{\ell - 1}{\sqrt{n}} \exp \left[i \left(\frac{2\pi q}{2\ell - 1} - \frac{s - 1}{\ell - 1} \right) \right],$$

and so

$$\mathcal{N} \left(A_{\ell-1, \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}} \right) \geq k = (\ell - 1)^2 + q.$$

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Since

$$\frac{n}{\pi} |A_{j,\theta}| = j^2 + \frac{\theta}{2\pi}(2j+1),$$

we have

$$\frac{n}{\pi} \left| A_{\ell-1, \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}} \right| = k - \frac{2\ell-1}{2\pi(\ell-1)}(s-1) \leq k - \frac{s-1}{\pi},$$

and so Proposition 4.2 implies that

$$\begin{aligned} & \mathbb{P} \left[\mathcal{N}(A_{\ell-1, \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}}) \geq k \right] \\ & \leq \mathbb{P} \left[\mathcal{N}(A_{\ell-1, \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}}) - \frac{n}{\pi} \left| A_{\ell-1, \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}} \right| \geq \frac{s-1}{\pi} \right] \\ & \leq \exp \left[- \min \left\{ \frac{(s-1)^2}{64\pi^2(\ell-1)}, \frac{s-1}{2\pi} \right\} \right]. \end{aligned}$$

Now suppose that $\frac{2\pi q}{2\ell-1} \leq \frac{s-1}{\ell-1} \leq \pi$ (note that $\frac{s-1}{\ell-1}$ is a lower bound for the length of a shortest path on the circle, hence the upper bound of π , and that the interval in question is non-empty only if $q \leq \frac{2\ell-1}{2}$). Then

$$\lambda_k \prec \frac{\ell-2}{\sqrt{n}} \exp \left[i \left(2\pi + \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1} \right) \right],$$

and so

$$\mathcal{N}(A_{\ell-2, 2\pi + \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}}) \geq k.$$

Now

$$\frac{n}{\pi} \left| A_{\ell-2, 2\pi + \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}} \right| \leq k - \frac{2\ell-3}{2\pi(\ell-1)}(s-1) \leq k - \frac{s-1}{2\pi}$$

for $\ell \geq 2$. Thus in this range of s Proposition 4.2 implies that

$$\begin{aligned} & \mathbb{P} \left[\mathcal{N}(A_{\ell-2, 2\pi + \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}}) \geq k \right] \\ & \leq \mathbb{P} \left[\mathcal{N}(A_{\ell-2, 2\pi + \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}}) - \frac{n}{\pi} \left| A_{\ell-2, 2\pi + \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}} \right| \geq \frac{s-1}{2\pi} \right] \\ & \leq \exp \left[- \min \left\{ \frac{(s-1)^2}{256\pi^2(\ell-1)}, \frac{s-1}{4\pi} \right\} \right]. \end{aligned}$$

As observed above, the estimates above cover the entire possible range of s , and so

$$\mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| \geq \frac{s}{\sqrt{n}} \text{ and } \lambda_k \prec \tilde{\lambda}_k \right] \leq \exp \left[- \min \left\{ \frac{(s-1)^2}{256\pi^2(\ell-1)}, \frac{s-1}{4\pi} \right\} \right]$$

for all $s \geq 1$.

Case 2: $\lambda_k \succ \tilde{\lambda}_k$.

With probability 1, $\lambda_k \succ \tilde{\lambda}_k$ implies that either

$$\frac{\ell-1}{\sqrt{n}} \leq |\lambda_k| < \frac{\ell}{\sqrt{n}} \quad \text{and} \quad \arg \lambda_k > \arg \tilde{\lambda}_k = \frac{2\pi q}{2\ell-1}.$$

or

$$|\lambda_k| \geq \frac{\ell}{\sqrt{n}}.$$

Observe that if $|\lambda_k - \tilde{\lambda}_k| \geq s/\sqrt{n}$ and $|\lambda_k| \in \left[\frac{\ell-1}{\sqrt{n}}, \frac{\ell+1}{\sqrt{n}}\right)$, then as above,

$$a\left(\arg \lambda_k, \frac{2\pi q}{2\ell-1}\right) \geq \frac{s-2}{\ell-1} \quad (4.4)$$

for $s \geq 2$. We will need to make different arguments depending on the value of $\frac{s-2}{\ell-1}$.

1. Suppose first that $\frac{s-2}{\ell-1} < 2\pi - \frac{2\pi q}{2\ell-1}$.

Since either $|\lambda_k| \geq \frac{\ell}{\sqrt{n}}$ or $\arg \lambda_k > \frac{2\pi q}{2\ell-1}$, Inequality (4.4) implies that

$$\lambda_k \succ \frac{\ell-1}{\sqrt{n}} \exp\left[i\left(\frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1}\right)\right],$$

and so

$$\mathcal{N}(A_{\ell-1, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1}}) < k.$$

Since

$$\frac{n}{\pi} \left| A_{\ell-1, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1}} \right| = k + \frac{2\ell-1}{2\pi(\ell-1)}(s-2) \geq k + \frac{s-2}{\pi},$$

Proposition 4.2 implies that

$$\begin{aligned} & \mathbb{P}\left[\mathcal{N}(A_{\ell-1, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1}}) < k\right] \\ & \leq \mathbb{P}\left[\frac{n}{\pi} \left| A_{\ell-1, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1}} \right| - \mathcal{N}(A_{\ell-1, \frac{2\pi q}{2\ell-1} - \frac{s-1}{\ell-1}}) > \frac{s-2}{\pi}\right] \\ & \leq 3 \exp\left[-\min\left\{\frac{(s-2)^2}{64\pi^2(\ell-1)}, \frac{s-2}{2\pi}\right\}\right]. \end{aligned}$$

2. Next suppose that $2\pi - \frac{2\pi q}{2\ell-1} \leq \frac{s-2}{\ell-1} \leq \pi$ (note that this case only occurs when $q \geq \frac{2\ell-1}{2}$). Then we have that

$$\lambda_k \succ \frac{\ell}{\sqrt{n}} \exp\left[i\left(\frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1} - 2\pi\right)\right],$$

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and so

$$\mathcal{N}(A_{\ell, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1} - 2\pi}) < k.$$

Now

$$\frac{n}{\pi} \left| A_{\ell, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1} - 2\pi} \right| \geq k + \frac{2\ell+1}{2\pi(\ell-1)}(s-2) - 2 \geq k + \frac{s-2}{\pi} - 2 \geq k + \frac{s-9}{\pi}$$

for $s \geq 9$. Thus in this range Proposition 4.2 implies that

$$\begin{aligned} & \mathbb{P} \left[\mathcal{N}(A_{\ell, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1} - 2\pi) < k \right] \\ & \leq \mathbb{P} \left[\frac{n}{\pi} \left| A_{\ell, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1} - 2\pi} \right| - \mathcal{N}(A_{\ell, \frac{2\pi q}{2\ell-1} + \frac{s-2}{\ell-1} - 2\pi) > \frac{s-9}{\pi} \right] \\ & \leq 3 \exp \left[- \min \left\{ \frac{(s-9)^2}{64\pi^2(\ell-1)}, \frac{s-9}{2\pi} \right\} \right]. \end{aligned}$$

3. Now suppose that $\pi < \frac{s-2}{\ell-1} \leq \frac{\sqrt{n} - \sqrt{\log n} + \ell - 3}{\ell-1}$.

By the triangle inequality, $|\lambda_k| \geq \frac{s}{\sqrt{n}} - \left| \tilde{\lambda}_k \right| = \frac{s-\ell+1}{\sqrt{n}}$, and so

$$\mathcal{N} \left(\frac{s-\ell+1}{\sqrt{n}} D \right) = \mathcal{N}(A_{s-\ell, 2\pi}) < k.$$

The inequality $\frac{s-2}{\ell-1} \leq \frac{\sqrt{n} - \sqrt{\log n} + \ell - 3}{\ell-1}$ is equivalent to $s-\ell \leq \sqrt{n} - \sqrt{\log n} - 1$, and so the second estimate of Proposition 4.2 applies. Since $k = (\ell-1)^2 + q$ and $1 \leq q \leq 2\ell-1 \leq \frac{2(s-2)}{\pi} + 1$,

$$\begin{aligned} \frac{n}{\pi} |A_{s-\ell, 2\pi}| &= (s-\ell+1)^2 \\ &= s^2 - 2s(\ell-1) + k - q \geq s^2 \left(1 - \frac{2}{\pi} \right) - \frac{6s}{\pi} - \frac{4}{\pi} - 1 + k, \end{aligned}$$

and so

$$\begin{aligned} & \mathbb{P} \left[\mathcal{N}(A_{s-\ell, 2\pi}) < k \right] \\ & \leq \mathbb{P} \left[\frac{n}{\pi} |A_{s-\ell, 2\pi}| - \mathcal{N}(A_{s-\ell, 2\pi}) > \frac{n}{\pi} |A_{s-\ell, 2\pi}| - k \right] \\ & \leq \mathbb{P} \left[\frac{n}{\pi} |A_{s-\ell, 2\pi}| - \mathcal{N}(A_{s-\ell, 2\pi}) > s^2 \left(1 - \frac{2}{\pi} \right) + \frac{6s}{\pi} - \frac{4}{\pi} - 1 \right] \\ & \leq 3 \exp \left[- \min \left\{ \frac{s^3}{2304}, \frac{s^2}{12} \right\} \right], \end{aligned}$$

for $s \geq 9$.

4. Finally, suppose that $\frac{s-2}{\ell-1} > \frac{\sqrt{n}-\sqrt{\log n}+\ell-3}{\ell-1}$; that is, that $s - \ell > \sqrt{n} - \sqrt{\log n} - 1$. As in the previous case, $\lambda > \tilde{\lambda}_k$ and $|\lambda_k - \tilde{\lambda}_k| > \frac{s}{\sqrt{n}}$ implies that

$$\mathcal{N}\left(\frac{s-\ell+1}{\sqrt{n}}D\right) = \mathcal{N}(A_{s-\ell,2\pi}) < k,$$

but the second inequality of Proposition 4.2 does not apply. If $\frac{s-\ell+1}{\sqrt{n}} \geq 1$, then $\mathbb{P}[\mathcal{N}(A_{s-\ell,2\pi}) < k] = 0$; otherwise, one can use the weaker estimate of Proposition 3.1 for $\mathbb{E}\mathcal{N}(A_{s-\ell,2\pi})$ to get that

$$\begin{aligned} & \mathbb{P}[\mathcal{N}(A_{s-\ell,2\pi}) < k] \\ & \leq \mathbb{P}\left[\mathbb{E}\mathcal{N}(A_{s-\ell,2\pi}) - \mathcal{N}(A_{s-\ell,2\pi}) > \frac{n}{\pi}|A_{s-\ell,2\pi}| - e\sqrt{n} - k\right] \\ & \leq \mathbb{P}\left[\mathbb{E}\mathcal{N}(A_{s-\ell,2\pi}) - \mathcal{N}(A_{s-\ell,2\pi}) > s^2\left(1 - \frac{2}{\pi}\right) + \frac{6s}{\pi} - \frac{4}{\pi} - 1 - e\sqrt{n}\right] \end{aligned}$$

Since $s \geq \sqrt{n} - \sqrt{\log n}$, the lower bound above can be replaced, for n large enough, by cs^2 for any $c < \left(1 - \frac{2}{\pi}\right)$. Applying Bernstein's inequality and the variance estimate of Proposition 3.3 then yields

$$\mathbb{P}[\mathcal{N}(A_{s-\ell,2\pi}) < k] \leq C \exp\left[-\min\left\{c^2s^3, \frac{cs^2}{2}\right\}\right]. \quad \square$$

5. Distances in the circular law

In this section, we assemble the previous results to give quantitative versions of the circular law. We first note that our estimates for the means of the eigenvalue counting functions for balls already yield the correct order for the total variation distance between the averaged empirical spectral measure and the uniform measure on the disc. The fact that the mean spectral measure $\mathbb{E}\mu_n$ converges to the uniform measure ν in total variation can be deduced from Mehta's work [13, Chapter 15]. We would not be surprised to learn that the correct rate of convergence is known, but we have not found it in the literature.

PROPOSITION 5.1. — *For each positive integer n , $\frac{1}{e\sqrt{n}} \leq d_{TV}(\nu, \mathbb{E}\mu_n) \leq \frac{e}{\sqrt{n}}$.*

Proof. — For any Borel set $A \subseteq \mathbb{C}$, Proposition 3.1 implies that

$$\nu(A) - \frac{e}{\sqrt{n}} \leq \mathbb{E}\mu_n(A \cap D) \leq \nu(A),$$

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so

$$\nu(A) - \mathbb{E}\mu_n(A) \leq \nu(A) - \mathbb{E}\mu_n(A \cap D) \leq \frac{e}{\sqrt{n}}.$$

Furthermore,

$$\mathbb{E}\mu_n(\mathbb{C} \setminus D) = 1 - \mathbb{E}\mu_n(D) = \nu(D) - \mathbb{E}\mu_n(D) \leq \frac{e}{\sqrt{n}},$$

so

$$\mathbb{E}\mu_n(A) - \nu(A) \leq \mathbb{E}\mu_n(\mathbb{C} \setminus D) + \mathbb{E}\mu_n(A \cap D) - \nu(A) \leq \frac{e}{\sqrt{n}}.$$

Thus $d_{TV}(\nu, \mathbb{E}\mu_n) = \sup_A |\nu(A) - \mathbb{E}\mu_n(A)| \leq \frac{e}{\sqrt{n}}$.

On the other hand, the proof of Proposition 3.2 implies that

$$\mathbb{E}\mu_n(\mathbb{C} \setminus D) = \frac{e^{-n} n^n}{n!} \geq \frac{1}{e\sqrt{n}}$$

by Stirling's approximation. Since $\nu(\mathbb{C} \setminus D) = 0$, this provides the lower bound. \square

The deviation estimates in the previous section allow us to finally establish the stronger version of the circular law given in Theorem 1.1, via the following proposition.

PROPOSITION 5.2. — *For any positive integers $m \leq n$ and any $p \geq 1$,*

$$\mathbb{E}W_p(\mu_n, \nu) < \frac{8}{\sqrt{n}} + 2 \max_{1 \leq k \leq n-m} \left(\mathbb{E} \left| \lambda_k - \tilde{\lambda}_k \right|^p \right)^{1/p} + C \left(4 + \sqrt{\frac{p}{n}} \right) \left(\frac{m}{n} \right)^{1/p}.$$

Proof. — By Lemma 4.1,

$$\begin{aligned} \mathbb{E}W_p(\mu_n, \nu)^p &\leq \frac{1}{n} \left(\sum_{k=1}^{n-m} \mathbb{E} \left| \lambda_k - \tilde{\lambda}_k \right|^p + \sum_{k=n-m+1}^n \mathbb{E} \left| \lambda_k - u \right|^p \right) \\ &\leq \max_{1 \leq k \leq n-m} \mathbb{E} \left| \lambda_k - \tilde{\lambda}_k \right|^p + \left(4^p + \left(\frac{32}{9n} \right)^{\frac{p}{2}} \Gamma \left(1 + \frac{p}{2} \right) \right) \frac{m}{n}, \end{aligned}$$

where u is uniform in the outer part of the disc and independent of λ_k . Lemma 2.1 and the triangle inequality for W_p imply that

$$\begin{aligned} \mathbb{E}W_p(\mu_n, \nu) &< \frac{8}{\sqrt{n}} + 2 \max_{1 \leq k \leq n-m} \left(\mathbb{E} \left| \lambda_k - \tilde{\lambda}_k \right|^p \right)^{1/p} \\ &\quad + \left(4 + \sqrt{\frac{32}{9n}} \Gamma \left(1 + \frac{p}{2} \right)^{1/p} \right) \left(\frac{m}{n} \right)^{1/p}, \end{aligned}$$

and the proposition follows from Stirling's approximation. \square

Proof of Theorem 1.1. — Let m be such that $n - m$ is a perfect square, and such that if $1 \leq k \leq n - m$, then $\ell = \lceil \sqrt{k} \rceil \leq \sqrt{n} - \sqrt{\log n}$. By Fubini's theorem and Corollary 4.3, for $1 \leq k \leq n - m$,

$$\begin{aligned}
 \mathbb{E} \left| \lambda_k - \tilde{\lambda}_k \right|^p &= \int_0^\infty p t^{p-1} \mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| > t \right] dt \\
 &= \frac{p}{n^{p/2}} \int_0^\infty s^{p-1} \mathbb{P} \left[\left| \lambda_k - \tilde{\lambda}_k \right| > \frac{s}{\sqrt{n}} \right] ds \\
 &\leq \frac{p}{n^{p/2}} \left(\frac{9^p}{p} + \int_9^{2+\pi(\ell-1)} C s^{p-1} \exp \left[-\frac{(s-9)^2}{256\pi^2(\ell-1)} \right] ds \right. \\
 &\quad \left. + \int_{2+\pi(\ell-1)}^\infty C s^{p-1} e^{-cs^2} ds \right) \\
 &\leq \frac{p}{n^{p/2}} \left[\frac{9^p}{p} + \int_0^\infty C s^{p-1} e^{-\frac{cs^2}{\ell-1}} ds \right] \\
 &\leq \frac{p C^p (\ell-1)^{p/2} \Gamma \left(\frac{p+1}{2} \right)}{n^{p/2}} \\
 &\leq \frac{p C^p \Gamma \left(\frac{p+1}{2} \right)}{n^{p/4}},
 \end{aligned}$$

since $\ell \leq \sqrt{n}$.

Noting that we can take $m \leq c\sqrt{n \log n}$, it follows from Proposition 5.2 that

$$\begin{aligned}
 \mathbb{E} W_p(\mu_n, \nu) &\leq \frac{8}{\sqrt{n}} + 2 \frac{C \Gamma \left(\frac{p+1}{2} \right)^{\frac{1}{p}}}{n^{1/4}} + C \left(4 + \sqrt{\frac{p}{n}} \right) \left(\frac{m}{n} \right)^{\frac{1}{p}} \\
 &\leq C \max \left\{ \frac{\sqrt{p}}{n^{1/4}}, \left(\frac{\log n}{n} \right)^{\frac{1}{2p}} \right\}.
 \end{aligned}$$

□

Proof of Theorem 1.2. — By Lemma 2.1, up to the value of absolute constants it suffices to prove the theorem with ν_n in place of ν . Let m be as in the proof of Theorem 1.1. For any $t > 0$,

$$\begin{aligned}
\mathbb{P}[W_p(\mu_n, \nu_n) > t] &\leq \mathbb{P}\left[\frac{1}{n} \sum_{k=1}^n \left|\lambda_k - \tilde{\lambda}_k\right|^p > t^p\right] \\
&\leq \mathbb{P}\left[\sum_{k=1}^{n-m} \left|\lambda_k - \tilde{\lambda}_k\right|^p > \frac{nt^p}{2}\right] \\
&\quad + \mathbb{P}\left[\sum_{k=n-m+1}^n \left|\lambda_k - \tilde{\lambda}_k\right|^p > \frac{nt^p}{2}\right] \\
&\leq \sum_{k=1}^{n-m} \mathbb{P}\left[\left|\lambda_k - \tilde{\lambda}_k\right|^p > \frac{nt^p}{2(n-m)}\right] \\
&\quad + \sum_{k=n-m+1}^n \mathbb{P}\left[\left|\lambda_k - \tilde{\lambda}_k\right|^p > \frac{nt^p}{2m}\right] \\
&\leq \sum_{k=1}^{n-m} \mathbb{P}\left[\left|\lambda_k - \tilde{\lambda}_k\right| > \frac{t}{2}\right] \\
&\quad + \sum_{k=n-m+1}^n \mathbb{P}\left[\left|\lambda_k - \tilde{\lambda}_k\right| > \left(\frac{n}{m}\right)^{1/p} \frac{t}{2}\right].
\end{aligned}$$

Suppose first that $1 \leq p \leq 2$. If $K > 0$ is large enough, then for sufficiently large n , Theorem 4.3 implies that for $1 \leq k \leq n - m$,

$$\mathbb{P}\left[\left|\lambda_k - \tilde{\lambda}_k\right| > K \frac{\sqrt{\log n}}{n^{1/4}}\right] \leq \frac{1}{n^3}.$$

Moreover, for $k > n - m$, Lemma 4.1 implies that for sufficiently large n ,

$$\mathbb{P}\left[\left|\lambda_k - \tilde{\lambda}_k\right| > K \left(\frac{n}{m}\right)^{1/p} \frac{\sqrt{\log n}}{n^{1/4}}\right] \leq e^{-cn}.$$

It follows that

$$\sum_{n=1}^{\infty} \mathbb{P}\left[W_p(\mu_n, \nu_n) > 2K \frac{\sqrt{\log n}}{n^{1/4}}\right] < \infty,$$

and an application of the Borel–Cantelli lemma completes the proof.

Now suppose that $p > 2$. If $K > 0$ is large enough, then similar arguments show that

$$\sum_{n=1}^{\infty} \mathbb{P}\left[W_p(\mu_n, \nu_n) > 2K \left(\frac{\log n}{n}\right)^{1/2p}\right] < \infty,$$

and again the proof is completed by applying the Borel–Cantelli lemma. Note that the choice of $t = K \left(\frac{\log n}{n} \right)^{1/2p}$ is dictated entirely by the fact that the tail bound in Lemma 4.1 only applies when

$$\left(\frac{n}{m} \right)^{1/p} \frac{t}{2} > 4.$$

□

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