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Arrangements and Frobenius like structures

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Abstract. — We consider a family of generic weighted arrangements of n hyperplanes in $\mathbb{C}^k$ and show that the Gauss-Manin connection for the associated hypergeometric integrals, the contravariant form on the space of singular vectors, and the algebra of functions on the critical set of the master function define a Frobenius like structure on the base of the family. As a result of this construction we show that the matrix elements of the linear operators of the Gauss-Manin connection are given by the $2k+1$-st derivatives of a single function on the base of the family, the function called the potential of second kind, see formula (6.46).

Résumé. — On considère une famille d'arrangements pondérés génériques de n hyperplans dans C^k et montre que la connexion de Gauss - Manin pour les intégrales hypergéométriques associées, la forme contravariante sur l'espace des vecteurs singuliers et l'algébre de fonctions sur l'ensemble des points critiques définissent une structure du type Frobenius sur la base de la famille. Comme un résultat de cette construction nous montrons que les éléments matriciels des opérateurs linéaires de la connexion de Gauss - Manin sont donnés par les $(2k+1)$-mes dérivées d'une seule fonction sur la base de la famille, cf. la formule (6.46).

Contents

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1. Introduction

There are three places, where a flat connection depending on a parameter appears:

• KZ equations,

$$\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z) I(z), \quad z = (z_1, \ldots, z_n), \quad i = 1, \ldots, n. \qquad (1.1)$$

Here κ is a parameter, $I(z)$ a V-valued function, where V is a vector space from representation theory, $K_i(z) : V \to V$ are linear operators, depending on z. The connection is flat for all κ.

• Quantum differential equations,

$$\kappa \frac{\partial I}{\partial z_i}(z) = p_i *_z I(z), \quad z = (z_1, \ldots, z_n), \quad i = 1, \ldots, n. \qquad (1.2)$$

Here $p_1, \ldots, p_n$ are generators of some commutative algebra H with quantum multiplication $*_z$ depending on z. These equations are part of the Frobenius structure on the quantum cohomology of a variety.

• Differential equations for hypergeometric integrals associated with a family of weighted arrangements with parallelly transported hyperplanes,

$$\kappa \frac{\partial I}{\partial z_i}(z) = K_i(z) I(z), \quad z = (z_1, \ldots, z_n), \quad i = 1, \ldots, n. \qquad (1.3)$$

It is well known that KZ equations are closely related with the differential equations for hypergeometric integrals. According to [22] the KZ equations can be presented as equations for hypergeometric integrals for suitable arrangements. Thus (1.1) and (1.3) are related. Recently it was realized that in some cases the KZ equations appear as quantum differential equations, see [2] and [11], and therefore the KZ equations are related to the Frobenius structures. On Frobenius structures see, for example, [4, 5, 14]. Hence (1.1) and (1.2) are related. In this paper I argue how a Frobenius like structure may appear on the base of a family of weighted arrangements. The goal is to make equations (1.3) related to Frobenius structures.

The main ingredients of a Frobenius structure are a flat connection depending on a parameter, a constant metric, a multiplication on tangent spaces. In our case, the connection comes from the differential equations for the associated hypergeometric integrals, the flat metric comes from the contravariant form on the space of singular vectors and the multiplication comes from the multiplication in the algebra of functions on the critical set of the master function. In this paper I consider the families of generic weighted arrangements.

The organization of the paper is as follows. In Section 2, objects associated with a weighted arrangement are recalled (Orlik-Solomon algebra, space of singular vectors, contravariant form, master function, canonical isomorphism of the space of singular vectors and the algebra of functions on the critical set of the master function). In Section 3, a family of arrangements with parallelly transported hyperplanes is considered. The construction of a Frobenius like structure on the base of the family is given. Conjectures 3.7, 3.8, 3.14 are formulated and corollaries of the conjectures are discussed. In Sections 4 and 5 the conjectures are proved for the family of points on the line and for a family of generic arrangements of lines on plane. The corresponding Frobenius like structures are described. Here are the corresponding potential functions of second kind:

$$\tilde{P}(z_1, \ldots, z_n) = \frac{1}{2} \sum_{1 \leqslant i < j \leqslant n} a_i a_j (z_i - z_j)^2 \log(z_i - z_j) \qquad (1.4)$$

for the family of arrangements of n points on line and

$$\tilde{P}(z_1, \ldots, z_n) = \frac{1}{4!} \sum_{1 \leqslant i < j < k \leqslant n} \frac{a_i a_j a_k}{d_{i,j}^2 d_{j,k}^2 d_{k,i}^2} (z_i d_{j,k} + z_j d_{k,i} + z_k d_{i,j})^4$$

$$\log(z_i d_{j,k} + z_j d_{k,i} + z_k d_{i,j}) \qquad (1.5)$$

for the family of arrangements of n generic lines on plane. The variables $z_1, \ldots, z_n$ are the parameters of the family, $a_1, \ldots, a_n$ are weights, $|a| = a_1 + \cdots + a_n$, the number $d_{k,\ell}$ is the oriented area of the parallelogram generated by the normal vectors to the k-th and ℓ-th lines, see formulas (4.36) and (6.44). Note that the potential $\tilde{P}$ from (1.4) appears in [5] for $a_1 = \cdots = a_n$ and in [19] for $a_1, \ldots, a_n \in \mathbb{Z}$.

In Section 6, the conjectures are proved for a family of generic arrangements of n hyperplanes in $\mathbb{C}^k$ for any k. The potential $\tilde{P}(z_1, \ldots, z_n)$ of second kind is defined by formula (6.44) similar to formulas (1.4) and (1.5). It is shown that the matrix elements of the operators $K_i(z_1, \ldots, z_n)$ of the Gauss-Manin connection for associated hypergeometric integrals are given by the $2k + 1$-st derivatives of the potential of second kind, see formula (6.46).

This fact that the Gauss-Manin differential equations for associated hypergeometric integrals can be described in terms of derivatives of a single function on the base of the family is an important application of our Frobenius like structure. One may expect that this is a manifestation of a much more general phenomenon.

It should be stressed that that somewhat technical constructions in Section 3 are explained in details in Sections 4, 5, and 6 for the particular situations discussed there. The reader may decide to read first the easiest Section 4.

In this paper I followed one of I.M. Gelfand's rules: for a new subject, choose the simplest nontrivial example and write down everything explicitly for this example, see the introduction to [6].

I thank V. Schechtman and V. Tarasov for useful discussions.

2. Arrangements

2.1. Affine arrangement

Let k, n be positive integers, $k < n$. Denote $J = \{1, \ldots, n\}$. Let $\mathcal{C} = (H_j)_{j \in J}$, be an arrangement of n affine hyperplanes in $\mathbb{C}^k$. Denote $U =$

$\mathbb{C}^k - \cup_{j \in J} H_j$, the complement. An edge $X_\alpha \subset \mathbb{C}^k$ of $\mathcal{C}$ is a nonempty intersection of some hyperplanes of $\mathcal{C}$. Denote by $J_\alpha \subset J$ the subset of indices of all hyperplanes containing X_α. Denote $l_\alpha = \text{codim}_{\mathbb{C}^k} X_\alpha$.

A subset $\{j_1, \ldots, j_p\} \subset J$ is called *independent* if the hyperplanes $H_{j_1}, \ldots, H_{j_p}$ intersect transversally.

We assume that $\mathcal{C}$ is essential, that is, $\mathcal{C}$ has a vertex. An edge is called *dense* if the subarrangement of all hyperplanes containing the edge is irreducible: the hyperplanes cannot be partitioned into nonempty sets so that, after a change of coordinates, hyperplanes in different sets are in different coordinates.

2.2. Orlik-Solomon algebra

Define complex vector spaces $\mathcal{A}^p(\mathcal{C})$, $p = 0, \ldots, k$. For $p = 0$ we set $\mathcal{A}^p(\mathcal{C}) = \mathbb{C}$. For $p \geqslant 1$, $\mathcal{A}^p(\mathcal{C})$ is generated by symbols $(H_{j_1}, ..., H_{j_p})$ with $j_i \in J$, such that

(i) $(H_{j_1}, ..., H_{j_p}) = 0$ if $H_{j_1}, ..., H_{j_p}$ are not in general position, that is, if the intersection $H_{j_1} \cap ... \cap H_{j_p}$ is empty or has codimension less than p;

(ii) $(H_{j_{\sigma(1)}}, ..., H_{j_{\sigma(p)}}) = (-1)^{|\sigma|}(H_{j_1}, ..., H_{j_p})$ for any element σ of the symmetric group Σ_p;

(iii) $\sum_{i=1}^{p+1} (-1)^i (H_{j_1}, ..., \widehat{H}_{j_i}, ..., H_{j_{p+1}}) = 0$ for any $(p+1)$-tuple $H_{j_1}, ..., H_{j_{p+1}}$ of hyperplanes in $\mathcal{C}$ which are not in general position and such that $H_{j_1} \cap ... \cap H_{j_{p+1}} \neq \emptyset$.

The direct sum $\mathcal{A}(\mathcal{C}) = \oplus_{p=1}^N \mathcal{A}^p(\mathcal{C})$ is the (Orlik-Solomon) algebra with respect to multiplication

$$(H_{j_1}, ..., H_{j_p}) \cdot (H_{j_{p+1}}, ..., H_{j_{p+q}}) = (H_{j_1}, ..., H_{j_p}, H_{j_{p+1}}, ..., H_{j_{p+q}}). \quad (2.1)$$

2.3. Orlik-Solomon algebra as an algebra of differential forms

For $j \in J$, fix a defining equation for the hyperplane H_j, $f_j = 0$, where f_j is a polynomial of degree one on $\mathbb{C}^k$. Consider the logarithmic differential form $\omega_j = df_j/f_j$ on $\mathbb{C}^k$. Let $\bar{\mathcal{A}}(\mathcal{C})$ be the exterior $\mathbb{C}$-algebra of differential forms generated by 1 and ω_j, $j \in J$. The map $\mathcal{A}(\mathcal{C}) \to \bar{\mathcal{A}}(\mathcal{C})$, $(H_j) \mapsto \omega_j$, is an isomorphism. We identify $\mathcal{A}(\mathcal{C})$ and $\bar{\mathcal{A}}(\mathcal{C})$.

2.4. Weights

An arrangement $\mathcal{C}$ is *weighted* if a map $a : J \to \mathbb{C}^\times$, $j \mapsto a_j$, is given; a_j is called the *weight* of H_j. For an edge X_α, define its weight $a_\alpha = \sum_{j \in J_\alpha} a_j$.

Denote $\nu(a) = \sum_{j \in J} a_j(H_j) \in \mathcal{A}^1(\mathcal{C})$. Multiplication by $\nu(a)$ defines a differential $d^{(a)} : \mathcal{A}^p(\mathcal{C}) \to \mathcal{A}^{p+1}(\mathcal{C})$, $x \mapsto \nu(a) \cdot x$, on $\mathcal{A}(\mathcal{C})$.

2.5. Space of flags, see [22]

For an edge X_α of codimension $l_\alpha = p$, a flag starting at X_α is a sequence

$$X_{\alpha_0} \supset X_{\alpha_1} \supset \cdots \supset X_{\alpha_p} = X_\alpha \tag{2.2}$$

of edges such that $l_{\alpha_j} = j$ for $j = 0, \ldots, p$. For an edge X_α, we define $\overline{\mathcal{F}}_\alpha$ as the complex vector space with basis vectors $\overline{F}_{\alpha_0,\ldots,\alpha_p = \alpha}$ labeled by the elements of the set of all flags starting at X_α.

Define $\mathcal{F}_\alpha$ as the quotient of $\overline{\mathcal{F}}_\alpha$ by the subspace generated by all the vectors of the form

$$\sum_{X_\beta,\ X_{\alpha_{j-1}} \supset X_\beta \supset X_{\alpha_{j+1}}} \overline{F}_{\alpha_0,\ldots,\alpha_{j-1},\beta,\alpha_{j+1},\ldots,\alpha_p = \alpha} \ .$$

Such a vector is determined by $j \in \{1, \ldots, p-1\}$ and an incomplete flag $X_{\alpha_0} \supset \ldots \supset X_{\alpha_{j-1}} \supset X_{\alpha_{j+1}} \supset \ldots \supset X_{\alpha_p} = X_\alpha$ with $l_{\alpha_i} = i$.

Denote by $F_{\alpha_0,\ldots,\alpha_p}$ the image in $\mathcal{F}_\alpha$ of the basis vector $\overline{F}_{\alpha_0,\ldots,\alpha_p}$. For $p = 0, \ldots, k$, we set

$$\mathcal{F}^p(\mathcal{C}) \ = \ \oplus_{X_\alpha,\, l_\alpha = p} \mathcal{F}_\alpha \ . \tag{2.3}$$

2.6. Duality, see [22]

The vector spaces $\mathcal{A}^p(\mathcal{C})$ and $\mathcal{F}^p(\mathcal{C})$ are dual. The pairing $\mathcal{A}^p(\mathcal{C}) \otimes \mathcal{F}^p(\mathcal{C}) \to \mathbb{C}$ is defined as follows. For $H_{j_1}, \ldots, H_{j_p}$ in general position, we set $F(H_{j_1}, \ldots, H_{j_p}) = F_{\alpha_0,\ldots,\alpha_p}$ where $X_{\alpha_0} = \mathbb{C}^k$, $X_{\alpha_1} = H_{j_1}$, $\ldots$, $X_{\alpha_p} = H_{j_1} \cap \cdots \cap H_{j_p}$. Then we define $\langle (H_{j_1}, \ldots, H_{j_p}), F_{\alpha_0,\ldots,\alpha_p} \rangle = (-1)^{|\sigma|}$, if $F_{\alpha_0,\ldots,\alpha_p} = F(H_{j_{\sigma(1)}}, \ldots, H_{j_{\sigma(p)}})$ for some $\sigma \in \Sigma_p$, and $\langle (H_{j_1}, \ldots, H_{j_p}), F_{\alpha_0,\ldots,\alpha_p} \rangle = 0$ otherwise.

Define a map $\delta^{(a)} : \mathcal{F}^p(\mathcal{C}) \to \mathcal{F}^{p-1}(\mathcal{C})$ to be the map adjoint to $d^{(a)} : \mathcal{A}^{p-1}(\mathcal{C}) \to \mathcal{A}^p(\mathcal{C})$. An element $v \in \mathcal{F}^k(\mathcal{C})$ is called *singular* if $\delta^{(a)} v = 0$. Denote by

$$\mathrm{Sing}\, \mathcal{F}^k(\mathcal{C}) \subset \mathcal{F}^k(\mathcal{C}) \tag{2.4}$$

the subspace of all singular vectors.

2.7. Contravariant map and form, see [22]

Weights $(a_j)_{j\in J}$ determine a contravariant map

$$\mathcal{S}^{(a)} : \mathcal{F}^p(\mathcal{C}) \to \mathcal{A}^p(\mathcal{C}), \quad F_{\alpha_0,\dots,\alpha_p} \mapsto \sum a_{j_1} \cdots a_{j_p} \, (H_{j_1}, \dots, H_{j_p}) \,, \quad (2.5)$$

where the sum is taken over all p-tuples $(H_{j_1}, \dots, H_{j_p})$ such that

$$H_{j_1} \supset X_{\alpha_1}, \quad \dots, \quad H_{j_p} \supset X_{\alpha_p} \,. \qquad (2.6)$$

Identifying $\mathcal{A}^p(\mathcal{C})$ with $\mathcal{F}^p(\mathcal{C})^*$, we consider this map as a bilinear form,

$$S^{(a)} : \mathcal{F}^p(\mathcal{C}) \otimes \mathcal{F}^p(\mathcal{C}) \to \mathbb{C}. \qquad (2.7)$$

The bilinear form is called the *contravariant form*. The contravariant form is symmetric. For $F_1, F_2 \in \mathcal{F}^p(\mathcal{C})$, we have

$$S^{(a)}(F_1, F_2) =$$

$$\sum_{\{j_1,\dots,j_p\}\subset J} a_{j_1} \cdots a_{j_p} \, \langle (H_{j_1}, \dots, H_{j_p}), F_1 \rangle \, \langle (H_{j_1}, \dots, H_{j_p}), F_2 \rangle \,, \qquad (2.8)$$

where the sum is over all unordered p-element subsets.

2.8. Arrangement with normal crossings

An essential arrangement $\mathcal{C}$ is *with normal crossings*, if exactly k hyperplanes meet at every vertex of $\mathcal{C}$. Assume that $\mathcal{C}$ is an essential arrangement with normal crossings only. A subset $\{j_1, \dots, j_p\} \subset J$ is called *independent* if the hyperplanes $H_{j_1}, \dots, H_{j_p}$ intersect transversally.

A basis of $\mathcal{A}^p(\mathcal{C})$ is formed by $(H_{j_1}, \dots, H_{j_p})$ where $\{j_1 < \cdots < j_p\}$ are independent ordered p-element subsets of J. The dual basis of $\mathcal{F}^p(\mathcal{C})$ is formed by the corresponding vectors $F(H_{j_1}, \dots, H_{j_p})$. These bases of $\mathcal{A}^p(\mathcal{C})$ and $\mathcal{F}^p(\mathcal{C})$ will be called *standard*.

We have

$$F(H_{j_1}, \dots, H_{j_p}) = (-1)^{|\sigma|} F(H_{j_{\sigma(1)}}, \dots, H_{j_{\sigma(p)}}), \qquad \sigma \in \Sigma_p. \qquad (2.9)$$

For an independent subset $\{j_1, \dots, j_p\}$, we have

$$S^{(a)}(F(H_{j_1}, \dots, H_{j_p}), F(H_{j_1}, \dots, H_{j_p})) = a_{j_1} \cdots a_{j_p} \qquad (2.10)$$

and

$$S^{(a)}(F(H_{j_1}, \dots, H_{j_p}), F(H_{i_1}, \dots, H_{i_k})) = 0 \qquad (2.11)$$

for distinct elements of the standard basis.

2.9. If the weights of dense edges are nonzero

THEOREM 2.1. — *Assume that the weights $(a_j)_{j \in J}$ are such that the weights of all dense edges of $\mathcal{C}$ are nonzero. Then*

(i) *the contravariant form is nondegenerate;*

(ii) *$H^p(\mathcal{A}^*(\mathcal{C}), d^{(a)}) = 0$ for $p < k$ and dim $H^k(\mathcal{A}^*, d^{(a)}) = |\chi(U)|$, where $\chi(U)$ is the Euler characteristics of U.*

In particular, these statements hold if all the weights are positive.

Part (i) is proved in [22]. Part (ii) is a straightforward corollary of results in [22] as explained in Theorem 2.2 in [28]. Part (ii) is proved in [30], [17].

2.10. Master function

Given weights $(a_j)_{j \in J}$, define the (multivalued) *master function* $\Phi : U \to \mathbb{C}$ by the formula:

$$\Phi = \Phi_{\mathcal{C},a} = \sum_{j \in J} a_j \log f_j. \qquad (2.12)$$

A point $t \in U$ is a *critical point* if $d\Phi|_t = \nu(a)|_t = 0$.

THEOREM 2.2 ([25, 16, 20]). — *For generic weights $(a_j)_{j \in J}$ all the critical points of Φ are nondegenerate and the number of critical points equals $|\chi(U)|$.* □

2.11. If the weights are unbalanced

Let $\mathcal{C} = (H_j)_{j \in J}$ be an essential arrangement in $\mathbb{C}^k$ with weights $(a_j)_{j \in J}$. Consider the compactification of the arrangement $\mathcal{C}$ in the projective space $\mathbb{P}^k$. Assign the weight $a_\infty = -\sum_{j \in J} a_j$ to the hyperplane $H_\infty = \mathbb{P}^k - \mathbb{C}^k$ and denote by $\hat{\mathcal{C}}$ the arrangement $(H_j)_{j \in J \cup \infty}$ in $\mathbb{P}^k$.

The weights of the arrangement $\mathcal{C}$ are called *unbalanced* if the weights of all the dense edges of $\hat{\mathcal{C}}$ are nonzero, see [28]. For example, if all the weights $(a_j)_{j \in J}$ are positive, then the weights are unbalanced. The unbalanced weights form a Zarisky open subset in the space of all weight systems on $\mathcal{C}$.

THEOREM 2.3 ([28]). — *If the weights $a = (a_j)_{j \in J}$ of $\mathcal{C}$ are unbalanced, then all the critical points of the master function of the weighted arrangement $(\mathcal{C}, a)$ are isolated and the sum of Milnor numbers of all the critical points equals $|\chi(U)|$.*

2.12. Hessian and residue bilinear form

Denote $\mathbb{C}(U)$ the algebra of rational functions on $\mathbb{C}^k$ regular on U and $I_\Phi = \langle \frac{\partial \Phi}{\partial t_i} \mid i = 1, \ldots, k \rangle \subset \mathbb{C}(U)$ the ideal generated by first derivatives of Φ. Let

$$A_\Phi = \mathbb{C}(U)/I_\Phi \qquad (2.13)$$

be the algebra of functions on the critical set and $[\,] : \mathbb{C}(t)_U \to A_\Phi, \ f \mapsto [f]$, the canonical homomorphism.

If all critical points are isolated, then the critical set is finite and the algebra A_Φ is finite-dimensional. In that case, A_Φ is the direct sum of local algebras corresponding to points p of the critical set,

$$A_\Phi = \oplus_p A_{p,\Phi} \ . \qquad (2.14)$$

The local algebra $A_{p,\Phi}$ can be defined as the quotient of the algebra of germs at p of holomorphic functions modulo the ideal $I_{p,\Phi}$ generated by first derivatives of Φ.

LEMMA 2.4 ([28]). — *The elements $[1/f_j]$, $j \in J$, generate A_Φ.* $\qquad \square$

We fix affine coordinates $t_1, \ldots, t_k$ on $\mathbb{C}^k$. Let

$$f_j = b_j^0 + b_j^1 t_1 + \cdots + b_j^k t_k. \qquad (2.15)$$

LEMMA 2.5. — *The identity element $[1] \in A_\Phi(z)$ satisfies the equation*

$$[1] = \frac{1}{|a|} \sum_{j \in J} b_j^0 \Big[\frac{a_j}{f_j} \Big], \qquad (2.16)$$

where $|a| = \sum_{j \in J} a_j$.

Proof. — The lemma follows from the equality

$$\sum_{i=1}^k t_i \frac{\partial \Phi}{\partial t_i} = |a| - \sum_{j \in J} b_j^0 \Big[\frac{a_j}{f_j} \Big]. \qquad (2.17)$$

$\qquad \square$

Surprisingly, formula (2.17) and Lemma 2.4 play central roles in the constructions of this paper.

We define the rational function Hess : $\mathbb{C}^k \to \mathbb{C}$, regular on U, by the formula

$$\mathrm{Hess}(t) \;=\; \det_{1 \leqslant i,j \leqslant k} \Big(\frac{\partial^2 \Phi}{\partial t_i \partial t_j} \Big)(t) \; . \tag{2.18}$$

The function is called the *Hessian* of Φ.

Let $\rho_p : A_{p,\Phi} \to \mathbb{C}$, be the *Grothendieck residue*,

$$f \;\mapsto\; \frac{1}{(2\pi\sqrt{-1})^k} \; \mathrm{Res}_p \; \frac{f}{\prod_{i=1}^k \frac{\partial\Phi}{\partial t_i}} = \frac{1}{(2\pi\sqrt{-1})^k} \int_{\Gamma_p} \frac{f \; dt_1 \wedge \cdots \wedge dt_k}{\prod_{i=1}^k \frac{\partial\Phi}{\partial t_i}} \; , \tag{2.19}$$

where Γ_p is the real k cycle in a small neighborhood of p, defined by the equations $|\frac{\partial\Phi}{\partial t_i}| = \epsilon_i$, $i = 1, \ldots, k$, and oriented by the condition $d \arg \frac{\partial\Phi}{\partial t_1} \wedge \cdots \wedge d \arg \frac{\partial\Phi}{\partial t_k} > 0$, here ϵ_s are positive numbers sufficiently small with respect to the size of the neighborhood, see [12, 1].

Let $(\, , \,)_p$ be the *residue bilinear form* on $A_{p,\Phi}$,

$$(f,g)_p \;=\; \rho_p(fg), \tag{2.20}$$

for $f, g \in A_{p,\Phi}$. This form is nondegenerate.

Let all the critical points of Φ be isolated and hence, $A_\Phi = \oplus_p A_{p,\Phi}$. We define the *residue bilinear form* $(\, , \,)$ on A_Φ as $\oplus_p (\, , \,)_p$. This form is nondegenerate and $(fg, h) = (f, gh)$ for all $f, g, h \in A_\Phi$. In other words, the pair $(A_\Phi, (\, , \,))$ is a *Frobenius algebra*.

2.13. Canonical isomorphism and algebra structures on $\mathrm{Sing}\,\mathcal{F}^k(\mathcal{C})$

Let $(F_m)_{m \in M}$ be a basis of $\mathcal{F}^k(\mathcal{C})$ and $(H^m)_{m \in M} \subset \mathcal{A}^k(\mathcal{C})$ the dual basis. Consider the element $\sum_m H^m \otimes F_m \in \mathcal{A}^k(\mathcal{C}) \otimes \mathcal{F}^k(\mathcal{C})$. We have $H^m = f^m dt_1 \wedge \cdots \wedge dt_k$ for some $f^m \in \mathbb{C}(U)$. The element

$$E = \sum_{m \in M} f^m \otimes F_m \;\in\; \mathbb{C}(U) \otimes \mathcal{F}^k(\mathcal{C}) \tag{2.21}$$

is called the *canonical element* of $\mathcal{C}$. Denote $[E]$ the image of the canonical element in $A_\Phi \otimes \mathcal{F}^k(\mathcal{C})$.

THEOREM 2.6 ([28]). — *We have* $[E] \in A_\Phi \otimes \mathrm{Sing}\,\mathcal{F}^k(\mathcal{C})$.

Assume that all critical points of Φ are isolated. Introduce the linear map

$$\alpha : A_\Phi \to \mathrm{Sing}\,\mathcal{F}^k(\mathcal{C}), \qquad [g] \mapsto ([g], [E]). \tag{2.22}$$

THEOREM 2.7 ([28]). — *If the weights $(a_j)_{j \in J}$ of $\mathcal{C}$ are unbalanced, then the canonical map α is an isomorphism of vector spaces. The isomorphism α identifies the residue form on A_Φ and the contravariant form on $\operatorname{Sing} \mathcal{F}^k(\mathcal{C})$ multiplied by $(-1)^k$, that is,*

$$(f, g) = (-1)^k S^{(a)}(\alpha(f), \alpha(g)) \qquad \text{for all } f, g \in A_\Phi. \tag{2.23}$$

The map α is called the *canonical* map or *canonical* isomorphism.

COROLLARY 2.8 ([28]). *The restriction of the contravariant form $S^{(a)}$ to the subspace $\operatorname{Sing} \mathcal{F}^k(\mathcal{C})$ is nondegenerate.* ☐

On the restriction of the contravariant form $S^{(a)}$ to the subspace $\operatorname{Sing} \mathcal{F}^k(\mathcal{C})$ see [7].

If all critical points p of the master function are nondegenerate, then

$$\alpha : [g] \mapsto \sum_p \sum_m \frac{g(p) f^m(p)}{\operatorname{Hess}(p)} F_m. \tag{2.24}$$

If the weights $(a_j)_{j \in J}$ of $\mathcal{C}$ are unbalanced, then the canonical isomorphism $\alpha : A_\Phi \to \operatorname{Sing} \mathcal{F}^k(\mathcal{C})$ induces a commutative associative algebra structure on $\operatorname{Sing} \mathcal{F}^k(\mathcal{C})$. Together with the contravariant form it is a Frobenius algebra structure.

2.14. Change of variables and canonical isomorphism

Assume that we change coordinates on $\mathbb{C}^n$, $t_i = \sum_{j=1}^k c_{i,j} s_j$ with $c_{i,j} \in \mathbb{C}$.

LEMMA 2.9. — *The canonical map (2.22) in coordinates $t_1, \ldots, t_k$ equals the canonical map (2.22) in coordinates $s_1, \ldots, s_k$ divided by $\det(c_{i,j})$, $\alpha_t = \frac{1}{\det(c_{i,j})} \alpha_s$.*

Proof. — We have $H^m = f^m dt_1 \wedge \cdots \wedge dt_k = \det(c_{i,j}) f^m ds_1 \wedge \cdots \wedge ds_k$ and $\operatorname{Hess}_t = \det^2(c_{i,j}) \operatorname{Hess}_s$. Now the lemma follows, for example, from (2.24). ☐

To make the map (2.22) independent of coordinates one needs to consider it as a map

$$A_\Phi \otimes dt_1 \wedge \cdots \wedge dt_k \to \operatorname{Sing} \mathcal{F}^k(\mathcal{C}), \quad [g] \otimes dt_1 \wedge \cdots \wedge dt_k \mapsto ([g], [E]). \tag{2.25}$$

3. A family of parallelly transported hyperplanes

This section contains the main constructions of the paper. These constructions are explained in details in Sections 4, 5, and 6 for the particular situations discussed there.

3.1. An arrangement in $\mathbb{C}^n \times \mathbb{C}^k$

Recall that $J = \{1, \ldots, n\}$. Consider $\mathbb{C}^k$ with coordinates $t_1, \ldots, t_k$, $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$, the projection $\mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$. Fix n nonzero linear functions on $\mathbb{C}^k$, $g_j = b_j^1 t_1 + \cdots + b_j^k t_k$, $j \in J$, where $b_j^i \in \mathbb{C}$. Define n linear functions on $\mathbb{C}^n \times \mathbb{C}^k$, $f_j = z_j + g_j = z_j + b_j^1 t_1 + \cdots + b_j^k t_k$, $j \in J$. In $\mathbb{C}^n \times \mathbb{C}^k$ we define the arrangement $\tilde{\mathcal{C}} = \{\tilde{H}_j \mid f_j = 0, \ j \in J\}$. Denote $\tilde{U} = \mathbb{C}^n \times \mathbb{C}^k - \cup_{j \in J} \tilde{H}_j$.

For every $z = (z_1, \ldots, z_n)$ the arrangement $\tilde{\mathcal{C}}$ induces an arrangement $\mathcal{C}(z)$ in the fiber of the projection over z. We identify every fiber with $\mathbb{C}^k$. Then $\mathcal{C}(z)$ consists of hyperplanes $H_j(z), j \in J$, defined in $\mathbb{C}^k$ by the equations $f_j = 0$. Denote $U(\mathcal{C}(z)) = \mathbb{C}^k - \cup_{j \in J} H_j(z)$, the complement to the arrangement $\mathcal{C}(z)$. We assume that for every z the arrangement $\mathcal{C}(z)$ has a vertex. This happens if and only if $\mathcal{C}(0)$ has a vertex.

A point $z \in \mathbb{C}^n$ is called *good* if $\mathcal{C}(z)$ has normal crossings only. Good points form the complement in $\mathbb{C}^n$ to the union of suitable hyperplanes called the *discriminant*.

3.2. Discriminant

The collection $(g_j)_{j \in J}$ induces a matroid structure on J. A subset $C = \{i_1, \ldots, i_r\} \subset J$ is a *circuit* if $(g_i)_{i \in C}$ are linearly dependent but any proper subset of C gives linearly independent g_i's.

For a circuit $C = \{i_1, \ldots, i_r\}$, let $(\lambda_i^C)_{i \in C}$ be a nonzero collection of complex numbers such that $\sum_{i \in C} \lambda_i^C g_i = 0$. Such a collection is unique up to multiplication by a nonzero number.

For every circuit C we fix such a collection and denote $f_C = \sum_{i \in C} \lambda_i^C z_i$. The equation $f_C = 0$ defines a hyperplane H_C in $\mathbb{C}^n$. It is convenient to assume that $\lambda_i^C = 0$ for $i \in J - C$ and write $f_C = \sum_{i \in J} \lambda_i^C z_i$.

For any $z \in \mathbb{C}^n$, the hyperplanes $(H_i(z))_{i \in C}$ in $\mathbb{C}^k$ have nonempty intersection if and only if $z \in H_C$. If $z \in H_C$, then the intersection has codimension $r - 1$ in $\mathbb{C}^k$.

Denote by $\mathfrak{C}$ the set of all circuits in J. Denote $\Delta = \cup_{C \in \mathfrak{C}} H_C$. The arrangement $\mathcal{C}(z)$ in $\mathbb{C}^k$ has normal crossings if and only if $z \in \mathbb{C}^n - \Delta$, see [28].

For example, if $k = 1$ and $f_j = t_1 + z_j, j \in J$, then the discriminant is the union of hyperplanes in $\mathbb{C}^n$ defined by the equations $z_i - z_j = 0$, $1 \leqslant i < j \leqslant n$.

3.3. Good fibers and combinatorial connection

For any $z^1, z^2 \in \mathbb{C}^n - \Delta$, the spaces $\mathcal{F}^p(\mathcal{C}(z^1)), \mathcal{F}^p(\mathcal{C}(z^2))$ are canonically identified. Namely, a vector $F(H_{j_1}(z^1), \ldots, H_{j_p}(z^1))$ of the first space is identified with the vector $F(H_{j_1}(z^2), \ldots, H_{j_p}(z^2))$ of the second. In other words, we identify the standard bases of these spaces.

Assume that nonzero weights $(a_j)_{j \in J}$ are given. Then each arrangement $\mathcal{C}(z)$ is weighted. The identification of spaces $\mathcal{F}^p(\mathcal{C}(z^1)), \mathcal{F}^p(\mathcal{C}(z^2))$ for $z^1, z^2 \in \mathbb{C}^n - \Delta$ identifies the corresponding subspaces $\operatorname{Sing} \mathcal{F}^k(\mathcal{C}(z^1))$, $\operatorname{Sing} \mathcal{F}^k(\mathcal{C}(z^2))$ and contravariant forms.

For a point $z \in \mathbb{C}^n - \Delta$, we denote $V = \mathcal{F}^k(\mathcal{C}(z)), \operatorname{Sing} V = \operatorname{Sing} \mathcal{F}^k(\mathcal{C}(z))$. The triple $(V, \operatorname{Sing} V, S^{(a)})$ does not depend on $z \in \mathbb{C}^n - \Delta$ under the above identification.

As a result of this reasoning we obtain the canonically trivialized vector bundle

$$\sqcup_{z \in \mathbb{C}^n - \Delta} \mathcal{F}^k(\mathcal{C}(z)) \to \mathbb{C}^n - \Delta, \tag{3.1}$$

with the canonically trivialized subbundle $\sqcup_{z \in \mathbb{C}^n - \Delta} \operatorname{Sing} \mathcal{F}^k(\mathcal{C}(z)) \to \mathbb{C}^n - \Delta$ and the constant contravariant form on the fibers. This trivialization identifies the bundle in (3.1) with

$$(\mathbb{C}^n - \Delta) \times V \to \mathbb{C}^n - \Delta \tag{3.2}$$

and the subbundle with

$$(\mathbb{C}^n - \Delta) \times (\operatorname{Sing} V) \to \mathbb{C}^n - \Delta. \tag{3.3}$$

The bundle in (3.3) will be called the *combinatorial bundle*, the flat connection on it will be called *combinatorial*.

LEMMA 3.1. — *If the weights $(a_j)_{j \in J}$ are unbalanced for the arrangement $\mathcal{C}(z)$ for some $z \in \mathbb{C}^n - \Delta$, then the weights $(a_j)_{j \in J}$ are unbalanced for $\mathcal{C}(z)$ for all $z \in \mathbb{C}^n - \Delta$.* $\square$

3.4. Bad fibers

Points of $\Delta \subset \mathbb{C}^n$ are called *bad*. Let $z^0 \in \Delta$ and $z \in \mathbb{C}^n - \Delta$. By definition, for any p the space $\mathcal{A}^p(\mathcal{C}(z^0))$ is obtained from $\mathcal{A}^p(\mathcal{C}(z))$ by adding new relations. Hence $\mathcal{A}^k(\mathcal{C}(z^0))$ is canonically identified with the quotient space of $V^* = \mathcal{A}^k(\mathcal{C}(z))$ and $\mathcal{F}^p(\mathcal{C}(z^0))$ is identified with a subspace of $V = \mathcal{F}^p(\mathcal{C}(z))$.

3.5. Operators $K_j(z) : V \to V$, $j \in J$

For any circuit $C = \{i_1, \ldots, i_r\} \subset J$, we define the linear operator $L_C : V \to V$ as follows.

For $m = 1, \ldots, r$, we define $C_m = C - \{i_m\}$. Let $\{j_1 < \cdots < j_k\} \subset J$ be an independent ordered subset and $F(H_{j_1}, \ldots, H_{j_k})$ the corresponding element of the standard basis. We define $L_C : F(H_{j_1}, \ldots, H_{j_k}) \mapsto 0$ if $|\{j_1, \ldots, j_k\} \cap C| < r - 1$. If $\{j_1, \ldots, j_k\} \cap C = C_m$ for some $1 \leqslant m \leqslant r$, then by using the skew-symmetry property (2.9) we can write

$$F(H_{j_1}, \ldots, H_{j_k}) = \pm F(H_{i_1}, H_{i_2}, \ldots, \widehat{H_{i_m}}, \ldots, H_{i_{r-1}} H_{i_r}, H_{s_1}, \ldots, H_{s_{k-r+1}}) \tag{3.4}$$

with $\{s_1, \ldots, s_{k-r+1}\} = \{j_1, \ldots, j_k\} - C_m$. We set

$$L_C \ : \ F(H_{i_1}, \ldots, \widehat{H_{i_m}}, \ldots, H_{i_r}, H_{s_1}, \ldots, H_{s_{k-r+1}}) \mapsto \tag{3.5}$$

$$(-1)^m \sum_{l=1}^{r} (-1)^l a_{i_l} F(H_{i_1}, \ldots, \widehat{H_{i_l}}, \ldots, H_{i_r}, H_{s_1}, \ldots, H_{s_{k-r+1}}).$$

Consider on $\mathbb{C}^n \times \mathbb{C}^k$ the logarithmic differential one-forms $\omega_C = \frac{df_C}{f_C}$, $C \in \mathfrak{C}$. Recall that $f_C = \sum_{j \in J} \lambda_j^C z_j$. We define

$$K_j(z) \ = \ \sum_{C \in \mathfrak{C}} \frac{\lambda_j^C}{f_C(z)} L_C \, , \qquad j \in J. \tag{3.6}$$

The operators $K_j(z)$ are rational functions on $\mathbb{C}^n$ regular on $\mathbb{C}^n - \Delta$ and

$$\sum_{C \in \mathfrak{C}} \omega_C \otimes L_C = \sum_{j \in J} dz_j \otimes K_j(z). \tag{3.7}$$

THEOREM 3.2 ([28]). — *For any $j \in J$ and $z \in \mathbb{C}^n - \Delta$, the operator $K_j(z)$ preserves the subspace $\mathrm{Sing}\, V \subset V$ and is a symmetric operator, $S^{(a)}(K_j(z)v, w) = S^{(a)}(v, K_j(z)w)$ for all $v, w \in V$.*

3.6. Gauss-Manin connection on $(\mathbb{C}^n - \Delta) \times (\operatorname{Sing} V) \to \mathbb{C}^n - \Delta$

Consider the master function

$$\Phi(z, t) = \sum_{j \in J} a_j \log f_j(z, t) \tag{3.8}$$

as a function on $\tilde{U} \subset \mathbb{C}^n \times \mathbb{C}^k$. Let $\kappa \in \mathbb{C}^\times$. The function $e^{\Phi(z,t)/\kappa}$ defines a rank one local system $\mathcal{L}_\kappa$ on $\tilde{U}$ whose horizontal sections over open subsets of $\tilde{U}$ are univalued branches of $e^{\Phi(z,t)/\kappa}$ multiplied by complex numbers, see for example [22, 24].

The vector bundle

$$\sqcup_{z \in \mathbb{C}^n - \Delta} H_k(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))}) \to \mathbb{C}^n - \Delta \tag{3.9}$$

will be called the *homology bundle*. The homology bundle has a canonical flat Gauss-Manin connection.

For a fixed z, choose any $\gamma \in H_k(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))})$. The linear map

$$\{\gamma\} \; : \; \mathcal{A}^k(\mathcal{C}(z)) \to \mathbb{C}, \qquad \omega \mapsto \int_\gamma e^{\Phi(z,t)/\kappa} \omega, \tag{3.10}$$

is an element of $\operatorname{Sing} \mathcal{F}^k(\mathcal{C}(z))$ by Stokes' theorem. It is known that for generic κ any element of $\operatorname{Sing} \mathcal{F}^k(\mathcal{C}(z))$ corresponds to a certain γ and in that case this construction gives an isomorphism

$$H_k(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))}) \to \operatorname{Sing} \mathcal{F}^k(\mathcal{C}(z)), \tag{3.11}$$

see [22]. This isomorphism will be called the *integration isomorphism*. The precise values of κ for which (3.11) is an isomorphism can be deduced from the determinant formula in [23].

For generic κ the fiber isomorphisms (3.11) defines an isomorphism of the homology bundle and the combinatorial bundle. The Gauss-Manin connection induces a flat connection on the combinatorial bundle. This connection on the combinatorial bundle will be also called the *Gauss-Manin connection*.

Thus, there are two connections on the combinatorial bundle: the combinatorial connection and the Gauss-Manin connection depending on κ. In this situation we can consider the differential equations for flat sections of the Gauss-Manin connection with respect to the combinatorially flat standard basis. Namely, let $\gamma(z) \in H_k(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))})$ be a flat section of the Gauss-Manin connection. Let us write the corresponding section $I_\gamma(z)$

of the bundle $\mathbb{C}^n \times \operatorname{Sing} V \to \mathbb{C}^n$ in the combinatorially flat standard basis,

$$I_\gamma(z) = \sum_{\substack{\text{independent} \\ \{j_1 < \cdots < j_k\} \subset J}} I_\gamma^{j_1, \ldots, j_k}(z) F(H_{j_1}, \ldots, H_{j_k}),$$

$$I_\gamma^{j_1, \ldots, j_k}(z) = \int_{\gamma(z)} e^{\Phi(z,t)/\kappa} \omega_{j_1} \wedge \cdots \wedge \omega_{j_k}. \tag{3.12}$$

For $I = \sum I^{j_1, \ldots, j_k} F(H_{j_1}, \ldots, H_{j_k})$ and $j \in J$, we denote

$$\frac{\partial I}{\partial z_j} = \sum \frac{\partial I^{j_1, \ldots, j_k}}{\partial z_j} F(H_{j_1}, \ldots, H_{j_k}). \tag{3.13}$$

THEOREM 3.3 ([24, 28]). — *The section $I_\gamma(z)$ satisfies the differential equations*

$$\kappa \frac{\partial I}{\partial z_j}(z) = K_j(z) I(z), \qquad j \in J, \tag{3.14}$$

where $K_j(z) : V \to V$ are the linear operators defined in (3.6).

From this formula we see, in particular, that the combinatorial connection on the combinatorial bundle is the limit of the Gauss-Manin connection as $\kappa \to \infty$.

3.7. Bundle of algebras

For $z \in \mathbb{C}^n$, denote $A_\Phi(z)$ the algebra of functions on the critical set of the master function $\Phi(z, \cdot) : U(\mathcal{C}(z)) \to \mathbb{C}$. Assume that the weights $(a_j)_{j \in J}$ are unbalanced for all $\mathcal{C}(z)$, $z \in \mathbb{C}^n - \Delta$. Then the dimension of $A_\Phi(z)$ does not depend on $z \in \mathbb{C}^n - \Delta$ and equals $\dim \operatorname{Sing} V$. Denote $|a| = \sum_{j \in J} a_j$.

LEMMA 3.4. — *The identity element $[1](z) \in A_\Phi(z)$ satisfies the equation*

$$[1](z) = \frac{1}{|a|} \sum_{j \in J} z_j \left[\frac{a_j}{f_j} \right]. \tag{3.15}$$

Proof. — The lemma follows from Lemma 2.5. $\qquad\square$

The vector bundle

$$\sqcup_{z \in \mathbb{C}^n - \Delta} A_\Phi(z) \to \mathbb{C}^n - \Delta \tag{3.16}$$

will be called the *bundle of algebras* of functions on the critical set. The fiber isomorphisms (2.22),

$$\alpha(z) : A_\Phi(z) \to \operatorname{Sing} V,$$

establish an isomorphism α of the bundle of algebras and the combinatorial bundle. The isomorphism α and the connections on the combinatorial bundle (combinatorial and Gauss-Manin connections) induce connections on the bundle of algebras which will be called also the *combinatorial and Gauss-Manin connections* on the bundle of algebras.

The canonical isomorphism $\alpha(z)$ induces a Frobenius algebra structure on $\mathrm{Sing}\,V$ which depends on z. The multiplication $*_z$ is described by the following theorem.

THEOREM 3.5 ([28]). — *The elements* $\alpha(z)\left[\frac{a_j}{f_j}\right] \in \mathrm{Sing}\,V$, $j \in J$, *generate the algebra. We have*

$$\alpha(z)\left[\frac{a_j}{f_j}\right] *_z v = K_j(z)v, \tag{3.17}$$

for all $v \in \mathrm{Sing}\,V$ *and* $j \in J$.

3.8. Quantum integrable model of the arrangement $(\mathcal{C}(z), a)$

For $z \in \mathbb{C}^n - \Delta$, the (commutative) subalgebra $\mathcal{B}(z) \subset \mathrm{End}(\mathrm{Sing}\,V)$ generated by $K_j(z), j \in J$, is called the *algebra of geometric Hamiltonians*, the triple $(\mathrm{Sing}\,V, S^{(a)}, \mathcal{B}(z))$ is called the *quantum integrable model of the weighted arrangement* $(\mathcal{C}(z), a)$, see [28].

The canonical isomorphism $\alpha(z)$ identifies the triple $(\mathrm{Sing}\,V, S^{(a)}, \mathcal{B}(z))$ with the triple $(A_\Phi(z), (-1)^k (\ ,\ )_z, A_\Phi(z))$, see Theorems 2.7 and 3.5.

Notice that the operators $K_j(z)$ are defined in combinatorial terms, see Section 3.5, while the algebra $A_\Phi(z)$ is an analytic object, see (2.13). C.f. Corollaries 5.28 and 6.21.

3.9. A remark. Asymptotically flat sections

Assume that the weights $(a_j)_{j \in J}$ are unbalanced for all $\mathcal{C}(z)$, $z \in \mathbb{C}^n - \Delta$. Let $B \subset \mathbb{C}^n - \Delta$ be an open real $2n$-dimensional ball. Let $\Psi : B \to \mathbb{C}$ be a holomorphic function. Let $s_j, j \in \mathbb{Z}_{\geq 0}$ be holomorphic sections over B of the bundle of algebras, see (3.16). We say that

$$s(z, \kappa) = e^{\Psi(z)/\kappa} \sum_{j \geq 0} \kappa^j s_j(z) \tag{3.18}$$

is an *asymptotically flat section* of the Gauss-Manin connection on bundle of algebras as $\kappa \to 0$ if $s(z, \kappa)$ satisfies the flat section equations formally, see, for example, [18, 27].

Assume that B is such that for any $z \in B$, all the critical points of $\Phi(z, \cdot) : U(\mathcal{C}(z)) \to \mathbb{C}$ are nondegenerate. Let us order them: $p_1(z), \dots, p_d(z)$, where $d = \dim A_\Phi(z) = \dim \operatorname{Sing} V$. We may assume that every $p_i(z)$ depends on z holomorphically. Then the function

$$z \mapsto \operatorname{Hess}(z, p_i(z)) = \det_{1 \leqslant i,j \leqslant k} \left(\frac{\partial^2 \Phi}{\partial t_i \partial t_j} \right)(z, p_i(z)) \tag{3.19}$$

is a nonzero holomorphic function on B. We fix a square root $\operatorname{Hess}(z, p_i(z))^{1/2}$. We denote $w_i(z)$ the element of $A_\Phi(z)$ which equals $\operatorname{Hess}(z, p_i(z))^{1/2}$ at $p_i(z)$ and equals zero at all other critical points. Let $(\, , \,)_z$ be the residue form on $A_\Phi(z)$. Then

$$(w_i(z), w_j(z))_z = \delta_{ij} \quad \text{and} \quad w_i(z) \cdot w_j(z) = \delta_{ij} \operatorname{Hess}(z, p_i(z))^{1/2} w_i(z) \tag{3.20}$$

for all i, j.

THEOREM 3.6. — *For every i, there exists a unique asymptotically flat section $s(z, \kappa) = e^{\Psi(z)/\kappa} \sum_{j \geqslant 0} \kappa^j s_j(z)$ of the Gauss-Manin connection on the bundle of algebras such that*

$$\Psi(z) = \Phi(z, p_i(z)) \qquad \text{and} \qquad s_0(z) = w_i(z). \tag{3.21}$$

Proof. — We first write asymptotically flat sections of the Gauss-Manin connection on the bundle $(\mathbb{C}^n - \Delta) \times \operatorname{Sing} V \to (\mathbb{C}^n - \Delta)$ by using the steepest descent method as in [18, 27] and then observe that the leading terms of those sections are nothing else but $\alpha(z)\big(e^{\Phi(z,p_i(z))/\kappa} w_i(z)\big)$. $\square$

3.10. Conformal blocks, period map, potential functions

Denote by

$$\{1\}(z) = \alpha(z)([1](z)) \tag{3.22}$$

the identity element of the algebra structure on $\operatorname{Sing} V$ corresponding to a point $z \in \mathbb{C}^n - \Delta$. An analog of this element was studied in [15] in a situation related to the geometric Langlands correspondence, see element v_1 in [15, Section 8].

For $r < k$ and $m_1, \dots, m_r \in J$, denote

$$I_{m_1,\dots,m_r}(z) = \frac{\partial^r \{1\}}{\partial z_{m_1} \dots \partial z_{m_r}}(z). \tag{3.23}$$

CONJECTURE 3.7. — *The* $\mathrm{Sing}\, V$*-valued function* $\{1\}(z)$ *satisfies the Gauss-Manin differential equations with parameter* $\kappa = \frac{|a|}{k}$,

$$\frac{|a|}{k}\frac{\partial\{1\}}{\partial z_j}(z) = K_j(z)\{1\}(z), \qquad j \in J, \tag{3.24}$$

where the derivatives are defined with respect to a combinatorially flat basis as in (3.13). More generally, for $r < k$ *and* $m_1, \ldots, m_r \in J$, *the* $\mathrm{Sing}\, V$*-valued function* $I_{m_1,\ldots,m_r}(z)$ *satisfies the Gauss-Manin differential equations with parameter* $\kappa = \frac{|a|}{k-r}$,

$$\frac{|a|}{k-r}\frac{\partial I_{m_1,\ldots,m_r}}{\partial z_j}(z) = K_j(z)I_{m_1,\ldots,m_r}(z), \qquad j \in J. \tag{3.25}$$

CONJECTURE 3.8. — *If we write the* $\mathrm{Sing}\, V$*-valued function* $\{1\}(z)$ *in coordinates with respect to a combinatorially flat basis, then* $\{1\}(z)$ *is a homogeneous polynomial in* z *of degree* k.

The conjectures describes the interrelations of four objects: the identity element in $A_\Phi(z)$, the canonical isomorphism, the integration isomorphism, and the Gauss-Manin connection on the homology bundle. In the next sections we will prove this conjecture for families of generic arrangements.

THEOREM 3.9. — *If Conjecture 3.7 holds, then for* $r \leqslant k$ *and* $m_1, \ldots, m_r \in J$, *we have*

$$\frac{\partial^r\{1\}}{\partial z_{m_1}\ldots\partial z_{m_r}}(z) = \frac{k(k-1)\ldots(k-r+1)}{|a|^r}\alpha(z)\Big(\prod_{i=1}^r\Big[\frac{a_{m_i}}{f_{m_i}}\Big]\Big). \tag{3.26}$$

Proof. — The proof is by induction on r. For $r = 0$, the statement is true: $\{1\} = \{1\}$. Assuming the statement is true for some r, we prove the statement for $r+1$. By (3.25) and Theorem 3.5, we have

$$\frac{\partial^{r+1}\{1\}}{\partial z_{m_1}\ldots\partial z_{m_r}\partial z_j}(z) = \frac{k-r}{|a|}K_j(z)\frac{\partial^r\{1\}}{\partial z_{m_1}\ldots\partial z_{m_r}}(z) \tag{3.27}$$

$$= \frac{k-r}{|a|}\alpha(z)\Big(\Big[\frac{a_j}{f_j}\Big]\Big) *_z \frac{k(k-1)\ldots(k-r+1)}{|a|^r}\alpha(z)\Big(\prod_{i=1}^r\Big[\frac{a_{m_i}}{f_{m_i}}\Big]\Big)$$

$$= \frac{k(k-1)\ldots(k-r+1)(k-r)}{|a|^{r+1}}\alpha(z)\Big(\Big[\frac{a_j}{f_j}\Big]\prod_{i=1}^r\Big[\frac{a_{m_i}}{f_{m_i}}\Big]\Big).$$

$\square$

For given $r < k$, the sections $I_{m_1,\ldots,m_r}(z)$, $m_1,\ldots,m_r \in J$, generate a subbundle of the combinatorial bundle. We will call it the *subbundle of conformal blocks* at level $\frac{|a|}{k-r}$ and denote by $CB_{\frac{|a|}{k-r}}$. The subbundle of conformal blocks at level $\frac{|a|}{k-r}$ is invariant with respect to the Gauss-Manin connection with $\kappa = \frac{|a|}{k-r}$. On conformal blocks in conformal field theory see, for example, [8, 9, 24] and Section 3.6 in [26].

One may show that

$$CB_{\frac{|a|}{k}} \subset CB_{\frac{|a|}{k-1}} \subset \cdots \subset CB_{\frac{|a|}{1}}. \tag{3.28}$$

Let us consider $\text{Sing} V$ as a complex manifold. At every point of $\text{Sing} V$, the tangent space is identified with the vector space $\text{Sing} V$. We will consider the manifold $\text{Sing} V$ with the constant holomorphic metric defined by the contravariant form $S^{(a)}$. We will denote this metric by the same symbol $S^{(a)}$.

Define the *period map* $q : \mathbb{C}^n - \Delta \to \text{Sing} V$ by the formula

$$q : z \mapsto \{1\}(z). \tag{3.29}$$

The period map is a polynomial map. Define the *potential function of first kind* $P : \mathbb{C}^n - \Delta \to \mathbb{C}$, by the formula

$$P(z) = S^{(a)}(q(z), q(z)). \tag{3.30}$$

The potential function of first kind is a polynomial.

3.11. Tangent bundle and a Frobenius like structure

Let $T(\mathbb{C}^n - \Delta) \to \mathbb{C}^n - \Delta$ be the tangent bundle on $\mathbb{C}^n - \Delta$. Denote $\partial_j = \frac{\partial}{\partial z_j}$ for $j \in J$. Consider the morphism β of the tangent bundle to the bundle of algebras defined by the formula,

$$\beta(z) \ : \ \partial_j \in T_z(\mathbb{C}^n - \Delta) \quad \mapsto \quad \left[\frac{\partial \Phi}{\partial z_j}\right] = \left[\frac{a_j}{f_j}\right] \in A_\Phi(z). \tag{3.31}$$

The morphism β will be called the *tangent morphism*.

The residue form on the bundle of algebras induces a holomorphic bilinear form η on fibers of the tangent bundle,

$$\begin{aligned}
\eta(\partial_i, \partial_j)_z &= (\beta(z)(\partial_i), \beta(z)(\partial_j))_z & \tag{3.32} \\
&= (-1)^k S^{(a)}(\alpha(z)\beta(z)(\partial_i), \alpha(z)\beta(z)(\partial_j)) \\
&= \left(\left[\frac{a_i}{f_i}\right], \left[\frac{a_j}{f_j}\right]\right)_z = (-1)^k S^{(a)}\left(\alpha(z)\left(\left[\frac{a_i}{f_i}\right]\right), \alpha(z)\left(\left[\frac{a_j}{f_j}\right]\right)\right).
\end{aligned}$$

THEOREM 3.10. — *If Conjecture 3.7 holds, then the bilinear form η is induced by the period map $q : \mathbb{C}^n - \Delta \to \operatorname{Sing} V$ from the flat metric $S^{(a)}$ multiplied by $(-1)^k \frac{|a|^2}{k^2}$,*

$$\eta(\partial_i, \partial_j)_z = \frac{|a|^2}{k^2}(-1)^k S^{(a)}(\frac{\partial q}{\partial z_i}(z), \frac{\partial q}{\partial z_j}(z)). \tag{3.33}$$

Proof. — By Theorem 3.9, we have $\frac{|a|}{k} \frac{\partial q}{\partial z_j} = \alpha(z)(\left[\frac{a_j}{f_j}\right])$. Hence

$$\frac{|a|^2}{k^2}(-1)^k S^{(a)}(\frac{\partial q}{\partial z_i}, \frac{\partial q}{\partial z_j}) = (-1)^k S^{(a)}(\alpha(z)(\left[\frac{a_i}{f_i}\right]), \alpha(z)(\left[\frac{a_j}{f_j}\right])) = \eta(\partial_i, \partial_j)_z. \tag{3.34}$$

$\square$

For $r \leqslant 2k$, introduce the constant $A_{k,r}$ by the formula

$$A_{k,r} = \sum_{i=0}^{r} \binom{r}{i} \frac{(k!)^2}{(k-i)!(k-r+i)!}, \qquad \text{if } r \leqslant k, \tag{3.35}$$

$$A_{k,r} = \sum_{i=r-k}^{k} \binom{r}{i} \frac{(k!)^2}{(k-i)!(k-r+i)!}, \qquad \text{if } r > k.$$

For example, $A_{2,3} = 24$ and $A_{k,2k} = (2k)!$.

THEOREM 3.11. — *If Conjectures 3.7 and 3.8 hold, then for any $r \leqslant 2k$, we have*

$$(\beta(z)(\partial_{m_1}) *_z \cdots *_z \beta(z)(\partial_{m_r}), [1](z))_z = \frac{(-1)^k |a|^r}{A_{k,r}} \frac{\partial^r P}{\partial z_{m_1} \ldots \partial z_{m_r}}(z), \tag{3.36}$$

for all $m_1, \ldots, m_r \in J$. Here $(\ ,\ )_z$ is the residue bilinear form on $A_\Phi(z)$.

Proof. — We have

$$(\beta(z)(\partial_{m_1}) *_z \cdots *_z \beta(z)(\partial_{m_r}), [1](z))_z = (\prod_{i=1}^{r} \left[\frac{a_{m_i}}{f_{m_i}}\right], [1](z))_z$$

$$= (-1)^k S^{(a)}(\alpha(z)(\prod_{i=1}^{r} \left[\frac{a_{m_i}}{f_{m_i}}\right]), \{1\}(z)). \tag{3.37}$$

Consider the example $r = 2, k \geqslant 2$. Then

$$\frac{\partial^2}{\partial z_i \partial z_j} S^{(a)}(q(z), q(z)) = S^{(a)}\left(\frac{\partial^2 q}{\partial z_i \partial z_j}, q\right) + S^{(a)}\left(\frac{\partial q}{\partial z_i}, \frac{\partial q}{\partial z_j}\right) \qquad (3.38)$$

$$+ S^{(a)}\left(\frac{\partial q}{\partial z_j}, \frac{\partial q}{\partial z_i}\right) + S^{(a)}\left(q, \frac{\partial^2 q}{\partial z_i \partial z_j}\right) = \frac{k(k-1)}{|a|^2} S^{(a)}\left(\alpha(z)\left(\left[\frac{a_i}{f_i}\right]\left[\frac{a_i}{f_i}\right]\right), q\right)$$

$$+ \frac{k^2}{|a|^2} S^{(a)}\left(\alpha(z)\left(\left[\frac{a_i}{f_i}\right]\right), \alpha(z)\left(\left[\frac{a_j}{f_j}\right]\right)\right) + \frac{k^2}{|a|^2} S^{(a)}\left(\alpha(z)\left(\left[\frac{a_j}{f_j}\right]\right), \alpha(z)\left(\left[\frac{a_i}{f_i}\right]\right)\right)$$

$$+ \frac{k(k-1)}{|a|^2} S^{(a)}\left(q, \alpha(z)\left(\left[\frac{a_i}{f_i}\right]\left[\frac{a_i}{f_i}\right]\right)\right) = \frac{A_{k,2}}{|a|^2} S^{(a)}\left(\alpha(z)\left(\left[\frac{a_i}{f_i}\right]\left[\frac{a_i}{f_i}\right]\right), q\right).$$

Here we used Theorem 3.9 and the fact that $S^{(a)}$ is constant with respect to the combinatorial connection. This calculation proves the theorem for $r = 2, k \geqslant 2$. The general case for $r \leqslant k$ is proved exactly in the same way. If $r > k$, then we need to take into account that $q(z)$ is a polynomial of degree k. $\qquad \square$

For $v \in \operatorname{Sing} V$, define the differential one-form ψ_v on $\mathbb{C}^n - \Delta$ by the formula

$$\psi_v : \partial_i \in T_z(\mathbb{C}^n - \Delta) \mapsto S^{(a)}(v, \alpha(z)\beta(z)(\partial_i)). \qquad (3.39)$$

THEOREM 3.12. — *If Conjecture 3.7 holds, then the differential form ψ_v is exact,*

$$\psi_v = \frac{|a|}{k} dS^{(a)}(v, q(z)). \qquad (3.40)$$

Proof. — By Theorem 3.9, we have $\frac{|a|}{k} \frac{\partial q}{\partial z_j} = \alpha(z)\left(\left[\frac{a_j}{f_j}\right]\right)$. Hence

$$\psi_v(\partial_i) = S^{(a)}(v, \alpha(z)\beta(z)(\partial_i)) = S^{(a)}\left(v, \alpha(z)\left(\left[\frac{a_i}{f_i}\right]\right)\right) = S^{(a)}\left(v, \frac{|a|}{k} \frac{\partial q}{\partial z_j}\right)$$

$$= \frac{|a|}{k} \frac{\partial}{\partial z_j} S^{(a)}(v, q).$$

$\qquad \square$

For $\kappa \in \mathbb{C}^\times$, let $I(z) \in \operatorname{Sing} V$ be a flat (multivalued) section of the Gauss-Manin connection with parameter κ. Define the (multivalued) differential one-form ψ_I on $\mathbb{C}^n - \Delta$ by the formula

$$\psi_I : \partial_i \in T_z(\mathbb{C}^n - \Delta) \mapsto S^{(a)}(I(z), \alpha(z)\beta(z)(\partial_i)). \qquad (3.41)$$

THEOREM 3.13. — *If Conjecture 3.7 holds and $\kappa \neq \frac{|a|}{k}$, then the differential form ψ_I is exact,*

$$\psi_I = \Big(\frac{1}{\kappa} + \frac{k}{|a|}\Big)^{-1} dS^{(a)}(I(z), q(z)). \tag{3.42}$$

Proof. — We have

$$\frac{\partial}{\partial z_j} S^{(a)}(I(z), q(z)) = S^{(a)}\big(\frac{\partial}{\partial z_j} I(z), q(z)\big) + S^{(a)}\big(I(z), \frac{\partial}{\partial z_j} q(z)\big)$$

$$= S^{(a)}\big(\frac{1}{\kappa} K_j(z) I(z), q(z)\big) + S^{(a)}\big(I(z), \frac{k}{|a|} K_j(z) q(z)\big)\big)$$

$$= \Big(\frac{1}{\kappa} + \frac{k}{|a|}\Big) S^{(a)}\Big(I(z), \alpha(z)\beta(z)\big(\big[\frac{a_j}{f_j}\big]\big)\Big).$$

$\square$

The functions $\mathbb{C}^n - \Delta \to \mathbb{C}, z \mapsto S^{(a)}(v, q(z))$, of Theorem 3.12 are nothing else but the coordinate functions of the period map. We will call them *flat periods*. The functions $\mathbb{C}^n - \Delta \to \mathbb{C}, z \mapsto S^{(a)}(I(z), q(z))$, of Theorem 3.13 will be called *twisted periods*.

CONJECTURE 3.14. — *There exists a function $\tilde{P}(z_1, \ldots, z_n)$ such that*

$$\tag{3.43}$$

$$\frac{\partial^{2k+1}\tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_{2k}}}(z) = (-1)^k (\beta(z)(\partial_{m_0}) *_z \cdots *_z \beta(z)(\partial_{m_{2k}}), [1](z))_z$$

for all $m_0, \ldots, m_{2k} \in J$.

The function $\tilde{P}(z)$ with this property will be called the *potential function of second kind*.

Notice that formula (3.36) does not hold for $r = 2k + 1$.

The potential function of second kind $\tilde{P}(z)$ determines the potential function of first kind $P(z)$. Indeed, by formula (3.4) we have

$$P(z) = \frac{1}{|a|^{2k+1}} \sum_{m_0, m_1, \ldots, m_{2k} \in J} z_{m_0} z_{m_1} \ldots z_{m_{2k}} \frac{\partial^{2k+1}\tilde{P}}{\partial z_{m_0} \partial z_{m_1} \ldots \partial z_{m_{2k}}}(z) \tag{3.44}$$

More generally, for any $r \leqslant 2k$, we have

$$\frac{\partial^r P}{\partial z_{m_0} \ldots \partial z_{m_{r-1}}}(z) = \frac{A_{k,r}}{|a|^{2k+1}} \sum_{m_r, \ldots, m_{2k} \in J} z_{m_r} \ldots z_{m_{2k}} \frac{\partial^{2k+1}\tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_{2k}}}(z). \tag{3.45}$$

We will call the collection of our objects– the combinatorial bundle $(\mathbb{C}^n - \Delta) \times (\operatorname{Sing} V) \to \mathbb{C}^n - \Delta$ with the contravariant form $S^{(a)}$ and connections (combinatorial and Gauss-Manin); the bundle of algebras $\sqcup_{z \in \mathbb{C}^n - \Delta} A_\Phi(z) \to \mathbb{C}^n - \Delta$; the period map $q : \mathbb{C}^n - \Delta \to \operatorname{Sing} V$, the potential functions $P(z)$ and $\tilde{P}(z)$, flat periods $S^{(a)}(v, q(z))$, twisted periods $S^{(a)}(I(z), q(z))$– *a Frobenius like structure* on $\mathbb{C}^n - \Delta$.

The situation here reminds the structure induced on a submanifold of a Frobenius manifold, cf. [21]. From that point of view one may expect that $\operatorname{Sing} V$ has an honest Frobenius structure and our Frobenius like structure on $\mathbb{C}^n - \Delta$ is what can be induced from the Frobenius structure on $\operatorname{Sing} V$ by the period map, cf. with constructions in [3, 13].

Numerous variations of the definition of the Frobenius structure see, for example, in [4, 5, 14, 21, 10].

In the next sections we will prove Conjectures 3.7, 3.8, 3.14 for families of generic arrangements and will describe our structure more precisely.

4. Points on line

4.1. An arrangement in $\mathbb{C}^n \times \mathbb{C}$

Recall that $J = \{1, \dots, n\}$. Consider $\mathbb{C}$ with coordinate t and $\mathbb{C}^n$ with coordinates $z_1, \dots, z_n$. Consider n linear functions on $\mathbb{C}^n \times \mathbb{C}$, $f_j = z_j + t$, $j \in J$. In $\mathbb{C}^n \times \mathbb{C}$ we define the arrangement $\tilde{\mathcal{C}} = \{\tilde{H}_j \mid f_j = 0, \ j \in J\}$.

For every $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ the arrangement $\tilde{\mathcal{C}}$ induces an arrangement $\mathcal{C}(z)$ in the fiber over z of the projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$. We identify the fiber with $\mathbb{C}$. The arrangement $\mathcal{C}(z)$ is the arrangement of points $\{-z_1, \dots, -z_n\}$. Denote $U(\mathcal{C}(z)) = \mathbb{C} - \{-z_1, \dots, -z_n\}$ the complement.

A point $z \in \mathbb{C}^n$ is *good* if the points $-z_1, \dots, -z_n$ are distinct. Good points form the complement in $\mathbb{C}^n$ to the *discriminant* Δ, which is the union of hyperplanes $H_{ij} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i = z_j\}$ labeled by two-element subsets $\{i, j\} \subset J$.

4.2. Good fibers

For any $z \in \mathbb{C}^n - \Delta$, the space $\mathcal{A}^1(\mathcal{C}(z))$ has the standard basis $H_1(z)$, $\dots$, $H_n(z)$, the space $\mathcal{F}^1(\mathcal{C}(z))$ has the standard dual basis $F(H_1(z))$, $\dots$, $F(H_n(z))$. For $z^1, z^2 \in \mathbb{C}^n - \Delta$, the combinatorial connection identifies the spaces $\mathcal{A}^1(\mathcal{C}(z^1))$, $\mathcal{F}^1(\mathcal{C}(z^1))$ with the spaces $\mathcal{A}^1(\mathcal{C}(z^2))$, $\mathcal{F}^1(\mathcal{C}(z^2))$, respectively, by identifying the corresponding standard bases.

Assume that nonzero weights $(a_j)_{j \in J}$ are given. Then each arrangement $\mathcal{C}(z)$ is weighted. For $z \in \mathbb{C}^n - \Delta$, the arrangement $\mathcal{C}(z)$ is unbalanced if $|a| = \sum_{j \in J} a_j \neq 0$. We assume $|a| \neq 0$.

For $z \in \mathbb{C}^n - \Delta$, we denote $V = \mathcal{F}^1(\mathcal{C}(z))$. We also denote $F_j = F(H_j(z))$ for $j \in J$. We have

$$S^{(a)}(F_i, F_j) = \delta_{ij} a_i, \qquad \mathrm{Sing}\, V = \Big\{ \sum_{j \in J} c_j F_j \mid \sum_{j \in J} c_j a_j = 0 \Big\}. \qquad (4.1)$$

For $j \in J$, we define the vector $v_j \in V$ by the formula

$$v_j = -F_j + \frac{a_j}{|a|} \sum_{i \in J} F_i. \qquad (4.2)$$

LEMMA 4.1. — *We have the following properties.*

(i) $\dim \mathrm{Sing}\, V = n - 1$.

(ii) *For $j \in J$, we have $v_j \in \mathrm{Sing}\, V$ and $\sum_{j \in J} v_j = 0$.*

(iii) *Any $n - 1$ vectors of $(v_j)_{j \in J}$ are linearly independent.*

(iv) *We have*

$$S^{(a)}(v_j, v_j) = a_j - \frac{a_j^2}{|a|}, \qquad j \in J, \qquad (4.3)$$

$$S^{(a)}(v_i, v_j) = -\frac{a_i a_j}{|a|}, \qquad i, j \in J, \ i \neq j.$$

$\square$

LEMMA 4.2. — *We have*

$$\det_{1 \leqslant i,j \leqslant n-1} (S^{(a)}(v_i, v_j)) = \frac{1}{|a|} \prod_{j \in J} a_j. \qquad (4.4)$$

Proof. — Denote M the transition matrix from the standard basis $F_1, \ldots, F_n$ of V to the basis $v_1, \ldots, v_{n-1}, \sum_{j \in J} F_j$. It is easy to see that $\det M = (-1)^{n-1}$. The vector $\sum_{j \in J} F_j$ is orthogonal to $\mathrm{Sing}\, V$ and $S^{(a)}(\sum_{j \in J} F_j, \sum_{j \in J} F_j) = |a|$. The determinant of $S^{(a)}$ on V with respect to the standard basis $F_1, \ldots, F_n$ equals $\prod_{j \in J} a_j$. These remarks imply (4.4). $\square$

4.3. Operators $K_j(z) : V \to V$

For any pair $\{i, j\} \subset J$, we define the linear operator $L_{i,j} : V \to V$ by the formula

$$F_i \mapsto a_j F_i - a_i F_j, \qquad F_j \mapsto a_i F_j - a_j F_i, \qquad F_m \mapsto 0, \quad \text{if}\ \ m \notin \{i, j\}, (4.5)$$

see formula (3.5). Define the operators $K_j(z) : V \to V$, $j \in J$, by the formula

$$K_j(z) = \sum_{i \neq j} \frac{L_{j,i}}{z_j - z_i}, \tag{4.6}$$

see formula (3.6). For any $j \in J$ and $z \in \mathbb{C}^n - \Delta$, the operator $K_j(z)$ preserves the subspace $\operatorname{Sing} V \subset V$ and is a symmetric operator, that is $S^{(a)}(K_j(z)v, w) = S^{(a)}(v, K_j(z)w)$ for all $v, w \in V$, see Theorem 3.2.

LEMMA 4.3. — *For $j \in J$, we have*

$$\begin{aligned}
K_j(z)v_i &= \frac{a_j}{z_j - z_i}v_i + \frac{a_i}{z_i - z_j}v_j, \qquad i \neq j, \tag{4.7} \\
K_j(z)v_j &= -\sum_{i \neq j} K_j(z)v_i.
\end{aligned}$$

$\square$

COROLLARY 4.4. — *We have $K_j(z)v_i = K_i(z)v_j$ for all i, j.*

The differential equations (3.14) for flat sections of the Gauss-Manin connection on $(\mathbb{C}^n - \Delta) \times \operatorname{Sing} V \to \mathbb{C}^n - \Delta$ take the form

$$\kappa \frac{\partial I}{\partial z_j}(z) = K_j(z)I(z), \qquad j \in J. \tag{4.8}$$

For generic κ all the flat sections are given by the formula

$$I_\gamma(z) = \sum_{i \in J} \Big(\int_{\gamma(z)} \prod_{m \in J} (z_m + t)^{a_m/\kappa} \frac{dt}{z_i + t} \Big) F_i, \tag{4.9}$$

see formula (3.12). More precisely, all the flat sections are given by (4.9) if $1 + \frac{|a|}{\kappa} \notin \mathbb{Z}_{\leqslant 0}$ and $1 + \frac{a_j}{\kappa} \notin \mathbb{Z}_{\leqslant 0}$ for all $j \in J$, see [23] or Theorem 3.3.5 in [26].

Notice that equations (4.8) are a particular case of the KZ equations, see Section 1.1-1.3 in [26].

4.4. Conformal blocks

LEMMA 4.5. — *If $\kappa = |a|$, then the Gauss-Manin connection has a one-dimensional invariant subbundle, generated by the section*

$$q : z \mapsto \frac{1}{|a|} \sum_{j \in J} z_j v_j = \frac{1}{|a|} \sum_{j \in J} q_j(z) F_j, \tag{4.10}$$

where

$$q_i(z) = -z_i + \sum_{j \in J} \frac{a_j}{|a|} z_j. \tag{4.11}$$

This section is flat.

Proof. — The lemma follows from formulas (4.7). $\qquad\square$

This one-dimensional subbundle will be called the bundle of *conformal blocks* at level $|a|$. A flat section of the subbundle of conformal blocks can be presented as an integral $I_\gamma(z)$, where γ is a small circle around infinity.

4.5. Canonical isomorphism and period map

The master function of the arrangement $\mathcal{C}(z)$ is

$$\Phi(z,t) = \sum_{j \in J} a_j \log f_j = \sum_{j \in J} a_j \log(z_j + t). \tag{4.12}$$

The critical point equation is $\frac{\partial \Phi}{\partial t} = \sum_{j \in J} \frac{a_j}{z_j + t} = 0$. The critical set is

$$C_\Phi(z) = \big\{ t \in U(\mathcal{C}(z)) \mid \sum_{j \in J} \frac{a_j}{z_j + t} = 0 \big\}. \tag{4.13}$$

The algebra functions on the critical set is

$$A_\Phi(z) = \mathbb{C}(U(\mathcal{C}(z))) / \big\langle \sum_{j \in J} \frac{a_j}{z_j + t} \big\rangle. \tag{4.14}$$

The identity element $[1](z) \in A_\Phi(z)$ equals $\frac{1}{|a|} \sum_{j \in J} z_j \big[\frac{a_j}{f_j} \big]$.

LEMMA 4.6. — *We have* $\dim A_\Phi(z) = n - 1$. *Any $n - 1$ elements of* $\big(\big[\frac{a_j}{z_j + t} \big] \big)_{j \in J}$ *are linearly independent.* $\qquad\square$

Let $p \in C_\Phi(z)$. The Grothendieck residue $\rho_p : A_{p,\Phi}(z) \to \mathbb{C}$ is given by

$$f \; \mapsto \; \frac{1}{2\pi\sqrt{-1}} \operatorname{Res}_p \frac{f}{\frac{\partial \Phi}{\partial t}} = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_p} \frac{f \, dt}{\frac{\partial \Phi}{\partial t}}, \tag{4.15}$$

where Γ_p is a small circle around the critical point p oriented clock-wise. The residue bilinear form $(\, ,\,)_z$ on $A_\Phi(z)$ is $\oplus_{p \in C_\Phi(z)} (\, ,\,)_p$.

LEMMA 4.7. — *For $f, g \in \mathbb{C}(U(\mathcal{C}(z)))$, we have*

$$([f], [g]) = -\frac{1}{2\pi\sqrt{-1}} \operatorname{Res}_{t=\infty} \frac{fg}{\frac{\partial\Phi}{\partial t}} - \frac{1}{2\pi\sqrt{-1}} \sum_{i \in J} \operatorname{Res}_{t=-z_i} \frac{fg}{\frac{\partial\Phi}{\partial t}}. \qquad (4.16)$$

$\square$

The canonical element is

$$[E] = \sum_{j \in J} \left[\frac{1}{z_j + t} \right] \otimes F_j \quad \in \quad A_\Phi(z) \otimes \operatorname{Sing} V. \qquad (4.17)$$

The canonical isomorphism $\alpha(z) : A_\Phi(z) \to \operatorname{Sing} V$ is given by the formula

$$[f] \mapsto -\frac{1}{2\pi\sqrt{-1}} \sum_{j \in J} \left(\operatorname{Res}_{t=\infty} \frac{f}{(z_j + t)\frac{\partial\Phi}{\partial t}} + \sum_{i \in J} \operatorname{Res}_{t=-z_i} \frac{f}{(z_j + t)\frac{\partial\Phi}{\partial t}} \right) F_j. \qquad (4.18)$$

THEOREM 4.8. — *For $k \in J$, we have*

$$\alpha(z) : \left[\frac{a_k}{z_k + t} \right] \mapsto v_k. \qquad (4.19)$$

Proof. — Denote

$$g_{kj} = \frac{a_k}{(z_k + t)(z_j + t)} \frac{1}{\frac{\partial\Phi}{\partial t}} = \frac{a_k}{(z_k + t)(z_j + t)} \frac{\prod_{m \in J}(z_m + t)}{\sum_{m \in J} a_m \prod_{\ell \neq m}(z_\ell + t)}. \qquad (4.20)$$

If $k \neq j$, then $\operatorname{Res}_{t=-z_i} g_{jk} = 0$ for all $i \in J$. If $k = j$, then $\operatorname{Res}_{t=-z_i} g_{jj} = 0$ for $i \neq j$ and $\operatorname{Res}_{t=-z_j} g_{jj} = 2\pi\sqrt{-1}$. We also have $\operatorname{Res}_{t=\infty} g_{kj} = -2\pi\sqrt{-1}\frac{a_k}{|a|}$ for all $j \in J$. We obtain the theorem by comparing these formulas with formula (4.2). $\square$

COROLLARY 4.9. — *Conjectures 3.7 and 3.8 hold for this family of arrangements.*

Proof. — By Theorem 4.8, we have $\alpha(z)([1](z)) = q(z)$, where $q(z)$ is given by (4.10). Lemma 4.5 implies Conjectures 3.7 and 3.8. $\square$

COROLLARY 4.10. — *For this family of arrangements the period map $q : \mathbb{C}^n - \Delta \to \operatorname{Sing} V$ is given by the formula*

$$q(z) = \frac{1}{|a|} \sum_{j \in J} z_j v_j = \frac{1}{|a|} \sum_{j \in J} q_j(z) F_j, \qquad (4.21)$$

the potential function of first kind is

$$P(z) = \frac{1}{|a|^2} \sum_{j \in J} a_j q_j^2(z) = \sum_{1 \leqslant i < j \leqslant n} \frac{a_1 a_2}{|a|^3} (z_i - z_j)^2. \qquad (4.22)$$

By Corollary 4.10, the period map extends to a linear map $\mathbb{C}^n \to \operatorname{Sing} V$. The linear map is an epimorphism. The kernel is generated by the vector $(1, \ldots, 1)$.

The standard basis $(H_j)_{j \in J} \in V^*$ induces linear functions on $\operatorname{Sing} V$,

$$h_j : v_i \mapsto \frac{a_i}{|a|}, \qquad \text{if } j \neq i, \qquad v_j \mapsto -1 + \frac{a_j}{|a|}. \tag{4.23}$$

We have $\sum_{j \in J} a_j h_j = 0$ and any $n - 1$ of these functions form a basis of $(\operatorname{Sing} V)^*$.

For $i \neq j$, define the hyperplane $\tilde{H}_{i,j} \subset \operatorname{Sing} V$ by the equation $h_i - h_j = 0$.

LEMMA 4.11. — *For all i, j we have $q^*(h_i - h_j) = z_j - z_i$ and $q(\Delta) = \cup_{i < j} \tilde{H}_{i,j}$.* $\square$

4.6. Contravariant map as the inverse to the canonical map

The canonical map $\alpha(z) : A_\Phi(z) \to \operatorname{Sing} V$ is the isomorphism described in Theorem 4.8. The contravariant map $\mathcal{S}^{(a)} : V \to V^*$ is defined by the formula $F_i \mapsto a_i(H_i)$. By identifying $a_i(H_i)$ with the differential form $\frac{a_i}{f_i} dt$ and then projecting the coefficient to $A_\Phi(z)$ we obtain the map

$$[\mathcal{S}^{(a)}] : V \to A_\Phi(z), \qquad F_i \mapsto \left[\frac{a_i}{f_i} \right]. \tag{4.24}$$

THEOREM 4.12. — *The composition $\alpha(z) \circ [\mathcal{S}^{(a)}] : V \to \operatorname{Sing} V$ is the orthogonal projection multiplied by -1. The composition $[\mathcal{S}^{(a)}] \circ \alpha(z) : A_\Phi(z) \to A_\Phi(z)$ is the identity map multiplied by -1.*

Proof. — The composition $\alpha(z) \circ [\mathcal{S}^{(a)}]$ sends F_i to v_i which is the orthogonal projection multiplied by -1 The composition $[\mathcal{S}^{(a)}] \circ \alpha(z)$ sends $\left[\frac{a_i}{f_i} \right]$ to

$$-\left[\frac{a_i}{f_i} \right] + \frac{a_i}{|a|} \sum_{j \in J} \left[\frac{a_j}{f_j} \right]. \tag{4.25}$$

The last sum is zero in $A_\Phi(z)$. $\square$

4.7. Multiplication on $\operatorname{Sing} V$ and $(\operatorname{Sing} V)^*$

THEOREM 4.13. — *The canonical isomorphism $\alpha(z) : A_\Phi(z) \to \operatorname{Sing} V$ defines an algebra structure on $\operatorname{Sing} V$,*

$$
\begin{aligned}
v_j *_z v_i &= \frac{a_j}{z_j - z_i} v_i + \frac{a_i}{z_i - z_j} v_j, \qquad i \neq j, &&(4.26) \\
v_j *_z v_j &= -\sum_{i \neq j} v_j *_z v_i.
\end{aligned}
$$

The element

$$
\frac{1}{|a|} \sum_{j \in J} z_j v_j \tag{4.27}
$$

is the identity element. □

The isomorphism $S^{(a)}|_{\operatorname{Sing} V} : \operatorname{Sing} V \to (\operatorname{Sing} V)^*$ induces an algebra structure on $(\operatorname{Sing} V)^*$.

LEMMA 4.14. — *The isomorphism $S^{(a)}|_{\operatorname{Sing} V} : \operatorname{Sing} V \to (\operatorname{Sing} V)^*$ is given by the formula $v_j \mapsto -a_j h_j$ for all j.*

Proof. The lemma follows from formulas (4.23) and (4.3). □

COROLLARY 4.15. — *The multiplication on $(\operatorname{Sing} V)^*$ is given by the formula*

$$
\begin{aligned}
h_j *_z h_i &= \frac{1}{z_i - z_j} h_i + \frac{1}{z_j - z_i} h_j, \qquad i \neq j, &&(4.28) \\
a_j h_j *_z h_j &= -\sum_{i \neq j} a_i h_i *_z h_j.
\end{aligned}
$$

The element

$$
-\frac{1}{|a|} \sum_{j \in J} a_j z_j h_j \tag{4.29}
$$

is the identity element. □

4.8. Tangent morphism

The tangent morphism β of the tangent bundle $T(\mathbb{C}^n - \Delta) \to \mathbb{C}^n - \Delta$ to the bundle of algebras $\sqcup_{z \in \mathbb{C}^n - \Delta} A_\Phi(z) \to \mathbb{C}^n - \Delta$ is given by the formula (3.31),

$$
\beta(z) \ : \ \partial_j \in T_z(\mathbb{C}^n - \Delta) \ \mapsto \ \left[\frac{\partial \Phi}{\partial z_j}\right] = \left[\frac{a_j}{z_j + t}\right] \in A_\Phi(z). \tag{4.30}
$$

LEMMA 4.16. — *The map $\beta(z)$ is an epimorphism. The kernel of $\beta(z)$ is generated by the vector $\sum_{j \in J} \partial_j$.* $\qquad\square$

The residue form on the bundle of algebras induces a holomorphic symmetric bilinear form η on $T(\mathbb{C}^n - \Delta)$, see formula (3.32). The bilinear form η has rank $n - 1$. Its kernel is generated by the vector $\sum_{j \in J} \partial_j$.

LEMMA 4.17. — *We have*

$$\eta(\partial_j, \partial_j) = -a_j + \frac{a_j^2}{|a|}, \qquad j \in J, \tag{4.31}$$

$$\eta(\partial_i, \partial_j) = \frac{a_i a_j}{|a|}, \qquad i, j \in J, \; i \neq j.$$

Proof. — The lemma follows from Lemmas 4.1, 4.8 and Theorem 2.7. It can be checked also by a straightforward calculation. $\qquad\square$

4.9. Multiplication and potential function of second kind

Let us define the multiplication on fibers of $T(\mathbb{C}^n - \Delta)$ by the formulas

$$\partial_i *_z \partial_j = \frac{a_i}{z_i - z_j} \partial_j + \frac{a_j}{z_j - z_i} \partial_i, \tag{4.32}$$

$$\partial_i *_z \partial_i = -\sum_{j \neq i} \partial_i *_z \partial_j,$$

cf. formula 5.25 in [5]. The vector $\sum_{i \in J} \partial_i$ has zero product with everything.

LEMMA 4.18. — *For every $z \in \mathbb{C}^n - \Delta$, the morphism $\beta(z)$ defines an algebra epimorphism of $T_z(\mathbb{C}^n - \Delta)$ to $A_\Phi(z)$, in particular, $\beta(z)(v) *_z \beta(z)(w) = \beta(z)(v *_z w)$ for all $v, w \in T_z(\mathbb{C}^n - \Delta)$.* $\qquad\square$

Consider the ideal of $T_z(\mathbb{C}^n - \Delta)$ generated by $\sum_{j \in J} \partial_j$. Denote $B(z)$ the quotient algebra. The morphism $\beta(z)$ induces an isomorphism $B(z) \simeq A_\Phi(z)$.

LEMMA 4.19. — *The element $\frac{1}{|a|} \sum_{i \in J} z_j \partial_j$ projects to the identity element of $B(z)$.* $\qquad\square$

The bilinear form η defines a morphism $\tilde{\eta}$ of the tangent bundle $T(\mathbb{C}^n - \Delta)$ to the cotangent bundle $T^*(\mathbb{C}^n - \Delta)$. For $j \in J$, denote $p_j(z) = a_j q_j(z) = a_j(-z_j + \sum_{i \in J} \frac{a_i}{|a|} z_i)$. We have $\sum_{j \in J} p_j = 0$.

LEMMA 4.20. — *The morphism $\tilde{\eta}$ is given by the formula $\partial_j \mapsto dp_j$ for all j. The kernel of $\tilde{\eta}$ is generated by the vector $\sum_{j \in J} \partial_j$.* $\qquad\square$

Consider the span of differential one-forms $(dp_j)_{j \in J}$. This span equals the span of differential one-forms $(dz_i - dz_j)_{1 \leqslant i < j \leqslant n}$. The spans in the fibers define the subbundle

$$\sqcup_{z \in \mathbb{C}^n - \Delta} B^*(z) \to \mathbb{C}^n - \Delta \qquad (4.33)$$

of the cotangent bundle $T^*(\mathbb{C}^n - \Delta)$. The subbundle has rank $n - 1$.

LEMMA 4.21. — *The form η induces the algebra structure on $B^*(z)$ given by the formula*

$$
\begin{aligned}
dp_i *_z dp_j &= \frac{a_i}{z_i - z_j} dp_j + \frac{a_j}{z_j - z_i} dp_i = \frac{a_i a_j}{z_i - z_j} d(z_i - z_j), && i \neq j, \\
dp_i *_z dp_i &= -\sum_{j \neq i} dp_j *_z dp_i, && (4.34)
\end{aligned}
$$

and the bilinear form

$$
\begin{aligned}
(dp_j, dp_j) &= -a_j + \frac{a_j^2}{|a|}, && j \in J, && (4.35) \\
(dp_i, dp_j) &= \frac{a_i a_j}{|a|}, && i, j \in J, \ i \neq j.
\end{aligned}
$$

$\square$

Introduce the *potential function of second kind*

$$\tilde{P}(z) = \frac{1}{2} \sum_{1 \leqslant i < j \leqslant n} a_i a_j (z_i - z_j)^2 \log(z_i - z_j). \qquad (4.36)$$

THEOREM 4.22. — *We have*

$$d\left(\frac{\partial^2 \tilde{P}}{\partial z_i \partial z_j}\right) = -\tilde{\eta}(\partial_i) *_z \tilde{\eta}(\partial_j) \qquad (4.37)$$

for all i, j.

Proof. — The theorem follows from Lemma 4.21. $\square$

Notice that equation (4.37) is the definition (3.5) in [5] of the potential function of an almost dual Frobenius structure.

The right hand side in (4.37) can be rewritten: $\tilde{\eta}(\partial_i) *_z \tilde{\eta}(\partial_j) = \tilde{\eta}(\partial_i *_z \partial_j)$. For all i, j, k, we have

$$\tilde{\eta}(\partial_i *_z \partial_j)(\partial_\ell) = \eta(\partial_i *_z \partial_j, \partial_\ell) = (\beta(z)(\partial_i) *_z \beta(z)(\partial_j) *_z \beta(z)(\partial_\ell), [1](z))_z, \qquad (4.38)$$

where $(\ ,\ )_z$ is the residue form on $A_\Phi(z)$. Formula (4.37) says that for all i, j, k, we have

$$\frac{\partial^3 \tilde{P}}{\partial z_i \partial z_j \partial_\ell}(z) = -(\beta(z)(\partial_i) *_z \beta(z)(\partial_j) *_z \beta(z)(\partial_\ell), [1](z))_z. \qquad (4.39)$$

Hence Conjecture 3.14 holds for this family of arrangements.

4.10. Connections on the bundle $\sqcup_{z \in \mathbb{C}^n - \Delta} B^*(z) \to \mathbb{C}^n - \Delta$ defined in (4.33)

The combinatorial and Gauss-Manin connections on $(\mathbb{C}^n - \Delta) \times \mathrm{Sing}\, V \to \mathbb{C}^n - \Delta$ induce the combinatorial and Gauss-Manin connections on bundle (4.33).

LEMMA 4.23. — *The differential one-forms* $(dp_j)_{j \in J}$ *are flat sections of the combinatorial connection on bundle* (4.33).

Proof. — The vectors $v_j \in \mathrm{Sing}\, V$ give flat sections of the combinatorial connection on $\mathbb{C}^n \times \mathrm{Sing}\, V \to \mathbb{C}^n$. By formula (4.19), the elements $\left[\frac{a_j}{z_j + t}\right] \in A_\Phi(z)$ give flat sections of bundle of algebras. Now formula (4.30) and Lemma 4.20 imply Lemma 4.23. $\qquad\square$

Let $I(z) = \sum_{j \in J} I^j(z) dp_j$ be a section of bundle (4.33). For $i \in J$, we denote $\frac{\partial I}{\partial z_i} = \sum \frac{\partial I^j}{\partial z_i} dp_j$.

LEMMA 4.24. — *The differential equations for flat sections of the Gauss-Manin connection take the from*

$$\kappa \frac{\partial I}{\partial z_i} = dp_i *_z I, \qquad i \in J, \qquad (4.40)$$

see formula (4.34). *For generic* κ *all the flat sections are given by the formula*

$$I_{\gamma,\kappa}(z) = \sum_{j \in J} \Big(\int_{\gamma(z)} \prod_{i \in J} (z_i + t)^{a_i/\kappa} \frac{dt}{z_j + t} \Big) dp_j, \qquad (4.41)$$

where $\gamma(z) \in H_1(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))})$ *is a flat section of the Gauss-Manin connection on* $\sqcup_{z \in \mathbb{C}^n - \Delta} H_k(U(\mathcal{C}(z)), \mathcal{L}_\kappa|_{U(\mathcal{C}(z))}) \to \mathbb{C}^n - \Delta$.

Proof. — The lemma follows from Theorem 3.3 and formula (4.9). $\qquad\square$

THEOREM 4.25. — *For every flat section* $I_{\gamma,\kappa}$, *we have* $I_{\gamma,\kappa} = -\kappa\, dp_{\gamma,\kappa}$ *where*

$$p_{\gamma,\kappa} = \int_{\gamma(z)} \prod_{i \in J} (z_i + t)^{a_i/\kappa} dt. \qquad (4.42)$$

Proof. — The theorem follows from two formulas:

$$\kappa \frac{\partial p_{\gamma,\kappa}}{\partial z_j} = \int_{\gamma(z)} \prod_{i \in J} (z_i + t)^{a_i/\kappa} \frac{a_j dt}{z_j + t} \qquad (4.43)$$

and $\sum_{j \in J} \frac{\partial p_{\gamma,\kappa}}{\partial z_j} = 0$. □

Following Dubrovin [4, 5], we will call the functions $p_{\gamma,\kappa}$ *twisted periods*. Notice that this definition agrees with the definition of twisted periods in Section 3.11, namely, the twisted periods of Theorem 4.25 can be also defined by formula (3.42) of Theorem 3.13.

LEMMA 4.26. — *Given $\kappa \in \mathbb{C}^\times$, let $I(z, \kappa)$ be a flat section of the Gauss-Manin connection with the parameter κ. Let $I(z, -\kappa)$ be a flat section of the Gauss-Manin connection with the parameter $-\kappa$. Then $(I(z, \kappa), I(z, -\kappa))_z$ does not depend on $z \in \mathbb{C}^n - \Delta$.* □

Proof. — The lemma follows from the fact that $B^*(z)$ is a Frobenius algebra. □

4.11. Functoriality

In this section we will discuss how our objects extend to strata of the discriminant $\Delta \subset \mathbb{C}^n$.

4.11.1. A stratum X of Δ is given by a partition $(J_1, \ldots, J_m)$ of J,

$$X = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i - z_j = 0 \quad \text{for } i, j \in J_\ell, \ \ell = 1, \ldots, m\}, \ (4.44)$$

$\dim X = m$. The coordinates on X are functions $x_1, \ldots, x_m$ where $x_\ell = z_j$ for $j \in J_\ell$. Let $\iota : X \hookrightarrow \mathbb{C}^n$ be the natural embedding. Then

$$\iota_* : \frac{\partial}{\partial x_\ell} \mapsto \sum_{j \in J_\ell} \frac{\partial}{\partial z_j}, \qquad \iota^* : dz_j \mapsto dx_\ell \quad \text{if } j \in J_\ell. \qquad (4.45)$$

The remaining strata of Δ cut on X the union of hyperplanes $x_i = x_j$, $1 \leqslant i < j \leqslant m$ which we denote Δ_X. For $\ell = 1, \ldots, m$, we denote $b_\ell = \sum_{j \in J_\ell} a_j$. We assume that $b_\ell \neq 0$ for all ℓ.

We restrict our family of arrangements $\mathcal{C}(z)$, $z \in \mathbb{C}^n$, to $X - \Delta_X$. For $x \in X - \Delta_X$ the corresponding arrangement $\mathcal{C}(x)$ consists of points $-x_1, \ldots, -x_m$ of weights $b_1, \ldots, b_m$, respectively. For this new family we will construct all the objects described in Sections 4.1-4.10 and relate them to the objects constructed for the arrangements $\mathcal{C}(z)$, $z \in \mathbb{C}^n - \Delta$. The objects corresponding to the new family will be provided with the index X.

4.11.2. For $x \in X - \Delta_X$, the space $(V_X)^* = \mathcal{A}^1(\mathcal{C}(x))$ has the standard basis $(H_{\ell,X}), \ell = 1, \ldots, m$. Recall that the space V^* of Sections 4.1-4.10 has the standard basis $(H_j), j \in J$. We have the canonical epimorphism

$$f^* : V^* \to (V_X)^*, \qquad (H_j) \mapsto (H_{\ell,X}) \quad \text{if} \quad j \in J_\ell. \qquad (4.46)$$

The space $V_X = \mathcal{F}^1(\mathcal{C}(x))$ has the standard basis $F_{\ell,X}, \ell = 1, \ldots, m$. The space V of Sections 4.1-4.10 has the standard basis $F_j, j \in J$. We have the canonical embedding

$$f : V_X \hookrightarrow V, \qquad F_{\ell,X} \mapsto \sum_{j \in J_\ell} F_j. \qquad (4.47)$$

The subspace of singular vector is defined by the formula

$$\operatorname{Sing} V_X = \Big\{ \sum_{\ell=1}^{m} c_\ell F_{\ell,X} \mid \sum_{\ell=1}^{m} b_\ell c_\ell = 0 \Big\}. \qquad (4.48)$$

We have $f(\operatorname{Sing} V_X) = f(V_X) \cap (\operatorname{Sing} V)$. Consider the embedding

$$\tilde{f} : \operatorname{Sing} V_X \hookrightarrow \operatorname{Sing} V, \qquad v \mapsto f(v). \qquad (4.49)$$

For the contravariant form on V_X we have

$$S_X^{(b)}(F_{\ell,X}, F_{k,X}) = S^{(a)}(f(F_{\ell,X}), f(F_{k,X})) = \delta_{\ell,k} b_k. \qquad (4.50)$$

For $\ell = 1, \ldots, m$, we define a vector $v_{\ell,X} \in \operatorname{Sing} V_X$ by the formula

$$v_{\ell,X} = -F_{\ell,X} + \frac{b_\ell}{|a|} \sum_{k=1}^{m} F_{k,X}. \qquad (4.51)$$

We have $f : v_{\ell,X} \mapsto \sum_{j \in J_\ell} v_j$.

The standard basis $(H_{\ell,X}), \ell = 1, \ldots, m$, induces linear functions on $\operatorname{Sing} V_X$,

$$h_{\ell,X} : v_{k,X} \mapsto \frac{b_k}{|a|} \quad \text{if} \quad k \neq \ell, \qquad v_{\ell,X} \mapsto -1 + \frac{b_\ell}{|a|}. \qquad (4.52)$$

We have $\tilde{f}^* : h_j \mapsto h_{\ell,X}$ if $j \in J_\ell$.

4.11.3. For $\ell = 1, \ldots, m$ and $x \in X - \Delta_X$, the operators $K_{\ell,X}(x)$: $\mathrm{Sing}\, V_X \to \mathrm{Sing}\, V_X$ are defined by formulas (4.7),

$$
\begin{aligned}
K_{\ell,X}(x)v_{k,X} &= \frac{b_\ell}{x_\ell - x_k}v_{k,X} + \frac{b_k}{x_k - x_\ell}v_{\ell,X} \quad \text{for} \quad \ell \neq k, \quad (4.53) \\
K_{\ell,X}(x)v_{\ell,X} &= -\sum_{k \neq \ell} K_{\ell,X}(x)v_{k,X}.
\end{aligned}
$$

For all ℓ, k, we have

$$
f(K_{\ell,X}(x)v_{k,X}) = \sum_{j \in J_\ell} K_j(x)f(v_{k,X}). \tag{4.54}
$$

Notice that the right hand side in (4.54) is well-defined despite the fact that $K_j(x)v_i$ is not well-defined for all v_i, see formula (4.7).

4.11.4. Multiplication on $\mathrm{Sing}\, V_X$ is defined by formulas (4.26),

$$
\begin{aligned}
v_{\ell,X} *_{x,X} v_{k,X} &= \frac{b_\ell}{x_\ell - x_k}v_{k,X} + \frac{b_k}{x_k - x_\ell}v_{\ell,X} \quad \text{for} \quad \ell \neq k, \quad (4.55) \\
v_{\ell,X} *_{x,X} v_{\ell,X} &= -\sum_{k \neq \ell} v_{\ell,X} *_{x,X} v_{k,X}.
\end{aligned}
$$

For all ℓ, k, we have

$$
f(v_{\ell,X} *_{x,X} v_{k,X}) = f(v_{\ell,X}) *_x f(v_{k,X}). \tag{4.56}
$$

Notice that the right hand side in (4.56) is well-defined despite the fact that $v_i *_x v_j$ is not well-defined for all v_i, v_j, see formula (4.26).

4.11.5. The multiplication on $(\mathrm{Sing}\, V_X)^*$ is given by formula (4.28),

$$
\begin{aligned}
h_{\ell,X} *_{x,X} h_{k,X} &= \frac{1}{x_\ell - x_k}h_{\ell,X} + \frac{1}{z_k - z_\ell}h_{k,X}, \quad \ell \neq k, \quad (4.57) \\
b_\ell h_{\ell,X} *_{x,X} h_{k,\ell} &= -\sum_{k \neq \ell} b_k\, h_{k,X} *_{x,X} h_{\ell,X}.
\end{aligned}
$$

If $\ell \neq k$, $i \in J_\ell, j \in J_k$, then

$$
\tilde{f}^*(h_i *_x h_j) = h_{\ell,X} *_{x,X} h_{k,X}. \tag{4.58}
$$

Notice that $h_i *_x h_j$ is well-defined despite the fact that $h_i *_x h_j$ is not well-defined for all h_i, h_j, see formula (4.26).

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4.11.6. For $x \in X - \Delta_X$ the residue form on $A_\Phi(x)$ induces a holomorphic bilinear form η_X on $T_x(X - \Delta_X)$,

$$\eta_X\Big(\frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_\ell}\Big) = -b_\ell + \frac{b_\ell^2}{|a|}, \qquad j \in J, \tag{4.59}$$

$$\eta_X\Big(\frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_k}\Big) = \frac{b_\ell b_k}{|a|}, \qquad \ell \neq k.$$

For all ℓ, k, have

$$\eta_X\Big(\frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_k}\Big) = \eta\Big(\sum_{i \in J_\ell} \frac{\partial}{\partial z_i}, \sum_{j \in J_\ell} \frac{\partial}{\partial z_j}\Big). \tag{4.60}$$

For $\ell = 1, \ldots, m$, we define a linear function on X,

$$q_{\ell,X}(x) = -x_\ell + \sum_{k \neq \ell} \frac{b_k}{|a|} x_k. \tag{4.61}$$

We have

$$q_{\ell,X}(x) = q_i(x), \qquad \text{if } i \in J_\ell. \tag{4.62}$$

The period map $q_X : X - \Delta_X \to \operatorname{Sing} V_X$ is defined by formula (4.21),

$$q_X(x) = \frac{1}{|a|} \sum_{\ell=1}^m q_{\ell,X}(x) F_{\ell,X} = \frac{1}{|a|} \sum_{\ell=1}^m x_\ell \, v_{\ell,X}. \tag{4.63}$$

THEOREM 4.27. — *For all $x \in X$, we have*

$$f(q_X(x)) = q(x) \tag{4.64}$$

and for the potential functions of first kind we have

$$P_X(x) = P(x). \tag{4.65}$$

$\square$

4.11.7. For $x \in X - \Delta_X$, the multiplication on $T_x(X - \Delta_X)$ is defined by formulas (4.32),

$$\frac{\partial}{\partial x_\ell} *_{x,X} \frac{\partial}{\partial x_k} = \frac{b_\ell}{x_\ell - x_k} \frac{\partial}{\partial x_k} + \frac{b_k}{x_k - x_\ell} \frac{\partial}{\partial x_\ell}, \tag{4.66}$$

$$\frac{\partial}{\partial x_\ell} *_{x,X} \frac{\partial}{\partial x_\ell} = -\sum_{k \neq \ell} \frac{\partial}{\partial x_k} *_{x,X} \frac{\partial}{\partial x_\ell}.$$

For all ℓ, k, we have

$$\frac{\partial}{\partial x_\ell} *_{x,X} \frac{\partial}{\partial x_k} = \Big(\sum_{i \in J_\ell} \frac{\partial}{\partial z_i} \Big) *_x \Big(\sum_{j \in J_k} \frac{\partial}{\partial z_j} \Big). \tag{4.67}$$

Notice that the right hand side in (4.67) is well-defined despite the fact that $\frac{\partial}{\partial z_i} *_x \frac{\partial}{\partial z_j}$ is not well-defined for all i, j, see (4.66).

4.11.8. For $\ell = 1, \ldots, m$, we denote $p_{\ell,X} = b_\ell q_{\ell,X}$. The map

$$\tilde{\eta}_X : T_x(X - \Delta_X) \to T_x^*(X - \Delta_X), \quad w \mapsto \eta_X(w, \cdot), \tag{4.68}$$

sends $\frac{\partial}{\partial x_\ell}$ to $dp_{\ell,X}$. We have

$$dp_{\ell,X} = \iota^* \Big(\sum_{i \in J_\ell} dp_i \Big). \tag{4.69}$$

Denote $B_X^*(x)$ the span of $(dp_{\ell,X})_{\ell=1}^m$ in $T_x^*(X - \Delta_X)$. This span equals the span of differential forms $(dx_\ell - dx_k)_{1 \leqslant \ell < k \leqslant m}$. The multiplication on $B_X^*(x)$ is given by the formulas (4.34),

$$
\begin{aligned}
dp_{\ell,X} *_{x,X} dp_{k,X} &= \frac{b_\ell}{x_\ell - x_k} dp_{k,X} + \frac{b_k}{x_k - x_\ell} dp_{\ell,X} \\
&= \frac{b_\ell b_k}{x_\ell - x_k} d(x_\ell - x_k), \ \ell \neq k, \\
dp_{\ell,X} *_{x,X} dp_{\ell,X} &= -\sum_{k \neq \ell} dp_{k,X} *_{x,X} dp_{\ell,X}.
\end{aligned}
\tag{4.70}
$$

For all ℓ, k, we have

$$\iota^* \Big(\sum_{i \in J_\ell} dp_i \Big) *_{x,X} \iota^* \Big(\sum_{j \in J_k} dp_j \Big) = \iota^* \Big(\sum_{i \in J_\ell} \sum_{j \in J_k} dp_i *_x dp_j \Big). \tag{4.71}$$

The potential function of second kind is defined by formula (4.36),

$$\tilde{P}_X(x_1, \ldots, x_m) = \frac{1}{2} \sum_{1 \leqslant \ell < k \leqslant m} b_\ell b_k (x_\ell - x_k)^2 \log(x_\ell - x_k). \tag{4.72}$$

By formula (4.37),

$$d \Big(\frac{\partial^2 \tilde{P}_X}{\partial x_\ell \partial x_k} \Big) = -\tilde{\eta}_X \Big(\frac{\partial}{\partial x_\ell} \Big) *_{x,X} \tilde{\eta}_X \Big(\frac{\partial}{\partial x_k} \Big) \tag{4.73}$$

for all ℓ, k. For all ℓ, k, we have

$$\frac{\partial^2 \tilde{P}_X}{\partial x_\ell \partial x_k}(x) = \lim_{\substack{z \to x \\ z \in \mathbb{C}^n - \Delta}} \sum_{i \in J_\ell} \sum_{j \in J_k} \frac{\partial^2 \tilde{P}}{\partial z_i \partial z_j}(z). \tag{4.74}$$

4.12. Frobenius like structure

Consider the quotient M of $\mathbb{C}^n$ by the one-dimensional subspace $\mathbb{C}(1,\ldots,1)$ and the natural projection $\pi : \mathbb{C}^n \to M$. Then all our objects – the combinatorial bundle $(\mathbb{C}^n - \Delta) \times (\operatorname{Sing} V) \to \mathbb{C}^n - \Delta$ with the contravariant form $S^{(a)}$ and connections (combinatorial and Gauss-Manin); the bundle of algebras $\sqcup_{z \in \mathbb{C}^n - \Delta} A_\Phi(z) \to \mathbb{C}^n - \Delta$; the period map $q : \mathbb{C}^n - \Delta \to \operatorname{Sing} V$, potential functions $P(z)$ and $\tilde{P}(z)$, flat periods $p_j(z)$, twisted periods $p_{\gamma,\kappa}(z)$ – descend to the quotient and form on $M - \pi(\Delta)$ a structure which we will also call a *Forbenius like structure*.

In particular, the functions $p_1,\ldots,p_{n-1}$ will form a coordinate systems on M and η will induce a holomorphic metric on M constant with respect to the coordinates $p_1,\ldots,p_{n-1}$. If $y_i = \sum_{j=1}^{n-1} c_{j,i}p_j$, $j = 1,\ldots,n-1$, is a linear change of coordinates with $c_{i,j} \in \mathbb{C}$ such that $\eta = \sum_{j=1}^{n-1} dy_j^2$. Then equation (4.37) will take the form

$$d\left(\frac{\partial^2 \tilde{P}}{\partial y_i \partial y_j}\right) \;=\; -dy_i *_y dy_j \tag{4.75}$$

for all i,j. The functions $-\frac{\partial^3 \tilde{P}}{\partial y_i \partial y_j \partial y_k}$ will become the structure constants of the multiplication on $T_y^*(M - \pi(\Delta))$ and the potential function $\tilde{P}$ will satisfies the WDVV equations with respect to the coordinates $y_1,\ldots,y_{n-1}$.

5. Generic lines on plane

5.1. An arrangement in $\mathbb{C}^n \times \mathbb{C}^2$

5mm Consider $\mathbb{C}^2$ with coordinates t_1, t_2, $\mathbb{C}^n$ with coordinates $z_1,\ldots,z_n$. Fix n linear functions on $\mathbb{C}^2$, $g_j = b_j^1 t_1 + b_j^2 t_2$, $j \in J$, $b_j^i \in \mathbb{C}$. We assume that

$$d_{i,j} = \det\begin{pmatrix} b_i^1 & b_i^2 \\ b_j^1 & b_j^2 \end{pmatrix} \neq 0 \qquad \text{for all } i \neq j. \tag{5.1}$$

We define n linear functions on $\mathbb{C}^n \times \mathbb{C}^2$, $f_j = z_j + g_j$, $j \in J$. In $\mathbb{C}^n \times \mathbb{C}^2$ we define the arrangement $\tilde{\mathcal{C}} = \{\tilde{H}_j \mid f_j = 0,\ j \in J\}$.

For every $z = (z_1,\ldots,z_n)$ the arrangement $\tilde{\mathcal{C}}$ induces an arrangement $\mathcal{C}(z)$ in the fiber of the projection $\mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}^n$ over z. We identify every fiber with $\mathbb{C}^2$. Then $\mathcal{C}(z)$ consists of lines $H_j(z)$, $j \in J$, defined in $\mathbb{C}^2$ by the equations $f_j = 0$. Denote $U(\mathcal{C}(z)) = \mathbb{C}^2 - \cup_{j \in J} H_j(z)$, the complement to the arrangement $\mathcal{C}(z)$.

The arrangement $\mathcal{C}(z)$ is with normal crossings if and only if $z \in \mathbb{C}^n - \Delta$, where $\Delta = \cup_{1 \leqslant i < j < k \leqslant n} H_{i,j,k}$ and the hyperplane $H_{i,j,k}$ is defined by the equation $f_{i,j,k} = 0$,

$$f_{i,j,k} = d_{j,k} z_i + d_{k,i} z_j + d_{i,j} z_k. \tag{5.2}$$

LEMMA 5.1. — *For any four distinct indices i, j, k, ℓ we have*

$$\frac{f_{i,j,k}}{d_{k,i} d_{i,j}} + \frac{f_{i,k,\ell}}{d_{\ell,i} d_{i,k}} + \frac{f_{i,\ell,j}}{d_{j,i} d_{i,\ell}} = 0, \tag{5.3}$$

$$\frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} - \frac{f_{j,k,\ell}^2}{d_{j,k} d_{k,\ell} d_{\ell,j}} + \frac{f_{k,\ell,i}^2}{d_{k,\ell} d_{\ell,i} d_{i,k}} - \frac{f_{\ell,i,j}^2}{d_{\ell,i} d_{i,j} d_{j,\ell}} = 0. \tag{5.4}$$

$\square$

5.2. Good fibers

For any $z \in \mathbb{C}^n - \Delta$, the space $\mathcal{A}^2(\mathcal{C}(z))$ has the standard basis $(H_i(z), H_j(z))$, $1 \leqslant i < j \leqslant n$. The space $\mathcal{F}^2(\mathcal{C}(z))$ has the standard dual basis $F(H_i(z), H_j(z))$, $1 \leqslant i < j \leqslant n$. For $z^1, z^2 \in \mathbb{C}^n - \Delta$, the combinatorial connection identifies the spaces $\mathcal{A}^2(\mathcal{C}(z^1))$, $\mathcal{F}^2(\mathcal{C}(z^1))$ with the spaces $\mathcal{A}^2(\mathcal{C}(z^2))$, $\mathcal{F}^2(\mathcal{C}(z^2))$, respectively, by identifying the standard bases.

Assume that nonzero weights $(a_j)_{j \in J}$ are given. Then the arrangement $\mathcal{C}(z)$ is weighted. For $z \in \mathbb{C}^n - \Delta$, the arrangement $\mathcal{C}(z)$ is unbalanced if $|a| \neq 0$. We assume that $|a| \neq 0$.

For $z \in \mathbb{C}^n - \Delta$, we denote $V = \mathcal{F}^2(\mathcal{C}(z))$, $V^* = (\mathcal{F}^2(\mathcal{C}(z))^* = \mathcal{A}^2(\mathcal{C}(z))$, $F_{i,j} = F(H_i(z), H_j(z))$. We have $F_{i,j} = -F_{j,i}$,

$$S^{(a)}(F_{i,j}, F_{k,\ell}) = 0, \quad \text{if } i < j, \ k < \ell \text{ and } (i,j) \neq (k,\ell), \tag{5.5}$$

$$S^{(a)}(F_{i,j}, F_{i,j}) = a_i a_j,$$

$$\operatorname{Sing} V = \Big\{ \sum_{1 \leqslant i < j \leqslant n} c_{i,j} F_{i,j} \mid \sum_{j=1}^{i-1} a_j c_{j,i} - \sum_{j=i+1}^{n} a_j c_{i,j} = 0, \ i = 1, \dots, n \Big\}. \tag{5.6}$$

By Corollary 2.8, the restriction of $S^{(a)}$ to $\operatorname{Sing} V$ is nondegenerate. Denote $(\operatorname{Sing} V)^\perp$ the orthogonal complement to $\operatorname{Sing} V$ with respect to $S^{(a)}$. Then $V = \operatorname{Sing} V \oplus (\operatorname{Sing} V)^\perp$. Denote $\pi : V \to \operatorname{Sing} V$ the orthogonal projection.

LEMMA 5.2. — *The space $(\operatorname{Sing} V)^\perp$ is generated by vectors*

$$\sum_{i \in J} F_{i,j}, \qquad j \in J. \tag{5.7}$$

$\square$

For $i \neq j$, we define the vector $v_{i,j} \in V$ by the formula

$$v_{i,j} = F_{i,j} - \frac{a_j}{|a|} \sum_{k \in J} F_{i,k} - \frac{a_i}{|a|} \sum_{\ell \in J} F_{\ell,j}. \tag{5.8}$$

We have $v_{i,j} = -v_{j,i}$. Set $v_{i,i} = 0$.

LEMMA 5.3. — *We have the following properties.*

(i) $\dim \operatorname{Sing} V = \binom{n-1}{2}$.

(ii) *We have* $v_{i,j} \in \operatorname{Sing} V$ *and* $v_{i,j} = \pi(F_{i,j})$.

(iii) *For* $j \in J$, *we have* $\sum_{i \in J} v_{i,j} = 0$.

(iv) *For any* $k \in J$, *the set* $v_{i,j}$, $1 \leqslant i < j \leqslant n$, $k \notin \{i,j\}$, *is a basis of* $\operatorname{Sing} V$.

$\square$

LEMMA 5.4. — *We have*

$$S^{(a)}(v_{i,j}, v_{k,\ell}) = 0, \qquad \text{if } i,j,k,\ell \text{ are distinct}, \tag{5.9}$$

$$S^{(a)}(v_{i,j}, v_{i,k}) = -\frac{a_i a_j a_k}{|a|}, \qquad \text{if } i,j,k \text{ are distinct},$$

$$S^{(a)}(v_{i,j}, v_{i,j}) = -\sum_{k \neq j} S^{(a)}(v_{i,j}, v_{i,k}) = a_i a_j - \frac{a_i a_j (a_i + a_j)}{|a|}.$$

$\square$

5.3. Operators $K_i(z) : V \to V$

For any subset $\{i,j,k\} \subset J$, we define the linear operator $L_{i,j,k} : V \to V$ by the formula

$$
\begin{aligned}
F_{i,j} &\mapsto a_k F_{i,j} + a_i F_{j,k} + a_j F_{k,i}, \tag{5.10}\\
F_{j,k} &\mapsto a_k F_{i,j} + a_i F_{j,k} + a_j F_{k,i},\\
F_{k,i} &\mapsto a_k F_{i,j} + a_i F_{j,k} + a_j F_{k,i},\\
F_{\ell,m} &\mapsto 0, \qquad \text{if } \{\ell,m\} \text{ is not a subset of } \{i,j,k\}.
\end{aligned}
$$

see formula (3.5). Notice that $L_{i,j,k}$ does not depend on the order of i,j,k.

We define the operators $K_i(z) : V \to V$, $i \in J$, by the formula

$$K_i(z) = \sum \frac{d_{j,k}}{f_{i,j,k}} L_{i,j,k}, \tag{5.11}$$

where the sum is over all unordered subsets $\{j,k\} \subset J - \{i\}$, see formula (3.6). For any $i \in J$ and $z \in \mathbb{C}^n - \Delta$, the operator $K_i(z)$ preserves the subspace $\text{Sing}\, V \subset V$ and is a symmetric operator, $S^{(a)}(K_i(z)v, w) = S^{(a)}(v, K_i(z)w)$ for all $v, w \in V$, see Theorem 3.2.

LEMMA 5.5. — *For $i \in J$, we have*

$$K_i(z)v_{j,k} \;=\; \frac{d_{j,k}}{f_{i,j,k}}\,(a_i v_{j,k} + a_j v_{k,i} + a_k v_{i,j}), \qquad if\ i \notin \{j,k\}, (5.12)$$

$$K_i(z)v_{j,i} \;=\; -\sum_{k \notin \{i,j\}} K_i(z)v_{j,k}. \tag{5.13}$$

Proof. — The restriction of $K_i(z)$ to $(\text{Sing}\, V)^\perp$ is zero by formula (5.10) and Lemma 5.2. We have

$$K_i(z)v_{j,k} = K_i(z)F_{j,k} = \frac{d_{j,k}}{f_{i,j,k}}\,(a_i F_{j,k} + a_j F_{k,i} + a_k F_{i,j}). \tag{5.14}$$

The right hand side in (5.14) equals the right hand side in (5.12) by Lemma 5.3. $\qquad\square$

The differential equations (3.14) for flat sections of the Gauss-Manin connection on $(\mathbb{C}^n - \Delta) \times \text{Sing}\, V \to \mathbb{C}^n - \Delta$ take the form

$$\kappa \frac{\partial I}{\partial z_j}(z) = K_j(z)I(z), \qquad j \in J. \tag{5.15}$$

For generic κ all the flat sections are given by the formula

$$I_\gamma(z) = \sum_{1 \leqslant i < j \leqslant n} \Big(\int_{\gamma(z)} \prod_{m \in J} f_m^{a_m/\kappa}\, \frac{d_{i,j}}{f_i f_j}\, dt_1 \wedge dt_2 \Big) F_{i,j}, \tag{5.16}$$

see formula (3.12). These generic κ can be determined more precisely from the determinant formula in [23].

5.4. Conformal blocks

Define the map $q : \mathbb{C}^n \to \text{Sing}\, V$ by the formula

$$q : z \mapsto -\frac{1}{|a|^2} \sum_{1 \leqslant i < j \leqslant n} \frac{(z_i b_j^1 - z_j b_i^1)^2}{d_{i,j} b_i^1 b_j^1}\, v_{i,j}. \tag{5.17}$$

LEMMA 5.6. — *For any $k \in J$, we have*

$$q(z) = \frac{1}{|a|^2} \sideset{}{'}\sum \frac{f_{i,j,k}^2}{d_{i,j}d_{j,k}d_{k,i}}\, v_{i,j}, \tag{5.18}$$

where the sum is over all pairs $i < j$ such that $k \notin \{i,j\}$.

Proof. — Denote $A_{i,j} = -\frac{(z_i b_j^1 - z_j b_i^1)^2}{d_{i,j} b_i^1 b_j^1}$, then

$$q(z) = \frac{1}{|a|^2} \sum_{1 \leqslant i < j \leqslant n} A_{i,j} v_{i,j}. \tag{5.19}$$

We replace in (5.19) each $v_{i,k}$ with $-\sum_{j \neq k} v_{i,j}$ and each $v_{k,j}$ with $-\sum_{i \neq k} v_{i,j}$. Then

$$q(z) = \frac{1}{|a|^2} \sideset{}{'}\sum (A_{i,j} + A_{j,k} + A_{k,i}) v_{i,j}, \tag{5.20}$$

where the sum is the same as in (5.18). The lemma follows from the identity

$$A_{i,j} + A_{j,k} + A_{k,i} = \frac{f_{i,j,k}^2}{d_{i,j}d_{j,k}d_{k,i}}. \tag{5.21}$$

$\square$

By Lemma 5.6, the map q can be defined in terms of the determinants $d_{i,j}$, $1 \leqslant i < j \leqslant n$, only without using the individual numbers b_i^1.

THEOREM 5.7. — *If $\kappa = |a|/2$, then the Gauss-Manin connection on $(\mathbb{C}^n - \Delta) \times (\operatorname{Sing} V) \to \mathbb{C}^n - \Delta$ has a one-dimensional invariant subbundle, generated by the section $q : z \mapsto q(z)$, see (5.18). This section is flat.*

This one-dimensional subbundle will be called the bundle of *conformal blocks* at level $|a|/2$.

Proof. — We check that

$$\frac{|a|}{2} \frac{\partial q}{\partial z_1}(z) = K_1(z) q(z). \tag{5.22}$$

The other differential equations are proved similarly. By Lemma 5.6 we have

$$q(z) = \frac{1}{|a|^2} \sum_{1 < i < j \leqslant n} \frac{f_{1,i,j}^2}{d_{i,j}d_{j,1}d_{1,i}}\, v_{i,j}. \tag{5.23}$$

Then

$$\frac{|a|}{2}\frac{\partial q}{\partial z_1}(z) = \frac{1}{|a|}\sum_{1<i<j\leqslant n}\frac{f_{1,i,j}}{d_{j,1}d_{1,i}}\,v_{i,j} \tag{5.24}$$

and

$$K_1(z)q(z) = \frac{1}{|a|^2}\sum_{1<i<j\leqslant n}\frac{f_{1,i,j}}{d_{j,1}d_{1,i}}\,(a_1 v_{i,j}+a_i v_{j,1}+a_j v_{1,i}) \tag{5.25}$$

by formula (5.12). By replacing $v_{j,1}$ with $-\sum_{i\neq 1}v_{j,i}$ and $v_{1,i}$ with $-\sum_{j\neq 1}v_{i,j}$ and using Lemma 5.1 we obtain

$$K_1(z)q(z) = \frac{1}{|a|}\sum_{1<i<j\leqslant n}\frac{f_{1,i,j}}{d_{j,1}d_{1,i}}\,v_{i,j}. \tag{5.26}$$

$\square$

5.5. Algebra $A_\Phi(z)$

The master function of the arrangement $\mathcal{C}(z)$ is

$$\Phi(z,t) = \sum_{j\in J}a_j\log f_j = \sum_{j\in J}a_j\log(z_j+b_j^1 t_1+b_j^2 t_2). \tag{5.27}$$

The critical point equations are

$$\frac{\partial\Phi}{\partial t_1} = \sum_{j\in J}a_j\frac{b_j^1}{f_j}=0, \qquad \frac{\partial\Phi}{\partial t_2} = \sum_{j\in J}a_j\frac{b_j^2}{f_j}=0. \tag{5.28}$$

Introduce H_i, $i=1,2$, by the formula

$$\frac{\partial\Phi}{\partial t_i} = \frac{H_i}{\prod_{j\in J}f_j}. \tag{5.29}$$

We have

$$t_1\frac{\partial\Phi}{\partial t_1}+t_2\frac{\partial\Phi}{\partial t_2} = |a|-\sum_{i\in J}z_i\frac{a_i}{f_i}. \tag{5.30}$$

In other words, we have

$$t_1 H_1+t_2 H_2 = |a|\prod_{j\in J}f_j-\sum_{i\in J}z_i a_i\prod_{j\neq i}f_j. \tag{5.31}$$

The critical set is

$$C_\Phi(z) = \left\{ t \in U(\mathcal{C}(z)) \big| \frac{\partial \Phi}{\partial t_1} = 0, \frac{\partial \Phi}{\partial t_2} = 0 \right\} = \{ t \in U(\mathcal{C}(z)) | H_1 = 0, H_2 = 0 \}. \tag{5.32}$$

The algebra of functions on the critical set is

$$A_\Phi(z) = \mathbb{C}(U(\mathcal{C}(z))) / \left\langle \frac{\partial \Phi}{\partial t_1}, \frac{\partial \Phi}{\partial t_2} \right\rangle = \mathbb{C}(U(\mathcal{C}(z))) / \langle H_1, H_2 \rangle. \tag{5.33}$$

LEMMA 5.8. — *We have* $\dim A_\Phi(z) = \binom{n-1}{2}$. $\qquad\qquad\square$

Introduce elements $w_{i,j} \in A_\Phi(z)$ by the formula

$$w_{i,j} = a_i a_j \left[\frac{d_{i,j}}{f_i f_j} \right]. \tag{5.34}$$

LEMMA 5.9. — *We have* $w_{i,j} = -w_{j,i}$ *and* $\sum_{j \in J} w_{i,j} = 0$.

Proof. — The lemma follows from the identity

$$a_i \frac{df_i}{f_i} \wedge \frac{d\Phi}{\Phi} = \sum_{j \in J} a_i a_j \frac{d_{i,j}}{f_i f_j} dt_1 \wedge dt_2 + \sum_{m \in J} dz_j \wedge \mu_j, \tag{5.35}$$

where μ_j are suitable one-forms. $\qquad\qquad\square$

The elements $\left[\frac{a_i}{f_i} \right]$, $i \in J$, generate $A_\Phi(z)$ by Lemma 2.4.

LEMMA 5.10. — *For* $j \in J$, *we have the following identity in* $A_\Phi(z)$,

$$\sum_{i \in J} d_{i,j} \left[\frac{a_i}{f_i} \right] = 0. \tag{5.36}$$

$$\qquad\qquad\square$$

LEMMA 5.11. — *We have*

$$\left[\frac{a_i}{f_i} \right] *_z \left[\frac{a_j}{f_j} \right] = \frac{1}{d_{i,j}} w_{i,j}, \qquad i \neq j, \tag{5.37}$$

$$\left[\frac{a_j}{f_j} \right] *_z \left[\frac{a_j}{f_j} \right] = \sum_{i \notin \{j,k\}} \frac{d_{k,i}}{d_{j,k} d_{i,j}} w_{i,j}, \qquad k \neq j.$$

$$\left[\frac{a_i}{f_i} \right] *_z w_{j,k} = \frac{d_{j,k}}{f_{i,j,k}} (a_i w_{j,k} + a_j w_{k,i} + a_k w_{i,j}), \;\; if \; i \notin \{j,k\}, \tag{5.38}$$

$$\left[\frac{a_i}{f_i} \right] *_z w_{j,i} = - \sum_{k \notin \{i,j\}} \left[\frac{a_i}{f_i} \right] *_z w_{j,k}.$$

$$\qquad\qquad\square$$

COROLLARY 5.12. — *The elements $w_{i,j}$, $1 \leqslant i < j \leqslant n$, span $A_\Phi(z)$.*

LEMMA 5.13. — *The identity element $[1](z) \in A_\Phi(z)$ satisfies the equations*

$$[1](z) = \frac{1}{|a|} \sum_{i \in J} z_i \left[\frac{a_i}{f_i} \right] = \frac{1}{|a|^2} \left(\sum_{i \in J} z_i \left[\frac{a_i}{f_i} \right] \right)^2 = -\frac{1}{|a|^2} \sum_{1 \leqslant i < j \leqslant n} \frac{(z_i b_j^1 - z_j b_i^1)^2}{d_{i,j} b_i^1 b_j^1} \, w_{i,j}.$$
(5.39)

Proof. — To obtain the last expression in (5.39) we replace $\left[\frac{a_i}{f_i} \right] \left[\frac{a_i}{f_i} \right]$ with $\left[-\frac{a_i}{f_i} \sum_{j \neq i} \frac{b_j^1}{b_i^1} \frac{a_j}{f_j} \right]$. $\qquad\square$

THEOREM 5.14. — *For any $k \in J$, the identity element $[1](z) \in A_\Phi(z)$ satisfies the equation*

$$[1](z) = \frac{1}{|a|^2} \sum{}' \frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} \, w_{i,j},$$
(5.40)

where the sum is over all pairs $i < j$ such that $k \notin \{i,j\}$.

Proof. — The proof is the same as the proof of Lemma 5.6. $\qquad\square$

The canonical element is

$$[E] = \sum_{1 \leqslant i < j \leqslant n} \left[\frac{d_{i,j}}{f_i f_j} \right] \otimes F_{i,j} \quad \in \quad A_\Phi(z) \otimes \operatorname{Sing} V.$$
(5.41)

THEOREM 5.15. — *The canonical isomorphism*

$$\alpha(z) : A_\Phi(z) \to \operatorname{Sing} V$$
(5.42)

is given by the formula

$$w_{i,j} \mapsto v_{i,j}.$$
(5.43)

COROLLARY 5.16. — *We have $\alpha(z)[1]) = q(z)$, where $q(z)$ is the conformal block of Theorem 5.7.*

5.6. Proof of Theorem 5.15

Introduce the coefficients $B_{i,j}$ by the formula

$$\alpha(z)(w_{k,\ell}) = \sum_{1 \leqslant i < j \leqslant n} B_{i,j} F_{i,j}.$$
(5.44)

We have

$$-\frac{4\pi^2}{a_k a_\ell d_{k,\ell} d_{i,j}} B_{i,j} = \sum_{p \in C_\Phi(z)} \operatorname{Res}_p \frac{1}{f_k f_\ell f_i f_j} \frac{\prod_{m \in J} f_m^2}{H_1 H_2}. \qquad (5.45)$$

LEMMA 5.17. — *We have $B_{i,j} = 0$, if $\{i, j\} \cap \{k, \ell\} = \emptyset$.*

Proof. — The differential form

$$\omega_{k,\ell,i,j} = \frac{\prod_{m \in J} f_m^2}{f_k f_\ell f_i f_j H_1 H_2} \, dt_1 \wedge dt_2 \qquad (5.46)$$

has poles only on the curves $H_1 = 0$ and $H_2 = 0$. The poles are of first order. To calculate the right hand side in (5.45), we need to take the residue $\operatorname{Res}_{H_1=0} \omega_{k,\ell,i,j}$ of the form $\omega_{k,\ell,i,j}$ at the curve $H_1 = 0$ and then take the residue of that form on the curve $H_1 = 0$ at the points where $H_2 = 0$. This is the same as if we took with minus sign the residue of $\operatorname{Res}_{H_1=0} \omega_{k,\ell,i,j}$ on the curve $H_1 = 0$ at infinity. That residue at infinity with minus sign could be obtained differently in two steps. First we may take the residue $\operatorname{Res}_\infty \omega_{k,\ell,i,j}$ of $\omega_{k,\ell,i,j}$ at the line at infinity and then take the residue of that one-form on the line at infinity at the points where $H_1 = 0$.

So to calculate the right hand side in (5.45) we first calculate $\operatorname{Res}_\infty \omega_{k,\ell,i,j}$. The coordinates at infinity are $u_1 = t_1/t_2$, $u_2 = 1/t_2$. We have $f_m = (b_m^1 u_1 + b_m^2 + u_2 z_m)/u_2$. Denote $\tilde{f}_m(u_1) = b_m^1 u_1 + b_m^2$. For $i = 1, 2$, we have $H_i(u_1/u_2, 1/u_2) = \hat{H}_i(u_1, u_2)/u_2^{n-1}$, where $\hat{H}_i(u_1, u_2)$ are some polynomials. Denote $\tilde{H}_i(u_1) = \hat{H}_i(u_1, 0)$. We have $dt_1 \wedge dt_2 = -\frac{1}{u_2^3} du_1 \wedge du_2$. Then the residue of $\omega_{k,\ell,i,j}$ at the line at infinity equals

$$2\pi\sqrt{-1}\,\tilde{\omega}_{k,\ell,i,j} = \frac{\prod_{m \in J} \tilde{f}_m(u)^2}{\tilde{f}_k(u)\tilde{f}_\ell(u)\tilde{f}_i(u)\tilde{f}_j(u)\tilde{H}_1(u)\tilde{H}_2(u)} \, du, \qquad (5.47)$$

where $u = u_1$. On the line at infinity this one-form is holomorphic at $u = \infty$. The number $\frac{2\pi\sqrt{-1}}{a_k a_\ell d_{k,\ell} d_{i,j}} B_{i,j}$ equals the sum of residues of the form $\tilde{\omega}_{k,\ell,i,j}$ at the points where $\tilde{H}_1(u) = 0$.

By formula (5.30), we have

$$u\tilde{H}_1(u) + \tilde{H}_2(u) = |a| \prod_{m \in J} \tilde{f}_m(u). \qquad (5.48)$$

Thus $\tilde{H}_2(u) = |a| \prod_{m \in J} \tilde{f}_m(u)$ at the point where $H_1(u) = 0$. Therefore, the sum of residues of the form $\tilde{\omega}_{k,\ell,i,j}$ at the points where $\tilde{H}_1(u) = 0$ equals

the sum of residues of the form

$$\tilde{\omega}_{k,\ell,i,j} = \frac{\prod_{m \notin \{k,\ell,i,j\}} \tilde{f}_m(u)}{|a|\,\tilde{H}_1(u)}\,du \tag{5.49}$$

at the points where $\tilde{H}_1(u) = 0$. This sum is zero. $\qquad\square$

LEMMA 5.18. — *We have* $B_{k,j} = -\frac{a_\ell}{|a|}$, *if* $j \notin \{k,\ell\}$.

Proof. — On the line at infinity we consider the differential one form

$$\tilde{\omega}_{k,\ell,i,j} = \frac{\prod_{m \in J} \tilde{f}_m(u)^2}{\tilde{f}_k^2(u)\tilde{f}_\ell(u)\tilde{f}_j(u)\tilde{H}_1(u)\tilde{H}_2(u)}\,du, \tag{5.50}$$

As in Lemma 5.17 we observe that $\frac{2\pi\sqrt{-1}}{a_k a_\ell d_{k,\ell} d_{k,j}} B_{k,j}$ equals the sum of residues of that one-form at the points where $H_1 = 0$. Consider the differential one-form

$$\mu = \frac{\prod_{m \notin \{k,\ell,j\}} \tilde{f}_m(u)}{|a|\,\tilde{f}_k(u)\tilde{H}_1(u)}\,du. \tag{5.51}$$

As in Lemma 5.17 we observe that $\frac{2\pi\sqrt{-1}}{a_k a_\ell d_{k,\ell} d_{k,j}} B_{k,j}$ equals the sum of residues of μ at the points where $\tilde{H}_1(u) = 0$ and this sum equals

$$-\operatorname{Res}_{\tilde{f}_k = 0} \mu = -\frac{2\pi\sqrt{-1}}{a_k|a|d_{k,\ell}d_{k,j}}. \tag{5.52}$$

The lemma is proved.

$\qquad\square$

By Lemmas 5.17 and 5.18 we know that $\alpha(z)(w_{k,\ell}) = B_{k,\ell}F_{k,\ell} - \frac{a_k}{|a|}\sum_{i \neq k} F_{i,\ell} - \frac{a_\ell}{|a|}\sum_{j \neq \ell} F_{k,j}$. From the condition that $\alpha(z)(w_{k,\ell}) \in \operatorname{Sing} V$ we conclude that $B_{k,\ell} = \frac{|a| - a_k - a_\ell}{|a|}$. The theorem is proved.

5.7. Contravariant map as the inverse to the canonical map

The canonical map $\alpha(z) : A_\Phi(z) \to \operatorname{Sing} V$ is the isomorphism described in Theorem 5.15. The contravariant map $\mathcal{S}^{(a)} : V \to V^*$ is defined by the formula $F_{i,j} \mapsto a_i a_j (H_i, H_j)$. By identifying $a_i a_j (H_i, H_j)$ with the differential form $a_i a_j \frac{d_{i,j}}{f_i f_j}\,dt_1 \wedge dt_2$ and then projecting the coefficient to $A_\Phi(z)$ we obtain the map

$$[\mathcal{S}^{(a)}] : V \to A_\Phi(z), \qquad F_{i,j} \mapsto w_{i,j} = a_i a_j \left[\frac{d_{i,j}}{f_i f_j}\right]. \tag{5.53}$$

THEOREM 5.19. — *The composition $\alpha(z) \circ [\mathcal{S}^{(a)}] : V \to \operatorname{Sing} V$ is the orthogonal projection. The composition $[\mathcal{S}^{(a)}] \circ \alpha(z) : A_\Phi(z) \to A_\Phi(z)$ is the identity map.*

Proof. — The composition $\alpha(z) \circ [\mathcal{S}^{(a)}]$ sends $F_{i,j}$ to $v_{i,j}$ which is the orthogonal projection by Lemma 5.3. The composition $[\mathcal{S}^{(a)}] \circ \alpha(z)$ sends $w_{i,j}$ to

$$w_{i,j} - \frac{a_i}{|a|} \sum_{k \in J} w_{k,j} - \frac{a_j}{|a|} \sum_{\ell \in J} w_{i,\ell}. \tag{5.54}$$

The last two sums are equal to zero in $A_\Phi(z)$ by Lemma 5.9. □

5.8. Corollaries of Theorem 5.15

THEOREM 5.20. — *For any $j \in J$, the $\operatorname{Sing} V$-valued function $\frac{\partial q}{\partial z_j}(z)$ satisfies the Gauss-Manin differential equations with $\kappa = |a|$,*

$$|a| \frac{\partial}{\partial z_i} \frac{\partial q}{\partial z_j}(z) = K_i(z) \frac{\partial q}{\partial z_j}(z), \qquad i \in J. \tag{5.55}$$

Proof. — By Theorems 5.14 and 5.15, we have

$$q(z) = \alpha(z)([1](z)) = \frac{1}{|a|^2} \alpha(z)\Big(\big(\sum_{m \in J} z_m \Big[\frac{a_m}{f_m}\Big]\big)^2\Big)$$

$$= \frac{1}{|a|^2} \sum_{m,\ell \in J} z_m z_\ell \alpha(z)\Big(\Big[\frac{a_m}{f_m}\Big]\Big[\frac{a_\ell}{f_\ell}\Big]\Big).$$

By Theorem 5.15, for any $m, \ell \in J$, the element $\alpha(z)\big(\big[\frac{a_m}{f_m}\big]\big[\frac{a_\ell}{f_\ell}\big]\big) \in \operatorname{Sing} V$ is a linear combination of vectors $v_{i,j}$ with constant coefficients. Hence

$$\frac{\partial^2 q}{\partial z_j \partial z_i}(z) = \frac{2}{|a|^2} \alpha(z)\Big(\Big[\frac{a_i}{f_i}\Big]\Big[\frac{a_j}{f_j}\Big]\Big),$$

$$\frac{\partial q}{\partial z_j}(z) = \frac{2}{|a|} \alpha(z)\Big(\Big[\frac{a_j}{f_j}\Big] *_z \frac{1}{|a|} \sum_{m \in J} z_m \Big[\frac{a_m}{f_m}\Big]\Big) = \frac{2}{|a|} \alpha(z)\Big(\Big[\frac{a_j}{f_j}\Big]\Big),$$

$$K_i(z) \frac{\partial q}{\partial z_j}(z) = \frac{2}{|a|} \alpha(z)\Big(\Big[\frac{a_i}{f_i}\Big]\Big[\frac{a_j}{f_j}\Big]\Big).$$

This implies the theorem. □

COROLLARY 5.21. — *Conjectures 3.7 and 3.8 hold for this family of arrangements.* □

The tangent morphism β and the residue form on the bundle of algebras induce a holomorphic bilinear form η on fibers of the tangent bundle,

$$
\begin{aligned}
\eta(\partial_i, \partial_j)_z &= (\beta(z)(\partial_i), \beta(z)(\partial_j))_z = (-1)^k S^{(a)}(\alpha(z)\beta(z)(\partial_i), \alpha(z)\beta(z)(\partial_j)) \\
&= \left(\left[\frac{a_i}{f_i} \right], \left[\frac{a_j}{f_j} \right] \right)_z = (-1)^k S^{(a)}\left(\alpha(z)\left(\left[\frac{a_i}{f_i} \right] \right), \alpha(z)\left(\left[\frac{a_j}{f_j} \right] \right) \right). \quad (5.56)
\end{aligned}
$$

By Theorem 3.10, we have

$$
\eta(\partial_i, \partial_j)_z = \frac{|a|^2}{4} S^{(a)}\left(\frac{\partial q}{\partial z_i}(z), \frac{\partial q}{\partial z_j}(z) \right). \quad (5.57)
$$

Theorems 3.12 and 3.13 also hold for this family of arrangements.

THEOREM 5.22. — *Recall the potential function of first kind $P(z) = S^{(a)}(q(z), q(z))$. We have*

$$
P(z) = \sum_{1 \leqslant i < j < k \leqslant n} \frac{a_i a_j a_k}{|a|^5} \frac{f_{i,j,k}^4}{d_{i,j}^2 d_{j,k}^2 d_{k,i}^2}. \quad (5.58)
$$

Proof. — By Theorem 3.11, for any $r \leqslant 4$, we have

$$
(\beta(z)(\partial_{m_1}) *_z \cdots *_z \beta(z)(\partial_{m_r}), [1](z))_z = \frac{|a|^r}{A_{2,r}} \frac{\partial^r P}{\partial z_{m_1} \ldots \partial z_{m_r}}(z), \quad (5.59)
$$

for all $m_1, \ldots, m_r \in J$. In particular,

$$
(\beta(z)(\partial_k) *_z \beta(z)(\partial_\ell) *_z \beta(z)(\partial z_m), [1](z))_z = \frac{|a|^3}{4!} \frac{\partial^3 P}{\partial z_k \partial z_\ell \partial z_m}(z) \quad (5.60)
$$

for all $k, \ell, m \in J$. Introduce the function

$$
\hat{P}(z) = \frac{1}{4!} \sum_{1 \leqslant i < j < k \leqslant n} \frac{a_i a_j a_k}{|a|^2} \frac{f_{i,j,k}^4}{d_{i,j}^2 d_{j,k}^2 d_{k,i}^2}. \quad (5.61)
$$

PROPOSITION 5.23. — *We have*

$$
(\beta(z)(\partial_j) *_z \beta(z)(\partial_\ell) *_z \beta(z)(\partial z_m), [1](z))_z = \frac{\partial^3 \hat{P}}{\partial z_j \partial z_\ell \partial z_m}(z) \quad (5.62)
$$

for all $k, \ell, m \in J$.

Proof. — For $k, \ell \in J$, we define the differential one-form $\psi_{k,\ell}$ on $\mathbb{C}^n - \Delta$ by the formula

$$\psi_{k,\ell}(\partial_m) = (\beta(z)(\partial_k) *_z \beta(z)(\partial_\ell), \beta(z)(\partial_m))_z. \tag{5.63}$$

The canonical isomorphism identifies the residue form and the contravariant form and therefore we may write

$$\psi_{k,\ell}(\partial_m) = S^{(a)}(\alpha(z)\beta(z)(\partial_k) *_z \alpha(z)\beta(z)(\partial_\ell), \alpha(z)\beta(z)(\partial_m)). \tag{5.64}$$

LEMMA 5.24. — *The form $\psi_{k,\ell}$ is the differential of the function*

$$\varphi_{k,\ell}(z) = \frac{|a|}{2} \frac{1}{d_{k,\ell}} S^{(a)}(v_{k,\ell}, q(z)), \tag{5.65}$$

if $k \neq \ell$, and of the function

$$\varphi_{k,\ell}(z) = \frac{|a|}{2} \sum_{i \notin \{j,k\}} \frac{d_{j,i}}{d_{k,j} d_{i,k}} S^{(a)}(v_{i,k}, v), \tag{5.66}$$

if $k = \ell$, where j is any number in J such that $j \neq k$.

Proof. — The vector $\alpha(z)\beta(z)(\partial_k) *_z \alpha(z)\beta(z)(\partial_\ell) = \alpha(z)(\left[\frac{a_k}{f_k}\right]\left[\frac{a_\ell}{f_\ell}\right]) \in \mathrm{Sing}\, V$ equals $\frac{1}{d_{k,\ell}} v_{k,\ell}$ if $k \neq \ell$ and equals $\sum_{i \notin \{j,k\}} \frac{d_{j,i}}{d_{k,j} d_{i,k}} v_{i,k}$ if $k = \ell$. We also have $\alpha(z)\beta(z)(\partial_m) = \frac{|a|}{2} \frac{\partial q}{\partial z_m}$. This implies the lemma. $\qquad\square$

Proposition 5.23 is equivalent to the formula

$$\frac{\partial^2 \hat{P}}{\partial z_k \partial z_\ell} = \varphi_{k,\ell} \tag{5.67}$$

for all $k, \ell \in J$. The proof of (5.67) is by direct verification. Namely, assume that $k < \ell$. Then

$$\tag{5.68}$$

$$\frac{\partial^2 \hat{P}}{\partial z_k \partial z_\ell} = \frac{a_i a_k a_\ell}{2|a|^2} \Big(\sum_{i<k} \frac{f_{i,k,l}^2}{d_{i,k} d_{k,\ell} d_{\ell,i}} \frac{1}{d_{k,\ell}} + \sum_{k<i<\ell} \frac{f_{k,i,l}^2}{d_{k,i} d_{i,\ell} d_{\ell,k}} \frac{1}{d_{\ell,k}}$$

$$+ \sum_{i>\ell} \frac{f_{k,\ell,i}^2}{d_{k,\ell} d_{\ell,i} d_{i,k}} \frac{1}{d_{k,\ell}} \Big).$$

We also have

$$(5.69)$$

$$
\begin{aligned}
\varphi_{k,\ell} &= \frac{1}{2|a|} S^{(a)} \Big(\frac{1}{d_{k,\ell}} v_{k,\ell}, \sum_{\substack{i<j \\ k \notin \{i,j\}}} \frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} v_{i,j} \Big) = \\
&= \frac{1}{2|a| d_{k,\ell}} S^{(a)} \Big(v_{k,\ell}, \sum_{i<k} \frac{f_{i,\ell,k}^2}{d_{i,\ell} d_{\ell,k} d_{k,i}} v_{i,\ell} + \sum_{k<i<\ell} \frac{f_{i,\ell,k}^2}{d_{i,\ell} d_{\ell,k} d_{k,i}} v_{i,\ell} \\
&+ \sum_{i>\ell} \frac{f_{\ell,i,k}^2}{d_{\ell,i} d_{i,k} d_{k,\ell}} v_{\ell,i} \Big) \\
&= \frac{a_i a_\ell a_k}{2|a|^2 d_{k,\ell}} \Big(-\sum_{i<k} \frac{f_{i,\ell,k}^2}{d_{i,\ell} d_{\ell,k} d_{k,i}} - \sum_{k<i<\ell} \frac{f_{i,\ell,k}^2}{d_{i,\ell} d_{\ell,k} d_{k,i}} + \sum_{i>\ell} \frac{f_{\ell,i,k}^2}{d_{\ell,i} d_{i,k} d_{k,\ell}} \Big).
\end{aligned}
$$

Comparing (5.68) and (5.68) we conclude that (5.67) holds if $k < \ell$. Assume that $k = \ell$. Then

$$
\frac{\partial^2 \hat{P}}{\partial z_k^2} = \sum_{\substack{i<j \\ k \notin \{i,j\}}} \frac{a_i a_j a_k}{2|a|^2} \frac{f_{i,j,k}^2}{d_{j,k}^2 d_{k,i}^2} \tag{5.70}
$$

We also have

$$
\begin{aligned}
\varphi_{k,\ell} &= \frac{1}{2|a|} S^{(a)} \Big(\sum_{m \notin \{k,\ell\}} \frac{d_{\ell,m}}{d_{k,\ell} d_{m,k}} v_{m,k}, \sum_{\substack{i<j \\ k \notin \{i,j\}}} \frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} v_{i,j} \Big) \quad (5.71) \\
&= \sum_{\substack{i<j \\ k \notin \{i,j\}}} \frac{1}{2|a|} \frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} S^{(a)} \Big(\frac{d_{\ell,i}}{d_{k,\ell} d_{i,k}} v_{i,k} + \frac{d_{\ell,j}}{d_{k,\ell} d_{j,k}} v_{j,k}, v_{i,j} \Big) \\
&= \sum_{\substack{i<j \\ k \notin \{i,j\}}} \frac{a_i a_j a_k}{2|a|^2 d_{k,\ell}} \frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} \Big(-\frac{d_{\ell,i}}{d_{i,k}} + \frac{d_{\ell,j}}{d_{j,k}} \Big) \\
&= \sum_{\substack{i<j \\ k \notin \{i,j\}}} \frac{a_i a_j a_k}{2|a|^2 d_{k,\ell}} \frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} \frac{d_{\ell,k} d_{i,j}}{d_{i,k} d_{j,k}} = \sum_{\substack{i<j \\ k \notin \{i,j\}}} \frac{a_i a_j a_k}{2|a|^2} \frac{f_{i,j,k}^2}{d_{j,k}^2 d_{k,i}^2}.
\end{aligned}
$$

Comparing (5.70) and (5.71) we conclude that (5.67) holds for $k = \ell$. The proposition is proved. □

Both functions $|a|^3 P(z)/4!$ and $\hat{P}(z)$ satisfy the same equation and both functions are homogeneous polynomials in z of degree four. Hence $|a|^3 P(z)/4! = \hat{P}(z)$. Thus the proposition implies the theorem. □

The period map $q : \mathbb{C}^n - \Delta \to \mathbb{C}^n - \Delta$ is a polynomial map in z of degree two with respect to the combinatorial connection.

The space $\operatorname{Sing} V$ has distinguished bases labeled by $k \in J$. The basis corresponding to k consists of the vectors $v_{i,j}$ such that $1 \leqslant i < j \leqslant n$ and $k \notin \{i, j\}$. Such a basis defines coordinate hyperplanes in $\operatorname{Sing} V$.

LEMMA 5.25. — *The period map sends the discriminant $\Delta \subset \mathbb{C}^n$ to the union $\Delta_V \subset \operatorname{Sing} V$ of all coordinate hyperplanes of all distinguished bases in $\operatorname{Sing} V$.*

Proof. — The period map is given by the formula $q(z) = \frac{1}{|a|^2} \sum' \frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}} v_{i,j}$, where the sum is over all pairs $i < j$ such that $k \notin \{i, j\}$. Thus the functions $\frac{f_{i,j,k}^2}{d_{i,j} d_{j,k} d_{k,i}}$ are the coordinate functions of the period map in this (combinatorially flat) basis. The lemma follows from this description of the coordinate functions. □

LEMMA 5.26. — *For $z \in \mathbb{C}^n - \Delta$, the kernel of the differential of the period map is two dimensional. The kernel is spanned by the vectors*

$$\sum_{j \neq i} d_{j,i} \partial_j, \qquad i \in J. \tag{5.72}$$

Any two of these vectors are linearly independent. □

Introduce the *potential function of second kind*

$$\tilde{P}(z_1, \ldots, z_n) = \frac{1}{4!} \sum_{1 \leqslant i < j < k \leqslant n} \frac{a_i a_j a_k}{d_{i,j}^2 d_{j,k}^2 d_{k,i}^2} f_{i,j,k}^4 \log f_{i,j,k}. \tag{5.73}$$

THEOREM 5.27. — *For any $m_0, \ldots, m_4 \in J$ we have*

$$\frac{\partial^5 \tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_4}}(z) = \left(\left[\frac{a_{m_0}}{f_{m_0}} \right] *_z \cdots *_z \left[\frac{a_{m_4}}{f_{m_4}} \right], [1](z) \right)_z. \tag{5.74}$$

Theorem 5.27 proves Conjecture 3.14 for this family of arrangements.

If $m_1 \neq m_2$ and $m_3 \neq m_4$, equation (5.74) takes the form

$$S^{(a)}(K_{m_0}(z) v_{m_1, m_2}, v_{m_3, m_4}) = d_{m_1, m_2} d_{m_3, m_4} \frac{\partial^5 \tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_4}}(z). \tag{5.75}$$

COROLLARY 5.28. — *The matrix elements of the operators $K_i(z)$ with respect to the (combinatorially constant) vectors $v_{i,j}$ are described by the fifth derivatives of the potential function of second kind.*

Notice that

$$S^{(a)}(v_{m_1,m_2}, v_{m_3,m_4}) = d_{m_1,m_2} d_{m_3,m_4} \frac{|a|^4}{4!} \frac{\partial^4 P}{\partial z_{m_1} \ldots \partial z_{m_4}}(z), \qquad (5.76)$$

where $P(z)$ is the potential function of first kind, see Theorem 3.11.

Proof of Theorem 5.27. — We have the relation $\sum_{j \in J} d_{i,j} \left[\frac{a_j}{f_j} \right] = 0$ for any $i \in J$, see (5.36), and the relation

$$\sum_{j \in J} d_{i,j} \frac{\partial}{\partial z_j} \frac{\partial^4 P}{\partial z_{m_1} \ldots \partial z_{m_4}}(z) = 0 \qquad (5.77)$$

for any $m_1, \ldots, m_4, i \in J$. By using these two relations and by reordering the set J if necessary, we can reduce formula (5.74) to three cases in which $(m_0, \ldots, m_4)$ equals $(5, 1, 2, 3, 4)$ or $(3, 1, 2, 3, 4)$ or $(3, 1, 2, 1, 2)$.

Let $(m_0, \ldots, m_4) = (5, 1, 2, 3, 4)$. Then $\frac{\partial^5 \tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_4}}(z) = 0$ and

$$\left(\left[\frac{a_{m_0}}{f_{m_0}} \right] *_z \cdots *_z \left[\frac{a_{m_4}}{f_{m_4}} \right], [1](z) \right)_z = \frac{1}{d_{1,2} d_{3,4}} S^{(a)}(K_5(z) v_{1,2}, v_{3,4})$$

$$= \frac{d_{1,2}}{d_{1,2} d_{3,4} f_{5,1,2}} S^{(a)}(a_5 v_{1,2} + a_1 v_{2,5} + a_2 v_{5,1}, v_{3,4}) = 0. \qquad (5.78)$$

Let $(m_0, \ldots, m_4) = (3, 1, 2, 3, 4)$. Then $\frac{\partial^5 \tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_4}}(z) = 0$ and

$$\left(\left[\frac{a_{m_0}}{f_{m_0}} \right] *_z \cdots *_z \left[\frac{a_{m_4}}{f_{m_4}} \right], [1](z) \right)_z = \frac{1}{d_{1,2} d_{3,4}} S^{(a)}(K_3(z) v_{1,2}, v_{3,4})$$

$$= \frac{d_{1,2}}{d_{1,2} d_{3,4} f_{3,1,2}} S^{(a)}(a_3 v_{1,2} + a_1 v_{2,3} + a_2 v_{3,1}, v_{3,4})$$

$$= \frac{d_{1,2}}{d_{1,2} d_{3,4} f_{3,1,2}} \left(0 + a_1 \frac{a_2 a_3 a_4}{|a|} - a_2 \frac{a_1 a_3 a_4}{|a|} \right) = 0. \qquad (5.79)$$

Let $(m_0, \ldots, m_4) = (3, 1, 2, 1, 2)$. Then $\frac{\partial^5 \tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_4}}(z) = \frac{a_1 a_2 a_3}{d_{1,2} f_{1,2,3}}$ and

$$\left(\left[\frac{a_{m_0}}{f_{m_0}} \right] *_z \cdots *_z \left[\frac{a_{m_4}}{f_{m_4}} \right], [1](z) \right)_z = \frac{1}{d_{1,2} d_{1,2}} S^{(a)}(K_3(z) v_{1,2}, v_{1,2})$$

$$= \frac{d_{1,2}}{d_{1,2} d_{1,2} f_{3,1,2}} S^{(a)}(a_3 v_{1,2} + a_1 v_{2,3} + a_2 v_{3,1}, v_{1,2})$$

$$= \frac{d_{1,2}}{d_{1,2} d_{1,2} f_{3,1,2}} \left(a_3 \frac{a_1 a_2 \sum_{j \notin \{1,2\}} a_j}{|a|} + a_1 \frac{a_1 a_2 a_3}{|a|} + a_2 \frac{a_1 a_2 a_3}{|a|} \right)$$

$$= \frac{a_1 a_2 a_3}{d_{1,2} f_{1,2,3}}. \qquad (5.80)$$

The theorem is proved. $\qquad\qquad \square$

5.9. Frobenius like structure

Consider the quotient M of $\mathbb{C}^n$ by the two-dimensional subspace, which is the kernel of the period map, see Lemma 5.26. Let $\pi : \mathbb{C}^n \to M$ be the natural projection. Then all our objects descend to the quotient and form on $M - \pi(\Delta)$ a structure which we will also call a *Forbenius like structure*.

6. Generic arrangements in $\mathbb{C}^k$

6.1. An arrangement in $\mathbb{C}^n \times \mathbb{C}^k$

Consider $\mathbb{C}^k$ with coordinates $t_1, \ldots, t_k$, $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$. Fix n linear functions on $\mathbb{C}^k$, $g_j = \sum_{m=1}^{k} b_j^m t_m$, $j \in J$, $b_j^m \in \mathbb{C}$. For $i_1, \ldots, i_k \subset J$, denote

$$d_{i_1,\ldots,i_k} = \det_{\ell,m=1}^{k}(b_{i_\ell}^m). \tag{6.1}$$

We assume that all the numbers $d_{i_1,\ldots,i_k}$ are nonzero if $i_1, \ldots, i_k$ are distinct. In other words we assume that the collection of functions $g_j, j \in J$, is generic. We define n linear functions on $\mathbb{C}^n \times \mathbb{C}^k$, $f_j = z_j + g_j$, $j \in J$. In $\mathbb{C}^n \times \mathbb{C}^k$ we define the arrangement $\tilde{\mathcal{C}} = \{\tilde{H}_j \mid f_j = 0, \ j \in J\}$.

For every $z = (z_1, \ldots, z_n)$ the arrangement $\tilde{\mathcal{C}}$ induces an arrangement $\mathcal{C}(z)$ in the fiber of the projection $\mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$ over z. We identify every fiber with $\mathbb{C}^k$. Then $\mathcal{C}(z)$ consists of hyperplanes $H_j(z), j \in J$, defined in $\mathbb{C}^k$ by the equations $f_j = 0$. Denote $U(\mathcal{C}(z)) = \mathbb{C}^k - \cup_{j \in J} H_j(z)$, the complement to the arrangement $\mathcal{C}(z)$.

The arrangement $\mathcal{C}(z)$ is with normal crossings if and only if $z \in \mathbb{C}^n - \Delta$,

$$\Delta = \cup_{\{i_1 < \cdots < i_{k+1}\} \subset J} H_{i_1,\ldots,i_{k+1}}, \tag{6.2}$$

where $H_{i_1,\ldots,i_{k+1}}$ is the hyperplane defined by the equation $f_{i_1,\ldots,i_{k+1}} = 0$,

$$f_{i_1,\ldots,i_{k+1}} = \sum_{m=1}^{k+1} (-1)^{m-1} z_{i_m} d_{i_1,\ldots,\widehat{i_m},\ldots,i_{k+1}}. \tag{6.3}$$

6.2. Good fibers

For any $z \in \mathbb{C}^n - \Delta$, the space $\mathcal{A}^k(\mathcal{C}(z))$ has the standard basis $(H_{i_1}(z), \ldots, H_{i_k}(z))$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$. The space $\mathcal{F}^k(\mathcal{C}(z))$ has the standard dual basis $F(H_{i_1}(z), \ldots, H_{i_k}(z))$. For $z^1, z^2 \in \mathbb{C}^n - \Delta$, the combinatorial connection identifies the spaces $\mathcal{A}^k(\mathcal{C}(z^1))$, $\mathcal{F}^k(\mathcal{C}(z^1))$ with the

spaces $\mathcal{A}^k(\mathcal{C}(z^2))$, $\mathcal{F}^k(\mathcal{C}(z^2))$, respectively, by identifying the corresponding standard bases.

Assume that nonzero weights $(a_j)_{j \in J}$ are given. Then each arrangement $\mathcal{C}(z)$ is weighted. For $z \in \mathbb{C}^n - \Delta$, the arrangement $\mathcal{C}(z)$ is unbalanced if $|a| \neq 0$. We assume that $|a| \neq 0$.

For $z \in \mathbb{C}^n - \Delta$, we denote $V = \mathcal{F}^k(\mathcal{C}(z))$, $V^* = (\mathcal{F}^k(\mathcal{C}(z)))^* = \mathcal{A}^k(\mathcal{C}(z))$, $F_{i_1,\ldots,i_k} = F(H_{i_1}(z), \ldots, H_{i_k}(z))$. For any permutation $\sigma \in \Sigma_k$, we have $F_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} = (-1)^\sigma F_{i_1,\ldots,i_k}$. If $v = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} c_{i_1,\ldots,i_k} F_{i_1,\ldots,i_k}$ is a vector of V, we introduce $c_{i_1,\ldots,i_k}$ for all $i_1, \ldots, i_k \in J$ by the rule: $c_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} = (-1)^\sigma c_{i_1,\ldots,i_k}$. The contravariant form on V is defined by

$$S^{(a)}(F_{i_1,\ldots,i_k}, F_{j_1,\ldots,j_k}) \quad = \quad 0, \qquad \text{if } \{i_1, \ldots, i_k\} \neq \{i_1, \ldots, i_k\}, \quad (6.4)$$

$$S^{(a)}(F_{i_1,\ldots,i_k}, F_{i_1,\ldots,i_k}) \quad = \quad \prod_{m=1}^{k} a_{i_m},$$

the singular subspace is defined by

$$\text{Sing}\, V = \Big\{ \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} c_{i_1,\ldots,i_k} F_{i_1,\ldots,i_k} \mid \sum_{j \in J} a_j\, c_{j,j_1,\ldots,j_{k-1}} = 0$$

$$\text{for all } \{j_1, \ldots, j_{k-1}\} \subset J \Big\}. \quad (6.5)$$

By Corollary 2.8, the restriction of $S^{(a)}$ to $\text{Sing}\, V$ is nondegenerate. Denote $(\text{Sing}\, V)^\perp$ the orthogonal complement to $\text{Sing}\, V$ with respect to $S^{(a)}$. Then $V = \text{Sing}\, V \oplus (\text{Sing}\, V)^\perp$. Denote $\pi : V \to \text{Sing}\, V$ the orthogonal projection.

LEMMA 6.1. — *The space* $(\text{Sing}\, V)^\perp$ *is generated by vectors*

$$\sum_{j \in J} F_{j,j_1,\ldots,j_{k-1}}, \quad (6.6)$$

labeled by subsets $\{j_1, \ldots, j_{k-1}\} \subset J.$ $\qquad \square$

For distinct $i_1, \ldots, i_k$, we define the vector $v_{i_1,\ldots,i_k} \in V$ by the formula

$$v_{i_1,\ldots,i_k} = F_{i_1,\ldots,i_k} - \sum_{m=1}^{k} \frac{a_{i_m}}{|a|} \sum_{j \in J} F_{i_1,\ldots,i_{m-1},j,i_{m+1},\ldots,i_k}. \quad (6.7)$$

We have $v_{i_{\sigma(1)},\ldots,i_{\sigma(k)}} = (-1)^\sigma v_{i_1,\ldots,i_k}$. Set $v_{i_1,\ldots,i_k} = 0$ if $i_1, \ldots, i_k$ are not distinct.

LEMMA 6.2. — *We have the following properties.*

(i) $\dim \operatorname{Sing} V = \binom{n-1}{k}$.

(ii) *For distinct* $i_1, \ldots, i_k$, *we have* $v_{i_1,\ldots,i_k} \in \operatorname{Sing} V$ *and* $v_{i_1,\ldots,i_k} = \pi(F_{i_1,\ldots,i_k})$.

(iii) *For* $\{j_1, \ldots, j_{k-1}\} \subset J$, *we have* $\sum_{j \in J} v_{j_1,\ldots,j_{k-1},j} = 0$.

(iv) *For any* $m \in J$, *the set* $v_{i_1,\ldots,i_k}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$, $m \notin \{i_1, \ldots, i_k\}$, *is a basis of* $\operatorname{Sing} V$.

$\square$

LEMMA 6.3. — *We have*

$$S^{(a)}(v_{i_1,\ldots,i_k}, v_{j_1,\ldots,j_k}) = 0, \quad \text{if } |\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_k\}| < k - 1,$$

$$S^{(a)}(v_{i_1,\ldots,i_{k-1},i_k}, v_{i_1,\ldots,i_{k-1},i_{k+1}}) = -\frac{\prod_{\ell=1}^{k+1} a_{i_\ell}}{|a|}$$

for distinct $i_1, \ldots, i_{k-1}, i_k, i_{k+1}$,

$$S^{(a)}(v_{i_1,\ldots,i_k}, v_{i_1,\ldots,i_k}) = \frac{(\sum_{\ell \notin \{i_1,\ldots,i_k\}} a_{i_\ell}) \prod_{m=1}^{k} a_{i_m}}{|a|}. \tag{6.8}$$

$\square$

Proof. — The lemma is a straightforward corollary of (6.4) and (6.7). $\square$

6.3. Operators $K_i(z) : V \to V$

For any subset $\{i_1, \ldots, i_{k+1}\} \subset J$, we define the linear operator $L_{i_1,\ldots,i_{k+1}} : V \to V$ by the formula

$$F_{i_1,\ldots,\widehat{i_m},\ldots,i_{k+1}} \mapsto (-1)^m \sum_{\ell=1}^{k+1} (-1)^\ell a_{i_\ell} F_{i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}}, \quad m = 1, \ldots, k+1, \tag{6.9}$$

$$F_{j_1,\ldots,j_k} \mapsto 0, \qquad \text{if } \{j_1, \ldots, j_k\} \text{ is not a subset of } \{i_1, \ldots, i_{k+1}\},$$

see formula (3.5). Notice that $L_{i_1,\ldots,i_{k+1}}$ does not depend on the order of $i_1, \ldots, i_{k+1}$.

We define the operators $K_i(z) : V \to V$, $i \in J$, by the formula

$$K_i(z) = \sum \frac{d_{i_1,\ldots,i_k}}{f_{i,i_1,\ldots,i_k}} L_{i,i_1,\ldots,i_k}, \tag{6.10}$$

where the sum is over all unordered subsets $\{i_1, \ldots, i_k\} \subset J - \{i\}$, see formula (3.6). For any $i \in J$ and $z \in \mathbb{C}^n - \Delta$, the operator $K_i(z)$ preserves the subspace $\operatorname{Sing} V \subset V$ and is a symmetric operator, $S^{(a)}(K_i(z)v, w) = S^{(a)}(v, K_i(z)w)$ for all $v, w \in V$, see Theorem 3.2.

LEMMA 6.4. — *We have*

$$K_{i_1}(z)v_{i_2,\dots,i_{k+1}} = \frac{d_{i_2,\dots,i_{k+1}}}{f_{i_1,i_2,\dots,i_{k+1}}} \sum_{\ell=1}^{k+1} (-1)^{\ell+1} a_{i_\ell} v_{i_1,\dots,\widehat{i_\ell},\dots,i_{k+1}},$$

$$\text{if } i_1 \notin \{i_2,\dots,i_{k+1}\},$$

$$K_{i_1}(z)v_{i_1,i_2,\dots,i_k} = -\sum_{m \notin \{i_1,\dots,i_k\}} K_{i_1}(z)v_{m,i_2,\dots,i_k}. \tag{6.11}$$

Proof. — The operator $K_i(z)$ preserve the decomposition $\operatorname{Sing} V \oplus (\operatorname{Sing} V)^\perp$. Hence

$$\begin{aligned}
K_{i_1}(z)v_{i_2,\dots,i_{k+1}} &= K_{i_1}(z)\pi(F_{i_2,\dots,i_{k+1}}) = \pi(K_{i_1}(z)F_{i_2,\dots,i_{k+1}}) \\
&= \pi\Big(\frac{d_{i_2,\dots,i_{k+1}}}{f_{i_1,i_2,\dots,i_{k+1}}} \sum_{\ell=1}^{k+1} (-1)^{\ell+1} a_{i_\ell} F_{i_1,\dots,\widehat{i_\ell},\dots,i_{k+1}}\Big) \\
&= \frac{d_{i_2,\dots,i_{k+1}}}{f_{i_1,i_2,\dots,i_{k+1}}} \sum_{\ell=1}^{k+1} (-1)^{\ell+1} a_{i_\ell} v_{i_1,\dots,\widehat{i_\ell},\dots,i_{k+1}}.
\end{aligned}$$

$\square$

The differential equations (3.14) for flat sections of the Gauss-Manin connection on $(\mathbb{C}^n - \Delta) \times \operatorname{Sing} V \to \mathbb{C}^n - \Delta$ take the form

$$\kappa \frac{\partial I}{\partial z_j}(z) = K_j(z)I(z), \qquad j \in J. \tag{6.12}$$

For generic κ all the flat sections are given by the formula

$$I_\gamma(z) = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant n} \Big(\int_{\gamma(z)} \prod_{m \in J} f_m^{a_m/\kappa} \frac{d_{i_1,\dots,i_k}}{f_{i_1}\dots f_{i_k}} \, dt_1 \wedge \dots \wedge dt_k\Big) F_{i_1,\dots,i_k}, \tag{6.13}$$

see formula (3.12). These generic κ can be determined more precisely from the determinant formula in [23].

6.4. Algebra $A_\Phi(z)$

The master function of the arrangement $\mathcal{C}(z)$ is

$$\Phi(z,t) = \sum_{j \in J} a_j \log f_j. \tag{6.14}$$

The critical point equations are

$$\frac{\partial \Phi}{\partial t_i} = \sum_{j \in J} b_j^i \frac{a_j}{f_j} = 0, \qquad i = 1,\dots,k. \tag{6.15}$$

Introduce $H_i, i = 1, \ldots, k$, by the formula

$$\frac{\partial \Phi}{\partial z_i} = \frac{H_i}{\prod_{j \in J} f_j}. \tag{6.16}$$

The critical set is

$$
\begin{aligned}
C_\Phi(z) &= \{t \in U(\mathcal{C}(z)) \mid \frac{\partial \Phi}{\partial z_i}(z, t) = 0, i = 1, \ldots, k\} = \tag{6.17} \\
&= \{t \in U(\mathcal{C}(z)) \mid H_i(z, t) = 0, i = 1, \ldots, k\}.
\end{aligned}
$$

The algebra of functions on the critical set is

$$A_\Phi(z) = \mathbb{C}(U(\mathcal{C}(z))) / \left\langle \frac{\partial \Phi}{\partial t_1}, \ldots, \frac{\partial \Phi}{\partial t_k} \right\rangle = \mathbb{C}(U(\mathcal{C}(z))) / \langle H_1, \ldots, H_k \rangle. \tag{6.18}$$

LEMMA 6.5. — *We have* $\dim A_\Phi(z) = \binom{n-1}{k}$. $\qquad\qquad\square$

Introduce elements $w_{i_1, \ldots, i_k} \in A_\Phi(z)$ by the formula

$$w_{i_1, \ldots, i_k} = a_{i_1} \ldots a_{i_k} \left[\frac{d_{i_1, \ldots, i_k}}{f_{i_1} \ldots f_{i_k}} \right]. \tag{6.19}$$

LEMMA 6.6. — *We have* $w_{i_{\sigma(1)}, \ldots, i_{\sigma(k)}} = (-1)^\sigma w_{i_1, \ldots, i_k}$ *for* $\sigma \in \Sigma_k$ *and*

$$\sum_{i \in J} w_{i_1, \ldots, i_{k-1}, i} = 0. \tag{6.20}$$

$\qquad\qquad\square$

Proof. — The lemma follows from the identity

$$\frac{a_{i_1} df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{a_{i_{k-1}} df_{i_{k-1}}}{f_{i_{k-1}}} \wedge \frac{d\Phi}{\Phi} = \tag{6.21}$$

$$= \sum_{i \in J} a_{i_1} \ldots a_{k-1} a_i \frac{d_{i_1, \ldots, i_{k-1}, i}}{f_{i_1} \ldots f_{k-1} f_i} dt_1 \wedge \cdots \wedge dt_k + \sum_{m \in J} dz_j \wedge \mu_j,$$

where μ_j are suitable $k-1$-forms. $\qquad\qquad\square$

The elements $\left[\frac{a_i}{f_i} \right]$, $i \in J$, generate $A_\Phi(z)$ by Lemma 2.4.

LEMMA 6.7. — *For* $i_1, \ldots, i_{k-1} \in J$, *we have the following identity in* $A_\Phi(z)$,

$$\sum_{i \in J} d_{i, i_1, \ldots, i_{k-1}} \left[\frac{a_i}{f_i} \right] = 0. \tag{6.22}$$

$\qquad\qquad\square$

Denote $I = \{i_1, \ldots, i_{k-1}\}$. Relation (6.22) will be called the *I-relation*.

LEMMA 6.8. — *We have in* $A_\Phi(z)$,

$$(6.23)$$

$$
\left[\frac{a_{i_1}}{f_{i_1}}\right] *_z w_{i_2,\ldots,i_{k+1}} = \frac{d_{i_2,\ldots,i_{k+1}}}{f_{i_1,i_2,\ldots,i_{k+1}}} \sum_{\ell=1}^{k+1} (-1)^{\ell+1} a_{i_\ell} w_{i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}},
$$

$$
\text{if } i_1 \notin \{i_2, \ldots, i_{k+1}\},
$$

$$
\left[\frac{a_{i_1}}{f_{i_1}}\right] *_z w_{i_1,i_2,\ldots,i_k} = - \sum_{m \notin \{i_1,\ldots,i_k\}} \left[\frac{a_{i_1}}{f_{i_1}}\right] *_z w_{m,i_2,\ldots,i_k}.
$$

$\square$

Proof. — The first formula follows from the identity

$$
\frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{df_{i_{k+1}}}{f_{i_{k+1}}} \tag{6.24}
$$

$$
= \frac{df_{i_1,\ldots,i_{k+1}}}{f_{i_{k+1}}} \wedge \sum_{m=1}^{k+1} (-1)^{m-1} \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\widehat{df_{i_m}}}{f_{i_m}} \wedge \cdots \wedge \frac{df_{i_{k+1}}}{f_{i_{k+1}}}.
$$

$\square$

LEMMA 6.9. — *Fix* $i_0 \in J$. *Then every monomial* $M = \prod_{j \in J} \left[\frac{a_j}{f_j}\right]^{s_j} \in A_\Phi(z)$ *with* $\sum_{j \in J} s_j = k$ *can be written as a linear combination of elements* $w_{i_1,\ldots,i_k}$ *where* $i_1 < \cdots < i_k$ *and* $i_0 \notin \{i_1,\ldots,i_k\}$ *with coefficients independent of* z.

Proof. — Let us write

$$
M = \left[\frac{a_{i_0}}{f_{i_0}}\right]^{\ell_{i_0}} \left[\frac{a_{j_1}}{f_{j_1}}\right]^{\ell_{j_1}} \cdots \left[\frac{a_{j_m}}{f_{j_m}}\right]^{\ell_{j_m}}, \tag{6.25}
$$

where $i_0, j_1, \ldots, j_m$ are distinct, $\ell_{i_0}, \ell_{j_1}, \ldots, \ell_{j_m}$ are positive and $\ell_{i_0} + \ell_{j_1} + \cdots + \ell_{j_m} = k$.

If $\ell_{i_0} > 0$, then let us decrease ℓ_{i_0} by one. For that let us use an *I*-relation of formula (6.22), where $I = \{p_1, \ldots, p_{k-1}\}$ is any subset which contains $j_1, \ldots, j_m$ but does not contain i_0. By using (6.22), we can write

$$
\left[\frac{a_{i_0}}{f_{i_0}}\right]^{\ell_{i_0}} = -\left[\frac{a_{i_0}}{f_{i_0}}\right]^{\ell_{i_0}-1} \Big(\sum_{i \notin \{i_0,p_1,\ldots,p_{k-1}\}} \frac{d_{i,p_1,\ldots,p_{k-1}}}{d_{i_0,p_1,\ldots,p_{k-1}}} \left[\frac{a_i}{f_i}\right] \Big). \tag{6.26}
$$

Substituting this expression into M, we will present M as a sum of monomials M' with the degree of $\left[\frac{a_{i_0}}{f_{i_0}}\right]$ equal to $\ell_{i_0} - 1$. In any monomial M'

the degrees of initial factors $\left[\frac{a_{j_s}}{f_{j_s}}\right]$ are the same and one new factor appears. Now to each of the monomials M' we will apply the same procedure until the monomial $\left[\frac{a_{i_0}}{f_{i_0}}\right]$ will not appear in each of the constructed monomial. Then we will decrease those degrees of $\ell_{j_1}, \ldots, \ell_{j_m}$ which are greater than one. In the end we will present M as a sum of monomials of the form $\left[\frac{a_{i_1}}{f_{i_1}}\right] \ldots \left[\frac{a_{i_k}}{f_{i_k}}\right]$, where $i_1 < \cdots < i_k$ and $i_0 \notin \{i_1, \ldots, i_k\}$. Such a monomial equals $\frac{1}{d_{i_1,\ldots,i_k}} w_{i_1,\ldots,i_k}$. $\qquad \square$

LEMMA 6.10. — *Fix $i_0 \in J$. Then every monomial $M = \prod_{j \in J} \left[\frac{a_j}{f_j}\right]^{s_j} \in A_\Phi(z)$ can be written as a linear combination of elements $w_{i_1,\ldots,i_k}$ where $i_1 < \cdots < i_k$ and $i_0 \notin \{i_1, \ldots, i_k\}$. If $\sum_{j \in J} s_j \neq k$, then the coefficients of the linear combination may depend on z.*

Proof. — The lemma follows from Lemmas 6.9, 6.8, 3.4. $\qquad \square$

THEOREM 6.11. — *Fix $i_0 \in J$. Then the $\binom{n-1}{k}$ elements $w_{i_1,\ldots,i_k}$ with $i_1 < \cdots < i_k$ and $i_0 \notin \{i_1, \ldots, i_k\}$, form a basis of $A_\Phi(z)$.*

Proof. — The elements $\left[\frac{a_i}{f_i}\right]$, $i \in J$, generate $A_\Phi(z)$ by Lemma 2.4. Each polynomial in $\left[\frac{a_i}{f_i}\right]$, $i \in J$, is a linear combination of $\binom{n-1}{k}$ elements $w_{i_1,\ldots,i_k}$ with $i_1 < \cdots < i_k$ and $i_0 \notin \{i_1, \ldots, i_k\}$ by Lemma 6.10. But $\dim A_\Phi(z) = \binom{n-1}{k}$. The theorem follows. $\qquad \square$

THEOREM 6.12. — *For $i_0 \in J$, the identity element $[1](z) \in A_\Phi(z)$ satisfies the equation*

$$[1](z) = \frac{1}{|a|^k} \sum_{\substack{i_1 < \cdots < i_k \\ i_0 \notin \{i_1,\ldots,i_k\}}} \frac{f_{i_0,i_1,\ldots,i_k}^k}{\prod_{m=0}^k (-1)^m d_{i_0,\ldots,\widehat{i_m},\ldots,i_k}} w_{i_1,\ldots,i_k} = \quad (6.27)$$

$$= \frac{1}{|a|^k} \sum_{\substack{i_1 < \cdots < i_k \\ i_0 \notin \{i_1,\ldots,i_k\}}} \frac{(\sum_{m=0}^k (-1)^m z_{i_m} d_{i_0,\ldots,\widehat{i_m},\ldots,i_k})^k}{\prod_{m=0}^k (-1)^m d_{i_0,\ldots,\widehat{i_m},\ldots,i_k}} w_{i_1,\ldots,i_k}.$$

Proof. — Our goal is to prove that the decomposition of

$$[1](z) = \frac{1}{|a|^k} \left(\sum_{j \in J} z_j \left[\frac{a_j}{f_j}\right] \right)^k = \frac{1}{|a|^k} \sum_{s_1 + \cdots + s_n = k} \binom{k}{s_1, \ldots, s_n} \prod_{j \in J} z_j^{s_j} \left[\frac{a_j}{f_j}\right]^{s_j}$$
$$(6.28)$$

with respect to the basis $(w_{i_1,\ldots,i_k}, i_1 < \cdots < i_k, i_0 \notin \{i_1, \ldots, i_k\})$ equals the right hand side in (6.27). For every monomial $\prod_{j \in J} z_j^{s_j}$ we need to show that $\binom{k}{s_1,\ldots,s_n} \prod_{j \in J} \left[\frac{a_j}{f_j}\right]^{s_j}$ equals the coefficient of that monomial in (6.27). For that we need to express $\prod_{j \in J} \left[\frac{a_j}{f_j}\right]^{s_j}$ as a linear combination of

basis vectors. To obtain this linear combination we will eliminate from this product the factor $\left[\frac{a_{i_0}}{f_{i_0}}\right]^{s_{i_0}}$ and will make the powers of all other factors not greater than 1. This will be done by using the I-relations of formula (6.22) like in the proof of Lemma 6.9. At every step of that simplification we will use one of the I-relations. Although the steps of this procedure are not unique, the resulting linear combination is unique. To prove that the linear combination of basis vectors representing $\binom{k}{s_1,\ldots,s_n}\prod_{j\in J}\left[\frac{a_j}{f_j}\right]^{s_j}$ equals the coefficient of $\prod_{j\in J}z_j^{s_j}$ in (6.27), we will fix an arbitrary basis vector $w_{i_1,\ldots,i_k}$ and choose a particular sequence of I-relations so that the coefficient of $w_{i_1,\ldots,i_k}$ in the decomposition of $\binom{k}{s_1,\ldots,s_n}\prod_{j\in J}\left[\frac{a_j}{f_j}\right]^{s_j}$ will be equal to the coefficient of $\prod_{j\in J}z_j^{s_j}w_{i_1,\ldots,i_k}$ in (6.27).

By comparing the coefficients of a monomial $\prod_{j\in J}z_j^{s_j}$ in (6.27) and (6.28), we observe that the coefficients have common factors $\frac{1}{|a|^k}\binom{k}{s_1,\ldots,s_k}$, so we will ignore these common factors in our next reasonings.

Before explaining the choice of the I-relations for an arbitrary pair $(\prod_{j\in J}z_j^{s_j}, w_{i_1,\ldots,i_k})$ let us consider two examples.

As the first example, we consider a monomial $M = z_{i_1}\ldots z_{i_k}$, $i_1 < \cdots < i_k$, $i_0 \notin \{i_1,\ldots z_{i_k}\}$. The coefficient of M in (6.28) is

$$\left[\frac{a_{i_1}}{f_{i_1}}\right]\cdots\left[\frac{a_{i_k}}{f_{i_k}}\right] = \frac{1}{d_{i_1,\ldots,i_k}}w_{i_1,\ldots,i_k} = \frac{1}{(-1)^0 d_{\widehat{i_0},i_1,\ldots,i_k}}w_{i_1,\ldots,i_k}$$

$$= \frac{\prod_{m=1}^k(-1)^m d_{i_0,\ldots,\widehat{i_m},\ldots,i_k}}{\prod_{m=0}^k(-1)^m d_{i_0,\ldots,\widehat{i_m},\ldots,i_k}}w_{i_1,\ldots,i_k}, \tag{6.29}$$

which is the coefficient of M in (6.27).

As the second example we consider a monomial $M = z_{i_0}z_{j_i}\ldots z_{j_{k-1}}$, where $i_0, j_i,\ldots,j_{k-1}$ are distinct. The monomial M appears in (6.27) in the coefficient of a basis vector $w_{i_1,\ldots,i_k}$ if $\{j_1,\ldots,j_{k-1}\} \subset \{i_1,\ldots,i_k\}$. So we may assume that for some $1 \leqslant \ell \leqslant k$, we have $M = z_{i_0}\prod_{\substack{m=1\\m\neq\ell}}^k z_{i_m}$. In (6.28), the coefficient of M is $\left[\frac{a_{i_0}}{f_{i_0}}\right]\prod_{\substack{m=1\\m\neq\ell}}^k\left[\frac{a_{i_m}}{f_{i_m}}\right]$. By using the I-relation for $I = \{\widehat{i_0},i_1,\ldots,\widehat{i_\ell},\ldots,i_k\}$, we transform it into

$$-\prod_{\substack{m=1\\m\neq\ell}}^k\left[\frac{a_{i_m}}{f_{i_m}}\right]\sum_{m\notin\{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_k\}}\frac{d_{m,i_1,\ldots,\widehat{i_\ell},\ldots,i_k}}{d_{i_0,i_1,\ldots,\widehat{i_\ell},\ldots,i_k}}\left[\frac{a_m}{f_m}\right]. \tag{6.30}$$

We choose the summand in (6.30) corresponding to $m = i_\ell$. This summand is

$$-\prod_{\substack{m=1 \\ m \neq \ell}}^{k} \left[\frac{a_{i_m}}{f_{i_m}}\right] \frac{d_{i_\ell, i_1, \ldots, \widehat{i_\ell}, \ldots, i_k}}{d_{i_0, i_1, \ldots, \widehat{i_\ell}, \ldots, i_k}} \left[\frac{a_{i_\ell}}{f_{i_\ell}}\right] = \prod_{m=1}^{k} \left[\frac{a_{i_m}}{f_{i_m}}\right] \frac{d_{i_1, \ldots, i_\ell, \ldots, i_k}}{(-1)^\ell d_{i_0, i_1, \ldots, \widehat{i_\ell}, \ldots, i_k}}$$

$$= \frac{\prod_{\substack{m=0 \\ m \neq \ell}}^{k} (-1)^m d_{i_0, \ldots, \widehat{i_m}, \ldots, i_k}}{\prod_{m=0}^{k} (-1)^m d_{i_0, \ldots, \widehat{i_m}, \ldots, i_k}} w_{i_1, \ldots, i_\ell, \ldots, i_k},$$

which is the coefficient of $M w_{i_1, \ldots, i_\ell, \ldots, i_k}$ in (6.27).

Now let M be an arbitrary monomial of degree k in variables $z_1, \ldots, z_n$. The monomial M appears in (6.27) in the coefficient of a vector $w_{i_1, \ldots, i_k}$ if there is a subset $\{p_1, \ldots, p_r\} \subset \{1, \ldots, k\}$ such that $M = z_{i_0}^{s_0} \prod_{m=1}^{r} z_{i_{p_m}}^{s_m}$, $\sum_{m=0}^{r} s_m = k$, and all numbers $s_1, \ldots, s_r$ are positive. Denote $\{q_1, \ldots, q_{k-r}\} = \{1, \ldots, k\} - \{p_1, \ldots, p_r\}$, the complement.

In (6.28), the coefficient of M is $P = \left[\frac{a_{i_0}}{f_{i_0}}\right]^{s_0} \prod_{m=1}^{r} \left[\frac{a_{i_{p_m}}}{f_{i_{p_m}}}\right]^{s_m}$. To express this product as a linear combination of the basis vectors we need to apply to this product I-relations $k - r$ times. To calculate the coefficient of $w_{i_1, \ldots, i_k}$ we first apply the I-relation with $I = \{\widehat{i_0}, i_1, \ldots, \widehat{i_{q_1}}, \ldots, i_k\} \subset \{i_0, \ldots, i_k\}$ and decrease the degree of $\left[\frac{a_{i_0}}{f_{i_0}}\right]$ by 1,

$$P = -\left[\frac{a_{i_0}}{f_{i_0}}\right]^{s_0-1} \prod_{m=1}^{r} \left[\frac{a_{i_{p_m}}}{f_{i_{p_m}}}\right]^{s_m} \sum_{m \notin \{i_0, \ldots, \widehat{i_{q_1}}, \ldots, i_k\}} \frac{d_{m, \widehat{i_0}, i_1, \ldots, \widehat{i_{q_1}}, \ldots, i_k}}{d_{i_0, \widehat{i_0}, i_1, \ldots, \widehat{i_{q_1}}, \ldots, i_k}} \left[\frac{a_m}{f_m}\right].$$

In the next steps we will simplify further the first factors of this expression. After the future simplifications the only term of this sum that can give $w_{i_1, \ldots, i_k}$ is the term with $m = i_{q_1}$, which is

$$-\left[\frac{a_{i_0}}{f_{i_0}}\right]^{s_0-1} \prod_{m=1}^{r} \left[\frac{a_{i_{p_m}}}{f_{i_{p_m}}}\right]^{s_m} \frac{d_{i_{q_1}, \widehat{i_0}, i_1, \ldots, \widehat{i_{q_1}}, \ldots, i_k}}{d_{i_0, \widehat{i_0}, i_1, \ldots, \widehat{i_{q_1}}, \ldots, i_k}} \left[\frac{a_{i_{q_1}}}{f_{i_{q_1}}}\right] \qquad (6.31)$$

$$= \left[\frac{a_{i_0}}{f_{i_0}}\right]^{s_0-1} \prod_{m=1}^{r} \left[\frac{a_{i_{p_m}}}{f_{i_{p_m}}}\right]^{s_m} \frac{d_{\widehat{i_0}, i_1, \ldots, i_k}}{(-1)^{q_1} d_{i_0, i_1, \ldots, \widehat{i_{q_1}}, \ldots, i_k}} \left[\frac{a_{i_{q_1}}}{f_{i_{q_1}}}\right].$$

We will call this term the *main term*. Now we apply the I-relation with $I = \{\widehat{i_0}, i_1, \ldots, \widehat{i_{q_2}}, \ldots, i_k\} \subset \{i_0, \ldots, i_k\}$ and again decrease the degree of $\left[\frac{a_{i_0}}{f_{i_0}}\right]$ by one. After the second step the only term of the obtained sum that may produce the vector $w_{i_1, \ldots, i_k}$ is the term

$$\left[\frac{a_{i_0}}{f_{i_0}}\right]^{s_0-1} \prod_{m=1}^{r} \left[\frac{a_{i_{p_m}}}{f_{i_{p_m}}}\right]^{s_m} \frac{d_{\widehat{i_0}, i_1, \ldots, i_k}}{(-1)^{i_{q_1}} d_{i_0, i_1, \ldots, \widehat{i_{q_1}}, \ldots, i_k}} \left[\frac{a_{i_{q_1}}}{f_{i_{q_1}}}\right] \frac{d_{\widehat{i_0}, i_1, \ldots, i_k}}{(-1)^{q_2} d_{i_0, i_1, \ldots, \widehat{i_{q_2}}, \ldots, i_k}} \left[\frac{a_{i_{q_2}}}{f_{i_{q_2}}}\right].$$

This will be our *main term* after two steps of the simplifying procedure. We will repeat this procedure to kill all factors $\left[\frac{a_{i_0}}{f_{i_0}}\right]$. After s_{i_0} steps the *main term* will be

$$\prod_{m=1}^{r}\left[\frac{a_{i_{p_m}}}{f_{i_{p_m}}}\right]^{s_m}\prod_{m=1}^{s_0}\frac{d_{\widehat{i_0},i_1,\ldots,i_k}}{(-1)^{q_m}d_{i_0,i_1,\ldots,\widehat{i_{q_m}},\ldots,i_k}}\left[\frac{a_{i_{q_m}}}{f_{i_{q_2}}}\right].$$

Now we will apply the I-relation with $I=\{i_0,\ldots,\widehat{i_{p_1}},\ldots,\widehat{i_{q_{s_0+1}}},\ldots,i_k\}\subset\{i_0,\ldots,i_k\}$ and decrease the degree of $\left[\frac{a_{i_{p_1}}}{f_{i_{p_1}}}\right]$ by one. After this procedure the main term will be

$$\left[\frac{a_{i_{p_1}}}{f_{i_{p_1}}}\right]^{s_1-1}\prod_{m=2}^{r}\left[\frac{a_{i_{p_m}}}{f_{i_{p_m}}}\right]^{s_m}\prod_{m=1}^{s_0}\frac{d_{\widehat{i_0},i_1,\ldots,i_k}}{(-1)^{q_m}d_{i_0,i_1,\ldots,\widehat{i_{q_m}},\ldots,i_k}}\left[\frac{a_{i_{q_m}}}{f_{i_{q_2}}}\right]$$
$$\times\frac{(-1)^{p_1}d_{i_0,i_1,\ldots,\widehat{i_{p_1}},\ldots,i_k}}{(-1)^{q_{s_0+1}}d_{i_0,i_1,\ldots,\widehat{i_{q_{s_0+1}}},\ldots,i_k}}\left[\frac{a_{i_{q_{s_0+1}}}}{f_{i_{q_{s_0+1}}}}\right].$$

Now we will be decreasing the degree of $\left[\frac{a_{j_1}}{f_{j_1}}\right]$ to make it 1. Then we will continue this procedure of simplification, which will end with the main term

$$\left(d_{\widehat{i_0},i_1,\ldots,i_k}\prod_{m=1}^{k}\left[\frac{a_{i_m}}{f_{i_m}}\right]\right)\frac{((-1)^0d_{\widehat{i_0},i_1,\ldots,i_k})^{s_0}\prod_{m=1}^{r}((-1)^{p_m}d_{i_0,i_1,\ldots,\widehat{i_{p_m}},\ldots,i_k})^{s_m}}{\prod_{m=0}^{k}(-1)^md_{i_0,\ldots,\widehat{i_m},\ldots,i_k}}.$$

After replacing the first factor $d_{\widehat{i_0},i_1,\ldots,i_k}\prod_{m=1}^{k}\left[\frac{a_{i_m}}{f_{i_m}}\right]$ with $w_{i_1,\ldots,i_k}$ we observe that the second factor equals the coefficient of $Mw_{i_1,\ldots,i_k}$ in (6.27). The theorem is proved. $\square$

6.5. Canonical isomoprhism

The set of vectors $v_{i_1,\ldots,i_k},1<i_1<\cdots<i_k\leqslant n$, is a basis of $\operatorname{Sing}V$, by Lemma 6.2. For $z\in\mathbb{C}^n-\Delta$, the set of vectors $w_{i_1,\ldots,i_k},1<i_1<\cdots<i_k\leqslant n$, is a basis of $A_\Phi(z)$, by Theorem 6.11.

THEOREM 6.13. — *For $z\in\mathbb{C}^n-\Delta$, the matrix of the canonical isomorphism $\alpha(z):A_\Phi(z)\to\operatorname{Sing}V$ with respect to these bases does not depend on z.*

Proof. — For $1<m_1<\cdots<m_k\leqslant n$ and $1\leqslant i_1<\cdots<i_k\leqslant n$ denote

$$B_{i_1,\ldots,i_k}=\sum_{p\in C_\Phi(z)}\operatorname{Res}_p\frac{1}{\prod_{j=1}^{k}f_{m_j}}\frac{1}{\prod_{j=1}^{k}f_{i_j}}\frac{\prod_{\ell\in J}f_\ell^k}{\prod_{i=1}^{k}H_i}.\qquad(6.32)$$

Then

$$\alpha(z)(w_{m_1,\ldots,m_k}) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \frac{d_{m_1,\ldots,m_k} d_{i_1,\ldots,i_k} \prod_{j=1}^{k} a_{m_j}}{(2\pi\sqrt{-1})^k} B_{i_1,\ldots,i_k} F_{i_1,\ldots,i_k},$$
(6.33)

see formulas (2.19), (2.22). In order to prove the theorem we need to show that every $B_{i_1,\ldots,i_k}$ does not depend on z.

The differential form

$$\omega = \frac{1}{\prod_{j=1}^{k} f_{m_j}} \frac{1}{\prod_{j=1}^{k} f_{i_j}} \frac{\prod_{\ell \in J} f_\ell^k}{\prod_{i=1}^{k} H_i} \, dt_1 \wedge \cdots \wedge dt_k$$
(6.34)

has poles only on the hypersurfaces $H_i = 0$, $i = 1, \ldots, k$. The poles are of first order. To calculate $B_{i_1,\ldots,i_k}$, we need to take the residue $\psi = \mathrm{Res}_{H_i=0, i=1,\ldots,k-1} \omega$ of the form ω at the curve $\mathcal{C} = \{H_i = 0, i = 1, \ldots, k-1\}$ and then take the residue of the form ψ on the curve $\mathcal{C}$ at the points where $H_k = 0$. This is the same as if we took with minus sign the residue at infinity of the form ψ on the curve $\mathcal{C}$. That residue at infinity (up to sign) can be obtained differently in two steps. First we may take the residue of ω at the hyperplane at infinity (denote the residue by φ) and then take the residue of φ at the points of the set $\{H_i = 0, i = 1, \ldots, k-1\}$.

So to calculate $B_{i_1,\ldots,i_k}$ we first calculate φ. The coordinates at infinity are $u_1 = t_1/t_k$, $\ldots$, $u_{k-1} = t_{k-1}/t_k$, $u_k = 1/t_k$. We have $f_m = (b_m^1 u_1 + \cdots + b_m^{k-1} u_{k-1} + b_m^k + z_m u_k)/u_k$. Denote $\tilde{f}_m(u_1, \ldots, u_{k-1}) = b_m^1 u_1 + \cdots + b_m^{k-1} u_{k-1} + b_m^k$. For $i = 1, \ldots, k$, we have $H_i(u_1/u_k, \ldots, u_{k-1}/u_k, 1/u_k) = \hat{H}_i(u_1, \ldots, u_{k-1}, u_k)/u_k^{n-1}$, where $\hat{H}_i(u_1, \ldots, u_{k-1}, u_k)$ is a polynomial. Denote $\tilde{H}_i(u_1, \ldots, u_{k-1}) = \hat{H}_i(u_1, \ldots, u_{k-1}, 0)$. The polynomial $\tilde{H}_i(u_1, \ldots, u_{k-1})$ does not depend on z.

We have $dt_1 \wedge \cdots \wedge dt_k = -\frac{1}{u_k^{k+1}} du_1 \wedge \cdots \wedge du_k$. By counting all orders of u_k in factors of ω we conclude that the form ω has the first order pole at the hyperplane at infinity. The residue φ of ω at the infinite hyperplane equals

$$\pm 2\pi\sqrt{-1} \frac{1}{\prod_{j=1}^{k} \tilde{f}_{m_j}} \frac{1}{\prod_{j=1}^{k} \tilde{f}_{i_j}} \frac{\prod_{\ell \in J} \tilde{f}_\ell^k}{\prod_{i=1}^{k} \tilde{H}_i} \, du_1 \wedge \cdots \wedge du_{k-1}.$$
(6.35)

This form does not depend on z. Now we are supposed to take the sum of residues of φ at the points of the set $\{\tilde{H}_i = 0, i = 1, \ldots, k-1\}$ and the polynomials $\tilde{H}_i$ also do not depend on z. Hence $B_{i_1,\ldots,i_k}$ does not depend on z. The theorem is proved. $\square$

Define the *naive isomorphism* $\nu(z) : A_\Phi(z) \to \operatorname{Sing} V$ by the formula

$$\nu(z) : w_{i_1,\ldots,i_k} \mapsto v_{i_1,\ldots,i_k} \tag{6.36}$$

for all $i_1, \ldots, i_k \in J$.

LEMMA 6.14. — *The map $\nu(z)$ is an isomorphism of vector spaces and for every $i \in J$ and $w \in A_\Phi(z)$ we have*

$$\nu(z)\left[\frac{a_i}{f_i}\right] *_z w = K_i(z)\nu(z)(w). \tag{6.37}$$

Proof. — The map $\nu(z)$ is an isomorphism by Lemma 6.2 and Theorem 6.11. Formula (6.37) holds by Lemmas 6.4 and 6.8. $\qquad\Box$

Introduce the linear isomorphism

$$\zeta = \alpha(z)\nu(z)^{-1} : \operatorname{Sing} V \to \operatorname{Sing} V. \tag{6.38}$$

By Theorem 6.13 and Lemma 6.14 the isomorphism ζ does not depend on z. By Theorems 6.13 and 3.5 the isomorphism ζ commutes with the action of operators $K_i(z)$ for all $i \in J$ and $z \in \mathbb{C}^n - \Delta$, $[K_i(z), \zeta] = 0$.

THEOREM 6.15. — *The isomorphism ζ is a scalar operator.*

Proof. — By Lemma 4.3 in [27], the eigenvalues of the operators $K_i(z)$ separate the eigenvectors. The theorem follows from the fact that the operators $K_i(z)$ have too many eigenvectors, and ζ must preserve all of them. More precisely, let $i_1, \ldots, i_{k+1} \in J$ be distinct. Assume that $z \in \mathbb{C}^n - \Delta$ tends to a generic point z^0 of the hyperplane defined by the equation $f_{i_1,\ldots,i_{k+1}} = 0$. It follows from Lemma 6.4 that the vector

$$x_{i_1,\ldots,i_{k+1}} = \sum_{\ell=1}^{k+1}(-1)^{\ell+1}a_{i_\ell}v_{i_1,\ldots,\widehat{i_\ell},\ldots,i_{k+1}} \in \operatorname{Sing} V \tag{6.39}$$

is the limit of an eigenvector of operators $K_i(z)$ as $z \to z^0$. Hence, $x_{i_1,\ldots,i_{k+1}}$ is an eigenvector of ζ. It is easy to see that the vectors $x_{i_1,\ldots,i_{k+1}}$ generate $\operatorname{Sing} V$ and for distinct $i_1, \ldots, i_{k+2}$ we have

$$\sum_{\ell=1}^{k+2}(-1)^{\ell}a_\ell x_{i_1,\ldots,\widehat{i_\ell},\ldots,i_{k=2}} = 0. \tag{6.40}$$

This equation implies that ζ is a scalar operator on the subspace generated by the vectors $x_{i_1,\ldots,\widehat{i_\ell},\ldots,i_{k=2}}$, $\ell = 1, \ldots, k+2$, and this fact implies that ζ is a scalar operator on $\operatorname{Sing} V$. $\qquad\Box$

COROLLARY 6.16. — *There exists $c \in \mathbb{C}^\times$ such that $\alpha(z) = c\,\nu(z)$, that is,*

$$\alpha(z) : w_{i_1,\dots,i_k} \mapsto c\,v_{i_1,\dots,i_k} \tag{6.41}$$

for all $i_1,\dots,i_k \in J$. $\qquad\qquad\square$

One may expect that $c = (-1)^k$, see Theorems 4.8 and 5.15*.

The canonical isomorphism $\alpha(z)$ induces an algebra structure on $\mathrm{Sing}\,V$ depending on $z \in \mathbb{C}^n - \Delta$.

COROLLARY 6.17. — *For any $i_0 \in J$, the identity element $\{1\}(z)$ of that algebra structure satisfies the equation*

$$\{1\}(z) = \frac{c}{|a|^k} \sum_{\substack{i_1 < \cdots < i_k \\ i_0 \notin \{i_1,\dots,i_k\}}} \frac{f^k_{i_0,i_1,\dots,i_k}}{\prod_{m=0}^{k}(-1)^m d_{i_0,\dots,\widehat{i_m},\dots,i_k}} v_{i_1,\dots,i_k}, \tag{6.42}$$

where c is defined in Corollary 6.16. $\qquad\qquad\square$

THEOREM 6.18. — *Conjectures 3.7 and 3.8 hold for this family of arrangements.*

Proof. — Conjecture 3.8 is a direct corollary of Theorem 6.13.

LEMMA 6.19. — *For $r \leqslant k$ and $m_1,\dots,m_r \in J$, we have*

$$\frac{\partial^r \{1\}}{\partial z_{m_1}\dots\partial z_{m_r}}(z) = \frac{k(k-1)\dots(k-r+1)}{|a|^r}\,\alpha(z)\Big(\prod_{i=1}^{r}\Big[\frac{a_{m_i}}{f_{m_i}}\Big]\Big). \tag{6.43}$$

Proof. — The proof is by induction on r. For $r = 0$, the statement is true: $\{1\} = \{1\}$. Assuming the statement is true for some r, we prove the statement for $r+1$. We have

$$\frac{\partial}{\partial z_j}\frac{\partial^r\{1\}}{\partial z_{m_1}\dots\partial z_{m_r}}(z) = \frac{\partial}{\partial z_j}\frac{k(k-1)\dots(k-r+1)}{|a|^r}\,\alpha(z)\Big(\prod_{i=1}^{r}\Big[\frac{a_{m_i}}{f_{m_i}}\Big]\Big)$$

$$= \frac{\partial}{\partial z_j}\frac{k(k-1)\dots(k-r+1)}{|a|^r}\,\alpha(z)\Big(\prod_{i=1}^{r}\Big[\frac{a_{m_i}}{f_{m_i}}\Big]\frac{1}{|a|^{k-r}}\Big(\sum_{i\in J}z_i\Big[\frac{a_i}{f_i}\Big]\Big)^{k-r}\Big)$$

$^{(*)}$ $c = (-1)^k$ by [29, Theorem 2.16].

$$= \frac{k-r}{|a|} \alpha(z) \left(\left[\frac{a_j}{f_j} \right] \right) *_z \frac{k(k-1) \dots (k-r+1)}{|a|^r} \alpha(z) \left(\prod_{i=1}^{r} \left[\frac{a_{m_i}}{f_{m_i}} \right] \right.$$

$$\times \frac{1}{|a|^{k-r-1}} \left(\sum_{i \in J} z_i \left[\frac{a_i}{f_i} \right] \right)^{k-r-1} \right)$$

$$= \frac{k(k-1) \dots (k-r+1)(k-r)}{|a|^{r+1}} \alpha(z) \left(\left[\frac{a_j}{f_j} \right] \prod_{i=1}^{r} \left[\frac{a_{m_i}}{f_{m_i}} \right] \right).$$

$\square$

Let us finish the proof of Conjecture 3.7. We have

$$\frac{\partial}{\partial z_j} \frac{\partial^r \{1\}}{\partial z_{m_1} \dots \partial z_{m_r}}(z)$$

$$= \frac{k-r}{|a|} \alpha(z) \left(\left[\frac{a_j}{f_j} \right] \right) *_z \frac{k(k-1) \dots (k-r+1)}{|a|^r} \alpha(z) \left(\prod_{i=1}^{r} \left[\frac{a_{m_i}}{f_{m_i}} \right] \right)$$

$$= \frac{k-r}{|a|} K_j(z) \frac{\partial^r \{1\}}{\partial z_{m_1} \dots \partial z_{m_r}}(z).$$

$\square$

Introduce the *potential function of second kind*

$$\tilde{P}(z_1, \dots, z_n) \tag{6.44}$$

$$= \frac{c^2}{(2k)!} \sum_{1 \leqslant i_1 < \dots < i_{k+1} \leqslant n} \frac{\prod_{\ell=1}^{k+1} a_{i_\ell}}{\prod_{\ell=1}^{k+1} d^2_{i_1, \dots, \widehat{i_\ell}, \dots, i_{k+1}}} f^{2k}_{i_1, \dots, i_{k+1}} \log f_{i_1, \dots, i_{k+1}},$$

where c is the constant defined in Corollary 6.16.

THEOREM 6.20. — *For any $m_0, \dots, m_{2k} \in J$, we have*

$$\frac{\partial^{2k+1} \tilde{P}}{\partial z_{m_0} \dots \partial z_{m_{2k}}}(z) = (-1)^k \left(\left[\frac{a_{m_0}}{f_{m_0}} \right] *_z \dots *_z \left[\frac{a_{m_{2k}}}{f_{m_{2k}}} \right], [1](z) \right)_z. \tag{6.45}$$

Theorem 6.20 proves Conjecture 3.14 for this family of arrangements.

If $m_1, \dots, m_k$ are distinct and $m_{k+1}, \dots, m_{2k}$ are distinct, equation (6.45) takes the form

$$c^2 S^{(a)}(K_{m_0}(z) v_{m_1, \dots, m_k}, v_{m_{k+1}, \dots, m_{2k}}) \tag{6.46}$$

$$= d_{m_1, \dots, m_k} d_{m_{k+1}, \dots, m_{2k}} \frac{\partial^{2k+1} \tilde{P}}{\partial z_{m_0} \dots \partial z_{m_{2k}}}(z).$$

COROLLARY 6.21. — *The matrix elements of the operators $K_i(z)$ with respect to the (combinatorially constant) vectors $v_{i_1,\ldots,i_k}$ are described by the $(2k+1)$-st derivatives of the potential function of second kind.*

Notice that

$$c^2 S^{(a)}(v_{m_1,\ldots,m_k}, v_{m_{k+1},\ldots,m_{2k}})$$

$$= d_{m_1,\ldots,m_k} d_{m_{k+1},\ldots,m_{2k}} \frac{|a|^{2k}}{(2k)!} \frac{\partial^{2k} P}{\partial z_{m_1} \ldots \partial z_{m_{2k}}}(z), \qquad (6.47)$$

where $P(z)$ is the potential function of first kind, see Theorem 3.11.

Proof. — We have the I-relation $\sum_{j \in J} d_{j,i_1,\ldots,i_{k-1}} \left[\frac{a_j}{f_j}\right] = 0$ for any $i_1, \ldots, i_{k-1} \in J$, see (6.22), and the relation

$$\sum_{j \in J} d_{i,i_1,\ldots,i_{k-1}} \frac{\partial}{\partial z_j} \frac{\partial^{2k} \tilde{P}}{\partial z_{m_1} \ldots \partial z_{m_{2k}}}(z) = 0 \qquad (6.48)$$

for any $m_1, \ldots, m_{2k}, i_1, \ldots, i_{k-1} \in J$. By using these two relations and by reordering the set J if necessary, we can reduce formula (6.45) to the case in which $(m_1, \ldots, m_k)$ are distinct, $(m_{k+1}, \ldots, m_{2k})$ are distinct, and $m_0 \notin \{m_1, \ldots, m_k\}$. After that we need to check identity (6.46). That is done by direct calculation of the left and right hand sides, cf. the proof of Theorem 5.27.

For example, the most difficult case is if $(m_0, \ldots, m_{2k}) = (k+1, 1, \ldots, k, 1, \ldots, k)$. Then

$$\frac{\partial^{2k+1} \tilde{P}}{\partial z_{m_0} \ldots \partial z_{m_{2k}}}(z) = c^2 \frac{\prod_{m=1}^{k+1} a_m}{(-1)^k d_{1,\ldots,k} f_{1,\ldots,k+1}} \qquad (6.49)$$

and

$$(-1)^k \left(\left[\frac{a_{m_0}}{f_{m_0}}\right] *_z \cdots *_z \left[\frac{a_{m_{2k}}}{f_{m_{2k}}}\right], [1](z) \right)_z$$

$$= c^2 \frac{1}{d_{1,\ldots,k}^2} S^{(a)}(K_{k+1}(z) v_{1,\ldots,k}, v_{1,\ldots,k})$$

$$= c^2 \frac{1}{d_{1,\ldots,k} f_{k+1,1,\ldots,k}} S^{(a)}\left(a_{k+1} v_{1,\ldots,k} + \sum_{\ell=1}^{k}(-1)^\ell a_\ell v_{k+1,1,\ldots,\hat{\ell},\ldots,k}, v_{1,\ldots,k}\right)$$

$$= c^2 \frac{1}{d_{1,\ldots,k} f_{k+1,1,\ldots,k}} \left(\frac{\prod_{m=1}^{k+1} a_m}{|a|} \sum_{\ell \notin \{1,\ldots,k\}} a_\ell + \frac{\prod_{m=1}^{k+1} a_m}{|a|} \sum_{\ell \in \{1,\ldots,k\}} a_\ell \right)$$

$$= c^2 \frac{1}{d_{1,\ldots,k} f_{k+1,1,\ldots,k}} \prod_{m=1}^{k+1} a_m. \qquad (6.50)$$

$\square$

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